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BENJAMIN KLOPSCH AND MARTYN QUICK

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Just infinite groups play a significant role in profinite group theory. For each $c \geqslant 0$, we consider more generally $\mathrm{JNN}_c\mathrm{F}$ profinite (or, in places, discrete) groups that are Fitting-free; these are the groups G such that every proper quotient of G is virtually class-c nilpotent whereas G itself is not, and additionally G does not have any nontrivial abelian normal subgroup. When $c = 1$, we obtain the just non-(virtually abelian) groups without nontrivial abelian normal subgroups.

Our first result is that a finitely generated profinite group is virtually class-c nilpotent if and only if there are only finitely many subgroups arising as the lower central series terms $\gamma_{c+1}(K)$ of open normal subgroups K of G. Based on this we prove several structure theorems. For instance, we characterize the $\mathrm{JNN}_c\mathrm{F}$ profinite groups in terms of subgroups of the above form $\gamma_{c+1}(K)$. We also give a description of $\mathrm{JNN}_c\mathrm{F}$ profinite groups as suitable inverse limits of virtually nilpotent profinite groups. Analogous results are established for the family of hereditarily $\mathrm{JNN}_c\mathrm{F}$ groups and, for instance, we show that a Fitting-free $\mathrm{JNN}_c\mathrm{F}$ profinite (or discrete) group is hereditarily $\mathrm{JNN}_c\mathrm{F}$ if and only if every maximal subgroup of finite index is $\mathrm{JNN}_c\mathrm{F}$. Finally, we give a construction of hereditarily $\mathrm{JNN}_c\mathrm{F}$ groups, which uses as an input known families of hereditarily just infinite groups.

1. Introduction and main results

If $\mathscr{P}$ is a property of groups, a group G is said to be *just non-*$\mathscr{P}$ when G does not have property $\mathscr{P}$ but all proper quotients of G do satisfy $\mathscr{P}$. In the case when G is a profinite group, we require instead that every quotient of G by a nontrivial *closed* normal subgroup has $\mathscr{P}$. The property $\mathscr{P}$ considered most often has been that of being finite and the more common term *just infinite* is then used. Just infinite groups are particularly important within the context of profinite — or more generally residually finite — groups, since infinite residually finite groups are never simple but instead just infinite groups can be viewed as those with all proper

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quotients *essentially* trivial from a 'residually finite' viewpoint (see, for example, the discussion in [Leedham-Green and McKay 2002, §12.1]). Important examples of just infinite groups include the Grigorchuk group [1984] and the Nottingham group [Klopsch 2000; Hegedűs 2001], but also families arising as quotients of arithmetic groups by their centers [Bass et al. 1967].

There is a dichotomy in the study of just non-$\mathscr{P}$ groups. One thread within the literature is concerned with the study of just non-$\mathscr{P}$ groups possessing a nontrivial normal abelian subgroup. In this context, a key idea is to exploit the structure of a maximal abelian normal subgroup when viewed as a module in the appropriate way. Studies of this type include [McCarthy 1968; 1970; De Falco 2002; Quick 2007] and we also refer to the monograph [Kurdachenko et al. 2002] for more examples. On the other hand, Wilson [1971; 2000] addresses the case of just infinite groups with no nontrivial abelian normal subgroup. He shows that such groups fall into two classes: (i) branch groups and (ii) certain subgroups of wreath products of a hereditarily just infinite group by a symmetric group of finite degree. The class of branch groups has been studied considerably (see, for example, [Grigorchuk 2000; Bartholdi et al. 2003], though many more articles on these groups have appeared since these surveys were written). It is known that every proper quotient of a branch group is virtually abelian (see the proof of [Grigorchuk 2000, Theorem 4]) and there are examples of branch groups that are not just infinite (see [Fink 2014], for example). It is interesting therefore to note that Wilson's methods extend to the class of groups with all proper quotients virtually abelian, as observed by Hardy in his PhD thesis [2002]. We shall use the abbreviation *JNAF groups* for these just non-(abelian-by-finite) groups.

More recently, Reid [2010a; 2010b; 2012; 2018] established various fundamental results concerning the structure and properties of just infinite groups. One might wonder to what extent JNAF groups have a similar structure to just infinite groups. In this article, we demonstrate how, for fixed $c \geqslant 0$, Reid's results may in fact be extended to the even larger class of groups with all proper quotients being virtually nilpotent of class at most c; that is, the just non-(class-c-nilpotent-by-finite) groups. We shall abbreviate this term to *JNN$_c$F group* in what follows. The case $c = 0$ essentially returns Reid's results, while the case $c = 1$ covers all JNAF groups and so, in particular, would apply to all branch groups.

We do require some additional, though rather mild, hypotheses to those appearing in Reid's work. First, the JNN$_c$F groups that we consider will be assumed to be *Fitting-free*; that is, to have no nontrivial abelian normal subgroup. This is consistent with Wilson's and Hardy's studies and with the viewpoint that says that the case with a nontrivial abelian normal subgroup should be studied through a module-theoretic lens. (As an aside, we emphasize that JNN$_c$F groups with some nontrivial abelian normal subgroup are, in particular, abelian-by-nilpotent-by-finite). Infinite

Fitting-free groups cannot be virtually nilpotent, so part of the definition of $\mathrm{JNN}_c\mathrm{F}$ group comes immediately. In addition, we shall frequently assume that the groups under consideration are finitely generated. This latter condition will enable us to control the structure of the quotients that arise.

It is interesting to note which parts of Reid's ideas adapt readily to the $\mathrm{JNN}_c\mathrm{F}$ setting and where differences occur. One example is that he implicitly uses the fact that a proper quotient of a just infinite group, being finite, has only finitely many subgroups. In contrast, any infinite (virtually nilpotent) quotient of a profinite group will necessarily have infinitely many open normal subgroups. We shall depend upon the following result as a key tool in our work. It means that, while a finitely generated virtually nilpotent profinite group typically has infinitely many closed normal subgroups, it only has finitely many that occur as corresponding lower central series subgroups of open normal subgroups.

Theorem A. *Let G be a finitely generated profinite group. Then G is virtually nilpotent of class at most c if and only if the set $\{\gamma_{c+1}(K) \mid K \trianglelefteq_{\mathrm{o}} G\}$ is finite.*

Theorem A is established as Theorem 2.10 in Section 2. In that section, we also give precise definitions and recall properties needed during the course of our work.

In Section 3, we fix an integer $c \geqslant 0$ and investigate the structure of $\mathrm{JNN}_c\mathrm{F}$ profinite groups G that are Fitting-free. We shall establish various descriptions that generalize those of just infinite groups in [Reid 2010a; 2012; 2018]. One point that can be noted is that the subgroups of the form $\gamma_{c+1}(K)$, for K an open normal subgroup of G, play a role in $\mathrm{JNN}_c\mathrm{F}$ groups analogous to that of open normal subgroups in just infinite groups. For example, we show that a directed graph Γ can be constructed from a suitable subcollection of $\{\gamma_{c+1}(K) \mid K \trianglelefteq_{\mathrm{c}} G\}$ that is locally finite. This enables us to establish our first characterization of $\mathrm{JNN}_c\mathrm{F}$ groups (established as Theorem 3.3 below), which is the following analogue of Reid's "generalized obliquity theorem" [Reid 2010a, Theorem A]. Specifying $c = 0$ results in a mild weakening of Reid's theorem.

Theorem B. *Let G be a finitely generated infinite profinite group that has no nontrivial abelian closed normal subgroup. Then G is $\mathrm{JNN}_c\mathrm{F}$ if and only if the set $\mathcal{A}_H = \{\gamma_{c+1}(K) \mid K \trianglelefteq_{\mathrm{o}} G \text{ with } \gamma_{c+1}(K) \nleqslant H\}$ is finite for every open subgroup H of G.*

This result is used to characterize, in Theorem 3.5, when a finitely generated Fitting-free profinite group is $\mathrm{JNN}_c\mathrm{F}$. The characterization is expressed in terms of properties of a descending chain of open normal subgroups H_i and the $(c+1)$-th term of their lower central series. In Theorem 3.7, we establish a further characterization of such a group as an inverse limit in a manner analogous to [Reid 2012, Theorem 4.1]. One important tool (see Lemma 3.1) that is used throughout Section 3 is that, if G is a Fitting-free $\mathrm{JNN}_c\mathrm{F}$ profinite group and N is a nontrivial closed normal subgroup, then the Melnikov subgroup $M(N)$ of N is nontrivial and so, via the Fitting-free assumption, $\gamma_i(M(N)) \neq \mathbf{1}$ for all $i \geqslant 1$.

Section 4 is concerned with the structure of profinite groups that are hereditarily JNN$_c$F. We establish there a similar suite of results, though the description of a finitely generated, Fitting-free hereditarily JNN$_c$F group as an inverse limit is more technical (see Theorem 4.7). It shares this level of technicality with Reid's characterization of hereditarily just infinite groups.

In Section 5, we establish the following (as Corollary 5.5) which is the analogue of the main result of [Reid 2010a]. The material in this section does not depend upon Theorem A and so is more directly developed from Reid's arguments.

Theorem C. *Let G be a JNN$_c$F profinite or discrete group that has no nontrivial abelian normal subgroup. Then G is hereditarily JNN$_c$F if and only if every maximal* (*open*) *subgroup of finite index is JNN$_c$F.*

One reasonable conclusion from the results described so far is that there is a similarity in the structure of JNN$_c$F groups when compared to just infinite groups. One might ask: just how closely are these classes linked? As JNN$_c$F groups have not yet been studied systematically, there are presently rather few examples to examine when considering these links. In the final section of the paper, Section 6, we take a first step and present one way to construct hereditarily JNN$_c$F groups from hereditarily just infinite groups as semidirect products and discuss some explicit examples. We give examples of hereditarily JNAF groups of the form $G \rtimes A$ where G can be a hereditarily just infinite group suitably built as an iterated wreath product or using Wilson's Construction B [2010] and A can be selected from a rather broad range of abelian groups (see Examples 6.10 and 6.16). By exploiting the fact that every countable pro-p group can be embedded in the Nottingham group, we construct a hereditarily JNN$_c$F group of the form $\mathrm{SL}_n^1(\mathbb{F}_p[\![T]\!]) \rtimes A$ where A can be any virtually nilpotent pro-p group (see Example 6.17). This last family of examples demonstrates that, for every possible choice of $c \geqslant 1$, there is a JNN$_c$F pro-p group that is not just non-(virtually nilpotent of smaller class).

Since the examples constructed are built using hereditarily just infinite groups, one is drawn back to the above question concerning the link between JNN$_c$F groups and just infinite groups. The results of Sections 3–5 suggest such a link and it is an open challenge to produce examples of hereditarily JNN$_c$F groups of a compellingly different flavor to those built in Section 6.

2. Preliminaries

In this section, we first give the precise definitions of the groups under consideration. We then recall some useful tools from [Reid 2012] and make a number of basic observations about JNN$_c$F groups. In the last part of the section we consider the behavior of finitely generated virtually nilpotent groups and establish Theorem A which is crucial for the sections that follow.

We shall write maps on the right throughout, so $H\phi$ denotes the image of a group H under a homomorphism ϕ and x^y is the conjugate $y^{-1}xy$. If G is a profinite group, we use the usual notation $H \trianglelefteq_{\mathrm{o}} G$ and $K \trianglelefteq_{\mathrm{c}} G$ for an open normal subgroup and a closed normal subgroup, respectively. If K and L are closed subgroups of G, then $[K, L]$ will denote the *closed* subgroup generated by all commutators $[x, y] = x^{-1}y^{-1}xy$ where $x \in K$ and $y \in L$. The *lower central series* of G is then defined by $\gamma_1(G) = G$ and $\gamma_{i+1}(G) = [\gamma_i(G), G]$ for each $i \geqslant 1$. As usual, we also use G' for the derived subgroup $\gamma_2(G)$ of G. These concepts will, in particular, be relevant for the instances of the following definition that concern us.

Definition 2.1. Let $\mathscr{P}$ be a property of groups. A profinite (or discrete) group G is said to be *just non-$\mathscr{P}$* if G does not have property $\mathscr{P}$ but G/N does have $\mathscr{P}$ for every nontrivial closed normal subgroup N of G. It is *hereditarily just non-$\mathscr{P}$* if every closed subgroup of finite index in G is just non-$\mathscr{P}$.

(When G is discrete, the word "closed" can and should be ignored. Note that a closed subgroup of finite index is necessarily open, but the definition is phrased to enable that for discrete groups to be readily extracted).

In this paper we consider three options for the property $\mathscr{P}$:

(1) When $\mathscr{P}$ is the property of being finite, we use the more common term *just infinite* for an infinite group with every proper quotient finite.

(2) We use the abbreviation *JNAF* for just non-$\mathscr{P}$ when $\mathscr{P}$ is the property of being virtually abelian, which is the same as being abelian-by-finite. A profinite group has an abelian subgroup of finite index if and only if it has an abelian open subgroup (as the topological closure of any abelian subgroup is again abelian), so we use the term virtually abelian in this situation also.

(3) If c is an integer with $c \geqslant 0$, we use the abbreviation *JNN$_c$F* for just non-$\mathscr{P}$ when $\mathscr{P}$ is the property that there is a subgroup H of finite index such that $\gamma_{c+1}(H) = \mathbf{1}$. A profinite group has a class-c nilpotent subgroup of finite index if and only if it has an open class-c nilpotent subgroup (as the topological closure of any class-c nilpotent subgroup is again class-c nilpotent).

The case $c = 1$ for a JNN$_c$F group is then identical to it being JNAF. We shall speak of a group G being *virtually class-c nilpotent* when it has a subgroup H of finite index satisfying $\gamma_{c+1}(H) = \mathbf{1}$. More precisely such a group is "virtually (nilpotent of class at most c)". The JNN$_c$F groups G considered will usually be assumed to not have a nontrivial abelian closed normal subgroup. Consequently, such G will itself not be virtually nilpotent (of any class) and so we are studying groups that are just non-(virtually nilpotent) with an additional bound upon the nilpotency class occurring in the proper quotients. In particular, when $c = 1$ we are considering groups that are not virtually metabelian but where every proper quotient is virtually abelian.

Let G be a profinite group. In line with [Reid 2012, Definition 2.1], a *chief factor* of G is a quotient K/L where K and L are closed normal subgroups of G such that there is no closed normal subgroup M of G with $L < M < K$. Accordingly, we do not require that K be open in G in this definition, though necessarily L is open in K and hence K/L is isomorphic (under an isomorphism that commutes with the action of G) to a chief factor K_0/L_0 with K_0 an open normal subgroup of G.

The *Melnikov subgroup* $M(G)$ of G is the intersection of all maximal open normal subgroups of G. Provided G is nontrivial, this is a topologically characteristic proper closed subgroup of G. As usual, to say a subgroup of G is *topologically characteristic* means that it is invariant under all automorphisms of G that are also homeomorphisms. We follow [Reid 2012, Definition 3.1] and, for a nontrivial closed normal subgroup A of G, define $M_G(A)$ to be the intersection of all maximal G-invariant open subgroups of A. This satisfies $M(A) \leqslant M_G(A) < A$. We call A a *narrow subgroup* of G if A has a unique maximal G-invariant open subgroup (that is, when $M_G(A)$ is this unique subgroup). The first part of the following lemma is a consequence of the correspondence theorem, while the other two are, respectively, Lemmas 3.2 and 3.3 in Reid's paper [2012].

Lemma 2.2. *Let G be a profinite group.*

(i) *Let K and L be closed normal subgroups of G such that $L \leqslant M_G(K)$. Then $M_{G/L}(K/L) = M_G(K)/L$.*

(ii) *Let K and L be closed normal subgroups of G. Then $K \leqslant LM_G(K)$ if and only if $K \leqslant L$.*

(iii) *If K/L is a chief factor of G, there is a closed normal subgroup A which is narrow in G and is contained in K but not in L. This narrow subgroup satisfies $A \cap L = M_G(A)$.*

It is well-known that a finitely generated finite-by-abelian discrete group is center-by-finite. This is established by ideas related to FC-groups (see [Robinson 1996, Section 14.5], in particular, the proof of (14.5.11)). In the case of profinite groups, however, the hypothesis of finite generation is unnecessary, as observed by Detomi, Morigi and Shumyatsky (see [Detomi et al. 2020, Lemma 2.7]). In fact, a similar argument establishes the following result needed in our context and that only needs residual finiteness as a hypothesis:

Lemma 2.3. *Let G be a residually finite group with a finite normal subgroup N. If G/N is virtually class-c nilpotent, for some $c \geqslant 0$, then G is also virtually class-c nilpotent.*

Proof. For each nonidentity element x of N, there exists a normal subgroup of finite index in G that does not contain x. By intersecting these, we produce a normal subgroup K of finite index in G such that $N \cap K = \mathbf{1}$. Then G embeds in the direct product $G/N \times G/K$ of G/N and a finite group, so the result follows. □

Corollary 2.4. *Let G be a profinite group that is JNN$_c$F. Then G has no nontrivial finite normal subgroup.*

Sections 3 and 4 are concerned with profinite JNN$_c$F groups, whereas the last two sections consider both profinite and abstract JNN$_c$F groups. To state efficiently the results in Section 5, we shall adopt there the convention that "subgroup" for a profinite group means "closed subgroup" so that it remains in the same category. For the results in the current section that will be used in the discrete case, we simply bracket the word "closed" to indicate it is unnecessary in such a situation. The following lemma illustrates this convention. It is a standard elementary fact about just non-$\mathscr{P}$ groups when $\mathscr{P}$ is a property that is inherited by both finite direct products and subgroups.

Lemma 2.5. *Let G be a profinite group or discrete group that is JNN$_c$F. If K and L are nontrivial* (*closed*) *normal subgroups of G, then $K \cap L \neq \mathbf{1}$.*

Just as Reid [2010b] does, we use Wilson's concept [2000] of a basal subgroup:

Definition 2.6. A subgroup B of a group G is called *basal* if B is nontrivial, has finitely many conjugates $B_1, B_2, \dots, B_n$ in G and the normal closure of B in G is the direct product of these conjugates: $B^G = B_1 \times B_2 \times \cdots \times B_n$.

