SEMIGROUP PRESENTATIONS

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Contents

Declarations

I, Nikola Ruškuc, declare that this thesis has been composed by myself, that it is a record of my own work, and that it has not been accepted in any previous application for any degree.

Signature [redacted] **Name** Nikola Ruškuc **Date** 24/02/95

I was admitted to the Faculty of Science of the University of St Andrews under Ordinance General No. 12 on 1st October 1992 and as a candidate for the degree of Doctor of Philosophy on 1st October 1993.

Signatur[redacted]. **Name** Nikola Ruškuc **Date** 24/02/95

We certify that Nikola Ruškuc has satisfied the conditions of the Resolution and Regulations and is thus qualified to submit the accompanying thesis in application for the degree of Doctor of Philosophy.

Signature [redacted] **Name** John M. Howie **Date** 24/02/95
Signature [redacted]. **Name** Edmund F. Robertson **Date** 24/02/95

I agree that access to my thesis in the University Library shall be unrestricted.

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Preface and acknowledgements

Although every group is a semigroup, groups are by no means typical examples of semigroups. For this reason it is not customary to date the beginnings of semigroup theory together with the beginnings of group theory. Neverthless, the concept of a semigroup presentation is almost identical to the concept of a group presentation. In this sense semigroup presentations are older than semigroup theory itself.

Both the theory of group presentations and semigroup theory have had a dynamic development over the years. Slightly surprisingly, the theory of semigroup presentations has stayed somewhat behind. The majority of results about semigroup presentations have been obtained in connection with so called decidability problems, most notably with the word problem. Other achievements include Adian's embeddability results, Aĭzenštat's and Popova's presentations for some semigroups of mappings, Rédei's treatment of commutative semigroups and the modification of the Todd—Coxeter enumeration procedure by Neumann and Jura.

The original results presented in this thesis constitute an attempt at a further development of the theory of semigroup presentations, which is built on the above mentioned foundations. This research has been done during my research studentship at the School of Mathematical and Computational Sciences of the University of St Andrews, under the supervision of Professor John M. Howie and Dr Edmund F. Robertson. At first the research started in two separate areas: finding presentations for common semigroups and investigating the structure of semigroups defined by presentations. Soon, however, some ideas common for both approaches emerged. The most important ones are the use of computational methods and semigroup constructions.

Most of the results presented here have been or will be published in papers Campbell, Robertson, Ruškuc and Thomas (1994), (1994a), (1995), (1995a), (1995b), (1995c), (1995d), Campbell, Robertson, Ruškuc, Thomas and Ünlü (1995), Howie and Ruškuc (1994), Ruškuc (1994), (1995). However, the thesis is not a compilation of these papers. The main reason for this is that the papers have been written as the research progressed, so that special results have been followed by more general ones. Here this order is completely reversed: the special results are proved as consequences of the general ones. I hope that this will make more apparent the strength of various results and connections that exist between different areas of research in semigroup presentations. However, I have tried to retain some of the flavour of the original research by commenting frequently on the history and motivation for the main ideas.

The thesis begins with an introduction to semigroup presentations and general problems that are considered in the text. The first of these problems—finding generators and defining relations for common semigroups—is considered in Chapters 4 and 5. Presentations for various semigroup constructions are given in Chapter 6, and a detailed discussion on presentations for subsemigroups is given in Chapters 7 and 10. Chapters 8, 11, 12 and 13 contain applications of these general results to various semigroups defined by presentations, while in Chapter 10 connections between the semigroup and the group defined by the same presentation are explored. Chapter 14 contains a discussion of Todd—Coxeter type computational methods for semigroup presentations. The elementary facts of semigroup theory which we use in the main text are given in Appendix A, and Appendix B contains a list of all open problems posed in the main text.

Mathematical statements in this thesis are classified as theorems, corollaries, propositions and lemmas. Theorems are the main original results of the thesis. Corollaries are consequences of theorems. Propositions are relevant results by other authors, included here for the reasons of completeness, or general well-known results which cannot be attributed to any particular author. Lemmas are technical results that are needed for the proof of a theorem. The black square symbol (■) denotes the end of a mathematical argument, and is typically found at the end of a proof or an example. The black square at the end of a theorem, corollary, proposition or lemma means that the statement will have no proof. With few exceptions, mappings are written on the right. The list of open problems in Appendix B is not intended as a complete list of open problems in the field; it is rather a list of the author's unsuccessful attempts in proving theorems.

There are several reasons for the success of the research described in this thesis. First of all I would like to mention the contribution of my supervisors Professor John M. Howie and Dr Edmund F. Robertson, whose ideas, guidance, patience and support has been essential at all stages. I would also like to thank my supervisors for giving me an insight into their perception of mathematics, for hours of mathematical and non-mathematical conversation, for careful reading of the drafts of various papers and this thesis, and for trying unsuccessfully to teach me how to use the words 'the' and 'a' in the English language.

Next I would like to thank Dr Colin M. Campbell, Professor J.M. Howie, Dr Edmund F. Robertson and Dr Richard M. Thomas, who are co-authors of the various results appearing here, for most enjoyable time I have had working with them, and for enabling me to use what is a result of their work as well as mine in this thesis. I sincerely hope that this thesis is not the end of our cooperation, but merely a stage of it (see Appendix B).

I also gratefully acknowledge the financial support of the University of St Andrews Research Scholarship and ORS award.

Although the results presented here have all been obtained in the last two and a half years during my stay in St Andrews, my fascination with mathematics started much earlier. Since then it has been one of the most valuable parts of my life. Therefore I would like to express my deepest gratitude for those who have most contributed to this fascination. They are: Mr Petar Kulešević, Mr Dušan Kovačević, Professor Siniša Crvenković, Professor John M. Howie and Dr Edmund F. Robertson.

This thesis also marks the end of a chapter of my life, and I feel obliged to mentioned the people that I have lived with over the years: my mother Mrs Margarita Ruškuc, my father Mr Vladimir Ruškuc, my grandmother Mrs Anka Piperkova, my aunt Mrs Vera Miler, my uncle Mr Ottokar Miler and my wife Ester. I have had an extraordinary fortune that these are also the people that I have loved and love most. I am grateful to all of them for standing by me even in the most difficult moments in my life, and for bearing me when I was 'doing mathematics'. I wish I could give them in return something more valuable, or at least something that they could enjoy reading, than this thesis is.

Nikola Ruškuc

Abstract

In this thesis we consider in detail the following two fundamental problems for semigroup presentations:

1. Given a semigroup find a presentation defining it.
2. Given a presentation describe the semigroup defined by it.

We also establish two links between these two approaches: semigroup constructions and computational methods.

After an introduction to semigroup presentations in Chapter 3, in Chapters 4 and 5 we consider the first of the two approaches. The semigroups we examine in these two chapters include completely 0-simple semigroups, transformation semigroups, matrix semigroups and various endomorphism semigroups. In Chapter 6 we find presentations for the following semigroup constructions: wreath product, Bruck—Reilly extension, Schützenberger product, strong semilattices of monoids, Rees matrix semigroups, ideal extensions and subsemigroups. We investigate in more detail presentations for subsemigroups in Chapters 7 and 10, where we prove a number of Reidemeister—Schreier type results for semigroups. In Chapter 9 we examine the connection between the semigroup and the group defined by the same presentation. The general results from Chapters 6, 7, 9 and 10 are applied in Chapters 8, 11, 12 and 13 to subsemigroups of free semigroups, Fibonacci semigroups, semigroups defined by Coxeter type presentations and one relator products of cyclic groups. Finally, in Chapter 14 we describe the Todd—Coxeter enumeration procedure and introduce three modifications of this procedure.

Chapter 1

Introduction: semigroups and their presentations

1. Semigroups

Semigroup theory, in its half century or so of history, has undergone a big change: from a study of various generalisations of groups it has become a separate scientific subject with a great number of published results, its own journal, a growing number of monographs and many open research fronts. The main reason for this development is that semigroups appear naturally in almost all mathematical contexts, and information about semigroup(s) related to a mathematical object yields some information about the object itself.

The monograph Howie (1976) is an excellent introduction to semigroup theory. Parts of this theory which we will need in this text can be found in Appendix A, while in this section we content ourselves with listing some important examples of semigroups. The main objective in doing this is to introduce specific types of semigroups that we will be dealing with throughout the text. However, we hope that the examples will also illustrate different mathematical contexts in which semigroups appear.

Transformation semigroups

The trivial fact that the composition of functions is associative gives rise to one of the most important families of semigroups—transformation semigroups. For a set X, the set of all mappings $X \longrightarrow X$ is denoted by T_X, and called the *full transformation semigroup.* If $X = \{1, \ldots, n\}$ then we write T_n for T_X. The importance of the full transformation semigroups lies in the fact that every semigroup is isomorphic to a subsemigroup of T_X. The proof of this is trivial, and is based on the fact that any semigroup S acts faithfully by postmultiplication on S with an identity adjoined; the corresponding representation is usually called the *right regular representation.* This parallels the Cayley theorem for groups which asserts that every group is isomorphic to a subgroup of a suitable symmetric group

$\mathcal{S}_X$. In the following chapters we shall encounter various important subsemigroups of T_n.

Some generalisations of the full transformation semigroup are:

- the *partial transformation semigroup* PT_X, consisting of all partial mappings on X;
- the *symmetric inverse semigroup* I_X, consisting of all one-one partial mappings on X;
- the semigroup B_X of all binary relations on X.

Richer algebraic structures as semigroups

Probably the simplest examples of semigroups are additive and multiplicative semigroups of various number sets. Also, as we already mentioned, every group is a semigroup. There is a strong link between group theory and semigroup theory. Group theory, being an older discipline, has a larger body of results and methods. Generalising these results for semigroups is a widespread direction of research in semigroup theory, and we shall frequently embark on it. For instance in Chapters 6 (Section 7), 7, 8, 9 we shall show how Reidemeister—Schreier results for groups (see Magnus, Karrass and Solitar (1966)) can be generalised to semigroups.

On the other hand, the fact that groups are much better understood than semigroups justifies attempts to describe semigroups in terms of groups. Almost all structure theories for various classes of semigroups have this idea in common. We shall exploit this idea in the later chapters, where we shall describe the structure of various finitely presented semigroups by examining their subgroups.

Slightly more surprisingly, methods of semigroup theory are becoming increasingly relevant for some aspects of group theory. Examples of this link can be found in Baumslag et al. (1991), Epstein et al. (1992) and Sims (1994), where the connection between semigroups, formal languages and rewriting systems has been exploited for investigations about group presentations.

Semigroup theory is also strongly related to ring theory. The theory of ideals of semigroups (the beginnings of which can be found in Appendix A) is directly influenced by the corresponding theory for rings. On the other hand the family of so called semigroup rings is an important and interesting family of rings; see Okninski (1990).

Endomorphisms of mathematical structures

Just as the set of all automorphisms of a mathematical structure forms a group, so the set of all endomorphisms of such a structure forms a semigroup—the *endomorphism semigroup* of the structure. Endomorphism semigroups are the main link between semigroup theory and other branches of mathematics, and are very common objects of investigation in semigroup theory. The best understood ones are

- endomorphisms of vector spaces over fields; see Dawlings (1980), (1982);
- endomorphisms of linearly ordered sets; see Aĭzenštat (1962), Howie (1971), Howie and Schein (1973), Schein (1975/76) and Gomes and Howie (1990);
- endomorphisms of boolean algebras; see Magill (1970) and Howie and Schein (1985);
- endomorphisms of independence algebras; see Gomes and Howie (1995) and Fountain and Lewin (1992), (1995).

Note that the full transformation semigroup T_X is also an example of an endomorphism semigroup: it is the endomorphism semigroup of the set X with no operations defined on it, or, alternatively, of the set X with one trivial unary operation (i.e. $x \mapsto x$).

Free semigroups

For a set A (often called alphabet), we denote by A^* the set of all (finite) words over A, and by A^+ the set of all non-empty words in A^*. Thus $A^* = A^+ \cup \{\epsilon\}$, where ϵ denotes the empty word. The sets A^* and A^+ can be made into semigroups if we define multiplication of words to be the concatenation.

The semigroup A^+ is obviously generated by A. Moreover, it is *freely generated* by this set in the following sense:

Proposition 1.1. *Let A be a set, and let S be any semigroup. Then any mapping $\phi : A \longrightarrow S$ can be extended in a unique way to a homomorphism $\overline{\phi} : A^+ \longrightarrow S$, and A^+ is determined up to isomorphism by these properties.* ■

We say that A^+ is the free semigroup on A. Proposition 1.1 has the following important consequence:

Proposition 1.2. *Every (finitely generated) semigroup S is a homomorphic image of a (finitely generated) free semigroup.* ■

The proofs for both Proposition 1.1 and Proposition 1.2 are standard, and can be found in Lallement (1979).

In a similar way, A^* is a free object in the category of all *monoids* (i.e. algebraic structures with one associative binary operation and one constant acting as an identity). If we adjoin a zero to A^+ or to A^*, we obtain a *free semigroup with zero* A_0^+ and a *free monoid with zero* A_0^*, with obvious meanings. Results analogous to Proposition 1.1 and Proposition 1.2 hold for these free objects as well, and are straightforward to prove.

So far we have seen various examples of semigroups. It is clear, however, that in itself the listing of specific examples is not a particularly hopeful method for studying wider classes of semigroups. There are three main approaches that one might adopt in an attempt at such a study. The first is to try and *construct* new semigroups out of the known ones. This approach is usually called *structure theory.* The other possibility is to investigate subsemigroups of full transformation semigroups, since every semigroup is isomorphic to such a semigroup. This is particularly fruitful in the case of finite semigroups, since in this case it is possible to use various computational techniques; see Lallement and McFadden (1990) and Howie and McFadden (1990). Finally, one can consider homomorphic images of free semigroups, again because of the fact that every semigroup is isomorphic to such a homomorphic image (Proposition 1.2). In this thesis we concentrate on this last approach, and in the next section we shall see that semigroup presentations are a very natural tool in this type of investigation.

2. Presentations

Let A be an alphabet. A *semigroup presentation* is an ordered pair $\langle A \mid \mathfrak{R}\rangle$, where $\mathfrak{R} \subseteq A^+ \times A^+$. An element a of A is called a *generating symbol*, while an element (u,v) of $\mathfrak{R}$ is called a *defining relation*, and is usually written as $u = v$. Also if $A = \{a_1, \dots, a_m\}$ and $\mathfrak{R} = \{u_1 = v_1, \dots, u_n = v_n\}$, we write $\langle a_1, \dots, a_m \mid u_1 = v_1, \dots, u_n = v_n\rangle$ for $\langle A \mid \mathfrak{R}\rangle$. It is important to note at this stage that a presentation is a purely syntactical device—it is just a sequence of symbols.

The *semigroup defined by a presentation* $\langle A \mid \mathfrak{R}\rangle$ is A^+/ρ, where ρ is the smallest congruence on A^+ containing $\mathfrak{R}$. More generally, a semigroup S is said to be defined by the presentation $\langle A \mid \mathfrak{R}\rangle$ if $S \cong A^+/\rho$. Thus, elements of S are in one-one correspondence with the congruence classes of words from A^+, or, to put it differently, each word from A^+ *represents* an element of S. It is customary, although sometimes confusing, to identify words and elements they represent. To lessen the likelihood of confusion, for $w_1, w_2 \in A^+$ we write $w_1 \equiv w_2$ if w_1 and w_2 are identical words, and $w_1 = w_2$ if they represent the same element of S (i.e. if $(w_1, w_2) \in \rho$). Thus, for example, if $A = \{a, b\}$ and $\mathfrak{R}$ is $\{ab = ba\}$, then $aba = a^2b$ but $aba \not\equiv a^2b$.

Let T be any semigroup, let B be a generating set for T, and let $\phi : A \longrightarrow B$ be an onto mapping. By Proposition 1.1 the mapping ϕ can be extended in a unique way to an epimorphism $\overline{\phi} : A^+ \longrightarrow T$. The semigroup T is said to *satisfy relations* $\mathfrak{R}$ if for each $(u,v) \in \mathfrak{R}$ we have $u\overline{\phi} = v\overline{\phi}$, or, in other words, if $\mathfrak{R} \subseteq \ker(\overline{\phi})$. Note that by definition the semigroup S defined by $\langle A \mid \mathfrak{R}\rangle$ satisfies relations $\mathfrak{R}$. Actually, from the minimality of ρ, we obtain the maximality of S among all semigroups satisfying $\mathfrak{R}$, in the following sense:

Proposition 2.1. *Let $\langle A \mid \mathfrak{R}\rangle$ be a presentation, let S be the semigroup defined by this presentation, and let T be a semigroup satisfying $\mathfrak{R}$. Then T is a natural homomorphic image of S.* ■

Let $w_1, w_2 \in A^+$ be two words. We say that w_2 *is obtained from* w_1 *by one application of one relation from* $\mathfrak{R}$ if there exist $\alpha, \beta \in A^*$ and $(u,v) \in \mathfrak{R}$ such that $w_1 \equiv \alpha u \beta$ and $w_2 \equiv \alpha v \beta$. We say that w_2 *can be deduced from* w_1 if there is a sequence $w_1 \equiv \alpha_1, \alpha_2, \ldots, \alpha_{k-1}, \alpha_k \equiv w_2$ of words from A^+ such that α_{i+1} is obtained from α_i by one application of one relation from $\mathfrak{R}$; alternatively, we say that $w_1 = w_2$ is a *consequence* of $\mathfrak{R}$. A straightforward modification of Proposition 1.5.9 in Howie (1976) gives the following

Proposition 2.2. *Let* $\langle A \mid \mathfrak{R} \rangle$ *be a presentation, let* S *be the semigroup defined by it, and let* $w_1, w_2 \in A^+$. *Then* $w_1 = w_2$ *in* S *if and only if* w_2 *can be deduced from* w_1. ■

Actually, a stronger result holds:

Proposition 2.3. *Let* S *be a semigroup generated by a set* A, *and let* $\mathfrak{R} \subseteq A^+ \times A^+$. *Then* $\langle A \mid \mathfrak{R} \rangle$ *is a presentation for* S *if and only if the following two conditions are satisfied:*

(i) S *satisfies all the relations from* $\mathfrak{R}$; *and*

(ii) if $u, v \in A^+$ *are any two words such that* S *satisfies the relation* $u = v$, *then* $u = v$ *is a consequence of* $\mathfrak{R}$.

PROOF. ($\Rightarrow$) If $\langle A \mid \mathfrak{R} \rangle$ is a presentation for S, then S satisfies all the relations from $\mathfrak{R}$ by the definition, while (ii) follows from Proposition 2.2.

($\Leftarrow$) Let $\phi : A^+ \longrightarrow S$ be the epimorphism extending the identity mapping $\mathrm{id} : A \longrightarrow A$, and let η be the smallest congruence on A^+ containing $\mathfrak{R}$. Since S satisfies all the relations from $\mathfrak{R}$ we have $\mathfrak{R} \subseteq \ker\phi$, and hence $\eta \subseteq \ker\phi$. On the other hand, if $(u,v) \in \ker\phi$, then S satisfies the relation $u = v$, and so $u = v$ is a consequence of $\mathfrak{R}$, which implies $(u,v) \in \eta$ by Proposition 2.2. Therefore, $\eta = \ker\phi$, and hence $S \cong A^+/\ker\phi = A^+/\eta$ is defined by $\langle A \mid \mathfrak{R} \rangle$. ■

As we said before, presentation is a syntactical concept. Sometimes it is fruitful to manipulate with presentations in a syntactical manner, i.e. without referring to the semigroups defined by them. Proposition 2.2 is the key result in this approach.

Now we give some examples of semigroup presentations and semigroups defined by them.

Example 2.4. The smallest congruence on A^+ containing the empty set is the diagonal relation $\{(w,w) \mid w \in A^+\}$, and thus the semigroup defined by the presentation $\langle A \mid \ \rangle$ is the free semigroup A^+. ■

Example 2.5. The presentation $\langle a \mid a^2 = a \rangle$ defines the trivial semigroup; the corresponding congruence is obviously the full congruence on $\{a\}^+$. ■

Example 2.6. The presentation $\langle a \mid a^{n+1} = a \rangle$ defines the cyclic group of order n. Obviously, each word a^k can be transformed by using $a^{n+1} = a$ into a word from $\{a, a^2, \ldots, a^n\}$, and so the semigroup S defined by $\langle a \,|\, a^{n+1} = a \rangle$ has at most n elements. On the other hand, if we note that, modulo n, the exponent of a is an invariant of the relation $a^{n+1} = a$ and recall Proposition 2.2, we see that all $a, a^2, \ldots, a^n$ represent distinct elements of S, and thus S is the cyclic group of order n. More generally, the presentation $\langle a \mid a^{n+r} = a^r \rangle$ defines the monogenic semigroup of order $n + r - 1$ and period n. ■

Example 2.7. Let T be any semigroup. Let $A = \{a_t \,|\, t \in T\}$ be an alphabet, let $\mathfrak{R}$ be the set of all relations of the form $a_x a_y = a_{xy}$, where $x, y \in T$, and let S denote the semigroup defined by $\langle A \mid \mathfrak{R} \rangle$. The semigroup T satisfies relations from $\mathfrak{R}$ by the choice of our relations, and so there is a natural epimorphism $\phi : S \longrightarrow T$, $a_t \mapsto t$, by Proposition 2.2. Let $w_1, w_2 \in A^+$, and assume that $w_1\phi = w_2\phi$. It is clear that there are $x, y \in T$ such that $w_1 = a_x$ and $w_2 = a_y$ hold in S. Thus we have $a_x\phi = a_y\phi$, i.e. $x = y$, so that $a_x = a_y$. Therefore, ϕ is an isomorphism, and T is defined by the presentation $\langle A \mid \mathfrak{R} \rangle$. ■

Example 2.7 in effect asserts that *every* semigroup can be defined by a presentation. Besides, if the semigroup is finite, both the set of generators and the set of defining relations can be chosen to be finite.

One should note the difference between Examples 2.6 and 2.7. To prove that certain words represent different elements in Example 2.6 we used certain (syntactic) invariants of the presentation, while in Example 2.7 we found a semigroup satisfying all the relations, in which the words represented different elements.

If in the definitions at the beginning of this section we replace A^+ by A^*, A_0^+ or A_0^* we obtain notions of *monoid presentations*, *presentations of semigroups with zero* and *presentations of monoids with zero* respectively. Both Propositions 2.1 and 2.2 have straightforward modifications in all these cases.

All these four types of presentations are closely related one to each other. For example, every semigroup presentation is a monoid presentation as well. If S is the semigroup defined by $\langle A \,|\, \mathfrak{R} \rangle$, then the monoid defined by $\langle A \,|\, \mathfrak{R} \rangle$ is S with an identity adjoined to it. If S possesses an identity, represented by a word $e \in A^+$, then $\langle A \mid \mathfrak{R},\ e = 1 \rangle$ is a monoid presentation for S. If M is the monoid defined by a monoid presentation $\langle B \mid \mathfrak{S} \rangle$, then M can be defined as a semigroup by $\langle B, e \,|\, \overline{\mathfrak{S}},\ e^2 = e,\ eb = be = b\ (b \in B) \rangle$, where $\overline{\mathfrak{S}}$ is obtained from $\mathfrak{S}$ by replacing every relation of the form $w = 1$ by the relation $w = e$. Similar connections hold between any other two types of presentations.

Most of the material in this thesis will be presented in terms of semigroup presentations. Sometimes, however, we will use other types, usually without explicitly stating this, but having in mind the above-mentioned connections.

Frequently we shall also encounter another type of presentations—group presentations. We assume the reader's familiarity with the basic concepts and results of combinatorial group theory; see Magnus, Karrass and Solitar (1966). If a group G is defined by a group presentation $\langle A \,|\, \mathfrak{R}\rangle$, then G can be defined by the monoid presentation $\langle A, A^{-1} \mid \mathfrak{R}, aa^{-1} = a^{-1}a = 1 \ \ (a \in A)\rangle$, where $A^{-1} = \{a^{-1} \mid a \in A\}$ is a new alphabet in one-one correspondence to A, and disjoint from A. On the other hand each semigroup presentation (and each monoid presentation) is a group presentation, but the connection between the group and the semigroup defined by the same semigroup presentation is much harder to determine. Chapter 10 will be devoted to this task.

3. Combinatorial semigroup theory

In the previous section we have seen that presentations are a means of defining semigroups as homomorphic images of free semigroups. The main advantage of presentations when compared to other ways of defining semigroups (such as Cayley tables or transformation semigroups) is that they allow us to study a larger class of semigroups, including various infinite semigroups. However, one should note that, although theoretically every semigroup can be defined by a presentation (see Example 2.7), not every semigroup can be defined 'nicely' in such a way. The most plausible class of semigroups for analysis via presentations are so called *finitely presented semigroups*, i.e. semigroups which can be defined by presentations $\langle A \mid \mathfrak{R}\rangle$, where both A and $\mathfrak{R}$ are finite. We have already encountered examples of finitely presented semigroups: finite semigroups (Example 2.7), free semigroups (Example 2.4), and monogenic semigroups (Example 2.6). Less trivially, every finitely generated commutative semigroup is finitely presented; see Rédei (1965) or Clifford and Preston (1967).

It is important to note that the property of being finitely presented does not depend on the choice of a generating set.

Proposition 3.1. *Let S be a semigroup, and let A and B be two finite generating sets for S. If S can be defined by a finite presentation in terms of generators A, then S can be defined by a finite presentation in terms of generators B as well.*

PROOF. Since B is a generating set for S, for each a in A there exists $a\zeta \in B^+$ such that a and $a\zeta$ represent the same element of S. The mapping $a \mapsto a\zeta$ can be extended to a homomorphism $\zeta : A^+ \longrightarrow B^+$ by Proposition 2.1. This homomorphism obviously has the property that w and $w\zeta$ represent the same element of S for each $w \in A^+$. Similarly, for each b in B there exists $b\eta \in A^+$ such that b and $b\eta$ represent the same element of S, and this mapping can be extended to a homomorphism $\eta : B^+ \longrightarrow A^+$ with the property that w and $w\eta$ represent the same element of S for each $w \in B^+$.

Let $\langle A \mid \mathfrak{R}\rangle$ be a finite presentation defining S in terms of the generating set A, and let $\mathfrak{R}\zeta$ denote the set $\{u\zeta = v\zeta \mid (u = v) \in \mathfrak{R}\}$. We shall show that the (finite) presentation

$$\mathfrak{P} = \langle B \mid \mathfrak{R}\zeta,\ b = b\eta\zeta\ (b \in B)\rangle$$

defines S in terms of the generating set B. The semigroup S certainly satisfies the relations from the above presentation because of the definitions of ζ and η. By Proposition 2.3, to finish the proof, it is enough to prove that if a relation $w_1 = w_2$ holds in S then it is a consequence of $\mathfrak{P}$.

Let w_1, w_2 be words from B^+ representing the same element of S. Then the words $w_1\eta$ and $w_2\eta$ belong to A^+ and represent the same element of S. Since $\langle A \mid \mathfrak{R}\rangle$ is a presentation for S, $w_2\eta$ can be obtained from $w_1\eta$ by applying relations from $\mathfrak{R}$. Hence $w_2\eta\zeta$ can be obtained from $w_1\eta\zeta$ by applying relations from $\mathfrak{R}\zeta$. Now suppose that $w_1 \equiv b_1b_2\dots b_k$, where $b_i \in B$, $1 \le i \le k$. Then $w_1\eta\zeta \equiv (b_1\eta\zeta)(b_2\eta\zeta)\dots(b_k\eta\zeta)$, since both η and ζ are homomorphisms, and we see that the relation $w_1 = w_1\eta\zeta$ is a consequence of relations $b = b\eta\zeta$ $(b \in B)$. Similarly, we can obtain the relation $w_2 = w_2\eta\zeta$, and so the relation $w_1 = w_2$ is a consequence of $\mathfrak{P}$ as required. ∎

However, the class of finitely presented semigroups is far from covering all semigroups. First of all, every finitely presented semigroup is finitely generated, but there are even finitely generated semigroups which are not finitely presented, as the next example shows.

Example 3.2. Let S be the semigroup defined by the presentation

$$\mathfrak{P} = \langle a, b \mid ab^ia = aba\ \ (i \in \mathbb{N})\rangle.$$

Suppose that S is finitely presented. Then, by Proposition 3.1, S can be defined by a finite presentation $\langle a, b \mid \mathfrak{R}\rangle$ in terms of the generators a and b. Since the relation $ab^ia = aba$ holds in S for each $i \in \mathbb{N}$, aba can be obtained from ab^ia by applying relations from $\mathfrak{R}$. On the other hand, it is not possible to apply any relation from $\mathfrak{P}$ to the word ab^i. Therefore, the word ab^i does not satisfy any non-trivial relations, in the sense that $ab^i = w$ in S implies $ab^i \equiv w$. Similarly, the word b^ia does not satisfy any non-trivial relations. The conclusion from all this is that for each $i > 1$, $\mathfrak{R}$ must contain a relation whose left-hand side or right hand side is identical to ab^ia, so that $\mathfrak{R}$ is infinite, a contradiction. ∎

The situation is complicated further by various *undecidability results*. Here are the best known of them:

- there exists no algorithm for testing if two presentations define isomorphic semigroups; see Markov (1951);
- there exists no algorithm for deciding if a given presentation defines a finite semigroup; see Markov (1951a);
- there exists a finitely presented semigroup for which there is no algorithm which decides equality of two words in the semigroup; Markov (1947) and Post(1947);.

