SEMIGROUPS OF ORDER-DECREASING TRANSFORMATIONS

Abdullahi Umar

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IN THE NAME OF ALLAH, MOST MERCIFUL, MOST COMPASSIONATE

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I declare that the accompanying thesis has been composed by myself and that it is a record of my own work. No part of this thesis has been accepted in any previous application for a higher degree.

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JOHN M. HOWIE

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ABSTRACT

Let X be a totally ordered set and consider the semigroups of order-decreasing(increasing) full (partial, partial one-to-one) transformations of X. In this Thesis the study of order-increasing full (partial, partial one-to-one) transformations has been reduced to that of order-decreasing full (partial, partial one-to-one) transformations and the study of order-decreasing partial transformations to that of order-decreasing full transformations for both the finite and infinite cases.

For the finite order-decreasing full (partial one-to-one) transformation semigroups, we obtain results analogous to Howie (1971) and Howie and McFadden (1990) concerning products of idempotents (quasi-idempotents), and concerning combinatorial and rank properties. By contrast with the semigroups of order-preserving transformations and the full transformation semigroup, the semigroups of order-decreasing full (partial one-to-one) transformations and their Rees quotient semigroups are not regular. They are, however, abundant (type A) semigroups in the sense of Fountain (1982, 1979). An explicit characterisation of the minimum semilattice congruence on the finite semigroups of order-decreasing transformations and their Rees quotient semigroups is obtained.

If X is an infinite chain then the semigroup S of order-decreasing full transformations need not be abundant. A necessary and sufficient condition on X is obtained for S to be abundant. By contrast, for every chain X the semigroup of order-decreasing partial one-to-one transformations is type A.

The ranks of the nilpotent subsemigroups of the finite semigroups of order-decreasing full (partial one-to-one) transformations have been investigated.

GLOSSARY OF NOTATION

$\langle A \rangle$	the semigroup generated by A
A^*	the upper saturation of A
$\aleph_0$	the smallest infinite cardinal
AQE(S)	the set of amenable elements of S
B_n	the nth Bell's exponential number
$\mathcal{D}$	Green's relation defined by $\mathcal{L}$ join $\mathcal{R}$
$\mathcal{D}^*$	starred Green's relation defined by $\mathcal{L}^*$ join $\mathcal{R}^*$
dom α	the domain of the mapping α
E(S)	the set of idempotents of S
E(n, r)	the two-sided ideal of $E(I_n)$ consisting of elements α for which $\lvert \mathrm{im}\, \alpha \rvert \leq r$
EP(n, r)	the Rees quotient semigroup $E(n, r) / E(n, r-1)$
$F(\alpha)$	the set of fixed points of the mapping α
$f(\alpha)$	the cardinal of $F(\alpha)$
$\mathcal{H}$	Green's relation defined by $\mathcal{L} \cap \mathcal{R}$
$\mathcal{H}^*$	starred Green's relation defined by $\mathcal{L}^* \cap \mathcal{R}^*$
id_A	the partial identity on A
idrank S	the idempotent rank of S
$IJ^*(n, r)$	the cardinal of J_r^* in $(I_n^-)^1$
im α	the image set or range of the mapping α
I_n	the symmetric inverse semigroup of X_n
I(X)	the symmetric inverse semigroup of X
I_n^-	the semigroup of order-decreasing partial one-to-one maps of X_n
I_n^+	the semigroup of order-increasing partial one-to-one maps of X_n
$\mathcal{J}$	Green's relation defined by equality of principal ideals
$\mathcal{J}^*$	starred Green's relation defined by equality of principal *-ideals

$J^*(a)$	the principal *-ideal generated by a
$J^*(n, r)$	the cardinal of J_r^* in $(S_n^-)^1$
J_r^*	the $\mathcal{J}^*$-class consisting of elements α for which $\lvert\mathrm{im}\ \alpha\rvert = r$
$K(n, r)$	the two-sided ideal of T_n consisting of elements α for which $\lvert\mathrm{im}\ \alpha\rvert \le r$
$K^-(n, r)$	the two-sided ideal of S_n^- consisting of elements α for which $\lvert\mathrm{im}\ \alpha\rvert \le r$
$K^+(n, r)$	the two-sided ideal of S_n^+ consisting of elements α for which $\lvert\mathrm{im}\ \alpha\rvert \le r$
$K(\lvert X\rvert, \xi)$	the two-sided ideal of $T(X)$ consisting of elements α for which $\lvert\mathrm{im}\ \alpha\rvert \le \xi$
$K^-(\lvert X\rvert, \xi)$	the two-sided ideal of $S^-(X)$ consisting of elements α for which $\lvert\mathrm{im}\ \alpha\rvert \le \xi$
$\mathcal{L}$	Green's relation defined by equality of principal left ideals
$\mathcal{L}^*$	starred Green's relation defined by equality of principal left *-ideals
L_α^*	the $\mathcal{L}^*$-class containing α
$L(n, r)$	the two-sided ideal of I_n consisting of elements α for which $\lvert\mathrm{im}\ \alpha\rvert \le r$
$L^-(n, r)$	the two-sided ideal of I_n^- consisting of elements α for which $\lvert\mathrm{im}\ \alpha\rvert \le r$
$L^+(n, r)$	the two-sided ideal of I_n^+ consisting of elements α for which $\lvert\mathrm{im}\ \alpha\rvert \le r$
$L(\lvert X\rvert, \xi)$	the two-sided ideal of $I(X)$ consisting of elements α for which $\lvert\mathrm{im}\ \alpha\rvert \le \xi$
$L^-(\lvert X\rvert, \xi)$	the two-sided ideal of $I^-(X)$ consisting of elements α for which $\lvert\mathrm{im}\ \alpha\rvert \le \xi$
$\mathbb{N}$	the set of natural numbers
O_n	the semigroup of order-preserving maps of X_n
$PJ^*(n, r)$	the cardinal of J_r^* in $(P_n^-)^1$
$PK(n, r)$	the two-sided ideal of P_n consisting of elements α for which $\lvert\mathrm{im}\ \alpha\rvert \le r$
$PK^-(n, r)$	the two-sided ideal of P_n^- consisting of elements α for which $\lvert\mathrm{im}\ \alpha\rvert \le r$

$PK^+(n, r)$	the two-sided ideal of P_n^+ consisting of elements α for which $\lvert\text{im } \alpha\rvert \le r$
P_n	the partial transformation semigroup of X_n
PP_r	the Rees quotient semigroup $PK(n, r) / PK(n, r-1)$
PP_r^-	the Rees quotient semigroup $PK^-(n, r) / PK^-(n, r-1)$
$P(X)$	the partial transformation semigroup of X
$P(\xi_i)$	the Rees quotient semigroup $K(\lvert X\rvert, \xi_i) / K(\lvert X\rvert, \xi_{i-1})$
$q(n, r)$	the cardinal of $AQE(L^-(n, r))$
Q_r	the Rees quotient semigroup $L(n, r) / L(n, r-1)$
Q_r^-	the Rees quotient semigroup $L^-(n, r) / L^-(n, r-1)$
quaidrank S	the quasi-idempotent rank of S
$Q(\xi_i)$	the Rees quotient semigroup $L(\lvert X\rvert, \xi_i) / L(\lvert X\rvert, \xi_{i-1})$
$\mathbb{R}$	the set of real numbers
$\mathcal{R}$	Green's relation defined by equality of principal right ideals
$\mathcal{R}^*$	starred Green's relation defined by equality of principal right *- ideals
R_α^*	the $\mathcal{R}^*$-class containing α
rank S	the rank of S
S	a semigroup
S^1	a semigroup with identity
$S(\alpha)$	the set of shifting points of the mapping α
$s(\alpha)$	the cardinal of $S(\alpha)$
$sh(n, r)$	the cardinal of the set $\{\alpha \in (S_n^-)^1 : s(\alpha) = r-1\}$
$\underline{sh}(n, r)$	the cardinal of the set $\{\alpha \in (P_n^-)^1 : s(\alpha) = r\}$
$\lvert s(n, r)\rvert$	the signless or absolute Stirling number of the first kind
$Sing_n$	the subsemigroup of singular elements of T_n
S_n^-	the subsemigroup of order-decreasing elements of $Sing_n$
S_n^+	the subsemigroup of order-increasing elements of $Sing_n$
$S(n, r)$	the Stirling number of the second kind
$S^-(X)$	the subsemigroup of order-decreasing elements of $T(X)$
$S^+(X)$	the subsemigroup of order-increasing elements of $T(X)$

T_n	the full transformation semigroup of X_n
T_n^0	the full transformation semigroup of X_n^0
$T(X)$	the full transformation semigroup of X
X	a totally ordered set
$\|X\|$	the cardinal of the set X
$[x]_n$	the ascending factorial of x (of degree n)
ξ	a cardinal number
ξ_i	the immediate successor of ξ_{i-1} in the set of cardinal numbers
X^0	a totally ordered set with a minimum element 0
X_n	the set $\{1, 2, \ldots, n\}$ of first n natural numbers
X_n^0	$X_n \cup \{0\}$
$\mathbb{Z}$	the set of integers

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CHAPTER 1

INTRODUCTION AND BACKGROUND

1. Introduction

Just as the study of (finite) symmetric and alternating groups forms an important part of group theory, so the study of various (finite) semigroups of transformations makes a significant contribution to semigroup theory. For one thing such semigroups are a rich source of examples. But it is also clear that they are worth studying in their own right as 'naturally occurring' objects. In fact the study of semigroups of full (partial, partial one-to-one) transformations has been fruitful over the years. (See for example, Clifford and Preston [**5**, **6**], Howie [**19**, **21-24**], Howie, Robertson and Schein [**27**], Howie and McFadden [**26**], Gomes and Howie [**15**, **16**], Garba [**11**, **12**], Schein [**31**] and Tainiter [**33**].)

Another important class which has aroused much interest in recent years is the class of semigroups of order-preserving transformations of a totally ordered set. (See for example, Howie [**20**], Howie and Schein [**28**], Schein [**32**], Gomes and Howie [**17**] and Garba [**13**, **14**].)

In this introductory chapter a survey of some results concerning the full (partial, partial one-to-one) transformation semigroups and the semigroups of order-preserving transformations as well as abundant semigroups is presented. For elementary concepts and propositions as well as standard notation in Semigroup Theory see [**5**, **6**, **21**].

It has been known for many years that Sing_n, the semigroup of singular mappings of a finite set is generated by idempotents, and indeed that it is generated by the $n(n-1)$ idempotents of defect $d(\alpha) = |X \setminus \text{im } \alpha| = 1$. The rank (idempotent rank) of a finite semigroup S is defined as the cardinality of a minimal generating set (of idempotents). The question of the rank of Sing_n does not seem to have been raised until the mid-eighties, when Gomes and Howie [**16**] proved that

$$\text{rank Sing}_n = \text{idrank Sing}_n = n(n-1)/2.$$

This result was later generalised by Howie and McFadden [**26**] to the semigroup

$$K(n, r) = \{\alpha \in \text{Sing}_n : |\text{im } \alpha| \leq r\}, \ (2 \leq r \leq n-1)$$

where they showed that

$$\text{rank } K(n, r) = \text{idrank } K(n, r) = S(n, r),$$

where $S(n, r)$ is the Stirling number of the second kind. Garba [**11**] considered the semigroup P_n of all partial transformations of X_n and showed that for the semigroup

$$K'(n, r) = \{\alpha \in P_n : |\text{im } \alpha| \leq r\}$$

both the rank and the idempotent rank are equal to $S(n+1, r+1)$.

Gomes and Howie [**15**], also examined the symmetric inverse semigroup I_n ($= I(X_n)$) consisting of all partial one-to-one maps of X_n and showed that the rank (as an inverse semigroup) of the inverse semigroup

$$SI_n = \{\alpha \in I_n : |\text{im } \alpha| \leq n-1\}$$

is $n+1$. Garba [**13**] generalised this by showing that for $r = 3, \ldots, n-1$ the rank of

$$L(n, r) = \{\alpha \in I_n : |\text{im } \alpha| \leq r\}$$

is

$$\binom{n}{r} + 1.$$

In [**20**] Howie studied the semigroup O_X of singular order-preserving mappings of a totally ordered set X. In the finite case where we may assume that O_n is the set of singular order-preserving mappings of $X_n = \{1, 2, \ldots, n\}$, Howie showed that O_n is a regular idempotent-generated subsemigroup of the full transformation semigroup on X_n. Howie also showed that

$$|O_n| = \binom{2n-1}{n-1} - 1,$$

$$|E(O_n)| = f_{2n} - 1,$$

where, for $m \geq 1$, f_m denotes the mth Fibonacci number.

In another paper [**17**], Gomes and Howie investigated the rank of the semigroups O_n, PO_n, SPO_n (the semigroup of order-preserving full transformations, order-preserving partial transformations and order-preserving strictly partial transformations on X_n). They showed that the rank of O_n is n, that of PO_n is $2n - 1$ and SPO_n has rank $2n - 2$. The idempotent rank of O_n is $2n - 2$, PO_n is idempotent generated and its idempotent rank is $3n - 2$. The semigroup SPO_n is not idempotent-generated and so the question of its idempotent rank does not arise. These results have been generalised by Garba [**14**], (in line with Howie and McFadden [**26**]). Let

$$L(n, r) = \{\alpha \in S : |\mathrm{im}\ \alpha| \leq r \text{ and } r \leq n - 2\}.$$

Then Garba showed that for $S = O_n$, the rank and the idempotent rank of $L(n, r)$ are both equal to $\binom{n}{r}$. For $S = PO_n$ the rank and the idempotent rank are both equal to

$$\sum_{k=r}^{n} \binom{n}{k}\binom{k-1}{r-1}$$

and for $S = SPO_n$ the rank and the idempotent rank are both equal to

$$\sum_{k=r}^{n-1} \binom{n}{k}\binom{k-1}{r-1}.$$

As remarked by Howie [**25**] another way of taking note of the order on a totally ordered set X is to consider the semigroups of order-decreasing (increasing) transformations. In [**34, 35**] I considered S_n^- and I_n^-, the semigroups of order-decreasing full and partial one-to-one transformations of X_n respectively. Pin [**29**] has also considered both the monoids (i.e. semigroup with an identity) of order-preserving and order-increasing transformations of a finite totally ordered set in connection with formal languages. However, he referred to the order-preserving transformations as increasing and our order-increasing transformations as extensive.

In this Thesis we consider the semigroups of order-decreasing full (partial, partial one-to-one) transformations for both the finite and infinite cases.

In Chapter 2, we consider the order-decreasing(increasing) finite full transformation semigroup $K^-(n, r)$ ($K^+(n, r)$, for, $1 \le r \le n-1$) and first show that $K^-(n, r)$ and $K^+(n, r)$ are isomorphic. Then we obtain results analogous to those of Howie [**20**] concerning products of idempotents in $K^-(n, r)$. We characterise the nilpotent elements of $K^-(n, r)$ and show that they form an ideal. We also characterise the Green's and starred Green's relations on $K^-(n, r)$ and hence show that $K^-(n, r)$ is a non-regular abundant semigroup for which $\mathcal{D}^* = \mathcal{J}^*$. See the next section for a brief account of abundant semigroups. From the results obtained for the full transformation case we show how one can deduce the corresponding results for the partial case using Vagner's result in [**39**]. We obtain results analogous to those for $K^-(n, r)$ for the Rees quotient semigroup $P_r^- = K^-(n, r)/K^-(n, r-1)$. Finally, we characterise the minimum semilattice congruence on these semigroups.

In Chapter 3 we consider the semigroup of order-decreasing(increasing) finite partial one-to-one transformations $L^-(n, r)$ ($L^+(n, r)$, for, $1 \le r \le n-1$) and obtain results analogous to those for the order-decreasing finite full transformations case studied in the previous Chapter. In particular we show that $L^-(n, r)$ and its Rees quotient semigroup $Q_r^- = L^-(n, r)/L^-(n, r-1)$ are quasi-idempotent-generated type A semigroups. See the next section for a brief account of type A semigroups.

We devote Chapter 4 to enumerative problems of a combinatorial nature, where we trivially obtain the order for S_n^-, (the semigroup of all singular order-decreasing transformations on X_n), the number of nilpotent elements in S_n^- and, perhaps less trivially, a formula for the number of idempotent elements of S_n^-. We also obtain some recurrence relations satisfied by the equivalences

$$J^*(n, r) = |\{\alpha \in (S_n^-)^1 : \alpha \in J_r^*\}|$$

$$= |\{\alpha \in (S_n^-)^1 : |\mathrm{im}\,\alpha| = r\}|$$

$$sh(n, r) = |\{\alpha \in (S_n^-)^1 : s(\alpha) = r - 1\}|,$$

and hence show that $J^*(n, r)$ and $sh(n, r)$ are the eulerian number and the complementary signless Stirling number of the first kind respectively. Then we obtain analogous results for the partial and partial one-to-one cases, which involve the Bell's number (B_n) and the Stirling number of the second kind.

In Chapter 5 we obtain results analogous to those of Howie and McFadden [**26**] for the finite semigroups of full (partial, partial one-to-one) order-decreasing transformations. In particular, we show that

$$\mathrm{rank}\ K^-(n, r) = \mathrm{idrank}\ K^-(n, r) = S(n, r),$$

$$\mathrm{rank}\ PK^-(n, r) = \mathrm{idrank}\ PK^-(n, r) = S(n + 1, r + 1),$$

$$\mathrm{rank}\ L^-(n, r) = \mathrm{quaidrank}\ L^-(n, r) = \binom{n}{r-1}\frac{[(n - r)(r + 1) + 1]}{r}.$$

Moreover, we obtain the ranks of the nilpotent subsemigroups of the finite semigroups of order-decreasing full and partial one-to-one transformations as

$$\mathrm{rank}\ N(S_n^-) = (n - 2)!(n - 2),$$

$$\mathrm{rank}\ N(I_n^-) = B_n - B_{n-1}.$$

In Chapter 6 we consider the semigroup of order-decreasing infinite full transformation $S^-(X)$ and first characterise its Green's and starred Green's relations. Then we establish the Isomorphism Theorem between $S^-(X)$ and $S^-(Y)$ and hence show that $S^-(X)$ and $S^+(Y)$ are isomorphic if and only if X and Y are order anti-isomorphic. By contrast with the finite case, in general $S^-(X)$ need not be abundant for an arbitrary chain. Therefore we find a necessary and sufficient condition (on the chain X) for which $S^-(X)$ is abundant. Next we consider the subsemigroup $K^-(|X|, \xi)$ (and its Rees quotient semigroup $P^-(\xi)$, where $\xi \leq |X|$) of the abundant semigroup $S^-(X)$. We characterise their Green's and starred Green's relations and hence show that $\mathcal{D}^* = \mathcal{J}^*$. From the results obtained for the full transformation case we show how one can deduce the corresponding results for the partial case using Vagner's result in [**39**].

In Chapter 7 we consider $I^-(X)$, $(I^+(X))$ the semigroup of order-decreasing(increasing) infinite partial one-to-one transformations and certain Rees quotient semigroup and obtain analogous results to the order-decreasing infinite full transformation case studied in the previous Chapter. However, most of the results for the order-decreasing infinite partial one-to-one case follow from the proofs of the corresponding results for the finite case studied in Chapter 3. A notable exception is the proof of the Isomorphism Theorem between $I^-(X)$ and $I^-(Y)$.

2. Abundant semigroups

The general study of abundant semigroups was initiated by Fountain [**10**] and therefore most of the results of this section are from [**10**].

On a semigroup S the relation $\mathcal{L}^*$ $(\mathcal{R}^*)$ is defined by the rule that $(a, b) \in \mathcal{L}^*$ $(\mathcal{R}^*)$ if and only if the elements a, b are related by the Green's relation $\mathcal{L}$ $(\mathcal{R})$ in some oversemigroup of S. The join of the equivalences $\mathcal{L}^*$ and $\mathcal{R}^*$ is denoted by $\mathcal{D}^*$ and their intersection by $\mathcal{H}^*$. A semigroup S in which each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class contains an idempotent is called *abundant* . For any result concerning $\mathcal{L}^*$ there is, of course, a dual result for $\mathcal{R}^*$. Evidently, $\mathcal{L}^*$ is a right congruence on S. The following lemma gives an alternative characterization of $\mathcal{L}^*$.

Lemma 1.2.1.[**10**, Lemma 1.1] *Let* a, b *be elements of a semigroup* S. *Then the following are equivalent* :

(1) $(a, b) \in \mathcal{L}^*$;

(2) *for* $x, y \in S^1$, $ax = ay$ *if and only if* $bx = by$.

As in [**10**] we introduce *-ideals to obtain the starred analogue of the Green's relation $\mathcal{J}$. The $\mathcal{L}^*$-class containing the element a is denoted by L^*_a. The corresponding notation is used for the class of the other relations. We now define a *left (right) *-ideal* of a semigroup S to be a left (right) ideal I of S for which $L^*_a \subseteq I$ ($R^*_a \subseteq I$) for all elements a of I. A subset I of S is a **-ideal* if it is both a left *-ideal and a right *-ideal. The principal *-ideal $J^*(a)$ generated by the element a of S is the intersection of all *-ideals of S to which a belongs. The relation $\mathcal{J}^*$ is defined by the rule that : a $\mathcal{J}^*$ b *if and only if* $J^*(a) = J^*(b)$. From [**10**, Lemma 1.7] we have:

Lemma 1.2.2. *If* a, b *are elements of a semigroup* S, *then* $b \in J^*(a)$ *if and only if there are elements* $a_0, a_1, \dots, a_n \in S$, $x_1, \dots, x_n, y_1, \dots, y_n \in S^1$ *such that* $a = a_0$, $b = a_n$ *and* $(a_i, x_i a_{i-1} y_i) \in \mathcal{D}^*$ *for* $i = 1, \dots, n$.

Obviously, on any semigroup S we have $\mathcal{L} \subseteq \mathcal{L}^*$. It is well known and easy to see that for regular elements $a, b \in S$, $(a, b) \in \mathcal{L}^*$ if and only if $(a, b) \in \mathcal{L}$. In particular, if S is a regular semigroup then $(a, b) \in \mathcal{K}$ if and only if $(a, b) \in \mathcal{K}^*$, where $\mathcal{K}$ is any of $\mathcal{H}$, $\mathcal{L}$, $\mathcal{R}$, $\mathcal{D}$ or $\mathcal{J}$. Moreover, in any semigroup S, $\mathcal{K} \subseteq \mathcal{K}^*$. In fact the starred relations play a role in the theory of abundant semigroups analogous to that of Green's relations in the theory of regular semigroups.

In case of ambiguity we will denote the relation $\mathcal{K}^*$ on S by $\mathcal{K}^*(S)$. Clearly if U is a subsemigroup of S, then $\mathcal{L}^*(S) \cap (U \times U) \subseteq \mathcal{L}^*(U)$. In general, we do not have equality as the following example shows.

Example 1.2.3.[**10**, Example 1.3] Let A be the free monoid on two generators x, y with identity 1 and let $B = \{e, f\}$ be the two element left zero semigroup. Let $S = A \cup B$ and define a product on S which extends those on A and B by letting 1 be the identity for S and by putting $bw = b$; $wb = e$ if w begins with x, $wb = f$ if w begins with y where $b \in B$, $w \in A \setminus \{1\}$. It is routine to check that S is a monoid. Since $xe = e = xf$ but $1e \neq 1f$ we have $(x, 1) \notin \mathcal{L}^*(S)$ so that $\mathcal{L}^*(S) \cap (A \times A) \neq A \times A = \mathcal{L}^*(A)$.

However there are some cases where we do have equality.

Lemma 1.2.4.[**8**, Lemma 1.6]. *Let* U *be an abundant subsemigroup of an abundant semigroup* S *such that the idempotents of* U *form an order-ideal of those of* S. *Then*

$$\mathcal{L}^*(S) \cap (U \times U) = \mathcal{L}^*(U).$$

The lemma applies, in particular, to full subsemigroups and *-ideals of an abundant semigroup. By a *full* subsemigroup of a semigroup S we mean simply one which contains all the idempotents of S. Clearly, if S is abundant, then full subsemigroups and *-ideals are abundant.

Next we describe another important class of subsemigroups (from [**36**, Section 3] where we do have equality.

Definition 1.2.5. Let S be a semigroup and let U be a subsemigroup of S. Then U will be called an *inverse ideal* of S if for all $u \in U$, there exists $u' \in S$ such that $uu'u = u$ and $uu', u'u \in U$. Notice that an inverse ideal need not be an ideal as the next two examples show.

Examples 1.2.6. (1) Every regular subsemigroup is an inverse ideal, however not every regular subsemigroup is an ideal.

(2) Let B be the bicyclic semigroup and let $B^* = \{(m, n) \in B : m \geq n\}$. Then B^* is a full subsemigroup of B and it is an inverse ideal since

$$(m, n)(n, m) = (m, m) \in B^*$$

$$(n, m)(m, n) = (n, n) \in B^*$$

for all $(m, n) \in B^*$. Notice that B^* is non-regular and is not an ideal. In fact the only regular elements of B^* are its idempotents.

Lemma 1.2.7. *Every inverse ideal* U *of a semigroup* S *is abundant.*

Proof. Since for all $u \in U$,

$$(u, u'u) \in \mathcal{L}(S) \quad \text{and} \quad (u, uu') \in \mathcal{R}(S)$$

it follows that

$$(u, u'u) \in \mathcal{L}^*(U) \quad \text{and} \quad (u, uu') \in \mathcal{R}^*(U).$$

Hence every $\mathcal{L}^*$-class and every $\mathcal{R}^*$-class of U contains an idempotent since uu', $u'u$ are idempotents in U. Thus U is abundant. ■

Lemma 1.2.8. *Let* U *be an inverse ideal of a semigroup* S. *Then*

(1) $\mathcal{L}^*(U) = \mathcal{L}(S) \cap (U \times U)$

(2) $\mathcal{R}^*(U) = \mathcal{R}(S) \cap (U \times U)$

(3) $\mathcal{H}^*(U) = \mathcal{H}(S) \cap (U \times U)$.

Proof. (1) Certainly,

$$\mathcal{L}(S) \cap (U \times U) \subseteq \mathcal{L}^*(U).$$

Conversely, suppose that $(a, b) \in \mathcal{L}^*(U)$ and $aa', a'a \in U$ where a', b' are elements in S such that $aa'a = a$ and $bb'b = b$. Then

$$\begin{aligned}
& (a'a, a) \in \mathcal{L}(S) \quad \text{and} \quad (b, b'b) \in \mathcal{L}(S) \\
\Rightarrow \quad & (a'a, a) \in \mathcal{L}^*(U) \quad \text{and} \quad (b, b'b) \in \mathcal{L}^*(U) \\
\Rightarrow \quad & (a'a, b'b) \in \mathcal{L}^*(U) && \text{(by transitivity)} \\
\Leftrightarrow \quad & (a'a, b'b) \in \mathcal{L}(U) \\
\Rightarrow \quad & (a'a, b'b) \in \mathcal{L}(S) && (\text{since } \mathcal{L}(U) \subseteq \mathcal{L}(S) \cap (U \times U)),
\end{aligned}$$

and hence,

$$(a, b) \in \mathcal{L}(S)$$

so that

$$\mathcal{L}^*(U) \subseteq \mathcal{L}(S) \cap (U \times U)$$

and the result follows.

(2) The proof is similar to that of (1).

(3) This is a simple set-theoretic consequence of (1) & (2). ∎

Corollary 1.2.9. $\mathcal{L}^*(U) = \mathcal{L}^*(S) \cap (U \times U)$, $\mathcal{R}^*(U) = \mathcal{R}^*(S) \cap (U \times U)$ and $\mathcal{H}^*(U) = \mathcal{H}^*(S) \cap (U \times U)$.

The following example [**10**, Example 1.11] shows that in general $\mathcal{L}^* \circ \mathcal{R}^* \neq \mathcal{R}^* \circ \mathcal{L}^*$.

Example 1.2.10. Let A be the infinite cyclic semigroup with generator *a* and let B be the infinite cyclic monoid with generator *b* and identity *e*. Let $S = A \cup B \cup \{1\}$ and define a product on S which extends those on A and B and has 1 as the identity by putting $a^m b^n = b^{m+n}$, $b^n a^m = a^{n+m}$ for integers $m > 0$ and $n \geq 0$ where $b^0 = e$.

It is routine to check that S is a monoid with idempotents 1, *e*. The $\mathcal{L}^*$-classes of S are $A \cup \{1\}$ and B, and the $\mathcal{R}^*$-classes of S are $A \cup B$ and $\{1\}$. Thus S is an abundant monoid. Furthermore, it is clear that on S, $\mathcal{D}^*$ is the universal relation, while the $\mathcal{H}^*$-classes are $\{1\}$, A, B. We note that $\mathcal{L}^* \circ \mathcal{R}^* \neq \mathcal{R}^* \circ \mathcal{L}^*$ since, for example, $1 \, \mathcal{L}^* \, a \, \mathcal{R}^* \, b$ but $R_1^* \cap L_b^*$ is empty.