Lemma 2.8 below is based on [Reid 2010b, Lemma 5]. The hypothesis that K has only finitely many conjugates is sufficient to adapt the proof of Reid's lemma to our needs. In its statement, and in many that follow, we shall say that a (profinite or discrete) group G is *Fitting-free* when it has no nontrivial abelian (closed) normal subgroup. This is immediately equivalent to the requirement that the Fitting subgroup $F(G)$ be trivial. Furthermore, if G is a JNN$_c$F group, one observes that G is Fitting-free if and only if G is not virtually soluble. If K is a normal subgroup of G, then $Z(K) = K \cap C_G(K)$ and we deduce the following characterization of the Fitting-free condition in JNN$_c$F groups using Lemma 2.5.

Lemma 2.7. *Let G be a profinite or discrete group that is JNN$_c$F. Then G is Fitting-free if and only if $C_G(K) = \mathbf{1}$ for every nontrivial normal* (*closed*) *subgroup K of G.*

Lemma 2.8. *Let G be a profinite or discrete group that is Fitting-free. Let K be a nontrivial* (*closed*) *subgroup of G whose conjugates $\{K_i \mid i \in I\}$ are parametrized by the finite set I and which satisfies $K \trianglelefteqslant K^G$. Then there exists some $J \subseteq I$ such that $\bigcap_{j \in J} K_j$ is basal.*

Proof. For $J \subseteq I$, define $K_J = \bigcap_{j \in J} K_j$. Let $\mathcal{I}$ be the set of subsets J of I such that $K_J \neq \mathbf{1}$. Certainly $\mathcal{I}$ is nonempty since it contains all singletons as $K \neq \mathbf{1}$. Choose $J \in \mathcal{I}$ of largest size and define $B = K_J$. Then B also has finitely many conjugates in G and we denote these by $B_1, B_2, \dots, B_n$. Two distinct conjugates intersect trivially, $B_i \cap B_j = \mathbf{1}$ when $i \neq j$, since this is the intersection of more than $|J|$ conjugates of K. Since each K_i is normal in K^G, it follows that each $B_j \trianglelefteq K^G$ and therefore $[B_i, B_j] \leqslant B_i \cap B_j = \mathbf{1}$ when $i \neq j$. Set $L = B^G = B_1 B_2 \cdots B_n$. Then the center of L is the product of the centers of the B_j. Our hypothesis that G has no nontrivial abelian (closed) subgroup then forces $Z(B_j) = \mathbf{1}$ for each j. Now if $j \in \{1, 2, \dots, n\}$, set $P_j = B_1 \cdots B_{j-1} B_{j+1} \cdots B_n$. Then $[B_j, P_j] = \mathbf{1}$ and so $P_j \cap B_j \leqslant Z(B_j) = \mathbf{1}$. Since this holds for each j, we conclude that $L = B_1 \times B_2 \times \cdots \times B_n$; that is, B is basal. □

Properties of virtually nilpotent profinite groups. If N is a closed normal subgroup of a profinite group G, we define the commutator subgroup $[N,_i G] \trianglelefteq_{\mathrm{c}} G$ recursively by $[N,_0 G] = N$ and $[N,_i G] = [[N,_{i-1} G], G]$ for $i \geqslant 1$. Thus, using left-normed commutator notation,

$$[N,_c G] = [N, \underbrace{G, G, \dots, G}_{c \text{ times}}].$$

We also write $Z_i(G)$ for the i-th term of the upper central series of a group G.

Lemma 2.9. *Let G be a finitely generated profinite group and N be an open normal subgroup of G such that $\gamma_{c+1}(N) = \mathbf{1}$ for some $c \geqslant 0$. Then $[N,_i G]$ is an open subgroup of $\gamma_{c+1}(G)$ for all $i \geqslant c$.*

Proof. Define $k = |G/N|$. It follows from the definitions that $[N,_c G]$ is a closed normal subgroup of G with $N/[N,_c G] \leqslant Z_c(G/[N,_c G])$. Hence, this term of the upper central series is open in $G/[N,_c G]$ and a theorem of Baer — see [Robinson 1996, (14.5.1)] — shows that $\gamma_{c+1}(G/[N,_c G])$ is finite. Hence $[N,_c G]$ is an open subgroup of $\gamma_{c+1}(G)$.

Now suppose that we have shown $[N,_i G]$ is open in $\gamma_{c+1}(G)$ for some $i \geqslant c$. This subgroup is generated, modulo $[N,_{i+1} G]$, by all left-normed commutators $[x, y_1, y_2, \dots, y_i]$ where x is selected from some finite generating set for N and $y_1, y_2, \dots, y_i$ from a finite generating set for G. In particular, $[N,_i G]/[N,_{i+1} G]$ is a finitely generated abelian profinite group. Furthermore, standard commutator calculus shows that, modulo $[N,_{i+1} G]$,

$$[x, y_1, y_2, \dots, y_i]^{k^i} \equiv [x, y_1^k, y_2^k, \dots, y_i^k] \in \gamma_{i+1}(N) = \mathbf{1}.$$

Hence, every generator of $[N,_i G]/[N,_{i+1} G]$ has finite order and so this abelian group is finite. It follows that $[N,_{i+1} G]$ is an open subgroup of $[N,_i G]$. The lemma then follows by induction on $i \geqslant c$. □

The next result establishes, in particular, Theorem A, stated in the introduction.

Theorem 2.10. (i) *Let G be a finitely generated virtually class-c nilpotent profinite group, for some $c \geqslant 0$. Then the set $\{\gamma_{c+1}(K) \mid K \trianglelefteq_{\mathrm{c}} G\}$ is finite.*

(ii) *Conversely, if G is a profinite group such that $\{\gamma_{c+1}(K) \mid K \trianglelefteq_{\mathrm{o}} G\}$ is finite, for some $c \geqslant 0$, then G is virtually class-c nilpotent.*

Proof. (i) Let N be an open normal subgroup of G such that $\gamma_{c+1}(N) = \mathbf{1}$. Let K be any closed normal subgroup of G and set $L = KN$. By standard commutator calculus, any element of $[N,_{2c} KN]$ can be expressed as a product of commutators $[x, y_1, y_2, \ldots, y_{2c}]$ where $x \in N$ and each y_i belongs either to K or to N. Since such a commutator involves either at least $c+1$ entries from K or at least $c+1$ entries from N, we deduce

$$[N,_{2c} L] = [N,_{2c} KN] \leqslant \gamma_{c+1}(N)\, [N,_{c+1} K] \leqslant \gamma_{c+1}(K) \leqslant \gamma_{c+1}(L).$$

Furthermore, upon applying Lemma 2.9 to the profinite group L, we conclude that $[N,_{2c} L]$ is an open subgroup of $\gamma_{c+1}(L)$. Therefore, for each open normal subgroup L of G that contains N, there are at most finitely many possibilities for $\gamma_{c+1}(K)$ as K ranges over all closed normal subgroups of G with $KN = L$. Finally, since there are only finitely many possibilities for L, we conclude that $\{\gamma_{c+1}(K) \mid K \trianglelefteq_{\mathrm{c}} G\}$ is indeed finite.

(ii) Let G be a profinite group and suppose that $\mathcal{A} = \{\gamma_{c+1}(K) \mid K \trianglelefteq_{\mathrm{o}} G\}$ is finite. If N is any open normal subgroup of G, then the set $\mathcal{L}_{G/N} = \{\gamma_{c+1}(H) \mid H \trianglelefteq G/N\}$ is the image of $\mathcal{A}$ under the map induced by the natural homomorphism $G \to G/N$. In particular, there exists some open normal subgroup M of G such that $|\mathcal{L}_{G/M}|$ is maximal. If N is an open normal subgroup of G contained in M, then $|\mathcal{L}_{G/N}| = |\mathcal{L}_{G/M}|$ and so, in particular, $\gamma_{c+1}(M/N)$ must coincide with $\gamma_{c+1}(N/N)$; that is, $\gamma_{c+1}(M) \leqslant N$. As this holds for all such open normal subgroups N, we conclude that $\gamma_{c+1}(M) = \mathbf{1}$. This shows that G is virtually class-c nilpotent. □

The following example demonstrates that the assumption of finite generation is necessary in Theorem 2.10(i). We construct a countably-based virtually abelian pro-p group such that the set $\{K' \mid K \trianglelefteq_{\mathrm{o}} G\}$ contains infinitely many subgroups.

Example 2.11. Let p be a prime and, for each $i \geqslant 0$, set V_i to be the direct product of p copies of the cyclic group C_p of order p. Take $H = C_p$ and let H act on each V_i by cyclically permuting the factors. Define $W_i = V_i \rtimes H \cong C_p \operatorname{wr} C_p$, the standard wreath product. Then $[V_i, H]$ and $[V_i, H, H]$ are normal subgroups of W_i of indices p^2 and p^3, respectively. Now take $G = \left(\prod_{i=0}^{\infty} V_i\right) \rtimes H$. This is a virtually abelian pro-p group, indeed $G = \varprojlim G_n$ where $G_n = \left(\prod_{i=0}^{n} V_i\right) \rtimes H$. Certainly G

is not finitely generated. Observe that, for each finite subset S of $\mathbb{N}_0$,

$$U_S = \Big(\prod_{i \in S} [V_i, H] \times \prod_{i \notin S} V_i \Big) \rtimes H$$

is an open normal subgroup of G and

$$U'_S = \prod_{i \in S} [V_i, H, H] \times \prod_{i \notin S} [V_i, H].$$

In particular, the set $\{K' \mid K \trianglelefteq_o G\}$ is infinite for this group G.

3. Characterization of JNN$_c$F profinite groups

We fix the integer $c \geqslant 0$ throughout this section. In order to establish Theorem B that characterizes Fitting-free JNN$_c$F profinite groups, we shall associate a directed graph Γ to the set $\mathcal{C}_H$ that appears in the statement of Theorem 3.3 below. This graph is similar to that used by Reid [2010a]. A key difference is that the vertices of Γ correspond only to closed subgroups that have the form $\gamma_{c+1}(K)$ (where K is a closed normal subgroup of the profinite group under consideration) rather than any other nontrivial closed subgroups that the group may have. We begin by describing this graph and establishing that it is locally finite.

In the following, recall that the Melnikov subgroup $M(N)$ of N is the intersection of the maximal open normal subgroups of N.

Lemma 3.1. *Let G be a Fitting-free JNN$_c$F profinite group and let N be a nontrivial closed normal subgroup of G. Then $\gamma_i(M(N)) \neq \mathbf{1}$ for all $i \geqslant 1$.*

Proof. We shall show that the normal subgroup $M(N)$ is nontrivial, for the hypothesis that G is Fitting-free then ensures it cannot be nilpotent. Suppose for a contradiction that $M(N) = \mathbf{1}$. Let $\mathcal{L}$ be the set of open normal subgroups M of N such that N/M is cyclic of prime order and $\mathcal{M}$ be the set of open normal subgroups M of N such that N/M is a nonabelian finite simple group. Then $\big(\bigcap \mathcal{L}\big) \cap \big(\bigcap \mathcal{M}\big) = M(N) = \mathbf{1}$. By Lemma 2.5, either $\bigcap \mathcal{L} = \mathbf{1}$ or $\bigcap \mathcal{M} = \mathbf{1}$. If $\bigcap \mathcal{L} = \mathbf{1}$, then N embeds in a Cartesian product of cyclic groups of prime order and so N would be abelian, contrary to hypothesis.

Hence $\bigcap \mathcal{M} = \mathbf{1}$. Then [Ribes and Zalesskii 2000, Corollary 8.2.3] tells us that N is a Cartesian product of nonabelian finite simple groups S_R indexed by the set $\mathcal{M}$, say $N = \prod_{R \in \mathcal{M}} S_R$. Now there exists some open normal subgroup K of G such that $N \cap K < N$. Define

$$M_1 = \prod_{S_R \leqslant K} S_R \quad \text{and} \quad M_2 = \prod_{S_R \nleqslant K} S_R,$$

the products of those factors S_R contained in K and not contained in K, respectively. Any closed normal subgroup of N is the product of the factors S_R that it contains, so $M_1 = N \cap K$. Hence M_2 is nontrivial and finite. Furthermore, since K is normal in G, for $g \in G$, $S_R \nleqslant K$ if and only if $S_R^g \nleqslant K$. Therefore M_2 is a normal subgroup of G and we have a contradiction by Corollary 2.4. $\square$

Let G be a finitely generated Fitting-free JNN$_c$F profinite group and H be an open subgroup of G. Construct a directed graph $\Gamma = \Gamma(H)$ whose vertices are the members of the set

$$\mathcal{C}_H = \{\gamma_{c+1}(K) \mid K \trianglelefteqslant_{\mathrm{c}} G \text{ with } \gamma_{c+1}(K) \nleqslant H\}$$

and where there is an edge from a member A of $\mathcal{C}_H$ to another member B when $B < A$ and there is no $C \in \mathcal{C}_H$ with $B < C < A$.

Lemma 3.2. *Let G be a finitely generated Fitting-free JNN$_c$F profinite group, let H be an open subgroup of G and let $\Gamma = \Gamma(H)$ be the graph defined above.*

(i) *If K and L are closed normal subgroups of G such that $\gamma_{c+1}(K), \gamma_{c+1}(L) \in \mathcal{C}_H$ and there is an edge from $\gamma_{c+1}(K)$ to $\gamma_{c+1}(L)$ in Γ, then $\gamma_{c+1}(M(\gamma_{c+1}(K))) \leqslant \gamma_{c+1}(L)$.*

(ii) *If K is a closed normal subgroup of G such that $\gamma_{c+1}(K) \in \mathcal{C}_H$, then there are at most finitely many $\gamma_{c+1}(L) \in \mathcal{C}_H$ such that there is an edge from $\gamma_{c+1}(K)$ to $\gamma_{c+1}(L)$ in Γ.*

Proof. (i) Suppose that there is an edge from $\gamma_{c+1}(K)$ to $\gamma_{c+1}(L)$ in Γ. Then $\gamma_{c+1}(L)$ is a proper subgroup of $\gamma_{c+1}(K)$, so the intersection R of the maximal open normal subgroups of $\gamma_{c+1}(K)$ that contain $\gamma_{c+1}(L)$ satisfies $R < \gamma_{c+1}(K)$. By definition, $M(\gamma_{c+1}(K)) \leqslant R$ and so $M(\gamma_{c+1}(K))\gamma_{c+1}(L) \leqslant R < \gamma_{c+1}(K)$. Take $J = M(\gamma_{c+1}(K))L$. Then J is a closed normal subgroup of G and $\gamma_{c+1}(L) \leqslant \gamma_{c+1}(J) \leqslant M(\gamma_{c+1}(K))\gamma_{c+1}(L) < \gamma_{c+1}(K)$. Since there is an edge in Γ from $\gamma_{c+1}(K)$ to $\gamma_{c+1}(L)$, this forces $\gamma_{c+1}(J) = \gamma_{c+1}(L)$ and hence $\gamma_{c+1}(M(\gamma_{c+1}(K))) \leqslant \gamma_{c+1}(L)$.

(ii) Define $M = \gamma_{c+1}(M(\gamma_{c+1}(K)))$. By Lemma 3.1, $M \neq \mathbf{1}$ and hence $Q = G/M$ is virtually class-c nilpotent. If there is an edge from $\gamma_{c+1}(K)$ to $\gamma_{c+1}(L)$ in Γ then, by part (i), $\gamma_{c+1}(L)$ corresponds to $\gamma_{c+1}(L/M)$ and here L/M is a closed normal subgroup of Q. Consequently, there are only finitely many possibilities for $\gamma_{c+1}(L)$ by Theorem 2.10(i). $\square$

Theorem 3.3. *Let G be a finitely generated infinite profinite group that is Fitting-free and let c be a nonnegative integer. Then the following conditions are equivalent*:

(i) *The group G is JNN$_c$F.*

(ii) *The set $\mathcal{A}_H = \{\gamma_{c+1}(K) \mid K \trianglelefteqslant_{\mathrm{o}} G \text{ with } \gamma_{c+1}(K) \nleqslant H\}$ is finite for every open subgroup H of G.*

(iii) *The set $\mathcal{C}_H = \{\gamma_{c+1}(K) \mid K \trianglelefteqslant_{\mathrm{c}} G \text{ with } \gamma_{c+1}(K) \nleqslant H\}$ is finite for every open subgroup H of G.*

Observe that if H is any open subgroup of G with $C = \mathrm{Core}_G(H)$, then $\mathcal{A}_H = \mathcal{A}_C$ and $\mathcal{C}_H = \mathcal{C}_C$. Hence each of the conditions (ii) and (iii) is equivalent to the requirement that the given set be finite for every open *normal* subgroup H of G.

Proof. Assume that G is JNN$_c$F. Suppose that $\mathcal{C}_H$ is infinite for some open subgroup H of G. As described above, construct the graph $\Gamma = \Gamma(H)$ whose vertices are the members of $\mathcal{C}_H$. Lemma 3.2(ii) tells us that each vertex of Γ has finite out-degree. Furthermore, if $\gamma_{c+1}(K) \in \mathcal{C}_H$, then $G/\gamma_{c+1}(K)$ is a proper quotient of G and so is virtually class-c nilpotent. Hence, by Theorem 2.10(i), $G/\gamma_{c+1}(K)$ contains only finitely many subgroups of the form $\gamma_{c+1}(\bar{L})$ where $\bar{L}$ is a closed normal subgroup; that is, there are only finitely many members of $\mathcal{C}_H$ that contain $\gamma_{c+1}(K)$. Consequently there is a path of finite length in Γ from $\gamma_{c+1}(G)$ to $\gamma_{c+1}(K)$.

Thus Γ is a connected, locally finite, infinite directed graph. By König's lemma (see, for example, [Diestel 2017, Lemma 8.1.2]), Γ has an infinite directed path and this corresponds to an infinite descending chain $\gamma_{c+1}(K_1) > \gamma_{c+1}(K_2) > \cdots$ of members of $\mathcal{C}_H$. An application of [Reid 2010a, Lemma 2.4], taking $O = H$, shows that $J = \bigcap_{i=1}^{\infty} \gamma_{c+1}(K_i) \neq \mathbf{1}$. Then G/J is finitely generated and virtually class-c nilpotent but it has infinitely many subgroups of the form $\gamma_{c+1}(K_i)/J$ with $K_i \trianglelefteq_{\mathrm{c}} G$. This contradicts Theorem 2.10(i). We conclude therefore that $\mathcal{C}_H$ is finite for every open subgroup H of G.