We say that the *isomorphism problem*, the *finiteness problem* and the *word problem* all are not soluble for semigroup presentations.

On the other hand, the experience of combinatorial group theory, where the same anomalies are present, suggests that despite these problems semigroup presentations might be a powerful tool for handling semigroups, and we hope that the results of this thesis will add some evidence for this view. These results can be grouped into two main types:

1. given a semigroup find a presentation defining it;
2. given a presentation describe the semigroup defined by it.

Actually, we have already encountered both types of results. In Examples 2.4, 2.5, 2.6 and 2.7 we found presentations for some well known semigroups, while in Example 3.2 we defined a semigroup by a presentation, and then went on to prove some properties of this semigroup.

It should be noted that both 1 and 2 are far too general to be considered as mathematical problems; they rather mark two relatively separated areas of research in the theory of semigroup presentations. The purpose of this thesis is to present some new work in both these areas, and link it to the already existing results. Thus, Chapters 4, 5 and 8 will mainly contain results related to 1, while Chapters 10, 11, 12 and 13 will be concerned with finding the structure of various semigroups defined by presentations.

We shall also seek to establish connections between these two approaches to semigroup presentations. Connections are many, but we shall concentrate on two of them: presentations for semigroup constructions, and the use of computational methods. We review existing computational methods for finitely presented semigroups, and present some new ones in Chapter 14. These methods have been frequently used during this research, although the statements and proofs of the final results do not depend on the computational evidence. Nevertheless, we shall give a number of examples which, we hope, will illustrate how computational techniques were used to obtain various results.

When semigroup constructions are concerned, in Chapters 6, 8 and 9 we shall consider various special cases of the following general problem:

3. Suppose that the semigroups S_i, $i \in I$, are defined by presentations $\langle A_i | \mathfrak{R}_i \rangle$, and that a semigroup S is obtained from the family S_i, $i \in I$, by applying a semigroup construction. Find a presentation for S.

The connection between 1 and 3 is clear, since constructions are means for creating new semigroups out of existing ones. We also link 2 and 3 via subsemigroups and Rees matrix semigroups. The main idea here is to investigate a finitely presented semigroup S by investigating some distinguished subsemigroups of S, which have a simpler structure (e.g. are groups, or Rees matrix semigroups). We first find a presentation for such a substructure, and then try to derive information about the substructure from its presentation.

Chapter 2

Generating sets for common semigroups

The first step in finding a presentation for a semigroup S is to find a generating set for S. In this chapter we give generating sets for various semigroups that will be considered in the next two chapters. In doing so it is natural to seek the 'best possible' generating set. Since, unlike for vector spaces, minimal generating sets of a semigroup do not necessarily have the same cardinality, it is reasonable to look for a generating set with the minimal possible number of elements; the cardinality of such a set is usually called the *rank* of S, and is denoted by $\text{rank}(S)$. Ranks of various transformation semigroups have been calculated by Gomes and Howie (1986), (1990), and Howie and McFadden (1990).

After introducing in Section 1 a class of completely 0-simple semigroups, which we call connected completely 0-simple semigroups, we find formulae for the rank of such a semigroup in Sections 2 and 3. Sections 4, 5 and 6 contain various applications of these formulae. In Section 4 we find the minimal rank of a finite completely 0-simple semigroup, given the rank of its Schützenberger group. In Section 5 we show how the results of Howie and McFadden (1990) about the ranks of principal ideals of a full transformation semigroup can be obtained as corollaries of our results about connected 0-simple semigroups, and in Section 6 we obtain similar results for matrix semigroups. Finally, in Section 7 we consider some interesting semigroups which are not finitely generated.

The results of Sections 1 to 5 have been published in Ruškuc (1994), and the results of Section 6 will appear in Ruškuc (1995). The results of Section 7 appear here for the first time.

1. Connected completely 0-simple semigroups

We shall introduce the class of connected completely 0-simple semigroups by a number of equivalent conditions (Theorem 1.2). All the background facts about completely 0-simple semigroups that we need can be found in Appendix A. A more complete introduction to these semigroups can be found in Howie (1976).

We use the notation for completely 0-simple semigroups introduced in Section 2 of Appendix A. Thus, for a completely 0-simple semigroup S, R_i', $i \in I$, denote the 0-minimal right ideals of S, and L_λ', $\lambda \in \Lambda$, denote 0-minimal left ideals of S. The semigroup S has two $\mathcal{D}$-classes $S - \{0\}$ and $\{0\}$. The non-zero $\mathcal{R}$-classes of S are $R_i = R_i' - \{0\}$, $i \in I$, the non-zero $\mathcal{L}$-classes of S are $L_\lambda = L_\lambda' - \{0\}$, $\lambda \in \Lambda$, and we have

$$S - \{0\} = \bigcup_{i \in I} R_i = \bigcup_{\lambda \in \Lambda} L_\lambda.$$

When I is finite we assume that $I = \{1, \ldots, m\}$. Similarly, if Λ is finite, we write $\Lambda = \{1, \ldots n\}$. Nevertheless, sometimes we will need to assume that $I \cap \Lambda = \emptyset$ and in that case we will distinguish elements of I and Λ by putting appropriate subscripts, e.g. $1_I \in I$, $1_\Lambda \in \Lambda$ etc. Any two distinct R_i and R_j, as well as any two distinct L_λ and L_μ, are disjoint. The intersection of R_i and L_λ is denoted by $H_{i\lambda}$ ($i \in I$, $\lambda \in \Lambda$). Each $H_{i\lambda}$ ($i \in I$, $\lambda \in \Lambda$) is either a group or it is a set with the zero multiplication by Proposition A.2.1 (vi). If $H_{i\lambda}$ is a group, then $e_{i\lambda}$ will denote its identity. All the (non-zero) group $\mathcal{H}$-classes are isomorphic. The group to which they are all isomorphic is called the *Schützenberger group* of S. When considering completely simple semigroups, we sometimes assume (without explicitly stating this) that such a semigroup has a zero adjoined to it, making it into a completely 0-simple semigroup. $E(S)$ and $F(S)$ denote the set of all idempotents of S, and the subsemigroup generated by that set respectively, for an arbitrary semigroup S.

In what follows we shall frequently refer to the following properties of completely 0-simple semigroups.

Lemma 1.1. *Let S be a completely 0-simple semigroup.*

(i) If $x \in H_{i\lambda}$ and $y \in H_{j\mu}$ then $xy \neq 0$ if and only if $H_{j\lambda}$ is a group, in which case $xy \in H_{i\mu}$.

(ii) If $xa \neq 0$ for some $x \in L_\lambda$ then $ya \neq 0$ for all $y \in L_\lambda$. Dually, if $ax \neq 0$ for some $x \in R_i$ then $ay \neq 0$ for all $y \in R_i$.

(iii) $a_1 a_2 \ldots a_t = 0$ if and only if at least one of products $a_1a_2, a_2a_3, a_{t-1}a_t$ is equal to zero.

(iv) If $p \in F(S)$ then every inverse of p also belongs to $F(S)$.

PROOF. (i), (ii) and (iii) can be easily deduced from Proposition A.2.1 (ix) and (x). For (iv) see Fitz-Gerald (1972). ∎

For an arbitrary completely 0-simple semigroup S we define a graph $\Gamma(S)$ as follows. The set of vertices of $\Gamma(S)$ is

$$\{(i, \lambda) \in I \times \Lambda \mid H_{i\lambda} \text{ is a group}\}.$$

Two vertices (i, λ) and (j, μ) are adjacent if and only if $i = j$ or $\lambda = \mu$.

Theorem 1.2. *The following conditions are equivalent for any completely 0-simple semigroup S:*

(i) $\Gamma(S)$ *is a connected graph.*
(ii) $L_\lambda F(S) R_i = S$ *for every* $i \in I$ *and every* $\lambda \in \Lambda$.
(iii) $F(S) \cap H_{i\lambda} \neq \emptyset$ *for every* $i \in I$ *and every* $\lambda \in \Lambda$.
(iv) For every $i, j \in I$ *and every* $\lambda, \mu \in \Lambda$ *there exist* $p(i, \lambda, j, \mu)$, $q(i, \lambda, j, \mu) \in F(S)$ *such that the mapping* $\phi(i, \lambda, j, \mu) : H_{i\lambda} \longrightarrow H_{j\mu}$ *defined by*

$$\phi(i, \lambda, j, \mu)(x) = p(i, \lambda, j, \mu) x q(i, \lambda, j, \mu) \tag{1}$$

is a bijection.

If both $H_{i\lambda}$ *and* $H_{j\mu}$ *are groups, then the elements* $p(i, \lambda, j, \mu)$, $q(i, \lambda, j, \mu)$ *can be chosen so that*

$$\phi(i, \lambda, j, \mu)^{-1} = \phi(j, \mu, i, \lambda)$$

and $\phi(i, \lambda, j, \mu)$ *is a group isomorphism.*

PROOF. (i)$\Rightarrow$(ii) $L_\lambda F(S) R_i$ is a two-sided ideal, and therefore is equal to $\{0\}$ or S. By Proposition A.2.1 (viii), L_λ contains at least one group $\mathcal{H}$-class, say $H_{j\lambda}$. Similarly, there exists a group $H_{i\mu}$ within R_i. Since $\Gamma(S)$ is connected there exists a path $(j, \lambda) \to (k_1, \nu_1) \to \ldots \to (k_t, \nu_t) \to (i, \mu)$ connecting (j, λ) and (i, μ). Now, by Proposition A.2.1 (ix), we have

$$e_{j\lambda} e_{k_1\nu_1} \neq 0, e_{k_1\nu_1} e_{k_2\nu_2} \neq 0, \ldots, e_{k_t\nu_t} e_{i\mu} \neq 0,$$

because in every case the two factors are either in the same $\mathcal{R}$-class or in the same $\mathcal{L}$-class. By Lemma 1.1(iii) we have $e_{j\lambda} e_{k_1\nu_1} \ldots e_{i\mu} \neq 0$. However, $e_{j\lambda} e_{k_1\nu_1} \ldots e_{i\mu} \in L_\lambda F(S) R_i$, and thus $L_\lambda F(S) R_i \neq \{0\}$.

(ii)$\Rightarrow$(iii) Both R_i and L_λ contain at least one group $\mathcal{H}$-class, say $H_{i\mu}$ and $H_{j\lambda}$. From $L_\mu F(S) R_j = S$ and Lemma 1.1(ii) it follows that there exists $p \in F(S)$ such that $xpy \neq 0$ for every $x \in L_\mu$ and any $y \in R_j$. In particular $e_{i\mu} p e_{j\lambda} \neq 0$, and therefore $e_{i\mu} p e_{j\lambda} \in H_{i\lambda} \cap F(S)$.

(iii)$\Rightarrow$(iv) Each of $R_i, R_j, L_\lambda, L_\mu$ must contain at least one group $\mathcal{H}$-class, say $H_{i\nu}$, $H_{j\pi}$, $H_{k\lambda}$, $H_{l\mu}$ (see Figure 1, in which group $\mathcal{H}$-classes are shaded). By (ii) there exist $p(i, \lambda, j, \mu) \in H_{j\nu} \cap F(S)$ and $q(i, \lambda, j, \mu) \in H_{k\mu} \cap F(S)$. Proposition A.2.1 (xi) and Lemma 1.1(iv) imply the existence of $p(j, \mu, i, \lambda) \in H_{i\pi} \cap F(S)$ and $q(j, \mu, i, \lambda) \in H_{l\lambda} \cap F(S)$ which are inverses of $p(i, \lambda, j, \mu)$ and $q(i, \lambda, j, \mu)$ respectively.

If we now define mappings $\phi(i, \lambda, j, \mu)$ and $\phi(j, \mu, i, \lambda)$ by the rule (1), it is obvious that they map $H_{i\lambda}$ into $H_{j\mu}$ and $H_{j\mu}$ into $H_{i\lambda}$ respectively, and that they are inverses one of each other.

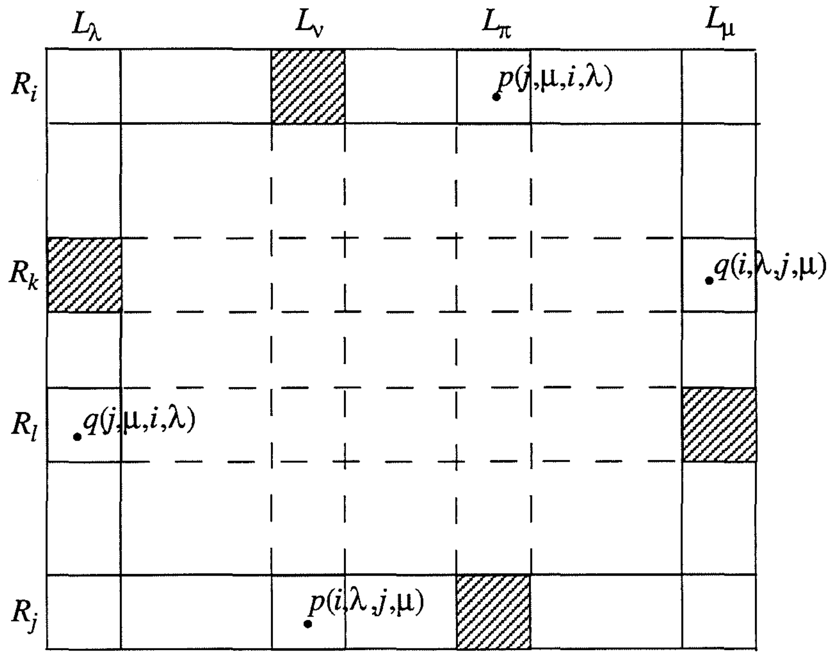


Figure 1.

If both $H_{i\lambda}$ and $H_{j\mu}$ are groups, $p(i,\lambda,j,\mu)$ can be chosen from $H_{j\lambda}$, and then for $q(i,\lambda,j,\mu)$ we can choose its inverse from $H_{i\mu}$. By Green's theorem $\phi(i,\lambda,j,\mu)$ is a group isomorphism from $H_{i\lambda}$ onto $H_{j\mu}$.

(iv)$\Rightarrow$(i) Let (i,λ) and (j,μ) be any two vertices of the graph $\Gamma(S)$. This means that $H_{i\lambda}$ and $H_{j\mu}$ are groups. Let $p(i,\lambda,j,\mu), q(i,\lambda,j,\mu) \in F(S)$ be such that

$$\phi(i,\lambda,j,\mu) : x \mapsto p(i,\lambda,j,\mu)xq(i,\lambda,j,\mu)$$

is a bijection. If $p(i,\lambda,j,\mu) = e_{i_1\lambda_1}e_{i_2\lambda_2}\dots e_{i_s\lambda_s}$ then $i_1 = j$ because $p \in R_j$. Also, all $H_{i_2\lambda_1}, H_{i_3\lambda_2}, \dots, H_{i_s\lambda_{s-1}}, H_{i\lambda_s}$ are groups, because of $p(i,\lambda,j,\mu)H_{i\lambda} \neq \{0\}$ and Lemma 1.1(i). But then

$$\begin{aligned}(j,\mu) &\to (j,\lambda_1) \to (i_2,\lambda_1) \to (i_2,\lambda_2) \to \dots \\ &\to (i_s,\lambda_{s-1}) \to (i_s,\lambda_s) \to (i,\lambda_s) \to (i,\lambda)\end{aligned}$$

is a path connecting (j,μ) and (i,λ) in the graph $\Gamma(S)$. ■

Definition 1.3. If a completely 0-simple semigroup S satisfies any of the (equivalent) conditions of Theorem 1.2 we say that S is connected.

We finish this section by giving examples of both connected and non-connected completely 0-simple semigroups.

Example 1.4. If S is a completely simple semigroup, then all $H_{i\lambda}$ are groups; see Proposition A.2.2 (iv). Therefore $e_{i\lambda} \in F(S) \cap H_{\lambda}$, so that S is connected by Theorem 1.2. The set of vertices of the graph $\Gamma(S)$ is the whole set $I \times \Lambda$. ■

Example 1.5. The second principal factor P_2 of the full transformation semigroup T_4 is a completely 0-simple semigroup. The egg-box picture of this completely 0-simple semigroup is given in Figure 2; see also Table 3 in Clifford and Preston (1961). From this egg-box picture it is easy to compute the graph $\Gamma(P_2)$. Figure 3 represents this graph with loops omitted. In particular, P_2 is connected. This is actually a special case of a more general situation: any principal factor of any finite full transformation semigroup is a connected 0-simple semigroup; see Lemma 5.5. ■

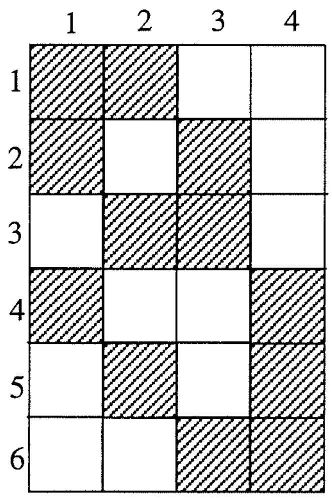


Figure 2.

Example 1.6. A *Brandt semigroup* is defined as a completely 0-simple inverse semigroup; see Petrich (1984). The product of any two distinct idempotents of a Brandt semigroup is zero (Lemma II.3.2 in Petrich (1984)), and therefore a Brandt semigroup is not connected, unless it is a group with zero adjoined. If S is a Brandt semigroup with $|I|$ non-zero $\mathcal{R}$-classes, then $\Gamma(S)$ has $|I|$ vertices and all the edges are loops. ■

2. Ranks of connected completely 0-simple semigroups

The main purpose of this section is to establish a formula for the rank of a finite connected 0-simple semigroup. First, however, we find a generating set for a general completely 0-simple semigroup, which we will use in Chapter 6.

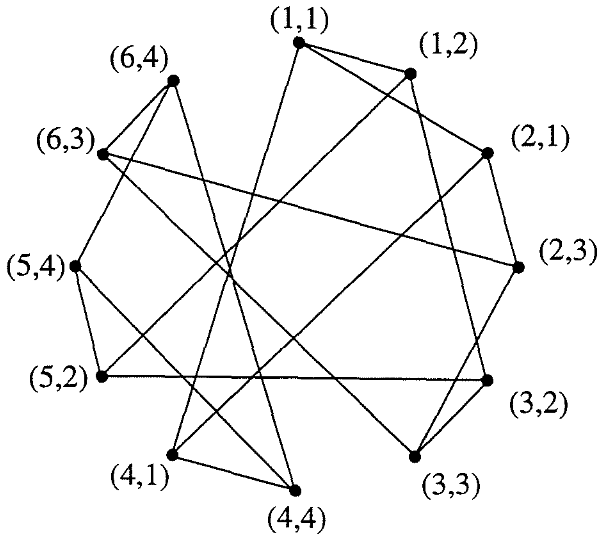


Figure 3.

Theorem 2.1. *Let S be a completely 0-simple semigroup, and let $i_0 \in I$, $\lambda_0 \in \Lambda$ be such that $H_{i_0\lambda_0}$ is a group. If $A \subseteq H_{i_0\lambda_0}$ generates $H_{i_0\lambda_0}$ as a semigroup, and if $b_\lambda \in H_{i_0\lambda}$, $\lambda \in \Lambda - \{\lambda_0\}$, and $c_i \in H_{i\lambda_0}$, $i \in I - \{i_0\}$, are arbitrary, then the set*

$$X = A \cup \{b_\lambda \mid \lambda_0 \neq \lambda \in \Lambda\} \cup \{c_i \mid i_0 \neq i \in I\} \cup \{0\}$$

generates S.

In order to prove Theorem 2.1 we need the following

Lemma 2.2. *Let S be a completely 0-simple semigroup, let T be a subsemigroup of S, and assume that $H_{i_0\lambda_0}$ is a group. If $0 \in T$, $H_{i_0\lambda_0} \subseteq T$ and $T \cap H_{i\lambda} \neq \emptyset$ for all $i \in I$, $\lambda \in \Lambda$, then $T = S$.*

Proof. Let $i \in I$, $\lambda \in \Lambda$ be arbitrary. For $x \in T \cap H_{i_0\lambda}$ we have $H_{i_0\lambda_0}x = H_{i_0\lambda}$, since $H_{i_0\lambda_0}$ is a group (Proposition A.2.1 (x)), so that $H_{i_0\lambda} \subseteq T$. A similar argument shows that $H_{i\lambda_0} \subseteq T$. Now $H_{i\lambda_0}H_{i_0\lambda} \neq \{0\}$, since $H_{i_0\lambda_0}$ is a group (Lemma 1.1); hence $H_{i\lambda} = H_{i\lambda_0}H_{i_0\lambda} \subseteq T$ by Proposition A.2.1 (x). Finally, we have

$$S = \{0\} \cup (\bigcup_{i\in I, \lambda\in\Lambda} H_{i\lambda}) \subseteq T,$$

which completes the proof. ■

Proof of Theorem 2.1. Since $H_{i_0\lambda_0}$ is a group, we have $c_ib_\lambda \in H_{i\lambda}$, $i \in I$, $\lambda \in \Lambda$, by Lemma 1.1(i). Therefore $\langle X\rangle \cap H_{i\lambda} \neq \emptyset$ for all $i \in I$ and all $\lambda \in \Lambda$, and the theorem follows from Lemma 2.2. ■

As a corollary we obtain an upper bound and a lower bound for the rank of an arbitrary completely 0-simple semigroup.

Corollary 2.3. *If S is a completely 0-simple semigroup, then*

$$\max(|I|, |\Lambda|) \leq \mathrm{rank}(S) \leq \mathrm{rank}(G) + |I| + |\Lambda| - 1,$$

where G is the Schützenberger group of S.

PROOF. The first inequality has been proved in Howie and McFadden (1990), although they state it with less generality. We prove it here again for the sake of completeness.

First note that, if $a \in R_i$ and $b \in S$, then either $ab \in R_i$ or $ab = 0$ by Lemma 1.1(i). Therefore, each generating set A of S must have a non-empty intersection with each non-zero $\mathcal{R}$-class, and thus has at least $|I|$ elements. Similarly, $|A| \geq |\Lambda|$, and hence $|A| \geq \max(|I|, |\Lambda|)$, as required.

The second inequality is a direct consequence of Theorem 2.1. ∎

The following two examples show that both bounds can be attained.

Example 2.4. Let S be the five element semigroup defined by the following multiplication table:

	0	a	b	c	d
0	0	0	0	0	0
a	0	a	b	0	0
b	0	a	b	a	b
c	0	c	d	0	0
d	0	c	d	c	d

Alternatively, S can be considered as the semigroup consisting of the following five matrices

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

It is easy to see that S is 0-simple, and thus completely 0-simple, since it is finite. It has two non-zero $\mathcal{R}$-classes $R_1 = \{a, b\}$, $R_2 = \{c, d\}$, as well as two non-zero $\mathcal{L}$-classes $L_1 = \{a, c\}$ and $L_2 = \{b, d\}$. S obviously is not monogenic, but it is generated by $\{b, c\}$, so that

$$\mathrm{rank}(S) = 2 = \max(|I|, |\Lambda|). \ \blacksquare$$

Example 2.5. Let G be a group, and let S be G with a zero adjoined. S is obviously a completely 0-simple semigroup with $|I| = |\Lambda| = 1$. Every generating set for S has the form $A \cup \{0\}$, where A is a (semigroup) generating set for G, so that

$$\mathrm{rank}(S) = \mathrm{rank}(G) + 1 = \mathrm{rank}(G) + |I| + |\Lambda| - 1. \ \blacksquare$$

Now we continue to work towards an exact formula for the rank of a finite connected completely 0-simple semigroup, and first we establish some technical lemmas. In all of them we assume the notation for completely 0-simple semigroups and connected completely 0-simple semigroups from Section 1. In particular, if S is a connected completely 0-simple semigroup, we assume that the elements $p(i,\lambda,j,\mu)$, $q(i,\lambda,j,\mu)$, as well as the mappings $\phi(i,\lambda,j,\mu)$ are fixed in accord with Theorem 1.2.

Lemma 2.6. *Let S be a connected completely 0-simple semigroup. Then for all $i,j \in I$, $\lambda,\mu \in \Lambda$, and all $a \in H_{i\lambda}$,*

$$a \in F(S)[\phi(i,\lambda,j,\mu)(a)]F(S).$$

PROOF. Since $\phi(i,\lambda,j,\mu)^{-1} = \phi(j,\mu,i,\lambda)$ we have

$$\begin{aligned} a &= \phi(j,\mu,i,\lambda)(\phi(i,\lambda,j,\mu)(a)) \\ &= p(j,\mu,i,\lambda)[\phi(i,\lambda,j,\mu)(a)]q(j,\mu,i,\lambda) \\ &\in F(S)[\phi(i,\lambda,j,\mu)(a)]F(S), \end{aligned}$$

as required. ∎

Lemma 2.7. *Let S be a completely 0-simple semigroup. If $H_{i\lambda}$ is a group then*

$$e_{i\lambda}F(S)e_{i\lambda} - \{0\} = H_{i\lambda} \cap F(S).$$

PROOF. ($\supseteq$) If $p \in H_{i\lambda} \cap F(S)$ then $p = e_{i\lambda}pe_{i\lambda}$ (since $e_{i\lambda}$ is the identity of the group $H_{i\lambda}$) and therefore p belongs to $e_{i\lambda}F(S)e_{i\lambda} - \{0\}$.

($\subseteq$) This inclusion follows from Lemma 1.1(i). ∎

Lemma 2.8. *Let S be a connected completely 0-simple semigroup, let $A = \{a_1, \ldots, a_r\} \subseteq S$ with $a_j \in H_{i_j\lambda_j}$, $j = 1,\ldots,r$, and let $H_{i\lambda}$ be a group. If we write*

$$B = \{\phi(i_1,\lambda_1,i,\lambda)(a_1),\ldots,\phi(i_r,\lambda_r,i,\lambda)(a_r)\} \subseteq H_{i\lambda}$$

then

$$\langle F(S) \cup A\rangle \cap H_{i\lambda} = \langle (F(S) \cap H_{i\lambda}) \cup B\rangle.$$

PROOF. ($\supseteq$) From $B \subseteq H_{i\lambda}$ and $F(S) \cap H_{i\lambda} \subseteq H_{i\lambda}$ it follows that

$$\langle (F(S) \cap H_{i\lambda}) \cup B\rangle \subseteq H_{i\lambda}.$$

On the other hand, we have

$$F(S) \cap H_{i\lambda} \subseteq \langle F(S) \cup A\rangle,$$

which together with

$$B \subseteq F(S)AF(S) \subseteq \langle F(S) \cup A \rangle$$

gives

$$\langle (F(S) \cap H_{i\lambda}) \cup B \rangle \subseteq \langle F(S) \cup A \rangle.$$

($\subseteq$) By Lemma 2.6 we have

$$A \subseteq F(S)BF(S) \subseteq \langle F(S) \cup B \rangle,$$

and therefore

$$\begin{array}{rcll} \langle F(S) \cup A \rangle \cap H_{i\lambda} & \subseteq & \langle F(S) \cup \langle F(S) \cup B \rangle \rangle \cap H_{i\lambda} & \\ & = & \langle F(S) \cup B \rangle \cap H_{i\lambda} & \\ & = & e_{i\lambda}(\langle F(S) \cup B \rangle \cap H_{i\lambda})e_{i\lambda} & \\ & \subseteq & e_{i\lambda}\langle F(S) \cup B \rangle e_{i\lambda} \cap H_{i\lambda} & \\ & \subseteq & \langle e_{i\lambda}F(S)e_{i\lambda} \cup B \rangle \cap H_{i\lambda} & (B \subseteq H_{i\lambda}) \\ & = & \langle (e_{i\lambda}F(S)e_{i\lambda} - \{0\}) \cup B \rangle \cap H_{i\lambda} & (0 \notin H_{i\lambda}) \\ & = & \langle (H_{i\lambda} \cap F(S)) \cup B \rangle \cap H_{i\lambda} & (\text{Lemma 2.7}) \\ & = & \langle (H_{i\lambda} \cap F(S)) \cup B \rangle, & \end{array}$$

since $\langle (H_{i\lambda} \cap F(S)) \cup B \rangle \subseteq H_{i\lambda}$. ■

Lemma 2.9. *Let S be a connected completely 0-simple semigroup, let $i, j \in I$ and let $\lambda, \mu \in \Lambda$. If both $H_{i\lambda}$ and $H_{j\mu}$ are groups then $\phi(i, \lambda, j, \mu)$ maps $F(S) \cap H_{i\lambda}$ onto $F(S) \cap H_{j\mu}$ isomorphically.*

Proof. Certainly, $\phi(i, \lambda, j, \mu)$ is a homomorphism by Theorem 1.2. It is clear that

$$\phi(i, \lambda, j, \mu)(F(S) \cap H_{i\lambda}) \subseteq F(S) \cap H_{j\mu},$$

since both $p(i, \lambda, j, \mu)$ and $q(i, \lambda, j, \mu)$ are products of idempotents. The converse inclusion follows from

$$p = \phi(i, \lambda, j, \mu)(\phi(j, \mu, i, \lambda)(p)),$$

and the result follows. ■

Definition 2.10. Let S be a semigroup and let T be a subsemigroup of S. The *rank of S modulo T*, denoted by $\operatorname{rank}(S : T)$, is the least possible cardinality of all sets $A \subseteq S$ for which $\langle A \cup T \rangle = S$.