A semigroup S with zero is *0-$\mathcal{J}^*$-simple* if the only *-ideals of S are S, $\{0\}$, and $S^2 \neq \{0\}$. Note that this does not entail $S^2 = S$ since the ideal S^2 need not be a *-ideal of S. In fact the six element semigroup $S = \{0, a, a^2, b, c, d\}$, where a.a = a.b = a.c = a^2; d.a = d.b = d.c = a^2; and all other products being 0 has $\mathcal{L}^*$-classes: $\{0\}$, $\{a^2, b, c\}$, $\{a, d\}$ and $\mathcal{R}^*$-classes: $\{0\}$, $\{a^2, d\}$, $\{a, b, c\}$. Thus S has $\mathcal{J}^*$-classes: $\{0\}$, $\{a^2, a, b, c, d\}$. However, $S^2 = \{0, a^2\} \neq S$. A semigroup S is $\mathcal{J}^*$-*simple* if it has no proper *-ideals. Clearly S is $\mathcal{J}^*$-simple if and only if $\mathcal{J}^*$ is the universal relation on S; S is 0-$\mathcal{J}^*$-simple if and only if $S^2 \neq \{0\}$, and $\{0\}$, $S\backslash\{0\}$ are the only $\mathcal{J}^*$-classes of S. In a regular semigroup S all ideals are *-ideals so that in this case a (0)-$\mathcal{J}^*$-simple semigroup is just a (0)-simple semigroup.

In a completely 0-simple semigroup the existence of a primitive idempotent implies regularity and that every non-zero idempotent is primitive. In contrast to this a semigroup may be 0-$\mathcal{J}^*$-simple and primitive but not abundant. An example is provided by adjoining a zero to a left cancellative monoid which is not cancellative. In fact as pointed out in [**10**] there are several analogues of complete 0-simplicity for 0-$\mathcal{J}^*$-simple semigroups.

It is also the case that in general, on a semigroup S, $\mathcal{D}^* \neq \mathcal{J}^*$. However we do have

Lemma 1.2.11.[**10**, Proposition 4.1] *On a primitive abundant* 0-$\mathcal{J}^*$-*simple semigroup S,* $\mathcal{D}^* = \mathcal{J}^*$.

An abundant semigroup S in which E(S) is a semilattice is called *adequate* [**9**]. Of course inverse semigroups are adequate since in this case $\mathcal{L}^* = \mathcal{L}$ and $\mathcal{R}^* = \mathcal{R}$. As in [**9**] for an element a of an adequate semigroup S, the (unique) idempotent in the $\mathcal{L}^*$-class ($\mathcal{R}^*$-class) containing a will be denoted by a^* (a^+). An adequate semigroup S is said to be *type A* if $ea = a(ea)^*$ and $ae = (ae)^+a$ for all elements a in S and all idempotents e in S. Notice that if S is adequate (type A), then full subsemigroups and *-ideals are adequate (type A).

CHAPTER 2

FINITE ORDER-DECREASING FULL TRANSFORMATION SEMIGROUPS

1. Preliminaries

Let X_n be a finite totally ordered set, so that effectively we may identify X_n with the set $\{1, 2, \dots, n\}$ of the first n natural numbers. Let T_n be the full transformation semigroup on X_n and let

$$\mathrm{Sing}_n = \{\alpha \in T_n : |\mathrm{im}\,\alpha| \leq n-1\}$$

be the subsemigroup of all singular self-maps of X_n. Let

$$S_n^- = \{\alpha \in \mathrm{Sing}_n : (\text{for all } x \in X_n)\ x\alpha \leq x\} \qquad (1.1)$$

$$S_n^+ = \{\alpha \in \mathrm{Sing}_n : (\text{for all } x \in X_n)\ x\alpha \geq x\} \qquad (1.2)$$

be the subsets of Sing_n consisting of all decreasing and increasing singular self-maps of X_n respectively. For $1 \leq r \leq n$, let

$$K(n, r) = \{\alpha \in T_n : |\mathrm{im}\,\alpha| \leq r\} \qquad (1.3)$$

$$K^-(n, r) = \{\alpha \in (S_n^-)^1 : |\mathrm{im}\,\alpha| \leq r\} \qquad (1.4)$$

$$K^+(n, r) = \{\alpha \in (S_n^+)^1 : |\mathrm{im}\,\alpha| \leq r\} \qquad (1.5)$$

Thus $K^-(n, n) = (S_n^-)^1$, $K^-(n, n-1) = S_n^-$ and each $K^-(n, r)$ is a two-sided ideal of $(S_n^-)^1$.

Lemma 2.1.1. $K^-(n, r)$ *and* $K^+(n, r)$ *are isomorphic subsemigroups of* Sing_n.

Proof. Let $\alpha, \beta \in K^-(n, r)$. Then for all $x \in X_n$

$$(x\alpha)\beta \leq x\alpha \leq x$$

so that $\alpha\beta \in K^-(n, r)$ as required. Similarly we can show that $\alpha\beta \in K^+(n, r)$ for all

$\alpha, \beta \in K^+(n, r)$. Thus $K^-(n, r)$ and $K^+(n, r)$ are subsemigroups of $Sing_n$. Now define a map $\theta : K^-(n, r) \to K^+(n, r)$ by

$$\theta(\alpha) = \bar{\alpha} \quad (\alpha \in K^-(n, r), \bar{\alpha} \in K^+(n, r))$$

where

$$i\bar{\alpha} = n - (n - i + 1)\alpha + 1 \qquad (i = 1, 2, \dots , n).$$

Clearly θ is a bijection and

$$i\bar{\alpha} = n - (n - i + 1)\alpha + 1 \geq n - (n - i + 1) + 1 = i.$$

Moreover,

$$\begin{aligned}(i\bar{\alpha})\bar{\beta} &= n - (n - i\bar{\alpha} + 1)\beta + 1 \\ &= n - \{n - [n - (n - i + 1)\alpha + 1] + 1\}\beta + 1 \\ &= n - (n - i + 1)\alpha\beta + 1 \\ &= i\overline{\alpha\beta}.\end{aligned}$$

Thus θ is an isomorphism. ■

Remark 2.1.2. Notice that in view of the above lemma we can deduce the results for $K^+(n, r)$ from those for $K^-(n, r)$ respectively, in an obvious manner.

Let us denote by $f(\alpha)$ the cardinal of the set

$$F(\alpha) = \{ x \in X_n : x\alpha = x \},$$

and establish the following basic lemma.

Lemma 2.1.3. *Let* $\alpha, \beta \in K^-(n, r)$. *Then*

(1) $F(\alpha\beta) = F(\alpha) \cap F(\beta)$;

(2) $F(\alpha\beta) = F(\beta\alpha)$.

Proof. (1) First notice that for all $\alpha, \beta \in K^-(n, r)$

$$F(\alpha) \cap F(\beta) \subseteq F(\alpha\beta).$$

Conversely, let $x \in F(\alpha\beta)$. Then

$$x\alpha\beta = x, \qquad x\alpha\beta \leq x\alpha \leq x$$

so that $x\alpha = x$ and $x\beta = (x\alpha)\beta = x$. Hence

$$F(\alpha\beta) \subseteq F(\alpha) \cap F(\beta).$$

Thus

$$F(\alpha\beta) = F(\alpha) \cap F(\beta).$$

(2) $$F(\alpha\beta) = F(\alpha) \cap F(\beta) = F(\beta) \cap F(\alpha) = F(\beta\alpha) \qquad \text{(by (1)).} \blacksquare$$

Let us denote by $s(\alpha)$ the cardinal of the set

$$S(\alpha) = \{ x \in X_n : x\alpha \neq x \},$$

and if we denote an element $\alpha \in K^{-}(n, r)$ by

$$\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_k \\ a_1 & a_2 & \dots & a_k \end{pmatrix},$$

then $A_1, A_2, \dots, A_k$ are called the blocks of α, where $A_i = a_i\alpha^{-1}$ (or $A_i\alpha = a_i$).

Lemma 2.1.4. *Let* $\alpha \in K^{-}(n, r)$. *Then*

(1) α *is an idempotent if and only if* $(\forall\ t \in \text{im}\,\alpha)$, $t = \min\{x : x \in t\alpha^{-1}\}$,

(2) α *is an idempotent if and only if* $f(\alpha) = |\text{im}\,\alpha|$,

(3) α *is an idempotent if* $s(\alpha) = 1$,

(4) α *is an idempotent if* $f(\alpha) = r$.

Proof. (1) First recall from [**27**, Section 2] that $\alpha \in Sing_n$ is an idempotent if and only if every block of α is stationary,

$$\text{i.e., iff } (\forall t \in \text{im}\,\alpha)\ t \in t\alpha^{-1},$$

$$\text{i.e., iff } (\forall t \in \text{im}\,\alpha)\ t = \min\{x : x \in t\alpha^{-1}\},$$

since $t \in t\alpha^{-1}$ and $x \in t\alpha^{-1}$ implies that $x \geq x\alpha = t\alpha = t$.

(2) Again from [**27**, Section 2], $\alpha \in Sing_n$ is an idempotent if and only if every block of α is stationary,

$$\text{i.e., iff } (\forall t \in \text{im}\,\alpha)\ t \in t\alpha^{-1},$$

$$\text{i.e., iff } f(\alpha) = |\text{im}\,\alpha|.$$

(3) Let $S(\alpha) = \{u\}$. Then $x\alpha = x$ for all $x \notin S(\alpha)$, and since $u\alpha \neq u$ we must have $u\alpha \notin S(\alpha)$. Hence $u\alpha^2 = u\alpha$.

(4) Since $f(\alpha) = r \geq |\text{im}\,\alpha| \geq f(\alpha)$, it follows from (2) that α is an idempotent. ∎

Theorem 2.1.5. *Let* $K^-(n, r)$ *be as defined in* (1.4). *Then every* $\alpha \in K^-(n, r)$ *is expressible as a product of idempotents in* $K^-(n, r)$.

Proof. Suppose that

$$\alpha = \begin{pmatrix} A_1 & A_2 & \dots & A_k \\ a_1 & a_2 & \dots & a_k \end{pmatrix} \in K^-(n, r)$$

and let $V(\alpha) = S(\alpha) \cap \text{im}\,\alpha$ with $v_0 = \min V(\alpha)$. Now define ε, β (respectively) by

$$A_i\varepsilon = t_i = \min A_i \qquad (i = 1, \dots, k\},$$

$$x\beta = \begin{cases} v_0 & (\text{if } x \in v_0\alpha^{-1} \cup \{v_0\}) \\ x\alpha & (\text{otherwise.}) \end{cases}$$

It is then clear that $\varepsilon^2 = \varepsilon$, $\beta \in K^-(n, r)$ and $s(\beta) = s(\alpha) - 1$. Moreover,

$$A_i\varepsilon\beta = t_i\beta = \begin{cases} v_0 & (\text{if } t_i \in v_0\alpha^{-1}) \\ a_i & (\text{otherwise.}) \end{cases}$$

since $v_0 \neq t_i$ (for all i). For if $v_0 = t_i$, then we must have

$$v_0 = v_0\alpha \quad \text{or} \quad v_0 > v_0\alpha \in V(\alpha)$$

which is a contradiction in either case. Thus $\varepsilon\beta = \alpha$, since

$$A_i\alpha = v_0 = A_i\varepsilon\beta \quad (\text{if } t_i \in v_0\alpha^{-1}),$$

$$A_i\alpha = a_i = t_i\beta = A_i\varepsilon\beta \quad (\text{otherwise}).$$

The result now follows by induction. ∎

Define a map ζ (in $K^-(n, r)$) by

$$x\zeta = 1 \qquad \text{for all } x \in X_n.$$

Then for every α in $K^-(n, r)$

$$\alpha\zeta = \zeta\alpha = \zeta;$$

so ζ is the *zero* element of $K^-(n, r)$.

An element α of $K^-(n, r)$ is called *nilpotent* if $\alpha^k = \zeta$ (= 0) for some $k \geq 1$. The next lemma gives us a complete characterization of nilpotent elements in $K^{-}(n, r)$.

Lemma 2.1.6. *Let* $\alpha \in K^{-}(n, r)$. *Then* α *is nilpotent if and only if* $F(\alpha) = \{1\}$.

Proof. First suppose that α is nilpotent, with $\alpha^k = \zeta$, and that $b \in F(\alpha)$. Then

$$b = b\alpha = \ldots = b\alpha^k = 1.$$

Conversely, suppose that $F(\alpha) = \{1\}$. Then $x\alpha < x$ for all $x \in X_n \setminus \{1\}$ and the descent

$$x > x\alpha > x\alpha^2 > \ldots$$

must terminate at 1; i.e.

$$x\alpha^{k_x} = 1 \quad \text{for some } k_x > 0.$$

Let $k = \max\{k_x : x \in X_n\}$. Then

$$x\alpha^k = 1 \qquad \text{for all } x \in X_n,$$

and hence α is nilpotent. ■

Lemma 2.1.7. *Let* $N(K^{-}(n, r)) = \{\alpha \in K^{-}(n, r) : F(\alpha) = \{1\}\}$. *Then* $N(K^{-}(n, r))$ *is an ideal of* $K^{-}(n, r)$.

Proof. Let $\alpha \in N(K^{-}(n, r))$ and $\beta \in K^{-}(n, r)$. Then since $x\alpha\beta \leq x\alpha < x$ for all $x \in X_n \setminus \{1\}$, it follows that $\alpha\beta \in N(K^{-}(n, r))$. To show that $\beta\alpha \in N(K^{-}(n, r))$, again consider $x \in X_n \setminus \{1\}$. If $x\beta = 1$ then

$$x\beta\alpha = 1 < x;$$

if $x\beta \neq 1$ then

$$x\beta\alpha < x\beta \leq x.$$

In both cases $x\beta\alpha < x$. Hence $F(\beta\alpha) = \{1\}$ and so $\beta\alpha \in N(K^{-}(n, r))$ as required. ■

2. Green's and starred Green's relations

To avoid excessive use of notation $K^{-}(n, r)$ might be abbreviated to K^{-}, and similar abbreviations will occur throughout the remainder of the text.

Lemma 2.2.1. *Let* $\alpha, \beta \in K^{-}(n, r)$. *Then*

(1) $(\alpha, \beta) \in \mathcal{R}$ *if and only if* $\alpha = \beta$;

(2) $(\alpha, \beta) \in \mathcal{L}$ *if and only if* $\operatorname{im}\alpha = \operatorname{im}\beta$ *and* $\min z\alpha^{-1} = \min z\beta^{-1}$ *for all* z *in* $\operatorname{im}\alpha$.

Proof. (1) Suppose that $(\alpha, \beta) \in \mathcal{R}$. Then there exists δ, γ in $(K^{-}(n, r))^1$ such that $\alpha\delta = \beta$ and $\beta\gamma = \alpha$. However, for all x in X_n,

$$x\beta = (x\alpha)\delta \leq x\alpha, \qquad x\alpha = (x\beta)\gamma \leq x\beta.$$

Thus $x\alpha = x\beta$ for all x in X_n and so $\alpha = \beta$.

(2) Suppose that $\operatorname{im}\alpha = \operatorname{im}\beta$ and that $\min z\alpha^{-1} = \min z\beta^{-1}$ for all z in $\operatorname{im}\alpha$. Let $u_z = \min z\alpha^{-1}$ $(= \min z\beta^{-1})$ and define δ, γ by

$$x\delta = u_z \qquad (x \in z\alpha^{-1},\ z \in \operatorname{im}\alpha)$$

$$y\gamma = u_z \qquad (y \in z\beta^{-1},\ z \in \operatorname{im}\beta).$$

Clearly, $\delta, \gamma \in K^{-}(n, r)$ and $\alpha = \delta\beta$, $\beta = \gamma\alpha$. Thus $(\alpha, \beta) \in \mathcal{L}$.

Conversely, suppose that $(\alpha, \beta) \in \mathcal{L}(K^{-})$. Then since

$$\mathcal{L}(K^{-}) \subseteq \mathcal{L}(T_n) \cap (K^{-} \times K^{-})$$

we have $\operatorname{im}\alpha = \operatorname{im}\beta$, by [**20**, Ex.II.10]. Moreover there exists δ, γ in $(K^{-}(n, r))^1$ such that $\alpha = \gamma\beta$ and $\beta = \delta\alpha$. Let $z \in \operatorname{im}\alpha = \operatorname{im}\beta$ and let $y = \min z\alpha^{-1}$. Then

$$y\gamma\beta = y\alpha = z,$$

and so

$$y\gamma \in z\beta^{-1}.$$

Hence

$$y \geq y\gamma \geq \min z\beta^{-1}.$$

That is, $\min z\alpha^{-1} \geq \min z\beta^{-1}$, and we can similarly show that

$$\min z\beta^{-1} \geq \min z\alpha^{-1} .$$

Thus

$$\min z\alpha^{-1} = \min z\beta^{-1},$$

as required. ■

Since $\mathcal{D} = \mathcal{J}$ on any finite semigroup (by [**20**, Proposition II.1.5]), then we immediately deduce the following corollary:

Corollary 2.2.2. *On the semigroup* $K^-(n, r)$, $\mathcal{H} = \mathcal{R}$ *and* $\mathcal{L} = \mathcal{D} = \mathcal{J}$.

Remark 2.2.3. Notice that since $K^-(n, r)$ contains some non-idempotent elements then by Lemma 2.2.1 above not every $\mathcal{R}$-class of $K^-(n, r)$ contains an idempotent. Hence $K^-(n, r)$ is non-regular.

Recall that a subsemigroup U (of a semigroup S) is said to be an *inverse ideal* of S if for all $u \in U$, there exists $u' \in S$ such that $uu'u = u$ and $uu', u'u \in U$.

We now have

Lemma 2.2.4. $K^-(n, r)$ *is an inverse ideal of* T_n.

Proof. For a given $\alpha \in K^-(n, r)$ define α' by

$$x\alpha' = \min\{y : y \in x\alpha^{-1}\} \quad (\text{for all } x \in \text{im}\,\alpha),$$
$$x\alpha' = 1 \quad (\text{otherwise}).$$

Then clearly $\alpha\alpha'\alpha = \alpha$. Moreover, for all $x \in \text{im}\,\alpha$

$$x\alpha'\alpha = x \quad \text{and} \quad y\alpha'\alpha = 1 < y \quad (\text{for all } y \notin \text{im}\,\alpha).$$

And for all x

$$x\alpha\alpha' = \min\{y : y \in (x\alpha)\alpha^{-1}\} \leq x.$$

Thus $\alpha'\alpha, \alpha\alpha' \in K^-(n, r)$ since $|\text{im}\,\alpha'| = |\text{im}\,\alpha|$ (from the construction of α'). It now follows that $K^-(n, r)$ is an inverse ideal as required. ■

Hence by Lemmas 1.2.7 & 1.2.8 and [**20**, Proposition II. 4.5 and Ex.II.10] we obtain the next two results:

Theorem 2.2.5. *For* $r > 2$ *let* $K^-(n, r)$ *be as defined in* (1.4). Then $K^-(n, r)$ *is a non-regular abundant semigroup.*

Lemma 2.2.6. *Let* $\alpha, \beta \in K^-(n, r)$. *Then*

(1) $(\alpha, \beta) \in \mathcal{L}^*$ *if and only if* $\operatorname{im} \alpha = \operatorname{im} \beta$;

(2) $(\alpha, \beta) \in \mathcal{R}^*$ *if and only if* $\alpha \circ \alpha^{-1} = \beta \circ \beta^{-1}$;

(3) $(\alpha, \beta) \in \mathcal{H}^*$ *if and only if* $\operatorname{im} \alpha = \operatorname{im} \beta$ *and* $\alpha \circ \alpha^{-1} = \beta \circ \beta^{-1}$.

Now since for regular elements a, b in a semigroup S

$$(a, b) \in \mathcal{R} \text{ if and only if } (a, b) \in \mathcal{R}^*,$$

then by Lemma 2.2.1(1) we deduce that:

Lemma 2.2.7. *Every* $\mathcal{R}^*$*-class of* $K^-(n, r)$ *contains a unique idempotent.*

Now we begin the characterization of $\mathcal{D}^*$ on $K^-(n, r)$ by first proving these two simple but rather technical lemmas.

Lemma 2.2.8. *Let* $\alpha \in K^-(n, r)$ *with* $|\operatorname{im} \alpha| = k$. *Then there exists* β *in* $K^-(n, r)$ *with* $\operatorname{im} \beta = \{1, \dots, k\}$ *such that* $(\alpha, \beta) \in \mathcal{R}^*$.

Proof. Suppose that

$$\alpha = \begin{pmatrix} A_1 & . & . & . & A_k \\ a_1 & . & . & . & a_k \end{pmatrix} \in K^-(n, r)$$

and that $\min A_{i+1} > \min A_i$ for all $i = 1, \dots, k-1$. Then it is clear that $\min A_i \geq i$ (for $i = 1, 2, \dots, k$). Define β by

$$A_i\beta = i \qquad (i = 1, \dots, k).$$

Then $\beta \in K^-(n, r)$ and $(\alpha, \beta) \in \mathcal{R}^*$ (by Lemma 2.2.6). ■

Let σ_k be the equivalence on X_n whose classes are $\{1, \ldots, n-k+1\}$, $\{n-k+2\}, \ldots, \{n\}$. Note that $|X_n/\sigma_k| = k$.

Lemma 2.2.9. *Let* $\alpha \in K^-(n, r)$ *with* $\operatorname{im} \alpha = \{a_1, \ldots, a_k\}$ *such that* $1 = a_1 < a_2 < \ldots < a_k$. *Then there exists* β *in* $K^-(n, r)$ *with* $\beta \circ \beta^{-1} = \sigma_k$ *such that* $(\alpha, \beta) \in \mathcal{L}^*$.

Proof. Define β (in $K^-(n, r)$) by

$$(n-i)\beta = a_{k-i} \qquad (i = 0, 1, \ldots, k-2),$$

$$j\beta = a_1 = 1 \qquad (j \le n-k+1).$$

Then clearly $\beta \circ \beta^{-1} = \sigma_k$ and $(\alpha, \beta) \in \mathcal{L}^*$ (by Lemma 2.2.6). ■

On the semigroup $K^-(n, r)$, define a relation $\mathcal{K}$ by the rule that

$$(\alpha, \beta) \in \mathcal{K} \text{ if and only if } |\operatorname{im} \alpha| = |\operatorname{im} \beta|.$$

Then clearly $\mathcal{K}$ is an equivalence relation containing both $\mathcal{L}^*$ and $\mathcal{R}^*$. In fact $\mathcal{D}^* \subseteq \mathcal{K}$ since $\mathcal{D}^*$ is the smallest equivalence relation containing both $\mathcal{L}^*$ and $\mathcal{R}^*$.

Lemma 2.2.10. $\mathcal{K} = \mathcal{R}^* \circ \mathcal{L}^* \circ \mathcal{R}^* = \mathcal{L}^* \circ \mathcal{R}^* \circ \mathcal{L}^* = \mathcal{D}^*$.

Proof. Suppose that $(\alpha, \beta) \in \mathcal{K}$, so that $|\operatorname{im} \alpha| = |\operatorname{im} \beta| = k$ (say). Then there exists $\delta, \gamma \in K^-(n, r)$ with $\operatorname{im} \delta = \operatorname{im} \gamma = \{1, \ldots, k\}$ such that $(\alpha, \delta) \in \mathcal{R}^*$ and $(\gamma, \beta) \in \mathcal{R}^*$ (by Lemma 2.2.8). However, since $\operatorname{im} \delta = \operatorname{im} \gamma$ implies $(\delta, \gamma) \in \mathcal{L}^*$ (by Lemma 2.2.6) we have $\alpha \,\mathcal{R}^*\, \delta \,\mathcal{L}^*\, \gamma \,\mathcal{R}^*\, \beta$. Thus

$$\mathcal{K} \subseteq \mathcal{R}^* \circ \mathcal{L}^* \circ \mathcal{R}^*. \tag{1}$$

Conversely, suppose that $(\alpha, \beta) \in \mathcal{R}^* \circ \mathcal{L}^* \circ \mathcal{R}^*$. Then there exist $\delta, \gamma \in K^-(n, r)$ such that $\alpha \,\mathcal{R}^*\, \delta \,\mathcal{L}^*\, \gamma \,\mathcal{R}^*\, \beta$. Hence

$$|\operatorname{im} \alpha| = |\operatorname{im} \delta|, \operatorname{im} \delta = \operatorname{im} \gamma \text{ and } |\operatorname{im} \gamma| = |\operatorname{im} \beta|,$$

and so $|\operatorname{im} \alpha| = |\operatorname{im} \beta|$. Thus

$$\mathcal{R}^* \circ \mathcal{L}^* \circ \mathcal{R}^* \subseteq \mathcal{H} \tag{2}$$

From (1) and (2) we deduce that

$$\mathcal{H} = \mathcal{R}^* \circ \mathcal{L}^* \circ \mathcal{R}^*.$$

Similarly, suppose that $(\alpha, \beta) \in \mathcal{H}$, so that $|\mathrm{im}\ \alpha| = |\mathrm{im}\ \beta| = k$ (say). Then there exists $\delta, \gamma \in K^-(n, r)$ with $\delta \circ \delta^{-1} = \sigma_k = \gamma \circ \gamma^{-1}$, $(\alpha, \delta) \in \mathcal{L}^*$ and $(\gamma, \beta) \in \mathcal{L}^*$ (by Lemma 2.2.9). However, since $\delta \circ \delta^{-1} = \gamma \circ \gamma^{-1}$ implies $(\delta, \gamma) \in \mathcal{R}^*$ (by Lemma 2.2.6) then we have $\alpha \mathcal{L}^* \delta \mathcal{R}^* \gamma \mathcal{L}^* \beta$. Thus

$$\mathcal{H} \subseteq \mathcal{L}^* \circ \mathcal{R}^* \circ \mathcal{L}^*. \tag{3}$$

Conversely, suppose that $(\alpha, \beta) \in \mathcal{L}^* \circ \mathcal{R}^* \circ \mathcal{L}^*$. Then there exist $\delta, \gamma \in K^-(n, r)$ such that $\alpha \mathcal{L}^* \delta \mathcal{R}^* \gamma \mathcal{L}^* \beta$. Hence

$$\mathrm{im}\ \alpha = \mathrm{im}\ \delta,\ |\mathrm{im}\ \delta| = |\mathrm{im}\ \gamma| \text{ and } \mathrm{im}\ \gamma = \mathrm{im}\ \beta,$$

and so $|\mathrm{im}\ \alpha| = |\mathrm{im}\ \beta|$. Thus

$$\mathcal{L}^* \circ \mathcal{R}^* \circ \mathcal{L}^* \subseteq \mathcal{H} \tag{4}$$

From (3) and (4) we deduce that

$$\mathcal{H} = \mathcal{L}^* \circ \mathcal{R}^* \circ \mathcal{L}^*.$$

To complete the proof of the lemma, note that from the inequalities

$$\mathcal{D}^* \subseteq \mathcal{H} = \mathcal{R}^* \circ \mathcal{L}^* \circ \mathcal{R}^* = \mathcal{L}^* \circ \mathcal{R}^* \circ \mathcal{L}^* \subseteq \mathcal{D}^*$$

we deduce that $\mathcal{D}^* = \mathcal{H} = \mathcal{R}^* \circ \mathcal{L}^* \circ \mathcal{R}^* = \mathcal{L}^* \circ \mathcal{R}^* \circ \mathcal{L}^*$. ■

Corollary 2.2.11. $(\alpha, \beta) \in \mathcal{D}^*$ *if and only if* $|\mathrm{im}\ \alpha| = |\mathrm{im}\ \beta|$ *for all* $\alpha, \beta \in K^-(n, r)$.

Recall that the relation $\mathcal{J}^*$ is defined by the rule that $a \mathcal{J}^* b$ if and only if $J^*(a) = J^*(b)$, where $J^*(a)$ is the principal *-ideal generated by a.

Lemma 2.2.12. *Let* $\alpha, \beta \in K^-(n, r)$. *If* $\alpha \in J^*(\beta)$ *then* $|\mathrm{im}\ \alpha| \leq |\mathrm{im}\ \beta|$.

Proof. Suppose that $\alpha \in J^*(\beta)$. Then (by Lemma 1.2.2) there exist $\beta_0, \beta_1, \ldots, \beta_m \in K^-(n, r)$, $\delta_1, \ldots, \delta_m, \gamma_1, \ldots, \gamma_m \in (K^-(n, r))^1$ such that $\beta = \beta_0$,

$\alpha = \beta_m$ and $(\beta_i, \delta_i\beta_{i-1}\gamma_i) \in \mathcal{D}^*$ for $i = 1, \ldots, m$. So by Corollary 2.2.11

$$|\text{im } \beta_i| = |\text{im}(\delta_i\beta_{i-1}\gamma_i)| \leq |\text{im } \beta_{i-1}| \quad \text{for all } i = 1, \ldots, m.$$

Hence

$$|\text{im } \alpha| \leq |\text{im } \beta|.$$ ■

Lemma 2.2.13 *On the semigroup* $K^-(n, r)$, $\mathcal{D}^* = \mathcal{J}^*$.