Since $\mathcal{A}_H \subseteq \mathcal{C}_H$ for every H, it is certainly the case that the third condition in the statement implies the second.

Suppose finally that $\mathcal{A}_H$ is finite for every open subgroup H of G. As G is Fitting-free, it is not virtually nilpotent. Let N be a nontrivial closed normal subgroup of G. Then $\gamma_{c+1}(N) \neq \mathbf{1}$ and so there exists an open normal subgroup H of G such that $\gamma_{c+1}(N) \nleqslant H$. By hypothesis, $\mathcal{A}_H = \{\gamma_{c+1}(L_1), \gamma_{c+1}(L_2), \ldots, \gamma_{c+1}(L_r)\}$ for some open normal subgroups $L_1, L_2, \ldots, L_r$ of G. Set $L = \bigcap_{i=1}^{r} L_i$. If K is an open normal subgroup of G with $N \leqslant K$, then necessarily $\gamma_{c+1}(K) \nleqslant H$ and so $\gamma_{c+1}(K) = \gamma_{c+1}(L_i)$ for some i. Therefore

$$\gamma_{c+1}(L) \leqslant \bigcap\{\gamma_{c+1}(K) \mid N \leqslant K \trianglelefteq_{\mathrm{o}} G\} \leqslant \bigcap\{K \mid N \leqslant K \trianglelefteq_{\mathrm{o}} G\} = N.$$

Hence LN/N is a class-c nilpotent open subgroup of G/N, as required. □

We shall now use Theorem 3.3 to establish further information about finitely generated Fitting-free JNN$_c$F groups, including a description of them as inverse limits of suitable virtually nilpotent groups (see Theorem 3.7 below).

Suppose that G is a finitely generated Fitting-free JNN$_c$F group. We start with any open normal subgroup H_0. Then certainly $\gamma_{c+1}(H_0) \neq \mathbf{1}$ since G is Fitting-free. Now assume, as an inductive hypothesis, that we have constructed a sequence of open normal subgroups $G \geqslant H_0 > H_1 > \cdots > H_{n-1}$ such that for each $i \in \{1, 2, \ldots, n-1\}$ the following holds: $\gamma_{c+1}(H_i) \leqslant M_G(\gamma_{c+1}(H_{i-1}))$ and if N is an open normal subgroup of G either $\gamma_{c+1}(N) \leqslant H_{i-1}$ or $\gamma_{c+1}(H_i) \leqslant \gamma_{c+1}(N)$. By Theorem 3.3, the set $\mathcal{A}_{H_{n-1}} = \{\gamma_{c+1}(K) \mid K \trianglelefteq_o G \text{ with } \gamma_{c+1}(K) \nleqslant H_{n-1}\}$ is finite. Also $M_G(\gamma_{c+1}(H_{n-1})) \neq \mathbf{1}$ since it contains $M(\gamma_{c+1}(H_{n-1}))$ which is nontrivial by Lemma 3.1. Let

$$R = M_G(\gamma_{c+1}(H_{n-1})) \cap \bigcap \mathcal{A}_{H_{n-1}}.$$

Since this is a finite intersection of nontrivial closed normal subgroups, R is also a nontrivial closed normal subgroup of G by Lemma 2.5. Then G/R is virtually class-c nilpotent, so there exists an open normal subgroup S of G with $\gamma_{c+1}(S) \leqslant R$. Take $H_n = H_{n-1} \cap S$, so that H_n is open in G, $H_n \leqslant H_{n-1}$ and

$$\gamma_{c+1}(H_n) \leqslant R \leqslant M_G(\gamma_{c+1}(H_{n-1})) < \gamma_{c+1}(H_{n-1}).$$

Furthermore, if N is an open normal subgroup of G, then either $\gamma_{c+1}(N) \leqslant H_{n-1}$ or $\gamma_{c+1}(N) \in \mathcal{A}_{H_{n-1}}$. In the latter case, $\gamma_{c+1}(H_n) \leqslant R \leqslant \gamma_{c+1}(N)$ according to our definition of R.

By repeated application of these steps, we obtain a descending sequence of open normal subgroups H_n. Let $J = \bigcap_{n=0}^{\infty} H_n$. If $J \neq \mathbf{1}$, then necessarily $\gamma_{c+1}(J) \neq \mathbf{1}$ so $G/\gamma_{c+1}(J)$ is virtually class-c nilpotent. By Theorem 2.10(i), the set $\{\gamma_{c+1}(K) \mid K \trianglelefteq_o G/\gamma_{c+1}(J)\}$ is finite but each term $\gamma_{c+1}(H_i)/\gamma_{c+1}(J)$ is a member of this set. This contradiction shows that $J = \mathbf{1}$.

In conclusion, we have established the following observation:

Lemma 3.4. *Let G be a finitely generated profinite group that is Fitting-free and JNN$_c$F. Then there is a descending sequence $G \geqslant H_0 > H_1 > H_2 > \cdots$ of open normal subgroups such that*:

(i) *For each $n \geqslant 1$, $\gamma_{c+1}(H_n) \leqslant M_G(\gamma_{c+1}(H_{n-1})) < \gamma_{c+1}(H_{n-1})$.*

(ii) $\bigcap_{n=0}^{\infty} H_n = \mathbf{1}$.

(iii) *If N is an open normal subgroup of G and $n \geqslant 1$, then either $\gamma_{c+1}(N) \leqslant H_{n-1}$ or $\gamma_{c+1}(H_n) \leqslant \gamma_{c+1}(N)$.*

The conditions appearing in the lemma are sufficient to ensure that the group G is JNN$_c$F. In fact, we can make them marginally weaker as the following shows:

Theorem 3.5. *Let G be a finitely generated Fitting-free profinite group and let c be a nonnegative integer. Then G is JNN$_c$F if and only if there is a descending sequence $G \geqslant H_0 > H_1 > H_2 > \cdots$ of open normal subgroups such that*:

(i) $\bigcap_{n=0}^{\infty} H_n = \mathbf{1}$.

(ii) *If N is an open normal subgroup of G and $n \geqslant 1$, then either $\gamma_{c+1}(N) \leqslant H_{n-1}$ or $\gamma_{c+1}(H_n) \leqslant \gamma_{c+1}(N)$.*

When these conditions are satisfied, the group G is just infinite if and only if $\gamma_{c+1}(H_n)$ has finite index in G for all $n \geqslant 0$.

Proof. If G is JNN$_c$F, the existence of the descending sequence of open subgroups H_n is provided by Lemma 3.4. Suppose conversely that G possesses a descending chain H_n, $n \geqslant 0$, of open normal subgroups satisfying (i) and (ii). Since G is Fitting-free, it cannot be virtually nilpotent. Let K be a nontrivial closed normal subgroup of G. Then, for the same reason, $\gamma_{c+1}(K) \neq \mathbf{1}$. Therefore, since condition (i) holds, there exists some $m \geqslant 0$ such that $\gamma_{c+1}(K) \nleqslant H_m$. Let N be any open normal subgroup of G with $K \leqslant N$. Since $\gamma_{c+1}(N) \nleqslant H_m$, condition (ii) shows that $\gamma_{c+1}(H_{m+1}) \leqslant \gamma_{c+1}(N) \leqslant N$. It follows that

$$\gamma_{c+1}(H_{m+1}) \leqslant \bigcap\{N \mid K \leqslant N \trianglelefteqslant_{\mathrm{o}} G\} = K,$$

and hence G/K is virtually nilpotent of class c, as required.

Finally, observe that if $|G : \gamma_{c+1}(H_n)|$ is infinite for some $n \geqslant 0$, then $G/\gamma_{c+1}(H_n)$ is an infinite quotient, and therefore G is not just infinite. On the other hand, if $|G : \gamma_{c+1}(H_n)| < \infty$ for all $n \geqslant 0$, then in the previous paragraph it follows that if K is a nontrivial closed normal subgroup of G then $\gamma_{c+1}(H_{m+1}) \leqslant K$ for some $m \geqslant 0$, and so K has finite index. Hence G is in fact just infinite under this assumption. □

Reid [2012, Theorem 3.6] presents a condition which guarantees the existence of a just infinite quotient of a profinite group. The condition is expressed in terms of the relation $\succ_{\mathrm{nar}}$ concerning chief factors of the profinite group G under consideration. Notice, however, that with use of [Reid 2012, Proposition 3.5(iii)], the assumption that $K_1/L_1 \succ_{\mathrm{nar}} K_2/L_2 \succ_{\mathrm{nar}} \cdots$ is a descending sequence of open chief factors (as appears in [Reid 2012, Theorem 3.6]) is equivalent to the existence of open normal subgroups $G \geqslant K_1 > L_1 \geqslant K_2 > L_2 \geqslant \cdots$ with $L = \bigcap_{n=1}^{\infty} L_n$ such that, for each n, K_n/L is a narrow subgroup of G/L and $M_{G/L}(K_n/L) = L_n/L$. Theorem 3.6 below can consequently be viewed as an analogous result for the existence of JNN$_c$F quotients of a profinite group.

The application of Zorn's Lemma in our proof is more delicate than for Reid's result. Under the hypotheses and notation of [Reid 2012, Theorem 3.6], the quotient G/L_n would be finite and so would have only finitely many subgroups. However, our corresponding quotient G/L_n is a finitely generated virtually nilpotent group and this does not necessarily even possess the ascending chain condition on closed normal subgroups.

Theorem 3.6. *Let G be a finitely generated profinite group and let c be a nonnegative integer.*

(i) *For each $n \geqslant 1$, let K_n and L_n be closed normal subgroups of G and define $L = \bigcap_{n=1}^{\infty} L_n$. Suppose that*

$$G \geqslant K_1 > L_1 \geqslant \gamma_{c+1}(L_1)L \geqslant K_2 > L_2 \geqslant \gamma_{c+1}(L_2)L \geqslant \cdots$$

and that, for each n, K_n/L is a narrow subgroup of G/L with $M_{G/L}(K_n/L) = L_n/L$ and G/L_n is virtually class-c nilpotent. Then there exists a closed normal subgroup K of G that is maximal subject to the conditions that $K \geqslant L$ and $K_n \nleqslant L_n K$ for all n. Furthermore, such a closed normal subgroup K has the property that G/K is JNN$_c$F.

(ii) *Every Fitting-free JNN$_c$F quotient G/K of G arises in the manner described in (i) with $L = K$.*

Proof. (i) When $c = 0$, this follows from [Reid 2012, Theorem 3.6(i)]. We shall assume that $c \geqslant 1$ in the following argument. Let $\mathcal{N}$ be the set of all closed normal subgroups N of G which contain L and such that $K_n \nleqslant L_n N$ for all $n \geqslant 1$. We shall order $\mathcal{N}$ by inclusion. Observe that $L \in \mathcal{N}$ since $L_n L = L_n < K_n$. Let $\mathcal{C}$ be a chain in $\mathcal{N}$ and define $R = \overline{\bigcup \mathcal{C}}$. Suppose that $R \notin \mathcal{N}$. Then there exists some $m \geqslant 1$ such that $K_m \leqslant L_m R$. If $C \in \mathcal{C}$, then $K_m \nleqslant L_m C$, so $(K_m \cap C)L_m = K_m \cap L_m C < K_m$ and therefore $K_m \cap C \leqslant L_m$ since L_m is maximal among G-invariant open subgroups of K_m. Hence $[C, K_m] \leqslant L_m$ and so $C \leqslant C_G(K_m/L_m)$ for all $C \in \mathcal{C}$. Since this centralizer is an open subgroup of G, it follows that $R \leqslant C_G(K_m/L_m)$. Hence $K_m \leqslant L_m R \leqslant C_G(K_m/L_m)$ and so the chief factor K_m/L_m is abelian; that is, it is an elementary abelian q-group for some prime q.

Since G/L_m is virtually class-c nilpotent, there is an open normal subgroup A with $L_m \leqslant A$ such that $\gamma_{c+1}(A) \leqslant L_m$. If $K_m \nleqslant A$, then $K_m \cap A = L_m$ and so $K_m A/L_m \cong K_m/L_m \times A/L_m$ is also class-c nilpotent. Consequently, if necessary, we may replace A by $K_m A$ and hence assume $K_m \leqslant A$. For each prime p, write $A[p]/L_m$ for the Sylow pro-p subgroup of A/L_m. Then A/L_m is the product $\prod_p A[p]/L_m$ of these pro-p groups. Furthermore, for each $C \in \mathcal{C}$ and prime p, let $C[p]/L_m$ be the Sylow pro-p subgroup of $(C \cap A)L_m/L_m$. Since $\mathcal{C}$ is a chain, so is the set $\mathcal{S}_p = \{C[p]/L_m \mid C \in \mathcal{C}\}$. As a finitely generated nilpotent pro-p group, $A[p]/L_m$ satisfies the ascending chain condition on closed subgroups and so there exists some maximal member $M[p]/L_m$ of $\mathcal{S}_p$.

If it were the case that $K_m/L_m \leqslant M[q]/L_m$, then $K_m \leqslant (C \cap A)L_m \leqslant C L_m$ for some $C \in \mathcal{C}$, contrary to the fact that $C \in \mathcal{N}$. Define M to be the closed subgroup of A defined by $M/L_m = \prod_p M[p]/L_m$. Then $K_m \nleqslant M$ since we have observed that K_m/L_m is not contained in the Sylow pro-q subgroup of M/L_m.

On the other hand, $C \cap A \leqslant M$ for all $C \in \mathcal{C}$, since by construction $C[p]/L_m \leqslant M[p]/L_m$ for each prime p. Furthermore, since A is a clopen subset of G,

$$\overline{\bigcup_{C \in \mathcal{C}} (C \cap A)} = \overline{\left(\bigcup \mathcal{C}\right)} \cap A = R \cap A$$

and so we conclude that $R \cap A \leqslant M$. Therefore, $K_m \leqslant L_m R \cap A = (R \cap A)L_m \leqslant M$, which contradicts our previous observation.

In conclusion, we have shown that $R = \overline{\bigcup \mathcal{C}} \in \mathcal{N}$ and so every chain in $\mathcal{N}$ has an upper bound. Therefore, by Zorn's lemma, there is a maximal member $K \in \mathcal{N}$; that is, K is maximal subject to the condition that $K_n \nleqslant L_n K$ for all $n \geqslant 1$. Suppose that G/K is virtually class-c nilpotent. By Theorem 2.10(i), the set $\{\gamma_{c+1}(J) \mid J \trianglelefteq_{\mathrm{c}} G/K\}$ is finite. Hence $\gamma_{c+1}(L_m)K = \gamma_{c+1}(L_{m+1})K$ for some $m \geqslant 1$ and so

$$K_{m+1} \leqslant \gamma_{c+1}(L_m)L \leqslant \gamma_{c+1}(L_m)K = \gamma_{c+1}(L_{m+1})K \leqslant L_{m+1}K,$$

contrary to the fact that $K \in \mathcal{N}$. We deduce that G/K is not virtually class-c nilpotent.

Now let N be a closed normal subgroup of G that strictly contains K. Then $N \notin \mathcal{N}$ by maximality of K, so there exists some $m \geqslant 1$ such that $K_m \leqslant L_m N$; that is, $K_m/L \leqslant M_{G/L}(K_m/L) \cdot (N/L)$. Lemma 2.2(ii) then tells us that $K_m \leqslant N$. Hence G/N is a quotient of G/L_m and so is virtually class-c nilpotent. This shows that G/K is indeed JNN$_c$F.

(ii) Assume that G/K is a JNN$_c$F quotient of G and that it is Fitting-free. We define the sequences of closed normal subgroups K_n and L_n as follows. First take any chief factor of G/K and let K_1/K be a narrow subgroup as provided by Lemma 2.2(iii) and define L_1 by $L_1/K = M_{G/K}(K_1/K)$. Note that $L_1 > K$ by use of Corollary 2.4 and hence $\gamma_{c+1}(L_1)K > K$ by the hypothesis that G/K is Fitting-free. Assuming that, for some $n \geqslant 2$, we have defined K_{n-1} and L_{n-1} with $\gamma_{c+1}(L_{n-1})K > K$, use Lemma 2.2 again to produce a narrow subgroup K_n/K of G/K with $K_n \leqslant \gamma_{c+1}(L_{n-1})K$. Define L_n by $L_n/K = M_{G/K}(K_n/K)$ and note $\gamma_{c+1}(L_n)K > K$. This produces the required chain of closed normal subgroups

$$G \geqslant K_1 > L_1 \geqslant \gamma_{c+1}(L_1)K \geqslant K_2 > L_2 \geqslant \gamma_{c+1}(L_2)K \geqslant \cdots .$$

Now $L = \bigcap_{n=1}^{\infty} L_n$ certainly contains K, while the quotient G/L cannot be virtually class-c nilpotent by Theorem 2.10(i) as the subgroups $\gamma_{c+1}(L_n/L)$ are distinct. Hence $L = K$. Finally, if N is a closed normal subgroup of G with $N > K$, then G/N is virtually class-c nilpotent and so, by use of Theorem 2.10(i), there exists $m \geqslant 1$ such that $\gamma_{c+1}(L_m)N = \gamma_{c+1}(L_{m+1})N$. The same argument as used in part (i) shows that $K_{m+1} \leqslant L_{m+1}N$. This shows that, amongst closed normal subgroups, K is indeed maximal subject to $K_n \nleqslant L_n K$ for all n; that is, arises as in part (i). □

Our final result of this section is a characterization of finitely generated Fitting-free JNN$_c$F profinite groups as inverse limits. The natural inverse system to associate to such a group is of virtually nilpotent profinite groups rather than of some class of finite groups. The properties possessed by this inverse system are analogous to those in [Reid 2018, Theorem 4.1].