Lemma 2.11. *Let S be a connected completely 0-simple semigroup, let $i \in I$ and let $\lambda \in \Lambda$. If $H_{i\lambda}$ is a group then*

$$\operatorname{rank}(S) \geq \operatorname{rank}(H_{i\lambda} : H_{i\lambda} \cap F(S)).$$

PROOF. Let A be any generating set of S, and define $B \subseteq H_{i\lambda}$ as in Lemma 2.8. Then Lemma 2.8 implies that

$$H_{i\lambda} = S \cap H_{i\lambda} = \langle A \rangle \cap H_{i\lambda} = \langle F(S) \cup A \rangle \cap H_{i\lambda} = \langle (F(S) \cap H_{i\lambda}) \cup B \rangle,$$

and therefore

$$|A| \geq |B| \geq \operatorname{rank}(H_{i\lambda} : F(S) \cap H_{i\lambda}),$$

as required. ■

Remark 2.12. Although $H_{i\lambda}$ is a group, $\operatorname{rank}(H_{i\lambda} : H_{i\lambda} \cap F(S))$ retains its semigroup meaning as in Definition 2.10. In other words, it is the minimal cardinal number of all sets A such that $A \cup (H_{i\lambda} \cap F(S))$ generates $H_{i\lambda}$ *as a semigroup.* However, this possible source of confusion is going to disappear in a minute, when we are going to assume that S, and hence $H_{i\lambda}$, is finite: a subset of a finite group G generates G as a group if and only if it generates G as a semigroup.

Lemma 2.13. *Let S be a finite completely 0-simple semigroup, let $H_{i\lambda}$ be a group, and let*

$$t = \max(|I|, |\Lambda|, \operatorname{rank}(H_{i\lambda} : H_{i\lambda} \cap F(S)).$$

Then there exists $A \subseteq S$ with $|A| = t$ such that $S - \{0\} \subseteq \langle A \rangle$.

PROOF. First we prove that there exist $i_1, \ldots, i_t \in I$ and $\lambda_1, \ldots, \lambda_t \in \Lambda$ such that for any choice of $q_k \in H_{i_k\lambda_k}$, $k = 1, \ldots, t$, and any $j \in I$, $\mu \in \Lambda$, the intersection $H_{j\mu} \cap \langle q_1, \ldots, q_t \rangle$ is non-empty. Let $\{(j_1, \mu_1), \ldots, (j_s, \mu_s)\}$ be a maximal set of ordered pairs having the property that all $H_{j_1\mu_1}, \ldots, H_{j_s\mu_s}$ are groups, $j_1, \ldots, j_s$ are all different, and $\mu_1, \ldots, \mu_s$ are all different. It is clear that this set contains at least one pair and that for any group $H_{j\mu}$ either $j \in \{j_1, \ldots, j_s\}$ or $\mu \in \{\mu_1, \ldots, \mu_s\}$. Since $t \geq \max(|I|, |\Lambda|)$ we can choose $i_1, \ldots, i_t \in I$ and $\lambda_1, \ldots, \lambda_t \in \Lambda$ so that the following conditions hold:

$$i_1 = j_s,\ i_2 = j_1,\ \ldots\ , i_s = j_{s-1},\ \{i_{s+1}, \ldots, i_t\} = I - \{i_1, \ldots, i_s\},$$
$$\lambda_1 = \mu_1,\ \lambda_2 = \mu_2,\ \ldots\ , \lambda_s = \mu_s,\ \{\lambda_{s+1}, \ldots, \lambda_t\} = \Lambda - \{\lambda_1, \ldots, \lambda_s\}.$$

Let $q_k \in H_{i_k\lambda_k}$, $k = 1, \ldots, t$, be arbitrary and let $j = i_l \in I$ and $\lambda = \lambda_m \in \Lambda$. If $l \leq s$ and $m \leq s$ the product $q_l q_{l+1} \ldots q_m$ (with the subscripts reduced modulo s) is non-zero because $H_{j_l\mu_l}, \ldots, H_{j_{m-1}\mu_{m-1}}$ are groups, and so $q_l q_{l+1} \cdots q_m \in H_{j\mu} \cap \langle q_1, \ldots, q_t \rangle$. Consider now the case when $l \geq s+1$ and $m \leq s$. The $\mathcal{L}$-class L_{λ_l} contains at least one group $H_{i_k\lambda_l}$. But now $k \leq s$ and $m \leq s$ so that $H_{i_k\lambda_m}$ contains an element q from $\langle q_1, \ldots, q_t \rangle$. The product $q_l q$ is non-zero since $H_{i_k\lambda_l}$ is a group, and so $q_l q \in H_{j\mu} \cap \langle q_1, \ldots, q_t \rangle$. The case when $l \leq s$ and $m \geq s+1$ can be treated similarly. Finally, if $l \geq s+1$ and $m \geq s+1$ we choose any group $H_{i_k\lambda_r}$ with $k, r \leq s$. Then there exist $q_1 \in H_{i_l\lambda_r} \cap \langle q_1, \ldots, q_t \rangle$ and $q_2 \in H_{i_k\lambda_m} \cap \langle q_1, \ldots, q_t \rangle$, so that $q_1 q_2 \in H_{i\lambda} \cap \langle q_1, \ldots, q_t \rangle$.

Since $\operatorname{rank}(H_{i\lambda} : F(S) \cap H_{i\lambda}) \leq t$, there exists a set $B = \{b_1, \ldots, b_t\} \subseteq H_{i\lambda}$ such that

$$\langle (F(S) \cap H_{i\lambda}) \cup B \rangle = H_{i\lambda}. \tag{2}$$

Now choose $a_k \in H_{i_k \lambda_k}$, $k = 1, \ldots, t$, such that

$$\phi(i_k, \lambda_k, i, \lambda)(a_k) = b_k, \quad k = 1, \ldots, t, \tag{3}$$

and denote $\{a_1, \ldots, a_t\}$ by A. The choice of $i_1, \ldots, i_t$ and $\lambda_1, \ldots, \lambda_t$ gives

$$\langle A \rangle \cap H_{j\mu} \neq \emptyset, \quad \text{for all } j \in I,\ \mu \in \Lambda. \tag{4}$$

Since S is finite so are all non-zero group $\mathcal{H}$ classes, and (4) implies $F(S) \subseteq \langle A \rangle$, which, together with (3), gives $B \subseteq \langle A \rangle$. Now $H_{i\lambda} \subseteq \langle A \rangle$ by (2), and the result follows from (4) and Lemma 2.2. ■

Now we are in the position to prove the main result of this section.

Theorem 2.14. *Let S be a finite connected 0-simple semigroup, let $\{R_i \mid i \in I\}$ and $\{L_\lambda \mid \lambda \in \Lambda\}$ be the sets of all non-zero $\mathcal{R}$- and $\mathcal{L}$-classes respectively, and let $H_{i\lambda} = R_i \cap L_\lambda$ be any non-zero $\mathcal{H}$-class which is a group.*

(i) If S has divisors of zero then

$$\operatorname{rank}(S) = \max(|I|, |\Lambda|, \operatorname{rank}(H_{i\lambda} : H_{i\lambda} \cap F(S))).$$

(ii) If S has no divisors of zero then

$$\operatorname{rank}(S) = \max(|I|, |\Lambda|, \operatorname{rank}(H_{i\lambda} : H_{i\lambda} \cap F(S))) + 1.$$

PROOF. (i) In this case $S - \{0\} \subseteq \langle A \rangle$ implies $\langle A \rangle = S$ for any $A \subseteq S$, so that result follows from Corollary 2.3 and Lemmas 2.11 and 2.13.

(ii) In this case $S - \{0\}$ is a semigroup of rank $\max(|I|, |\Lambda|, \operatorname{rank}(H_{i\lambda} : H_{i\lambda} \cap F(S)))$ by Corollary 2.3 and Lemmas 2.11 and 2.13, and zero cannot be avoided as a generating element. ■

Remark 2.15. Lemma 2.9 implies that the number $\max(|I|, |\Lambda|, \operatorname{rank}(H_{i\lambda} : H_{i\lambda} \cap F(S)))$ does not depend on the choice of $H_{i\lambda}$.

Remark 2.16. From the proof of Lemma 2.13 it is clear that the condition that S is finite in Theorem 2.14 can be replaced by the weaker condition that the Schützenberger group of S is periodic.

Remark 2.17. If we consider a completely 0-simple semigroup S as a semigroup with zero, so that the zero is automatically included in any subsemigroup, then Theorem 2.14 simply says that

$$\operatorname{rank}(S) = \max(|I|, |\Lambda|, \operatorname{rank}(H_{i\lambda} : H_{i\lambda} \cap F(S))).$$

An important consequence of Theorem 2.14 is a formula for the rank of any finite simple semigroup.

Corollary 2.18. *If S is a finite simple semigroup then*

$$\mathrm{rank}(S) = \max(|I|, |\Lambda|, \mathrm{rank}(H_{i\lambda} : H_{i\lambda} \cap F(S))),$$

where $|I|$ and $|\Lambda|$ are the numbers of minimal right ideals and minimal left ideals respectively, and $H_{i\lambda}$ is the intersection of any two of them.

PROOF. Since S is finite and simple, it is necessarily completely simple. Adjoining a zero to S results in a completely 0-simple semigroup T, which is connected (see Example 1.4), and has no divisors of zero. The rank of such a semigroup is given in Theorem 2.14(ii). The corollary now follows from the observation that each generating set of T necessarily has the form $A \cup \{0\}$, where A is a generating set for S. ■

The formula from Corollary 2.18 takes a particularly pleasing form when $H_{i\lambda} \cap F(S)$ is a normal subgroup of $H_{i\lambda}$:

Theorem 2.19. *If S is a finite simple semigroup such that $H_{i\lambda} \cap F(S) \lhd H_{i\lambda}$ for some $\mathcal{H}$-class $H_{i\lambda}$, then*

$$\mathrm{rank}(S) = \max(|I|, |\Lambda|, \mathrm{rank}(S/\rho(S)))$$

where $|I|$ and $|\Lambda|$ are the number of minimal right ideals and the number of minimal left ideals respectively, and $\rho(S)$ is the least group congruence on S.

Remark 2.20. Every completely simple semigroup is regular, and thus possesses a least group congruence. However, non-regular semigroups do not necessarily have a least group congruence; see Exercise 5.26 in Howie (1976).

In order to prove Theorem 2.19 we have to introduce some more notation and technical results.

Following Higgins (1992) we say that a subsemigroup T of an arbitrary semigroup S is *full* if it contains all the idempotents of S. Also, T is said to be *self-conjugate* if $aTa' \subseteq T$ for any pair of mutually inverse elements a, a' (i.e. $aa'a = a,\ a'aa' = a'$). Since the intersection of any family of full self-conjugate subsemigroups of any semigroup S is again full and self-conjugate, there exists a least such subsemigroup, which we will denote by $C(S)$.

Lemma 2.21. *Let S be a completely 0-simple semigroup, let $H_{i\lambda}$ be a group, and let $N_{i\lambda}$ be the normal subsemigroup of $H_{i\lambda}$ generated by $H_{i\lambda} \cap F(S)$. Then*

$$H_{i\lambda} \cap \langle N_{i\lambda} \cup F(S) \rangle = N_{i\lambda}.$$

PROOF. ($\supseteq$) This inclusion is obvious.

($\subseteq$) Let $a \in H_{i\lambda} \cap \langle N_{i\lambda} \cup F(S) \rangle$ be arbitrary. Then there exist a natural number l and elements $p_1, \ldots, p_l, p_{l+1} \in F(S)$, $n_1, \ldots, n_l \in N_{i\lambda}$ such that $a = p_1 n_1 p_2 n_2 \ldots p_l n_l p_{l+1}$. Since $e_{i\lambda}$ is the identity of $H_{i\lambda}$, and since $a, n_1, \ldots, n_l \in H_{i\lambda}$, we have

$$a = e_{i\lambda} a e_{i\lambda},\ n_1 = e_{i\lambda} n_1 e_{i\lambda}, \ldots, n_l = e_{i\lambda} n_l e_{i\lambda},$$

so that

$$a = e_{i\lambda} a e_{i\lambda} = e_{i\lambda} p_1 e_{i\lambda} n_1 e_{i\lambda} p_2 e_{i\lambda} n_2 e_{i\lambda} \ldots p_l e_{i\lambda} n_l e_{i\lambda} p_{l+1} e_{i\lambda}.$$

Now if we note that

$$e_{i\lambda} p_1 e_{i\lambda}, \ldots, e_{i\lambda} p_{l+1} e_{i\lambda} \in F(S) \cap H_{i\lambda} \subseteq N_{i\lambda},$$

we obtain $a \in N_{i\lambda}$, as required. ■

Lemma 2.22. *Let S be a connected completely 0-simple semigroup, and let $H_{i\lambda}$ and $N_{i\lambda}$ be as in the previous lemma. Then*

$$C(S) = \langle N_{i\lambda} \cup F(S) \rangle.$$

PROOF. Let us denote $\langle N_{i\lambda} \cup F(S) \rangle$ by T. Clearly, T is full. We want to show that T is self-conjugate as well. Let $a, a' \in S$ be any pair of mutually inverse elements and let $t \in T$. If $ata' = 0$ then, clearly, $ata' \in T$. Therefore the nontrivial case is when $ata' \neq 0$. If $a \in H_{j\nu}$ and $a' \in H_{k\mu}$, then both $H_{k\nu}$ and $H_{j\mu}$ are groups by Proposition A.2.2 (vii), and we have

$$ata' = ae_{k\nu} t e_{k\nu} a' = at_1 a' \tag{5}$$

for $t_1 = e_{k\nu} t e_{k\nu} \in H_{k\nu} \cap T$. By Theorem 1.2 we have

$$t_1 = p(j, \mu, k, \nu) t_2 q(j, \mu, k, \nu),$$

for some $t_2 \in H_{j\mu}$; also $t_2 = p(k, \nu, j, \mu) t_1 q(k, \nu, j, \mu) \in T$. Hence

$$\begin{aligned} at_1 a' &= ap(j, \mu, k, \nu) t_2 q(j, \mu, k, \nu) a' \\ &= ap(j, \mu, k, \nu) e_{j\mu} t_2 e_{j\mu} q(j, \mu, k, \nu) a' = bt_2 c, \end{aligned} \tag{6}$$

for $b = ap(j, \mu, k, \nu) e_{j\mu}$, $c = e_{j\mu} q(j, \mu, k, \nu) a'$. Clearly $b, c \in H_{j\mu}$ and

$$bc = ap(j, \mu, k, \nu) e_{j\mu} e_{j\mu} q(j, \mu, k, \nu) a' = ae_{k\nu} a' = aa' = e_{j\mu}$$

(since $\phi(j, \mu, k, \nu)$ is an isomorphism), so that b and c are inverses of each other in the group $H_{j\mu}$. By Theorem 1.2

$$\begin{aligned} bt_2 c &= \phi(i, \lambda, j, \mu)(\phi(j, \mu, i, \lambda)(bt_2 c)) \\ &= p(i, \lambda, j, \mu)[\phi(j, \mu, i, \lambda)(b) \cdot \phi(j, \mu, i, \lambda)(t_2) \cdot \\ &\qquad \cdot \phi(j, \mu, i, \lambda)(c)] q(i, \lambda, j, \mu) \\ &= p(i, \lambda, j, \mu)[\phi(j, \mu, i, \lambda)(b) \cdot p(j, \mu, i, \lambda) t_2 \\ &\qquad q(j, \mu, i, \lambda) \cdot \phi(j, \mu, i, \lambda)(c)] q(i, \lambda, j, \mu). \end{aligned} \tag{7}$$

By Lemma 2.21, the product $p(j,\mu,i,\lambda)t_2q(j,\mu,i,\lambda)$ belongs to $N_{i\lambda}$. The elements $\phi(j,\mu,i,\lambda)(b)$ and $\phi(j,\mu,i,\lambda)(c)$ are inverses of each other in the group $H_{i\lambda}$, since $\phi(j,\mu,i,\lambda)$ is an isomorphism. Therefore

$$\phi(j,\mu,i,\lambda)(b)\cdot p(j,\mu,i,\lambda)t_2q(j,\mu,i,\lambda)\cdot\phi(j,\mu,i,\lambda)(c)\in N_{i\lambda}$$

(because $N_{i\lambda}$ is normal), and thus

$$\begin{aligned}p(i,\lambda,j,\mu)[\phi(j,\mu,i,\lambda)(b)\cdot p(j,\mu,i,\lambda)t_2\\ q(j,\mu,i,\lambda)\cdot\phi(j,\mu,i,\lambda)(c)]q(i,\lambda,j,\mu)\in T.\end{aligned}\tag{8}$$

From (5), (6), (7) and (8) it follows that T is self conjugate.

To prove that T is the least self conjugate subsemigroup of S, let T_1 be any other such semigroup. T_1 contains $F(S)$, and therefore contains $H_{i\lambda}\cap F(S)$. Since it is closed under conjugation, T_1 must contain all conjugates of $H_{i\lambda}\cap F(S)$ in $H_{i\lambda}$, i.e. it must contain $N_{i\lambda}$, and thus $\langle N_{i\lambda}\cup F(S)\rangle = T$. ∎

The following lemma gives a description of the least group congruence $\rho(S)$ on a regular semigroup S. It was proved in Feigenbaum (1979).

Lemma 2.23. *If S is a regular semigroup then*

$$\rho(S)=\{(a,b)\in S\times S\mid xa=by \text{ for some } x,y\in C(S)\}.\ ∎$$

As our next lemma, we prove the following relation among $S,\rho(S)$, $H_{i\lambda}$ and $N_{i\lambda}$ for any completely simple semigroup S:

Lemma 2.24. *If S is a completely simple semigroup then*

$$S/\rho(S)\cong H_{i\lambda}/N_{i\lambda}.$$

PROOF. Let G be the group $S/\rho(S)$ and let $\xi : S\longrightarrow G$ be the natural epimorphism. Since $H_{i\lambda}=e_{i\lambda}Se_{i\lambda}$ it is easy to show that $\xi\restriction_{H_{i\lambda}}$ is also onto. Now we show that $\ker(\xi\restriction_{H_{i\lambda}})=N_{i\lambda}$ and the lemma then follows from the first isomorphism theorem for groups. Clearly, $\xi(F(S))=\{1_G\}$, and therefore $F(S)\cap H_{i\lambda}\subseteq\ker(\xi\restriction_{H_{i\lambda}})$ so that $N_{i\lambda}\subseteq\ker(\xi\restriction_{H_{i\lambda}})$. On the other hand, if $a\in\ker(\xi\restriction_{H_{i\lambda}})$, then $(a,e_{i\lambda})\in\rho(S)$, and by Lemmas 2.22 and 2.23 there exist $x,y\in C(S)=\langle N_{i\lambda}\cup F(S)\rangle$ such that $xa=e_{i\lambda}y$. The standard $\mathcal{L}$-$\mathcal{R}$ argument proves that $x\in R_i$ and $y\in L_\lambda$. But then $xa=e_{i\lambda}y$ implies $(xe_{i\lambda})a=e_{i\lambda}y$. Since $xe_{i\lambda},e_{i\lambda}y\in\langle N_{i\lambda}\cup F(S)\rangle\cap H_{i\lambda}$, Lemma 2.21 implies that $xe_{i\lambda},e_{i\lambda}y\in N_{i\lambda}$, and thus $a\in N_{i\lambda}$. Therefore $\ker(\xi\restriction_{H_{i\lambda}})\subseteq N_{i\lambda}$. ∎

Now we are in the position to prove Theorem 2.19.

PROOF OF THEOREM 2.19. Since $H_{i\lambda}\cap F(S)\lhd H_{i\lambda}$, we have $N_{i\lambda}=H_{i\lambda}\cap F(S)$, and Corollary 2.18 implies

$$\operatorname{rank}(S)=\max(|I|,|\Lambda|,\operatorname{rank}(H_{i\lambda}:N_{i\lambda})).$$

If we note that $\operatorname{rank}(H_{i\lambda}:N_{i\lambda})=\operatorname{rank}(H_{i\lambda}/N_{i\lambda})$ and apply Lemma 2.24, the result follows. ∎

Corollary 2.25. *If the Schützenberger group of a finite simple semigroup S is abelian (or, more generally, hamiltonian) then*

$$\operatorname{rank}(S) = \max(|I|, |\Lambda|, \operatorname{rank}(S/\rho(S))).$$

PROOF. The proof follows from Theorem 2.19 and the fact that every subgroup of an abelian (hamiltonian) group is normal. ■

Remark 2.26. No result similar to the previous ones holds for 0-simple semigroups, even if zero is adjoined, since $S/\rho(S)$ is always trivial in that case.

We finish this section by posing the following

Open Problem 1. Find a formula for the rank of a general (finite) completely 0-simple semigroup.

In addition to Corollary 2.3 and Theorem 2.14, the following result of Gomes and Howie (1987) is highly relevant for this problem.

Proposition 2.27. *If S is a Brandt semigroup with n non-zero $\mathcal{R}$-classes, and if G is the Schützenberger group of S, then*

$$\operatorname{rank}(S) = n + \operatorname{rank}(G) - 1. \ ■$$

3. Rees matrix semigroups

By the Rees—Suschkewitsch Theorem (Proposition A.2.3) every completely 0-simple semigroup is isomorphic to some Rees matrix semigroup $\mathcal{M}^0[G; I, \Lambda; P]$ with the regular Rees matrix $P = (p_{\lambda i})_{\lambda\in\Lambda, i\in I}$. Regularity means that there is no column or row consisting entirely of zeros. Using this representation we are able to obtain a more concrete formulation for Theorem 2.14 and Corollary 2.18.

Let S be a Rees matrix semigroup $\mathcal{M}^0[G; I, \Lambda; P]$, with P regular. Howie (1978) defined a bipartite graph on the set $I \cup \Lambda$ (assuming I and Λ are disjoint) in which $i \in I$ and $\lambda \in \Lambda$ are adjacent if and only if $p_{\lambda i} \neq 0$. We shall denote this graph by $\Gamma(P)$. Two vertices x and y of $\Gamma(P)$ are connected ($x \sim y$) if there exists an oriented path starting in x and ending in y. If $\pi = z_1 \to z_2 \to \ldots \to z_t$ is a path in $\Gamma(P)$ then the value of π is defined as

$$V(\pi) = (z_1, z_2)\phi \cdot (z_2, z_3)\phi \cdot \ldots \cdot (z_{t-1}, z_t)\phi,$$

where

$$(i, \lambda)\phi = p_{\lambda i}^{-1}, \ \ (\lambda, i)\phi = p_{\lambda i} \ \ (i \in I, \ \lambda \in \Lambda).$$

The value of the zero path from z to z is 1, the identity of G. Define P_{xy} to be the set of all paths connecting x and y, and and then let

$$V_{xy} = \{V(\pi) | \pi \in P_{xy}\}.$$

With this notation the following result was proved in Howie (1978):

Lemma 3.1. *The subsemigroup generated by the set of all idempotents of a Rees matrix semigroup* $S = \mathcal{M}^0[G; I, \Lambda; P]$ *is*

$$F(S) = \{(i, a, \lambda) \mid i \sim \lambda, a \in V_{i\lambda}\}. \blacksquare$$

It is possible to express the connectedness of S (Definition 1.3) as a condition on the graph $\Gamma(P)$:

Theorem 3.2. *Let* S *be the Rees matrix semigroup* $\mathcal{M}^0[G; I, \Lambda; P]$. *The semigroup* S *is connected if and only if the graph* $\Gamma(P)$ *is connected.*

PROOF. By Theorem 1.2 the semigroup S is connected if and only if $H_{i\lambda} \cap F(S) \neq \emptyset$ for any $i \in I, \lambda \in \Lambda$. However, this is true if and only if $i \sim \lambda$ in $\Gamma(P)$, by Lemma 3.1. $\blacksquare$

Now we start with establishing the facts needed to prove the main result of this section (Theorem 3.6). In the following lemmas S will denote a finite connected 0-simple Rees matrix semigroup $\mathcal{M}^0[G; I, \Lambda; P]$. Without loss of generality we also assume that $p_{11} \neq 0$, so that H_{11} is a group. (Notice that here p_{11} and H_{11} stand for $p_{1_\Lambda 1_I}$ and $H_{1_I 1_\Lambda}$ respectively, where $1_I \in I$ and $1_\Lambda \in \Lambda$.)

Lemma 3.3. *The mapping* $\psi : H_{11} \longrightarrow G$ *defined by*

$$(1, g, 1)\psi = gp_{11}$$

is a group isomorphism. It maps $H_{11} \cap F(S)$ *onto* $V_{11}p_{11}$.

PROOF. Since

$$((1, g, 1)(1, h, 1))\psi = (1, gp_{11}h, 1)\psi = gp_{11}hp_{11} = (1, g, 1)\psi \cdot (1, h, 1)\psi,$$

ψ is a homomorphism. It is easy to see that it is bijective. The second part is a consequence of Lemma 3.1. $\blacksquare$

Since S is connected, for any $\lambda \in \Lambda$ there exists a path π_λ connecting 1_I and λ. Analogously, for each $i \in I$ there exists a path π_i connecting i and 1_Λ. Certainly, we can choose both π_{1_I} and π_{1_Λ} to be equal to the path $1_I \to 1_\Lambda$. For any $i \in I, \lambda \in \Lambda$ we define

$$q_{\lambda i} = V(\pi_\lambda)p_{\lambda i}V(\pi_i)p_{11}.$$

Note that $q_{\lambda i} = 0$ if and only if $p_{\lambda i} = 0$.

Lemma 3.4. *If*

$$\pi = 1_I \to \lambda_1 \to i_2 \to \lambda_2 \to \ldots \to i_t \to 1_\Lambda$$

is a path from $P_{1_I 1_\Lambda}$, *then*

$$V(\pi)p_{11} = q_{\lambda_1 1_I}^{-1} q_{\lambda_1 i_2} q_{\lambda_2 i_2}^{-1} \cdots q_{1_\Lambda i_t}^{-1}.$$

PROOF. Since π is a path, all $p_{\lambda_1 1_I}, p_{\lambda_1 i_2}, \ldots, p_{1_\Lambda i_t}$ are different from zero, so that all the corresponding q's are different from zero as well.