Proof. Note that we need only show that $\mathcal{J}^* \subseteq \mathcal{D}^*$ (since $\mathcal{D}^* \subseteq \mathcal{J}^*$). So suppose that $(\alpha, \beta) \in \mathcal{J}^*$, so that $J^*(\alpha) = J^*(\beta)$. Then $\alpha \in J^*(\beta)$ and $\beta \in J^*(\alpha)$, and by Lemma 2.2.12 this implies that

$$|\text{im } \alpha| \leq |\text{im } \beta|, \quad |\text{im } \beta| \leq |\text{im } \alpha|.$$

Thus

$$|\text{im } \alpha| = |\text{im } \beta|,$$

and so $(\alpha, \beta) \in \mathcal{D}^*$ by Corollary 2.2.11. ■

We observe that $K^-(n, r)$ is a *-ideal since it is a union of $\mathcal{J}^*$-classes (of S_n^-)

$$J_1^*, J_2^*, \ldots, J_r^*$$

where

$$J_k^* = \{ \alpha \in K^-(n, r) : |\text{im } \alpha| = k \}.$$

Finally (in this section) we show by an example that in the semigroup $K^-(n, r)$, ($n \geq 3$ & $r \geq 2$) $\mathcal{R}^* \circ \mathcal{L}^* \neq \mathcal{D}^* \neq \mathcal{L}^* \circ \mathcal{R}^*$.

Example 2.2.14. Let $\alpha, \beta \in K^-(n, r)$ be defined (respectively) by

$$x\alpha = \begin{cases} 1 & (\text{if } x = 1) \\ 2 & (\text{if } x \neq 1) \end{cases}; \quad x\beta = \begin{cases} n & (\text{if } x = n) \\ 1 & (\text{if } x \neq n). \end{cases}$$

Then clearly $(\alpha, \beta) \in \mathcal{D}^*$ and if $(\alpha, \beta) \in \mathcal{R}^* \circ \mathcal{L}^*$, then there must exist γ in $K^-(n, r)$ such that $\alpha \, \mathcal{R}^* \, \gamma \, \mathcal{L}^* \, \beta$. However, by Lemma 2.2.6 γ can either be

$$\begin{pmatrix} 1 & \{2, \dots, n\} \\ 1 & n \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & \{2, \dots, n\} \\ n & 1 \end{pmatrix}$$

and in both cases $\gamma \notin K^-(n, r)$ (since $n \geq 3$). Hence $\mathcal{D}^* \neq \mathcal{R}^* \circ \mathcal{L}^*$.

Similarly, $(\beta, \alpha) \in \mathcal{D}^*$, and if $(\beta, \alpha) \in \mathcal{L}^* \circ \mathcal{R}^*$ then there must exist $\delta \in K^-(n, r)$ such that $\beta \,\mathcal{L}^*\, \delta \,\mathcal{R}^*\, \alpha$. Again by Lemma 2.2.6 δ can either be

$$\begin{pmatrix} 1 & \{2, \dots, n\} \\ 1 & n \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & \{2, \dots, n\} \\ n & 1 \end{pmatrix}$$

and in both cases $\delta \notin K^-(n, r)$ (since $n \geq 3$). Hence $\mathcal{D}^* \neq \mathcal{L}^* \circ \mathcal{R}^*$.

Remark 2.2.15. Notice that the results obtained for $K^-(n, r)$ in this section and the previous one are extensions of the results obtained for S_n^- in [34].

3. Rees quotient semigroups

Now since $K^-(n, r)$ is a two-sided ideal let

$$P_r = K(n, r)/K(n, r-1) \tag{3.1}$$

$$P_r^- = K^-(n, r)/K^-(n, r-1) \tag{3.2}$$

be the Rees quotient semigroups on the two-sided ideals $K(n, r)$, $K^-(n, r)$ respectively. Then P_r is an idempotent-generated completely [0-] simple semigroup whose non-zero elements may be thought of as the elements of T_n of rank r precisely. The product of two elements of P_r is 0 whenever their product in T_n is of rank strictly less than r. Similarly P_r^- is an idempotent-generated (by Theorem 2.1.5) Rees quotient semigroup whose non-zero elements may be thought of as the elements of S_n^- of rank r precisely. The product of two elements of P_r^- is 0 whenever their product in S_n^- is of rank strictly less than r.

Since there are several analogues of complete 0-simplicity for 0-$\mathcal{J}^*$-simple semigroups we content ourselves with showing that P_r^- is a primitive abundant 0-$\mathcal{J}^*$-simple semigroup. But first let us characterize the Green's relations on P_r^-.

Lemma 2.3.1. *Let* $\alpha, \beta \in P_r^-$. *Then*

(1) $(\alpha, \beta) \in \mathcal{R}$ *if and only if* $\alpha = \beta$;

(2) $(\alpha, \beta) \in \mathcal{L}$ *if and only if* $\operatorname{im} \alpha = \operatorname{im} \beta$ *and* $\min z\alpha^{-1} = \min z\beta^{-1}$ *for all* z *in* $\operatorname{im} \alpha$.

Proof. (1) The proof is similar to that of Lemma 2.2.1(1).
(2) The proof is similar to that of Lemma 2.2.1(2). ■

Corollary 2.3.2. *On the semigroup* P_r^-, $\mathcal{H} = \mathcal{R}$ *and* $\mathcal{L} = \mathcal{D} = \mathcal{J}$.

Lemma 2.3.3. P_r^- *is an inverse ideal of* P_r.

Proof. The proof is similar to that of Lemma 2.2.4. ■

Hence by Lemmas 1.2.7 & 1.2.8 and [5, Lemmas 10.55 & 10.56] we deduce the following result:

Theorem 2.3.4. *Let* P_r^- *be as defined in* (3.2). *Then* P_r^- *is a non-regular abundant semigroup. Moreover, for* $\alpha, \beta \in P_r^-$ *we have*

(1) $(\alpha, \beta) \in \mathcal{L}^*$ *if and only if* $\operatorname{im} \alpha = \operatorname{im} \beta$;

(2) $(\alpha, \beta) \in \mathcal{R}^*$ *if and only if* $\alpha \circ \alpha^{-1} = \beta \circ \beta^{-1}$;

(3) $(\alpha, \beta) \in \mathcal{H}^*$ *if and only if* $\operatorname{im} \alpha = \operatorname{im} \beta$ *and* $\alpha \circ \alpha^{-1} = \beta \circ \beta^{-1}$.

Using the same techniques as in Section 2 we obtain a characterization of the relation $\mathcal{D}^*$ on P_r^-. In fact the proof is exactly the same as the proof of Lemma 2.2.10.

Lemma 2.3.5. *On the semigroup* P_r^-, *we have the following:*

(1) $\mathcal{D}^* = \mathcal{R}^* \circ \mathcal{L}^* \circ \mathcal{R}^* = \mathcal{L}^* \circ \mathcal{R}^* \circ \mathcal{L}^*$,

(2) $(\alpha, \beta) \in \mathcal{D}^*$ *if and only if* $|\operatorname{im} \alpha| = |\operatorname{im} \beta|$ *for all* $\alpha, \beta \in P_r^-$.

Theorem 2.3.6. *Let* P_r^- *be as defined in* (3.2). *Then* P_r^- *is a primitive abundant* 0^**-bisimple semigroup.*

Proof. Notice that it only remains to show that $E(P_r^-)$ is primitive. However, since $E(P_r)$ is primitive and $E(P_r^-) \subseteq E(P_r)$, it follows that $E(P_r^-)$ is primitive. ■

Theorem 2.3.7. *Let* P_r^- *be as defined in* (3.2) *and let* $N(P_r^-) = \{\alpha \in P_r^- : f(\alpha) < r\}$. *Then*

(1) α *is nilpotent if and only if* $\alpha \in N(P_r^-)$;

(2) $N(P_r^-)$ *is an ideal of* P_r^-.

Proof. First notice that for all $\alpha, \beta \in P_r^-$

$$F(\alpha\beta) = F(\alpha) \cap F(\beta) = F(\beta) \cap F(\alpha) = F(\beta\alpha).$$

(1) Let $\alpha \in P_r^-$ be such that $f(\alpha) < r$. Then α^k is an idempotent (for some $k > 1$) since P_r^- is finite. Moreover, $f(\alpha^k) = f(\alpha) < r$, so that $\alpha^k = 0$ (since 0 is the only idempotent in P_r^- for which $f(\alpha) < r$). Hence α is nilpotent.

(2) Let $\alpha \in N(P_r^-)$ and $\beta \in P_r^-$. Then clearly $f(\alpha\beta) = f(\beta\alpha) < r$, so that $\alpha\beta, \beta\alpha \in N(P_r^-)$ as required. ■

4. Finite order-decreasing partial transformations.

From the results obtained for the full transformation case it is shown in this section that we can deduce the corresponding results for the partial case using Vagner's result in [**39**]. Let P_n^* be the zero stabilizer subsemigroup of T_n^0, the full transformation semigroup on X_n^0 $(= X_n \cup \{0\})$ excluding the identity transformation, let P_n be the semigroup of all partial transformations on X_n and let

$$SP_n = \{\alpha \in P_n : |\mathrm{im}\,\alpha| \le n - 1\}$$

be the subsemigroup of strictly partial transformations on X_n. Let $(S_n^0)^-$ be the semigroup of order-decreasing full transformations of X_n^0 and let

$$P_n^- = \{\alpha \in SP_n :(\forall x \in \text{dom } \alpha)\ x\alpha \le x\} \cup \{\emptyset\} \tag{4.1}$$

$$P_n^+ = \{\alpha \in SP_n :(\forall x \in \text{dom } \alpha)\ x\alpha \ge x\} \cup \{\emptyset\} \tag{4.2}$$

be the subsets of SP_n consisting of all order-decreasing strictly partial transformations of X_n (including the empty or zero transformation) respectively. Also let

$$PK(n, r) = \{ \alpha \in P_n : |\text{im } \alpha| \le r\} \tag{4.3}$$

$$PK^-(n, r) = \{ \alpha \in (P_n^-)^1 : |\text{im } \alpha| \le r\} \tag{4.4}$$

$$PK^+(n, r) = \{ \alpha \in (P_n^+)^1 : |\text{im } \alpha| \le r\} \tag{4.5}$$

Thus $PK^-(n, n) = (P_n^-)^1$, $PK^-(n, n-1) = P_n^-$ and each $PK^-(n, r)$ is a two-sided ideal of $(P_n^-)^1$. Now since $PK^-(n, r)$ is a two-sided ideal let

$$PP_r(n) = PK(n, r)/PK(n, r-1) \tag{4.6}$$

$$PP_r^-(n)= PK^-(n, r)/PK^-(n, r-1) \tag{4.7}$$

be the Rees quotient semigroups on the two-sided ideals PK(n, r), $PK^-(n, r)$ respectively.

Lemma 2.4.1. *Let* $P^*(n, r) = \{\alpha \in P_n^* : |\text{im } \alpha| \le r\}$ *and let* $P_r^*(n) = P^*(n, r)/P^*(n, r-1)$. *Then* $K^-(n+1, r) \subseteq P^*(n, r)$ *and* $P_r^-(n+1) \subseteq P_r^*(n)$.

As in **[11]** we now record the result of Vagner **[39]** (also to be found in **[6**, p. 254]) restricted to the finite case.

Theorem 2.4.2. *For each* $\alpha \in P_n$, *define the transformation* α^* of X_n^0 *by*

$$x\alpha^* = \begin{cases} x\alpha & (\text{if } x \in \text{dom } \alpha) \\ 0 & (\text{if } x \notin \text{dom } \alpha) \end{cases}$$

Then α^* *belongs to the subsemigroup* P_n^* *of* T_n^0.

Conversely, if $\beta \in P_n^*$, *then its restriction to* X_n, $\beta|_{X_n} = \beta \cap (X_n \times X_n)$, *is a partial transformation of* X_n. *The domain of* $\beta|_{X_n}$ *is the set of all* x *in* X_n *for which* $x\beta \ne 0$. *Then the mappings* $\alpha \to \alpha^*$ *and* $\beta \to \beta|_{X_n}$ *are mutually inverse isomorphisms of* P_n *onto* P_n^* *and vice-versa.*

From Lemma 2.4.1 and Theorem 2.4.2 we observe that the isomorphism $\alpha \to \alpha^*$ maps $PK^-(n, r)$ onto $K^-(n + 1, r + 1)$; and $PP_r^-(n)$ onto $P_{r+1}^-(n + 1)$ since for all $x \in \text{dom}\, \alpha$

$$x\alpha^* \leq x \quad \text{if and only if} \quad x\alpha \leq x$$

and

$$x\alpha^* = 0 \leq x \quad (\text{for all } x \notin \text{dom}\, \alpha).$$

For convenience we record this as a corollary.

Corollary 2.4.3. *Let* $\theta : \alpha \to \alpha^*$ *be the isomorphism defined in Theorem 2.4.2. Then* $(PK^-(n, r))\theta = K^-(n + 1, r + 1)$, $(K^-(n + 1, r + 1))\theta^{-1} = PK^-(n, r)$, $(PP_r^-(n))\theta = P_{r+1}^-(n + 1)$ *and* $(P_{r+1}^-(n + 1))\theta^{-1} = PP_r^-(n)$.

Now let $PS^-(n, r)$ be any of the semigroups $PK^-(n, r)$ or $PP_r^-(n)$ for $1 \leq r \leq n - 1$. Immediate consequences of Corollary 2.4.3 are the following:

Theorem 2.4.4. *Let* $PS^-(n, r)$ *be any of the semigroups* $PK^-(n, r)$ *or* $PP_r^-(n)$ *for* $1 \leq r \leq n - 1$. *Then*

(1) $PS^-(n, r)$ *is idempotent-generated;*

(2) $PS^-(n, r)$ *is* $\mathcal{R}$*-trivial;*

(3) $(\alpha, \beta) \in \mathcal{L}$ *if and only if* $\text{im}\, \alpha = \text{im}\, \beta$ *and* $\min z\alpha^{-1} = \min z\beta^{-1}$ *for all* $z \in \text{im}\, \alpha$ $(\alpha, \beta \in PS^-(n, r))$.

Proof. (1) follows from Theorem 2.1.5, while (2) & (3) follow Lemma 2.2.1. ■

Corollary 2.4.5. *On the semigroup* $PS^-(n, r)$, $\mathcal{H} = \mathcal{R}$ *and* $\mathcal{L} = \mathcal{D} = \mathcal{J}$.

Lemma 2.4.6. *Let* $\alpha \in PK^-(n, r)$ *or* $PP_r^-(n)$. *Then* α *is nilpotent if and only if* $f(\alpha) = 0$ *or* $f(\alpha) < r$ *respectively.*

Proof. These follow from Lemma 2.1.6 and Theorem 2.3.7(1) respectively. ■

Lemma 2.4.7. *Let* $N(PK^-(n, r)) = \{\alpha \in PK^-(n, r) : f(\alpha) = 0\}$ *and let* $N(PP_r^-(n)) = \{\alpha \in PP_r^-(n) : f(\alpha) < r\}$. *Then* $N(PK^-(n, r))$ *is an ideal of* $PK^-(n, r)$ and $N(PP_r^-(n))$ *is an ideal of* $PP_r^-(n)$.

Proof. These follow from Lemmas 2.1.7 and Theorem 2.3.7(2) respectively. ■

Theorem 2.4.8. *Let* $PS^-(n, r)$ *be any of the semigroups* $PK^-(n, r)$ *or* $PP_r^-(n)$ *for* $1 < r \leq n - 1$. *Then* $PS^-(n, r)$ *is a non-regular abundant semigroup and for* $\alpha, \beta \in PS^-(n, r)$ *we have*

(1) $(\alpha, \beta) \in \mathcal{R}^*$ *if and only if* $\alpha \circ \alpha^{-1} = \beta \circ \beta^{-1}$;

(2) $(\alpha, \beta) \in \mathcal{L}^*$ *if and only if* $\operatorname{im} \alpha = \operatorname{im} \beta$;

(3) $(\alpha, \beta) \in \mathcal{H}^*$ *if and only if* $\alpha \circ \alpha^{-1} = \beta \circ \beta^{-1}$ *and* $\operatorname{im} \alpha = \operatorname{im} \beta$;

(4) $(\alpha, \beta) \in \mathcal{D}^*$ *if and only if* $|\operatorname{im} \alpha| = |\operatorname{im} \beta|$;

(5) $\mathcal{D}^* = \mathcal{J}^*$.

Proof. That $PS^-(n, r)$ is a non-regular abundant semigroup follows from Theorems 2.2.5 & 2.3.4; (1) & (2) follow from Lemma 2.2.6 and Theorem 2.3.4; (3) follows from (1) & (2) while (4) & (5) follow from Corollary 2.2.11 and Lemmas 2.2.13 & 2.3.5. ■

Remarks 2.4.9. (a) Results for $PS^+(n, r)$ could be deduced from the corresponding results for $PS^-(n, r)$.

(b) The results obtained in this section and the previous one are to appear in [37].

5. The minimum semilattice congruence

Let $S^{-}(n, r)$ be any of the semigroups $K^{-}(n, r)$ or $P_r^{-}(n)$ for $1 \le r \le n$ and define a relation $\rho^{\#}$ on $S^{-}(n, r)$ by the rule that

$$\alpha \, \rho^{\#} \, \beta \quad \text{iff} \quad F(\alpha) = F(\beta) \tag{5.1}$$

Lemma 2.5.1. Let $\rho^{\#}$ be as defined in (5.1). Then $\rho^{\#}$ is a semilattice congruence on $S^{-}(n, r)$.

Proof. Clearly $\rho^{\#}$ is an equivalence relation. To show that $\rho^{\#}$ is left compatible, let $\alpha \, \rho^{\#} \, \beta$, i.e. $F(\alpha) = F(\beta)$. However for all $\gamma \in S^{-}(n, r)$

$$F(\gamma\alpha) = F(\gamma) \cap F(\alpha) = F(\gamma) \cap F(\beta) = F(\gamma\beta)$$

(by Lemma 2.1.3). We can similarly show that $\rho^{\#}$ is right compatible. Hence $\rho^{\#}$ is a congruence. Moreover, $\rho^{\#}$ is a semilattice congruence since (by Lemma 2.1.3(2))

$$F(\alpha\beta) = F(\beta\alpha) \quad (\text{for all } \alpha, \beta \in K^{-}(n, r)). \qquad \blacksquare$$

Lemma 2.5.2. Let $\varepsilon, \eta \in E(S^{-}(n, r))$. Then $(\varepsilon, \eta) \in \rho^{\#}$ if and only if $(\varepsilon, \eta) \in \mathfrak{L}^{*}$.

Proof. Let $\varepsilon, \eta \in E(S^{-}(n, r))$. Then

$$\begin{aligned} (\varepsilon, \eta) \in \rho^{\#} \quad & \text{iff} \quad F(\varepsilon) = F(\eta) \\ \text{i.e.} \quad & \text{iff} \quad \operatorname{im} \varepsilon = \operatorname{im} \eta \\ \text{i.e.} \quad & \text{iff} \quad (\varepsilon, \eta) \in \mathfrak{L}^{*}. \end{aligned} \qquad \blacksquare$$

Let $E(I_{n-1})$ be the semilattice of all partial identities on X_{n-1} and let

$$E(n-1, r) = \{ \alpha \in E(I_{n-1}) : |\operatorname{im} \alpha| \le r \}.$$

Then writing $EP(n-1, r)$ for $E(n-1, r) / E(n-1, r-1)$, we have

Lemma 2.5.3. $K^{-}(n, r) / \rho^{\#} \cong E(n-1, r)$ and $P_r^{-}(n) / \rho^{\#} \cong EP(n-1, r)$.

Proof. From each $\alpha\rho^{\#}$ (in $K^{-}(n, r)/\rho^{\#}$) choose an element ε_α, defined by

$$x\varepsilon_\alpha = \begin{cases} x & (\text{if } x \in F(\alpha)) \\ 1 & (\text{otherwise}) \end{cases} .$$

It is clear that (for all $\alpha, \beta \in K^{-}(n, r)$)

$$\varepsilon_\alpha\varepsilon_\beta = \varepsilon_{\alpha\beta} = \varepsilon_{\beta\alpha} = \varepsilon_\beta\varepsilon_\alpha \qquad \text{(by Lemma 2.1.3)}$$

so that $U = \{\varepsilon_\alpha : \alpha \in K^{-}(n, r)\}$ is a semilattice isomorphic to $E(n-1, r)$ (since $1 \in F(\alpha)$, for all $\alpha \in K^{-}(n, r)$). Moreover, the map $\alpha\rho^{\#} \to \varepsilon_\alpha$ from $K^{-}(n, r)/\rho^{\#}$ onto U is an isomorphism. Hence

$$S^{-}(n, r)/\rho^{\#} \cong U \cong E(n-1, r).$$

Similarly we can show that

$$P_r^{-}(n)/\rho^{\#} \cong EP(n-1, r).$$

Thus the proof is complete. ■

Now for a given $\alpha \in S$ (any finite semigroup), let $\alpha^k = \varepsilon_\alpha \in E(S)$ for some $k \geq 1$. Then we have the following result:

Lemma 2.5.4. Let ρ be a semilattice congruence on a semigroup S. Then $(\alpha, \varepsilon_\alpha) \in \rho$ for all $\alpha \in S$.

Proof. Since ρ is a semilattice congruence, then for all $\alpha \in S$

$$\alpha\rho = (\alpha\rho)^2 = \alpha^2\rho$$

$$\Rightarrow \quad \alpha\rho = \alpha^k\rho \qquad (\forall k \geq 1.)$$

Therefore $(\alpha, \varepsilon_\alpha) \in \rho$ for all $\alpha \in S$. ■

Thus we now have the main result of this section:

Theorem 2.5.5. Let $S^{-}(n, r)$ be any of the semigroups $K^{-}(n, r)$ or $P_r^{-}(n)$ for $1 \leq r \leq n$ and let $\rho^{\#}$ be as defined in (5.1). Then $\rho^{\#}$ is the minimum semilattice congruence on $S^-(n, r)$. Moreover, the maximum semilattice image of $S^-(n, r)$ $(P_r^-(n))$ is $E(n-1, r)$ $(EP(n-1, r))$.

Proof. Since we have already shown that $\rho^\#$ is a semilattice congruence it now remains to show that for any semilattice congruence ρ on $S^-(n, r))$, $\rho^\# \subseteq \rho$. So suppose that $(\alpha, \beta) \in \rho^\#$. Then $(\varepsilon_\alpha, \varepsilon_\beta) \in \rho^\#$ (by Lemma 2.5.4) and $(\varepsilon_\alpha, \varepsilon_\beta) \in \mathfrak{L}^*$ (by Lemma 2.5.2), so that

$$(\varepsilon_\alpha)\rho = (\varepsilon_\alpha\varepsilon_\beta)\rho = (\varepsilon_\alpha)\rho(\varepsilon_\beta)\rho = (\varepsilon_\beta)\rho(\varepsilon_\alpha)\rho = (\varepsilon_\beta\varepsilon_\alpha)\rho = (\varepsilon_\beta)\rho.$$

Hence

$$(\alpha, \beta) \in \rho$$

since

$$(\alpha, \varepsilon_\alpha) \in \rho\,,\ (\varepsilon_\alpha, \varepsilon_\beta) \in \rho\,,\ (\varepsilon_\beta, \beta) \in \rho$$

by Lemma 2.5.4. Thus

$$\rho^\# \subseteq \rho$$

as required.

The last statement of the theorem that the maximum semilattice image of $S^-(n, r)$ $(P_r^-(n))$ is $E(n-1, r)$ $(EP(n-1, r))$ follows from Lemma 2.5.3. ■

CHAPTER 3

FINITE ORDER-DECREASING PARTIAL ONE-TO-ONE TRANSFORMATION SEMIGROUPS

1. Preliminaries

Let $X_n = \{1, \dots, n\}$, let I_n be the symmetric inverse semigroup on X_n, and let

$$SI_n = \{ \alpha \in I_n : |\mathrm{dom}\ \alpha| \leq n - 1\}$$

be the subsemigroup of strictly partial one-to-one self-maps of X_n. Also let

$$I_n^- = \{ \alpha \in SI_n : (\text{for all } x \in X_n)\ x\alpha \leq x\} \cup \{\emptyset\} \tag{1.1}$$

$$I_n^+ = \{ \alpha \in SI_n : (\text{for all } x \in X_n)\ x\alpha \geq x\} \cup \{\emptyset\} \tag{1.2}$$

be the subsets of SI_n consisting of all order-decreasing and order-increasing partial one-to-one maps both including the empty or zero map of X_n respectively. For $1 \leq r \leq n$, let

$$L(n, r) = \{ \alpha \in I_n : |\mathrm{im}\ \alpha| \leq r\} \tag{1.3}$$

$$L^-(n, r) = \{ \alpha \in (I_n^-)^1 : |\mathrm{im}\ \alpha| \leq r\} \tag{1.4}$$

$$L^+(n, r) = \{ \alpha \in (I_n^+)^1 : |\mathrm{im}\ \alpha| \leq r\} \tag{1.5}$$

Thus $L^-(n, n) = (I_n^-)^1$, $L^-(n, n - 1) = I_n^-$ and each $L^-(n, r)$ is a two-sided ideal of $(I_n^-)^1$.

Lemma 3.1.1. $L^-(n, r)$ *and* $L^+(n, r)$ *are isomorphic subsemigroups of* SI_n.

Proof. Let $\alpha, \beta \in L^-(n, r)$. Then for all $x \in \mathrm{dom}\ \alpha\beta$

$$(x\alpha)\beta \leq x\alpha \leq x$$

so that $\alpha\beta \in L^-(n, r)$ as required. Similarly we can show that $\alpha\beta \in L^+(n, r)$ for all $\alpha, \beta \in L^+(n, r)$. Thus $L^-(n, r)$ and $L^+(n, r)$ are subsemigroups of SI_n. Now define a map $\theta: L^-(n, r) \to L^+(n, r)$ by

$$\theta(\alpha) = \bar{\alpha} \quad (\alpha \in L^-(n, r), \bar{\alpha} \in L^+(n, r))$$

where

$$i\bar{\alpha} = n - (n - i + 1)\alpha + 1 \qquad (i \in \operatorname{dom} \bar{\alpha}).$$

Clearly θ is a bijection and

$$i\bar{\alpha} = n - (n - i + 1)\alpha + 1 \geq n - (n - i + 1) + 1 = i.$$

Moreover,

$$\begin{aligned}(i\bar{\alpha})\bar{\beta} &= n - (n - i\bar{\alpha} + 1)\beta + 1\\ &= n - \{n - [n - (n - i + 1)\alpha + 1] + 1\}\beta + 1\\ &= n - (n - i + 1)\alpha\beta + 1\\ &= i\overline{\alpha\beta}.\end{aligned}$$

Thus θ is an isomorphism. ■

Remark 3.1.2.(a) In view of the above Lemma results for $L^+(n, r)$ can be deduced from those for $L^-(n, r)$ in an obvious manner.

(b) It is easy to see that $L^-(n, r)$ is a *full* subsemigroup of $L(n, r)$. Hence $E(L^-(n, r))$ is a semilattice.

Recall that $f(\alpha)$ is the cardinal of the set

$$F(\alpha) = \{ x \in X_n : x\alpha = x \}.$$

Lemma 3.1.3. *Let* $\alpha, \beta \in L^-(n, r)$. *Then*

(1) $F(\alpha\beta) = F(\alpha) \cap F(\beta)$;

(2) $F(\alpha\beta) = F(\beta\alpha)$.

Proof. The proof is similar to that of Lemma 2.1.3. ■

It is clear that $L^-(n, r)$ cannot be idempotent-generated, since $E(L^-(n, r))$ is a semilattice and it is not nilpotent-generated as we shall see later. However if we define a *quasi-idempotent* as an element α for which $(\alpha^2)^2 = \alpha^2$; i.e., α^2 is an idempotent, then we see that $L^-(n, r)$ is generated by its set of quasi-idempotents. Clearly all idempotents are quasi-idempotents but not vice-versa. Notice also that in $L^-(n, r)$, $\alpha^4 = \alpha^2$ implies that $\alpha^3 = \alpha^2$ since

$$x\alpha^2 \geq x\alpha^3 \geq x\alpha^4 \quad (x \in \text{dom } \alpha^2.)$$

Now we begin our investigation by characterising quasi-idempotents in $L^-(n, r)$. But first recall that $s(\alpha)$ is the cardinal of the set

$$S(\alpha) = \{ x \in X_n : x\alpha \neq x \}.$$

Lemma 3.1.4. *Let* $\alpha \in L^-(n, r)$. *Then the following statements are equivalent*

(1) α *is a quasi-idempotent*;

(2) $x \in S(\alpha)$ *implies* $x\alpha \notin \text{dom } \alpha$;

(3) $S(\alpha)\alpha \cap \text{dom } \alpha$ *is empty;*

(4) $S(\alpha) \cap \text{im } \alpha$ *is empty*.