Theorem 3.7. *Let G be a finitely generated profinite group that is Fitting-free and let c be a nonnegative integer. If G is JNN$_c$F, then it is the inverse limit of a family G_n, for $n \geqslant 0$, of profinite groups with respect to surjective continuous homomorphisms $\rho_n : G_{n+1} \to G_n$ with the following properties. For every $n \geqslant 0$, G_n has an open normal subgroup P_n such that, upon setting $Q_n = P_{n+1}\rho_n$:*

(i) *G_n is virtually class-c nilpotent.*

(ii) *$P_n > Q_n$.*

(iii) *$\gamma_{c+1}(P_n) > M_{G_n}(\gamma_{c+1}(P_n)) \geqslant \ker \rho_{n-1} \geqslant \gamma_{c+1}(Q_n) > \mathbf{1}$.*

(iv) *If N is an open normal subgroup of G_n, then either*

$$\gamma_{c+1}(N) \leqslant P_n \quad \textit{or} \quad \gamma_{c+1}(Q_n) \leqslant \gamma_{c+1}(N).$$

Conversely, suppose, for some integer $d \geqslant 1$, that $G = \varprojlim G_n$ is an inverse limit of a countable family of d-generator profinite groups with respect to surjective continuous homomorphisms ρ_n such that G is Fitting-free and the above conditions hold. For each n, let $\pi_n : G \to G_n$ be the natural map associated to the inverse limit. Then if K is a nontrivial closed normal subgroup of G, there exists $n_0 \geqslant 0$ such that $\ker \pi_{n_0} \leqslant K$. In particular, G is JNN$_c$F.

In the case of finitely generated profinite groups, it is known that *any* homomorphism is necessarily continuous. Consequently, the word "continuous" could be omitted from the statement without affecting its validity. For arbitrary finitely generated profinite groups, this follows by the work of Nikolov and Segal [2007] (and depends upon the classification of finite simple groups). However, as the groups G_n are assumed to be virtually nilpotent, it is easy to reduce to the case of finitely generated (nilpotent) pro-p groups which was already covered by Serre; compare with [Anderson 1976].

Proof. Suppose G is JNN$_c$F. Then, as observed in Lemma 3.4, there is a descending sequence $G \geqslant H_0 > H_1 > H_2 > \cdots$ of open normal subgroups such that $\bigcap_{n=0}^{\infty} H_n = \mathbf{1}$ and, for each $n \geqslant 1$, $\gamma_{c+1}(H_n) \leqslant M_G(\gamma_{c+1}(H_{n-1})) < \gamma_{c+1}(H_{n-1})$ and if $N \trianglelefteqslant_o G$ either $\gamma_{c+1}(N) \leqslant H_{n-1}$ or $\gamma_{c+1}(H_n) \leqslant \gamma_{c+1}(N)$. To simplify notation, write $M_n = M_G(\gamma_{c+1}(H_{n+1}))$ for each $n \geqslant 0$. Then define $G_n = G/M_n$, $P_n = H_n/M_n$ and $Q_n = H_{n+1}/M_n$. Let $\rho_n : G_{n+1} \to G_n$ be the natural map. Since $\bigcap_{n=0}^{\infty} M_n \leqslant \bigcap_{n=0}^{\infty} H_n = \mathbf{1}$, it follows that $G = \varprojlim G_n$, while the conditions stated in the theorem all hold. Indeed, using Lemma 2.2(i), $\ker \rho_{n-1} = M_{n-1}/M_n = M_{G_n}(\gamma_{c+1}(P_n))$.

Conversely, suppose that $G = \varprojlim G_n$ is an inverse limit of d-generator profinite groups G_n, for $n \geqslant 0$, with respect to surjective continuous homomorphisms $\rho_n : G_{n+1} \to G_n$ such that G is Fitting-free and conditions (i)–(iv) hold where $P_n \trianglelefteqslant_o G_n$ and $Q_n = P_{n+1}\rho_n$. Then G is also d-generated (by [Ribes and Zalesskii 2000, Lemma 2.5.3]). Let $\pi_n : G \to G_n$ be the natural maps associated to the inverse limit.

Observe first that the open normal subgroups P_1 and Q_1 of G_1 satisfy $\gamma_{c+1}(P_1) > \gamma_{c+1}(Q_1) > \mathbf{1}$. Suppose that G_n, for some $n \geqslant 1$, possesses open normal subgroups $C_0, C_1, \ldots, C_n$ such that the subgroups $\gamma_{c+1}(C_i)$ are distinct and nontrivial. Upon taking the inverse images under the homomorphism ρ_n, we obtain open normal subgroups $C_0\rho_n^{-1}, C_1\rho_n^{-1}, \ldots, C_n\rho_n^{-1}$ with $\gamma_{c+1}(C_i\rho_n^{-1}) \nleqslant \ker\rho_n$. When taken together with Q_{n+1}, these give $n+1$ open normal subgroups K of G_{n+1} such that the corresponding $\gamma_{c+1}(K)$ are distinct and nontrivial. By induction, we conclude that $\{\gamma_{c+1}(K) \mid K \trianglelefteqslant_{\mathrm{o}} G_n\}$ contains at least $n+1$ subgroups for all n. The corresponding set for G must therefore be infinite and hence G is not virtually class-c nilpotent by Theorem 2.10(i).

Now let K be a nontrivial closed normal subgroup of G. Since G is Fitting-free, $\gamma_{c+1}(\gamma_{c+1}(K)) \neq \mathbf{1}$. If $\gamma_{c+1}(K)\pi_{n+2} \leqslant P_{n+2}$ for some $n \geqslant 1$, then we see that $\gamma_{c+1}(\gamma_{c+1}(K))\pi_n = \mathbf{1}$ because $P_{n+2}\rho_{n+1} = Q_{n+1}$ and $\gamma_{c+1}(Q_{n+1}) \leqslant \ker\rho_n$. Hence there exists $n_0 \geqslant 1$ such that $\gamma_{c+1}(K)\pi_n \nleqslant P_n$ for all $n \geqslant n_0$. Let N be any open normal subgroup of G with $K \leqslant N$. If $n \geqslant n_0$, then $N\pi_n$ is an open normal subgroup of G_n with $\gamma_{c+1}(N\pi_n)) \nleqslant P_n$ and so $\gamma_{c+1}(Q_n) \leqslant \gamma_{c+1}(N\pi_n)$ by condition (iv). Hence

$$\gamma_{c+1}(P_{n+1}) \leqslant \gamma_{c+1}(N\pi_{n+1}) \ker\rho_n \leqslant \gamma_{c+1}(N\pi_{n+1})\, M_{G_{n+1}}(\gamma_{c+1}(P_{n+1})),$$

and so we deduce $\gamma_{c+1}(P_{n+1}) \leqslant \gamma_{c+1}(N\pi_{n+1})$ by Lemma 2.2(ii). Consequently $\ker\rho_n \leqslant \gamma_{c+1}(N\pi_{n+1})$ for all $n \geqslant n_0$; that is, $\ker\pi_n \leqslant \gamma_{c+1}(N)\ker\pi_{n+1}$ for all $n \geqslant n_0$. This implies

$$\ker\pi_{n_0} \leqslant \bigcap_{n \geqslant n_0} \gamma_{c+1}(N)\ker\pi_n = \gamma_{c+1}(N) \leqslant N$$

since $\gamma_{c+1}(N)$ is closed. Now K is the intersection of all such open normal subgroups N and therefore $\ker\pi_{n_0} \leqslant K$. Consequently, G/K is a quotient of G_{n_0} and so is virtually class-c nilpotent. This demonstrates that G is JNN$_c$F, as required. □

4. Characterization of hereditarily JNN$_c$F profinite groups

In this section, we fix again the integer $c \geqslant 0$ and we shall provide various descriptions of Fitting-free profinite groups that are hereditarily JNN$_c$F. The results that we present parallel those of the previous section and indicate what additional properties ensure that not only is the group itself JNN$_c$F, but also every open subgroup is JNN$_c$F.

Let G be a profinite group. Analogous to the sets appearing in Theorem 3.3, we define, for an open subgroup H of G,

$$\mathcal{A}_H^* = \{\gamma_{c+1}(K) \mid K \leqslant_{\mathrm{o}} G \text{ with } H \leqslant N_G(K) \text{ and } \gamma_{c+1}(K) \nleqslant H\},$$
$$\mathcal{C}_H^* = \{\gamma_{c+1}(K) \mid K \leqslant_{\mathrm{c}} G \text{ with } H \leqslant N_G(K) \text{ and } \gamma_{c+1}(K) \nleqslant H\}.$$

If H and L are open subgroups of G with $H \leqslant L$, we also set

$$\mathcal{A}_H(L) = \{\gamma_{c+1}(K) \mid K \trianglelefteqslant_{\mathrm{o}} L \text{ with } \gamma_{c+1}(K) \nleqslant H\},$$
$$\mathcal{C}_H(L) = \{\gamma_{c+1}(K) \mid K \trianglelefteqslant_{\mathrm{c}} L \text{ with } \gamma_{c+1}(K) \nleqslant H\}.$$

The following observation is straightforward:

Lemma 4.1. *Let G be a profinite group and H be an open subgroup of G. Then*

(i) $\mathcal{A}_H^* = \bigcup\{\mathcal{A}_H(L) \mid L \leqslant_{\mathrm{o}} G \textit{ with } H \leqslant L\}$;

(ii) $\mathcal{C}_H^* = \bigcup\{\mathcal{C}_H(L) \mid L \leqslant_{\mathrm{o}} G \textit{ with } H \leqslant L\}$.

In order to establish Theorem 4.4, which is the analogue of Theorem 3.3 for hereditarily JNN$_c$F groups, we need to know that the condition that the group is Fitting-free is inherited by open subgroups. We establish this in Lemma 4.3 below. We shall use the following analogue of an observation made in the proof of [Wilson 2010, (2.1)]. The argument is similar but included for completeness.

Lemma 4.2. *Let G be a Fitting-free JNN$_c$F profinite group. Then*:

(i) *Every nonidentity element of G has infinitely many conjugates in G.*

(ii) *If H is a nontrivial finite subgroup of G, then H has infinitely many conjugates in G.*

Proof. (i) Suppose that x is a nonidentity element of G with finitely many conjugates in G. Let X be the closed normal subgroup of G generated by the conjugates of x and C be the intersection of the centralizers in G of each conjugate of x. Since x has finitely many conjugates, C is open in G and, in particular, nontrivial. Since $[C, X] = \mathbf{1}$, it follows that $C \cap X$ is an abelian closed normal subgroup of G and so $C \cap X = \mathbf{1}$ by assumption. This contradicts Lemma 2.5.

(ii) Let H be a nontrivial finite subgroup of G with finitely many conjugates in G. If x is a nonidentity element of H, then every conjugate of x belongs to one of the conjugates of H. It follows that x has finitely many conjugates in G, which contradicts (i). □

Lemma 4.3. *Let G be a Fitting-free JNN$_c$F profinite group. If H is any open subgroup of G, then H is also Fitting-free.*

Proof. Suppose that A is an abelian closed normal subgroup of H. Let $B = A \cap \mathrm{Core}_G(H)$. Note that B has finitely many conjugates in G and each of them is a normal subgroup of $\mathrm{Core}_G(H)$. Hence the normal closure B^G is the product of these subgroups and this is nilpotent by Fitting's Theorem. Since G is Fitting-free, it follows that $B = \mathbf{1}$. Therefore A is finite and so, by Lemma 4.2(ii), $A = \mathbf{1}$. □

Theorem 4.4. *Let G be a finitely generated infinite profinite group that is Fitting-free and let c be a nonnegative integer. Then the following conditions are equivalent*:

(i) *The group G is hereditarily JNN$_c$F.*

(ii) *The set $\mathcal{A}_H^*$ is finite for every open subgroup H of G.*

(iii) *The set $\mathcal{C}_H^*$ is finite for every open subgroup H of G.*

Proof. Suppose first that $\mathcal{A}_H^*$ is finite for every open subgroup H of G. Since $\mathcal{A}_H \subseteq \mathcal{A}_H^*$, it follows that G is JNN$_c$F by Theorem 3.3. Let L be an open subgroup of G. Then L is Fitting-free by Lemma 4.3 and $\mathcal{A}_H(L)$ is finite for every open subgroup H of L as it is contained in $\mathcal{A}_H^*$. Hence L is also JNN$_c$F by Theorem 3.3. This establishes (ii) $\Rightarrow$ (i).

Since $\mathcal{A}_H^* \subseteq \mathcal{C}_H^*$, certainly (iii) $\Rightarrow$ (ii). Finally assume that G is hereditarily JNN$_c$F and let H be an open subgroup of G. There are finitely many open subgroups L of G with $H \leqslant L$. If L is such an open subgroup, then L is JNN$_c$F, so $\mathcal{C}_H(L)$ is finite by Theorem 3.3 together with Lemma 4.3. Hence $\mathcal{C}_H^*$ is a finite union of finite sets, by Lemma 4.1, and so is finite. This establishes the final implication (i) $\Rightarrow$ (iii). □

Wilson [2010, (2.1)] characterizes when a just infinite group is not hereditarily just infinite. The following is our analogue for JNN$_c$F groups. The same method is used to construct the basal subgroup K and a few additional steps establish its properties.

Proposition 4.5. *Let G be a Fitting-free JNN$_c$F profinite group that is not hereditarily JNN$_c$F. Then G has an infinite closed basal subgroup K such that $N_G(K)/K$ is not virtually class-c nilpotent and K has no nontrivial abelian closed subgroup that is topologically characteristic in K. In particular, K is not normal in G.*

Proof. Since G is not hereditarily JNN$_c$F, there is an open subgroup H of G and a nontrivial closed normal subgroup L of H such that H/L is not virtually class-c nilpotent. Let C be the core of H in G. If $C \cap L = \mathbf{1}$, then L is finite, which is a contradiction by Lemma 4.2(ii). Hence $C \cap L \neq \mathbf{1}$. Note that CL/L is a subgroup of finite index in H/L and is isomorphic to $C/(C \cap L)$. If $C/(C \cap L)$ were virtually class-c nilpotent, then as H/L is a finite extension we would obtain another contradiction. Hence $C/(C \cap L)$ is not virtually class-c nilpotent and we may replace H and L by C and $C \cap L$, respectively, and assume that H is an open *normal* subgroup of G with a nontrivial closed normal subgroup L such that H/L is not virtually class-c nilpotent.

Now L has finitely many conjugates in G and these are all contained in H. Hence $L \trianglelefteq L^G$ and Lemma 2.8 tells us that we can construct a basal subgroup K of G by intersecting a suitable collection of the conjugates of L. We may assume that L is one of these conjugates so that $K \leqslant L$. Note that K is infinite by use of Lemma 4.2(ii). If $N_G(K)/K$ were virtually class-c nilpotent, then so would be H/L since $K \leqslant L \leqslant H \leqslant N_G(K)$, contrary to our hypothesis. Note then that K cannot be normal in G since if it were then $N_G(K)/K = G/K$ would be virtually class-c nilpotent. Finally if A were a nontrivial abelian closed subgroup that is topologically characteristic in K, then as conjugation is a homeomorphism there would be precisely one G-conjugate of A in each conjugate of K. Hence A would also be basal and its normal closure A^G would be a nontrivial abelian normal subgroup of G, contrary to assumption. This establishes the claimed conditions. □

Using the characterization given in Theorem 4.4, we are able to give a description of hereditarily JNN$_c$F groups of a similar form to our earlier Theorem 3.5.

Theorem 4.6. *Let G be a finitely generated profinite group that is Fitting-free and let c be a nonnegative integer. Then G is hereditarily JNN$_c$F if and only if there is a descending sequence $G \geqslant H_0 > H_1 > H_2 > \cdots$ of open normal subgroups such that*:

(i) $\bigcap_{n=0}^{\infty} H_n = \mathbf{1}$.

(ii) *If L is an open subgroup of G that is normalized by H_{n-1} for some $n \geqslant 1$, then either $\gamma_{c+1}(L) \leqslant H_{n-1}$ or $\gamma_{c+1}(H_n) \leqslant \gamma_{c+1}(L)$.*

Proof. Suppose first that G is hereditarily JNN$_c$F. We start with any open normal subgroup H_0 of G. Suppose then, as an inductive hypothesis, that we have constructed open normal subgroups $G \geqslant H_0 > H_1 > \cdots > H_{n-1}$ such that, for each $i \in \{1, \ldots, n-1\}$, $\gamma_{c+1}(H_{i-1}) > \gamma_{c+1}(H_i)$ and if L is normalized by H_{i-1} then either $\gamma_{c+1}(L) \leqslant H_{i-1}$ or $\gamma_{c+1}(H_i) \leqslant \gamma_{c+1}(L)$. By Theorem 4.4, the set

$$\mathcal{A}^*_{H_{n-1}} = \{\gamma_{c+1}(K) \mid K \leqslant_{\mathrm{o}} G \text{ with } H_{n-1} \leqslant N_G(K) \text{ and } \gamma_{c+1}(K) \nleqslant H_{n-1}\}$$

is finite. Use of Lemma 3.1 shows that $M_G(\gamma_{c+1}(H_{n-1})) \neq \mathbf{1}$. Hence

$$R = M_G(\gamma_{c+1}(H_{n-1})) \cap \bigcap \mathcal{A}^*_{H_{n-1}}$$

is a nontrivial closed normal subgroup of G (by Lemma 2.5). The quotient G/R is then virtually class-c nilpotent, so there exists an open normal subgroup S with $\gamma_{c+1}(S) \leqslant R$. Take $H_n = H_{n-1} \cap S$, so that H_n is an open normal subgroup of G contained in H_{n-1} with $\gamma_{c+1}(H_n) \leqslant R < \gamma_{c+1}(H_{n-1})$. If L is an open subgroup normalized by H_{n-1}, then either $\gamma_{c+1}(L) \leqslant H_{n-1}$ or $\gamma_{c+1}(L) \in \mathcal{A}^*_{H_{n-1}}$. In the latter case, $\gamma_{c+1}(H_n) \leqslant R \leqslant \gamma_{c+1}(L)$.

Repeating this process constructs a descending sequence of open normal subgroups H_n such that condition (ii) holds. If the intersection $J = \bigcap_{n=0}^{\infty} H_n$ were nontrivial, then $G/\gamma_{c+1}(J)$ would be virtually class-c nilpotent, but would have infinitely many distinct subgroups $\gamma_{c+1}(H_n/\gamma_{c+1}(J))$ contrary to Theorem 2.10. Hence condition (i) also holds.