Now we have

$$\begin{aligned} & q_{\lambda_1 1_I}^{-1} q_{\lambda_1 i_2} q_{\lambda_2 i_2}^{-1} \cdots q_{1_\Lambda i_t}^{-1} \\ = \; & p_{11}^{-1} V(\pi_1)^{-1} p_{\lambda_1 1}^{-1} V(\pi_{\lambda_1})^{-1} V(\pi_{\lambda_1}) p_{\lambda_1 i_2} V(\pi_2) p_{11} p_{11}^{-1} V(\pi_{i_2})^{-1} p_{\lambda_2 i_2}^{-1} \\ & V(\pi_{\lambda_2})^{-1} \ldots V(\pi_{\lambda_{t-1}}) p_{\lambda_{t-1} i_t} V(\pi_{i_t}) p_{11} p_{11}^{-1} V(\pi_{i_t})^{-1} p_{i_t}^{-1} V(\pi_1)^{-1} \\ = \; & p_{\lambda_1 1}^{-1} p_{\lambda_1 i_2} p_{\lambda_2 i_2}^{-1} \cdots p_{\lambda_{t-1} i_t} p_{1 i_t}^{-1} p_{11} = V(\pi) p_{11}. \end{aligned}$$

as required. ■

Lemma 3.5. *$V_{11} p_{11}$ is generated by the set $\{q_{\lambda i} | i \in I, \lambda \in \Lambda\}$.*

PROOF. The lemma is a direct consequence of Lemma 3.4. ■

Now we can prove the main result of this section:

Theorem 3.6. *Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be a finite connected Rees matrix semigroup, with regular matrix P, $p_{11} \neq 0$ and $p_{\mu j} = 0$ for some $j \in I$, $\mu \in \Lambda$. Let also π_i and π_λ ($i \in I, \lambda \in \Lambda$) be paths connecting i and 1_Λ, and 1_I and λ respectively, with $\pi_{1_I} = \pi_{1_\Lambda} = 1_I \to 1_\Lambda$ and let*

$$q_{\lambda i} = V(\pi_\lambda) p_{\lambda i} V(\pi_i) p_{11}.$$

If H is the subgroup of G generated by the set $\{q_{\lambda i} \mid \lambda \in \Lambda, i \in I, \, q_{\lambda i} \neq 0\}$ then

$$\mathrm{rank}(S) = \max(|I|, |\Lambda|, \mathrm{rank}(G : H)).$$

If $p_{\mu j} \neq 0$ for all $j \in I$ and $\mu \in \Lambda$ (the completely simple case) the rank of S is greater by one, while

$$\mathrm{rank}(S - \{0\}) = \max(|I|, |\Lambda|, \mathrm{rank}(G : H)).$$

PROOF. The theorem is a direct consequence of Theorem 2.14 and Lemmas 3.3 and 3.5. ■

We can obtain an even nicer result for the completely simple case. In that case the matrix P can be chosen to have normal form, i.e.

$$p_{\lambda 1} = p_{1i} = 1, \;\; i \in I, \lambda \in \Lambda$$

(Proposition A.2.6), and to have no zero entries.

Theorem 3.7. *Let $S = \mathcal{M}[G; I, \Lambda; P]$ be a finite Rees matrix semigroup with P in normal form. Then*

$$\mathrm{rank}(S) = \max(|I|, |\Lambda|, \mathrm{rank}(G : H))$$

where H is the subgroup of G generated by $\{p_{\lambda i} \mid 1 \neq i \in I, 1 \neq \lambda \in \Lambda\}$.

PROOF. Since all entries in P are non-zero, we can choose $\pi_i = i \to 1_\Lambda$ and $\pi_\lambda = 1_I \to \lambda$, so that

$$q_{\lambda i} = V(\pi_\lambda)p_{\lambda i}V(\pi_i)p_{11} = p_{\lambda 1}^{-1}p_{\lambda i}p_{i1}^{-1}p_{11} = p_{\lambda i}$$

because P is normal. By Theorem 3.6

$$\operatorname{rank}(S) = \max(|I|, |\Lambda|, \operatorname{rank}(G : H))$$

where H is the subgroup of G generated by

$$\{q_{\lambda i} \mid \lambda \in \Lambda, i \in I\} = \{p_{\lambda i} \mid \lambda \in \Lambda, i \in I\}.$$

However, $p_{\lambda 1} = p_{1i} = 1$, and the theorem follows. ∎

4. An extremal problem

In Section 3 we established a formula for the rank of certain types of finite completely 0-simple Rees matrix semigroups. This formula expressed the rank of $S = \mathcal{M}^0[G; I, \Lambda; P]$ as a function of $|I|$, $|\Lambda|$, G and P. It is therefore interesting to ask how each of these factors influences the rank. For example, if we fix I, then

$$\operatorname{rank}(S) \geq |I|$$

by Corollary 2.3. Actually, by choosing G to be trivial and varying Λ, we see that $\operatorname{rank}(S)$ can take any value $\geq |I|$. Similar considerations show that, if we fix Λ, $\operatorname{rank}(S)$ can take any value $\geq |\Lambda|$.

In this section we consider the case when we fix the group G. Our first observation is that $\operatorname{rank}(S)$ may well be less than $\operatorname{rank}(G)$.

Example 4.1. Let G be any finite group of rank 9 with generators $g_1, \ldots, g_9$, and let S be the Rees matrix semigroup $\mathcal{M}[G; I, \Lambda; P]$ where

$$I = \Lambda = \{1, 2, 3, 4\}$$

and

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & g_1 & g_2 & g_3 \\ 1 & g_4 & g_5 & g_6 \\ 1 & g_7 & g_8 & g_9 \end{bmatrix}.$$

Then the subgroup of G generated by the entries of P is G, so that, by Theorem 3.7,

$$\begin{aligned} \operatorname{rank}(S) &= \max(|I|, |\Lambda|, \operatorname{rank}(G : G)) \\ &= \max(4, 4, 1) = 4 < 9 = \operatorname{rank}(G). \ ∎ \end{aligned}$$

This can raise the following question: *For a given finite group G, what is the minimal possible rank of a (0-)simple Rees matrix semigroup $\mathcal{M}[G; I, \Lambda; P]$ ($\mathcal{M}^0[G; I, \Lambda; P]$)?*

First we consider the completely simple case. As remarked before, in this case we may suppose that P has normal form without loss of generality. For a real number x, $\lceil x \rceil$ denotes the least integer which is not less than x. Thus, for example, $\lceil \sqrt{2} \rceil = 1$, $\lceil 5 \rceil = 5$, etc.

Theorem 4.2. *If G is a finite group of rank r then for every Rees matrix semigroup $\mathcal{M}[G; I, \Lambda; P]$, with P normal,*

$$\operatorname{rank}(\mathcal{M}[G; I, \Lambda; P]) \geq \left\lceil \frac{1 + \sqrt{4r - 3}}{2} \right\rceil .$$

There exists a semigroup $\mathcal{M}[G; I, \Lambda; P]$ with rank $\left\lceil \frac{1+\sqrt{4r-3}}{2} \right\rceil$.

Proof. If

$$m = \left\lceil \frac{1 + \sqrt{4r - 3}}{2} \right\rceil$$

then

$$m - 1 < \frac{1 + \sqrt{4r - 3}}{2} \leq m,$$

which is equivalent to

$$m^2 - 3m + 3 < r \leq m^2 - m + 1. \tag{9}$$

For an arbitrary simple Rees matrix semigroup $S = \mathcal{M}[G; I, \Lambda; P]$ with P in normal form, let t denote its rank, and let H be the subgroup of G generated by $\{p_{\lambda i} | 1 \neq \lambda \in \Lambda, 1 \neq i \in I\}$, so that, by Theorem 3.7,

$$t = \max(|I|, |\Lambda|, \operatorname{rank}(G : H)).$$

Since

$$\operatorname{rank}(G : H) \leq t \text{ and } \operatorname{rank}(G) = r$$

we have

$$\operatorname{rank}(H) \geq r - t. \tag{10}$$

On the other hand H is generated by $(|I| - 1)(|\Lambda| - 1)$ elements, and, since $|I| \leq t$ and $|\Lambda| \leq t$, we have

$$(t - 1)^2 \geq \operatorname{rank}(H). \tag{11}$$

From (10) and (11) it follows

$$t^2 - t + 1 - r \geq 0.$$

Since $r > 0$ we deduce that

$$t \geq \frac{1+\sqrt{4r-3}}{2} > \frac{1+\sqrt{4m^2-12m+9}}{2} = m-1$$

((9) was used in proving the last inequality), so that $t \geq m$.

Let $\{g_1, \ldots, g_r\}$ be a generating set for G. Define $I = \Lambda = \{1, \ldots, m\}$ and let P be the matrix with ones in the first row and the first column, and elements $g_1, \ldots, g_{(m-1)^2}$ as remaining entries. If $r < (m-1)^2$, the entries $g_{r+1}, \ldots, g_{(m-1)^2}$ may be arbitrary. Then it is easy to see that

$$\operatorname{rank}(G:H) = \begin{cases} r-(m-1)^2 & \text{if } r > (m-1)^2 \\ 0 & \text{otherwise,} \end{cases}$$

where H is the subgroup of G generated by $\{g_1, \ldots, g_{(m-1)^2}\}$. Since

$$r-(m-1)^2 \leq m^2-m+1-(m-1)^2 = m$$

(because of (9)), we obtain

$$\operatorname{rank}(\mathcal{M}[G;I,\Lambda;P]) = \max(|I|, |\Lambda|, \operatorname{rank}(G:H)) = m,$$

as required. ∎

Surprisingly, if we consider 0-simple semigroups (rather than simple), the bound remains the same.

Corollary 4.3. *If G is a finite group of rank r then for every 0-simple Rees matrix semigroup $S = \mathcal{M}^0[G;I,\Lambda;P]$*

$$\operatorname{rank}(S) \geq \left\lceil \frac{1+\sqrt{4r-3}}{2} \right\rceil .$$

PROOF. If $\overline{P}$ is obtained from P by replacing all zero entries by 1, then

$$\operatorname{rank}(\mathcal{M}[G;I,\Lambda;\overline{P}]) \leq \operatorname{rank}(\mathcal{M}^0[G;I,\Lambda;P])$$

since every generating set of $\mathcal{M}^0[G;I,\Lambda;P]$ generates $\mathcal{M}[G;I,\Lambda;\overline{P}]$ as well. The corollary now follows from Theorem 4.2. ∎

5. Transformation semigroups

Transformation semigroups have been studied extensively over the years, and ranks for many important classes are known. In the following proposition we list some well known results; all of them have very easy proofs.

Proposition 5.1. *(i) The full transformation semigroup T_n, $n \geq 2$, has rank 3. It is generated by any set $\{a, b, t\}$, where a, b are two permutations generating the symmetric group S_n, and t is any mapping of rank $n-1$.*

(ii) The semigroup of partial transformations PT_n, $n \geq 2$, has rank 4. It is generated by any set $\{a, b, t, s\}$, where a, b, t are as above, and $|\mathrm{Dom}(s)| = |\mathrm{Im}\, s| = n-1$.

(iii) The symmetric inverse semigroup I_n, $n \geq 2$, has rank 3. It is generated by any set $\{a, b, s\}$, where a, b, s are as above. ■

The rank of the semigroup of singular mappings Sing_n is more difficult to determine; see Gomes and Howie (1987). Even more difficult are the ranks of the semigroups

$$K(n,r) = \{\alpha \in T_n \mid |\mathrm{Im}\, \alpha| \leq r\},\ 1 \leq r \leq n,$$

which were determined by Howie and McFadden (1990). In this section we show how our results about ranks of completely 0-simple semigroups can be used to obtain alternative proofs for the results of Gomes and Howie (1987) and Howie and McFadden (1990). The following is the key lemma for doing this.

Lemma 5.2. *Let S be a semigroup, let J be a maximal $\mathcal{J}$-class of S, and let $\overline{J}$ be the corresponding principal factor. If $\langle J \rangle = S$ then* $\mathrm{rank}(S) = \mathrm{rank}(\overline{J})$.

PROOF. Note that if $s_1, s_2 \in S^1$ and $t \in S - J$, then $s_1 t s_2 \notin J$, since J is maximal. Hence, for any generating set A of S, $J \subseteq \langle A \cap J \rangle$, which implies

$$\mathrm{rank}(S) \geq \mathrm{rank}(\overline{J}).$$

Assume now that B is a generating set of $\overline{J}$ having the minimal possible cardinality. Since $\langle J \rangle = S$ we see that either $S = J$, or $\overline{J}$ has divisors of zero. In any case we may assume that $0 \notin B$, or, equivalently, $B \subseteq J$. Now in S we have $S \subseteq \langle J \rangle \subseteq \langle B \rangle$, and thus

$$\mathrm{rank}(S) \leq \mathrm{rank}(\overline{J}),$$

which completes the proof. ■

Let us now recall the Green's structure of the full transformation semigroup. For more details and proofs the reader is referred to Clifford and Preston (1961). It is well known that

$$\alpha \mathcal{R} \beta \iff \mathrm{Ker}\, \alpha = \mathrm{Ker}\, \beta, \tag{12}$$

$$\alpha \mathcal{L} \beta \iff \mathrm{Im}\, \alpha = \mathrm{Im}\, \beta, \tag{13}$$

$$\alpha \mathcal{D} \beta \iff \alpha \mathcal{J} \beta \iff |\mathrm{Im}\, \alpha| = |\mathrm{Im}\, \beta|. \tag{14}$$

We see that T_n has n $\mathcal{J}$-classes:

$$J(n,r) = \{\alpha \mid |\mathrm{Im}\, \alpha| = r\},\ r = 1, \ldots, n,$$

which form a chain. The corresponding principal ideals are precisely the semigroups $K(n,r)$, $r = 1,\ldots,n$, while the corresponding principal factors will be denoted by $\overline{J}(n,r)$, $r = 1,\ldots,n$. Note that $K(n,n) = T_n$ and $K(n,n-1) = \mathrm{Sing}_n$.

Each $J(n,r)$ is a $\mathcal{D}$-class as well. The number of $\mathcal{R}$-classes in $J(n,r)$ is equal to the number of equivalence relations on the set $\{1,\ldots,n\}$ with exactly r equivalence classes; this number is known to be the Stirling number of the second kind $S(n,r)$; see Riordan (1958). The number of $\mathcal{L}$-classes in $J(n,r)$ is equal to the number of r-element subsets of $\{1,\ldots,n\}$; this number is $\binom{n}{r}$.

An important observation about $K(n,r)$ is that it inherits the Green's structure from T_n.

Lemma 5.3. *Green's relations $\mathcal{R}$, $\mathcal{L}$, $\mathcal{D}$ and $\mathcal{J}$ in $K(n,r)$ are given by* (12), (13), (14).

PROOF. We prove the lemma just for $\mathcal{R}$; the assertions about $\mathcal{L}$, $\mathcal{D}$ and $\mathcal{J}$ can be proved similarly. In one direction the lemma is obvious: if two mappings are $\mathcal{R}$-equivalent in $K(n,r)$, they are $\mathcal{R}$-equivalent in T_n as well, and so they have the same kernel. For the converse suppose that $\mathrm{Ker}\,\alpha = \mathrm{Ker}\,\beta$. This means that $\alpha\mathcal{R}\beta$ in T_n, or, in other words, $\alpha\gamma = \beta$ and $\beta\delta = \alpha$ for some $\gamma,\delta \in T_n$. Now note that

$$|(\mathrm{Im}\,\alpha)\gamma| = |\mathrm{Im}\,\beta| = |\mathrm{Im}\,\alpha|,$$

so if we define γ_1 by

$$x\gamma_1 = x \text{ if } x \in \mathrm{Im}\,\alpha,\ x\gamma_1 \in \mathrm{Im}\,\alpha \text{ if } x \notin \mathrm{Im}\,\alpha,$$

we have $\alpha\gamma_1 = \alpha$ and

$$|\mathrm{Im}\,(\gamma_1\gamma)| \leq |\mathrm{Im}\,\gamma_1| = |\mathrm{Im}\,\alpha| \leq r.$$

Therefore, for $\gamma_2 = \gamma_1\gamma$, we have $\gamma_2 \in K(n,r)$ and $\alpha\gamma_2 = \beta$. Similarly, there exists δ_2 such that $\delta_2 \in K(n,r)$ and $\beta\delta_2 = \alpha$, and hence $\alpha\mathcal{R}\beta$ in $K(n,r)$. ∎

Now we show that $K(n,r)$ satisfies the conditions of Lemma 5.2.

Lemma 5.4. *The semigroup $K(n,r)$, $r = 1,\ldots,n-1$, is generated by its maximal $\mathcal{J}$-class $J(n,r)$.*

PROOF. The lemma is a direct consequence of the fact that any mapping α with $|\mathrm{Im}\,\alpha| = r \leq n-2$ can be written as a composition of two mappings of rank $r+1$; see Lemma 4 in Howie and McFadden (1990). ∎

Our next goal is to prove that the principal factor $\overline{J}(n,r)$ is a connected completely 0-simple semigroup. To do this we recall the following result of Hall (1973).

Lemma 5.5. *A regular semigroup S is idempotent generated if and only if every principal factor of S is idempotent generated.* ∎

Lemma 5.6. *$\overline{J}(n,r)$, $r = 1, \ldots, n-1$, is a connected completely 0-simple semigroup.*

PROOF. The semigroup Sing_n is generated by its idempotents (Howie (1966)), and $\overline{J}(n,r)$ is a principal factor of Sing_n (Lemma 5.3), so that $\overline{J}(n,r)$ is idempotent-generated as well by Lemma 5.5.

A principal factor of a semigroup is either 0-simple or is a null semigroup; see Proposition A.4.7. Since $\overline{J}(n,r)$ contains idempotents, it is 0-simple, and since it is finite it has to be completely 0-simple. Moreover, since $\overline{J}(n,r)$ is idempotent-generated, each non-zero $\mathcal{H}$-class has a non-empty intersection with $F(\overline{J}(n,r))$, so that $\overline{J}(n,r)$ is connected. ∎

Now we have all the ingredients for an alternative proof of the following theorem from Howie and McFadden (1990):

Theorem 5.7. $\mathrm{rank}(K(n,r)) = S(n,r)$, $r = 1, \ldots, n-1$.

PROOF. By Lemmas 5.2 and 5.4 we have $\mathrm{rank}(K(n,r)) = \mathrm{rank}(\overline{J}(n,r))$. By Lemma 5.6 $\overline{J}(n,r)$ is a finite connected completely 0-simple semigroup, and hence its rank is given in Theorem 2.14. The number of $\mathcal{R}$-classes in $\overline{J}(n,r)$ is $S(n,r)$, while the number of $\mathcal{L}$-classes is $\binom{n}{r}$. Finally, the subsemigroup of $\overline{J}(n,r)$ generated by the idempotents is $\overline{J}(n,r)$ itself by Lemma 5.5, and so we have

$$\mathrm{rank}(\overline{J}(n,r)) = \max(S(n,r), \binom{n}{r}).$$

It is well known that $S(n,r) \geq \binom{n}{r}$, and the result follows. ∎

Note that $S(n,n-1) = \binom{n}{2}$, and since $K(n,n-1) = \mathrm{Sing}_n$, Theorem 5.7 has as a consequence Theorem 2.1 from Gomes and Howie (1987).

We finish this section by mentioning that the rank of the semigroup O_n of endomorphisms of a finite chain was determined in Gomes and Howie (1990). The rank of the semigroup B_n of binary relations on a finite set is not yet known. Devadze (1968) proved that $\mathrm{rank}(B_n) \geq n+1$. Actually, as noted in Kim and Rousch (1977), $\mathrm{rank}(B_n)$ grows at least exponentially with n.

6. Matrix semigroups

In this section we consider semigroups of matrices over a field in a similar way as we did with transformation semigroups in Section 5.

Let us first introduce some notation. Let F be a field. The set of all $d \times d$ matrices over F with non-zero determinant is a group, with respect to the usual multiplication of matrices. We denote this group by $\mathrm{GL}(d, F)$ and call it the *general linear group.* Another important group of matrices is the *special linear group* $\mathrm{SL}(d, F)$, consisting of all $d \times d$ matrices with determinant 1. It is a normal subgroup of $\mathrm{GL}(d, F)$; it is the kernel of the *determinant homomorphism* $\det : \mathrm{GL}(d, F) \longrightarrow F$, so that $\mathrm{GL}(d, F)/\mathrm{SL}(d, F)$ is isomorphic to the multiplicative group $F - \{0\}$. For details on linear groups see Rotman (1965).

The set of all $d \times d$ matrices over F is a semigroup with respect to the multiplication of matrices; we denote this semigroup by $\mathrm{GLS}(d, F)$ and call it the *general linear semigroup.* It is a semigroup analogue of $\mathrm{GL}(d, F)$. The semigroup analogue of $\mathrm{SL}(d, F)$ is the *special linear semigroup* $\mathrm{SLS}(d, F)$, consisting of all matrices having determinant 0 or 1. We also note that $\mathrm{GL}(d, F)$ and $\mathrm{GLS}(d, F)$ are isomorphic to the automorphism group and the endomorphism semigroup respectively of the vector space F^d.

The Green's structure of $\mathrm{GLS}(d, F)$ closely resembles that of T_n. In particular, for any two matrices A, B, we have

$$A \mathcal{R} B \iff \operatorname{Ker} A = \operatorname{Ker} B, \tag{15}$$

$$A \mathcal{L} B \iff \operatorname{Im} A = \operatorname{Im} B, \tag{16}$$

$$A \mathcal{D} B \iff \dim(\operatorname{Im} A) = \dim(\operatorname{Im} B) \tag{17}$$

(see Exercise 2.2.6 in Clifford and Preston (1967)). We also have

Theorem 6.1. *$\mathcal{J} = \mathcal{D}$ in* $\mathrm{GLS}(d, F)$.

PROOF. The vector space F^d, being finite-dimensional, satisfies the descending chain condition on subspaces. Therefore, both the set of $\mathcal{R}$-classes and the set of $\mathcal{L}$-classes satisfy the descending chain condition, and thus $\mathcal{J} = \mathcal{D}$ by Proposition 1.11 of Howie (1976). ∎

$\mathrm{GLS}(d, F)$ has $d + 1$ principal two-sided ideals

$$I(r, d, F) = \{A \mid \dim(\operatorname{Im} A) \leq r\}, \ r = 0, \dots, d.$$

The corresponding $\mathcal{J}$-classes are

$$J(r, d, F) = \{A \mid \dim(\operatorname{Im} A) = r\}, \ r = 0, \dots, d,$$

and the corresponding principal factors will be denoted by $\overline{J}(r, d, F)$. The semigroup $I(d-1, d, F)$ consists of all singular matrices, and we denote it by $\mathrm{Sing}(d, F)$. It is clear that

$$\mathrm{GLS}(d, F) = \mathrm{GL}(d, F) \cup \mathrm{Sing}(d, F),$$
$$\mathrm{SLS}(d, F) = \mathrm{SL}(d, F) \cup \mathrm{Sing}(d, F).$$

The main technical result needed for determining ranks of various matrix semigroups is the following:

Lemma 6.2. *The semigroup* $\mathrm{Sing}(d,F)$ *is generated by its maximal* $\mathcal{J}$*-class* $J(d-1,d,F)$.

PROOF. We show that each matrix A of rank r, $r \leq d-2$, can be written as BC, where B and C have rank $r+1$.

Let $\{a_1,\dots,a_d\}$ be a basis for the vector space F^d, and let us write $a_iA = b_i$, $i = 1,\dots,d$. The subspace V of F^d generated by $\{b_1,\dots,b_d\}$ has dimension r, and therefore $\{b_1,\dots,b_d\}$ contains an r-element basis for V. Without loss of generality we assume that $b_1,\dots,b_r$ are linearly independent; hence $b_{r+1},\dots,b_d$ are linear combinations of $b_1,\dots,b_r$. Let $c_{r+1},\dots,c_d$ be such that $\{b_1,\dots,b_r,c_{r+1},\dots,c_d\}$ is a basis for F^d. Let B and C be matrices such that

$$a_iB = b_i \text{ if } i \neq r+1,\ a_{r+1}B = c_{r+1};$$
$$b_iC = b_i \text{ for } 1 \leq i \leq r,\ c_{r+1}C = b_{r+1},\ c_{r+2}C = c_{r+2},\ c_jC = 0 \text{ for } j \geq r+3.$$

$\mathrm{Im}\, B$ is generated by $\{b_1,\dots,b_r,c_{r+1},b_{r+2},\dots,b_d\}$ and has dimension $r+1$; $\mathrm{Im}\, C$ is generated by $\{b_1,\dots,b_r,b_{r+1},c_{r+2}\}$ and has dimension $r+1$ as well. Finally, it is clear that $A = BC$, which completes the proof. ■

Now we are in the position to find the ranks of $\mathrm{GLS}(d,F)$ and $\mathrm{SLS}(d,F)$.

Theorem 6.3. *If* $S \in \mathrm{GLS}(d,F)$ *is any matrix of rank* $d-1$*, then*

$$\langle \mathrm{GL}(d,F) \cup \{S\} \rangle = \mathrm{GLS}(d,F).$$

In particular

$$\mathrm{rank}(\mathrm{GLS}(d,F)) = \mathrm{rank}(\mathrm{GL}(d,F)) + 1.$$

PROOF. For any matrix T of rank $d-1$ there exist non-singular matrices P and Q such that $PTQ = \mathrm{diag}(1,\dots,1,0)$. In particular, $PTQ = P_1SQ_1$ ($=\mathrm{diag}(1,\dots,1,0)$) for some non-singular matrices P_1 and Q_1, so that we have $T = P^{-1}P_1SQ_1Q^{-1}$, and the first part of the theorem follows from Lemma 6.2. The second part now follows from the fact that $\mathrm{Sing}(d,F)$ is an ideal in $\mathrm{GLS}(d,F)$, so that each generating set for $\mathrm{GLS}(d,F)$ necessarily contains a generating set for $\mathrm{GL}(d,F)$. ■

Slightly more surprisingly we have

Theorem 6.4. *If* $S \in \mathrm{GLS}(d,F)$ *is any matrix of rank* $d-1$ *then*

$$\langle \mathrm{SL}(d,F) \cup \{S\} \rangle = \mathrm{SLS}(d,F).$$

In particular

$$\mathrm{rank}(\mathrm{SLS}(d,F)) = \mathrm{rank}(\mathrm{SL}(d,F)) + 1.$$

PROOF. Let T be any matrix of rank $d-1$, so that $PTQ = \text{diag}(1,\ldots,1,0)$ for some non-singular matrices P and Q. Then

$$\begin{aligned} T &= P^{-1}\text{diag}(1,\ldots,1,0)Q^{-1} \\ &= P^{-1}\text{diag}(1,\ldots,1,\det(P))\cdot\text{diag}(1,\ldots,1,0)\cdot\text{diag}(1,\ldots,1,\det(Q))Q^{-1} \\ &= \overline{P}\text{diag}(1,\ldots,1,0)\overline{Q}, \end{aligned}$$

where $\overline{P} = P^{-1}\text{diag}(1,\ldots,1,\det(P))$ and $\overline{Q} = \text{diag}(1,\ldots,1,\det(Q))\ Q^{-1}$ are matrices of determinant 1. The rest of the proof is the same as in Theorem 6.3. ∎

Now we turn our attention to the principal ideals $I(r,d,F)$ of $\text{GLS}(d,F)$. Again, as in the full transformation semigroup, the principal ideals inherit the Green's structure from the semigroup.

Lemma 6.5. *Green's relations $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{D}$ in $I(r,d,F)$ are given by* (15), (16), (17), *and $\mathcal{J} = \mathcal{D}$.*

PROOF. The lemma can be proved by an argument analogous to the proof of Lemma 5.3. ∎

As an immediate corollary of Lemmas 5.2, 6.2 and 6.5 we obtain

Lemma 6.6. $\text{rank}(I(r,d,F)) = \text{rank}(\overline{J}(r,d,F))$. ∎

Now, however, we have to distinguish the case when F is an infinite field from the case when F is finite.

Theorem 6.7. *If F is an infinite field then the semigroup $I(r,d,F)$, $r = 1,\ldots,d-1$, is not finitely generated.*

PROOF. The semigroup $\overline{J}(r,d,F)$, being a principal ideal, is either 0-simple, or it is a null semigroup. It is, however, easy to see that $\overline{J}(r,d,F)$ contains idempotents, and hence it is 0-simple. Moreover, it is completely 0-simple, since F^d satisfies the descending chain condition on subspaces. By Corollary 2.3 we have

$$\text{rank}(\overline{J}(r,d,F)) \geq \max(|I|,|\Lambda|),$$

where I is the number of non-zero $\mathcal{R}$-classes, and $|\Lambda|$ is the number of non-zero $\mathcal{L}$-classes in $\overline{J}(r,d,F)$. By (16), the number of non-zero $\mathcal{L}$-classes in $\overline{J}(r,d,F)$ is equal to the number of subspaces of F^d of dimension r; we shall show that this number is infinite. The theorem will then follow from Lemma 6.6.

For f in F, consider the set

$$B(f) = \{a_1(f),\ldots,a_r(f)\} \subseteq F^d,$$

where

$$a_i(f) = (\underbrace{0,\ldots,0}_{i-1},1,\underbrace{0,\ldots,0}_{d-i-1},f).$$

It is obvious that $a_1(f),\ldots,a_r(f)$ are linearly independent; let $V(f)$ denote the r-dimensional subspace of F^d generated by $B(f)$.

We claim that, for $f \neq g$, we have $V(f) \neq V(g)$. Indeed, if $V(f) = V(g)$, then $a_1(g)$ is a linear combination of $a_1(f),\ldots,a_r(f)$:

$$a_1(g) = \alpha_1 a_1(f) + \ldots + \alpha_r a_r(f),$$

which gives

$$\alpha_1 = 1,\ \alpha_2 = 0,\ldots,\ \alpha_r = 0,\ (\alpha_1 + \ldots + \alpha_r)f = g,$$

or, in other words, $f = g$. ∎

Now we turn our attention to the case where F is a finite field. For the information on finite fields the reader is referred to Lidl and Niederreiter (1983). In particular, each finite field has order p^n, where p is a prime, and, up to isomorphism, there exists exactly one field of order p^n; this field is usually denoted by $\mathrm{GF}(p^n)$. If $n = 1$ then $\mathrm{GF}(p^n)$ is simply $\mathbb{Z}_p$, the field of integers modulo p.