Proof. (1) $\Rightarrow$ (2) First notice that if α is an idempotent in $L^-(n, r)$ then $S(\alpha)$ is empty and there is nothing to prove. On the other hand if α is a (non-idempotent) quasi-idempotent in $L^-(n, r)$ then for all x in $S(\alpha)$, $x\alpha \neq x$. If $x\alpha$ is in dom α then $x\alpha \neq x\alpha^2$ (since α is one-to-one) and

$$x > x\alpha > x\alpha^2,$$

which is a contradiction as $x\alpha^2 = x$, so $x\alpha \notin \text{dom } \alpha$.

(2) $\Rightarrow$ (3) This is clear.

(3) $\Rightarrow$ (4) If $y \in S(\alpha) \cap \text{im } \alpha$ then $y = x\alpha$ for some x. Since $y \in \text{dom } \alpha$ we must have that $y \notin S(\alpha)\alpha$ (by (3)). Therefore $y = x$ and so $y = y\alpha$ which contradicts the assumption that $y \in S(\alpha)$.

(4) $\Rightarrow$ (1) Suppose that $S(\alpha) \cap \text{im } \alpha$ is empty, and let $y \in \text{dom } \alpha$. Then $y\alpha \in \text{im } \alpha$ and so $y\alpha \notin S(\alpha)$. Hence either $y\alpha \in \text{dom } \alpha$ and $y\alpha^2 = y\alpha$ or $y\alpha \notin \text{dom } \alpha$. In the former case we have $y \in \text{dom } \alpha^2$ and $y\alpha^2 = y$. In the latter case we

have $y \notin \operatorname{dom} \alpha^2$. Thus $y\alpha^2 = y$ for all y in $\operatorname{dom} \alpha^2$, and so α^2 is idempotent. ■

Corollary 3.1.5. *Let* $\alpha \in L^-(n, r)$. *Then* α *is a quasi-idempotent if* $s(\alpha) \leq 1$.

Lemma 3.1.6. *Let* $\alpha \in L^-(n, r)$ *such that* $|\operatorname{im} \alpha| = k \leq r$. *Then* α *is expressible as a product of quasi-idempotents* η_i (in $L^-(n, r)$) *for which* $|\operatorname{im} \eta_i| = k$ *and* $s(\eta_i) = 1$.

Proof. Suppose that

$$\alpha = \begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ b_1 & b_2 & \cdots & b_k \end{pmatrix} \in L^-(n, r)$$

with $S(\alpha) = \{a_1, \ldots, a_t\}$ and $a_1 < a_2 < \ldots < a_t$ for some $0 < t \leq k$. Define η_1 by

$$a_1\eta_1 = b_1 \ (< a_1), \quad a_j\eta_1 = a_j \ (j \neq 1).$$

Recursively, for $i = 2, \ldots, t$ we define η_i to have domain equal to

$$\operatorname{im} \eta_{i-1} = \{b_1, \ldots, b_{i-1}, a_i, \ldots, a_k\}$$

and

$$a_i\eta_i = b_i, \quad a_j\eta_i = a_j \ (j > i), \quad b_j\eta_i = b_j \ (j \leq i-1).$$

Then clearly $\alpha = \eta_1\eta_2 \ldots \eta_t$. Moreover, $\eta_i \in L^-(n, r)$ and $s(\eta_i) = 1$ for all i. Thus η_i is a quasi-idempotent (by Corollary 3.1.5.) ■

Lemma 3.1.7. *Let* η *be a quasi-idempotent in* $L^-(n, r)$ *such that* $|\operatorname{im} \eta| = k \leq r$ *and* $s(\eta) = 1$. *Then* η *is expressible as a product of quasi-idempotents* γ_i *in* $L^-(n, r)$ *such that* $|\operatorname{im} \gamma_i| = r$ *and* $s(\gamma_i) \leq 1$.

Proof. Suppose that

$$\eta = \begin{pmatrix} a_1 \ldots a_t & \cdots & a_k \\ a_1 \ldots b_t & \cdots & .a_k \end{pmatrix} \quad (a_t > b_t.)$$

Let $Y_r \subseteq X_n$ such that $|Y_r| = r$, $b_t \notin Y_r$ and $\operatorname{dom} \eta \subseteq Y_r$. Now define γ_1 by

$$x\gamma_1 = x\eta \quad (x \in \operatorname{dom} \eta), \quad y\gamma_1 = y \quad (y \in Y_r \setminus \operatorname{dom} \eta).$$

Then clearly $|\operatorname{im} \gamma_1| = r$ and $s(\gamma_1) = 1$. Moreover, $\eta = \operatorname{id}_{\operatorname{dom} \eta} \cdot \gamma_1$. However, since

$id_{\mathrm{dom}\,\eta}$ is expressible as a product of idempotents γ_i for which $|\mathrm{im}\ \gamma_i| = r$, then the result follows. ■

Hence by Lemma 3.1.6 it follows that $L^-(n, r)$ is generated by its set of quasi-idempotents. However, we are going to show that this result can be sharpened. But first we introduce a new concept. For a given α in $L^-(n, r)$, let

$$A(\alpha) = \{\ y \in X_n : (\exists x \in X_n)\ x\alpha < y < x\}$$
$$= \{\ y \in X_n : (\exists x \in S(\alpha))\ x\alpha < y < x\}.$$

An element η in $L^-(n, r)$ is called *amenable* if $s(\eta) \leq 1$ and $A(\eta) \subseteq \mathrm{dom}\ \eta$. Observe that all amenable elements are quasi-idempotents; all idempotents are amenable since $A(\eta)$ is empty if η is an idempotent; and all the quasi-idempotents for which $|\mathrm{im}\ \alpha| = n - 1$ are amenable.

Lemma 3.1.8. *Let* $\eta \in L^-(n, r)$ *be a quasi-idempotent such that* $|\mathrm{im}\ \eta| = r$ *and* $s(\eta) = 1$. *Then* η *is expressible as a product of amenable elements* $\varepsilon_i \in L^-(n, r)$ *such that* $|\mathrm{im}\ \varepsilon_i| = r$.

Proof. Suppose that

$$\eta = \begin{pmatrix} a_1 \ldots a_k \ldots a_r \\ a_1 \ldots b_k \ldots a_r \end{pmatrix}$$

is a quasi-idempotent in $L^-(n, r)$. Let $B = \{y \in X_n : b_k \leq y < a_k\} \setminus F(\eta)$ $(= \{y_1, \ldots, y_t = b_k\}$ such that $y_1 > y_2 > \ldots > y_t$, say.) Define ε_i (recursively) by

$$y_{i-1}\varepsilon_i = y_i, \quad a_j\varepsilon_i = a_j \quad (j \neq k),$$

where $y_0 = a_k$, $\mathrm{dom}\ \varepsilon_1 = \mathrm{dom}\ \eta$ and $\mathrm{dom}\ \varepsilon_i = \mathrm{im}\ \varepsilon_{i-1}$ $(i \geq 2)$. Then clearly ε_i is a decreasing amenable element for all $i \in \{1, \ldots, t\}$. Moreover,

$$\eta = \varepsilon_1\varepsilon_2\ldots\varepsilon_t.$$

Hence the proof. ■

Recall that an element α in $L^-(n, r)$ is called *nilpotent* if $\alpha^k = 0$ $(= \emptyset$, in this case) for some $k > 0$ and $F(\alpha) = \{x \in X_n : x\alpha = x\}$. As in Section II.1 we obtain a characterization of nilpotents in $L^-(n, r)$.

Lemma 3.1.9. *Let* $\alpha \in L^-(n, r)$. *Then* α *is nilpotent if and only if* $F(\alpha)$ *is empty.*

Proof. First notice that if α is nilpotent then it is fairly obvious that $F(\alpha)$ is empty.

Conversely, suppose that $F(\alpha)$ is empty. Then $x\alpha < x$ for all x in $\operatorname{dom} \alpha$ and the descent

$$x > x\alpha > x\alpha^2 > \ldots,$$

must terminate at some stage; i.e.

$$x\alpha^{k_x} \notin \operatorname{dom} \alpha \quad \text{for some } k_x > 0.$$

Now let $k = \max\{ k_x : x \in X_n\}$. Then

$$x\alpha^k \notin \operatorname{dom} \alpha \quad (\text{for all } x \in X_n)$$

and hence α is nilpotent. ■

Lemma 3.1.10. *Let* $N(L^-(n, r)) = \{ \alpha \in L^-(n, r) : F(\alpha)$ *is empty*$\}$. *Then* $N(L^-(n, r))$ *is an ideal of* $L^-(n, r)$.

Proof. Let $\alpha \in N(L^-(n, r))$ and $\beta \in L^-(n, r)$. Then since $x\alpha\beta \leq x\alpha < x$ for all $x \in \operatorname{dom} \alpha\beta$, it follows that $\alpha\beta \in N(L^-(n, r))$. Also, since $x\beta\alpha < x\beta \leq x$ for all $x \in \operatorname{dom}\beta\alpha$, it follows that $\beta\alpha \in N(L^-(n, r))$. Thus $N(L^-(n, r))$ is an ideal of $L^-(n, r)$ as required. ■

2. Green's and starred Green's relations

Lemma 3.2.1. $L^-(n, r)$ *is* $\mathcal{J}$*-trivial.*

Proof. Notice that since $\mathcal{D} = \mathcal{J}$ on $L^-(n, r)$ (by [**20**, Proposition II.1.5]), then it suffices to show that $L^-(n, r)$ is both $\mathcal{R}$-trivial and $\mathcal{L}$-trivial. However, since $L^-(n, r)$ is a subsemigroup of P_n^-, then (by Theorem 2.4.4) $L^-(n, r)$ is $\mathcal{R}$-trivial and for all $z \in \operatorname{im} \alpha$

$$\begin{aligned}(\alpha, \beta) \in \mathcal{L}(L^-) &\Rightarrow \operatorname{im} \alpha = \operatorname{im} \beta \text{ and } \min z\alpha^{-1} = \min z\beta^{-1} \\ &\Rightarrow \operatorname{im} \alpha = \operatorname{im} \beta \text{ and } z\alpha^{-1} = z\beta^{-1} \\ &\Rightarrow \alpha = \beta\end{aligned}$$

since α, β are one-to-one. Hence the proof is complete. ■

Now since $L^-(n, r)$ $(n \geq 2)$ contains some non-idempotents we immediately deduce

Corollary 3.2.2. $L^-(n, r)$ *(for,* $n \geq 2$*) is irregular.*

For the starred Green's relations we have:

Lemma 3.2.3. *Let* α, β *be elements in* $L^-(n, r)$. *Then*

(1) $(\alpha,\beta) \in \mathcal{L}^*$ *if and only if* $\operatorname{im} \alpha = \operatorname{im} \beta$;

(2) $(\alpha,\beta) \in \mathcal{R}^*$ *if and only if* $\operatorname{dom} \alpha = \operatorname{dom} \beta$;

(3) $(\alpha,\beta) \in \mathcal{H}^*$ *if and only if* $\operatorname{im} \alpha = \operatorname{im} \beta$ *and* $\operatorname{dom} \alpha = \operatorname{dom} \beta$.

Proof. (1) Certainly if $\operatorname{im} \alpha = \operatorname{im} \beta$ then $(\alpha, \beta) \in \mathcal{L}(I_n)$ and so $(\alpha, \beta) \in \mathcal{L}^*(L^-)$.

Conversely, if $(\alpha, \beta) \in \mathcal{L}^*$ then by Lemma 1.2.1

$$\alpha\delta = \alpha\gamma \text{ if and only if } \beta\delta = \beta\gamma \quad (\text{for all } \delta, \gamma \in (L^-(n, r))^1).$$

However, if we denote the (partial) identity map in $L^-(n, r)$ on a set A by id_A then

$$\begin{aligned}x \notin \operatorname{im} \alpha &\quad \text{if and only if} \quad \alpha.\mathrm{id}_{\{x\}} = \alpha.\emptyset \\ \text{i.e.} &\quad \text{if and only if} \quad \beta.\mathrm{id}_{\{x\}} = \beta.\emptyset \qquad (\text{since } \alpha\, \mathcal{L}^*\, \beta) \\ \text{i.e.} &\quad \text{if and only if} \quad x \notin \operatorname{im} \beta.\end{aligned}$$

Thus $\operatorname{im} \alpha = \operatorname{im} \beta$.

(2) Certainly if $\operatorname{dom}\alpha = \operatorname{dom}\beta$ then $(\alpha, \beta) \in \mathcal{R}(I_n)$ and so $(\alpha, \beta) \in \mathcal{R}^*(L^-)$.

Conversely, if $(\alpha, \beta) \in \mathcal{R}^*$ then by Lemma 1.2.1

$$\delta\alpha = \gamma\alpha \quad \text{if and only if} \quad \delta\beta = \gamma\beta \quad (\text{for all } \delta, \gamma \in (L^-(n, r))^1).$$

And

$$\begin{array}{llll} x \notin \operatorname{dom}\alpha & \text{if and only if} & id_{\{x\}}.\alpha = \emptyset.\alpha & \\ \text{i.e.} & \text{if and only if} & id_{\{x\}}.\beta = \emptyset.\beta & (\text{since } \alpha\ \mathcal{L}^*\ \beta) \\ \text{i.e.} & \text{if and only if} & x \notin \operatorname{dom}\beta. & \end{array}$$

Thus $\operatorname{dom}\alpha = \operatorname{dom}\beta$.

(3) This follows directly from (1) & (2). ■

Remark 3.2.4. Alternatively, since $L^-(n, r)$ is a full subsemigroup of an abundant semigroup, then Lemma 3.2.3 follows directly from Lemma 1.2.3.

Recall that a subsemigroup U (of a semigroup S) is said to be an *inverse ideal* of S if for all $u \in U$, there exists $u' \in S$ such that $uu'u = u$ and $uu', u'u \in U$.

Lemma 3.2.5. $L^-(n, r)$ *is an inverse ideal of* I_n.

Proof. For a given $\alpha \in L^-(n, r)$ define α' by

$$x\alpha' = x\alpha^{-1} \quad (\text{for all } x \in \operatorname{im}\alpha).$$

Then clearly $\alpha\alpha'\alpha = \alpha$. Moreover, for all $x \in \operatorname{im}\alpha$

$$x\alpha'\alpha = x.$$

And for all x

$$x\alpha\alpha' = (x\alpha)\alpha^{-1} = x.$$

Thus $\alpha'\alpha, \alpha\alpha' \in L^-(n, r)$ since $\operatorname{im}\alpha' = \operatorname{dom}\alpha$ and $\operatorname{dom}\alpha' = \operatorname{im}\alpha$. It now follows that $L^-(n, r)$ is an inverse ideal as required. ■

Notice that since a full subsemigroup of a type A semigroup is itself type A, we deduce the following result:

Theorem 3.2.6. *Let* $L^-(n, r)$ *be as defined in* (1.4). *Then* $L^-(n, r)$ *(for,* $n \geq 2$ *) is an irregular type A semigroup.*

However, we can show that $L^-(n, r)$ is type A directly from the definition. First recall that an abundant semigroup S in which E(S) is a semilattice is called *adequate*. For an element a of an adequate semigroup S, the (unique) idempotent in the $\mathcal{L}^*$-class ($\mathcal{R}^*$-class) containing a will be denoted by a^* (a^+). An adequate semigroup S is said to be *type A* if $ea = a(ea)^*$ and $ae = (ae)^+a$ for all elements a in S and all idempotents e in S.

Let α, ε be elements in $L^-(n, r)$ such that $\varepsilon^2 = \varepsilon$. Define $\pi_{\varepsilon,\alpha}$ in $L^-(n, r)$ by

$$x\pi_{\varepsilon,\alpha} = x \quad (\text{for all } x \in \text{im } \varepsilon\alpha), \qquad \text{dom } \pi_{\varepsilon,\alpha} = \text{im } \varepsilon\alpha.$$

Then $\pi_{\varepsilon,\alpha}$ is an idempotent and $(\pi_{\varepsilon,\alpha}, \varepsilon\alpha) \in \mathcal{L}^*$ (by Lemma 3.2.3). Thus $\pi_{\varepsilon,\alpha} = (\varepsilon\alpha)^*$ and

$$\varepsilon\alpha = \alpha\, \pi_{\varepsilon,\alpha} = \alpha(\varepsilon\alpha)^*.$$

Next define $\phi_{\alpha,\varepsilon}$ in $L^-(n, r)$ by

$$x\phi_{\alpha,\varepsilon} = x \quad (\text{for all } x \in \text{dom } \alpha\varepsilon), \qquad \text{dom } \phi_{\varepsilon,\alpha} = \text{dom } \alpha\varepsilon.$$

Then $\phi_{\alpha,\varepsilon}$ is an idempotent and $(\phi_{\alpha,\varepsilon}, \alpha\varepsilon) \in \mathcal{R}^*$ (by Lemma 3.2.3). Thus $\phi_{\alpha,\varepsilon} = (\alpha\varepsilon)^+$ and

$$\alpha\varepsilon = \phi_{\alpha,\varepsilon}\, \alpha = (\alpha\varepsilon)^+\alpha\,.$$

Hence Theorem 3.2.6 follows from Remarks 3.1.2(b) & 3.2.4 and Corollary 3.2.2.

To characterize the relation $\mathcal{D}^*$ on $L^-(n, r)$ we use the same techniques as in Section II.2. We therefore begin by proving the analogues of Lemmas 2.2.8 & 2.2.9 .

Lemma 3.2.7. *Let* $\alpha \in L^-(n, r)$ *with* $|\text{im } \alpha| = k$. *Then there exists* β *in* $L^-(n, r)$ *with* $\text{im } \beta = \{1, \ldots, k\}$ *such that* $(\alpha, \beta) \in \mathcal{R}^*$.

Proof. Suppose that

$$\alpha = \begin{pmatrix} b_1 & b_2 & \cdots & b_k \\ a_1 & a_2 & \cdots & a_k \end{pmatrix} \in L^-(n, r),$$

with $1 \leq a_1 < a_2 < \ldots < a_k \leq n$. Define β by

$$b_i\beta = i \qquad (i = 1, \dots, k).$$

Then $\beta \in L^-(n, r)$ and $(\alpha, \beta) \in \mathfrak{R}^*$ (by Lemma 3.2.3). ■

Lemma 3.2.8. *Let* $\alpha \in L^-(n, r)$ *with* $\operatorname{im} \alpha = \{a_1, a_2, \dots, a_k\}$ *such that* $a_1 < a_2 < \dots < a_k \le n$. *Then there exists* β *in* $L^-(n, r)$ *with* $\operatorname{dom} \beta = \{n - k + 1, n - k + 2, \dots, n\}$ *such that* $(\alpha, \beta) \in \mathfrak{L}^*$.

Proof. Define β (in $L^-(n, r)$) by

$$(n - i)\beta = a_{k-i} \qquad (i = 0, 1, \dots, k - 1),$$

Then clearly $n - i \ge a_{k-i}$ (for all $i = 0, 1, \dots, k - 1$). Moreover, $\operatorname{dom} \beta = \{n - k + 1, n - k + 2, \dots, n\}$ and $(\alpha, \beta) \in \mathfrak{L}^*$ (by Lemma 3.2.3). ■

On the semigroup $L^-(n, r)$, define a relation $\mathfrak{K}$ by the rule that

$$(\alpha, \beta) \in \mathfrak{K} \text{ if and only if } |\operatorname{im} \alpha| = |\operatorname{im} \beta|.$$

Then clearly $\mathfrak{D}^* \subseteq \mathfrak{K}$ and we obtain the following:

Lemma 3.2.9. $\mathfrak{K} = \mathfrak{R}^* \circ \mathfrak{L}^* \circ \mathfrak{R}^* = \mathfrak{L}^* \circ \mathfrak{R}^* \circ \mathfrak{L}^* = \mathfrak{D}^*$.

Proof. Suppose that $(\alpha, \beta) \in \mathfrak{K}$, so that $|\operatorname{im} \alpha| = |\operatorname{im} \beta| = k$ (say). Then there exists $\delta, \gamma \in L^-(n, r)$ with $\operatorname{im} \delta = \operatorname{im} \gamma = \{1, \dots, k\}$ such that $(\alpha, \delta) \in \mathfrak{R}^*$ and $(\gamma, \beta) \in \mathfrak{R}^*$ (by Lemma 3.2.7), and since $\operatorname{im} \delta = \operatorname{im} \gamma$ implies $(\delta, \gamma) \in \mathfrak{L}^*$ (by Lemma 3.2.3) we have $\alpha \, \mathfrak{R}^* \, \delta \, \mathfrak{L}^* \, \gamma \, \mathfrak{R}^* \, \beta$. Thus

$$\mathfrak{K} \subseteq \mathfrak{R}^* \circ \mathfrak{L}^* \circ \mathfrak{R}^*. \tag{1}$$

Conversely, suppose that $(\alpha, \beta) \in \mathfrak{R}^* \circ \mathfrak{L}^* \circ \mathfrak{R}^*$. Then there exist $\delta, \gamma \in L^-(n, r)$ such that $\alpha \, \mathfrak{R}^* \, \delta \, \mathfrak{L}^* \, \gamma \, \mathfrak{R}^* \, \beta$. Hence

$$|\operatorname{im} \alpha| = |\operatorname{im} \delta|, \operatorname{im} \delta = \operatorname{im} \gamma \text{ and } |\operatorname{im} \gamma| = |\operatorname{im} \beta|,$$

and so $|\operatorname{im} \alpha| = |\operatorname{im} \beta|$. Thus

$$\mathfrak{R}^* \circ \mathfrak{L}^* \circ \mathfrak{R}^* \subseteq \mathfrak{K} \tag{2}$$

From (1) and (2) we deduce that

$$\mathcal{K} = \mathcal{R}^* \circ \mathcal{L}^* \circ \mathcal{R}^*.$$

Similarly, suppose that $(\alpha, \beta) \in \mathcal{K}$, so that $|\text{im}\ \alpha| = |\text{im}\ \beta| = k$ (say). Then there exists $\delta, \gamma \in L^-(n, r)$ with $\text{dom}\ \delta = \text{dom}\ \gamma$ such that $(\alpha, \delta) \in \mathcal{L}^*$ and $(\gamma, \beta) \in \mathcal{L}^*$ (by Lemma 3.2.8). However, since $\text{dom}\ \delta = \text{dom}\ \gamma$ implies $(\delta, \gamma) \in \mathcal{R}^*$ (by Lemma 3.2.3) then we have $\alpha\ \mathcal{L}^*\ \delta\ \mathcal{R}^*\ \gamma\ \mathcal{L}^*\ \beta$. Thus

$$\mathcal{K} \subseteq \mathcal{L}^* \circ \mathcal{R}^* \circ \mathcal{L}^*. \tag{3}$$

Conversely, suppose that $(\alpha, \beta) \in \mathcal{L}^* \circ \mathcal{R}^* \circ \mathcal{L}^*$. Then there exist $\delta, \gamma \in L^-(n, r)$ such that $\alpha\ \mathcal{L}^*\ \delta\ \mathcal{R}^*\ \gamma\ \mathcal{L}^*\ \beta$. Hence

$$\text{im}\ \alpha = \text{im}\ \delta,\ |\text{im}\ \delta| = |\text{im}\ \gamma| \ \text{ and }\ \text{im}\ \gamma = \text{im}\ \beta,$$

and so $|\text{im}\ \alpha| = |\text{im}\ \beta|$. Thus

$$\mathcal{L}^* \circ \mathcal{R}^* \circ \mathcal{L}^* \subseteq \mathcal{K} \tag{4}$$

From (3) and (4) we deduce that

$$\mathcal{K} = \mathcal{L}^* \circ \mathcal{R}^* \circ \mathcal{L}^*.$$

To complete the proof of the lemma, note that from the inequalities

$$\mathcal{D}^* \subseteq \mathcal{K} = \mathcal{R}^* \circ \mathcal{L}^* \circ \mathcal{R}^* = \mathcal{L}^* \circ \mathcal{R}^* \circ \mathcal{L}^* \subseteq \mathcal{D}^*$$

we deduce that $\mathcal{D}^* = \mathcal{K} = \mathcal{R}^* \circ \mathcal{L}^* \circ \mathcal{R}^* = \mathcal{L}^* \circ \mathcal{R}^* \circ \mathcal{L}^*$. ■

Corollary 3.2.10. *Let* $\alpha, \beta \in L^-(n, r)$. *Then* $(\alpha, \beta) \in \mathcal{D}^*$ *if and only if* $|\text{im}\ \alpha| = |\text{im}\ \beta|$.

The next lemma is the analogue of Lemma 2.2.11 and it enables us to characterize the relation $\mathcal{J}^*$ on the semigroup $L^-(n, r)$.

Lemma 3.2.11. *Let* $\alpha, \beta \in L^-(n, r)$. *If* $\alpha \in J^*(\beta)$ *then* $|\text{im}\ \alpha| \leq |\text{im}\ \beta|$.

Proof. Suppose that $\alpha \in J^*(\beta)$. Then (by Lemma 1.2.2) there exist $\beta_0, \beta_1, \ldots, \beta_m \in L^-(n, r), \delta_1, \ldots, \delta_m, \gamma_1, \ldots, \gamma_m \in (L^-(n, r))^1$ such that $\beta = \beta_0$, $\alpha = \beta_m$ and $(\beta_i, \delta_i\beta_{i-1}\gamma_i) \in \mathcal{D}^*$ for $i = 1, \ldots, m$. So by Corollary 3.2.10

$$|\text{im}\ \beta_i| = |\text{im}(\delta_i\beta_{i-1}\gamma_i)| \leq |\text{im}\ \beta_{i-1}| \quad \text{for all } i = 1, \ldots, m.$$

Hence

$$|\text{im}\,\alpha| \leq |\text{im}\,\beta|. \qquad \blacksquare$$

Lemma 3.2.12. *On the semigroup* $L^-(n, r)$, $\mathcal{D}^* = \mathcal{J}^*$.

Proof. Note that we need only show that $\mathcal{J}^* \subseteq \mathcal{D}^*$ (since $\mathcal{D}^* \subseteq \mathcal{J}^*$). So suppose that $(\alpha, \beta) \in \mathcal{J}^*$, so that $J^*(\alpha) = J^*(\beta)$. Then $\alpha \in J^*(\beta)$ and $\beta \in J^*(\alpha)$, and by Lemma 3.2.11 this implies that

$$|\text{im}\,\alpha| \leq |\text{im}\,\beta|, \quad |\text{im}\,\beta| \leq |\text{im}\,\alpha|.$$

Thus

$$|\text{im}\,\alpha| = |\text{im}\,\beta|,$$

and so $(\alpha, \beta) \in \mathcal{D}^*$ by Corollary 3.2.10. $\blacksquare$

We observe that $L^-(n, r)$ is a *-ideal since it is a union of $\mathcal{J}^*$-classes (of I_n^-)

$$J_0^*, J_1^*, \ldots, J_r^*$$

where

$$J_k^* = \{\, \alpha \in L^-(n, r) \colon |\text{im}\,\alpha| = k \}.$$

Finally (in this section), we note that on the semigroup $L^-(n, r)$ $(n \geq 2\ \&\ r \geq 1)$, $\mathcal{L}^* \circ \mathcal{R}^* \neq \mathcal{D}^* \neq \mathcal{R}^* \circ \mathcal{L}^*$. To see this let

$$\alpha = \binom{1}{1}, \ \beta = \binom{2}{2}.$$

Then clearly $(\alpha, \beta) \in \mathcal{D}^*$ and if $(\alpha, \beta) \in \mathcal{R}^* \circ \mathcal{L}^*$ then there must exist $\gamma \in L^-(n, r)$ such that

$$\alpha\, \mathcal{R}^*\, \gamma\, \mathcal{L}^*\, \beta.$$

However by Lemma 3.2.3

$$\gamma = \binom{1}{2} \notin L^-(n, r) \text{ so that } \mathcal{D}^* \neq \mathcal{R}^* \circ \mathcal{L}^*.$$

Similarly, $(\beta, \alpha) \in \mathcal{D}^*$, and if $(\beta, \alpha) \in \mathcal{L}^* \circ \mathcal{R}^*$ then there must exist $\delta \in L^-(n, r)$ such that

$$\beta\, \mathcal{L}^*\, \delta\, \mathcal{R}^*\, \alpha.$$

Again by Lemma 3.2.3

$$\delta = \binom{1}{2} \notin L^-(n, r) \text{ so that } \mathcal{D}^* \neq \mathcal{L}^* \circ \mathcal{R}^*.$$

Remark 3.2.13. Notice that the results obtained for $L^-(n, r)$ in this section and the previous one are extensions of the results obtained for I_n^- in [**35**].