Conversely suppose that G is a finitely generated profinite group that has no nontrivial abelian closed normal subgroup with a descending sequence of open normal subgroups H_n satisfying conditions (i) and (ii). In particular, G satisfies the conditions appearing in Theorem 3.5 and so is $\mathrm{JNN}_c\mathrm{F}$. Suppose that it is not hereditarily $\mathrm{JNN}_c\mathrm{F}$. By Proposition 4.5, G has a closed basal subgroup K with no nontrivial abelian topologically characteristic subgroup such that $N_G(K)/K$ is not virtually class-c nilpotent. Then $\gamma_{c+1}(K) \neq \mathbf{1}$, so there exists $m \geqslant 0$ such that $\gamma_{c+1}(K) \nleqslant H_m$. Since $\bigcap_{n=0}^{\infty} H_n = \mathbf{1}$, it follows that every open subgroup of G contains some H_n. Hence, by increasing m if necessary, we can assume $H_m \leqslant N_G(K)$. Let U be any open normal subgroup of G and $L = KU$. Then H_m normalizes L and $\gamma_{c+1}(L) \nleqslant H_m$. Hence, by condition (ii), $\gamma_{c+1}(H_{m+1}) \leqslant \gamma_{c+1}(L)$. It follows that

$$\gamma_{c+1}(H_{m+1}) \leqslant \bigcap_{U \trianglelefteqslant_{\mathrm{o}} G} KU = K.$$

Therefore $N_G(K)/K$ is isomorphic to a quotient of a subgroup of $G/\gamma_{c+1}(H_{m+1})$ and hence is virtually class-c nilpotent. This is a contradiction and we conclude that G is indeed hereditarily $\mathrm{JNN}_c\mathrm{F}$, as claimed. □

We complete the section by giving a suitable description of a hereditarily $\mathrm{JNN}_c\mathrm{F}$ profinite group as an inverse limit of virtually nilpotent groups in a manner analogous to the description appearing in Theorem 3.7.

Theorem 4.7. *Let G be a finitely generated profinite group that is Fitting-free and let c be a nonnegative integer. If G is hereditarily $\mathrm{JNN}_c\mathrm{F}$, then it is the inverse limit of a family G_n, for $n \geqslant 0$, of profinite groups with respect to surjective continuous homomorphisms $\rho_n : G_{n+1} \to G_n$ with the following properties. For every $n \geqslant 0$, G_n has an open normal subgroup P_n such that, upon setting $Q_n = P_{n+1}\rho_n$:*

(i) *G_n is virtually class-c nilpotent.*

(ii) *$P_n > Q_n$.*

(iii) *$\gamma_{c+1}(P_n) > M_{G_n}(\gamma_{c+1}(P_n)) \geqslant \ker \rho_{n-1} \geqslant \gamma_{c+1}(Q_n) > \mathbf{1}$.*

(iv) *If N is an open normal subgroup of G_n, then either*

$$\gamma_{c+1}(N) \leqslant P_n \quad \text{or} \quad \gamma_{c+1}(Q_n) \leqslant \gamma_{c+1}(N).$$

(v) *There is no nonnormal closed subgroup V of G_n with at most n conjugates such that any pair of distinct conjugates of V centralize each other and such that the normal closure $W = V^G$ satisfies $\gamma_{c+1}(P_n) \leqslant \gamma_{c+1}(\gamma_{c+1}(W))$.*

Conversely, if, for some integer $d \geqslant 1$, $G = \varprojlim G_n$ is an inverse limit of a countable family of d-generator profinite groups with respect to surjective continuous homomorphisms ρ_n such that G is Fitting-free and the above conditions hold, then G is hereditarily $\mathrm{JNN}_c\mathrm{F}$.

Proof. Suppose that G is hereditarily $\mathrm{JNN}_c\mathrm{F}$. Since G is finitely generated it has finitely many open subgroups of each index and so we can enumerate a sequence of open normal subgroups U_n of G such that, for each $n \geqslant 1$, every open subgroup of index at most n contains U_n. Take H_0 to be any open normal subgroup of G. Certainly $\gamma_{c+1}(H_0) \neq \mathbf{1}$. Now assume, as an inductive hypothesis, that we have constructed a sequence of open normal subgroups $G \geqslant H_0 > H_1 > \cdots > H_{n-1}$. By Theorem 4.4, the set $\mathcal{A}^*_{H_{n-1}}$ is finite while $M_G(\gamma_{c+1}(H_{n-1}))$ is nontrivial by Lemma 3.1. Hence, by Lemma 2.5,

$$R = M_G(\gamma_{c+1}(H_{n-1})) \cap \big(\bigcap \mathcal{A}^*_{H_{n-1}}\big)'$$

is a nontrivial closed subgroup of G, so G/R is virtually class-c nilpotent and there exists an open normal subgroup S of G with $\gamma_{c+1}(S) \leqslant R$. Take $H_n = H_{n-1} \cap U_n \cap S$. In particular, $\gamma_{c+1}(H_n) \leqslant R \leqslant M_G(\gamma_{c+1}(H_{n-1})) < \gamma_{c+1}(H_{n-1})$. By repeated application, we conclude there is a descending sequence of open normal subgroups $G \geqslant H_0 > H_1 > H_2 > \cdots$ such that $H_n \leqslant U_n$ and

$$\gamma_{c+1}(H_n) \leqslant M_G(\gamma_{c+1}(H_{n-1})) \cap \big(\bigcap \mathcal{A}^*_{H_{n-1}}\big)' < \gamma_{c+1}(H_{n-1})$$

for all $n \geqslant 1$. Since $H_n \leqslant U_n$ for each n, it immediately follows that $\bigcap_{n=0}^{\infty} H_n = \mathbf{1}$.

Now, for $n \geqslant 0$, write $M_n = M_G(\gamma_{c+1}(H_{2n+2}))$ and define $G_n = G/M_n$, $P_n = H_{2n}/M_n$ and $Q_n = H_{2n+2}/M_n$. Let $\rho_n : G_{n+1} \to G_n$ be the natural map. Since $\bigcap_{n=0}^{\infty} M_n = \mathbf{1}$, it is the case that $G = \varprojlim G_n$. Since each $M_n \neq \mathbf{1}$, the assumption that G is JNN$_c$F ensures each G_n is virtually class-c nilpotent and conditions (ii) and (iii) follow immediately from the construction. Indeed $\ker \rho_{n-1} = M_{n-1}/M_n = M_{G_n}(\gamma_{c+1}(P_n))$ using Lemma 2.2(i). If $N \trianglelefteqslant_{\mathrm{o}} G_n$, say $N = K/M_n$, such that $\gamma_{c+1}(N) \nleqslant P_n$, then $\gamma_{c+1}(K) \in \mathcal{A}_{H_{2n}} \subseteq \mathcal{A}^*_{H_{2n}}$. Hence $\gamma_{c+1}(H_{2n+2}) < \gamma_{c+1}(H_{2n+1}) \leqslant \bigcap \mathcal{A}^*_{H_{2n}} \leqslant \gamma_{c+1}(K)$ and this establishes condition (iv).

Suppose there is a nonnormal closed subgroup V of G_n with at most n conjugates such that $[V^g, V^h] = \mathbf{1}$ when $gh^{-1} \notin N_{G_n}(V)$ and such that the normal closure $W = V^{G_n}$ satisfies $\gamma_{c+1}(P_n) \leqslant \gamma_{c+1}(\gamma_{c+1}(W))$. Since elements from distinct conjugates of V commute, $\gamma_{c+1}(\gamma_{c+1}(W))$ is the product of the conjugates of $\gamma_{c+1}(\gamma_{c+1}(V))$. Write $V = K/M_n$ and $W = L/M_n$. Then observe $L = K^G$, $\gamma_{c+1}(\gamma_{c+1}(L)) \leqslant \gamma_{c+1}(\gamma_{c+1}(K))^G M_n$ and $\gamma_{c+1}(H_{2n}) \leqslant \gamma_{c+1}(\gamma_{c+1}(L))M_n$, which implies $\gamma_{c+1}(H_{2n}) \leqslant \gamma_{c+1}(\gamma_{c+1}(L))$ with use of Lemma 2.2(ii). Also K has at most n conjugates in G, so it must be the case that $H_{2n+1} \leqslant U_n \leqslant N_G(K)$. Now $\gamma_{c+1}(H_{2n+1}) < \gamma_{c+1}(\gamma_{c+1}(L))$, so $\gamma_{c+1}(\gamma_{c+1}(K)) \nleqslant \gamma_{c+1}(H_{2n+1})$ and therefore $\gamma_{c+1}(K) \nleqslant H_{2n+1}$. In conclusion, for each $i \geqslant 0$, KH_i is an open subgroup of G with the property that $\gamma_{c+1}(KH_i) \in \mathcal{A}^*_{H_{2n+1}}$. Thus

$$\bigcap \mathcal{A}^*_{H_{2n+1}} \leqslant \bigcap_{i \geqslant 0} \gamma_{c+1}(K)H_i = \gamma_{c+1}(K).$$

Since $\bigcap \mathcal{A}^*_{H_{2n+1}}$ is a normal subgroup, it is contained in all conjugates of K and therefore

$$\gamma_{c+1}(H_{2n+2}) \leqslant \big(\bigcap \mathcal{A}^*_{H_{2n+1}}\big)' \leqslant [K^g, K^h]$$

for all $g, h \in G$. Consequently, $\mathbf{1} \neq \gamma_{c+1}(Q_n) \leqslant [V^g, V^h]$ for all $g, h \in G_n$. However, as V is not normal in G_n there exists $g, h \in G_n$ such that V^g and V^h are distinct and these satisfy $[V^g, V^h] = \mathbf{1}$. This contradiction establishes condition (v).

Conversely, suppose that $G = \varprojlim G_n$ is an inverse limit of d-generator profinite groups G_n, for $n \geqslant 0$, with respect to surjective continuous homomorphisms $\rho_n : G_{n+1} \to G_n$ such that G has no nontrivial abelian closed normal subgroup and that conditions (i)–(v) hold where $P_n \trianglelefteq_{\mathrm{o}} G_n$ and $Q_n = P_{n+1}\rho_n$. In particular, the conditions of Theorem 3.7 are satisfied and so G is JNN$_c$F. Let $\pi_n : G \to G_n$ be the natural maps associated to the inverse limit. Suppose that G is not hereditarily JNN$_c$F. Then by Proposition 4.5, G has some closed nonnormal basal subgroup K. Take n_0 to be a positive integer such that K has fewer than n_0 conjugates in G and set $L = K^G$, the direct product of the conjugates of K.

Since $\gamma_{c+1}(\gamma_{c+2}(L)) \neq \mathbf{1}$, it is the case that $\ker \pi_n \leqslant \gamma_{c+1}(\gamma_{c+2}(L))$ for all sufficiently large n by Theorem 3.7. Hence, increasing n_0 if necessary, we may assume that $\ker \pi_n < \gamma_{c+1}(\gamma_{c+2}(L)) \leqslant L'$ for all $n \geqslant n_0$. The subgroup K has at least two conjugates in G and any distinct pair commutes as K is basal. If $K\pi_n$ were normal in G_n, then the images of these conjugates would coincide and so $L\pi_n = K\pi_n$ would be abelian. This is impossible since $\ker \pi_n < L'$. Since the number of conjugates cannot increase in the image, we deduce that, when $n \geqslant n_0$, $K\pi_n$ is a closed subgroup of G_n that is not normal and has at most n_0 conjugates in G_n. For such n, if $x \in \gamma_{c+1}(P_{n+2})$, write $x = g\pi_{n+2}$ for some $g \in G$. Using the fact that $\gamma_{c+1}(Q_{n+1}) \leqslant \ker \rho_n$, one observes $g \in \ker \pi_n \leqslant \gamma_{c+1}(\gamma_{c+1}(L))$ and therefore $\gamma_{c+1}(P_{n+2}) \leqslant \gamma_{c+1}(\gamma_{c+1}(L\pi_{n+2}))$ for $n \geqslant n_0$. In particular, for such n, taking $V = K\pi_{n+2}$ and $W = L\pi_{n+2}$ in G_{n+2} contradicts the hypothesis in condition (v). $\square$

When comparing the above description of hereditarily JNN$_c$F groups with the corresponding result of Reid [2018, Theorem 5.2] for hereditarily just infinite groups, one notices the bound on the number of conjugates appearing in our condition (v). There seems to be no analogue in the corresponding description of hereditarily just infinite groups. However, note that the bound of n for the number of conjugates could, with only minor adjustment to the proof, be replaced by some bound $f(n)$ where $f : \mathbb{N} \to \mathbb{N}$ is any strictly increasing function. In [Reid 2018], the hereditarily just infinite group is isomorphic to an inverse limit $G = \varprojlim G_n$ of *finite* groups and there is therefore an implicit bound on the number of conjugates for subgroups of G_n. Consequently, this condition is quite reasonable.

5. Subgroups of finite index in JNN$_c$F groups

In this section we shall establish Theorem C (see Corollary 5.5) and so consider both profinite groups and discrete groups. We shall adopt the common convention that, in the case of profinite groups, all subgroups are assumed to be within the same category and so “subgroup” means “closed subgroup” in this case. This enables our results to be more streamlined in their statement and the proofs correspondingly cleaner. We fix the integer $c \geqslant 0$ throughout and begin with an observation that is, modulo our standard assumption about abelian normal subgroups, an improvement on Corollary 2.4.

Lemma 5.1. *Let G be a profinite group or discrete group that is JNN$_c$F and Fitting-free. Then G has no nontrivial normal subgroup that is virtually nilpotent.*

Proof. Suppose that N is a nontrivial normal subgroup of G with a nilpotent normal subgroup of finite index in N. The Fitting subgroup $F(N)$ of N is then a product of finitely many nilpotent normal subgroups of N and so is a nilpotent normal subgroup of G. Since G is Fitting-free, it follows that N is finite. Then $C_G(N)$ has finite index in G, which contradicts Lemma 2.7. □

Lemma 5.2. *Let G be a profinite group or a discrete group that is Fitting-free. Suppose that every normal subgroup of finite index is JNN$_c$F. Then G is hereditarily JNN$_c$F.*

Proof. Suppose that H is a subgroup of finite index in G and that N is a nontrivial normal subgroup of H. Let $K = \mathrm{Core}_G(H)$, so that K is a normal subgroup of G also of finite index and hence JNN$_c$F by hypothesis. If it were the case that $K \cap N = \mathbf{1}$, then $[K, N] = \mathbf{1}$ since both K and N are normal subgroups of H. Then $N \leqslant C_G(K)$, in contradiction to Lemma 2.7. We deduce therefore that $K \cap N \neq \mathbf{1}$. Then H/N is a finite extension of $KN/N \cong K/(K \cap N)$, which is virtually class-c nilpotent. Hence H is JNN$_c$F, as required. □

Recall that the *finite radical* $\mathrm{Fin}(G)$ of a group G is the union of all finite normal subgroups of G. The following is a JNN$_c$F analogue of [Reid 2010b, Lemma 4].

Lemma 5.3. (i) *Let G be a group with* $\mathrm{Fin}(G) = \mathbf{1}$. *If H is a subgroup of finite index, then* $\mathrm{Fin}(H) = \mathbf{1}$.

(ii) *Let G be a profinite or discrete group with* $\mathrm{Fin}(G) = \mathbf{1}$ *and H be a subgroup of finite index that is JNN$_c$F. Then every subgroup of G containing H is JNN$_c$F.*

Proof. (i) This is established in [Reid 2010b, Lemma 4].

(ii) Suppose that $H \leqslant L \leqslant G$. First note that L is not virtually class-c nilpotent as it contains H. Let K be a nontrivial normal subgroup of L. Since $\mathrm{Fin}(L) = \mathbf{1}$ by part (i), K is infinite. As $H \cap K$ has finite index in K, it follows that $H \cap K$ is nontrivial and so $H/(H \cap K)$ is virtually class-c nilpotent. We conclude that L/K is a finite extension of $HK/K \cong H/(H \cap K)$, so L/K is virtually class-c nilpotent. Hence L is JNN$_c$F. □

We are now in a position to establish a theorem for JNN$_c$F groups that is an analogue of the main theorem of [Reid 2010b]:

Theorem 5.4. *Let G be a profinite group or a discrete group and let c be a non-negative integer. Suppose that G is JNN$_c$F and Fitting-free, and that H is a normal subgroup of finite index in G. Then the following are equivalent*:

(i) *The subgroup H is JNN$_c$F.*

(ii) *Every subgroup of G containing H is JNN$_c$F.*

(iii) *Every maximal subgroup of G containing H is JNN$_c$F.*

Proof. By Lemma 5.1, $\mathrm{Fin}(G) = \mathbf{1}$. Hence an application of Lemma 5.3(ii) shows that condition (i) implies condition (ii). It is trivial that condition (ii) implies condition (iii).

Now assume condition (iii). Let K be a nontrivial normal subgroup of H. Since H is a normal subgroup of G, we observe that $K^g \trianglelefteqslant H \leqslant N_G(K)$ for all $g \in G$ and hence $K \trianglelefteqslant K^G$. By Lemma 2.8, there is a basal subgroup B that is an intersection of some conjugates of K and, conjugating if necessary, we may assume $B \leqslant K$. Note also that $H \leqslant N_G(B)$ since each conjugate of K is normal in H. We shall show that B is normal in G. For then, G/B is virtually class-c nilpotent by hypothesis and hence H/K is also virtually class-c nilpotent since $B \leqslant K$. This will establish that H is indeed JNN$_c$F.

Suppose, for a contradiction, that B is not a normal subgroup of G. Consequently, $N_G(B)$ is a proper subgroup of G and there is some maximal subgroup M of G with $N_G(B) \leqslant M$. Now B^G is the direct product of the conjugates of B and it is not virtually nilpotent by Lemma 5.1. Observe that B has fewer conjugates in M than in the group G, so B^G/B^M is isomorphic to a direct product of some copies of B and so is not virtually nilpotent. On the other hand, M is JNN$_c$F by assumption, so the quotient M/B^M of M by the normal closure of B in M is a virtually nilpotent group. Hence $(M \cap B^G)/B^M$ is virtually nilpotent and this implies B^G/B^M is also virtually nilpotent since $M \cap B^G$ has finite index in B^G. This is a contradiction and completes the proof of the theorem. □

With use of Lemma 5.2, we then immediately conclude:

Corollary 5.5. *Let G be a profinite or discrete group that is JNN$_c$F and Fitting-free. Then G is hereditarily JNN$_c$F if and only if every maximal* (*open*) *subgroup of finite index is JNN$_c$F.*

6. A construction of hereditarily JNN$_c$F groups

The work of the preceding sections suggests that JNN$_c$F groups are quite closely related to just infinite groups. Similarly, Wilson's classification [1971; 2000] of just infinite groups has the same dichotomy as Hardy's [2002] for JNAF groups, namely branch groups and subgroups of wreath products built from hereditarily just infinite or JNAF groups. To fully investigate the class of JNN$_c$F groups, one would like a good supply of examples of hereditarily $\mathrm{JNN}_c\mathrm{F}$ groups. In Theorem 6.2, we provide one method for constructing such a group. At first sight the construction may appear somewhat unspectacular since it merely consists of a semidirect product of a hereditarily just infinite group H by some group A of (outer) automorphisms. However, by applying it to a variety of known hereditarily just infinite groups H and observing that the range of possible groups A that could be used is rather wide, we manufacture interesting examples of $\mathrm{JNN}_c\mathrm{F}$ groups. In both Examples 6.10 and 6.16, we shall observe that, with suitable choices of ingredients for H, among abelian profinite groups the options for A are about as wide as could be hoped for. For example, one can take A to be any closed subgroup of the Cartesian product of countably many copies of the profinite completion $\hat{\mathbb{Z}}$ of the integers. In Example 6.17, we are able to take A to be any finitely generated virtually nilpotent pro-p group and so again this permits a wide range of possible choices.