If $F = \mathrm{GF}(p^n)$, we will write $\mathrm{GLS}(d,p^n)$ for $\mathrm{GLS}(d,F)$. We will also use analogous notation for all other matrix groups and semigroups introduced so far. With this notation we have

Theorem 6.8. *The semigroup $I(r,d,p^n)$, $r = 1,\ldots,d-1$, has rank*

$$N(r,d,p^n) = \frac{(p^{nd}-1)(p^{n(d-1)}-1)\ldots(p^{n(d-r+1)}-1)}{(p^{nr}-1)(p^{n(r-1)}-1)\ldots(p^n-1)}.$$

In particular, $\mathrm{Sing}(d,p^n)$ *has rank* $(p^{nd}-1)/(p^n-1)$.

Proof. By Lemma 6.6 the rank of $I(r,d,p^n)$ is equal to the rank of the principal factor $\overline{J}(r,d,p^n)$. As before, $\overline{J}(r,d,p^n)$ is a finite completely 0-simple semigroup. Actually, it is connected as well. To prove this we recall that the semigroup $\mathrm{Sing}(d,p^n)$ is idempotent generated; see Erdos (1967). By Lemma 5.5, $\overline{J}(r,d,p^n)$ is also idempotent-generated, and so the intersection of any non-zero $\mathcal{H}$-class with $F(\overline{J}(r,d,p^n))$ is the whole $\mathcal{H}$-class. By Theorem 2.14, $\overline{J}(r,d,p^n)$ has rank $\max(\rho,\lambda)$, where ρ and λ are respectively the number of non-zero $\mathcal{R}$-classes and the number of non-zero $\mathcal{L}$-classes in $\overline{J}(r,d,p^n)$ (or, equivalently, in $J(r,d,p^n)$). As in Theorem 6.7, the number of $\mathcal{R}$-classes of $J(r,d,p^n)$ is equal to the number of subspaces of $\mathrm{GF}(p^n)^d$ of dimension $d-r$, and the number of $\mathcal{L}$-classes of $J(r,d,p^n)$ is equal to the number of subspaces of $\mathrm{GF}(p^n)^d$ of dimension r.

Any subspace of $\mathrm{GF}(p^n)^d$ of dimension r is generated by r linearly independent vectors, $v_1,\ldots,v_r$ say. Since v_1 can be any non-zero vector, there are $p^{nd}-1$

possible choices for v_1. Having chosen v_1, v_2 can be any vector which is not a scalar multiple of v_1; there are $p^{nd} - p^n$ such vectors. By repeating this argument we see that there are exactly $(p^{nd}-1)(p^{nd}-p^n)\dots(p^{nd}-p^{n(r-1)})$ different linearly independent sets of r vectors in $\mathrm{GF}(p^n)^d$. By the same argument, each subspace of dimension r has exactly $(p^{nr}-1)(p^{nr}-p^n)\dots(p^{nr}-p^{n(r-1)})$ different bases. Therefore there are exactly

$$\frac{(p^{nd}-1)(p^{nd}-p^n)\dots(p^{nd}-p^{n(r-1)})}{(p^{nr}-1)(p^{nr}-p^n)\dots(p^{nr}-p^{n(r-1)})} = N(r,d,p^n)$$

different subspaces of dimension r. Finally, note that $N(r,d,p^n) = N(d-r,d,p^n)$, and the result follows. ■

Remark 6.9. Dawlings (1980) and (1982) proved that the idempotent rank (i.e. the minimal number of idempotent generators) of $\mathrm{Sing}(d,p^n)$ is $(p^{nd}-1)/(p^n-1)$. Therefore, the rank and the idempotent rank of $\mathrm{Sing}(d,p^n)$ are equal. This parallels the results of Gomes and Howie (1987) for the semigroup Sing_n.

7. Some infinitely generated semigroups

On the basis of Proposition 5.1 and Theorem 6.3 one might be tempted to conjecture that the endomorphism semigroup of an algebraic structure is always generated by the automorphism group together with one additional endomorphism. Of course, this is not so, and in this section we show that the conjecture fails even for some 'tame' algebraic structures, such as free semigroups, free abelian groups and free groups.

In what follows, $\mathrm{Aut}(\mathfrak{A})$ and $\mathrm{End}(\mathfrak{A})$ denote the automorphism group and the endomorphism semigroup of an algebraic structure $\mathfrak{A}$.

Theorem 7.1. *Let A be a finite set and let $F = A^+$ be the free semigroup on A. The group $\mathrm{Aut}(F)$ is isomorphic to the symmetric group on A, while the semigroup $\mathrm{End}(F)$ is not finitely generated.*

PROOF. Let $f : F \longrightarrow F$ be an automorphism, and assume that $Af \not\subseteq A$. Then there exists $a \in A - Af$. However, since $|xf| \geq |x|$ for each $x \in F$, we have $a \notin \mathrm{Im}\, f$, a contradiction. Therefore, each automorphism of F induces a permutation on A. On the other hand, each permutation on A induces an automorphism on F by Proposition 3.1.1.

For the second part of the theorem, consider the set $B(p)$, p prime, consisting of all endomorphisms $f : F \longrightarrow F$, such that $af = b^p$ for some $a, b \in A$ and $(A - \{a\})f = A - \{b\}$. We prove that if an element f of $B(p)$ is a product of two endomorphisms, then one of those endomorphisms also belongs to $B(p)$. As a consequence we obtain that each generating set of $\mathrm{End}(F)$ has a non-empty intersection with each set $B(p)$, so that $\mathrm{End}(F)$ is not finitely generated.

So, let us assume that $f \in B(p)$ with

$$af = b^p,\ (A - \{a\})f = A - \{b\},$$

and that $f = gh$. Then clearly $|xg| = 1$ for every $x \in A - \{a\}$, so that $(A - \{a\})g = A - \{c\}$ for some $c \in A$, since A is finite. Similarly, $(A - \{c\})h = A - \{b\}$, so that ag does not contain any letter from $A - \{c\}$; hence $ag = c^m$ for some m. For a similar reason, ch is a power of b; let us say $ch = b^n$. But then

$$b^p = af = agh = c^m h = b^{mn}.$$

Since p is a prime, we have either $m = p$ or $n = p$, so that either $g \in B(p)$ or $h \in B(p)$. ■

The proof for free abelian groups is even simpler.

Theorem 7.2. *Let $F = \mathbb{Z}^n$ be a finitely generated free abelian group. Then the group* $\mathrm{Aut}(F)$ *is finitely generated, but* $\mathrm{End}(F)$ *is not finitely generated.*

PROOF. F is a free $\mathbb{Z}$-module, and therefore $\mathrm{End}(F)$ is isomorphic to the semigroup $\mathrm{GLS}(n, \mathbb{Z})$ of all $n \times n$ integer matrices, while $\mathrm{Aut}(F)$ is isomorphic to the group $\mathrm{GL}(n, \mathbb{Z})$ of all such matrices with determinant 1 or -1; see Blyth (1990). It is well known that $\mathrm{GL}(n, \mathbb{Z})$ is finitely generated—this fact can be deduced from the fact that each integer matrix can be transformed to a diagonal matrix by applying elementary transformations; see again Blyth (1990).

On the other hand, the determinant homomorphism $\det : \mathrm{GLS}(d, \mathbb{Z}) \longrightarrow \mathbb{Z}$ maps $\mathrm{GLS}(d, \mathbb{Z})$ onto $\mathbb{Z}$. The multiplicative semigroup of $\mathbb{Z}$ is not finitely generated since there are infinitely many primes, and therefore $\mathrm{GLS}(n, \mathbb{Z})$ is not finitely generated either. ■

Theorem 7.3. *Let F be the free group on a finite set A. The group* $\mathrm{Aut}(F)$ *is finitely generated, while the semigroup* $\mathrm{End}(F)$ *is not finitely generated.*

PROOF. The proof of the first part of the theorem can be found in Magnus, Karrass, Solitar (1966).

For the second part assume that $A = \{x_1, \ldots, x_n\}$, and let $\{y_1, \ldots, y_n\}$ be the natural free basis for the free abelian group $\mathbb{Z}^n$. Let $\phi : F \longrightarrow \mathbb{Z}^n$ be the natural epimorphism induced by $x_i \mapsto y_i$. Define a mapping

$$\psi : \mathrm{End}(F) \longrightarrow \mathrm{End}(\mathbb{Z}^n)$$

by

$$y_i(f\psi) = (x_i f)\phi. \tag{18}$$

We shall prove that ψ is an epimorphism, and the second part of the theorem then follows from Theorem 7.2.

Obviously (18) defines a homomorphism $(f\psi) \in \mathrm{End}(\mathbb{Z}^n)$, since $\mathbb{Z}^n$ is free, and $y_1, \ldots, y_n$ is its basis. We want to prove that ψ is a homomorphism. Let $f, g \in \mathrm{End}(F)$, and let f act on the generating set $\{x_1, \ldots, x_n\}$ as follows:

$$x_i f = w_i(x_1, \ldots, x_n),\ 1 \le i \le n,$$

where w_i is a (group) word. Then

$$\begin{aligned} y_i((fg)\psi) &= (x_i(fg))\phi = ((x_i f)g)\phi = (w_i(x_1, \ldots, x_n)g)\phi \\ &= (w_i(x_1 g, \ldots, x_n g))\phi = w_i((x_1 g)\phi, \ldots, (x_n g)\phi), \end{aligned}$$

and

$$\begin{aligned} y_i((f\psi)(g\psi)) &= (y_i(f\psi))(g\psi) = ((x_i f)\phi)(g\psi) = ((w_i(x_1, \ldots, x_n))\phi)(g\psi) \\ &= (w_i(y_1, \ldots, y_n))(g\psi) = w_i(y_1(g\psi), \ldots, y_n(g\psi)) \\ &= w_i((x_1 g)\phi, \ldots, (x_n g)\phi), \end{aligned}$$

proving that ψ is a homomorphism.

To prove that ψ is an epimorphism let $h \in \mathrm{End}(\mathbb{Z}^n)$ and let

$$y_i h = w_i(y_1, \ldots, y_n),\ 1 \le i \le n.$$

Define $f \in \mathrm{End}(F)$ by

$$x_i f = w_i(x_1, \ldots, x_n), 1 \le i \le n.$$

Then

$$y_i(f\psi) = (x_i f)\phi = (w_i(x_1, \ldots, x_n))\phi = w_i(y_1, \ldots, y_n) = y_i h,$$

so that $f\psi = h$. ∎

Chapter 3

Defining relations for common semigroups

In this chapter we will work on the first of our two fundamental problems for semigroup presentations: *given a semigroup find a presentation for it.*

After reviewing already known results for groups and semigroups in Section 1, in Section 2 we describe different methods for approaching the above problem. Two of these methods are then used in Sections 3 and 4 to find presentations for the special linear semigroup $\mathrm{SLS}(2,p)$ and the general linear semigroup $\mathrm{GLS}(2,p)$. Finally, in Section 5 we describe how computational methods can be used in this area, and consider the interdependence of our defining relations for $\mathrm{SLS}(2,p)$ and $\mathrm{GLS}(2,p)$.

Sections 1 and 2 have an introductory character. The results of Sections 3 and 4 and a shorter version of the discussion from Section 5 will appear in Ruškuc 1995.

1. Known presentations

'Nice' presentations for many interesting groups have been known for a long time. Here we give some examples, which we will need later. More examples can be found in Coxeter and Moser (1980).

One of the earliest presentations for the symmetric group $\mathcal{S}_n$ is the following presentation given by Moore (1897); see also Coxeter and Moser (1980).

Proposition 1.1. *The presentation*

$$\langle a,b \mid a^2 = b^n = (ba)^{n-1} = (b^{-1}ab)^3 = (ab^{-j}ab^j)^2 = 1 \ \ (2 \le j \le n-2)\rangle$$

defines $\mathcal{S}_n$ in terms of generators $(1\,2)$, $(1\,2\,\dots\,n)$. ∎

If we consider the larger generating set $(1\,2)$, $(2\,3)$,..., $(n-1\;n)$ we obtain a larger, but more symmetrical set of defining relations; see again Coxeter and Moser (1980):

Proposition 1.2. *The presentation*

$$\langle a_1, \ldots, a_n \mid a_i^2 = (a_j a_{j+1})^3 = (a_k a_l)^2 = 1$$
$$(1 \le i \le n-1,\ 1 \le j \le n-2,\ 1 \le k \le l-2 \le n-3)\rangle$$

defines $\mathcal{S}_n$ *in terms of generators* $(1\,2)$, $(2\,3)$,..., $(n-1\,n)$. ■

The first presentation for the alternating group $\mathcal{A}_n$ was given by Moore(1897):

Proposition 1.3. *The presentation*

$$\langle a_1, \ldots, a_{n-2} \mid a_1^3 = a_j^2 = (a_{i-1} a_i)^3 = (a_j a_k)^2 = 1$$
$$(1 < i \le n-2,\ 1 \le j < k-1 \le n-3)\rangle$$

defines $\mathcal{A}_n$ *in terms of generators* $(1\,2)(i+1\;i+2)$, $i = 1, \ldots, n-2$. ■

Another presentation for $\mathcal{A}_n$ is due to Carmichael (1923):

Proposition 1.4. *The presentation*

$$\langle a_1, \ldots, a_{n-2} \mid a_i^3 = (a_j a_k)^2 = 1,\ (1 \le i \le n-2,\ 1 \le j < k \le n-2)\rangle$$

defines $\mathcal{A}_n$ *in terms of generators* $(i\;n-1\;n)$, $i = 1, \ldots, n-2$. ■

Linear groups have particularly nice presentations in dimension 2.

Proposition 1.5. *If* $p > 2$ *is a prime then the presentation*

$$\langle a, b \mid a^p = 1,\ b^2 = (ab)^3 = (a^4 b a^{\frac{p+1}{2}} b)^2\rangle$$

defines $\mathrm{SL}(2,p)$ *in terms of generators*

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \ ■$$

The above presentation is due to Sunday (1972), and an alternative presentation can be found in Campbell and Robertson (1980). A presentation for $\mathrm{GL}(2,p)$ can be obtained from Proposition 1.5 by adjoining another generator

$$\begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix},$$

where ξ is a primitive root of 1 modulo p, and specifying the action of this generator on a and b:

Proposition 1.6. *If $p > 2$ is a prime, then the presentation*

$$\langle a, b, c \mid a^p = 1,\ b^2 = (ab)^3 = (a^4 b a^{\frac{p+1}{2}} b)^2,\ c^{p-1} = 1,\\ c^{-1}ac = c^{\xi},\ c^{-1}bc = ba^{-\frac{1}{\xi}} ba^{-\xi} ba^{-\frac{1}{\xi}} b\rangle$$

defines $\mathrm{GL}(2, p)$. ∎

A more general (and necessarily more complicated) presentation for $\mathrm{SL}(n, F)$, where F is a division ring, was given by Green (1977). Corollary 10.3 in Milnor (1971) gives a finite presentation for the special group $\mathrm{SL}(n, \mathbb{Z})$ over the ring of integers. Since $\mathrm{SL}(n, \mathbb{Z})$ has index 2 in $\mathrm{GL}(n, \mathbb{Z})$, this means that $\mathrm{GL}(n, \mathbb{Z})$ is also finitely presented. This is in contrast to the case of the semigroups $\mathrm{GLS}(n, \mathbb{Z})$ which are not finitely generated by results from Section 4.7, let alone finitely presented.

Turning our attention to 'proper' semigroups, we see that much fewer presentations are known. Aĭzenštat (1958) gives the following presentation for the full transformation semigroup T_n:

Proposition 1.7. *Assume that $\langle a, b \mid \mathfrak{R}\rangle$ is any (semigroup) presentation for the symmetric group $\mathcal{S}_n$ in terms of generators $\alpha = (1\,2)$ and $\beta = (1\,2\,\ldots\,n)$. Then the presentation*

$$\langle a, b, t \mid \mathfrak{R},\ ct = b^{n-2}ab^2tb^{n-2}ab^2 = bab^{n-1}abtb^{n-1}abab^{n-1} = (tbab^{n-1})^2 = t,\\ (b^{n-1}abt)^2 = tb^{n-1}abt = (tb^{n-1}ab)^2,\ (tbab^{n-2}ab)^2 = (bab^{n-2}ata)^2\rangle$$

defines the full transformation semigroup T_n in terms of generators α, β and

$$\tau = \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ 1 & 1 & 3 & \ldots & n \end{pmatrix}. \ \blacksquare$$

A nice feature of the above presentation is that, in addition to the 'group relations' $\mathfrak{R}$, it requires just seven 'semigroup relations'. It is natural to ask if this is the minimal number of relations:

Open Problem 2. Find the minimal number k such that there exists a presentation of the form $\langle A, B \mid \mathfrak{R},\ \mathfrak{S}\rangle$ for the full transformation semigroup T_n, where the presentation $\langle A \mid \mathfrak{R}\rangle$ defines the symmetric group $\mathcal{S}_n$ and $|\mathfrak{S}| = k$.

Another interesting question concerning the full transformation semigroup is to find presentations for its principal ideals (generating sets of these semigroups were considered in Section 4.5):

Open Problem 3. Find presentations for the semigroups Sing_n and $K(n, r)$.

A presentation for the semigroup PT_n of partial transformations in terms of the generating set given in Proposition 4.5.1(ii), was found by Popova (1961). The following presentation for the symmetric inverse semigroup is given in Meakin (1993), where it is attributed independently to Popova, Lipscomb and Easdown and Meakin:

Proposition 1.8. *Let* $\langle a_1, \ldots, a_{n-1} \,|\, \mathfrak{R}\rangle$ *be any (semigroup) presentation for the symmetric group* $\mathcal{S}_n$ *in terms of generators* $(1\,2), \ldots, (n-1\,n)$*. Then the presentation*

$$\langle a_1, \ldots a_{n-1}, t \,|\, \mathfrak{R},\ t^2 = t,\ ta_i = a_i t,\ ta_{n-1}t = ta_{n-1}ta_{n-1},$$
$$ta_{n-1}t = a_{n-1}ta_{n-1}t \ \ (1 \le i \le n-2)\rangle,$$

defines the symmetric inverse semigroup in terms of generators $(1\,2), \ldots, (n-1\,n)$ *and*

$$\begin{pmatrix} 1 & 2 & \ldots & n-1 & n \\ 1 & 2 & \ldots & n-1 & - \end{pmatrix}. \ \blacksquare$$

A similar presentation for the alternating inverse semigroup was given in Lipscomb (1991).

In contrast with Aĭzenštat's presentation for T_n, the number of 'semigroup' relations in the above presentation grows with n. So it is sensible to ask the following question:

Open Problem 4. Can I_n be defined by a presentation of the form $\langle A, t | \mathfrak{R}_n, \mathfrak{S}_n\rangle$, where $\langle A \,|\, \mathfrak{R}_n\rangle$ is a presentation for $\mathcal{S}_n$, and $|\mathfrak{S}_n|$ does not depend on n? If yes, what is the minimal possible cardinality for $\mathfrak{S}_n$?

Finally, let us mention that presentations for some semigroups of endomorphisms of linearly ordered sets were given by Aĭzenštat (1962) and Popova (1962).

2. General methods for finding presentations

There are three main general methods for finding presentations for a semigroup S:

- direct method (guessing and proving);
- Tietze transformations;
- using semigroup constructions.

The direct method is most commonly used—all the presentations mentioned in Section 1 have been obtained by using this method. Although there are slight variations, it usually consists of the following steps:

- find a generating set A for S;

- find a set $\mathfrak{R}$ of relations which are satisfied by the generators A, and which seem to be sufficient to define S;
- find a set $W \subseteq A^+$, such that each word from A^+ can be transformed to a word from W by applying relations from $\mathfrak{R}$;
- prove that distinct words from W represent distinct elements in S.

In the following theorem we prove that $\langle A \mid \mathfrak{R} \rangle$ is indeed a presentation for S. The set W satisfying last two conditions is often called a *set of canonical* (or *normal) forms* for S.

Proposition 2.1. *Let S be a semigroup, let A be a generating set for S, let $\mathfrak{R} \subseteq A^+ \times A^+$ be a set of relations, and let $W \subseteq A^+$. Assume that the following conditions are satisfied:*

(i) the generators A of S satisfy all the relations from $\mathfrak{R}$;
(ii) for each word $w \in A^+$ there exists a word $\overline{w} \in W$ such that $w = \overline{w}$ is a consequence of $\mathfrak{R}$;
(iii) if $u, v \in W$, $u \not\equiv v$, then $u \neq v$ in S.

Then the presentation $\langle A \mid \mathfrak{R} \rangle$ defines S in terms of generators A.

PROOF. Let $w_1, w_2 \in A^+$ be any two words such that the relation $w_1 = w_2$ holds in S. By condition (ii) we have that $w_1 = \overline{w}_1$ and $w_2 = \overline{w}_2$ are consequences of $\mathfrak{R}$, and by condition (iii) we have $\overline{w}_1 \equiv \overline{w}_2$. Hence $w_1 = w_2$ is a consequence of $\mathfrak{R}$, and it now follows from Proposition 3.2.3 that the presentation $\langle A \mid \mathfrak{R} \rangle$ defines S. ■

A very common modification of the described method, which can be applied when S is finite, has the same first three steps, and the fourth step consists of proving that $|W| \leq |S|$.

Proposition 2.2. *Let S be a finite semigroup, let A be a generating set for S, let $\mathfrak{R} \subseteq A^+ \times A^+$ be a set of relations, and let $W \subseteq A^+$. Assume that the following conditions are satisfied:*

(I) the generators A of S satisfy all the relations from $\mathfrak{R}$;
(II) for each word $w \in A^+$ there exists a word $\overline{w} \in W$ such that $w = \overline{w}$ is a consequence of $\mathfrak{R}$;
(III) $|W| \leq |S|$.

Then $\langle A \mid \mathfrak{R} \rangle$ is a presentation for S.

PROOF. We prove that conditions (I), (II) and (III) imply conditions (i), (ii) and (iii) of Proposition 2.1, and the result will follow. Since (I) and (II) are identical to (i) and (ii) respectively, we are left with proving that (iii) follows from (I),

(II) and (III). Since the generators A satisfy relations $\mathfrak{R}$, each element of S is represented by a word from W by (II). Hence $|W| \geq |S|$, so that (III) implies $|W| = |S|$, which means, since S is finite, that distinct elements of W represent distinct elements of S, as required. ■

Remark 2.3. It is worth pointing out again that finding a generating set for S, the problem we discussed in detail in Chapter 4, is the first step in the direct method for finding a presentation for S. ■

Remark 2.4. The described method has obvious modifications if we consider monoid presentations, or presentations of semigroups with zero, or presentations of monoids with zero. ■

The second method, Tietze transformations, is possible to apply only if we already know a presentation $\langle A \mid \mathfrak{R} \rangle$ for S, and then it yields alternative presentations for S. The idea is to transform the presentation $\langle A \mid \mathfrak{R} \rangle$ by applying some elementary moves, but without changing the semigroup defined by the presentation. These elementary moves, usually called *elementary Tietze transformations*, are:

(T1) adding a new relation $u = v$ to $\langle A|\mathfrak{R} \rangle$, providing that $u = v$ is a consequence of $\langle A \mid \mathfrak{R} \rangle$;

(T2) deleting a relation $(u = v) \in \mathfrak{R}$ from $\langle A \mid \mathfrak{R} \rangle$, providing that $u = v$ is a consequence of $\langle A \mid \mathfrak{R} - \{u = v\} \rangle$;

(T3) adding a new generating symbol b and a new relation $b = w$ for any non-empty word $w \in A^+$;

(T4) if $\langle A \mid \mathfrak{R} \rangle$ possesses a relation of the form $b = w$, where $b \in A$, and $w \in (A - \{b\})^+$, then deleting b from the list of generating symbols, deleting the relation $b = w$, and replacing all remaining appearances of b by w.

Proposition 2.5. *Two finite presentations define the same semigroup if and only if one can be obtained from the other by a finite number of applications of elementary Tietze transformations* (T1), (T2), (T3), (T4).

Proof. First we show that a single application of an elementary Tietze transformation to a presentation does not change the semigroup it defines. We prove this for transformations (T1) and (T3); the proofs for (T2) and (T4) are very similar.

Let $\langle A \mid \mathfrak{R} \rangle$ be a presentation, let η be the corresponding smallest congruence on A^+, and assume that the relation $u = v$ is a consequence of $\mathfrak{R}$. Consider the presentation $\langle A \mid \mathfrak{R},\ u = v \rangle$, and let ζ be the corresponding smallest congruence on A^+. Since $\mathfrak{R} \subseteq \zeta$, we have $\eta \subseteq \zeta$. On the other hand, $(u, v) \in \eta$, since $u = v$ is a consequence of $\mathfrak{R}$, and thus $\zeta \subseteq \eta$.

Consider now two presentations $\langle A \,|\, \mathfrak{R}\rangle$ and $\langle A, b \,|\, \mathfrak{R},\ b = w\rangle$, where $w \in A^+$, and let S and T denote the semigroups defined by these two presentations. Then the identity mapping id : $A \longrightarrow A$ induces a homomorphism $\phi : S \longrightarrow T$ by Proposition 3.2.1. Actually, ϕ is onto since b is a redundant generator in T. Now let $w_1, w_2 \in (A \cup \{b\})^+$ be two words such that $w_1 = w_2$ in T. This means that the relation $w_1 = w_2$ can be deduced by using relations $\mathfrak{R} \cup \{b = w\}$ by Proposition 3.2.2. However, if we omit all the applications of the relation $b = w$ from this deduction, we obtain a deduction of $w_1 = w_2$ by using just $\mathfrak{R}$. Therefore, $w_1 = w_2$ holds in S, and hence ϕ is an isomorphism.

Now we prove the converse: if two finite presentations $\langle A \mid \mathfrak{R}\}$ and $\langle B \mid \mathfrak{S}\rangle$ define isomorphic semigroups S and T then $\langle B|\mathfrak{S}\rangle$ can be obtained from $\langle A|\mathfrak{R}\rangle$ by applying elementary Tietze transformations. Let $\xi : S \longrightarrow T$ be an isomorphism. For $a \in A$ let $\overline{a} \in B^+$ be such that $a\xi = \overline{a}$ in T. Similarly, for $b \in B$ let $b\xi^{-1} = \overline{b}$, $\overline{b} \in A^+$. Let $\phi : A^+ \longrightarrow B^+$ and $\psi : B^+ \longrightarrow A^+$ be homomorphisms extending $a \mapsto \overline{a}$ and $b \mapsto \overline{b}$ respectively. (The homomorphisms ϕ and ψ can be thought of as *rewriting mappings*: they rewrite each word in one generating set into a corresponding word from the other generating set.)

Now, start from the presentation $\langle A \mid \mathfrak{R}\rangle$. All the relations from the set

$$\mathfrak{S}\psi = \{u\psi = v\psi \mid (u = v) \in \mathfrak{S}\}$$

certainly hold in S, and therefore are consequences of $\mathfrak{R}$. Similarly, the relations $a\phi\psi = a$ are consequences of $\mathfrak{R}$. By adding all these relations to $\langle A \mid \mathfrak{R}\rangle$ (transformation (T1)) we obtain

$$\langle A \mid \mathfrak{R},\ \mathfrak{S}\psi,\ a = a\phi\psi\rangle.$$

Now we introduce new generating symbols B by means of the relations $b = b\psi$ (transformation (T3)), and by using these relations and $\mathfrak{S}\psi$ we obtain $\mathfrak{S}$ (transformation (T1)). Now we eliminate $\mathfrak{S}\psi$, all relations of which are consequences of $\mathfrak{R}$ (transformation (T2)), and obtain

$$\langle A, B \mid \mathfrak{R},\ \mathfrak{S},\ a = a\phi\psi,\ b = b\psi\rangle.$$

This presentation defines S in terms of generators $A \cup B\xi^{-1}$ (or, equivalently, it defines T in terms of generators $A\xi \cup B$), so that the relations $a\phi = a$, $a \in A$ are consequences of it, and we can add these relations to the presentation transformation (T1)):

$$\langle A, B \mid \mathfrak{R},\ \mathfrak{S},\ a = a\phi\psi,\ b = b\psi,\ a = a\phi\rangle.$$

Now we eliminate generators A by using $a = a\phi$ (transformation (T4)), and obtain

$$\langle B \mid \mathfrak{R}\phi,\ \mathfrak{S},\ b = b\psi\phi\rangle.$$

Finally we note that all the relations from $\mathfrak{R}\phi$ as well as all the relations $b = b\psi\phi$, $b \in B$, hold in T, and therefore are consequences of $\mathfrak{S}$, so that we can eliminate them by using (T2), leaving the presentation $\langle B \mid \mathfrak{S} \rangle$ as required. ∎

The third method for finding a presentation for a semigroup S is via semigroup constructions. Here we try to express S in terms of some other (usually simpler) semigroups T_i, $i \in I$. Then we find presentations for the semigroups T_i, $i \in I$, and the way they combine to give a presentation for S.