3. Rees quotient semigroups

Now since $L^-(n, r)$ is a two-sided ideal let

$$Q_r = L(n, r)/L(n, r-1) \tag{3.1}$$

$$Q_r^- = L^-(n, r)/L^-(n, r-1) \tag{3.2}$$

be the Rees quotient semigroups on the two-sided ideals $L(n, r)$, $L^-(n, r)$ respectively. Then Q_r is a completely [0-] simple inverse semigroup whose non-zero elements may be thought of as the elements of I_n of rank r precisely. The product of two elements of Q_r is 0 whenever their product in I_n is of rank strictly less than r. Similarly Q_r^- is a Rees quotient semigroup whose non-zero elements may be thought of as the elements of I_n^- of rank r precisely. The product of two elements of Q_r^- is 0 whenever their product in I_n^- is of rank strictly less than r. Notice also that Q_r^- is generated by its set of amenable elements (Lemma 3.1.8). As in Section II.3 let us begin by characterizing the Green's relations on Q_r^-.

Lemma 3.3.1. *Q_r^- is $\mathcal{J}$-trivial.*

Proof. The proof is similar to that of Lemma 3.2.1 with P_n^- replaced by $PP_r^-(n)$. ■

Lemma 3.3.2. *Q_r^- is an inverse ideal of Q_r.*

Proof. The proof of Lemma 3.2.5 applies to this case. ■

Hence by Lemmas 1.2.7 & 1.2.8 and the analogues (to the partial one-to-one case) of [5, Lemmas 10.55 & 10.56] we deduce the following result:

Theorem 3.3.3. *Let* Q_r^- *be as defined in* (3.2). *Then* Q_r^- *is an (irregular, for* $n \geq 2$ & $r \geq 1$) *type* A *semigroup. Moreover for all* $\alpha, \beta \in Q_r^-$ *we have:*

(1) $(\alpha, \beta) \in \mathcal{L}^*$ *if and only if* $\operatorname{im} \alpha = \operatorname{im} \beta$;

(2) $(\alpha, \beta) \in \mathcal{R}^*$ *if and only if* $\operatorname{dom} \alpha = \operatorname{dom} \beta$;

(3) $(\alpha, \beta) \in \mathcal{H}^*$ *if and only if* $\operatorname{im} \alpha = \operatorname{im} \beta$ *and* $\operatorname{dom} \alpha = \operatorname{dom} \beta$.

Remark 3.3.4. Alternatively, since Q_r^- is a full subsemigroup of the inverse semigroup Q_r, then Theorem 3.3.3 follows directly from Lemma 1.2.4.

Using the same techniques as in Section 2 we obtain a characterization of the relation $\mathcal{D}^*$ on Q_r^-.

Lemma 3.3.5. *On the semigroup* Q_r^-, *we have the following:*

(1) $\mathcal{D}^* = \mathcal{R}^* \circ \mathcal{L}^* \circ \mathcal{R}^* = \mathcal{L}^* \circ \mathcal{R}^* \circ \mathcal{L}^*$;

(2) $(\alpha, \beta) \in \mathcal{D}^*$ *if and only if* $|\operatorname{im} \alpha| = |\operatorname{im} \beta|$ *for all* $\alpha, \beta \in Q_r^-$.

Theorem 3.3.6. *Let* Q_r^- *be as defined in* (3.2). *Then* Q_r^- *is a primitive non-regular* 0 **-bisimple type A semigroup.*

Proof. Notice that it only remains to show that $E(Q_r^-)$ is primitive. However, since $E(Q_r)$ is primitive and $E(Q_r^-) \subseteq E(Q_r)$, it follows that $E(Q_r^-)$ is primitive. ■

Theorem 3.3.7. *Let* Q_r^- *be as defined in* (3.2)*and let* $N(Q_r^-) = \{\alpha \in Q_r^- : f(\alpha) < r\}$. *Then*

(1) α *is nilpotent if and only if* $f(\alpha) < r$;

(2) $N(Q_r^-)$ *is an ideal of* Q_r^-.

Proof. First notice that for all $\alpha, \beta \in Q_r^-$

$$F(\alpha\beta) = F(\alpha) \cap F(\beta) = F(\beta) \cap F(\alpha) = F(\beta\alpha).$$

(1) Let $\alpha \in Q_r^-$ be such that $f(\alpha) < r$. Then α^k is an idempotent (for some $k > 1$) since Q_r^- is finite. Moreover, $f(\alpha^k) = f(\alpha) < r$, so that $\alpha^k = 0$ (since 0 is the only idempotent in Q_r^- for which $f(\alpha) < r$). Hence α is nilpotent.

(2) Let $\alpha \in N(Q_r^-)$ and $\beta \in Q_r^-$. Then clearly $f(\alpha\beta) = f(\beta\alpha) < r$, so that $\alpha\beta, \beta\alpha \in N(Q_r^-)$ as required. ■

Remark 3.3.8 The results obtained in this section are to appear in [**37**].

4. The minimum semilattice congruence

Let $I^-(n, r)$ be any of the semigroups $L^-(n, r)$ or $Q_r^-(n)$ for $1 \leq r \leq n$ and define a relation $\rho^\#$ on $I^-(n, r)$ by the rule that

$$\alpha \, \rho^\# \, \beta \quad \text{iff} \quad F(\alpha) = F(\beta) \tag{4.1}$$

Lemma 3.4.1. Let $\rho^\#$ be as defined in (4.1). Then $\rho^\#$ is a semilattice congruence.

Proof. Clearly $\rho^\#$ is an equivalence relation. To show that $\rho^\#$ is left compatible, let $\alpha \, \rho^\# \, \beta$, i.e. $F(\alpha) = F(\beta)$. However for all $\gamma \in I^-(n, r)$

$$F(\gamma\alpha) = F(\gamma) \cap F(\alpha) = F(\gamma) \cap F(\beta) = F(\gamma\beta)$$

(by Lemma 3.1.3). We can similarly show that $\rho^\#$ is right compatible. Hence $\rho^\#$ is a congruence. Moreover, $\rho^\#$ is a semilattice congruence since (by Lemma 3.1.3(2))

$$F(\alpha\beta) = F(\beta\alpha) \quad (\text{for all } \alpha, \beta \in L^-(n, r)). \qquad ■$$

Lemma 3.4.2. Let $\varepsilon, \eta \in E(I^-(n, r))$. Then $(\varepsilon, \eta) \in \rho^\#$ if and only if $(\varepsilon, \eta) \in \mathfrak{D}^*$.

Proof. Let $\varepsilon, \eta \in E(I^-(n, r))$. Then

$$(\varepsilon, \eta) \in \rho^\# \quad \text{iff} \quad F(\varepsilon) = F(\eta)$$

i.e. iff $\operatorname{im} \varepsilon = \operatorname{im} \eta$

i.e. iff $(\varepsilon, \eta) \in \mathfrak{L}^*$. ■

Recall that $E(I_n)$ is the semilattice of all partial identities on X_n and

$$E(n, r) = \{ \alpha \in E(I_n) : |\operatorname{im} \alpha| \le r \},$$

$$EP(n, r) = E(n, r) / E(n, r-1).$$

Then we have:

Lemma 3.4.3. $L^-(n, r) / \rho^\# \cong E(n, r)$ and $Q_r^-(n) / \rho^\# \cong EP(n, r)$.

Proof. From each $\alpha\rho^\#$ (in $L^-(n, r)/\rho^\#$) choose an element $id_{F(\alpha)}$, the partial identity on $F(\alpha)$. It is then clear that the map θ from $L^-(n, r)/\rho^\#$ onto $\{id_{F(\alpha)} : \alpha \in L^-(n, r)\} = E(n, r)$ defined by

$$\theta(\alpha\rho^\#) = id_{F(\alpha)} \quad (\alpha \in L^-(n, r))$$

is an isomorphism.

Similarly we can show that

$$Q_r^-(n) / \rho^\# \cong EP(n, r).$$

Hence the proof. ■

Thus we now have the main result of this section:

Theorem 3.4.4. Let $\Gamma^-(n, r)$ be any of the semigroups $L^-(n, r)$ or $Q_r^-(n)$ for $1 \le r \le n$ and let $\rho^\#$ be as defined in (4.1). Then $\rho^\#$ is the minimum semilattice congruence on $\Gamma^-(n, r)$. Moreover, the maximum semilattice image of $L^-(n, r)$ ($Q_r^-(n)$) is $E(n, r)$ ($EP(n, r)$).

Proof. Since we have already shown that $\rho^\#$ is a semilattice congruence it now remains to show that for any semilattice congruence ρ on $L^-(n, r)$), $\rho^\# \subseteq \rho$. So suppose that $(\alpha, \beta) \in \rho^\#$. Then $(\varepsilon_\alpha, \varepsilon_\beta) \in \rho^\#$ (by Lemma 2.5.5) and $(\varepsilon_\alpha, \varepsilon_\beta) \in \mathfrak{L}^*$

(by Lemma 3.4.2), so that

$$(\varepsilon_\alpha)\rho = (\varepsilon_\alpha\varepsilon_\beta)\rho = (\varepsilon_\alpha)\rho(\varepsilon_\beta)\rho = (\varepsilon_\beta)\rho(\varepsilon_\alpha)\rho = (\varepsilon_\beta\varepsilon_\alpha)\rho = (\varepsilon_\beta)\rho.$$

Hence

$$(\alpha, \beta) \in \rho$$

since

$$(\alpha, \varepsilon_\alpha) \in \rho \ , (\varepsilon_\alpha, \varepsilon_\beta) \in \rho \ , \ (\varepsilon_\beta, \beta) \in \rho$$

by Lemma 2.5.5. Thus

$$\rho^{\#} \subseteq \rho$$

as required.

The last statement of the theorem that the maximum semilattice image of $L^-(n, r)$ $(Q_r^-(n))$ is $E(n, r)$ $(EP(n, r))$ follows from Lemma 3.4.3. ■

CHAPTER 4

COMBINATORIAL RESULTS

We devote this chapter to enumerative problems of an essentially combinatorial nature, and determine the cardinalities of the order-decreasing semigroups considered in the previous chapters. We also obtain formulae for the number of idempotent and nilpotent elements as well as some recurrence relations involving some equivalences on these semigroups.

1. Finite order-decreasing full transformation semigroups

The first two results of this section are on the order of the semigroups S_n^- and $N(S_n^-)$.

Lemma 4.1.1. $|S_n^-| = n! - 1$.

Lemma 4.1.2. $|N(S_n^-)| = (n-1)!$.

The *Stirling number of the second kind* denoted by $S(n, r)$ is usually defined as the number of partitions of an n-element set into r (non-empty) subsets. It satisfies the recurrence relations

$$S(0, 0) = S(n, 1) = S(n, n) = 1, \ S(n, r) = S(n-1, r-1) + rS(n-1, r),$$

and the *Bell's exponential number* denoted by B_n is defined as

$$B_n = \sum_{r=0}^{n} S(n, r).$$

Hence by Lemma 2.2.6(1) we deduce that S_n^- has $S(n, r)$ $\mathcal{R}^*$-classes in each J_r^*. Thus we now have

Lemma 4.1.3. $|E(S_n^-)| = B_n - 1$.

Proof. From Lemma 2.2.7 we deduce that

$$|E(J_r^*)| = S(n, r).$$

Hence the result follows. ∎

Let

$$\begin{aligned} J^*(n, r) &= |\{\alpha \in (S_n^-)^1 : \alpha \in J_r^*\}| \\ &= |\{\alpha \in (S_n^-)^1 : |\operatorname{im} \alpha| = r\}| \end{aligned} \tag{1.1}$$

Then $J^*(n, 1) = 1$ and $J^*(n, n) = 1$. Moreover,

$$\sum_{r=1}^{n} J^*(n, r) = n!$$

Lemma 4.1.4. $J^*(n, r) = rJ^*(n-1, r) + (n-r+1)\ J^*(n-1, r-1)$.

Proof. Maps α in J_r^* divide naturally into two classes, depending upon whether

$$\operatorname{im}(\alpha|\{1, \dots, n-1\}) = \operatorname{im} \alpha \qquad (1)$$

or

$$\operatorname{im}(\alpha|\{1, \dots, n-1\}) \subset \operatorname{im} \alpha \qquad (2).$$

In case (1) n must map to one of the r elements in $\operatorname{im}(\alpha|\{1, \dots, n-1\})$, and so there are $rJ^*(n-1, r)$ elements of this kind. In case (2), $|\operatorname{im}(\alpha|\{1, \dots, n-1\})| = r - 1$ and n must map to one of the $n - r + 1$ elements not in $\operatorname{im}(\alpha|\{1, \dots, n-1\})$. Hence there are $(n-r+1)J^*(n-1, r-1)$ elements of this kind. Thus

$$J^*(n, r) = rJ^*(n-1, r) + (n-r+1)J^*(n-1, r-1),$$

as required. ∎

The above recurrence relation has been obtained in [**40**] and the following result is in Anderson [**1**, Ex. 4.4(4)].

Theorem 4.1.5. *Let* $J^*(n, r)$ *be as defined in* (1.1). *Then*

$$J^*(n, r) = \sum_{k=0}^{r-1} (-1)^k \binom{n+1}{k} (r-k)^n.$$

Remark 4.1.6. In fact $J^*(n, r)$ is known as the *Eulerian number* [30].

Now, let

$$sh(n, r) = |\{\alpha \in (S_n^-)^1 : s(\alpha) = r - 1\}| \tag{1.2}$$

Then $sh(n, 1) = 1$ and $\sum_{r=1}^{n} sh(n, r) = n!$. Moreover, it will be convenient to let $sh(n, r) = 0$ if $n = 0$ or $r = 0$ or $n < r$.

Lemma 4.1.7. $sh(n, r) = (n-1)sh(n-1, r-1) + sh(n-1, r)$.

Proof. Suppose that the shifting (or moving) points are $x_1, x_2, \ldots, x_{r-1}$ with $x_1 \leq x_2 \leq \ldots \leq x_{r-1}$. Then we have $x_i - 1$ choices for $x_i\alpha$, i.e., $(x_1 - 1)(x_2 - 1) \ldots (x_{r-1} - 1)$ choices in all. Write $\underline{x}$ for $(x_1, x_2, \ldots, x_{r-1})$ and $\underline{V}$ for $\{\underline{x} : 2 \leq x_1 \leq \ldots \leq x_{r-1} \leq n\}$. Then

$$sh(n,r) = \sum_{\underline{x} \in \underline{V}} (x_1 - 1)(x_2 - 1) \ldots (x_{r-1} - 1),$$

Now $\underline{V}$ is a disjoint union of

$$\underline{V}_1 = \{\underline{x} \in \underline{V} : x_{r-1} \leq n - 1) \text{ and } \underline{V}_2 = \{\underline{x} \in \underline{V} : x_{r-1} = n\},$$

and

$$\sum_{\underline{x} \in \underline{V}_1} (x_1 - 1)(x_2 - 1) \ldots (x_{r-1} - 1) = sh(n-1, r),$$

$$\sum_{\underline{x} \in \underline{V}_2} (x_1 - 1)(x_2 - 1) \ldots (x_{r-1} - 1) = (n-1)sh(n-1, r-1),$$

since $x_{r-1} = n$ implies that there are $n - 1$ choices for $x_{r-1}\alpha$. The result follows. ■

The next result gives us a generating function for $sh(n, r)$.

Theorem 4.1.8. *Let* sh(n, r) *be as defined in* (1.2). *Then*

$$\sum_{r=1}^{n} sh(n, r)x^{n-r+1} = [x]_n, \qquad n = 1, 2...$$

where $[x]_n$ *is the ascending factorial of* x *(of degree* n*) defined by*

$$[x]_n = x(x+1) \dots (x + n - 1), \ [x]_0 = 1. \qquad (1.3)$$

Proof. First note that if n = 1, then the result is clear. Suppose now that

$$\sum_{r=1}^{n} sh(n, r)x^{n-r+1} = [x]_n.$$

Then

$$\sum_{r=1}^{n+1} sh(n+1, r)x^{n-r+2} = \sum_{r=1}^{n+1} \{n\ sh(n, r-1)+sh(n, r)\}\, x^{n-r+2}$$

$$= n \sum_{r=1}^{n+1} sh(n, r-1)x^{n-r+2} + \sum_{r=1}^{n} sh(n, r)x^{n-r+2}$$

$$= n \sum_{r=2}^{n+1} sh(n, r-1)x^{n-r+2} + x \sum_{r=1}^{n} sh(n, r)x^{n-r+1}$$

$$= n \sum_{t=1}^{n} sh(n, t)x^{n-t+1} + x \sum_{r=1}^{n} sh(n, r)x^{n-r+1}$$

(where t = r − 1)

$$= n[x]_n + x[x]_n = [x]_n(x + n) = [x]_{n+1}.$$

Hence the result follows by induction. ■

Let |s(n, r)| be defined by

$$\sum_{r=1}^{n} |s(n, r)|x^r = [x]_n,$$

where $[x]_n$ is as defined in (1.3). Then |s(n, r)| *is known as the signless or absolute Stirling number of the first kind* [4]. The following result is now immediate.

Corollary 4.1.9. sh(n, r) = |s(n, n − r + 1)|.

Some special cases of sh(n, r) may be worth recording:

$$sh(n, 2) = n(n-1)/2, \qquad sh(n, 3) = (n-2)(n-1)n(3n-1)/24,$$

$$sh(n, 4) = (n-3)(n-2)(n-1)^2 n^2/48, \qquad sh(n, n) = (n-1)!$$

2. Finite order-decreasing partial transformation semigroups

Lemma 4.2.1. $|P_n^-| = (n + 1)! - 1$.

Proof. The proof follows from Corollary 2.4.3 and Lemma 4.1.1. ■

Lemma 4.2.2. $|N(P_n^-)| = n!$.

Proof. Since the nilpotent elements of P_n^- are mapped onto the nilpotent elements of S_{n+1}^- by every isomorphism between P_n^- and S_{n+1}^-, then the result follows from Corollary 2.4.3 and Lemma 4.1.2. ■

Lemma 4.2.3. $|E(P_n^-)| = B_{n+1} - 1$.

Proof. Since the idempotent elements of P_n^- are mapped onto the idempotent elements of S_{n+1}^- by every isomorphism between P_n^- and S_{n+1}^-, then the result follows from Corollary 2.4.3 and Lemma 4.1.3. ■

Let

$$\begin{aligned} PJ^*(n, r) &= |\{\alpha \in (P_n^-)^1 : \alpha \in J_r^*\}| \\ &= |\{\alpha \in (P_n^-)^1 : |\text{im } \alpha| = r\}| \end{aligned} \tag{2.1}$$

Then $PJ^*(n, 0) = 1$ and $PJ^*(n, n) = 1$. Moreover,

$$\sum_{r=0}^{n} PJ^*(n, r) = (n + 1)!$$

Hence by Corollary 2.4.3 we deduce

Lemma 4.2.4. $PJ^*(n, r) = J^*(n + 1, r + 1)$, $(n \geq r \geq 0)$.

Lemma 4.2.5. $PJ^*(n, r) = (r + 1)PJ^*(n - 1, r) + (n - r + 1)PJ^*(n - 1, r - 1)$.

Proof.
$$\begin{aligned} PJ^*(n, r) &= J^*(n + 1, r + 1) && \text{(by Lemma 4.2.4)} \\ &= (r + 1)J^*(n, r + 1) + (n - r + 1)J^*(n, r) && \text{(by Lemma 4.1.4)} \\ &= (r + 1)PJ^*(n - 1, r) + (n - r + 1)PJ^*(n - 1, r - 1), \end{aligned}$$

as required. ■

Theorem 4.2.6. *Let* $PJ^*(n, r)$ *be as defined in* (2.1). *Then*
$$PJ^*(n, r) = J^*(n + 1, r + 1) = \sum_{k=0}^{r} (-1)^k \binom{n+2}{k} (r + 1 - k)^{n+1}.$$

Recall that for a given $\alpha \in P_n^-$, $s(\alpha)$ is the cardinal of the set
$$S(\alpha) = \{x \in \text{dom } \alpha : x\alpha \neq x\}$$

and let
$$\underline{sh}(n, r) = |\{\alpha \in (P_n^-)^1 : s(\alpha) = r\}|. \tag{2.2}$$

Then $\underline{sh}(n, 0) = 2^n$, $\underline{sh}(n, n) = 0$ $(n \geq 1)$ and $\sum_{r=1}^{n} \underline{sh}(n, r) = (n + 1)!$. Moreover, it will be convenient to let $\underline{sh}(n, r) = 0$ if $n < r$.

However, despite Corollary 2.4.3, we observe that results obtained for $sh(n, r)$ (in the previous section) could not be used (directly) to deduce the corresponding results for $\underline{sh}(n,r)$, since the isomorphism $\alpha \to \alpha^*$ (from P_n^- onto S_{n+1}^-) does not necessarily imply that $S(\alpha) = S(\alpha^*)$. In fact $S(\alpha^*) = S(\alpha) \cup (X_n \setminus \text{dom } \alpha)$. But the same technique as used in Section 1 could be employed to obtain a similar result.

Lemma 4.2.7. $\underline{sh}(n, r) = (n - 1)\underline{sh}(n - 1, r - 1) + 2\underline{sh}(n - 1, r)$.

Proof. Suppose that the shifting (or moving) points are $x_1, x_2, \ldots, x_r$ with $x_1 \leq x_2 \leq \ldots \leq x_r$. Then we have $x_i - 1$ choices for $x_i\alpha$, i.e., $(x_1 - 1)(x_2 - 1) \ldots (x_r - 1)$ choices in all. Write $\underline{x}$ for $(x_1, x_2, \ldots, x_r)$ and $\underline{V}$ for $\{\underline{x} : 2 \leq x_1 \leq \ldots \leq x_r \leq n\}$. Then
$$\underline{sh}(n, r) = \sum_{\underline{x} \in \underline{V}} (x_1 - 1)(x_2 - 1) \ldots (x_r - 1).$$

Now $\underline{V}$ is a disjoint union of

$$\underline{V}_1 = \{\underline{x} \in \underline{V} : x_r \le n-1) \quad \text{and} \quad \underline{V}_2 = \{\underline{x} \in \underline{V} : x_r = n\},$$

and

$$\sum_{\underline{x} \in \underline{V}_1} (x_1 - 1)(x_2 - 1) \dots \ (x_{r-1} - 1) = 2\underline{sh}(n-1, r),$$

$$\sum_{\underline{x} \in \underline{V}_2} (x_1 - 1)(x_2 - 1) \dots \ (x_{r-1} - 1) = (n-1)\underline{sh}(n-1, r-1),$$

since $x_r \le n - 1$ implies that either $n\alpha = n$ or $n \notin \operatorname{dom} \alpha$ and $x_r = n$ implies that there are $n - 1$ choices for $x_r\alpha$. The result follows. ∎

Corollary 4.2.8. $\underline{sh}(n, 1) = 2^{n-1}n(n-1)/2$ *and* $\underline{sh}(n, n-1) = 2(n-1)!$.

Lemma 4.2.9. $\underline{sh}(n, r) = 2^{n-r}sh(n, r+1)$, $(n \ge r \ge 0)$.

Proof. First notice that the result is true for all n if $r = 0$. So suppose that the result is true for all $0 \le k < n$. Then

$$\begin{aligned}
\underline{sh}(n, r) &= (n-1)\underline{sh}(n-1, r-1) + 2\underline{sh}(n-1, r) && \text{(by Lemma 4.2.7)}\\
&= (n-1).2^{n-r}sh(n-1, r) + 2.2^{n-r-1}sh(n-1, r+1) && \text{(by supposition)}\\
&= 2^{n-r}\{(n-1)sh(n-1, r) + sh(n-1, r+1)\}\\
&= 2^{n-r}sh(n, r+1) && \text{(by Lemma 4.1.7).}
\end{aligned}$$ ∎

A generating function for $\underline{sh}(n, r)$ is given by

Theorem 4.2.10. *Let* $\underline{sh}(n, r)$ *be as defined in* (2.2). *Then*

$$\sum_{r=0}^{n} \underline{sh}(n, r)x^{n-r} = 2[2x+1]_{n-1}, \qquad n = 1, 2, \dots$$

where $[x]_m$ *is the ascending factorial of* x *(of degree* m*) defined by*

$$[x]_m = x(x+1) \dots (x+m-1), \ \ [x]_0 = 1.$$

Proof. First note that if $n = 1$, then the result is clear. Suppose now that

$$\sum_{r=0}^{n} \underline{sh}(n, r)x^{n-r} = 2[2x + 1]_{n-1}.$$

Then

$$\sum_{r=0}^{n+1} \underline{sh}(n + 1, r)x^{n-r+1} = \sum_{r=0}^{n+1} \{n\ \underline{sh}(n, r - 1)+2\underline{sh}(n, r)\}x^{n-r+1}$$

$$= n \sum_{r=0}^{n+1} \underline{sh}(n, r - 1)x^{n-r+1} + 2 \sum_{r=0}^{n} \underline{sh}(n, r)x^{n-r+1}$$

$$= n \sum_{r=1}^{n+1} \underline{sh}(n, r - 1)x^{n-r+1} + 2x \sum_{r=0}^{n} \underline{sh}(n, r)x^{n-r}$$

$$= n \sum_{t=0}^{n} \underline{sh}(n, t)x^{n-t} + 2x \sum_{r=0}^{n} \underline{sh}(n, r)x^{n-r}$$

(where $t = r - 1$)

$$= n.2[2x + 1]_{n-1} + 2x.2[2x + 1]_{n-1}$$

$$= 2[2x + 1]_{n-1}(2x + n) = 2[2x + 1]_n.$$

Hence the result. ■

3. Finite order-decreasing partial one-to-one transformation semigroups

Let

$$\begin{aligned} IJ^*(n, r) &= |\{\alpha \in (I_n^-)^1 : \alpha \in J_r^*\}| \\ &= |\{\alpha \in (I_n^-)^1 : |\mathrm{im}\ \alpha| = r\}| \end{aligned} \tag{3.1}$$

Theorem 4.3.1.[3, Proposition 3.1 and Remark 3.6]. *Let* $IJ^*(n, r)$ *be as defined in* (3.1). *Then* $IJ^*(n, r) = S(n + 1, n - r + 1)$, $(n \geq r \geq 0.)$

Theorem 4.3.2.[3, Proposition 3.1 and Remark 3.6]. *Let* I_n^- *be as defined in* (III.1.1). *Then* $|(I_n^-)^1| = B_{n+1}$, *where* B_{n+1} *is the Bell's number defined above.*

Theorem 4.3.3. *Let* $N(I_n^-) = \{\ \alpha \in I_n^- : F(\alpha)$ is empty$\}$. *Then* $|N(I_n^-)| = B_n$.

Proof. Notice that by virtue of Theorem 4.3.2 it suffices to establish a bijection between $N(I_n^-)$ and $(I_{n-1}^-)^1$. So for every $\alpha_n \in N(I_n^-)$ we associate an $\alpha \in (I_{n-1}^-)^1$ by

$$\theta(\alpha_n) = \alpha$$

where

$$i\alpha_n = (i-1)\alpha, \qquad (i \in \text{dom } \alpha_n).$$

Now since $1 \notin \text{dom } \alpha_n$ and $n \notin \text{dom } \alpha$ then clearly θ is a bijection. Thus the proof is complete. ■

Finally to determine the number of amenable elements in $L^-(n, r)$ first recall that an element η in $L^-(n, r)$ is called *amenable* if $s(\eta) \le 1$ and $A(\eta) \subseteq \text{dom } \eta$, where

$$A(\eta) = \{\, y \in X_n : (\exists x \in X_n)\ x\eta < y < x \}$$
$$= \{\, y \in X_n : (\exists x \in S(\eta))\ x\eta < y < x \}.$$

Now if we denote by $AQE(J_r^*)$ the set of amenable elements in J_r^* and the cardinal $|AQE(J_r^*)|$ by *q(n, r)* then the next Lemma gives us an expression for $q(n, r)$.

Lemma 4.3.4. $q(n, r) = \binom{n}{r} + \sum_{i=1}^{r} (n-i)\binom{n-i-1}{r-i} \qquad (r \ge 0).$

Proof. Clearly there are $\binom{n}{r}$ idempotents in J_r^*. And since for every (non-idempotent) amenable element η, $s(\eta) = 1$, then we may express η as

$$\eta = \binom{x}{y}, \quad (x > y)$$

where $x\eta = y$ and $z\eta = z$ for all z in $\text{dom } \eta \setminus \{x\}$. Now notice that there are $(n-i)$ pairs of the type $(x, x+i)$, $(x, x+i \in X_n.)$ However since $\{x+1, \dots, x+i-1\} \subseteq \text{dom } \eta$ then there are $\binom{n-i-1}{r-i}$ ways of choosing the remaining elements of dom η. Thus the number of amenable elements in J_r^* is

$$\binom{n}{r} + \sum_{i=1}^{r} (n-i)\binom{n-i-1}{r-i}$$

as required. ■

However, it is possible to obtain an explicit expression for q(n, r). To do this we require these two certainly known simple results:

Lemma 4.3.5. $\sum_{i=1}^{r} \binom{n-i}{r-i} = \binom{n}{r-1}$.