Lemma 6.1. *Let H be a group and A be a group of automorphisms of H such that $A \cap \operatorname{Inn} H = \mathbf{1}$. Define $G = H \rtimes A$ to be the semidirect product of H by A via its natural action on H. Then $C_G(H) = Z(H)$.*

Proof. Let $x = h\alpha \in C_G(H)$ with $h \in H$ and $\alpha \in A$. If τ_h denotes the inner automorphism of H induced by h on H, then we observe $\tau_h\alpha = 1$ in $\operatorname{Aut} H$, so $\alpha \in \operatorname{Inn} H$. Hence $\alpha = 1$, so $x = h$ and necessarily $h \in Z(H)$. The reverse inclusion is trivial. □

Theorem 6.2. *Let H be a hereditarily just infinite (discrete or profinite) group that is Fitting-free. Let A be a (discrete or profinite, respectively) group of (continuous) automorphisms of H that is virtually class-c nilpotent, for some $c \geqslant 0$, and satisfies $A \cap \operatorname{Inn} H = \mathbf{1}$. Then the semidirect product of H by A is hereditarily $\mathrm{JNN}_c\mathrm{F}$.*

The only discrete hereditarily just infinite groups that are virtually abelian are the infinite cyclic group and the infinite dihedral group. The only profinite hereditarily just infinite groups that are virtually abelian are semidirect products of the p-adic integers by a finite (and consequently cyclic) subgroup of its automorphism group. Consequently, the hypothesis that H is Fitting-free in the above theorem excludes only a small number of possibilities. Moreover, this hypothesis on H is also necessary since the semidirect product $H \rtimes A$ can otherwise be virtually abelian.

Proof. Let H be a hereditarily just infinite discrete group that is Fitting-free and $A \leqslant \operatorname{Aut} H$ be virtually class-c nilpotent with $A \cap \operatorname{Inn} H = \mathbf{1}$. We shall first show that the semidirect product $G = H \rtimes A$ is $\mathrm{JNN}_c\mathrm{F}$. We shall view H and A as subgroups of G in the natural way. Note that as H is Fitting-free, it is not virtually nilpotent and therefore neither is G.

Let N be a nontrivial normal subgroup of G. If $H \cap N = \mathbf{1}$, then $[H, N] = \mathbf{1}$, so $N \leqslant C_G(H) = Z(H)$ by use of Lemma 6.1. This is a contradiction and so $H \cap N \neq \mathbf{1}$. Thus $H \cap N$ is of finite index in H. Then $G/(H \cap N)$ has a copy of the group A

as a subgroup of finite index and is therefore also virtually class-c nilpotent. We deduce that G/N is virtually class-c nilpotent and hence G is $\mathrm{JNN}_c\mathrm{F}$, as claimed.

Let L be a normal subgroup of finite index in G and let N be a nontrivial normal subgroup of L. If $H \cap N = \mathbf{1}$, then $[H \cap L, N] \leqslant H \cap N = \mathbf{1}$, so $N \leqslant C_G(H \cap L)$. By Lemma 2.7, this is impossible since $H \cap L$ is a normal subgroup of G that is nontrivial (since it has finite index in H) and we have already observed G is $\mathrm{JNN}_c\mathrm{F}$.

Therefore $H \cap N \neq \mathbf{1}$. Since H is hereditarily just infinite, $H \cap N$ has finite index in $H \cap L$. Moreover, $H \cap N$ is normalized by L and hence has finitely many conjugates in G, each of which also has finite index in H. We deduce that $R = \mathrm{Core}_G(H \cap N)$ is nontrivial, so G/R is virtually class-c nilpotent. Since $R \leqslant N$, we conclude that L/N is virtually class-c nilpotent.

We have shown that every normal subgroup of finite index in G is $\mathrm{JNN}_c\mathrm{F}$ and therefore G is hereditarily $\mathrm{JNN}_c\mathrm{F}$ by Lemma 5.2.

The situation when H is profinite and A consists of continuous automorphisms of H is established by the same argument. The only difference is that one needs A to have the structure of a profinite group under the topology induced from the group $\mathrm{Aut}_{\mathrm{c}}\, H$ of topological automorphisms of H so that $G = H \rtimes A$ is a profinite group. □

Hereditarily $\mathrm{JNN}_c\mathrm{F}$ groups via iterated wreath products. We shall now construct abelian groups of automorphisms of some just infinite groups that arise as iterated wreath products of nonabelian finite simple groups. We permit two possible options for the action used for the permutational wreath product at each step. The just infinite groups constructed are closely related to those in Wilson's Construction A [2010], though he uses two applications of the permutational wreath product at each stage. If one employs the product action option (P) at each step of our construction, then the inverse limit constructed would be a special case of what Vannacci terms a *generalized Wilson group* (see [Matteo 2016, Definition 3]). Vannacci makes use of [Reid 2012, Theorem 6.2] to determine that the profinite groups concerned are hereditarily just infinite (and his groups also satisfy the hypotheses of the corrected version in [Reid 2018]). Since we also wish to construct discrete examples of hereditarily just infinite groups via a direct limit, we shall present a direct verification as the discrete and profinite cases are closely linked. This verification is somewhat general since it only requires the action employed to be transitive and subprimitive (in the sense of [Reid 2012]). We shall then specialize to regular actions and product actions in Example 6.6 when constructing automorphisms of the resulting hereditarily just infinite groups so as to apply Theorem 6.2.

We first recall the definition of what is meant by a subprimitive action:

Definition 6.3 [Reid 2012, Definition 1.4]. Let Ω be a set and H be a permutation group on Ω. We shall say that H acts *subprimitively* on Ω if every normal subgroup K of H acts faithfully on every K-orbit.

Let X_0, X_1, X_2, ... be a sequence of nonabelian finite simple groups. Define $W_0 = X_0$. Suppose that for some $n \geqslant 1$, we have constructed a group W_{n-1} and choose some faithful, transitive and subprimitive action of W_{n-1} on a finite set Ω_{n-1}. We define $W_n = X_n \operatorname{wr}_{\Omega_{n-1}} W_{n-1}$ to be the wreath product of X_n by W_{n-1} and write $B_n = X_n^{\Omega_{n-1}}$ for its base group. We shall assume at this point that such an action always exists, while in Example 6.6 below we describe possible examples. Write $\rho_n : W_n \to W_{n-1}$ for the natural surjective homomorphism associated to the wreath product and also note that W_{n-1} occurs as a subgroup of W_n so we have a chain of inclusions $W_0 \leqslant W_1 \leqslant W_2 \leqslant \cdots$. We shall write W to denote the direct limit $\varinjlim W_n$ of these wreath products and $\widehat{W}$ to denote the inverse limit $\varprojlim W_n$. It will be convenient to view W as the union of the groups W_n.

The following is the key observation required to show that W is a hereditarily just infinite (discrete) group and $\widehat{W}$ is a hereditarily just infinite profinite group.

Lemma 6.4. *Let X be a nonabelian simple group and H be a permutation group on a finite set Ω that acts transitively and subprimitively. Define $W = X \operatorname{wr}_\Omega H$ to be the wreath product of X by H with respect to this action and B to be the base group of W. Let K be a normal subgroup of W and N be a normal subgroup of K such that $N \nleqslant B$. Then $B \leqslant N$.*

Proof. Write $\pi : W \to H$ for the natural map associated to the wreath product. Since H acts transitively and faithfully on Ω, it easily follows that B is the unique minimal normal subgroup of W. Therefore $B \leqslant K$, so we may write $K = B \rtimes L$ where L is a normal subgroup of H. Write $\Omega = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_k$ as the disjoint union of the orbits of L. Since H is assumed to act subprimitively, L acts faithfully on each Γ_i.

Since $N \nleqslant B$ by hypothesis, $M = N\pi$ is a nontrivial normal subgroup of L, so the orbits of M on Γ_i form a block system for L. Consequently, M must act without fixed points on each Γ_i, as otherwise M would fix all points of Γ_i and then lie in the kernel of the action of L on Γ_i. Therefore M acts without fixed points on Ω. Let us write

$$B = Q_1 \times Q_2 \times \cdots \times Q_m,$$

where each $Q_j = X^{\Delta_j}$ corresponds to an orbit Δ_j of M on Ω. Let us suppose, for a contradiction, that $B \nleqslant N$. Then $Q_j \nleqslant N$ for some j. Since M permutes the factors of Q_j transitively, Q_j is a minimal normal subgroup of $BM = BN$. However, $B \leqslant K$ so B normalizes N and hence $Q_j \cap N$ is normal in BN. We deduce that $Q_j \cap N = \mathbf{1}$ and hence $[Q_j, N] = \mathbf{1}$. This implies that BN fixes all the direct factors of Q_j, which is a contradiction. This establishes that $B \leqslant N$, as claimed. □

Corollary 6.5. (i) *The group $W = \varinjlim W_n$ is hereditarily just infinite.*

(ii) *The profinite group $\widehat{W} = \varprojlim W_n$ is hereditarily just infinite.*

Proof. (i) Let K be a normal subgroup of finite index in W and N be a nontrivial normal subgroup of K. Then $N \cap W_k \neq \mathbf{1}$ for some k. Consequently $N \cap W_n \nleqslant B_n$ for all $n \geqslant k+1$. Applying Lemma 6.4 with $W = W_n$, we deduce $B_n \leqslant N \cap W_n$ for each $n \geqslant k+1$. Hence $\langle B_{k+1}, B_{k+2}, \dots \rangle$ is contained in N and the former is the kernel of the surjective homomorphism $W \to W_k$. It follows that K/N is finite and this shows that W is hereditarily just infinite.

(ii) We shall write $\pi_n : \widehat{W} \to W_n$ for the surjective homomorphisms associated with the inverse limit. Let K be an open normal subgroup of $\widehat{W}$ and N be a nontrivial closed normal subgroup of K. Then $N\pi_k \neq \mathbf{1}$ for some k. Now $N\pi_n \trianglelefteqslant K\pi_n \trianglelefteqslant W_n$ and $N\pi_n \nleqslant B_n$ for all $n \geqslant k+1$. Hence by Lemma 6.4, $B_n \leqslant N\pi_n$ for all $n \geqslant k+1$; that is, $\ker \rho_{n-1} \leqslant N\pi_n$ for all $n \geqslant k+1$. It follows that $\ker \pi_{n-1} \leqslant N \ker \pi_n$ for all $n \geqslant k+1$. As the kernels form a neighborhood base for the identity in $\widehat{W}$, we conclude that

$$\ker \pi_k \leqslant \bigcap_{n=0}^{\infty} N \ker \pi_n = \overline{N} = N.$$

Since $\widehat{W}/\ker \pi_k \cong W_k$ is finite, it follows that K/N is finite. This establishes that $\widehat{W}$ is hereditarily just infinite. □

We now specify the examples of subprimitive actions that we shall use and construct abelian groups of automorphisms of the iterated wreath products.

Example 6.6. As before, let $X_0, X_1, X_2, \dots$ be a sequence of nonabelian finite simple groups. Define $W_0 = X_0$, $\Omega_0 = X_0$, and let W_0 act regularly on Ω_0. We also define $B_0 = W_0$ for use later. Suppose that, for $n \geqslant 1$, we have constructed W_{n-1} with a specified action on a set Ω_{n-1}. As above, define $W_n = X_n \operatorname{wr}_{\Omega_{n-1}} W_{n-1}$ and write $B_n = X_n^{\Omega_{n-1}}$ for its base group. There are then two options for the action of W_n on some set Ω_n:

(R) take $\Omega_n = W_n$ and let W_n act regularly upon Ω_n; or

(P) let X_n act regularly on itself and use the *product action* of W_n on $\Omega_n = B_n = X_n^{\Omega_{n-1}}$.

For more information upon the product action of a wreath product, see, for example, [Dixon and Mortimer 1996, Section 2.7]. In the case (P) of the product action, the elements of B_n act regularly on the set Ω_n while the elements of W_{n-1} act to permute the factors; that is, the action of W_{n-1} on Ω_n coincides with the conjugation action of W_{n-1} on the base group B_n of W_n. It is immediate from the definition that the regular action of W_n is subprimitive. The product action is faithful and transitive and the following ensures shows that it is a valid choice for our construction.

Lemma 6.7. *Let X be a nonabelian finite simple group acting regularly upon itself and H be a transitive permutation group on a finite set Ω. Then the product action of $W = X \operatorname{wr}_\Omega H$ on the base group $B = X^\Omega$ is subprimitive.*

Proof. By transitivity of H on Ω, B is the unique minimal normal subgroup of W. Consequently, if K is a normal subgroup of W then $B \leqslant K$. In the product action, B acts regularly and hence K is transitive on B. Thus, as the product action is faithful, it follows that the action of K on the only K-orbit is also faithful. □

Corollary 6.5 therefore applies and tells us that $W = \varinjlim W_n$ and $\widehat{W} = \varprojlim W_n$ are hereditarily just infinite. We shall now construct some examples of abelian subgroups of the automorphism groups of these groups. There has been much study of automorphism groups of wreath products (see, for example, [Mohammadi Hassanabadi 1978]), but our requirement is simply to produce some automorphisms that commute and so we choose not to use the full power of such studies.

Suppose that, for each $i \geqslant 0$, ϕ_i is an automorphism of the simple group X_i. We take $\psi_0 = \phi_0$. Suppose that at stage $n-1$, we have constructed an automorphism ψ_{n-1} of W_{n-1}. Since the action of W_{n-1} on Ω_{n-1} is either regular or the product action (with $\Omega_{n-1} = B_{n-1}$ in the latter case), ψ_{n-1} induces a permutation of Ω_{n-1} (that we also denote by ψ_{n-1}) with the property that

$$(\omega^y)\psi_{n-1} = (\omega\psi_{n-1})^{y\psi_{n-1}} \tag{1}$$

for all $\omega \in \Omega_{n-1}$ and $y \in W_{n-1}$. We define a bijection $\psi_n : W_n \to W_n$ by

$$\psi_n : (x_\omega)y \mapsto ((x_{\omega\psi_{n-1}^{-1}})\phi_n)(y\psi_{n-1})$$

where $x_\omega \in X_n$ for each $\omega \in \Omega_{n-1}$ and $y \in W_{n-1}$. (Here we are writing elements of the base group B_n as sequences (x_ω) indexed by Ω_{n-1} with $x_\omega \in X_n$ in the ω-coordinate). Thus the effect of ψ_n on elements in the base group is to apply ϕ_n to each coordinate *and* permute the coordinates using the permutation ψ_{n-1} of Ω_{n-1}, while we simply apply the previous automorphism ψ_{n-1} to elements in the complement W_{n-1}. It is a straightforward calculation to verify that the resulting map is an automorphism of W_n and by construction it restricts to ψ_{n-1} on the subgroup W_{n-1}. (Indeed, in the case (R), the group W_n is the standard wreath product of X_n by W_{n-1}. If we write $\phi = \phi_n$ and $\beta = \psi_{n-1}$, then $\psi_n = \phi^*\beta^*$ is the composite of the automorphisms ϕ^* and β^* introduced on pages 474 and 476, respectively, of [Neumann and Neumann 1959]. The verification for the product action case (P) is similarly straightforward and depends primarily on (1)).

The final result is that, for each n, we have constructed an automorphism ψ_n of W_n that extends all the previous automorphisms. As a consequence, we certainly have determined an automorphism ψ of W whose restriction to each W_n coincides with ψ_n and an automorphism $\hat{\psi}$ of the group $\widehat{W}$ such that $\hat{\psi}\pi_n = \pi_n\psi_n$ for each

n (where, as above, we write $\pi_n : \widehat{W} \to W_n$ for the surjective homomorphism determined by the inverse limit). The key properties of the automorphisms that we have constructed are as follows:

Lemma 6.8. *Let (ϕ_i), (ϕ_i') be sequences of automorphisms with $\phi_i, \phi_i' \in \operatorname{Aut} X_i$ for each i. Define ψ and $\hat{\psi}$ to be the automorphisms of W and $\widehat{W}$ determined by the sequence (ϕ_i) and ψ' and $\hat{\psi}'$ those determined by (ϕ_i'). Then*:

(i) *$\hat{\psi}$ is a continuous automorphism of $\widehat{W}$.*

(ii) *$\psi\psi'$ and $\hat{\psi}\hat{\psi}'$ are the automorphisms of W and $\widehat{W}$ determined by the sequence $(\phi_i\phi_i')$.*

(iii) *If, for some $n \geqslant 0$, $\phi_0, \phi_1, \dots, \phi_{n-1}$ are the identity maps and ϕ_n is an outer automorphism of X_n, then ψ is an outer automorphism of W and $\hat{\psi}$ is an outer automorphism of $\widehat{W}$.*

Proof. (i) By construction, $\hat{\psi}$ fixes the kernels $\ker \pi_n$ associated to the inverse limit. These form a neighborhood base for the identity and so we deduce that $\hat{\psi}$ is continuous.

(ii) For each n, write ψ_n and ψ_n' for the automorphisms of W_n determined by the sequences (ϕ_i) and (ϕ_i'). One computes that, for $n \geqslant 1$, the composite $\psi_n\psi_n'$ is given by

$$(x_\omega)y \mapsto ((x_{\omega(\psi_{n-1}')^{-1}\psi_{n-1}^{-1}})\phi_n\phi_n')(y\psi_{n-1}\psi_{n-1}') = ((x_{\omega(\psi_{n-1}\psi_{n-1}')^{-1}})\phi_n\phi_n')(y\psi_{n-1}\psi_{n-1}').$$

A straightforward induction argument then shows that $\psi_n\psi_n'$ is the automorphism of W_n determined by the sequence $(\phi_i\phi_i')$. The claim appearing in the lemma then follows.