In the next two sections we will use the first two methods to find presentations for some general and special linear semigroups, while in Chapter 6 we will concentrate on the developing the third method.

3. Special linear semigroups $\mathbf{SLS}(2, p)$

Let us recall that the special linear semigroup $\mathrm{SLS}(2, p)$ consists of all 2×2 matrices over the field $\mathbb{Z}_p$ which have determinant 0 or 1. By Theorem 4.6.4 it is generated by any generating set for the special linear group $\mathrm{SL}(2, p)$ together with any matrix of rank 1. It is well known that any special linear group is generated by transvections, i.e. by matrices having ones on the diagonal and exactly one other non-zero entry; see Rotman (1965). Hence, $\mathrm{SL}(2, p)$ is generated by the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Two natural choices for the additional matrix of rank 1 are

$$S = \begin{pmatrix} \xi & 0 \\ 0 & 0 \end{pmatrix} \text{ or } T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

where $\xi = \xi(p)$ is a primitive root of 1 modulo p (i.e. ξ is a generator for the cyclic multiplicative group $\mathbb{Z}_p - \{0\}$). As the main results in this section we are going to prove the following two theorems.

Theorem 3.1. *Let $\langle a, b \mid \mathfrak{R} \rangle$ be any (monoid) presentation for* $\mathrm{SL}(2, p)$ *with respect to generators A and B. Then*

$$\mathfrak{P}_1 = \langle a, b, s | \mathfrak{R},\ bs = sa = s,\ sba^{p-1}s = 0,\ ba^{\xi-1}s = a^{1-\xi^{-1}}s^2,\ sb^{\xi-1}a = s^2 b^{1-\xi^{-1}} \rangle$$

is a presentation for $\mathrm{SLS}(2, p)$ *with respect to generators A, B and S.*

Theorem 3.2. *Let $\langle a, b \mid \mathfrak{R} \rangle$ be any (monoid) presentation for* $\mathrm{SL}(2, p)$ *with respect to generators A and B. Then*

$$\mathfrak{P}_2 = \langle a, b, t \mid \mathfrak{R},\ t^2 = bt = ta = t,\ tba^{p-1}t = 0,\ b^{\xi-1}atb = a^{\xi^{-1}} t b^{\xi} a^{1-\xi^{-1}} \rangle$$

is a presentation for $\mathrm{SLS}(2, p)$ *with respect to generators A, B and T.*

Remark 3.3. By Proposition 1.5 the presentation

$$\langle a, b_1 \mid a^p = 1,\ b_1^2 = (ab_1)^3 = (a^4 b_1 a^{\frac{p+1}{2}} b_1)^2 \rangle$$

defines $\mathrm{SL}(2,p)$ as a group in terms of the generators A and

$$B_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

However, if we note that

$$B_1 = BA^{-1}B,\ B = AB_1A,$$

we can use (group) Tietze transformations to obtain the following presentation in terms of A and B:

$$\langle a, b \mid a^p = 1,\ b = aba^{-1}ba,\ (ba^{-1}b)^2 = (aba^{-1}b)^3 = (a^4ba^{-1}ba^{\frac{p+1}{2}}ba^{-1}b)^2 \rangle.$$

A semigroup (or monoid) presentation for $\mathrm{SL}(2,p)$ can be obtained from this presentation as is described in Section 3.3. A simpler presentation is

$$\langle a, b | a^p = b^p = 1,\ b = aba^{p-1}ba,\ (ba^{p-1}b)^2 = (aba^{p-1}b)^3 = (a^4ba^{p-1}ba^{\frac{p+1}{2}}ba^{p-1}b)^2 \rangle.$$

The plan of the proof of Theorem 3.1 and Theorem 3.2 is to prove first Theorem 3.1 by using the direct method, and then to prove Theorem 3.2 by applying Tietze transformations to the presentation $\mathfrak{R}_1$. Before that, however, we consider the case $p = 2$, which is somewhat exceptional.

Theorem 3.4. *Let $\langle a, b \,|\, \mathfrak{R}\rangle$ be a presentation for $\mathrm{GL}(2,2)$ $(= \mathrm{SL}(2,2))$ in terms of generators A and B. Then*

$$\langle a, b, s \mid \mathfrak{R},\ s^2 = bs = sa = s, sbas = 0\rangle \tag{1}$$

is a presentation for $\mathrm{GLS}(2,2)$ $(= \mathrm{SLS}(2,2))$ in terms of the generators A, B and S $(= T)$.

PROOF. We prove the theorem by using the direct method, or, more precisely, its variant described in Proposition 2.2. By Theorem 4.6.4 the matrices A, B, S generate $\mathrm{SLS}(2,p)$, and it is a routine matter to verify that these generators satisfy relations (1). The relations $a^2 = b^2 = 1$ and $aba = bab$ hold in $\mathrm{GL}(2,p)$, and therefore are consequences of (1). Hence each word from $\{a, b, s\}^* \cup \{0\}$ can be transformed by using (1) to a word from the set

$$W = \{1, a, b, ab, ba, aba, s, sb, sba, as, asb, asba, bas, basb, basba, 0\}.$$

Now we have

$$|\mathrm{GLS}(2,p)| = 16 = |W|,$$

and the result follows by Proposition 2.2. ■

Note that, for $p = 2$, both $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are equivalent to the presentation (1). Thus for the rest of this section we concentrate on the case $p \geq 3$. As we already mentioned, we aim to apply Proposition 2.1, so that we have to check that the conditions (i), (ii) and (iii) of that theorem are satisfied. The condition (i) requires that A, B and S generate $\mathrm{SLS}(2,p)$, which we proved in Theorem 4.6.4. The second condition does not pose any problems either:

Lemma 3.5. *The generators A, B, S of* $\mathrm{SLS}(2,p)$ *satisfy all the relations from* $\mathfrak{P}_1$.

PROOF. The lemma can be proved by a straightforward matrix calculation. ■

Now we need a set of canonical forms. Let $W_1 \subseteq \{a,b\}^*$ be a set of canonical forms for $\mathrm{SL}(2,p)$. Since $\langle a, b \mid \mathfrak{R}\rangle$ is a presentation for $\mathrm{SL}(2,p)$, this means that for each $w \in \{a,b\}^*$ there exists $\overline{w} \in W_1$ such that $w = \overline{w}$ is a consequence of $\mathfrak{R}$. On the other hand, distinct elements of W_1 represent distinct elements of $\mathrm{SLS}(2,p)$, since they represent distinct elements of $\mathrm{SL}(2,p)$, and $\mathrm{SL}(2,p)$ is a subsemigroup of $\mathrm{SLS}(2,p)$. Therefore we have:

Lemma 3.6. *For each $w \in \{a,b\}^*$ there exists $\overline{w} \in W_1$ such that $w = \overline{w}$ is a consequence of $\mathfrak{P}_1$. Distinct words of W_1 represent distinct elements of* $\mathrm{SLS}(2,p)$.
■

Let us now consider the following words:

$$
\begin{array}{ll}
\mathrm{cf}_1(i,j,k) = a^i s^j b^k, & i,k = 1,\dots,p,\ j = 1,\dots,p-1,\\
\mathrm{cf}_2(i,j) = b^{p-1} a s^i b^j, & i = 1,\dots,p-1,\ j = 1,\dots,p,\\
\mathrm{cf}_3(i,j) = a^i s^j b a^{p-1}, & i = 1,\dots,p,\ j = 1,\dots,p-1,\\
\mathrm{cf}_4(i) = b^{p-1} a s^i b a^{p-1}, & i = 1,\dots,p-1,\\
\mathrm{cf}_5 = 0, &
\end{array}
$$

and let W_2 be the set of all these words. We aim to prove that $W = W_1 \cup W_2$ is a set of canonical forms, i.e. that each word from $\{a,b,s\}^* \cup \{0\}$ is equal to a word from W, and that distinct words from W represent distinct elements of $\mathrm{SLS}(2,p)$. The latter of these assertions is easier to prove:

Lemma 3.7. *The elements of W represent distinct matrices in the semigroup* $\mathrm{SLS}(2,p)$.

PROOF. We have already proved the assertion for W_1 (Lemma 3.6). The words $\mathrm{cf}_1(i,j,k)$, $\mathrm{cf}_2(i,j)$, $\mathrm{cf}_3(i,j)$, $\mathrm{cf}_4(i)$ and cf_5 in $\mathrm{SLS}(2,p)$ respectively represent matrices

$$
\begin{pmatrix} \xi^j & k\xi^j \\ i\xi^j & ik\xi^j \end{pmatrix},
\begin{pmatrix} 0 & 0 \\ \xi^i & j\xi^i \end{pmatrix},
\begin{pmatrix} 0 & \xi^j \\ 0 & i\xi^j \end{pmatrix},
\begin{pmatrix} 0 & 0 \\ 0 & \xi^i \end{pmatrix},
\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
$$

which are clearly all distinct. Finally, since any $w_1 \in W_1$ represents a non-singular matrix, while any $w_2 \in W_2$ represents a singular matrix, we see that they cannot represent the same element in $\mathrm{SLS}(2,p)$. ■

In order to prove the former assertion we first need to establish some consequences of $\mathfrak{P}_1$, which we shall then use to transform any word to its canonical form.

Lemma 3.8. *If $u, v \in \{a,b\}^*$ are such that $u = v$ holds in $\mathrm{SL}(2,p)$, then $u = v$ is consequence of $\mathfrak{P}_1$.*

PROOF. Since $\langle a, b \mid \mathfrak{R}\rangle$ is a presentation for $\mathrm{SL}(2,p)$, $u = v$ is a consequence of $\mathfrak{R}$. On the other hand, $\mathfrak{P}_1$ contains $\mathfrak{R}$, and the result follows. ■

Lemma 3.9. *The following relations are consequences of $\mathfrak{P}_1$:*

(i) $bab^{p-1}a^2b = b^{p-1}a$;

(ii) $ab^{p-1}ab = b^{p-1}a$;

(iii) $ba^{\xi^{m+1}-\xi^m}b^{p-1}a^{-\xi+2-\xi^{-m}}b^{p-1}a^{1-\xi^{-1}}b^{\xi^{m+2}-\xi^{m+1}+\xi} = a^{1-\xi^{-m-1}}$, $m \geq 1$.

PROOF. By Lemma 3.8 it is enough to check that the matrices A and B satisfy relations (i), (ii), (iii), which can be done by a straightforward matrix calculation. ■

Lemma 3.10. *The following relations are consequences of $\mathfrak{P}_1$:*

(i) $ba^{\xi^m-1}s = a^{1-\xi^{-m}}s^{1+m}$, *for all* $m \geq 1$;

(ii) $s^p = s$;

(iii) $ba^{p-1}s = b^{p-1}as^{\frac{p+1}{2}}$;

(iv) $ab^{p-1}as = b^{p-1}as$;

(v) $sb^{p-1}as = 0$.

PROOF. (i) We prove this relation by induction on m. For $m = 1$ it is the relation $ba^{\xi-1}s = a^{1-\xi^{-1}}s^2$, which is in $\mathfrak{P}_1$. Let us assume that it is true for some $m \geq 1$. Then

$$
\begin{aligned}
& ba^{\xi^{m+1}-1}s \\
= \;& ba^{\xi^{m+1}-\xi^m}a^{\xi^m-1}s \\
= \;& ba^{\xi^{m+1}-\xi^m}b^{p-1}ba^{\xi^m-1}s \\
= \;& ba^{\xi^{m+1}-\xi^m}b^{p-1}a^{1-\xi^{-m}}s^{1+m} && \text{(the hypothesis)} \\
= \;& ba^{\xi^{m+1}-\xi^m}b^{p-1}a^{1-\xi^{-m}}a^{1-\xi}a^{\xi-1}ss^m \\
= \;& ba^{\xi^{m+1}-\xi^m}b^{p-1}a^{-\xi+2-\xi^{-m}}b^{p-1}ba^{\xi-1}ss^m \\
= \;& ba^{\xi^{m+1}-\xi^m}b^{p-1}a^{-\xi+2-\xi^{-m}}b^{p-1}a^{1-\xi^{-1}}s^2s^m && (\text{by } ba^{\xi-1}s = a^{1-\xi^{-1}}s^2) \\
= \;& ba^{\xi^{m+1}-\xi^m}b^{p-1}a^{-\xi+2-\xi^{-m}}b^{p-1}a^{1-\xi^{-1}}b^{\xi^{m+2}-\xi^{m+1}+\xi}s^{2+m} \\
& && (\text{relation } bs = s) \\
= \;& a^{1-\xi^{-m-1}}s^{2+m}. && \text{(by Lemma 3.9(iii))}
\end{aligned}
$$

(ii) This relation can be obtained from relation (i) for $m = p-1$.

(iii) In this case we have

$$\begin{aligned} & ba^{p-1}s \\ = \;& ba^{p-1}a^{-\xi^{\frac{p-1}{2}}+1}b^{p-1}ba^{\xi^{\frac{p-1}{2}}-1}s \\ = \;& ba^{p-1}a^{-\xi^{\frac{p-1}{2}}+1}b^{p-1}a^{1-\xi^{-\frac{p-1}{2}}}s^{1+\frac{p-1}{2}} && \text{(by (i))} \\ = \;& ba^{p-1}a^{2}b^{p-1}a^{2}s^{\frac{p+1}{2}} && (\xi^{\frac{p-1}{2}} = \xi^{-\frac{p-1}{2}} = -1) \\ = \;& bab^{p-1}a^{2}bs^{\frac{p+1}{2}} && \text{(relation } bs = s) \\ = \;& b^{p-1}as^{\frac{p+1}{2}}. && \text{(by Lemma 3.9(i))} \end{aligned}$$

(iv) Now we use the relation $bs = s$ and Lemma 3.9(ii) to obtain

$$ab^{p-1}as = ab^{p-1}abs = b^{p-1}as.$$

(v) Using (ii) and (iii) we obtain

$$sb^{p-1}as = sb^{p-1}as^{\frac{p+1}{2}}s^{\frac{p-1}{2}} = sba^{p-1}ss^{\frac{p-1}{2}} = 0s^{\frac{p-1}{2}} = 0,$$

which completes the proof of the lemma. ∎

Let us now define a mapping

$$\phi : \{a, b, s\}^* \cup \{0\} \longrightarrow \{a, b, s\}^* \cup \{0\}$$

by

$$\phi(a) = b,\ \phi(b) = a,\ \phi(s) = s,\ \phi(0) = 0,\ \phi(\epsilon) = \epsilon,$$
$$\phi(\alpha_1\alpha_2\ldots\alpha_k) = \phi(\alpha_k)\ldots\phi(\alpha_2)\phi(\alpha_1),$$

where $\alpha_1, \ldots, \alpha_k \in \{a, b, s\}$. Since it is obvious that ϕ^2 is the identity mapping, we will refer to w and $\phi(w)$ as *dual* words. It is obvious that the matrix represented by the word $\phi(w)$ is the transpose of the matrix represented by the word w.

Lemma 3.11. *Let $u, v \in \{a, b, s\}^* \cup \{0\}$. If $u = v$ is a consequence of $\mathfrak{P}_1$ then so is $\phi(u) = \phi(v)$.*

PROOF. It is obviously enough to prove the lemma when $u = v$ is any relation from the presentation $\mathfrak{P}_1$. If $u = v$ belongs to $\mathfrak{R}$, then u and v represent the same matrix X in $\mathrm{SL}(2,p)$. But then both $\phi(u)$ and $\phi(v)$ represent X^T. Therefore the relation $\phi(u) = \phi(v)$ holds in $\mathrm{SL}(2,p)$, and is a consequence of $\mathfrak{R}$. The relations $bs = s$ and $sa = s$, and $ba^{\xi-1}s = a^{1-\xi^{-1}}s^2$ and $sb^{\xi-1}a = s^2b^{1-\xi^{-1}}$ are dual in pairs. Finally, the relation $sba^{p-1}s = 0$ is dual to the relation $sb^{p-1}as = 0$, which is a consequence of $\mathfrak{P}_1$ by Lemma 3.10(v). ∎

Lemma 3.12. *For any word* $w \in \{a, b, s\}^* \cup \{0\}$ *there exists a word* $\overline{w} \in W$ *such that* $w = \overline{w}$ *holds in* $\mathrm{SLS}(2, p)$.

PROOF. If $w \in \{a, b\}^*$ then this is Lemma 3.6. So we assume that w contains s, and prove the lemma by induction on the length of w.

If $|w| = 1$ then $w \equiv s = \mathrm{cf}_1(p, 1, p) \in W$. For $|w| > 1$, w can be written in one of the forms $w'a$, $w'b$, $w's$, aw', bw' or sw', where w' also contains s. Therefore, it is enough to show that the product of any element of W and any generator a, b or s is equal to another element of W. Note that, with respect to the anti-isomorphism ϕ, the words $\mathrm{cf}_1(i, j, k)$ and $\mathrm{cf}_1(k, j, i)$, and $\mathrm{cf}_2(i, j)$ and $\mathrm{cf}_3(j, i)$ are dual in pairs, and that the words $\mathrm{cf}_4(i)$ and cf_5 are self-dual. It follows that it is enough to show that premultiplying of a word from W by a generator yields a word which is equal in $\mathcal{S}$ to another word from W—the result for postmultiplying follows by applying ϕ and Lemma 3.11. Using Lemma 3.10 and relations from $\mathfrak{P}_1$ it is easy to see that:

$$
\begin{aligned}
&a \cdot \mathrm{cf}_1(i, j, k) = \mathrm{cf}_1(i + 1 \pmod p, j, k),\\
&b \cdot \mathrm{cf}_1(\xi^m - 1 \pmod p, j, k) = \mathrm{cf}_1(1 - \xi^{-m} \pmod p, j + m \pmod{p - 1}, k),\\
&b \cdot \mathrm{cf}_1(p - 1, j, k) = \mathrm{cf}_2(j + \frac{p-1}{2} \pmod{p - 1}, k),\\
&s \cdot \mathrm{cf}_1(i, j, k) = \mathrm{cf}_1(p, j + 1 \pmod{p - 1}, k),\\
&a \cdot \mathrm{cf}_2(i, j) = \mathrm{cf}_2(i, j),\\
&b \cdot \mathrm{cf}_2(i, j) = \mathrm{cf}_1(1, i, j),\\
&s \cdot \mathrm{cf}_2(i, j) = \mathrm{cf}_5,\\
&a \cdot \mathrm{cf}_3(i, j) = \mathrm{cf}_3(i + 1 \pmod p, j),\\
&b \cdot \mathrm{cf}_3(\xi^m - 1 \pmod p, j) = \mathrm{cf}_3(1 - \xi^{-m} \pmod p, j + m \pmod{p - 1}),\\
&b \cdot \mathrm{cf}_3(p - 1, j) = \mathrm{cf}_4(j + \frac{p-1}{2} \pmod{p - 1}),\\
&s \cdot \mathrm{cf}_3(i, j) = \mathrm{cf}_3(p, j + 1 \pmod{p - 1}),\\
&a \cdot \mathrm{cf}_4(i) = \mathrm{cf}_4(i),\\
&b \cdot \mathrm{cf}_4(i) = \mathrm{cf}_3(1, i),\\
&s \cdot \mathrm{cf}_4(i) = \mathrm{cf}_5,\\
&a \cdot \mathrm{cf}_5 = b \cdot \mathrm{cf}_5 = s \cdot \mathrm{cf}_5 = \mathrm{cf}_5.
\end{aligned}
$$

The calculations modulo p and modulo $p-1$ have been performed with the values in the sets $\{1, \ldots, p\}$ and $\{1, \ldots, p - 1\}$ respectively, and the fact that

$$\{\xi^m - 1 \pmod p \mid m \in \mathbb{N}\} = \{1, \ldots, p - 2, p\}$$

has been used.■

Now we have all the ingredients for proving the first of our main theorems.

PROOF OF THEOREM 3.1. For $p = 2$ the result follows from Theorem 3.4 and the remark after it. For $p \geq 3$ the matrices A, B, S generate $\mathrm{SLS}(2, p)$ by Theorem 4.6.4, and satisfy all the relations from $\mathfrak{P}_1$ by Lemma 3.5. Each word from $\{a, b, s\}^* \cup \{0\}$ is equal to a word from W by Lemma 3.12, and distinct elements of W represent distinct elements of $\mathrm{SLS}(2,p)$ by Lemma 3.7. Therefore, conditions (i), (ii), (iii) of Proposition 2.1 are satisfied, and the result follows by that theorem. ■

We prove Theorem 3.2 by applying Tietze transformations to the presentation $\mathfrak{P}_1$. First, however, we need to establish some consequences of the presentation $\mathfrak{P}_2$.

Lemma 3.13. *If $u = v$, $u, v \in \{a, b\}^*$, is a relation which holds in $\mathrm{SLS}(2, p)$, then $u = v$ is a consequence of $\mathfrak{P}_2$.*

PROOF. The proof is exactly the same as the proof of Lemma 3.8. ■

Lemma 3.14. *The following relations are consequences of $\mathfrak{P}_2$:*

(i) $a^{-\xi^{-1}+\xi^{-2}+\xi^{-i-1}-\xi^{-i-2}} b^{\xi} a^{1-\xi^{-1}} b^{\xi^{i}-2} a b^{\xi^{-i}-\xi^{-i-1}} = b^{\xi^{i+1}-1} a$, *for* $i \geq 0$;

(ii) $b a^{\xi-2} b^{-\xi+1} a^{\xi^{-1}} b^{\xi^2-2\xi} = a^{1-\xi^{-1}}$;

(iii) $a^{\xi^{-1}-\xi^{-2}} b^{\xi^2-\xi} a^{\xi^{-1}} b^{-\xi+2-\xi^{-1}} = b^{\xi-1} a$;

(iv) $a^{\xi^{-2}} b^{\xi} a^{1-\xi^{-1}} b^{-\xi^{-1}} = b^{\xi-1} a$.

PROOF. By Lemma 3.13 it is enough to check that the matrices A and B satisfy all the relations (i) to (iv). ■

Lemma 3.15. *The following relations are consequences of $\mathfrak{P}_2$:*

(i) $(tb^{\xi-1}at)^k = tb^{\xi^k-1}at$ *for* $k \geq 1$;

(ii) $ba^{\xi-1}tb^{\xi-1}at = a^{1-\xi^{-1}}(tb^{\xi-1}at)^2$;

(iii) $tb^{\xi-1}atb^{\xi-1}a = (tb^{\xi-1}at)^2 b^{1-\xi^{-1}}$.

PROOF. (i) We prove this relation by induction on k, the case $k = 1$ being obvious. Suppose the relation is true for some $k \geq 1$. Then

$$\begin{aligned}
& (tb^{\xi-1}at)^{k+1} \\
= \;& tb^{\xi-1}atb^{\xi^k-1}at && \text{(by the hypothesis and } t^2 = t) \\
= \;& tb^{\xi}a^{1-\xi^{-1}}b^{\xi^k-2}at && \text{(since } b^{\xi-1}atb = a^{\xi^{-1}}tb^{\xi}a^{1-\xi^{-1}} \text{ and } ta = t) \\
= \;& ta^{-\xi^{-1}+\xi^{-2}+\xi^{-k-1}-\xi^{-k-2}}b^{\xi}a^{1-\xi^{-1}}b^{\xi^k-2}ab^{\xi^{-k}-\xi^{-k-1}}t && \\
& && \text{(since } ta = bt = t) \\
= \;& tb^{\xi^{k+1}-1}at. && \text{(by Lemma 3.14(i))}
\end{aligned}$$

(ii) Now we have

$$\begin{array}{lll}
 & ba^{\xi-1}tb^{\xi-1}at & \\
= & ba^{\xi-2}(atb)b^{\xi-2}at & \\
= & ba^{\xi-2}b^{-\xi+1}a^{\xi^{-1}}tb^{\xi}a^{1-\xi^{-1}}b^{\xi-2}at & (\text{since } b^{\xi-1}atb = a^{\xi^{-1}}tb^{\xi}a^{1-\xi^{-1}}) \\
= & ba^{\xi-2}b^{-\xi+1}a^{\xi^{-1}}b^{\xi^2-2\xi}ta^{-\xi^{-1}+\xi^{-2}+\xi^{-2}-\xi^{-3}}b^{\xi}a^{1-\xi^{-1}}b^{\xi-2}ab^{\xi^{-1}-\xi^{-2}}t & \\
 & & (\text{since } ta = bt = t) \\
= & a^{1-\xi^{-1}}tb^{\xi^2-1}at & (\text{by Lemma 3.14(i) for } i = 1 \\
 & & \text{and Lemma 3.14(ii)}) \\
= & a^{1-\xi^{-1}}(tb^{\xi-1}at)^2. & (\text{by (i)})
\end{array}$$

(iii) In this case

$$\begin{array}{lll}
 & (tb^{\xi-1}at)^2b^{1-\xi^{-1}} & \\
= & tb^{\xi^2-1}atb^{1-\xi^{-1}} & (\text{by (i)}) \\
= & tb^{\xi^2-1}(atb)b^{-\xi^{-1}} & \\
= & tb^{\xi^2-1}b^{-\xi+1}a^{\xi^{-1}}tb^{\xi}a^{1-\xi^{-1}}b^{-\xi^{-1}} & (\text{since } b^{\xi-1}atb = a^{\xi^{-1}}tb^{\xi}a^{1-\xi^{-1}}) \\
= & ta^{\xi^{-1}-\xi^{-2}}b^{\xi^2-\xi}a^{\xi^{-1}}b^{-\xi+2-\xi^{-1}}ta^{\xi^{-2}}b^{\xi}a^{1-\xi^{-1}}b^{-\xi^{-1}} & \\
 & & (\text{since } ta = bt = t) \\
= & tb^{\xi-1}atb^{\xi-1}a, &
\end{array}$$

where Lemma 3.14(iii) and (iv) have been used. ■

PROOF OF THEOREM 3.2. Let us add to the presentation $\mathfrak{P}_1$ a new generating symbol t, and the relation $t = s^{p-1}$ (elementary Tietze transformation (T3)). The matrix represented by t in $\mathrm{SLS}(2,p)$ is

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since we know (Theorem 3.1) that $\mathfrak{P}_1$ defines $\mathrm{SLS}(2,p)$, and since all the relations from $\mathfrak{P}_2$ hold in $\mathrm{SLS}(2,p)$, we can also add all these relations to $\mathfrak{P}_1$ (transformation (T1)). Finally we add the valid relation $s = tb^{\xi-1}at$ and obtain the presentation:

$$\begin{gathered}
\langle a, b, s, t \mid \mathcal{R},\ bs = sa = s,\ sba^{p-1}s = 0, \\
ba^{\xi-1}s = a^{1-\xi^{-1}}s^2,\ sb^{\xi-1}a = s^2b^{1-\xi^{-1}},\ t = s^{p-1},\ s = tb^{\xi-1}at, \\
t^2 = bt = ta = t,\ tba^{p-1}t = 0,\ b^{\xi-1}atb = a^{\xi^{-1}}tb^{\xi}a^{1-\xi^{-1}}\rangle.
\end{gathered}$$

Now we eliminate s by using $s = tb^{\xi-1}at$ (transformation (T4)) and obtain the presentation:

$$\begin{gathered}
\langle a, b, t \mid \mathfrak{R},\ btb^{\xi-1}at = tb^{\xi-1}ata = tb^{\xi-1}at,\ tb^{\xi-1}atba^{p-1}tb^{\xi-1}at = 0, \\
ba^{\xi-1}tb^{\xi-1}at = a^{1-\xi^{-1}}(tb^{\xi-1}at)^2,\ tb^{\xi-1}atb^{\xi-1}a = (tb^{\xi-1}at)^2b^{1-\xi^{-1}}, \\
t = (tb^{\xi-1}at)^{p-1},\ t^2 = bt = ta = t,\ tba^{p-1}t = 0,\ b^{\xi-1}atb = a^{\xi^{-1}}tb^{\xi}a^{1-\xi^{-1}}\rangle.
\end{gathered}$$

Then we use the transformation (T2) to eliminate the first six relations (after $\mathfrak{R}$) as consequences of the remaining five. The first three of them are easy consequences of $bt = t$, $ta = t$ and $tba^{p-1}t = 0$ respectively. The next two follow from Lemma 3.15 (ii) and (iii) respectively, while the relation $t = (tb^{\xi-1}at)^{p-1}$ follows from Lemma 3.15 (i) for $k = p - 1$. The obtained presentation

$$\langle a, b, t \mid \mathfrak{R},\ t^2 = t,\ bt = ta = t,\ tba^{p-1}t = 0,\ b^{\xi-1}atb = a^{\xi^{-1}}tb^{\xi}a^{1-\xi^{-1}} \rangle$$

is obviously $\mathfrak{P}_2$. ■

4. General linear semigroups GLS$(2, p)$

As mentioned before SL$(2, p)$ is a normal subgroup of GL$(2, p)$, and is in fact the kernel of the determinant homomorphism det : GL$(2, p) \longrightarrow \mathbb{Z}_p$. Therefore GL$(2, p)$/SL$(2, p)$ is isomorphic to the multiplicative group of $\mathbb{Z}_p$. As $\mathbb{Z}_p$ is a finite field, its multiplicative group is cyclic, and we denote by ξ an arbitrary generator of this group. Thus, GL$(2, p)$ is generated by SL$(2, p)$ and another matrix of determinant ξ. Here it will be convenient to choose this matrix to be

$$C = \begin{pmatrix} 1 & 0 \\ 0 & \xi \end{pmatrix}.$$

The main result of this section is the following

Theorem 4.1. *Let $\langle a, b, c \mid \mathfrak{R} \rangle$ be any (monoid) presentation for* GL$(2, p)$ *in terms of generators A, B and C. Then the presentations*

$$\mathfrak{P}_3 = \langle a, b, c, s \mid \mathfrak{R},\ cs = sc = s,\ sba^{p-1}s = 0,$$
$$ba^{\xi-1}s = a^{1-\xi^{-1}}s^2,\ sb^{\xi-1}a = s^2b^{1-\xi^{-1}} \rangle$$

and

$$\mathfrak{P}_4 = \langle a, b, c, t \mid \mathfrak{R},\ t^2 = ct = tc = t,\ tba^{p-1}t = 0,\ b^{\xi-1}atb = a^{\xi^{-1}}tb^{\xi}a^{1-\xi^{-1}} \rangle$$

define the semigroup GLS$(2, p)$ *in terms of generators A, B, C, S and A, B, C, T respectively.*

Remark 4.2. By Proposition 1.6 the presentation

$$\langle a, b, c \mid a^p = 1,\ b^2 = (ab)^3 = (a^4ba^{\frac{p+1}{2}}b)^2,\ c^{p-1} = 1,$$
$$c^{-1}ac = a^{\xi}, c^{-1}bc = ba^{-\xi^{-1}}ba^{-\xi}ba^{-\xi^{-1}}b \rangle.$$

defines GL$(2, p)$ in terms of generators

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \overline{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \overline{C} = \begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix}.$$

It is not difficult, however, to modify this presentation into a presentation which would define GL$(2, p)$ in terms of A, B and C. ■

PROOF OF THEOREM 4.1. For $p = 2$ both presentations $\mathfrak{P}_3$ and $\mathfrak{P}_4$ are equivalent to the presentation (1), and the theorem follows by Theorem 3.4. To prove the theorem for $p \geq 3$ we first show that the presentation

$$\overline{\mathfrak{P}}_3 = \langle a, b, c, s \mid \mathfrak{R},\ bs = sa = cs = sc = s,\ sba^{p-1}s = 0,\\ ba^{\xi-1}s = a^{1-\xi^{-1}}s^2,\ sb^{\xi-1}a = s^2b^{1-\xi^{-1}} \rangle,$$

which has been obtained from $\mathfrak{P}_1$ by adding a new generating symbol c and two new relations $cs = s$ and $sc = s$, defines $\mathrm{GLS}(2,p)$. The matrices A, B, C, S generate $\mathrm{GLS}(2,p)$ by Theorem 4.6.4, and it is easy to check that they satisfy all the relations from $\overline{\mathfrak{P}}_3$.