Proof. The proof is by repeated application of the Pascal's triangular identity, i.e.,

$$\binom{n}{r-1} = \binom{n-1}{r-1} + \binom{n-1}{r-2}$$

$$= \binom{n-1}{r-1} + \binom{n-2}{r-2} + \binom{n-2}{r-3}$$

$$= \dots = \sum_{i=1}^{r-1} \binom{n-i}{r-i} + \binom{n-r+1}{0}$$

$$= \sum_{i=1}^{r} \binom{n-i}{r-i}. \qquad \blacksquare$$

Lemma 4.3.6. $\sum_{i=1}^{r} (n-i)\binom{n-i-1}{r-i} = (n-r)\binom{n}{r-1}$.

Proof. $$\sum_{i=1}^{r} (n-i)\binom{n-i-1}{r-i} = \sum_{i=1}^{r} \frac{(n-i-1)!(n-i)}{(n-r-1)!(r-i)!}$$

$$= \sum_{i=1}^{r} \frac{(n-i)!(n-r)}{(n-r)!(r-i)!} = (n-r)\sum_{i=1}^{r} \binom{n-i}{r-i}$$

$$= (n-r)\binom{n}{r-1} \qquad \text{(by Lemma 4.3.5).} \qquad \blacksquare$$

Hence we have this result

Theorem 4.3.7. $q(n, r) = \binom{n}{r-1}\frac{[(n-r)(r+1)+1]}{r}$.

Proof. $q(n, r) = \binom{n}{r} + \sum_{i=1}^{r} (n-i)\binom{n-i-1}{r-i}$ (by Lemma 4.3.4)

$$= \binom{n}{r} + (n-r)\binom{n}{r-1} \qquad \text{(by Lemma 4.3.6)}$$

$$= \frac{(n-r+1)}{r}\binom{n}{r-1} + (n-r)\binom{n}{r-1}$$

$$= \binom{n}{r-1}\frac{[(n-r)(r+1)+1]}{r}. \qquad \blacksquare$$

See the Appendix for some computed values of $J^*(n, r)$, sh(n, r), S(n, r), <u>sh</u>(n, r) and q(n, r).

CHAPTER 5

RANK PROPERTIES

The *rank* of a finite semigroup is usually defined by

$$\text{rank } S = \min\{|A| : A \subseteq S\,,\ \langle A\rangle = S\}.$$

If S is generated by its set E of idempotents, then the *idempotent rank* of S is defined by

$$\text{idrank } S = \min\{|A| : A \subseteq E\,,\ \langle A\rangle = S\}.$$

The questions of the ranks, idempotent ranks and nilpotent ranks of certain finite transformation semigroups have been considered by Howie [**22**], Gomes and Howie [**15** - **17**], Howie and McFadden [**26**] and Garba [**11** - **14**]. The results obtained in this chapter are to appear in [**38**].

1. Finite order-decreasing full transformation semigroups

In this section we investigate the rank and idempotent rank of $K^-(n, r)$ along the lines of Howie and McFadden [**26**].

It has been shown in Section II.2 that $K^-(n, r)$ is an abundant subsemiband of T_n and that

$$\alpha\, \mathcal{L}^*\, \beta \quad \text{if and only if} \quad \text{im}\,\alpha = \text{im}\,\beta,$$

$$\alpha\, \mathcal{R}^*\, \beta \quad \text{if and only if} \quad \alpha\circ\alpha^{-1} = \beta\circ\beta^{-1},$$

$$\alpha\, \mathcal{J}^*\, \beta \quad \text{if and only if} \quad |\text{im}\,\alpha| = |\text{im}\,\beta|.$$

Thus $K^-(n, r)$, like T_n itself, is the union of $\mathcal{J}^*$-classes

$$J_1^*, J_2^*, \dots, J_r^*$$

where

$$J_k^* = \{\, \alpha \in K^-(n, r) : |\text{im}\,\alpha| = k\}.$$

Moreover, $K^-(n, r)$ has $S(n, k)$ $\mathcal{R}^*$-classes and $\binom{n-1}{k-1}$ $\mathcal{L}^*$-classes in each J_k^*.

Now paying particular attention to the $\mathcal{J}^*$-class J_r^* at the top of the semigroup $K^-(n, r)$ we begin our investigation by recalling the following lemma from Section II.1.

Lemma 2.1.3. *Let* $\alpha, \beta \in K^-(n, r)$. *Then*

(1) $F(\alpha\beta) = F(\alpha) \cap F(\beta)$;

(2) $F(\alpha\beta) = F(\beta\alpha)$.

Lemma 5.1.1. *Let* $\varepsilon \in E(K^-(n, r))$. *Then* ε *is expressible as a product of idempotents in* J_r^*.

Proof. Suppose that

$$\varepsilon = \begin{pmatrix} A_1 & A_2 & \dots & A_k \\ a_1 & a_2 & \dots & a_k \end{pmatrix} \in K^-(n, r).$$

We may assume without loss of generality that $k \leq r - 1 < n - 1$. Essentially we can either have $|A_i| \geq 2$ and $|A_j| \geq 2$; or $|A_i| \geq 3$ for some $i, j \in \{1, \dots, k\}$. In the former case we choose an element $a_i' \neq a_i$ in A_i and an element $a_j' \neq a_j$ in A_j; in the latter case we choose two distinct elements a_i', a_i'' in $A_i \setminus \{a_i\}$. Then in the former case we define

$$a_i'f_1 = a_i', \quad xf_1 = x\varepsilon \quad (x \neq a_i')$$
$$a_j'f_2 = a_j', \quad yf_2 = y\varepsilon \quad (y \neq a_j');$$

in the latter we define

$$a_i'f_1 = a_i', \quad xf_1 = x\varepsilon \quad (x \neq a_i')$$
$$a_i''f_2 = a_i'', \quad yf_2 = y\varepsilon \quad (y \neq a_i'').$$

In both cases it is clear that f_1, f_2 are idempotents, and $\varepsilon = f_1f_2$. Moreover, $|\text{im } f_1| = |\text{im } f_2| = k + 1$. Hence the result follows by induction. ■

From Lemma 2.1.4(2) we easily deduce the result of the next lemma

Lemma 5.1.2. *Let* $\alpha \in J_k^*$. *Then* $\alpha^2 = \alpha$ *if and only if* $f(\alpha) = k$.

We shall henceforth use Lemmas 2.1.3 & 5.1.2 without reference.

Lemma 5.1.3. *Let* $\alpha, \beta \in J^*_k$ $(1 \leq k \leq r)$. *Then the following are equivalent:*

(1) $\alpha\beta \in E(J^*_k)$;

(2) $\alpha, \beta \in E(J^*_k)$ and $(\alpha,\beta) \in \mathfrak{L}^*$.

Proof. (1) $\Rightarrow$ (2). Suppose that $\alpha\beta \in E(J^*_k)$. Then

$$k = f(\alpha\beta) \leq f(\alpha) \leq |\text{im } \alpha| = k$$

$$k = f(\alpha\beta) \leq f(\beta) \leq |\text{im } \beta| = k$$

which implies that

$$f(\alpha) = k = f(\beta)$$

so that $\alpha, \beta \in E(J^*_k)$. Moreover, since

$$F(\alpha\beta) = \text{im } \alpha\beta = \text{im } \beta = F(\beta)$$

then

$$\text{im } \beta = F(\beta) = F(\alpha) = \text{im } \alpha$$

so that

$$(\alpha, \beta) \in \mathfrak{L}^*.$$

(2) $\Rightarrow$ (1) is clear. ■

Now since the set of $\mathfrak{L}^*$- related idempotents is a left zero semigroup then we deduce from Lemmas 5.1.1 and 5.1.3 that $E(J^*_r)$ is the unique minimal generating set for $K^-(n, r)$. Thus we now have the main result of this section:

Theorem 5.1.4. *Let* $K^-(n, r)$ *be as defined in* (II.1.4). *Then*

$$\text{rank } K^-(n, r) = \text{idrank } K^-(n, r) = S(n, r).$$

Proof. The result follows from the above remarks and the remarks preceeding Lemma 4.1.3. ■

Theorem 5.1.5. *Let* $PK^-(n, r) = \{\ \alpha \in (P^-_n)^1 : |\mathrm{im}\ \alpha| \le r\}$. *Then*

$$\mathrm{rank}\ PK^-(n, r) = \mathrm{idrank}\ PK^-(n, r) = S(n + 1, r + 1).$$

Proof. The proof follows from Corollary 2.4.3 and Theorem 5.1.4. ■

2. Finite strictly order-decreasing full transformation semigroups

In this section we investigate the rank of

$$\begin{aligned} N(S^-_n) &= \{\ \alpha \in S^-_n : F(\alpha) = \{1\}\} \\ &= \{\ \alpha \in S^-_n : (\forall x \in X_n \setminus \{1\})\ x\alpha < x\} \qquad (2.1) \end{aligned}$$

the semigroup of all strictly order-decreasing mappings of X_n. It has been shown (in Lemmas 2.1.6 & 2.1.7) that $N(S^-_n)$ is an ideal of S^-_n consisting of all the nilpotent elements of S^-_n. Now let

$$G = \{\ \alpha \in N(S^-_n) : (\text{there exists } x \ge 3)\ x\alpha = x - 1\} \qquad (2.2)$$

$$G' = \{\ \alpha \in N(S^-_n) : (\text{for all } x \ge 3)\ x\alpha < x - 1\} \qquad (2.3)$$

Then clearly $G \cup G' = N(S^-_n)$ and $G \cap G'$ is empty. Also observe that $(N(S^-_n))^2 = G'$. Our aim is to show that G is the unique minimal generating set for $N(S^-_n)$.

Proposition 5.2.1. *Let* $\alpha \in G'$. *Then* α *is expressible as a product of exactly two elements in* G.

Proof. For every $\alpha \in G$, let

$$\delta_1 = \begin{pmatrix} 1 & 2 & 3 & \dots & k & \dots & n \\ 1 & 1 & 2 & \dots & k\alpha + 1 & \dots & n\alpha + 1 \end{pmatrix} \quad (4 \le k \le n)$$

$$\delta_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \dots & n \\ 1 & 1 & 2 & 3 & 4 & \dots & n - 1 \end{pmatrix}$$

Clearly $\delta_1, \delta_2 \in G$ and $\alpha = \delta_1\delta_2$. Hence the proof. ■

Corollary 5.2.2. $\langle G \rangle = N(S_n^-)$.

It is now fairly obvious from Corollary 5.2.2 and the next two lemmas that G is the unique minimal generating set for $N(S_n^-)$.

Lemma 5.2.3. $G^2 \subseteq G'$.

Proof. Let $\alpha, \beta \in G$. Then clearly if $x\alpha \neq 1$

$$(x\alpha)\beta < x\alpha \leq x - 1;$$

if $x\alpha = 1$

$$(x\alpha)\beta = 1 < x - 1$$

for all $x \geq 3$. Thus in either case $\alpha\beta \in G'$. Hence $G^2 \subseteq G'$ as required. ■

Lemma 5.2.4. G' *is an ideal of* $N(S_n^-)$.

Proof. Let $\alpha \in G'$ and $\beta \in N(S_n^-)$. Since

$$(x\alpha)\beta < x\alpha \leq x - 1 \quad \text{and} \quad (x\beta)\alpha < x\beta \leq x - 1$$

for all $x \geq 3$, it follows that $\alpha\beta$, $\beta\alpha \in G'$. Hence G' is an ideal of $N(I_n^-)$ as required. ■

Notice that Lemmas 5.2.3 & 5.2.4 also follow from the fact that $(N(S_n^-))^2 = G'$. Thus we now have the main result of this section:

Theorem 5.2.5. *Let* $N(S_n^-)$ *and* G *be as defined in* (2.1) *and* (2.2) *respectively. Then*

$$\text{rank } N(S_n^-) = |G| = |N(S_n^-)| - |G'| = (n-2)!(n-2).$$

Proof. First notice that rank $N(S_n^-) = |G|$ has already been established. It is also not difficult to see that $|N(S_n^-)| = (n-1)!$ and $|G'| = (n-2)!$ from which the result follows. ■

3. Finite order-decreasing partial one-to-one transformation semigroups

In this section we investigate the rank and quasi-idempotent rank of the type A semigroup (Theorem 3.2.6) $L^-(n, r)$ along the lines of Howie and McFadden [**26**].

It has been shown in Section III.2 that on the semigroup $L^-(n, r)$

$$\alpha\, \mathcal{L}^*\, \beta \quad \text{if and only if} \quad \operatorname{im}\alpha = \operatorname{im}\beta,$$

$$\alpha\, \mathcal{R}^*\, \beta \quad \text{if and only if} \quad \operatorname{dom}\alpha = \operatorname{dom}\alpha$$

$$\alpha\, \mathcal{J}^*\, \beta \quad \text{if and only if} \quad |\operatorname{im}\alpha| = |\operatorname{im}\beta|.$$

Thus $L^-(n, r)$, like I_n itself, is the union of $\mathcal{J}^*$-classes

$$J_0^*, J_1^*, \dots, J_r^*$$

where

$$J_k^* = \{\, \alpha \in L^-(n, r) : |\operatorname{im}\alpha| = k \}.$$

Again paying particular attention to the $\mathcal{J}^*$-class J_r^* at the top of the semigroup $L^-(n, r)$ we begin our investigation by recalling some basic facts from Chapter 3.

First recall that an element η in $L^-(n, r)$ is called *amenable* if $s(\eta) \le 1$ and $A(\eta) \subseteq \operatorname{dom}\eta$, where

$$A(\eta) = \{\, y \in X_n : (\exists x \in X_n)\; x\eta < y < x \}$$
$$= \{\, y \in X_n : (\exists x \in S(\eta))\; x\eta < y < x \}.$$

We have already shown (in Lemma 3.1.6) that $L^-(n, r)$ is generated by $AQE(J_r^*)$, its set of amenable elements whose cardinal is denoted by $q(n, r)$. If we denote by *quaidrank* S the quasi-idempotent rank of S then the following is now immediate:

Corollary 5.3.1. quaidrank $L^-(n, r) \le q(n, r)$.

Now we are going to show that $AQE(J_r^*)$ is a minimal generating set for $L^-(n, r)$. However, first we establish

Lemma 5.3.2. *Let* $\alpha, \beta, \alpha\beta \in J_k^*$ $(0 \le k \le n-1)$. *Then* $\alpha\beta$ *is an idempotent if and only if* $\alpha = \alpha\beta = \beta$.

Proof. First notice that since α, β are one-to-one then $\text{im}\,\alpha = \text{dom}\,\beta$, $\text{dom}\,\alpha = \text{dom}\,\alpha\beta$ and $\text{im}\,\beta = \text{im}\,\alpha\beta$. Suppose that $\alpha\beta$ is idempotent. Then for all x in $\text{dom}\,\alpha\beta$ ($= \text{dom}\,\alpha$), $x = x\alpha\beta$ and since

$$x \ge x\alpha \ge x\alpha\beta$$

then $x = x\alpha$ and $y = y\beta$ for all $x \in \text{dom}\,\alpha$ and for all $y \in \text{im}\,\alpha$ ($= \text{dom}\,\beta$). So α, β are idempotents if $\alpha\beta$ is. Further, since

$$\alpha\, \mathcal{R}^*\, \alpha\beta, \quad \alpha\beta\, \mathcal{L}^*\, \beta \qquad \text{(by Lemma 3.2.3)}$$

and $L^-(n, r)$ is adequate then $\alpha = \alpha\beta = \beta$. The converse is clear. ■

An immediate consequence of Lemma 5.3.2 is that any generating set for $L^-(n, r)$ must contain $E(L^-(n, r))$. Next we are going to show that if γ, η are two (non-idempotent) amenable elements in J_r^* such that their product δ is in J_r^* also, then δ is NOT amenable. Thus, again any generating set for $L^-(n, r)$ must contain $AQE(J_r^*)$ since idempotents are partial identities in this case.

Lemma 5.3.3. *Let* $\gamma, \eta \in AQE(J_r^*)$ *such that* $s(\gamma) = s(\eta) = 1$ *and* $\gamma\eta \in J_r^*$. *Then* $\gamma\eta$ *is not amenable.*

Proof. Let $\gamma, \eta \in AQE(J_r^*)$ such that $s(\gamma) = s(\eta) = 1$ and $\gamma\eta \in J_r^*$. First notice that $\text{im}\,\gamma = \text{dom}\,\eta$ and $\text{dom}\,\gamma = \text{dom}\,\gamma\eta$. Now suppose that $\text{dom}\,\gamma = W$ and

$$g\gamma = h \;\; (g \in W), \quad x\gamma = x \;\; (x \ne g).$$

Then

$$\text{im}\,\gamma = (W \setminus \{g\}) \cup \{h\} = \text{dom}\,\eta,$$

and there are two possibilities for η: i.e., $h \in S(\eta)$ or $h \notin S(\eta)$. In the former we have

$$g\gamma\eta = k < h = g\gamma < g \;\; (k = h\eta)$$

and $s(\gamma\eta) = 1$. However, $h \notin \text{dom}\,\gamma$ ($= \text{dom}\,\gamma\eta$) since γ is 1-1. Thus $\gamma\eta$ is not amenable.

In the latter if we let $h' \in S(\eta)$ and $g' = h'\gamma^{-1}$ we have

$$g'\gamma\eta = h'\eta < h' \leq g' \quad \text{and} \quad g\gamma\eta = h \neq g,$$

so that $g, g' \in S(\gamma\eta)$. Again $\gamma\eta$ is not amenable. ■

Remark 5.3.4. In fact since $\langle AQE(J_r^*)\rangle = L^-(n, r)$ (by Lemma 3.1.7) then what we have shown is that $AQE(J_r^*)$ is the unique minimal generating set for $L^-(n, r)$. However this is not a coincidence since Doyen [7] has shown that every periodic $\mathcal{J}$- trivial monoid has a unique minimal generating set.

Thus we now have the main result of this section:

Theorem 5.3.5. *Let* $L^-(n, r)$ *be as defined in* (3.1). *Then*

$$\text{rank}\, L^-(n, r) = \text{quaidrank}\, L^-(n, r) = \binom{n}{r-1}\frac{[(n-r)(r+1)+1]}{r}.$$

Proof. First notice that by Remark 5.3.4 we have

$$\text{rank}\, L^-(n, r) = \text{quaidrank}\, L^-(n, r) = q(n, r).$$

However, since by Theorem 4.3.7 we have

$$q(n, r) = \binom{n}{r-1}\frac{[(n-r)(r+1)+1]}{r},$$

then the result follows. ■

4. Finite strictly order-decreasing partial one-to-one transformation semigroups

In this section we investigate the rank of

$$\begin{aligned} N(I_n^-) &= \{\, \alpha \in I_n^- : F(\alpha) \text{ is empty}\} \\ &= \{\, \alpha \in I_n^- : (\forall x \in \text{dom}\,\alpha)\; x\alpha < x\} \end{aligned} \tag{4.1}$$

the semigroup of all strictly order-decreasing partial one-to-one mappings of X_n. It has been shown (in Lemmas 3.1.8 & 3.1.9) that $N(I_n^-)$ is an ideal of I_n^- consisting of all the nilpotent elements of I_n^-. Now let

$$T = \{\ \alpha \in N(I_n^-) : (\exists x \in \mathrm{dom}\ \alpha)\ x\alpha = x - 1\} \tag{4.2}$$

$$T' = \{\ \alpha \in N(I_n^-) : (\forall x \in \mathrm{dom}\ \alpha)\ x\alpha < x - 1\} \tag{4.3}$$

Then clearly $T \cup T' = N(I_n^-)$ and $T \cap T'$ is empty. Also observe that $(N(I_n^-))^2 = T'$. Our aim is to show that T is the unique minimal generating set for $N(I_n^-)$.

Proposition 5.4.1. *Let* $\alpha \in T'$. *Then* α *is expressible as a product of exactly two elements in* T.

Proof. First notice that (for all $\beta \in N(I_n^-)$) $2 \in \mathrm{dom}\ \beta$ implies that $\beta \in T$. Now for a given $\alpha \in T'$ suppose that $\mathrm{dom}\ \alpha = \{x_1, x_2, \ldots, x_k\}$ (for some $1 \le k \le n - 2$), and let

$$\delta_1 = \begin{pmatrix} 2 & x_1 & x_2 & \cdots & x_k \\ 1 & x_1\alpha + 1 & x_2\alpha + 1 & \ldots & x_k\alpha + 1 \end{pmatrix}$$

$$\delta_2 = \begin{pmatrix} x_1\alpha + 1 & x_2\alpha + 1 & \ldots & x_k\alpha + 1 \\ x_1\alpha & x_2\alpha & \cdots & x_k\alpha \end{pmatrix}.$$

Clearly $\delta_1, \delta_2 \in T$ and $\alpha = \delta_1\delta_2$. Hence the proof. ■

Corollary 5.4.2. $\langle T \rangle = N(I_n^-)$.

It is now fairly obvious from Corollary 5.4.2 and the next two lemmas that T is the unique minimal generating set for $N(I_n^-)$.

Lemma 5.4.3. $T^2 \subseteq T'$.

Proof. Let $\alpha, \beta \in T$. First observe that $2\alpha = 1 \notin \mathrm{dom}\ \alpha$ (for $\alpha \in T$). Now clearly if $x\alpha \neq 1$

$$(x\alpha)\beta < x\alpha \le x - 1 \quad \text{(for all } x \in \operatorname{dom} \alpha\beta);$$

if $x\alpha \in \operatorname{dom} \alpha$

$$(x\alpha)\beta = 1 < x - 1$$

for all $x \geq 3$, it follows that in either case $\alpha\beta \in T'$. Hence $T^2 \subseteq T'$ as required. ■

Lemma 5.4.4. T' *is an ideal of* $N(I_n^-)$.

Proof. Let $\alpha \in T'$ and $\beta \in N(I_n^-)$. Since

$$(x\alpha)\beta < x\alpha < x \quad \text{and} \quad (x\beta)\alpha < x\beta < x$$

for all $x \in \operatorname{dom} \alpha\beta$ such that $x \geq 3$, it follows that $\alpha\beta, \beta\alpha \in T'$. Hence T' is an ideal of $N(I_n^-)$ as required. ■

Notice that Lemmas 5.4.3 & 5.4.4 also follow from the fact that $(N(I_n^-))^2 = T'$. Thus we now have the main result of this section:

Theorem 5.4.5. *Let* $N(I_n^-)$ *and* T *be as defined in* (4.1) *and* (4.2) *respectively. Then*

$$\operatorname{rank} N(I_n^-) = |T| = |N(I_n^-)| - |T'| = B_n - B_{n-1}.$$

Proof. First notice that $\operatorname{rank} N(I_n^-) = |T|$ has already been established. Also from Theorem 4.3.3 we see that $|N(I_n^-)| = B_n$. And using the same technique as in the proof of Theorem 4.3.3 we can easily establish a bijection between T' and $(I_{n-2}^-)^1$ so that again (from Theorem 4.3.3) we have $|T'| = B_{n-1}$. Hence the result follows. ■

CHAPTER 6

INFINITE ORDER-DECREASING FULL TRANSFORMATION SEMIGROUPS

1. Preliminaries

Let X be a totally ordered set or a chain and let T(X) be the full transformation semigroup on X. Consider the subsets of T(X)

$$S^-(X) = \{ \alpha \in T(X) : (\forall x \in X)\, x\alpha \le x \} \tag{1.1}$$

$$S^+(X) = \{ \alpha \in T(X) : (\forall x \in X)\, x\alpha \ge x \} \tag{1.2}$$

consisting of all order-decreasing and order-increasing selfmaps of X respectively. Then

Lemma 6.1. $S^-(X)$ *and* $S^+(X)$ *are subsemigroups of* T(X).

Proof. Let $\alpha, \beta \in S^-(X)$. Then for all $x \in X$

$$(x\alpha)\beta \le x\alpha \le x$$

so that $\alpha\beta \in S^-(X)$ as required. Similarly we can show that $\alpha\beta \in S^+(X)$ for all $\alpha, \beta \in S^+(X)$. ■

2. Green's and starred Green's relations

Lemma 6.2.1. $S^-(X)$ *is* $\mathcal{R}$*-trivial.*

Proof. Suppose that $(\alpha, \beta) \in \mathcal{R}$. Then there exists δ, γ in $S^-(X)$ such that $\alpha\delta = \beta$ and $\beta\gamma = \alpha$. However, for all x in X,

$$x\beta = (x\alpha)\delta \le x\alpha, \qquad x\alpha = (x\beta)\gamma \le x\beta.$$

Thus $x\alpha = x\beta$ for all x in X and so $\alpha = \beta$. ■

Now for a given subset U of X, let

$$U^* = \{x \in X : (\exists u \in U)\ x \geq u\}.$$

U^* is sometimes called the *upper saturation* of U or the smallest *filter* of X containing U.

Lemma 6.2.2. *Let* $\alpha, \beta \in S^-(X)$ *. Then* $(\alpha, \beta) \in \mathfrak{L}$ *if and only if* $\operatorname{im}\alpha = \operatorname{im}\beta$ *and* $(z\alpha^{-1})^* = (z\beta^{-1})^*$ *for all* z in $\operatorname{im}\alpha (= \operatorname{im}\beta)$.

Proof. Let $(\alpha, \beta) \in \mathfrak{L}$. Then certainly $\operatorname{im}\alpha = \operatorname{im}\beta$ and there exist δ, γ in $S^-(X)$ such that

$$\delta\alpha = \beta \quad \text{and} \quad \gamma\beta = \alpha.$$

Let $z \in \operatorname{im}\alpha = \operatorname{im}\beta$ and let $y \in (z\alpha^{-1})^*$. Then $y \geq y'$ for some $y' \in z\alpha^{-1}$ and

$$y'\gamma\beta = y'\alpha = z$$

so that

$$y \geq y' \geq y'\gamma \in z\beta^{-1}.$$

Hence $y \in (z\beta^{-1})^*$ or $(z\alpha^{-1})^* \subseteq (z\beta^{-1})^*$. Similarly we can show that

$$(z\beta^{-1})^* \subseteq (z\alpha^{-1})^*.$$

Therefore

$$(z\alpha^{-1})^* = (z\beta^{-1})^*$$

as required.

Conversely, suppose that $\operatorname{im}\alpha = \operatorname{im}\beta$ and $(z\alpha^{-1})^* = (z\beta^{-1})^*$ for all z in $\operatorname{im}\alpha$ ($= \operatorname{im}\beta$.) Then we have to find δ, γ in $S^-(X)$ such that

$$\delta\alpha = \beta, \quad \gamma\beta = \alpha.$$

Therefore we are required to show that $\exists y \in (x\beta)\alpha^{-1}$ such that $y \leq x$. Suppose not. Then

$$x < y \ \text{ for all } y \in (x\beta)\alpha^{-1}$$

$$\Rightarrow x < y \ \text{ for all } y \in ((x\beta)\alpha^{-1})^* = ((x\beta)\beta^{-1})^*,$$

a contradiction as $x \in ((x\beta)\beta^{-1})^*$. So $\exists y \in (x\beta)\alpha^{-1}$ such that $y \le x$. Now choose such a y and define $x\delta = y$. Then clearly $\delta\alpha = \beta$. Similarly we can define γ in $S^-(X)$ such that $\gamma\beta = \alpha$. Thus $(\alpha, \beta) \in \mathcal{L}$. ■

Corollary 6.2.3. *On the semigroup* $S^-(X)$, $\mathcal{H} = \mathcal{R}$ *and* $\mathcal{L} = \mathcal{D}$.

Recall that on a semigroup S the relation $\mathcal{L}^*$ ($\mathcal{R}^*$) is defined by the rule that $(a, b) \in \mathcal{L}^*$ ($\mathcal{R}^*$) if and only if the elements a,b are related by the Green's relation $\mathcal{L}$ ($\mathcal{R}$) in some oversemigroup of S.

Lemma 6.2.4. *Let* $\alpha, \beta \in S^-(X)$. *Then*

(1) $(\alpha, \beta) \in \mathcal{R}^*$ *if and only if* $\alpha \circ \alpha^{-1} = \beta \circ \beta^{-1}$;

(2) $(\alpha, \beta) \in \mathcal{L}^*$ *if and only if* $\operatorname{im} \alpha = \operatorname{im} \beta$;

(3) $(\alpha, \beta) \in \mathcal{H}^*$ *if and only if* $\alpha \circ \alpha^{-1} = \beta \circ \beta^{-1}$ *and* $\operatorname{im} \alpha = \operatorname{im} \beta$.

Proof. (1) If $\alpha \circ \alpha^{-1} = \beta \circ \beta^{-1}$ then $(\alpha, \beta) \in \mathcal{R}(T(X))$ and so $(\alpha, \beta) \in \mathcal{R}^*(S^-(X))$.