(iii) Suppose that ϕ_0, ϕ_1, $\dots$, ϕ_{n-1} are the identity and that $\phi_n \notin \operatorname{Inn} X_n$. We claim that $\psi_m \notin \operatorname{Inn} W_m$ for all $m \geqslant n$. The first of these automorphisms is given by $((x_\omega)_{\omega\in\Omega_{n-1}} \cdot y)\psi_n = (x_\omega\phi_n)_{\omega\in\Omega_{n-1}} \cdot y$ for $x_\omega \in X_n$ and $y \in W_{n-1}$. Suppose that ψ_n is produced by conjugating by the element bz where $b \in B_n$ and $z \in W_{n-1}$. Note that ψ_n fixes W_{n-1} and hence b normalizes W_{n-1}. Since $y^b = [b, y^{-1}]y$ for all $y \in W_{n-1}$, we determine that b centralizes W_{n-1}. Therefore $z \in Z(W_{n-1}) = \mathbf{1}$. We then determine that $b = (b_\omega)_{\omega\in\Omega_{n-1}}$ is the constant sequence and ϕ_n coincides with conjugation by the element b_ω, contrary to assumption. Hence ψ_n is an outer automorphism of W_n.

Now suppose, as an induction hypothesis, that $\psi_m \notin \operatorname{Inn} W_m$ for some $m \geqslant n$. Suppose that ψ_{m+1} is produced by conjugating by bz where $b \in B_{m+1}$ and $z \in W_m$. Then b fixes W_m and hence centralizes this subgroup. Consequently, ψ_m, which is the restriction of ψ_{m+1} to W_m is given by conjugating by z. This contradicts the inductive hypothesis. We conclude that ψ_m is an outer automorphism for all $m \geqslant n$. It now immediately follows that ψ is an outer automorphism of W and $\hat{\psi}$ is an outer automorphism of $\widehat{W}$. $\square$

Theorem 6.9. *Let $X_0, X_1, \ldots$ be a sequence of nonabelian finite simple groups. Define W to be the direct limit and $\widehat{W}$ to be the inverse limit of the wreath products W_n constructed as in Example 6.6. Suppose that, for each $i \geqslant 0$, ϕ_i is an automorphism of X_i such that $\langle \phi_i \rangle \cap \operatorname{Inn} X_i = \mathbf{1}$. Then the group $A = \prod_{i=0}^{\infty} \langle \phi_i \rangle$ embeds naturally*

(i) *as a subgroup of* $\operatorname{Aut} W$ *such that* $A \cap \operatorname{Inn} W = \mathbf{1}$;

(ii) *as a profinite subgroup of* $\operatorname{Aut_c} \widehat{W}$ *such that* $A \cap \operatorname{Inn} \widehat{W} = \mathbf{1}$.

Combining this theorem with Theorem 6.2 and Corollary 6.5 produces examples of hereditarily JNAF discrete and profinite groups.

Proof. (i) Each element g of $A = \prod_{i=1}^{\infty} \langle \phi_i \rangle$ is a sequence $(\phi_i^{k_i})$ of automorphisms. Let ψ_g denote the automorphism of W determined by this sequence. By Lemma 6.8(ii), the map $g \mapsto \psi_g$ is a homomorphism $\theta : A \to \operatorname{Aut} W$. It is clearly injective while part (iii) of the lemma ensures that the image satisfies $A\theta \cap \operatorname{Inn} W = \mathbf{1}$.

(ii) As with the first part, each g in $A = \prod_{i=1}^{\infty} \langle \phi_i \rangle$ determines a continuous automorphism $\hat{\psi}_g$ of $\widehat{W}$. Hence there is an injective homomorphism $\theta : A \to \operatorname{Aut_c} \widehat{W}$ given by $g \mapsto \hat{\psi}_g$. The subgroups $\Gamma_n = \{\gamma \in A\theta \mid [\widehat{W}, \gamma] \leqslant \ker \pi_n\}$, for $n \geqslant 0$, form a neighborhood base for the identity in the subspace topology on $A\theta$ (see [Dixon et al. 1999, Section 5.2]) and the inverse image of Γ_n under θ is $\prod_{i \geqslant n+1} \langle \phi_i \rangle$, which is open in the product topology on A. Hence θ is continuous and so its image is a profinite subgroup of $\operatorname{Aut_c} \widehat{W}$ that is topologically isomorphic to A and satisfies $A\theta \cap \operatorname{Inn} \widehat{W} = \mathbf{1}$ by Lemma 6.8(iii). □

Example 6.10. As a concrete example to finish our discussion of iterated wreath products, fix a prime number p and let (n_i) be a sequence of positive integers. Take $X_i = \mathrm{PSL}_2(p^{n_i})$, so that X_i has an outer automorphism ϕ_i of order n_i induced by the Frobenius automorphism of the finite field $\mathbb{F}_{p^{n_i}}$. Then Theorem 6.9 shows that the group $A = \prod_{i=0}^{\infty} C_{n_i}$ appears as a subgroup of the automorphism group of the direct limit W with $A \cap \operatorname{Inn} W = \mathbf{1}$ and as a profinite subgroup of $\operatorname{Aut_c} \widehat{W}$ with $A \cap \operatorname{Inn} \widehat{W} = \mathbf{1}$.

Many examples of profinite groups occur as closed subgroups of such a Cartesian product. For example, by taking a suitable enumeration (n_i) of prime-powers, we can embed the Cartesian product of countably many copies of the profinite completion $\hat{\mathbb{Z}}$ of the integers in some suitable product A and hence use Theorem 6.2 to construct a hereditarily JNAF profinite group of the form

$$(\varprojlim W_n) \rtimes \prod_{i=0}^{\infty} \hat{\mathbb{Z}}.$$

Hereditarily JNN$_c$F groups via Wilson's Construction B. The next examples of hereditarily just infinite groups that we shall consider are those introduced by Wilson [2010] in his Construction B. We recall this construction here in order that we can describe some automorphisms of these groups. We make one notational adjustment to Wilson's recipe. When constructing the group G_n, he defines $s = |U_{n-1}|$ and views $G_{n-1} = U_{n-1} \rtimes L_{n-1}$ as a subgroup of the symmetric group of degree s via its action upon U_{n-1}. Accordingly, various elements in his construction have an integer i as a parameter with $1 \leqslant i \leqslant s$. In our description, we shall index using the elements of U_{n-1} since this will aid our defining automorphisms of the constructed groups. We refer to [Wilson 2010] for justification of the assertions made when describing the construction.

Let (p_n), for $n \geqslant 1$, and (q_n), for $n \geqslant 0$, be two sequences of prime numbers such that, for every $n \geqslant 1$, $p_n \neq 2$, p_n divides $q_n - 1$ and $q_{n-1} \neq p_n$. Also let (t_n) be a sequence of positive integers. We now describe the construction of a sequence G_n of finite soluble groups.

First define $G_0 = U_0$ to be the additive group of the finite field $\mathbb{F}_{q_0}$ and take $L_0 = \mathbf{1}$. In particular, G_0 is cyclic of order q_0.

Now suppose that we have constructed a group $G_{n-1} = U_{n-1} \rtimes L_{n-1}$ where U_{n-1} is the unique minimal normal subgroup of G_{n-1} and U_{n-1} is an elementary abelian q_{n-1}-group. To simplify notation, write $U = U_{n-1}$ and let G_{n-1} act upon U by using the regular action of U_{n-1} upon itself and the conjugation action of L_{n-1} upon the normal subgroup U_{n-1}. Define

$$\Gamma = U \times \{1, 2, \dots, t_n\} = \{(u, k) \mid u \in U, 1 \leqslant k \leqslant t_n\}.$$

Let A be an elementary abelian p_n-group with basis $\{a_\gamma \mid \gamma \in \Gamma\}$ and V be the group algebra $\mathbb{F}_{q_n} A$. Let ζ be an element of order p_n in the multiplicative group of the field $\mathbb{F}_{q_n}$. Define invertible linear maps x_δ, y_δ (for $\delta \in \Gamma$) and z of V by $x_\delta : v \mapsto v a_\delta$ for $v \in V$, $y_\delta : \prod a_\gamma^{r_\gamma} \mapsto \zeta^{r_\delta} \prod a_\gamma^{r_\gamma}$ for each $\prod a_\gamma^{r_\gamma} \in A$, and $z : v \mapsto \zeta v$ for $v \in V$. Then define the following subgroups of $\mathrm{GL}(V)$: $X = \langle x_\gamma \mid \gamma \in \Gamma \rangle$, $Y = \langle y_\gamma \mid \gamma \in \Gamma \rangle$ and $E = \langle X, Y \rangle$. The action of G_{n-1} upon U induces an action on Γ and hence an action on the basis of A: $a_{(u,k)}^g = a_{(u^g,k)}$ for each $u \in U$, $1 \leqslant k \leqslant t_n$ and $g \in G_{n-1}$. Hence each $g \in G_{n-1}$ determines an invertible linear transformation of V and this normalizes both X and Y (see [Wilson 2010, (4.3)]).

Now fix some element $u_0 \in U$. Set $\widetilde{\Gamma} = \{(u, k) \in \Gamma \mid u \neq u_0\}$ and, for $(u, k) \in \widetilde{\Gamma}$, let

$$\tilde{a}_{(u,k)} = a_{(u_0,k)}^{-1} a_{(u,k)}, \quad \tilde{x}_{(u,k)} = x_{(u_0,k)}^{-1} x_{(u,k)}, \quad \tilde{y}_{(u,k)} = y_{(u_0,k)}^{-1} y_{(u,k)}.$$

Define $\tilde{A} = \langle \tilde{a}_\gamma \mid \gamma \in \widetilde{\Gamma} \rangle$, $\widetilde{X} = \langle \tilde{x}_\gamma \mid \gamma \in \widetilde{\Gamma} \rangle$, $\widetilde{Y} = \langle \tilde{y}_\gamma \mid \gamma \in \widetilde{\Gamma} \rangle$, and $D = \langle \widetilde{X}, \widetilde{Y} \rangle$. In [Wilson 2010, (4.2)], it is observed that

$$[x_\gamma, y_\delta] = \begin{cases} z & \text{if } \gamma = \delta, \\ 1 & \text{if } \gamma \neq \delta. \end{cases} \tag{2}$$

Since z is central, it follows that E is nilpotent of class 2 and that $D = \widetilde{X}\widetilde{Y}\langle z \rangle$. Also set $W = \mathbb{F}_{q_n} \tilde{A}$. Then W is an irreducible D-module and the group G_{n-1}, via its action on V, normalizes D and induces automorphisms of W; see [Wilson 2010, (4.4) and (4.5)]. Finally set $G_n = (W \rtimes D) \rtimes G_{n-1}$, $U_n = W$ and $L_n = D \rtimes G_{n-1}$. Associated to this semidirect product, there are surjective homomorphisms $G_n \to G_{n-1}$ and inclusions $G_{n-1} \hookrightarrow G_n$. Let $\widehat{G} = \varprojlim G_n$ and $G = \varinjlim G_n$ be the associated inverse and direct limits.

Proposition 6.11 [Wilson 2010, (4.7)]. *The inverse limit $\widehat{G}$ is a hereditarily just infinite profinite group and the direct limit G is a hereditarily just infinite* (*discrete*) *group.*

We need the following additional properties of the groups G_n that are not recorded in Wilson's paper:

Lemma 6.12. (i) *For $n \geqslant 1$, the center of G_n is trivial.*

(ii) *If $n = 1$ and p_1 divides $q_0 - 1$, then the center of $D \rtimes G_0$ is cyclic generated by z.*

(iii) *If $n \geqslant 1$ and p_n divides $q_{n-1} - 1$, then the center of D is cyclic generated by z.*

Proof. (i) It is observed in [Wilson 2010, (4.6)(b)] that $C_{G_n}(W) = W$. Hence $Z(G_n) \leqslant W$. However, note $z \in D'$ by [Wilson 2010, (4.4)(a)] and $w^z = \zeta w$ for all $w \in W$ and so only the identity (that is, the zero vector in W) commutes with all elements of G_n.

(ii) Suppose that p_1 divides $q_0 - 1$ and recall that $U = G_0$ when $n = 0$. Consider first the action of G_0 on $X = \langle x_\gamma \mid \gamma \in \Gamma \rangle$. The group X is an elementary abelian p_n-group and so as an $\mathbb{F}_{p_n} G_0$-module is a direct sum $X = \bigoplus_{k=1}^{t_1} X_k$ where X_k is isomorphic to the group algebra $\mathbb{F}_{p_n} G_0$ (since G_0 acts regularly on U in this case). There is a unique 1-dimensional submodule of X_k upon which G_0 acts trivially, namely that generated by the product $v_k = \prod_{u \in U} x_{(u,k)}$, and an element of X is fixed by G_0 if and only if it belongs to $P = \langle v_1, v_2, \dots, v_{t_1} \rangle$.

Now if v_k were an element of $\widetilde{X}$, it could be written as $v_k = \prod_{u \neq u_0} \tilde{x}_{(u,k)}^{r_u}$ for some values r_u; that is, $v_k = x_{(u_0,k)}^{-s} \prod_{u \neq u_0} x_{(u,k)}^{r_u}$ where $s = \sum_{u \neq u_0} r_u$. Hence $r_u = 1$ for all $u \neq u_0$, but then $s = |U| - 1 \equiv 0 \pmod{p_1}$ since p_1 divides $q_0 - 1$. This is a contradiction and so we conclude $v_k \notin \widetilde{X}$ for all k. Since the set of $\tilde{x}_\gamma$ for $\gamma \in \widetilde{\Gamma}$ forms a basis for X, we deduce that $\widetilde{X} \cap P = \mathbf{1}$; that is, only the identity element of $\widetilde{X}$ is fixed under the action of G_0. Similarly, only the identity element is fixed under the action of G_0 on $\widetilde{Y}$. From these observations, we deduce that if $a = ghz^t$ is centralized by G_0 where $g \in \widetilde{X}$ and $h \in \widetilde{Y}$, then necessarily $g = h = 1$. The claim that $Z(D \rtimes G_0) = \langle z \rangle$ then follows.

(iii) Suppose that p_n divides $q_{n-1} - 1$. Let $a = ghz^t$ be in the center of D where $g \in \widetilde{X}$ and $h \in \widetilde{Y}$. From (2), it follows that, for $\gamma = (u, k)$ and $\delta = (v, \ell)$ with

$u, v \neq u_0$,

$$[\tilde{x}_\gamma, \tilde{y}_\delta] = [x_{(u_0,k)}^{-1} x_{(u,k)}, y_{(u_0,\ell)}^{-1} y_{(v,\ell)}] = \begin{cases} z^2 & \text{if } \gamma = \delta, \\ z & \text{if } k = \ell \text{ and } u \neq v, \\ 1 & \text{if } k \neq \ell. \end{cases}$$

Suppose $g = \prod_{\gamma \in \widetilde{\Gamma}} \tilde{x}_\gamma^{r_\gamma}$. Then, for $\delta = (v, \ell) \in \widetilde{\Gamma}$, $[g, \tilde{y}_\delta] = z^{N_\ell + r_\delta}$ where $N_\ell = \sum_{u \neq u_0} r_{(u,\ell)}$. It follows that $r_\delta \equiv -N_\ell \pmod{p_n}$ for all $\delta = (u, \ell) \in \widetilde{\Gamma}$. Hence $N_\ell \equiv -(|U| - 1)N_\ell \equiv 0 \pmod{p_n}$ for $1 \leqslant \ell \leqslant t_n$, using the fact that U is an elementary abelian q_{n-1}-group and p_n divides $q_{n-1} - 1$. This shows $r_\delta \equiv 0 \pmod{p_n}$ for all $\delta \in \widetilde{\Gamma}$ and hence $g = 1$. It similarly follows that $h = 1$. We conclude that $a = z^t$ for some t and this establishes that $Z(D) = \langle z \rangle$. □

We shall now describe a method to construct some automorphisms of the groups G and $\widehat{G}$. For each $i \geqslant 0$, let λ_i be a nonzero scalar in the field $\mathbb{F}_{q_i}$. In particular, $\psi_0 : x \mapsto \lambda_0 x$ is an automorphism of the additive group $G_0 = \mathbb{F}_{q_0}$. Now suppose that we have constructed an automorphism ψ_{n-1} of G_{n-1}. Since U_{n-1} is the unique minimal normal subgroup of G_{n-1}, ψ_{n-1} induces an automorphism of $U = U_{n-1}$. Hence we induce a bijection $\psi_{n-1} : \Gamma \to \Gamma$ by $(u, k)\psi_{n-1} = (u\psi_{n-1}, k)$ and consequently determine an automorphism of A by $a_\gamma \mapsto a_{\gamma\psi_{n-1}}$ and this extends to an invertible linear map $\psi_{n-1} : V \to V$.