Let $W_1 \subseteq \{a, b, c\}^*$ be a set of canonical forms for $\mathrm{GL}(2,p)$, and let W_2 be the set of canonical forms for $\mathrm{Sing}(2,p)$ as defined before Lemma 3.7. We claim that the set $W = W_1 \cup W_2$ is a set of canonical forms for $\mathrm{GLS}(2,p)$. That distinct elements of W represent distinct elements of $\mathrm{GLS}(2,p)$ follows as in Lemmas 3.6 and 3.7. In order to prove that each word from $\{a, b, c, s\}^* \cup \{0\}$ is equal to a word from W, we first note that each word from $\{a, b, c\}^*$ is equal to a word from W_1 since W_1 is a set of canonical forms for $\mathrm{GL}(2,p)$, and $\overline{\mathfrak{P}}_3$ contains defining relations $\mathfrak{R}$ for $\mathrm{GL}(2,p)$. Also, each word from $\{a, b, s\}^*$ is equal to a word from W_2 by Lemma 3.12, since $\overline{\mathfrak{P}}_3$ contains defining relations for $\mathrm{SLS}(2,p)$.

If $w \in \{a, b\}^*$ is any word, then $wc = cw_1$ and $cw = w_2c$ for some $w_1, w_2 \in \{a, b\}^*$, because $\mathrm{SL}(2,p)$ is a normal subgroup of $\mathrm{GL}(2,p)$. But then $swc = scw_1 = sw_1$ and $cws = w_2cs = w_2s$. This can be used to prove that every word involving s can be transformed by using relations from $\overline{\mathfrak{P}}_3$ to a word not involving c, and hence can be transformed to a word from W_2. Therefore, $\overline{\mathfrak{P}}_3$ defines $\mathrm{GLS}(2,p)$ by Proposition 2.1.

Now we eliminate the relations $bs = s$ and $sa = s$. If we use the relation

$$s^2 = a^{-1+\xi^{-1}}ba^{\xi-1}s, \tag{2}$$

which is a consequence of $ba^{\xi-1}s = a^{1-\xi^{-1}}s^2$, together with the relations $ca = a^{\xi}c$ and $cb = b^{\xi^{-1}}c$, which are true in $\mathrm{GL}(2,p)$, we obtain

$$s^2 = cs^2 = a^{-\xi+1}b^{\xi^{-1}}a^{\xi^2-\xi}s. \tag{3}$$

From (2) and (3) it follows that

$$s = a^{-\xi+1}b^{-1}a^{-\xi+2-\xi^{-1}}b^{\xi^{-1}}a^{\xi^2-\xi}s = b^{-\xi^{-1}+\xi^{-2}}s, \tag{4}$$

since $a^{-\xi+1}b^{-1}a^{-\xi+2-\xi^{-1}}b^{\xi^{-1}}a^{\xi^2-\xi} = b^{-\xi^{-1}+\xi^{-2}}$ holds in $\mathrm{GL}(2,p)$. Since p is a prime number greater than 2, we have $-\xi^{-1} + \xi^{-2} \neq 0$, so that (4) implies $bs = s$, and this relation can be eliminated from $\overline{\mathfrak{P}}_3$ by using Tietze transformation (T2). Note that in eliminating the relation $bs = s$ we have not used the relation $sa = s$, so that $sa = s$ can be eliminated as well by a dual argument. Hence, $\mathfrak{P}_3$ is a presentation for $\mathrm{GLS}(2,p)$.

Now we prove that $\mathfrak{P}_4$ is a presentation for $\mathrm{GLS}(2,p)$. The argument proving that $\overline{\mathfrak{P}}_3$ is a presentation for $\mathrm{GLS}(2,p)$ can be essentially repeated to prove that the presentation

$$\overline{\mathfrak{P}}_4 = \langle a, b, c, t \mid R,\ t^2 = bt = ta = ct = tc = t,\\ tba^{p-1}t = 0,\ b^{\xi-1}atb = a^{\xi^{-1}}tb^{\xi}a^{1-\xi^{-1}} \rangle$$

is a presentation for $\mathrm{GLS}(2,p)$ as well. To eliminate the relation $bt = t$ we use the relation

$$t = a^{-1}b^{-\xi+1}a^{\xi^{-1}}tb^{\xi}a^{1-\xi^{-1}}b^{-1}, \tag{5}$$

which is a consequence of the last relations of $\overline{\mathfrak{P}}_4$. Premultiplying (5) by c gives

$$t = a^{-\xi}b^{-1+\xi^{-1}}atb^{\xi}a^{1-\xi^{-1}}b^{-1}. \tag{6}$$

Combining (5) and (6) gives

$$t = a^{-\xi^{-1}}b^{\xi-1}a^{-\xi+1}b^{-1+\xi^{-1}}at = b^{(\xi-1)^2}t, \tag{7}$$

since $a^{-\xi^{-1}}b^{\xi-1}a^{-\xi+1}b^{-1+\xi^{-1}}a = b^{(\xi-1)^2}$ holds in $\mathrm{GL}(2,p)$. The relation (7) implies $bt = t$, since $p > 2$. Again, we have not used the relation $ta = t$, so that a dual argument shows that this relation is redundant as well. ∎

Remark 4.3. Inclusion of 'group relations' $\mathfrak{R}$ in all presentations $\mathfrak{P}_1$, $\mathfrak{P}_2$, $\mathfrak{P}_3$ and $\mathfrak{P}_4$ does not cause any loss of generality. Since $\mathrm{Sing}(2,p)$ is an ideal in $\mathrm{GLS}(2,p)$ (respectively, in $\mathrm{SLS}(2,p)$), and since $\mathrm{GLS}(2,p) - \mathrm{Sing}(2,p) = \mathrm{GL}(2,p)$ (respectively $\mathrm{SLS}(2,p) - \mathrm{Sing}(2,p) = \mathrm{SL}(2,p)$) is a subsemigroup, any presentation for $\mathrm{GLS}(2,p)$ (respectively $\mathrm{SLS}(2,p)$) has the form $\langle A, B \mid \mathfrak{R},\ \mathfrak{S}\rangle$, where $\langle A \mid \mathfrak{R}\rangle$ is a presentation for $\mathrm{GL}(2,p)$ (respectively $\mathrm{SL}(2,p)$).

As in the case of full transformation semigroups it seems reasonable to pose a problem about finding minimal presentations.

Open Problem 5. Find the minimal number k such that there exists a presentation of the form $\langle A, B \mid \mathfrak{R},\ \mathfrak{S}\rangle$ for the special linear semigroup $\mathrm{SLS}(2,p)$ (respectively, for the general linear semigroup $\mathrm{GLS}(2,p)$) such that the presentation $\langle A \mid \mathfrak{R}\rangle$ defines the special linear group $\mathrm{SL}(2,p)$ (respectively, the general linear group $\mathrm{GL}(2,p)$) and $|\mathfrak{S}| = k$.

It would be also interesting to find presentations for some more general matrix semigroups.

Open Problem 6. Find presentations for the semigroup $\mathrm{GLS}(d,R)$ of all $d \times d$ matrices over the ring R for various d and various R. In particular, find presentations for $\mathrm{GLS}(d,R)$ in the following cases:

(i) $d = 2$, $R = \mathrm{GF}(p^n)$—a general finite field;

(ii) $d = 2$, $R = \mathbb{Z}_m$—the ring of integers modulo m;

(iii) $d > 2$, $R = \mathbb{Z}_p$, p prime.

5. Using computational tools

Computational techniques can be used very successfully as an aid in the type of investigation we have described in previous sections.

First of all these techniques can be used to predict results. The most important algorithm in this context is the *Todd—Coxeter enumeration procedure.* This procedure is described in detail in Chapter 14, but, for the moment it is sufficient to think about it as the simple program illustrated in Figure 4.

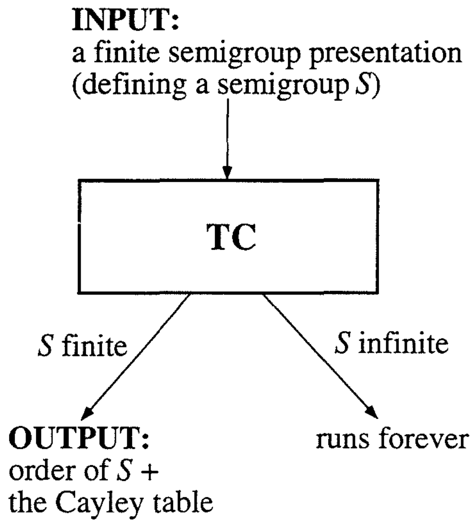


Figure 4.

So, if we are looking for a presentation for a finite semigroup S, and if we have a 'candidate' $\mathfrak{P}$, then we can test our conjecture by inputting $\mathfrak{P}$ into the Todd—Coxeter enumeration procedure. If the procedure terminates with the result equal to $|S|$, then $\mathfrak{P}$ is a presentation for S by Proposition 2.2; if the procedure terminates with the result different from $|S|$ then $\mathfrak{P}$ is not a presentation for S. However, if the procedure does not terminate after finite time we cannot draw any conclusions: we do not know if the procedure will never terminate or it will terminate eventually.

Actually, if we are looking for a presentation for a single semigroup, a satisfactory output from the program *is* a proof (if we trust the actual implementation). For example, this would be an alternative way to prove Theorem 3.4.

However, in Sections 3 and 4 we were dealing with infinite families of semigroups $\mathrm{SLS}(2,p)$ and $\mathrm{GLS}(2,p)$, p prime. In this case we cannot expect a computer to prove theorems for us, as it cannot check our presentation for infinitely many primes. We can, however, check our conjectured presentations for small values of p.

Suppose, for the sake of an example, that we are looking for a presentation for $\mathrm{SLS}(2,p)$ in terms of generators A, B and T (and without knowing the results of Section 3). It is well known that

$$|\mathrm{GL}(2,p)| = (p^2-1)(p^2-p)$$

(see Rotman (1965)), so that

$$|\mathrm{SL}(2,p)| = \frac{|GL|}{p-1} = p(p^2-1).$$

Also, $|\mathrm{GLS}(2,p)| = p^4$, and hence

$$|\mathrm{SLS}(2,p)| = |\mathrm{GLS}(2,p)| - |\mathrm{GL}(2,p)| + |\mathrm{SL}(2,p)| = p^4-(p^2-1)(p^2-p)+p(p^2-1).$$

We also know (see Remark 4.3) that any presentation for $\mathrm{SLS}(2,p)$ contains defining relations for $\mathrm{SL}(2,p)$; let us denote by $\mathfrak{R}(p)$ a set of such relations.

To begin with we choose $p=3$, so that $|\mathrm{SLS}(2,p)| = 57$. If we take obvious relations $t^2 = at = tb = t$ and input the presentation

$$\langle a,b,t \mid \mathfrak{R}(3),\ t^2 = at = tb = t\rangle, \tag{8}$$

the procedure does not terminate in a reasonable time. Although we cannot draw any conclusions from this, it is reasonable to expect that the semigroup defined by (8) is infinite. If we add the relation $tba^2t = 0$, we obtain the presentation

$$\langle a,b,t \mid \mathfrak{R}(3),\ t^2 = at = tb = t,\ tba^2t = 0\rangle, \tag{9}$$

which still seems to define an infinite semigroup. Finally, if we add the relation $batb = a^2tb^2a^2$ we obtain a semigroup of order 57. Thus

$$\langle a,b,t \mid \mathfrak{R}(3),\ t^2 = at = tb = t,\ tba^2t = 0,\ batb = a^2tb^2a^2\rangle \tag{10}$$

defines $\mathrm{SLS}(2,3)$.

Now we consider $p=5$. An analogue of (10) for $p=5$ might look like

$$\langle a,b,t \mid \mathfrak{R}(5),\ t^2 = at = tb = t,\ tba^4t = 0,\ batb = a^3tb^2a^3\rangle. \tag{11}$$

And indeed, (11) defines a semigroup of order 265, which is the same as $|\mathrm{SLS}(2,5)|$.

After this we might hope that

$$\langle a,b,t \mid \mathfrak{R}(p),\ t^2 = at = tb = t,\ tba^{p-1}t = 0,\ batb = a^{\frac{p+1}{2}}tb^2a^{\frac{p+1}{2}}\rangle \tag{12}$$

is a presentation for $\mathrm{SLS}(2,p)$.

However, for $p=7$ the presentation (12) becomes

$$\langle a,b,t \mid \mathfrak{R}(7),\ t^2 = at = tb = t,\ tba^6t = 0,\ batb = a^4tb^2a^4\rangle, \tag{13}$$

and if we input this presentation into the Todd—Coxeter program, the procedure does not terminate. Actually, we can prove that (13) does not define SLS$(2,7)$, for, if we add the relation

$$tb^2atb = tb^3a^3,$$

to (13) we obtain a semigroup of order 1105, while $|\mathrm{SLS}(2,7)| = 721$.

The first plausible explanation for this is that presentation (12) worked for $p = 3$ and $p = 5$ just because they are small enough, while, in general, some more relations are needed. However, if we test presentation (12) for $p = 11$ and $p = 13$, we obtain semigroups of orders 2761 and 4537 respectively, which is the same as the orders of SLS$(2,11)$ and SLS$(2,13)$ respectively. Presentation (12) fails again for $p = 17$.

On the other hand, if we replace the relation $batb = a^4tb^2a^4$ in (13) by $b^2atb = a^5tb^3a^3$ we obtain a semigroup of order $721 = |\mathrm{SLS}(2,7)|$. Similarly, we see that $batb = a^9tb^2a^9$ in the presentation (12) for $p = 17$ should be replaced by $b^2atb = a^6tb^3a^12$.

Let us now recall the fact that $\xi = 2$ is a primitive root of 1 modulo p, for $p = 3,5,11,13$, but is not a primitive root of 1 for $p = 7,17$, in which cases $\xi = 3$ is a primitive root. Therefore one might expect that the last relation of (12) should be replaced by a relation depending on ξ. In view of the above discussion, it seems that the left-hand side of the new relation should be $b^{\xi-1}atb$, and it is then easy to see that the right-hand side should be $a^{\xi^{-1}}tb^{\xi}a^{1-\xi^{-1}}$. Therefore, we arrive at the presentation

$$\langle a,b,t \mid \mathfrak{R}_p,\ t^2 = at = tb = t, tba^{p-1}t = 0,\ b^{\xi-1}atb = a^{\xi^{-1}}tb^{\xi}a^{1-\xi^{-1}} \rangle,$$

which indeed is a presentation for SLS$(2,p)$—the fact which we prove in Section 3.

Possibilities of using computational techniques do not stop here. In the process of proving a conjecture various computer algebra packages (such as GAP) can be used to perform calculations.

For instance, in the proof of Lemma 3.12 we have a list of relations, some of which are too complicated to be guessed by looking at the presentation $\mathfrak{P}_1$. However, the main point of this lemma is to show that multiplying of a canonical form by a generator results in a word equal to another canonical form. By means of a GAP routine writing a singular 2×2 matrix in its canonical form, it is possible to see what the resulting canonical form is, and therefore to obtain a list of relations which we needed to prove. For example, consider the canonical word $\mathrm{cf}_1(i,j,k) = a^is^jb^k$; it represents the matrix

$$\begin{pmatrix} \xi^j & k\xi^j \\ i\xi^j & ik\xi^j \end{pmatrix}$$

as noted in Lemma 3.7. Premultiplying this matrix by B gives

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \xi^j & k\xi^j \\ i\xi^j & ik\xi^j \end{pmatrix} = \begin{pmatrix} (i+1)\xi^j & (i+1)k\xi^j \\ i\xi^j & ik\xi^j \end{pmatrix} = X.$$

We see that X is equal either to $\mathrm{cf}_1(i,j,k)$ for some i,j,k, or to $\mathrm{cf}_2(i,j)$ for some i,j, depending on whether $i \neq p-1$ or $i = p-1$; thus we obtain the second and the third relation from the list in the proof of Lemma 3.12.

A similar GAP routine helped us to find canonical forms of elements of $\mathrm{Sing}(2,p)$ in terms of generators A, B, T. These canonical forms are

$$\begin{aligned}
&\overline{\mathrm{cf}}_1(i,j,k) = a^i b a^j t b^k, && i,k = 1,\ldots,p,\ j = 1,\ldots,p-1,\\
&\overline{\mathrm{cf}}_2(i,j) = b^{p-1} a t b^i a b^j, && i = 1,\ldots,p-1,\ j = 1,\ldots,p,\\
&\overline{\mathrm{cf}}_3(i,j) = a^i b a^j t b a^{p-1}, && i = 1,\ldots,p,\ j = 1,\ldots,p-1,\\
&\overline{\mathrm{cf}}_4(i) = b^i a^{i^{-1}} t b a^{p-1}, && i = 1,\ldots,p-1,\\
&\overline{\mathrm{cf}}_5 = 0.
\end{aligned}$$

Finally, after we have found presentations for our semigroups, computational techniques can be used to investigate the interdependence of our relations. Here we shall show that the presentations $\mathfrak{P}_1$, $\mathfrak{P}_2$, $\mathfrak{P}_3$, $\mathfrak{P}_4$ have no redundant relations for $p = 7$ (with $\xi = 3$).

The orders of $\mathrm{GLS}(2,7)$ and $\mathrm{SLS}(2,7)$ are 2401 and 721 respectively. Consider the presentation

$$\overline{\mathfrak{P}}_4 = \langle a,b,c,t \mid \mathfrak{R},\ t^2 = bt = ta = ct = tc = t,\ tba^6t = 0,\ b^2atb = a^5tb^3a^3 \rangle$$

from the proof of Theorem 4.1. Let Σ_1, Σ_6, Σ_7 be the presentations obtained from $\overline{\mathfrak{P}}_4$ by replacing $t^2 = t$, $tba^6t = 0$, $b^2atb = a^5tb^3a^3$ respectively by $t^3 = t$, $tba^6tb = tba^6t$, $tb^2atb = tb^3a^3$ respectively, and let Σ_2, Σ_3, Σ_4, Σ_5 be obtained from $\overline{\mathfrak{P}}_4$ by omitting the relations $bt = t$, $ta = t$, $ct = t$, $tc = t$ respectively. The orders of the semigroups defined by Σ_i, $i = 1,\ldots,7$, are 2786, 2401, 2401, 4321, 4321, 2408 and 4321 respectively. This shows that no relation from $\overline{\mathfrak{P}}_4$, except $bt = t$ and $ta = t$, is redundant. Consequently, the presentation $\mathfrak{P}_4$ has no redundant relations, and the presentation $\mathfrak{P}_2$ has no redundant relations except, possibly, $bt = t$ and $ta = t$. However, if we replace the relation $bt = t$ in $\mathfrak{P}_2$ by six relations $tb^it = t$, $i = 1,\ldots,6$, the resulting semigroup has order 3025, showing that $bt = t$ is not redundant. Similarly, replacing $ta = t$ by $ta^it = t$, $i = 1,\ldots,6$, yields a semigroup of order 3025, and $ta = t$ is not redundant either. Similar considerations show that, in general, there are no redundant relations in either $\mathfrak{P}_1$ or $\mathfrak{P}_3$.

Chapter 4

Constructions and presentations

In this chapter we work on the following general problem:

Problem. If $\langle A_i \mid \mathfrak{R}_i \rangle$, $i \in I$, are presentations for semigroups S_i, $i \in I$, and if the semigroup T is obtained from the family S_i, $i \in I$, by applying a *semigroup construction*, find a presentation for T.

For example, if the monoids S_1 and S_2 are given by presentations $\langle A_1 \mid \mathfrak{R}_1 \rangle$ and $\langle A_2 \mid \mathfrak{R}_2 \rangle$, and if T is the direct product $S_1 \times S_2$, then T has a presentation

$$\langle A_1, A_2 \mid \mathfrak{R}_1,\ \mathfrak{R}_2,\ a_1 a_2 = a_2 a_1\ (a_1 \in A_1,\ a_2 \in A_2) \rangle, \tag{1}$$

while the free product $S_1 * S_2$ has a presentation

$$\langle A_1, A_2 \mid \mathfrak{R}_1,\ \mathfrak{R}_2 \rangle. \tag{2}$$

It is worth noting that if S_1 and S_2 are assumed to be semigroups rather than monoids, then (1) fails to be a presentation for $S_1 \times S_2$ in general. For example if $S_1 = S_2 = \mathbb{N}$, the free monogenic semigroup, then both S_1 and S_2 can be defined by the presentation $\langle a \mid \ \rangle$. The direct product $S_1 \times S_2$, however, is not even finitely generated, let alone finitely presented: all the elements $(1, n)$, $n \in \mathbb{N}$, are indecomposable, and so every generating set for $S_1 \times S_2$ must contain all of them.

The constructions considered in this chapter are:

- the wreath product of two monoids (Section 1);
- Bruck—Reilly extension of a monoid (Section 2);
- Schützenberger product of two monoids (Section 3);
- strong semilattices of monoids (Section 4);
- Rees matrix semigroups over a monoid (Section 5);
- ideal extensions of a semigroup by another semigroup (Section 6);
- subsemigroups of semigroups (Section 7).

The results of the last section are of crucial importance for the rest of this thesis. They, together with the results of Sections 5 and 6, will be used in the following chapters to investigate the structure of semigroups defined by presentations.

It is also worth noting that the results of the first five sections do not generalise naturally to semigroups, for reasons similar to those mentioned for the direct product.

The results of Sections 1–5 will appear in Howie and Ruškuc (1994), the result of Section 6 will appear in Campbell, Robertson, Ruškuc and Thomas (1995d), and the results of Section 7 will appear in Campbell, Robertson, Ruškuc and Thomas (1995b).

1. The wreath product

In this section we are going to find a presentation for the (restricted) wreath product of two monoids. We shall do this by using the direct method described in Section 5.2 (Proposition 5.2.1). First, however, we recall the definition of the wreath product; for more details see Eilenberg (1976).

Let S and T be two monoids. The cartesian product of $|T|$ copies of the monoid S is denoted by $S^{\times T}$, while the corresponding direct product is denoted by $S^{\oplus T}$. One may think of $S^{\times T}$ as the set of all functions from T into S, and of $S^{\oplus T}$ as the set of all such functions f with finite support, that is to say, having the property that $xf = 1_S$ for all but finitely many $x \in T$. If we equip $S^{\times T}$ and $S^{\oplus T}$ with the componentwise multiplication, we obtain two monoids, both having the function

$$\overline{1} : T \longrightarrow S,\ x\overline{1} = 1_S,$$

as the identity.

The *unrestricted wreath product* of the monoid S by the monoid T, denoted by $S \operatorname{Wr} T$, is the set $S^{\times T} \times T$ with the multiplication defined by

$$(f,t)(g,t') = (fg^t, tt'),$$

where $g^t : T \longrightarrow S$ is defined by

$$xg^t = (xt)g,\ x \in T.$$

The *restricted wreath product* of the monoid S by the monoid T, denoted by $S \operatorname{wr} T$, is the set $S^{\oplus T} \times T$, with the same multiplication. Both $S \operatorname{Wr} T$ and $S \operatorname{wr} T$ are monoids with the identity $(\overline{1}, 1_T)$. Also, $S \operatorname{Wr} T = S \operatorname{wr} T$ if and only if $|S| = 1$ or T is finite.

We shall find a presentation for $S \operatorname{wr} T$, and the first step is to find a generating set. It is clear that the sets $\{(f, 1_T) \,|\, f \in S^{\oplus T}\}$ and $\{(\overline{1}, t) \,|\, t \in T\}$ are submonoids of $S \operatorname{wr} T$ isomorphic to $S^{\oplus T}$ and T respectively. Moreover, for $f \in S^{\oplus T}$ and $t \in T$,

$$(f, 1_T)(\overline{1}, t) = (f, t),$$

and so $S \operatorname{wr} T$ is generated by these two submonoids.

For $s \in S$ and $t \in T$ we now define $\overline{s}_t : T \longrightarrow S$ by

$$x\overline{s}_t = \begin{cases} s & \text{if } x = t \\ 1_S & \text{otherwise.} \end{cases}$$

Notice that if $f : T \longrightarrow S$ has finite support then

$$f = \prod_{t \in T} \overline{f(t)}_t.$$

Notice also that if the monoid S is generated by a set A, so that every $s \in S$ is expressible as a product $a^{(1)}a^{(2)} \dots a^{(n)}$ of elements of A, then

$$\overline{s}_t = \overline{a}_t^{(1)}\overline{a}_t^{(2)} \dots \overline{a}_t^{(n)},$$

for all $t \in T$. We therefore have

Lemma 1.1. *Suppose that the monoids S and T are generated by sets A and B respectively, and let*

$$\overline{A}_t = \{(\overline{a}_t, 1_T) \mid a \in A\},\ t \in T,$$
$$\overline{B} = \{(\overline{1}, b) \mid b \in B\}.$$

Then the set $(\bigcup_{t \in T} \overline{A}_t) \cup \overline{B}$ generates $A \operatorname{wr} B$. ■

Unlike the wreath product of groups (see Johnson (1980)), the above generating set is, in general, the best possible for monoids. If the identity 1_S of S is indecomposable in the sense that

$$s_1 s_2 = 1_S \Longrightarrow s_1 = s_2 = 1_S,$$

then the submonoid

$$T' = \{(\overline{1}, t) \mid t \in T\}$$

of $S \operatorname{wr} T$ has the property that

$$xy \in T' \Longrightarrow x \in T' \text{ and } y \in T',$$

and so every generating set for $S \operatorname{wr} T$ must contain a generating set for T'. Similarly, if 1_T is indecomposable in T, then each generating set for $S \operatorname{wr} T$ must contain a generating set for

$$\{(f, 1_T) \mid f \in S^{\oplus T}\} \cong S^{\oplus T},$$

and $\bigcup_{t \in T} \overline{A}_t$ is, in general, the smallest such set.