Conversely, if $(\alpha, \beta) \in \mathcal{R}^*(S^-(X))$ then (by Lemma 1.2.1)

$$\delta\alpha = \gamma\alpha \text{ iff } \delta\beta = \gamma\beta \qquad (\text{for all } \delta, \gamma \in S^-(X)).$$

Let $x, y \in X$, with $x \ge y$, and consider the map $\phi_{x,y} : X \to X$ defined by

$$x\phi_{x,y} = y\phi_{x,y} = y, \qquad z\phi_{x,y} = z \ \ (\text{otherwise}).$$

Then $\phi_{x,y} \in S^-(X)$ and

$$\begin{aligned} x\alpha = y\alpha \quad & \text{iff} \quad \phi_{x,y}\alpha = 1.\alpha \\ \text{i.e.,} \quad & \text{iff} \quad \phi_{x,y}\beta = 1.\beta \qquad (\text{since } \alpha \,\mathcal{R}^*\, \beta) \\ \text{i.e.,} \quad & \text{iff} \quad x\beta = y\beta. \end{aligned}$$

Thus $\alpha \circ \alpha^{-1} = \beta \circ \beta^{-1}$.

(2) Certainly if $\operatorname{im} \alpha = \operatorname{im} \beta$ then $(\alpha, \beta) \in \mathcal{L}(T(X))$ and so $(\alpha, \beta) \in \mathcal{L}^*(S^-(X))$.

Conversely, if $(\alpha, \beta) \in \mathcal{L}^*(S^-(X))$ then (by Lemma 1.2.1)

$$\alpha\delta = \alpha\gamma \quad \text{iff} \quad \beta\delta = \beta\gamma \quad (\forall \delta, \gamma \in S^-(X)).$$

Let $x \in X \setminus \{x_{min}\}$ (if x_{min} exists) and let π_x (in $S^-(X)$) be defined by

$$x\pi_x = x_0 < x, \qquad y\pi_x = y \quad (y \neq x).$$

Then

$$\begin{array}{lll} x \notin \operatorname{im} \alpha & \text{iff} & \alpha\pi_x = \alpha.1 \\ \text{i.e.} & \text{iff} & \beta\pi_x = \beta.1 \qquad\qquad (\text{since } \alpha\ \mathcal{L}^*\ \beta) \\ \text{i.e.} & \text{iff} & x \notin \operatorname{im} \beta. \end{array}$$

Hence $\operatorname{im} \alpha = \operatorname{im} \beta$, since $x_{min} \in \operatorname{im} \alpha \cap \operatorname{im} \beta$ (if x_{min} exists). ■

3. The isomorphism theorem

A natural partial order $\leq_p$ on $S^-(X)$ is given by

$$\alpha \leq_p \beta \text{ if and only if } x\alpha \leq x\beta \quad (\text{for all } x \in X).$$

For a given $x \in X$, define an element $\varepsilon_x \in S^-(X)$ by

$$z\varepsilon_x = z \quad (\text{for all } z \leq x) \quad \text{and} \quad z\varepsilon_x = x \quad (\text{otherwise}).$$

Then clearly $\varepsilon_x \in E(S^-(X))$ and $\operatorname{im} \varepsilon_x$ is a principal order-ideal of X (in the sense that $r \in \operatorname{im} \varepsilon_x$ and $q \leq r \Rightarrow q \in \operatorname{im} \varepsilon_x$) generated by x. Moreover, ε_x is the unique largest element in $L^*_{\varepsilon_x}$ under the partial order $\leq_p$. Notice that if $\operatorname{im} \alpha$ is a proper order-ideal which is not principal then L^*_{α} does not have a greatest element.

Let

$$B(X) = \{\varepsilon_x : x \in X\}.$$

Then the following result is evident

Lemma 6.3.1. $B(X) \cong X$.

Proof. Let $x, y \in X$ and $\varepsilon_x, \varepsilon_y \in B(X)$. Then clearly $\varepsilon_x \leq_p \varepsilon_y$ if and only if $x \leq y$, so that the map $\varepsilon_x \to x$ is an isomorphism. ■

Lemma 6.3.2. *Let* $\varepsilon_x \in B(X)$ *and let* ε *be an idempotent in* $L^*_{\varepsilon_x}$. *Then* $\varepsilon\eta \in E(S^-(X))$ *for all* $\eta \in E(S^-(X))$.

Proof. Let $\eta \in E(S^{-}(X))$; then $x\eta = x\eta^2$ for all $x \in X$. However, for all $z \in \operatorname{im} \varepsilon_X$

$$\begin{aligned} z\varepsilon\eta &= z\eta && (z\varepsilon = z) \\ &= (z\eta)\varepsilon && (z\eta \leq z) \\ &= (z\eta\varepsilon)\eta && (z\eta\varepsilon = z\eta) \\ &= (z\varepsilon)\eta\varepsilon\eta \\ &= z(\varepsilon\eta)^2; \end{aligned}$$

for all $z \notin \operatorname{im} \varepsilon_X$

$$\begin{aligned} z\varepsilon\eta &= y\eta && (z\varepsilon = y \leq x) \\ &= (y\eta)\varepsilon && (y\eta \leq y) \\ &= (y\eta\varepsilon)\eta && (y\eta\varepsilon = y\eta) \\ &= (z\varepsilon)\eta\varepsilon\eta \\ &= z(\varepsilon\eta)^2. \end{aligned}$$

Thus $\varepsilon\eta \in E(S^{-}(X))$ as required. ■

Lemma 6.3.3. *Let* $f : S^{-}(X) \to S^{-}(Y)$ *be an isomorphism and let* ε *be an idempotent in* $L^{*}_{\varepsilon_X}$. *Then* $\operatorname{im} \varepsilon$ *is an order-ideal of* X *if and only if* $\operatorname{im}(\varepsilon f)$ *is an order-ideal of* Y.

Proof. Suppose that $\operatorname{im} \varepsilon$ is an order-ideal and $\operatorname{im}(\varepsilon f)$ is not. Then there must exist $y_1, y_2 \in Y$ with $y_1 < y_2$ such that $y_2 \in \operatorname{im}(\varepsilon f)$ and $y_1 \notin \operatorname{im}(\varepsilon f)$. Define $\eta f \in E(S^{-}(Y))$ by

$$y_2(\eta f) = y_1 \text{ and } y(\eta f) = y \quad \text{(otherwise).}$$

Then clearly $(\varepsilon f)(\eta f) \notin E(S^{-}(Y))$ since

$$y_1(\varepsilon f)(\eta f) \leq y_1(\varepsilon f) < y_1 \quad \text{and} \quad y_1 \in \operatorname{im}(\varepsilon f)(\eta f).$$

However, $\varepsilon\eta$, the preimage of $(\varepsilon f)(\eta f))$ belongs to $E(S^{-}(X))$ (by Lemma 6.3.2), which is a contradiction as the image (under an isomorphism) of an idempotent must be an idempotent. Therefore if $\operatorname{im} \varepsilon$ is an order-ideal then $\operatorname{im}(\varepsilon f)$ must be an order-ideal also, and vice-versa. ■

Lemma 6.3.4. *Let* $f : S^-(X) \to S^-(Y)$ *be an isomorphism. Then* $B(X)f = B(Y)$.

Proof. By Lemmas 6.2.4 & 6.3.3 and the fact that ε_x is the unique largest element in $L^*_{\varepsilon_x}$, it is clear that

$$(L^*_{\varepsilon_x})f = L^*_{\varepsilon_y}$$

for some $\varepsilon_y \in B(Y)$. Hence $(B(X))f = B(Y)$ as required. ■

Thus we now have the main result of this section:

Theorem 6.3.5. *Let* $S^-(X)$ *and* $S^-(Y)$ *be as defined in* (1.1). *Then the following are equivalent:*

(1) X *and* Y *are isomorphic as ordered sets;*

(2) $S^-(X)$ *is isomorphic to* $S^-(Y)$.

Proof. (1) implies (2) is obvious.

(2) implies (1). Suppose that $S^-(X) \cong S^-(Y)$. Then from Lemmas 6.3.1 & 6.3.4 we have

$$X \cong B(X) \cong B(Y) \cong Y$$

as required. ■

An immediate consequence of this result is

Corollary 6.3.6. *Let* $S^-(X)$ *and* $S^+(Y)$ *be as defined in* (1.1) & (1.2) *respectively. Then the following are equivalent:*

(1) X *and* Y are *order anti-isomorphic;*

(2) $S^-(X)$ *is isomorphic to* $S^+(Y)$.

Remark 6.3.7. Results for $S^+(Y)$ could be deduced from those for $S^-(X)$, where X and Y are anti-isomorphic.

4. Abundant semigroups

The following lemma is proved for the finite case in [27] and no essential use is made of the finiteness of X.

Lemma 6.4.1.[27, Lemma 2.1]. *Let* $\alpha \in T(X)$. *Then* α *is an idempotent if and only if every block of* α *is stationary, i.e., if and only if* $t \in t\alpha^{-1}$ *for all* $t \in \operatorname{im} \alpha$.

Lemma 6.4.2. *Let* $\alpha \in S^{-}(X)$. *Then* α *is an idempotent if and only if (for all* $t \in \operatorname{im} \alpha$), $t = \min\{x : x \in t\alpha^{-1}\}$.

Proof. By Lemma 6.4.1 α is an idempotent if and only if $t \in t\alpha^{-1}$ for all $t \in \operatorname{im} \alpha$,

i.e., iff (for all $t \in \operatorname{im} \alpha$) $t = \min\{x : x \in t\alpha^{-1}\}$,

since $t \in t\alpha^{-1}$ and $x \in t\alpha^{-1}$ implies that $x \geq x\alpha = t\alpha = t$. ∎

Hence an immediate consequence of Lemmas 6.2.4 & 6.4.2 is that $S^{-}(X)$ need not be an abundant semigroup for an arbitrary chain since $x\alpha^{-1}$ need not contain a least element (for some $x \in \operatorname{im} \alpha$). Therefore our aim now is to find under what condition(s) is $S^{-}(X)$ abundant.

A subset B of X is said to be *left-bounded* if there exists $x \in X$ such that $x \leq b$ for all $b \in B$. A *right-bounded* set is defined dually. A totally ordered set Y is called *left properly ordered* if every left-bounded subset of Y is well-ordered.

Examples 6.4.3. (a) Every well-ordered set is left properly ordered but not vice-versa as the next example shows.

(b) The set of integers $\mathbb{Z}$ is left properly ordered but the set of reals $\mathbb{R}$ is not (under the usual ordering). Notice that the open interval $(0, 1)$ is left-bounded by 0 but it does not have a minimum element.

(c) The set $\mathbb{Z} \times \mathbb{N}$ (under the ordering $(a, b) \le (c, d)$ if and only if $a \le c$ or $a = c$ and $b \le d$) is left properly ordered but $\mathbb{Z} \times \mathbb{Z}$ is not under the corresponding (lexicographic) ordering.

It is worth remarking that if $\alpha \in S^{-}(X)$ then every $x\alpha^{-1}$ is left-bounded, by the element x itself. Hence if X is left properly ordered every non-empty set $x\alpha^{-1}$ has a least element. It is also the case that if X does not contain a least element then $\operatorname{im}\alpha$ must be infinite for all $\alpha \in S^{-}(X)$. Thus we now have

Theorem 6.4.4. *Let* $S^{-}(X)$ *be as defined in* (1.1). *Then* $S^{-}(X)$ *is abundant if and only if* X *is a left properly ordered set.*

Proof. To show the direct half that $S^{-}(X)$ is abundant if X is a left properly ordered set, consider $\alpha \in S^{-}(X)$ and define $\varepsilon, \eta \in S^{-}(X)$ by

$$x\varepsilon = \min\{t : t \in y\alpha^{-1}\} \qquad (x \in y\alpha^{-1}, y \in \operatorname{im}\alpha)$$

$$y\eta = x \qquad (x \in \operatorname{im}\alpha, y \in A(x))$$

where

$$A(x) = \{x\}^{*} \setminus \bigcup_{\substack{y \in \operatorname{im}\alpha \\ y > x}} \{y\}^{*}.$$

(Notice that $\bigcup_{x \in \operatorname{im}\alpha} A(x) = X$.) Then clearly $\varepsilon, \eta \in E(S^{-}(X))$. Moreover, $(\varepsilon, \alpha) \in \mathcal{R}^{*}$ and $(\eta, \alpha) \in \mathcal{L}^{*}$ (by Lemma 6.2.4). Thus $S^{-}(X)$ is abundant.

Conversely, if $S^{-}(X)$ is abundant then every $\mathcal{R}^{*}$-class contains an idempotent. Let B be a left-bounded subset of X. Let ρ be the equivalence whose only non-singleton class is B and consider the $\mathcal{R}^{*}$-class $\{\alpha \in S^{-}(X) : \alpha \circ \alpha^{-1} = \rho\}$. This contains an idempotent ε and $B\varepsilon = b$, with $b \in B$. Then $b = \min B$. Thus X is a left properly ordered set as required. ■

From now onwards X is a left properly ordered set. And for a given cardinal number $\xi \leq |X|$ let

$$K(|X|, \xi) = \{ \alpha \in T(X) : |\mathrm{im}\ \alpha| \leq \xi \} \tag{4.1}$$

$$K^{-}(|X|, \xi) = \{ \alpha \in S^{-}(X) : |\mathrm{im}\ \alpha| \leq \xi \} \tag{4.2}$$

be the ideals of $T(X)$ and $S^{-}(X)$ respectively. The next lemma is a genaralisation of Lemmas 6.2.1 & 6.2.2 and can be proved similarly.

Lemma 6.4.5. *Let* $(\alpha, \beta) \in K^{-}(|X|, \xi)$. *Then*

(1) $K^{-}(|X|, \xi)$ *is* $\mathcal{R}$*-trivial;*

(2) $(\alpha, \beta) \in \mathcal{L}$ *if and only if* $\mathrm{im}\ \alpha = \mathrm{im}\ \beta$ *and* $(z\alpha^{-1})^{*} = (z\beta^{-1})^{*}$ *for all* z in im $\alpha(= \mathrm{im}\ \beta)$;

(3) $\mathcal{H} = \mathcal{R}$ *and* $\mathcal{L} = \mathcal{D}$.

Since $K^{-}(|X|, \xi)$ contains some non idempotents then we deduce that

Corollary 6.4.6. $K^{-}(|X|, \xi)$ *is a non-regular semigroup.*

Recall that a subsemigroup U (of a semigroup S) is said to be an inverse ideal of S if for all $u \in U$, there exists $u' \in S$ such that $uu'u = u$ and $uu', u'u \in U$.

Lemma 6.4.7. $K^{-}(|X|, \xi)$ *is an inverse ideal of* $T(X)$.

Proof. The proof of Lemma 2.2.4 applies to this case since there was no essential use of the finiteness of X. ■

Hence by Lemmas 1.2.7 & 1.2.8 and [**20**, Proposition II.4.5 and Ex.II.10] we deduce the following result:

Theorem 6.4.8. $K^{-}(|X|, \xi)$ *is a non-regular abundant semigroup.*

Moreover, for all $\alpha, \beta \in K^{-}(|X|, \xi)$ *we have*

(1) $(\alpha, \beta) \in \mathfrak{L}^*$ *if and only if* $\operatorname{im} \alpha = \operatorname{im} \beta$;

(2) $(\alpha, \beta) \in \mathfrak{R}^*$ *if and only if* $\alpha \circ \alpha^{-1} = \beta \circ \beta^{-1}$;

(3) $(\alpha, \beta) \in \mathfrak{H}^*$ *if and only if* $\operatorname{im} \alpha = \operatorname{im} \beta$ *and* $\alpha \circ \alpha^{-1} = \beta \circ \beta^{-1}$.

To characterize $\mathfrak{D}^*$ on $K^{-}(|X|, \xi)$ we define the relation $\mathfrak{K}$ by the rule

$$(\alpha, \beta) \in \mathfrak{K} \quad \text{iff} \quad |\operatorname{im} \alpha| = |\operatorname{im} \beta|.$$

Then obviously $\mathfrak{R}^*$, $\mathfrak{L}^*$ and $\mathfrak{D}^* \subseteq \mathfrak{K}$. We now have the following result:

Lemma 6.4.9. $\mathfrak{K} = \mathfrak{L}^* \circ \mathfrak{R}^* \circ \mathfrak{L}^* = \mathfrak{R}^* \circ \mathfrak{L}^* \circ \mathfrak{R}^* = \mathfrak{D}^*$.

Proof. Suppose that $(\alpha, \beta) \in \mathfrak{K}$ so that $|\operatorname{im} \alpha| = |\operatorname{im} \beta|$. Let

$$\alpha = \begin{pmatrix} A_i \\ a_i \end{pmatrix}, \quad \beta = \begin{pmatrix} B_i \\ b_i \end{pmatrix} \qquad (i \in I)$$

and let θ be a bijection from $\operatorname{im} \alpha$ onto $\operatorname{im} \beta$. Also let $C = \{ c_i : c_i = \max(a_i, a_i\theta)\}$ and

$$A(c_i) = \{c_i\}^* \setminus \bigcup_{\substack{c \in \operatorname{im} \alpha \\ c > c_i}} \{c\}^*.$$

Now define δ, γ by

$$x\delta = a_i \qquad (x \in A(c_i))$$

$$x\gamma = a_i\theta \qquad (x \in A(c_i)).$$

Then clearly $\delta, \gamma \in K^{-}(|X|, \xi)$ and $\alpha \, \mathfrak{L}^* \, \delta \, \mathfrak{R}^* \, \gamma \, \mathfrak{L}^* \, \beta$ (by Lemma 6.2.4), so that

$$\mathfrak{K} \subseteq \mathfrak{L}^* \circ \mathfrak{R}^* \circ \mathfrak{L}^*.$$

It is clear that

$$\mathfrak{L}^* \circ \mathfrak{R}^* \circ \mathfrak{L}^* \subseteq \mathfrak{K}.$$

Thus

$$\mathfrak{K} = \mathfrak{L}^* \circ \mathfrak{R}^* \circ \mathfrak{L}^*.$$

On the other hand let $D = \{ d_i : d_i = \min(a_i, a_i\theta)\}$ and define δ', γ' by

$$A_i\delta' = d_i\,, \qquad B_i\gamma' = d_i$$

respectively. Then clearly $\delta', \gamma' \in K^-(|X|, \xi)$ and $\alpha\ \mathcal{R}^*\ \delta'\ \mathcal{L}^*\ \gamma'\ \mathcal{R}^*\ \beta$ (by Lemma 6.2.4), so that

$$\mathcal{K} \subseteq \mathcal{R}^* \circ \mathcal{L}^* \circ \mathcal{R}^*$$

It is clear that

$$\mathcal{R}^* \circ \mathcal{L}^* \circ \mathcal{R}^* \subseteq \mathcal{K}.$$

Thus

$$\mathcal{K} = \mathcal{R}^* \circ \mathcal{L}^* \circ \mathcal{R}^*.$$

Now from the inequalities

$$\mathcal{D}^* \subseteq \mathcal{K} = \mathcal{L}^* \circ \mathcal{R}^* \circ \mathcal{L}^* = \mathcal{R}^* \circ \mathcal{L}^* \circ \mathcal{R}^* \subseteq \mathcal{D}^*$$

we deduce the result of the lemma. ■

Corollary 6.4.10. *Let* $\alpha, \beta \in K^-(|X|, \xi)$. *Then* $(\alpha, \beta) \in \mathcal{D}^*$ *if and only if* $|\text{im}\ \alpha| = |\text{im}\ \beta|$.

Corollary 6.4.11. $K^-(|X|, \aleph_0)$ *is a *-bisimple abundant semigroup if* X *is not left-bounded.*

Proof. It follows from an ealier remark that if X is not left-bounded then $|\text{im}\ \alpha| \geq \aleph_0$, for all $\alpha \in K^-(|X|, \xi)$. ■

Recall that the relation $\mathcal{J}^*$ is defined by the rule that $a\ \mathcal{J}^*\ b$ if and only if $J^*(a) = J^*(b)$, where $J^*(a)$ is the principal *-ideal generated by a.

Lemma 6.4.12. *Let* $\alpha, \beta \in K^-(|X|, \xi)$. *If* $\alpha \in J^*(\beta)$ *then* $|\text{im}\ \alpha| \leq |\text{im}\ \beta|$.

Proof. The proof of Lemma 2.2.12 applies to this case since there was no essential use of the finiteness of X. ■

Lemma 6.4.13. *On the semigroup* $K^{-}(|X|, \xi)$, $\mathcal{D}^* = \mathcal{J}^*$.

Proof. The proof of Lemma 2.2.13 applies to this case since there was no essential use of the finiteness of X. ■

We also observe that $K^{-}(|X|, \xi)$ is a *-ideal since it is a union of $\mathcal{J}^*$-classes (of $S^{-}(X)$).

Finally, as in Section II.2, we show by an example that in the semigroup $K^{-}(|X|, \xi)$, $\mathcal{R}^* \circ \mathcal{L}^* \neq \mathcal{D}^* \neq \mathcal{L}^* \circ \mathcal{R}^*$.

Example 6.4.14. Let $y_1, y_2 \in X$ such that $y_1 < y_2$ and $|\{z : z \leq y_1\}| \geq \aleph_0$. Define $\alpha, \beta \in K^{-}(|X|, \xi)$ respectively by

$$x\alpha = \begin{cases} x & (\text{if } x < y_1) \\ y_1 & (\text{if } y_1 \leq x < y_2) \\ y_2 & (\text{if } y_2 \leq x) \end{cases}$$

$$x\beta = \begin{cases} x & (\text{if } x < y_1) \\ y_1 & (\text{if } y_1 \leq x). \end{cases}$$

Then clearly $|\text{im}\,\alpha| = |\text{im}\,\beta|$ so that $(\alpha, \beta) \in \mathcal{D}^*$.

On the other hand for all γ such that $\alpha\, \mathcal{L}^* \gamma\, \mathcal{R}^* \beta$ we must have (by Lemma 6.2.4)

$$x\gamma = y_2 \quad (\text{for some } x \leq y_1),$$

so that $\gamma \notin K^{-}(|X|, \xi)$. Thus $\mathcal{D}^* \neq \mathcal{L}^* \circ \mathcal{R}^*$.

Similarly, $(\beta, \alpha) \in \mathcal{D}^*$ and for all δ such that $\beta\, \mathcal{R}^* \delta\, \mathcal{L}^* \alpha$ we must have (by Lemma 6.2.4)

$$x\delta = y_2 \quad (\text{for some } x \leq y_1),$$

so that $\delta \notin K^{-}(|X|, \xi)$. Thus $\mathcal{D}^* \neq \mathcal{R}^* \circ \mathcal{L}^*$.

5. Rees quotient semigroups

For a given cardinal number $\xi \leq |X|$ let $K(|X|, \xi)$ and $K^{-}(|X|, \xi)$ be as defined in (4.1) & (4.2) (respectively) and let

$$P(X, \xi_i) = K(|X|, \xi_i)/ K(|X|, \xi_{i-1}) \tag{5.1}$$

$$P^{-}(X, \xi_i) = K^{-}(|X|, \xi_i)/ K^{-}(|X|, \xi_{i-1}) \tag{5.2}$$

where ξ_i is the immediate successor of ξ_{i-1}. (See **[18]** for a discussion about cardinal numbers.) Then $P(\xi_i)$ is a regular [0-] bisimple semigroup whose non-zero elements may be thought of as the elements of $T(X)$ of rank ξ_i precisely. The product of two elements of $P(\xi_i)$ is 0 whenever their product in $T(X)$ is of rank strictly less than ξ_i. Similarly $P^{-}(\xi_i)$ is a Rees quotient semigroup whose non-zero elements may be thought of as the elements of $S^{-}(X)$ of rank ξ_i precisely. The product of two elements of $P^{-}(\xi_i)$, is 0 whenever their product in $S^{-}(X)$ is of rank strictly less than ξ_i. We begin our investigation on the properties of $P^{-}(\xi_i)$ by first characterizing the Green's relations.

Lemma 6.5.1. *Let* $(\alpha, \beta) \in P^{-}(\xi_i)$. *Then*

(1) $P^{-}(\xi_i)$ *is* $\mathcal{R}$*-trivial;*

(2) $(\alpha, \beta) \in \mathcal{L}$ *if and only if* $\mathrm{im}\,\alpha = \mathrm{im}\,\beta$ *and* $(z\alpha^{-1})^* = (z\beta^{-1})^*$ *for all* z in $\mathrm{im}\,\alpha (= \mathrm{im}\,\beta)$;

(3) $\mathcal{H} = \mathcal{R}$ *and* $\mathcal{L} = \mathcal{D}$.

Proof. (1) & (2). The proof is similar to that of Lemmas 6.2.1 & 6.2.2 respectively.

(3) The proof follows directly from (1) & (2). ■

Lemma 6.5.2. $P^{-}(\xi_i)$ *is an inverse ideal of* $P(\xi_i)$.

Proof. The proof is similar to that of Lemma 2.2.4 since there was no essential use of the finiteness of X (in that proof). ■

Hence by Lemma 1.2.8 and [5, Lemmas 10.55 & 10.56] we deduce the following result:

Lemma 6.5.3. *Let* $\alpha, \beta \in P^{-}(\xi_i)$. *Then*

(1) $(\alpha, \beta) \in \mathcal{R}^*$ *if and only if* $\alpha \circ \alpha^{-1} = \beta \circ \beta^{-1}$;

(2) $(\alpha, \beta) \in \mathcal{L}^*$ *if and only if* $\operatorname{im} \alpha = \operatorname{im} \beta$;

(3) $(\alpha, \beta) \in \mathcal{H}^*$ *if and only if* $\alpha \circ \alpha^{-1} = \beta \circ \beta^{-1}$ *and* $\operatorname{im} \alpha = \operatorname{im} \beta$.

Using the same techniques as in Section 2 we obtain a characterization of the relation $\mathcal{D}^*$ on $P^{-}(\xi_i)$.

Lemma 6.5.4. *On the semigroup* $P^{-}(\xi_i)$, *we have the following:*

(1) $\mathcal{D}^* = \mathcal{R}^* \circ \mathcal{L}^* \circ \mathcal{R}^* = \mathcal{L}^* \circ \mathcal{R}^* \circ \mathcal{L}^*$;

(2) $(\alpha, \beta) \in \mathcal{D}^*$ *if and only if* $|\operatorname{im} \alpha| = |\operatorname{im} \beta|$ *for all* $\alpha, \beta \in P^{-}(\xi_i)$.

Hence by Lemma 1.2.7 & 6.5.2 we have

Theorem 6.5.5. *Let* $P^{-}(\xi_i)$ *be as defined in* (5.2). *Then* $P^{-}(\xi_i)$ *is a non-regular* 0^**-bisimple abundant semigroup.*

Corollary 6.5.6. *On the semigroup* $P^{-}(\xi_i)$, $\mathcal{D}^* = \mathcal{J}^*$.

6. Infinite order-decreasing partial transformations

From some of the results obtained for the full transformation case it is shown in this section that we can deduce the corresponding results for the partial case using Vagner's result in [39]. Let $P^*(X)$ be the zero stabilizer subsemigroup of $T^0(X)$, the full transformation semigroup on X^0 ($= X \cup \{0\}$ and $0 \leq x$ for all $x \in X$) and let $P(X)$ be the partial transformation semigroup on X. Consider the subsemigroups of $P(X)$

$$P^-(X) = \{ \alpha \in P(X) : (\forall x \in \operatorname{dom} \alpha)\ x\alpha \leq x\} \cup \{\varnothing\} \tag{6.1}$$

$$P^+(X) = \{ \alpha \in P(X) : (\forall x \in \operatorname{dom} \alpha)\ x\alpha \geq x\} \cup \{\varnothing\} \tag{6.2}$$

consisting of all order-decreasing and order-increasing partial selfmaps of X (including the empty or zero map) respectively. Also let

$$PK(\xi) = \{ \alpha \in P(X) : |\operatorname{im} \alpha| \leq \xi\} \tag{6.3}$$

$$PK^-(\xi) = \{ \alpha \in P^-(X) : |\operatorname{im} \alpha| \leq \xi\} \tag{6.4}$$

$$PK^+(\xi) = \{ \alpha \in P^+(X) : |\operatorname{im} \alpha| \leq \xi\} \tag{6.5}$$

Thus $PK^-(|X|) = P^-(X)$ and each $PK^-(\xi)$ is a two-sided ideal of $P^-(X)$. Now since $PK^-(\xi)$ is a two-sided ideal let

$$PP(X, \xi_i) = PK(\xi_i)/ PK(\xi_{i-1}) \tag{6.6}$$

$$PP^-(X, \xi_i) = PK^-(\xi_i)/ PK^-(\xi_{i-1}) \tag{6.7}$$

(where ξ_i is the immediate successor of ξ_{i-1}) be the Rees quotient semigroups on the two-sided ideals $PK(\xi_i)$ and $PK^-(\xi_i)$ respectively.