Lemma 6.13. *The induced linear map $\psi_{n-1} \in \mathrm{GL}(V)$ satisfies $\psi_{n-1}^{-1} x_\delta \psi_{n-1} = x_{\delta\psi_{n-1}}$ and $\psi_{n-1}^{-1} y_\delta \psi_{n-1} = y_{\delta\psi_{n-1}}$ for each $\delta \in \Gamma$.*

Proof. If $v \in V$, then $v\psi_{n-1}^{-1} x_\delta \psi_{n-1} = (v\psi_{n-1}^{-1} \cdot a_\delta)\psi_{n-1} = v \cdot a_{\delta\psi_{n-1}}$ and hence $\psi_{n-1}^{-1} x_\delta \psi_{n-1} = x_{\delta\psi_{n-1}}$. For an element $\prod a_\gamma^{r_\gamma} \in A$, we compute

$$\bigl(\prod a_\gamma^{r_\gamma}\bigr)\psi_{n-1}^{-1} y_\delta \psi_{n-1} = \bigl(\prod a_{\gamma\psi_{n-1}^{-1}}^{r_\gamma}\bigr) y_\delta \psi_{n-1} = \bigl(\zeta^{r_{\delta\psi_{n-1}}} \prod a_{\gamma\psi_{n-1}^{-1}}^{r_\gamma}\bigr)\psi_{n-1} = \zeta^{r_{\delta\psi_{n-1}}} \prod a_\gamma^{r_\gamma}$$

and hence $\psi_{n-1}^{-1} y_\delta \psi_{n-1} = y_{\delta\psi_{n-1}}$. □

As a consequence, we determine an automorphism ψ_{n-1}^* of the subgroup E of $\mathrm{GL}(V)$ given by conjugating by this linear map ψ_{n-1}. Notice furthermore that $D\psi_{n-1}^* = D$ since

$$\tilde{x}_{(u,k)} \psi_{n-1}^* = \psi_{n-1}^{-1} x_{(u_0,k)}^{-1} x_{(u,k)} \psi_{n-1} = x_{(u_0\psi_{n-1},k)}^{-1} x_{(u\psi_{n-1},k)} = \tilde{x}_{(u_0\psi_{n-1},k)}^{-1} \tilde{x}_{(u\psi_{n-1},k)}$$

and similarly for $\tilde{y}_{(u,k)}$. Finally, we determine a bijection $\psi_n : G_n \to G_n$ by applying ψ_{n-1}^* to elements in D and applying ψ_{n-1} to those in G_{n-1} and defining its effect on elements of $W = \mathbb{F}_{q_n}\tilde{A}$ by

$$\tilde{a}_{(u,k)} \psi_n = \lambda_n a_{(u_0\psi_{n-1},k)}^{-1} a_{(u\psi_{n-1},k)} = \lambda_n \tilde{a}_{(u_0\psi_{n-1},k)}^{-1} \tilde{a}_{(u\psi_{n-1},k)}$$

and extending by linearity. Thus, the effect of ψ_n on W is the composite of the linear map ψ_{n-1} defined above together with scalar multiplication by λ_n. Since each x_δ and y_δ is a linear map, it follows that ψ_n induces an automorphism of $W \rtimes D$. Also notice that, since the action of G_{n-1} on $U = U_{n-1}$ is given by the regular action of U_{n-1} on itself and the conjugation action of L_{n-1} on U_{n-1}, the automorphism ψ_{n-1} satisfies

$$(u^g)\psi_{n-1} = (u\psi_{n-1})^{g\psi_{n-1}}$$

for all $u \in U$ and $g \in G_{n-1}$ (and here exponentiation denotes the action). One determines, using Lemma 6.13, that $(x_\delta^g)\psi_{n-1}^* = (x_\delta \psi_{n-1}^*)^{g\psi_{n-1}}$ for $\delta \in \Gamma$ and $g \in G_{n-1}$. Similar formulae hold when we conjugate y_δ and a_δ by elements of G_{n-1} (in the latter case, we rely upon the fact that an element of G_{n-1} induces a linear map on V and so commutes with the operation of multiplying by the scalar λ_n). We conclude that ψ_n is indeed an automorphism of G_n that restricts to the previous one ψ_{n-1} on G_{n-1}. As a consequence, we determine an automorphism ψ of G whose restriction to each G_n equals ψ_n and an automorphism $\hat{\psi}$ of $\widehat{G}$ such that $\hat{\psi}\pi_n = \pi_n\psi_n$ for each n (where $\pi_n : \widehat{G} \to G_n$ is the surjective homomorphism associated to the inverse limit). The properties of this construction are analogous to those for the iterated wreath product and the first two parts of the following are established similarly to those of Lemma 6.8.

Lemma 6.14. *Let (λ_i), (μ_i) be sequences of scalars with $\lambda_i, \mu_i \in \mathbb{F}_{q_i}^*$. Define ψ and $\hat{\psi}$ to be the automorphisms of G and $\widehat{G}$ determined by the sequence (λ_i) and θ and $\hat{\theta}$ those determined by (μ_i). Then*:

(i) *$\hat{\psi}$ is a continuous automorphism of $\widehat{G}$.*

(ii) *$\psi\theta$ and $\hat{\psi}\hat{\theta}$ are the automorphisms of G and $\widehat{G}$ determined by the sequence $(\lambda_i\mu_i)$.*

(iii) *If p_i divides $q_{i-1} - 1$ for all $i \geqslant 1$ and, for some $n \geqslant 0$, $\lambda_i = 1$ in $\mathbb{F}_{q_i}$ for $0 \leqslant i \leqslant n-1$ and λ_n is not in the subgroup of order p_n in the multiplicative group of the field $\mathbb{F}_{q_n}$, then ψ is an outer automorphism of G and $\hat{\psi}$ is an outer automorphism of $\widehat{G}$.*

Proof. We prove part (iii). Suppose that p_i divides $q_{i-1} - 1$ for all $i \geqslant 1$ in addition to the original assumptions on the p_i and q_j. Suppose that $\lambda_i = 1$ for $0 \leqslant i \leqslant n-1$ and that λ_n is not a power of ζ where ζ is an element of order p_n in $\mathbb{F}_{q_n}^*$. Since $\lambda_i = 1$ for $0 \leqslant i \leqslant n-1$, the automorphism ψ_{n-1} of G_{n-1} is the identity map. We shall first show that $\psi_n \notin \operatorname{Inn} G_n$. We will need a different argument according to the value of n. If $n = 0$, then G_0 is abelian so ψ_0 cannot be an inner automorphism as it is not the identity.

Suppose that $n = 1$ and that ψ_1 is produced by conjugating by the element wdh where $w \in W$, $d \in D$ and $h \in G_0$. In this case, ψ_0 is the identity, so ψ_1 induces the identity on $D \rtimes G_0$ and hence $dh \in Z(D \rtimes G_0)$; that is, $h = 1$ and $d = z^k$ for some k by Lemma 6.12(ii). Now observe that w must normalize D since $D\psi_1 = D$ and it follows that $[w, g] = 1$ for all $g \in D$. Hence $\mathbb{F}_{q_n} w$ is a D-invariant subspace of W; so $w = 0$ as W is an irreducible D-module, by [Wilson 2010, (4.5)(c)]. In conclusion, ψ_1 is the inner automorphism of G_1 determined by conjugation by z^k. This means that $\lambda_1 = \zeta^k$, contrary to our assumption.

Suppose that $n \geqslant 2$ and that $\psi_n \in \operatorname{Inn} G_n$, and let conjugation by the element wdg (where $w \in W$, $d \in D$ and $g \in G_{n-1}$) achieve the same effect as applying ψ_n. In particular, wdg centralizes G_{n-1} and so wd normalizes G_{n-1}. It follows that $[wd, y] = 1$ for all $y \in G_{n-1}$ and hence w and d are both centralized by G_{n-1} and $g \in Z(G_{n-1})$. Therefore $g = 1$ by Lemma 6.12(i). Also necessarily $d \in Z(D)$, so $d = z^k$ for some k by Lemma 6.12(iii), while w spans a D-submodule of W and hence $w = 0$. We conclude, as in the previous case, that ψ_n is the inner automorphism of G_n determined by conjugation by z^k, which is impossible as $\lambda_n \notin \langle \zeta \rangle$ by assumption.

Now suppose that $\psi_m \notin \operatorname{Inn} G_m$ for some $m \geqslant n$. If it were the case that ψ_{m+1} is produced by conjugating by wdg where $w \in W$, $d \in D$ and $g \in G_m$, then ψ_m would coincide with conjugation by g, contrary to assumption. Hence $\psi_m \notin \operatorname{Inn} G_m$ for all $m \geqslant n$. It now follows that ψ is an outer automorphism of W and $\hat{\psi}$ is an outer automorphism of $\widehat{W}$. □

Theorem 6.15. *Let (p_n), for $n \geqslant 1$, and (q_n), for $n \geqslant 0$, be a sequence of prime numbers such that for every $n \geqslant 1$, $p_n \neq 2$, p_n divides both $q_{n-1} - 1$ and $q_n - 1$. Let (t_n) be any sequence of positive integers and define G to be the direct limit and $\widehat{G}$ to be the inverse limit of the semidirect products G_n built via Wilson's Construction B. Take $r_0 = q_0 - 1$ and, for each $i \geqslant 1$, write $q_i - 1 = r_i p_i^{m_i}$ where $p_i \nmid r_i$ and let C_{r_i} denote a cyclic group of order r_i. Then the group $A = \prod_{i=0}^{\infty} C_{r_i}$ embeds naturally*

(i) *as a subgroup of* $\operatorname{Aut} G$ *such that* $A \cap \operatorname{Inn} G = \mathbf{1}$;

(ii) *as a profinite subgroup of* $\operatorname{Aut_c} \widehat{G}$ *such that* $A \cap \operatorname{Inn} \widehat{G} = \mathbf{1}$.

Proof. The proof is similar to that of Theorem 6.9. For each i, let λ_i be an element of order r_i in the multiplicative group $\mathbb{F}_{q_i}^*$. Then, for $i \geqslant 1$, $\langle \lambda_i \rangle \cap \langle \zeta_i \rangle = \mathbf{1}$ where ζ_i denotes an element of order p_i in $\mathbb{F}_{q_i}^*$. Now if

$$g = (\lambda_i^{k_i}) \in \prod_{i=0}^{\infty} \langle \lambda_i \rangle \cong A,$$

write ψ_g for the automorphism ψ determined by the sequence $(\lambda_i^{k_i})$ as above. Lemma 6.14 ensures that $g \mapsto \psi_g$ is a homomorphism into $\operatorname{Aut} G$ whose image satisfies the conclusion of (i). The second part is established similarly: we determine an injective homomorphism $\theta : \prod_{i=0}^{\infty} \langle \lambda_i \rangle \to \operatorname{Aut_c} \widehat{G}$ and this is continuous since the inverse image under θ of the basic neighborhood of the identity comprising those automorphisms that act trivially on G_n is $\prod_{i \geqslant n+1} \langle \lambda_i \rangle$. □

Example 6.16. A specific example can be constructed as follows. Let (n_i) be any sequence of positive integers. Let (p_i), for $i \geqslant 1$, be any sequence of odd primes such that p_i does not divide n_i. When $i \geqslant 1$, take $a_i = \mathrm{lcm}(p_i n_i, p_{i+1})$ and $a_0 = \mathrm{lcm}(n_0, p_1)$. Now take, for $i \geqslant 0$, q_i to be any prime number of the form $a_i k + 1$ for some $k \in \mathbb{N}$. (The existence of such a prime number is guaranteed by Dirichlet's theorem). These choices of sequences then fulfill the requirements of Theorem 6.15 and the integer r_i appearing in the statement is divisible by n_i by construction. Consequently, we deduce that the Cartesian product $\prod_{i=0}^{\infty} C_{n_i}$ embeds in the subgroup A appearing in Theorem 6.15. We may use any closed subgroup of this Cartesian product as the choice of A in Theorem 6.2. In particular, there are many choices of abelian profinite groups A such that $\widehat{G} \rtimes A$ is hereditarily JNAF including, as with the iterated wreath product, a hereditarily JNAF example of the form

$$(\varprojlim G_n) \rtimes \prod_{i=0}^{\infty} \hat{\mathbb{Z}}.$$

Hereditarily JNN$_c$F groups by use of the Nottingham group. The following construction brings together two facets of the study of pro-p groups. As a first ingredient, we make use of the work of Lubotzky–Shalev [1994] on R-analytic groups, in the specific case when R is the formal powers series ring $\mathbb{F}_p[\![T]\!]$, to identify a specific hereditarily just infinite pro-p group G. Secondly, we use the fact that every countably based pro-p group embeds in the automorphism group $\mathrm{Aut}(R)$ to obtain a wide range of groups of automorphisms of our group G.

Example 6.17. Let p be a prime number and let n be a positive integer with $n \geqslant 2$ such that p does not divide n. Take $R = \mathbb{F}_p[\![T]\!]$, the pro-$p$ ring of all formal power series over the field of p elements, which is a local ring with unique maximal ideal $\mathfrak{m} = T\mathbb{F}_p[\![T]\!]$ generated by the indeterminate T. Then take $G = \mathrm{SL}_n^1(R)$, the first principal congruence subgroup of the special linear group of all $n \times n$ matrices of determinant 1 over R; that is,

$$G = \{g \in \mathrm{SL}_n(R) \mid g \equiv I \pmod{\mathfrak{m}}\},$$

where I denotes the $n \times n$ identity matrix. Using the techniques of [Lubotzky and Shalev 1994], it is straightforward to observe that G is a hereditarily just infinite pro-p group. First, G is R-perfect and so the terms of its lower central series are the congruence subgroups

$$\gamma_k(G) = G_k = \{g \in \mathrm{SL}_n(R) \mid g \equiv I \pmod{\mathfrak{m}^k}\},$$

for each $k \geqslant 1$. Adapting slightly the notation used in [Lubotzky and Shalev 1994], we see that the (completed) graded Lie ring associated to the lower central series of

G satisfies

$$L(G) = L_G(G) = \overline{\bigoplus_{i=1}^{\infty} G_i/G_{i+1}} \cong \prod_{i=1}^{\infty} T^i \mathfrak{sl}_n(\mathbb{F}_p) \cong \mathfrak{sl}_n(\mathfrak{m}),$$

the latter being the Lie algebra over $\mathbb{F}_p$ of $n \times n$ matrices with entries in $\mathfrak{m}$ and trace 0. To every closed subgroup H of G we associate a closed Lie subalgebra of $L(G)$ that we denote by $L_G(H)$ and whose properties are described in [Lubotzky and Shalev 1994, Lemma 2.13]. Using the isomorphism above we view $L_G(H)$ as a Lie subalgebra of $\mathfrak{sl}_n(\mathfrak{m})$. In particular, $L_G(G_k)$ corresponds to the Lie subalgebra $\prod_{i=k}^{\infty} T^i \mathfrak{sl}_n(\mathbb{F}_p) \cong \mathfrak{sl}_n(\mathfrak{m}^k)$. If W is a nonzero $\mathbb{F}_p$-subspace of $\mathfrak{sl}_n(R)$ satisfying $[W, \mathfrak{sl}_n(\mathfrak{m}^k)]_{\mathrm{Lie}} \subseteq W$ for some $k \geqslant 1$, then a direct computation shows there exists r such that $\mathfrak{sl}_n(\mathfrak{m}^r) \subseteq W$. (It is this computation that uses the fact that $p \nmid n$).

Now let H and N be closed subgroups of G such that $\mathbf{1} \neq N \trianglelefteqslant H \trianglelefteqslant G$. Then $L_G(H)$ is an ideal of the Lie algebra $\mathfrak{sl}_n(\mathfrak{m})$ and hence there exists r such that $\mathfrak{sl}_n(\mathfrak{m}^r) \subseteq L_G(H)$. Consequently, $G_r \leqslant H$, so that $[N, G_r] \leqslant N$ and one deduces $[L_G(N), \mathfrak{sl}_n(\mathfrak{m}^r)]_{\mathrm{Lie}} \subseteq L_G(N)$. It follows that $\mathfrak{sl}_n(\mathfrak{m}^s) \subseteq L_G(N)$ for some s and hence $G_s \leqslant N$ and so $|G : N| < \infty$. This shows that G is hereditarily just infinite.

Next we exploit properties of the Nottingham group $\mathcal{N}$ over $\mathbb{F}_p$ to produce groups of automorphisms of the above group G. The group $\mathcal{N}$ is the Sylow pro-p subgroup of the profinite group $\mathrm{Aut_c}(R) = \mathrm{Aut}(R)$; it coincides with the group $\mathrm{Aut}^1(R)$ of all automorphisms of the ring R that act trivially modulo $\mathfrak{m}^2$. Any element α of $\mathcal{N}$ is then uniquely determined by its effect upon the indeterminate T and, conversely, for any $f \in R$ with $f \equiv T \pmod{\mathfrak{m}^2}$ there is a unique element of $\mathcal{N}$ mapping T to f. (Thus $\mathcal{N}$ could alternatively be defined as a group of power series $T + \mathfrak{m}^2$ with the binary operation given by substitution of power series. For our construction, however, the behavior as automorphisms is more relevant). We refer to [Camina 2000] for background material concerning the Nottingham group, which plays a role also in number theory and dynamics.

The action of the Nottingham group $\mathcal{N}$ on R induces a faithful action upon the group $G = \mathrm{SL}_n^1(R)$ and hence we construct a subgroup $\dot{\mathcal{N}} \leqslant \mathrm{Aut_c}\, G$ isomorphic to $\mathcal{N}$. Suppose $\alpha \in \mathcal{N}$ is an element that induces an inner automorphism $\dot{\alpha}$ of the group G, and put $f = T\alpha \in T + \mathfrak{m}^2$. Then there exists a matrix $h \in G$ such that $hx^{\dot{\alpha}} = xh$ for all $x \in G$. In particular, upon taking $x = I + Te_{ij}$ for $1 \leqslant i, j \leqslant n$ with $i \neq j$, we conclude that h must be a diagonal matrix such that every pair of distinct diagonal entries a and b are linked by the relation $Ta = fb$ in R. It follows that $f^2 = T^2$ and hence, since $f \equiv T \pmod{\mathfrak{m}^2}$, that $f = T$ and $\dot{\alpha} = \mathrm{id}_G$. In conclusion, the copy of the Nottingham group in $\mathrm{Aut_c}\, G$ satisfies $\dot{\mathcal{N}} \cap \mathrm{Inn}\, G = \mathbf{1}$.

As the final step in our construction, we use the result of Camina [1997] that every countably-based pro-p group can be embedded as a closed subgroup in $\mathcal{N}$. Hence if A is any finitely generated pro-p group that is virtually nilpotent (say, of class c), then it may be embedded in $\mathrm{Aut_c}\, G$ in such a way that $A \cap \mathrm{Inn}\, G = \mathbf{1}$. Hence we have satisfied the conditions of Theorem 6.2 and the semidirect product $G \rtimes A$ is an example of a hereditarily $\mathrm{JNN}_c\mathrm{F}$ pro-p group.

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BENJAMIN KLOPSCH
MATHEMATISCHES INSTITUT
HEINRICH-HEINE-UNIVERSITAET
DUESSELDORF
GERMANY
klopsch@math.uni-duesseldorf.de

MARTYN QUICK
SCHOOL OF MATHEMATICS & STATISTICS
UNIVERSITY OF ST ANDREWS
ST ANDREWS
UNITED KINGDOM
mq3@st-andrews.ac.uk

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