Lemma 6.6.1. *Let* $P^*(\xi) = \{\alpha \in P^*(X) : |\operatorname{im} \alpha| \leq \xi\}$ *and let* $PP^*(\xi_i) = P^*(\xi_i)/P^*(\xi_{i-1})$. *Then* $K^-(|X^0|, \xi) \subseteq P^*(\xi)$ *and* $P^-(X^0, \xi_i) \subseteq PP^*(\xi_i)$.

As in Section II.4 we now record the result of Vagner [**39**] (also to be found in [**5**, p. 254]).

Theorem 6.6.2. *For each* $\alpha \in P(X)$, *define the transformation* α^* *of* X^0 *by*

$$x\alpha^* = \begin{cases} x\alpha & (\text{if } x \in \operatorname{dom} \alpha) \\ 0 & (\text{if } x \notin \operatorname{dom} \alpha) \end{cases}$$

Then α^* *belongs to the subsemigroup* $P^*(X)$ *of* $T^0(X)$.

Conversely, if $\beta \in P^*(X)$, *then its restriction to* X, $\beta|_X = \beta \cap (X \times X)$, *is a* *partial transformation of* X. *The domain of* $\beta|_X$ *is the set of all* x *in* X *for which* $x\beta \neq 0$. *Then the mappings* $\alpha \to \alpha^*$ *and* $\beta \to \beta|_X$ *are mutually inverse isomorphisms of* P(X) *onto* $P^*(X)$ *and vice-versa.*

From Lemma 6.6.1 and Theorem 6.6.2 we observe that the isomorphism $\alpha \to \alpha^*$ maps $PK^-(\xi)$ onto $P^*(\xi)$; and $PP^-(\xi_i)$ onto $PP^*(\xi_i)$ since for all $x \in$ dom α

$$x\alpha^* \leq x \quad \text{if and only if} \quad x\alpha \leq x$$

and

$$x\alpha^* = 0 \leq x \quad (\text{for all } x \notin \text{dom } \alpha).$$

For convenience we record this as a corollary

Corollary 6.6.3. *Let* $\theta : \alpha \to \alpha^*$ *be the isomorphism defined in Theorem 6.6.2. Then* $(PK^-(\xi))\theta = P^*(\xi)$, $(P^*(\xi))\theta^{-1} = PK^-(\xi)$, $(PP^-(\xi_i))\theta = PP^*(\xi_i)$ and $(PP^*(\xi_i)\theta^{-1} = PP^-(\xi_i)$.

Now let $PS^-(X)$ be any of the semigroups $PK^-(\xi)$ or $PP^*(\xi_i)$ for some cardinals $\xi_i, \xi \leq |X|$. Immediate consequences of Corollary 2.4.3 are the following:

Theorem 6.6.4. *Let* $PS^-(X)$ *be any of the semigroups* $PK^-(\xi)$ or $PP^*(\xi_i)$ *for some cardinals* $\xi_i, \xi \leq |X|$. *Then*

(1) $PS^-(X)$ *is* $\mathcal{R}$*-trivial;*

(2) $(\alpha, \beta) \in \mathcal{L}$ *if and only if* $\text{im } \alpha = \text{im } \beta$ *and* $(z\alpha^{-1})^* = (z\beta^{-1})^*$ *for all* $z \in \text{im } \alpha$ $(\alpha, \beta \in PS^-(X))$.

Proof. (1) These follow from Lemmas 6.2.1 & 6.2.2 respectively. ■

Corollary 6.6.5. *On the semigroup* $PS^-(X)$, $\mathcal{H} = \mathcal{R}$ *and* $\mathcal{L} = \mathcal{D}$.

Theorem 6.6.6. *Let* $PS^-(X)$ *be any of the semigroups* $PK^-(\xi)$ or $PP^*(\xi_i)$ *for some cardinals* $\xi_i, \xi \leq |X|$. *Then* $PS^-(X)$ *is a non-regular abundant semigroup and for* $\alpha, \beta \in PS^-(X)$ *we have*

(1) $(\alpha, \beta) \in \mathcal{R}^*$ *if and only if* $\alpha \circ \alpha^{-1} = \beta \circ \beta^{-1}$;

(2) $(\alpha, \beta) \in \mathcal{L}^*$ *if and only if* $\operatorname{im} \alpha = \operatorname{im} \beta$;

(3) $(\alpha, \beta) \in \mathcal{H}^*$ *if and only if* $\operatorname{im} \alpha = \operatorname{im} \beta$ *and* $\alpha \circ \alpha^{-1} = \beta \circ \beta^{-1}$;

(4) $(\alpha, \beta) \in \mathcal{D}^*$ *if and only if* $|\operatorname{im} \alpha| = |\operatorname{im} \beta|$;

(5) $\mathcal{D}^* = \mathcal{J}^*$.

Proof. That $PS^-(X)$ is a non-regular abundant semigroup follows from Theorems 6.4.8 & 6.5.5; (1) & (2) follow from Theorem 6.4.8 and Lemma 6.5.3; (3) follows from (1) & (2); while (3) & (4) follow from Corollary 6.4.10 and Lemmas 6.4.13 & 6.5.4. ∎

Remark 6.6.7. Results for $PS^+(X)$ could be deduced from the corresponding results for $PS^-(Y)$, where X and Y are order anti-isomorphic.

CHAPTER 7

ORDER-DECREASING PARTIAL ONE-TO-ONE TRANSFORMATION SEMIGROUPS

1. Preliminaries

Let X be a totally ordered set or a chain and let $I(X)$ be the symmetric inverse semigroup (i.e. the semigroup of partial one-to-one transformation semigroup on X). Consider the subsets of $I(X)$

$$I^-(X) = \{ \alpha \in I(X) : (\forall x \in \operatorname{dom} \alpha)\ x\alpha \leq x\} \cup \{\emptyset\} \tag{1.1}$$

$$I^+(X) = \{ \alpha \in I(X) : (\forall x \in \operatorname{dom} \alpha)\ x\alpha \geq x\} \cup \{\emptyset\} \tag{1.2}$$

consisting of all order-decreasing and order-increasing partial one-to-one selfmaps (including the empty or zero map) of X respectively. For a given cardinal number $\xi \leq |X|$, let

$$L(|X|, \xi) = \{ \alpha \in I(X) : |\operatorname{im} \alpha| \leq \xi\} \tag{1.3}$$

$$L^-(|X|, \xi) = \{ \alpha \in I^-(X) : |\operatorname{im} \alpha| \leq \xi\} \tag{1.4}$$

$$L^+(|X|, \xi) = \{ \alpha \in I^+(X) : |\operatorname{im} \alpha| \leq \xi\} \tag{1.5}$$

be the two-sided ideals of $I(X)$, $I^-(X)$ and $I^+(X)$ respectively. Then

Lemma 7.1. $L^-(|X|, \xi)$ *and* $L^+(|X|, \xi)$ *are subsemigroups of* $I(X)$.

Proof. The proof of Lemma 3.1.1 applies to this case since there was no essential use of the finiteness of X. ■

Remark 7.2. It is easy to see that $L^-(|X|, \xi)$ is a full subsemigroup of $L(|X|, \xi)$. Hence $E(L^-(|X|, \xi))$ is a semilattice.

2. Green's and starred Green's relations

Lemma 7.2.1. $L^-(|X|, \xi)$ *is* $\mathcal{D}$ *-trivial.*

Proof. First notice that since $L^-(|X|, \xi)$ is a subsemigroup of $P^-(X)$ then (by Theorem 6.4.4) $L^-(|X|, \xi)$ is $\mathcal{R}$-trivial and for all $z \in \operatorname{im}\alpha$

$$
\begin{aligned}
(\alpha, \beta) \in \mathcal{L}(L^-) &\Rightarrow \operatorname{im}\alpha = \operatorname{im}\beta \text{ and } (z\alpha^{-1})^* = (z\beta^{-1})^* \\
&\Rightarrow \operatorname{im}\alpha = \operatorname{im}\beta \text{ and } z\alpha^{-1} = z\beta^{-1} \\
&\Rightarrow \alpha = \beta
\end{aligned}
$$

since α, β are one-to-one. Thus $L^-(|X|, \xi)$ is $\mathcal{L}$ -trivial and the result now follows.

Now since $L^-(|X|, \xi)$ contains some non-idempotents we immediately deduce that

Corollary 7.2.2. $L^-(|X|, \xi)$ *is a non-regular semigroup.*

Lemma 7.2.3. *Let* $\alpha, \beta \in L^-(|X|, \xi)$. *Then*

(1) $(\alpha, \beta) \in \mathcal{L}^*$ *if and only if* $\operatorname{im}\alpha = \operatorname{im}\beta$;

(2) $(\alpha, \beta) \in \mathcal{R}^*$ *if and only if* $\operatorname{dom}\alpha = \operatorname{dom}\beta$;

(3) $(\alpha, \beta) \in \mathcal{H}^*$ *if and only if* $\operatorname{im}\alpha = \operatorname{im}\beta$ *and* $\operatorname{dom}\alpha = \operatorname{dom}\beta$.

Proof. The proof of Lemma 3.2.3 applies to this case since there was no essential use of the finiteness of X. ■

Remark 7.2.4. Alternatively, since $L^-(|X|, \xi)$ is a full subsemigroup of an abundant semigroup, then Lemma 7.2.3 follows directly from Lemma 1.2.3.

Lemma 7.2.5. $L^-(|X|, \xi)$ is an inverse ideal of $I(X)$.

Proof. The proof of Lemma 3.2.5 applies to this case. ■

Notice that since $L^-(|X|, \xi)$ is a full subsemigroup of $I(X)$ then we deduce the following result:

Theorem 7.2.6. *Let* $L^-(|X|, \xi)$ *be as defined in* (1.4). *Then* $L^-(|X|, \xi)$ *is a non- regular type* A *semigroup.*

To characterize the relation $\mathcal{D}^*$ we consider the relation $\mathcal{K}$ on $L^-(|X|, \xi)$ defined by the rule that

$$(\alpha,\beta) \in \mathcal{K} \text{ if and only if } |\mathrm{im}\,\alpha| = |\mathrm{im}\,\beta|,$$

$$\text{i.e., if and only if } |\mathrm{dom}\,\alpha| = |\mathrm{dom}\,\beta|.$$

Then clearly $\mathcal{D}^* \subseteq \mathcal{K}$. Now we show the following:

Lemma 7.2.7. $\mathcal{K} = \mathcal{R}^* \circ \mathcal{L}^* \circ \mathcal{R}^* = \mathcal{L}^* \circ \mathcal{R}^* \circ \mathcal{L}^* = \mathcal{D}^*$.

Proof. Suppose that $(\alpha, \beta) \in \mathcal{K}$ so that $|\mathrm{im}\,\alpha| = |\mathrm{im}\,\beta|$. Let

$$\alpha = \begin{pmatrix} a_i \\ a_i' \end{pmatrix}, \quad \beta = \begin{pmatrix} b_i \\ b_i' \end{pmatrix} \qquad (i \in I)$$

and let θ be a bijection from dom α onto dom β. Now let $C = \{ c_i : c_i = \min(a_i, a_i\theta)\}$ and define δ, γ with $\mathrm{im}\,\delta = \mathrm{im}\,\gamma = C$ by

$$a_i\delta = c_i \text{ and } a_i\theta\gamma = c_i$$

respectively. Then clearly $\delta, \gamma \in L^-(|X|, \xi)$ and $\alpha \,\mathcal{R}^*\, \delta \,\mathcal{L}^*\, \gamma \,\mathcal{R}^*\, \beta$ (by Lemma 7.2.3), so that

$$\mathcal{K} \subseteq \mathcal{R}^* \circ \mathcal{L}^* \circ \mathcal{R}^*.$$

It is clear that

$$\mathcal{R}^* \circ \mathcal{L}^* \circ \mathcal{R}^* \subseteq \mathcal{K}.$$

Thus

$$\mathcal{H} = \mathcal{R}^* \circ \mathcal{L}^* \circ \mathcal{R}^*.$$

On the other hand let θ' be a bijection from $\operatorname{im}\alpha$ onto $\operatorname{im}\beta$ and let $D = \{ d_i : d_i = \max(a_i', a_i'\theta')\}$. Now define δ, γ with $\operatorname{dom}\delta = \operatorname{dom}\gamma = D$ by

$$d_i\delta = a_i' \text{ and } d_i\gamma = a_i'\theta'$$

respectively. Then clearly $\delta, \gamma \in L^-(|X|, \xi)$ and $\alpha\, \mathcal{L}^*\, \delta\, \mathcal{R}^*\, \gamma\, \mathcal{L}^*\, \beta$ (by Lemma 7.2.3), so that

$$\mathcal{H} \subseteq \mathcal{L}^* \circ \mathcal{R}^* \circ \mathcal{L}^*.$$

It is clear that

$$\mathcal{L}^* \circ \mathcal{R}^* \circ \mathcal{L}^* \subseteq \mathcal{H}.$$

Thus

$$\mathcal{H} = \mathcal{L}^* \circ \mathcal{R}^* \circ \mathcal{L}^*.$$

Now from the inequalities

$$\mathcal{D}^* \subseteq \mathcal{H} = \mathcal{R}^* \circ \mathcal{L}^* \circ \mathcal{R}^* = \mathcal{L}^* \circ \mathcal{R}^* \circ \mathcal{L}^* \subseteq \mathcal{D}^*$$

we deduce the result of the lemma. ■

Corollary 7.2.8. *Let* $\alpha, \beta \in L^-(|X|, \xi)$. *Then* $(\alpha,\beta) \in \mathcal{D}^*$ *if and only if* $|\operatorname{im}\alpha| = |\operatorname{im}\beta|$.

Recall that the relation $\mathcal{J}^*$ is defined by the rule that $a\, \mathcal{J}^*\, b$ if and only if $J^*(a) = J^*(b)$, where $J^*(a)$ is the principal *-ideal generated by a.

Lemma 7.2.9. *Let* $\alpha, \beta \in L^-(|X|, \xi)$. *If* $\alpha \in J^*(\beta)$ *then* $|\operatorname{im}\alpha| = |\operatorname{im}\beta|$.

Proof. The proof of Lemma 3.2.12 applies to this case since there was no essential use of the finiteness of X. ■

Lemma 7.2.10. *On the semigroup* $L^-(|X|, \xi)$, $\mathcal{D}^* = \mathcal{J}^*$.

Proof. The proof of Lemma 3.2.13 applies to this case since there was no essential use of the finiteness of X. ■

We observe that $L^-(|X|, \xi)$ is a *-ideal since it is a union of $\mathfrak{J}^*$-classes (of $I^-(X)$).

Finally (in this section) we observe that on the semigroup $L^-(|X|, \xi)$ $\mathcal{L}^* \circ \mathcal{R}^* \neq \mathcal{D}^* \neq \mathcal{R}^* \circ \mathcal{L}^*$. For some $x, y \in X$ such that $x < y$, let

$$\alpha = \binom{x}{x}, \quad \beta = \binom{y}{y}.$$

Then clearly $(\alpha, \beta) \in \mathcal{D}^*$ and if $(\alpha, \beta) \in \mathcal{R}^* \circ \mathcal{L}^*$ then there must exist $\gamma \in L^-(|X|, \xi)$ such that

$$\alpha \, \mathcal{R}^* \, \gamma \, \mathcal{L}^* \, \beta.$$

However by Lemma 7.2.3

$$\gamma = \binom{x}{y} \notin L^-(|X|, \xi) \text{ so that } \mathcal{D}^* \neq \mathcal{R}^* \circ \mathcal{L}^*.$$

Similarly, $(\beta, \alpha) \in \mathcal{D}^*$, and if $(\beta, \alpha) \in \mathcal{L}^* \circ \mathcal{R}^*$ then there must exist $\delta \in L^-(|X|, \xi)$ such that

$$\beta \, \mathcal{L}^* \, \delta \, \mathcal{R}^* \, \alpha.$$

Again by Lemma 7.2.3

$$\delta = \binom{x}{y} \notin L^-(|X|, \xi) \text{ so that } \mathcal{D}^* \neq \mathcal{L}^* \circ \mathcal{R}^*.$$

3. The isomorphism theorem

A natural partial order $\leq_p$ on $I^-(X)$ is given by

$$\alpha \leq_p \beta \text{ if and only if } \operatorname{dom} \alpha \subseteq \operatorname{dom} \beta \text{ and } x\alpha \leq x\beta \text{ (for all } x \in \operatorname{dom} \alpha).$$

For a given $x \in X$, define an element $\varepsilon_x \in I^-(X)$ by

$$z\varepsilon_x = z \quad (\text{for all } z \leq x).$$

Then clearly $\varepsilon_x \in E(I^-(X))$ and $\operatorname{dom} \varepsilon_x = \operatorname{im} \varepsilon_x$ is a principal order-ideal of X generated by x.

Let

$$B(X) = \{\varepsilon_x : x \in X\}.$$

Then the following result is evident

Lemma 7.3.1. $B(X) \cong X$.

Proof. Let $x, y \in X$ and $\varepsilon_x, \varepsilon_y \in B(X)$. Then clearly $\varepsilon_x \leq_p \varepsilon_y$ if and only if $x \leq y$, so that the map $\varepsilon_x \to x$ is an isomorphism. ■

Proposition 7.3.2. *Let* R^*_α *be an* $\mathcal{R}^*$*-class of* $\Gamma(X)$. *Then the following are equivalent:*

(1) dom α *is an order-ideal of* X;

(2) R^*_α *is a subsemigroup of* $\Gamma(X)$.

Proof. (1) $\Rightarrow$ (2). Suppose that dom α is an order-ideal. Then clearly for all $\alpha, \beta \in R^*_\alpha$, im $\alpha \subseteq$ dom α = dom β (by Lemma 7.2.3), so that

$$\text{dom } \alpha\beta = (\text{im } \alpha \cap \text{dom } \beta)\alpha^{-1} = (\text{im } \alpha)\alpha^{-1} = \text{dom } \alpha.$$

Thus $\alpha\beta \in R^*_\alpha$ as required.

(2) $\Rightarrow$ (1). Suppose that dom α is not an order-ideal. Then there exist $x_1, x_2 \in X$, with $x_1 < x_2$ such that $x_2 \in$ dom α and $x_1 \notin$ dom α. However, β defined as

$$x_2\beta = x_1, \quad x\beta = x\alpha \quad (x \in \text{dom } \alpha \setminus \{x_2\})$$

(with dom β = dom α) is such that dom $\beta\alpha \neq$ dom β, since $x_2 \in$ dom β but $x_2 \notin$ dom $\beta\alpha$. Thus $\beta\alpha \notin R^*_\alpha$ if dom α is not an order-ideal. Hence the proof. ■

An immediate consequence of the above result is that if $f : \Gamma(X) \to \Gamma(Y)$ is an isomorphism then f induces an isotone (or order-preserving) bijection $\Phi : X \to Y$, where $\Phi(\text{dom } \alpha) = \Phi(\text{dom } (\alpha f))$, i. e., Φ maps the order-ideals of X onto the order-ideals of Y. In fact, it is also the case that Φ maps the principal order-ideals of X onto the principal order-ideals of Y. To see the latter from the former, let I_c be a principal order-ideal generated by c and let $J_c = \{x \in X : x < c\}$. (Note that J_c may or may not be principal.) Certainly $J_c \subset I_c$, so that $\Phi(J_c) \subset \Phi(I_c)$. Now suppose by way of contradiction that $\Phi(I_c) = K$ is not principal. Denote $\Phi(J_c)$ by L. Then $L \subset K$. Let $k_1 \in K \setminus L$ and

$$L_1 = \{ x : x \leq k_1 \} \supset L.$$

Since K is not a principal order-ideal, then there exist $k_2, k_3, \ldots$ such that $k_1 < k_2 < k_3 < \ldots$ and

$$L \subset L_1 \subset L_2 \subset \ldots \subset K.$$

On the other hand there is no order-ideal strictly between J_c and I_c. Thus we have a contradiction. Hence we now have

Lemma 7.3.3. *Let* $f : \Gamma(X) \rightarrow \Gamma(Y)$ *be an isomorphism. Then* $B(X)f = B(Y)$.

Proof. Suppose that $f : \Gamma(X) \rightarrow \Gamma(Y)$ is an isomorphism. Since

$$(R_\alpha^*)f = R_{\alpha f}^*$$

and R_α^* is a subsemigroup if and only if $R_{\alpha f}^*$ is a subsemigroup, it follows from Proposition 7.3.2 and the above remarks that

$$B(X)f = B(Y)$$

as required. ■

Thus we now have the main result of this section:

Theorem 7.3.4. *Let* $\Gamma(X)$ *and* $\Gamma(Y)$ *be as defined in* (1.1). *Then the following are equivalent:*

(1) X *and* Y *are order isomorphic;*

(2) $\Gamma(X)$ *is isomorphic to* $\Gamma(Y)$.

Proof. (1) implies (2) is obvious.

(2) implies (1). Suppose that $\Gamma(X) \cong \Gamma(Y)$. Then from Lemmas 7.3.1 & 7.3.3 we have

$$X \cong B(X) \cong B(Y) \cong Y$$

as required. ■

An immediate consequence of this result is

Corollary 7.3.5. *Let* $I^-(X)$ *and* $I^+(Y)$ *be as defined in* (1.1) & (1.2) *respectively. Then the following are equivalent:*

(1) X *and* Y *are order anti-isomorphic;*

(2) $I^-(X)$ *is isomorphic to* $I^+(Y)$.

Remark 7.3.6. Results for $I^+(Y)$ could be deduced from those for $I^-(X)$, where X and Y are order anti-isomorphic.

4. Rees quotient semigroups

For a given cardinal number $\xi \le |X|$ let $L(|X|\ \xi)$ and $L^-(|X|, \xi)$ be as defined in (1.3) & (1.4) (respectively) and let

$$Q(\xi_i) = L(|X|, \xi_i)/ L(|X|, \xi_{i-1}) \tag{4.1}$$

$$Q^-(\xi_i) = L^-(|X|, \xi_i)/ L^-(|X|, \xi_{i-1}) \tag{4.2}$$

be their Rees quotient semigroups respectively. Then $Q(\xi_i)$ is a [0-] bisimple inverse semigroup whose non-zero elements may be thought of as the elements of I(X) of rank ξ_i precisely. The product of two elements of $Q(\xi_i)$ is 0 whenever their product in I(X) is of rank strictly less than ξ_i. Similarly $Q^-(\xi_i)$ is a Rees quotient semigroup whose non-zero elements may be thought of as the elements of $I^-(X)$ of rank ξ_i precisely. The product of two elements of $Q^-(\xi_i)$, is 0 whenever their product in $I^-(X)$ is of rank strictly less than ξ_i. First we remark that

Remark 7.4.1. It is easy to see that $Q^-(\xi_i)$ is a full subsemigroup of $Q(\xi_i)$. Hence $E(Q^-(\xi_i))$ is a semilattice.

Lemma 7.4.2. $Q^-(\xi_i)$ *is* $\mathcal{D}$*-trivial.*

Proof. Suppose that $(\alpha, \beta) \in \mathcal{R}$. Then there exists δ, γ in $Q^{-}(\xi_i)$ such that $\alpha\delta = \beta$ and $\beta\gamma = \alpha$. However, for all x in X,

$$x\beta = (x\alpha)\delta \leq x\alpha, \qquad x\alpha = (x\beta)\gamma \leq x\beta.$$

Thus $x\alpha = x\beta$ for all x in X and so $\alpha = \beta$. Thus $Q^{-}(\xi_i)$ is $\mathcal{R}$-trivial.

Next suppose that $(\alpha, \beta) \in \mathcal{L}$. Then certainly $\operatorname{im} \alpha = \operatorname{im} \beta$ and there exist δ, γ in $Q^{-}(\xi_i)$ such that

$$\delta\alpha = \beta \quad \text{and} \quad \gamma\beta = \alpha.$$

Let $z \in \operatorname{im} \alpha = \operatorname{im} \beta$ and let $y \in (z\alpha^{-1})^*$. Then $y \geq y' = z\alpha^{-1}$ and

$$y'\gamma\beta = y'\alpha = z$$

so that

$$y \geq y' \geq y'\gamma \in z\beta^{-1} \subseteq (z\beta^{-1})^* .$$

Thus $(z\alpha^{-1})^* \subseteq (z\beta^{-1})^*$. Similarly we can show that

$$(z\beta^{-1})^* \subseteq (z\alpha^{-1})^*.$$

Therefore

$$(z\alpha^{-1})^* = (z\beta^{-1})^*,$$

and hence

$$z\alpha^{-1} = z\beta^{-1}$$

since α, β are one-to-one. However this implies that $\alpha = \beta$. Thus $Q^{-}(\xi_i)$ is $\mathcal{L}$-trivial. The result follows. ■

Lemma 7.4.3. $Q^{-}(\xi)$ *is an inverse ideal of* $Q(\xi)$.

Proof. For a given $\alpha \in Q^{-}(\xi_i)$ define α' by

$$x\alpha' = x\alpha^{-1} \quad (\text{for all } x \in \operatorname{im} \alpha).$$

Then clearly $\alpha\alpha'\alpha = \alpha$. Moreover, for all $x \in \operatorname{im} \alpha$

$$x\alpha'\alpha = x.$$

And for all x

$$x\alpha\alpha' = (x\alpha)\alpha^{-1} = x.$$

Thus $\alpha'\alpha, \alpha\alpha' \in Q^{-}(\xi_i)$ since $|\operatorname{im} \alpha'| = |\operatorname{im} \alpha|$. It now follows that $Q^{-}(\xi_i)$ is an inverse ideal of $Q(\xi_i)$ as required. ■

Hence by Lemmas 1.2.7 & 1.2.8 and the analogues (to the partial one-to-one case) of [5, Lemmas 10.55 & 10.56] we deduce the following result:

Theorem 7.4.4. *Let* $Q^-(\xi_i)$ *be as defined in* (4.4). *Then* $Q^-(\xi_i)$ *is a type A semigroup. Moreover, for all* $\alpha, \beta \in Q^-(\xi_i)$ *we have*

(1) $(\alpha, \beta) \in \mathcal{L}^*$ *if and only if* $\operatorname{im} \alpha = \operatorname{im} \beta$;

(2) $(\alpha, \beta) \in \mathcal{R}^*$ *if and only if* $\operatorname{dom} \alpha = \operatorname{dom} \beta$;

(3) $(\alpha, \beta) \in \mathcal{H}^*$ *if and only if* $\operatorname{im} \alpha = \operatorname{im} \beta$ *and* $\operatorname{dom} \alpha = \operatorname{dom} \beta$.

Using the same techniques as in Section 2 we obtain a characterization of the relation $\mathcal{D}^*$ on $Q^-(\xi_i)$.

Lemma 7.4.5. *On the semigroup* $Q^-(\xi_i)$, *we have the following:*

(1) $\mathcal{D}^* = \mathcal{R}^* \circ \mathcal{L}^* \circ \mathcal{R}^* = \mathcal{L}^* \circ \mathcal{R}^* \circ \mathcal{L}^*$;

(2) $(\alpha, \beta) \in \mathcal{D}^*$ *if and only if* $|\operatorname{im} \alpha| = |\operatorname{im} \beta|$ *for all* $\alpha, \beta \in Q^-(\xi_i)$.

Hence we have

Theorem 7.4.6. *Let* $Q^-(\xi_i)$ *be as defined in* (4.2). *Then* $Q^-(\xi_i)$ *is a non-regular* 0^**-bisimple type A semigroup.*

Corollary 7.4.7. *On the semigroup* $Q^-(\xi_i)$, $\mathcal{D}^* = \mathcal{J}^*$.

APPENDIX

n \ r	1	2	3	4	5	6	$\Sigma J^*(n, r)$
1	1						1
2	1	1					2
3	1	4	1				6
4	1	11	11	1			24
5	1	26	66	26	1		120
6	1	57	302	302	57	1	720

Table 1. Eulerian numbers

n \ r	1	2	3	4	5	6	$\Sigma sh(n, r)$
1	1						1
2	1	1					2
3	1	3	2				6
4	1	6	11	6			24
5	1	10	35	50	24		120
6	1	15	85	225	274	120	720

Table 2. Complementary signless Stirling number of the first kind

n \ r	1	2	3	4	5	6	$\Sigma S(n, r) = B_n$
1	1						1
2	1	1					2
3	1	3	1				5
4	1	7	6	1			15
5	1	15	25	10	1		52
6	1	31	90	75	15	1	203

Table 3. Stirling number of the second kind and the Bell's number

n \ r	0	1	2	3	4	5	$\Sigma \underline{sh}(n, r)$
1	2						2
2	4	2					6
3	8	12	4				24
4	16	48	44	12			120
5	32	160	280	200	48		720

Table 4.

n \ r	1	2	3	4	5	6	$\Sigma q(n, r)$
1	1						1
2	3	1					4
3	5	6	1				12
4	7	14	10	1			32
5	9	25	30	15	1		80
6	11	39	65	55	21	1	192

Table 5.

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