For any $t_1, t_2 \in T$ let $t_1 t_2^{-1}$ denote the set $\{t \in T \mid tt_2 = t_1\}$.

Theorem 1.2. *Suppose that the monoids S and T are defined by presentations $\langle A \mid \mathfrak{R} \rangle$ and $\langle B \mid \mathfrak{S} \rangle$ respectively. For each $t \in T$, let $A_t = \{a_t \mid t \in T\}$ be a copy of A, and let $\mathfrak{R}_t$ be the corresponding copy of $\mathfrak{R}$. The presentation having generators $(\bigcup_{t \in T} A_t) \cup B$ and relations*

$$\mathfrak{R}_t \ (t \in T); \ \mathfrak{S}; \tag{3}$$

$$a_t a'_u = a'_u a_t \ (a, a' \in A, \ u, t \in T, \ u \neq t); \tag{4}$$

$$b a_t = (\prod_{u \in tb^{-1}} a_u) b \ (a \in A, \ b \in B, \ t \in T); \tag{5}$$

defines $S \operatorname{wr} T$ in terms of the generators $(\bigcup_{t \in T} \overline{A}_t) \cup \overline{B}$.

Remark 1.3. Relations (4) imply that a product $a_{t_1} a_{t_2} \ldots a_{t_k}$, where $\{t_1, \ldots, t_k\} = X \subseteq T$ and all $t_1, \ldots, t_k$ are distinct, does not depend on the order of its factors. Therefore, we may use notation $\prod_{t \in X} a_t$, as in relations (5).

PROOF OF THEOREM 1.2. The correspondence $a \mapsto a_t$ between A and A_t can be extended to a bijection between A^* and A_t^*; for a word $w \in A^*$, w_t will denote its image in A_t^* under this bijection. Since the generators $\bigcup_{t \in T} \overline{A}_t$ generate the submonoid $\{(f, 1_T) \mid f \in S^{\oplus T}\} \cong S^{\oplus T}$, we see that, for each $w \in A^*$ and each $t \in T$, the word w_t represents the element $(\overline{w}_t, 1_T)$ of $S \operatorname{wr} T$, where $\overline{w}_t$ is given by

$$x\overline{w}_t = \begin{cases} w & \text{if } x = t \\ 1_S & \text{otherwise.} \end{cases}$$

Hence, these generators satisfy all the relations $\mathfrak{R}_t$, $t \in T$, as well as all the relations (4). Similarly, from the fact that the generators $\overline{B}$ generate the submonoid $\{(\overline{1}, t) \mid t \in T\} \cong T$, we deduce that they satisfy all the relations $\mathfrak{S}$.

To establish relation (5), note first that

$$(\overline{1}, b)(\overline{a}_t, 1_T) = (\overline{a}_t^b, b) = (\overline{a}_t^b, 1_T)(\overline{1}, b).$$

Now, for each x in T

$$\begin{aligned} x\overline{a}_t^b &= (xb)\overline{a}_t = \begin{cases} a & \text{if } xb = t \\ 1_S & \text{otherwise} \end{cases} = \begin{cases} a & \text{if } x \in tb^{-1} \\ 1_S & \text{otherwise} \end{cases} \\ &= \prod_{u \in tb^{-1}} x\overline{a}_u = x(\prod_{u \in tb^{-1}} \overline{a}_u). \end{aligned}$$

Thus

$$\overline{a}_t^b = \prod_{u \in tb^{-1}} \overline{a}_u,$$

and so

$$(\overline{1}, b)(\overline{a}_t, 1_T) = (\prod_{u \in tb^{-1}} (\overline{a}_u, 1_T))(\overline{1}, b),$$

as required.

Now let w be any word in the letters from the alphabet $(\bigcup_{t\in T} A_t) \cup B$. By using relations (5) it is possible to transform w into a word of the form $w'w''$, where $w' \in (\bigcup_{t\in T} A_t)^*$ and $w'' \in B^*$. Furthermore, by using relations (4), the word w' can be transformed into a product $\prod_{t\in T} w(t)_t$, where $w(t) \in A^*$, and only finitely many of the words $w(t)$, $t \in T$, are non-empty.

Let $W_1 \subseteq A^*$ denote a set of canonical forms for S, let $W_{1,t} \subseteq A_t^*$, $t \in T$, be the corresponding copy of W_1, and let W_2 be a set of canonical forms for T. Each word from A_t^*, $t \in T$, can be transformed into a word from $W_{1,t}$ by using relations $\mathfrak{R}_t$, and every word from B^* can be transformed into a word from W_2 by using relations $\mathfrak{S}$. We have just proved that each word from $((\bigcup_{t\in T} A_t) \cup B)^*$ can be transformed into a word from the set

$$W = \{(\textstyle\prod_{t\in T} w_1(t)_t)w_2 \mid w_2 \in W_2,\ w_1(t) \in W_1 \text{ for all } t \in T,$$
$$w_1(t) \equiv \epsilon \text{ for all but finitely many } t \in T\}$$

by using relations (3), (4), (5).

To finish the proof of the theorem we prove that different words from W represent different elements of $S \operatorname{wr} T$. (Here, again, words from W which differ only in order of terms $w_1(t)$ are considered equal.) So suppose that $(\prod_{t\in T} w_1(t)_t)w_2$ and $(\prod_{t\in T} w_1'(t)_t)w_2'$ represent the same element of $S \operatorname{wr} T$, that is to say

$$(\prod_{t\in T}(\overline{w_1(t)_t}, 1_T))(\overline{1}, w_2) = (\prod_{t\in T}(\overline{w_1'(t)_t}, 1_T))(\overline{1}, w_2'),$$

which is equivalent to

$$(\prod_{t\in T}(\overline{w_1(t)_t}, w_2) = (\prod_{t\in T}(\overline{w_1'(t)_t}, w_2').$$

Therefore

$$\prod_{t\in T} \overline{w_1(t)_t} = \prod_{t\in T} \overline{w_1'(t)_t} \text{ in } S^{\oplus T}, \tag{6}$$

$$w_2 = w_2' \text{ in } T. \tag{7}$$

Since W_2 is a set of canonical forms for T, we have $w_2 \equiv w_2'$, while from (6) we obtain, for $x \in T$,

$$x(\prod_{t\in T} \overline{w_1(t)_t}) = x(\prod_{t\in T} \overline{w_1'(t)_t}),$$

so that $w_1(x) = w_1'(x)$ holds in S. Since W_1 is a set of canonical forms for S we must have $w_1(x) \equiv w_1'(x)$. The result now follows from Proposition 5.2.1. ∎

We can obtain a much nicer presentation in the case where T is a group. It is a generalisation of the presentation for the wreath product of two groups given in Johnson (1980).

Corollary 1.4. *Let S be a monoid and let T be a group. If $\langle A \mid \mathfrak{R}\rangle$ and $\langle B \mid \mathfrak{S}\rangle$ are monoid presentations for S and T respectively, then the presentation*

$$\langle A, B \mid \mathfrak{R},\ \mathfrak{S},\ a(t^{-1}a't) = (t^{-1}a't)a\ (a, a' \in A,\ t \in T)\rangle \tag{8}$$

defines $S \operatorname{wr} T$.

PROOF. If we denote a_{1_T} by a then it is easy to see that the relations

$$a(t^{-1}a't) = (t^{-1}a't)a\ (a, a' \in A,\ t \in T), \tag{9}$$
$$a_t = t^{-1}at\ (a \in A,\ t \in T), \tag{10}$$

hold in $S \operatorname{wr} T$. Therefore, we can add these relations to the presentation (3), (4), (5). Next we eliminate a_t, $a \in A$, $t \in T - \{1_T\}$ by using (10). A general relation $u_t = v_t$ from $\mathfrak{R}_t$, $t \in T$, becomes $t^{-1}ut = t^{-1}vt$, and is equivalent to $u = v$. Therefore, all the relations $\mathfrak{R}_t$ can be replaced by $\mathfrak{R}$. The relations $\mathfrak{S}$ do not change as they involve only letters from B. Finally, relations (4) and (5) are consequences of (9) and (10) since

$$\begin{aligned} a_t a'_u &= t^{-1}atu^{-1}a'u = t^{-1}a(ut^{-1})^{-1}a'(ut^{-1})t = t^{-1}(ut^{-1})^{-1}a'(ut^{-1})at \\ &= u^{-1}a'ut^{-1}at, \end{aligned}$$

and

$$\Big(\prod_{u \in tb^{-1}} a_u\Big)b = a_{tb^{-1}}b = (tb^{-1})^{-1}atb^{-1}b = bt^{-1}at = ba_t.$$

Hence we can eliminate these relations from our presentation, thus obtaining presentation (8). ∎

2. The Schützenberger product

Let S and T be monoids. For $X \subseteq S \times T$, $s \in S$, $t \in T$, we define

$$sX = \{(sx_1, x_2) \mid (x_1, x_2) \in X\}$$
$$Xt = \{(x_1, x_2t) \mid (x_1, x_2) \in X\}.$$

The *Schützenberger product* of S and T, denoted by $S\Diamond T$, is the set $S \times \mathcal{P}(S \times T) \times T$, where $\mathcal{P}(S \times T)$ denotes the set of all subsets of $S \times T$, with the multiplication

$$(s_1, X_1, t_1)(s_2, X_2, t_2) = (s_1s_2, X_1t_2 \cup s_1X_2, t_1t_2).$$

$S\Diamond T$ is a monoid with identity $(1_S, \emptyset, 1_T)$. This construction plays an important role in language theory, especially in the theory of so called star-free languages; for details see Howie (1991).

In this section we find a presentation for $S\Diamond T$, when S and T are finite monoids. First, as usual, we find a generating set.

Lemma 2.1. *If finite monoids S and T are generated by sets A and B respectively, then $S\Diamond T$ is generated by the set*

$$\{(a,\emptyset,1_T)\mid a\in A\}\cup\{(1_S,\emptyset,b)\mid b\in B\}\cup\{(1_S,\{(s,t)\},1_T)\mid s\in S,\ t\in T\}.$$

PROOF. Let $s\in S$, $t\in T$ and let $X\subseteq S\times T$ be arbitrary. Since A is a generating set for S, s can be written as $s=a_1a_2\dots a_k$, with $a_1,\dots,a_k\in A$; similarly $t=b_1b_2\dots b_l$, with $b_1,\dots,b_l\in B$. Also, since both S and T are finite, X can be written as a union $X=\bigcup_{i\in I}\{(s_i,t_i)\}$, where I is a finite set and $s_i\in S$, $t_i\in T$ for all $i\in I$. Now we have

$$\begin{aligned}(s,\emptyset,1_T)&=(a_1,\emptyset,1_T)(a_2,\emptyset,1_T)\dots(a_k,\emptyset,1_T),\\(1_S,\emptyset,t)&=(1_S,\emptyset,b_1)(1_S,\emptyset,b_2)\dots(1_S,\emptyset,b_l),\\(1_S,X,1_T)&=\prod_{i\in I}(1_S,\{(s_i,t_i)\},1_T),\\(s,X,t)&=(1_S,\emptyset,t)(1_S,X,1_T)(s,\emptyset,1_T),\end{aligned}$$

and the lemma follows. ■

In general this is the best possible generating set for $S\Diamond T$. Indeed, if all the elements of the set $A\cup\{1_S\}$ and $B\cup\{1_T\}$ are indecomposable in S and T respectively, then all the generators for $S\Diamond T$ from Lemma 2.1 are indecomposable in $S\Diamond T$, and therefore must belong to every generating set.

Theorem 2.2. *Let S and T be finite monoids defined by presentations $\langle A\mid\mathfrak{R}\rangle$ and $\langle B\mid\mathfrak{S}\rangle$ respectively. The Schützenberger product $S\Diamond T$ is then defined by the presentation with generating symbols $C=A\cup B\cup\{c_{s,t}\mid s\in S,\ t\in T\}$ and relations*

$$\mathfrak{R};\ \mathfrak{S}; \tag{11}$$

$$c_{s,t}^2=c_{s,t};\ c_{s,t}c_{s_1,t_1}=c_{s_1,t_1}c_{s,t}; \tag{12}$$

$$ac_{s,t}=c_{as,t}a; \tag{13}$$

$$c_{s,t}b=bc_{s,tb}; \tag{14}$$

$$ab=ba; \tag{15}$$

where $a\in A$, $b\in B$, $s,s_1\in S$, $t,t_1\in T$.

PROOF. It is easy to see that the generators $\{(a,\emptyset,1_T)\mid a\in A\}$ generate the submonoid $\{(s,\emptyset,1_T)\mid s\in S\}$ which is isomorphic to S, and, since $\mathfrak{R}$ is a set of defining relations for S, they satisfy all the relations from $\mathfrak{R}$. Similar arguments show that the generators $\{(1_S,\emptyset,b)\mid b\in B\}$ satisfy relations $\mathfrak{S}$, and that generators $\{(1_S,\{(s,t)\},1_T)\mid s\in S,\ t\in T\}$ satisfy (12). Next we check that relations (13), (14), (15) hold in $S\Diamond T$:

$$\begin{aligned}&(a,\emptyset,1_T)(1_S,\{(s,t)\},1_T)=(a,\{(as,t)\},1_T)=(1_S,\{(as,t)\},1_T)(a,\emptyset,1_T),\\&(1_S,\{(s,t)\},1_T)(1_S,\emptyset,b)=(1_S,\{(s,tb)\},b)=(1_S,\emptyset,b)(1_S,\{(s,tb)\},1_T),\\&(a,\emptyset,1_T)(1_S,\emptyset,b)=(a,\emptyset,b)=(1_S,\emptyset,b)(a,\emptyset,1_T).\end{aligned}$$

Let W_1 and W_2 be sets of canonical forms for S and T respectively. We shall show that the set

$$W = \{w_2(\prod_{i\in I} c_{s_i,t_i})w_1 \mid w_1 \in W_1,\ w_2 \in W_2,\ I \text{ is finite},\ s_i \in S,\ t_i \in T\}$$

is a set of canonical forms for $S\Diamond T$.

First we show that each word $w \in C^*$ can be transformed by using relations (11)–(15) to a word from W. We do this by induction on $|w|$, the case $|w| \leq 1$ being obvious. If $|w| > 1$ then w has one of the forms $w'a$, $a \in A$, or $w'b$, $b \in B$, or $w'c_{s,t}$, $s \in S$, $t \in T$, where w' is of shorter length than w. By the inductive hypothesis w' can be transformed into a word $w'_2(\prod_{i\in I} c_{s_i,t_i})w'_1$ from W. If $w \equiv w'a$ we have

$$w = w'_2(\prod_{i\in I} c_{s_i,t_i})w'_1 a = w'_2(\prod_{i\in I} c_{s_i,t_i})w_1,$$

where $w_1 \in W_1$ is the canonical form for $w'_1 a$. In the case $w \equiv w'b$ we have

$$\begin{array}{rcll} w & = & w'_2(\prod_{i\in I} c_{s_i,t_i})w'_1 b & \\ & = & w'_2(\prod_{i\in I} c_{s_i,t_i})bw'_1 & \text{(by (15))} \\ & = & w'_2 b(\prod_{i\in I} c_{s_i,t_i b})w'_1 & \text{(by (14))} \\ & = & w_2(\prod_{i\in I} c_{s_i,t_i b})w'_1, & \end{array}$$

where $w_2 \in W_2$ is the canonical form for $w'_2 b$. Finally, if $w \equiv w'c_{s,t}$, we have

$$w \equiv w'_2(\prod_{i\in I} c_{s_i,t_i})w'_1 c_{s,t} = w'_2((\prod_{i\in I} c_{s_i,t_i})c_{w_1 s,t})w'_1,$$

by using (13).

Now we prove that different words from W represent different elements of $S\Diamond T$. So assume that

$$w'_2(\prod_{i\in I} c_{s_i,t_i})w'_1,\ w''_2(\prod_{j\in J} c_{s_j,t_j})w''_1 \in W$$

represent the same element of $S\Diamond T$. Then we have

$$(w'_1, \bigcup_{i\in I}\{(s_i,t_i)\}, w'_2) = (w''_1, \bigcup_{j\in J}\{(s_j,t_j)\}, w''_2).$$

Since W_1 is a set of canonical forms for S, from $w'_1 = w''_1$ in S it follows that $w'_1 \equiv w''_1$. Similarly, we have $w'_2 \equiv w''_2$. Finally, $\{(s_i,t_i) \mid i \in I\} = \{(s_j,t_j) \mid j \in J\}$ implies that $\prod_{i\in I} c_{s_i,t_i} = \prod_{j\in J} c_{s_j,t_j}$.

The theorem now follows by Proposition 5.2.1. ∎

3. The Bruck—Reilly extension

Let S be a monoid, and let $\theta : S \longrightarrow S$ be an endomorphism. Define a binary operation on the set $\mathbb{N}^0 \times S \times \mathbb{N}^0$ by

$$(m, s, n)(p, t, q) = (m - n + r, (s\theta^{r-n})(t\theta^{r-p}), q - p + r),$$

where $r = \max(n, p)$. It is an elementary exercise to show that this operation is associative and that $(0, 1_S, 0)$ is a neutral element. Hence $\mathbb{N}^0 \times S \times \mathbb{N}^0$ with this operation forms a monoid, which we denote by $BR(S, \theta)$.

If S is the trivial semigroup, $BR(S, \theta)$ is the *bicyclic monoid*; see Howie (1976). More generally, if S is not necessarily trivial, but the endomorphism θ satisfies $s\theta = 1_S$ for all $s \in S$, $BR(S, \theta)$ coincides with *Bruck extension* of S; see Bruck (1958). If S is a group then $BR(S, \theta)$ is known as *Reilly extension* of S; see Reilly (1966). Finally, if the image of θ is contained in the group of units of S, we obtain a general *Bruck—Reilly extension* of S, which is a generalisation of all previously mentioned extensions, and which was introduced by Munn (1970). All these constructions play important roles in the theory of inverse semigroups; see Howie (1976).

Lemma 3.1. *If A is a generating set for the monoid S, then*

$$\{(0, a, 0) \mid a \in A\} \cup \{(0, 1_S, 1), (1, 1_S, 0)\}$$

is a generating set for $BR(S, \theta)$.

PROOF. Let $m, n \in \mathbb{N}^0$, $s \in S$ be arbitrary. Since A is a generating set for S, s can be written as $s = a_1 a_2 \dots a_k$, where $a_1, \dots, a_k \in A$. Now we have

$$(0, s, 0) = (0, a_1 a_2 \dots a_k, 0) = (0, a_1, 0)(0, a_2, 0) \dots (0, a_k, 0),$$
$$(m, 1_s, 0) = (1, 1_S, 0)^m,\ (0, 1_S, n) = (0, 1_S, 1)^n,$$

so that

$$(m, s, n) = (m, 1_S, 0)(0, s, 0)(0, 1_S, n)$$

implies the lemma. ■

The obtained generating set is again the best possible in general. For example, if θ is the constant mapping $s \mapsto 1_S$, then $BR(S, \theta)$ is the direct product of S and the bicyclic monoid, and cannot be, in general, generated by fewer than $|A| + 2$ elements.

Theorem 3.2. *Let S be the monoid defined by a presentation $\langle A \mid \mathfrak{R} \rangle$, and let $\theta : S \longrightarrow S$ be an endomorphism. The monoid $BR(S, \theta)$ is then defined by the presentation*

$$\mathfrak{P} = \langle A, b, c \mid \mathfrak{R},\ bc = 1,\ ba = (a\theta)b,\ ac = c(a\theta)\ (a \in A) \rangle$$

in terms of the generators $\{(0, a, 0) \mid a \in A\} \cup \{(0, 1_S, 1), (1, 1_S, 0)\}$.

In order to prove the theorem we first deduce some consequences of the given relations.

Lemma 3.3. *The following relations are consequences of* $\mathfrak{P}$*:*

(i) $bw = (w\theta)b$, $w \in A^*$;

(ii) $wc = c(w\theta)$, $w \in A^*$;

(iii) $b^n a = (a\theta^n)b^n$, $n \in \mathbb{N}$, $a \in A$.

PROOF. (i) If $w \equiv a_1 \dots a_k$ then we have

$$bw \equiv ba_1a_2 \dots a_k = (a_1\theta)ba_2 \dots a_k = \dots = (a_1\theta)\dots(a_k\theta)b = (w\theta)b,$$

as required.

(ii) The proof is analogous to (i).

(iii) This follows by a multiple application of (i), since, for a word $w \in A^*$, the word $w\theta$ is in A^* as well. ∎

PROOF OF THEOREM 3.2. The set $\{(0, a, 0) \mid a \in A\}$ generates the submonoid $\{(0, s, 0) \mid s \in S\} \cong S$ of $BR(S, \theta)$, and, since $\mathfrak{R}$ are defining relations for S, we conclude that they hold in $BR(S, \theta)$ as well. Next, we check the remaining relations from $\mathfrak{P}$:

$$\begin{aligned}
&(0, 1_S, 1)(1, 1_S, 0) = (0 - 1 + 1, (1_S\theta^{1-1})(1_S\theta^{1-1}), 0 - 1 + 1) = (0, 1_S, 0),\\
&(0, 1_S, 1)(0, a, 0) = (0, (1_S\theta^0)(a\theta), 1) = (0, a\theta, 1) = (0, a\theta, 0)(0, 1_S, 1),\\
&(0, a, 0)(1, 1_S, 0) = (1, a\theta, 0) = (1, 1_S, 0)(0, a\theta, 0).
\end{aligned}$$

Let W_1 be a set of canonical forms for S. We shall show that the set

$$W = \{c^m w_1 b^n \mid m, n \in \mathbb{N}^0,\ w_1 \in W_1\}$$

is a set of canonical forms for $BR(S, \theta)$. First we show that each word $w \in (A \cup \{b, c\})^*$ can be transformed into a word from W by applying relations from $\mathfrak{P}$. We do this by induction on $|w|$, the case $|w| \le 1$ being obvious. If $|w| > 1$ then w can be written either as $w'a$, $a \in A$, or as $w'b$, or as $w'c$. By the inductive hypothesis w' can be transformed into a word $c^m w_1' b^n$ from W. Now, if $w \equiv w'a$, $a \in A$, then

$$\begin{aligned}
w &= c^m w' b^n a\\
&= c^m w'(a\theta^n)b^n \quad \text{(Lemma 3.3(iii))}\\
&= c^m w_1 b^n,
\end{aligned}$$

where $w_1 \in W_1$ is the canonical form for $w'(a\theta^n)$. In the case $w \equiv w'b$ we have

$$w = c^m w_1' b^{n+1} \in W.$$

Finally, for $w \equiv w'c$ we have

$$w = c^m w_1' b^n c.$$

Now, if $n = 0$ then

$$c^m w_1' c = c^{m+1}(w_1'\theta)$$

by Lemma 3.3(ii), and by reducing the word $w_1' \in A^*$ to a canonical form in W_1 we reduce w to a word from W. If $n \geq 1$, we use the relation $bc = 1$ and obtain

$$w = c^m w_1' b^{n-1} \in W.$$

To finish the proof we note that a typical element $c^m w_1 b^n$ of W represents the element (m, w_1, n) of $BR(S, \theta)$, from which it easily follows that different elements of W represent different elements of $BR(S, \theta)$. ∎

It is worth pointing out that the presentation $\mathfrak{P}$ from Theorem 3.2 has $|A|+2$ generating symbols and $|\mathfrak{R}| + 2|A| + 1$ relations, so that we have the following

Corollary 3.4. *If S is a finitely presented monoid, and if θ is an endomorphism of S, then the Bruck—Reilly extension $BR(S, \theta)$ is finitely presented.* ∎

4. Strong semilattices of monoids

Let Y be a semilattice, and let S_α, $\alpha \in Y$, be a family of (disjoint) monoids indexed by Y. Denote by 1_α the identity of the monoid S_α, $\alpha \in Y$. Suppose that for any two elements $\alpha, \beta \in Y$, with $\alpha \geq \beta$, there is a homomorphism $\phi_{\alpha,\beta} : S_\alpha \longrightarrow S_\beta$, and that these homomorphisms satisfy the following two conditions:

(a) $\phi_{\alpha,\alpha}$, $\alpha \in I$, is the identity mapping;

(b) $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$, for all $\alpha, \beta, \gamma \in Y$ with $\alpha \geq \beta \geq \gamma$.

The set $S = \bigcup_{\alpha \in Y} S_\alpha$ can be made into a semigroup by defining

$$ab = (a\phi_{\alpha,\alpha\beta})(b\phi_{\beta,\alpha\beta}),$$

for $a \in S_\alpha$ and $b \in S_\beta$. (Notice that the product on the right hand side of the above equation is a product within $S_{\alpha\beta}$.) We denote this semigroup by $S(Y; S_\alpha, \phi_{\alpha,\beta})$, and call it the *strong semilattice Y of monoids S_α with homomorphisms $\phi_{\alpha,\beta}$*. The multiplication in this semigroup extends multiplications from monoids S_α, $\alpha \in Y$, because of the condition (a), so that $S(Y; S_\alpha, \phi_{\alpha,\beta})$ is a disjoint union of monoids S_α.

Strong semilattices of semigroups provide one of the main tools for the structure theory of semigroups. Probably the best known result in this field is that a semigroup is a *Clifford semigroup* (i.e. a regular semigroup in which the idempotents are central) if and only if it is a strong semilattice of groups; see Clifford (1941) and Howie (1976).

Theorem 4.1. *Let Y be a semilattice, let S_α, $\alpha \in Y$, be a family of disjoint monoids, and let $\phi_{\alpha,\beta}$, $\alpha, \beta \in Y$, $\alpha \geq \beta$, be a family of homomorphisms satisfying (a) and (b). Suppose that the monoid S_α, $\alpha \in Y$, is defined by a* semigroup *presentation $\langle A_\alpha \mid \mathfrak{R}_\alpha \rangle$, with $A_\alpha \cap A_\beta = \emptyset$ for $\alpha \neq \beta$. Let $A = \bigcup_{\alpha \in Y} A_\alpha$, $\mathfrak{R} = \bigcup_{\alpha \in Y} \mathfrak{R}_\alpha$, and let $1_\alpha \in A_\alpha^+$ be a word representing the identity of S_α. Then*

$$\begin{aligned}\mathfrak{P} = \langle A \mid \mathfrak{R},\ & 1_\alpha 1_\beta = 1_\beta 1_\alpha,\ 1_\delta a = a 1_\delta = a\phi_{\gamma,\delta} \\ & (\alpha, \beta, \gamma, \delta \in Y,\ \alpha \neq \beta,\ \gamma > \delta,\ a \in A_\gamma)\rangle\end{aligned}$$

is a presentation for $S(Y; S_\alpha, \phi_{\alpha,\beta})$.

Remark 4.2. Although S_α, $\alpha \in Y$, are monoids, the semigroup $S(Y; S_\alpha, \phi_{\alpha,\beta})$, is not necessarily a monoid. This is the reason why in the above theorem we use semigroup presentations rather than monoid presentations.

Proof of Theorem 4.1. Let S denote the semigroup $S(Y; S_\alpha, \phi_{\alpha,\beta})$. Since S is a disjoint union of semigroups S_α, $\alpha \in Y$, it is certainly generated by the set $A = \bigcup_{\alpha \in Y} A_\alpha$. Also, since $\mathfrak{R}_\alpha$ is a set of defining relations for S_α, the generators A satisfy all the relations from $\mathfrak{R} = \bigcup_{\alpha \in Y} \mathfrak{R}_\alpha$. It is easy to check that the remaining relations from $\mathfrak{P}$ also hold in S:

$$\begin{aligned}1_\alpha 1_\beta &= (1_\alpha \theta_{\alpha,\alpha\beta})(1_\beta \theta_{\beta,\alpha\beta}) = 1_{\alpha\beta} 1_{\alpha\beta} = 1_{\alpha\beta} = 1_{\beta\alpha} = 1_\beta 1_\alpha, \\ 1_\delta a &= (1_\delta \theta_{\delta,\delta\delta})(a\theta_{\gamma,\gamma\delta}) = 1_\delta(a\theta_{\gamma,\delta}) = a\theta_{\gamma,\delta} = a 1_\delta.\end{aligned}$$

Let W_α, $\alpha \in Y$, denote a set of canonical forms for S_α. We shall show that the set

$$W = \bigcup_{\alpha \in Y} W_\alpha$$

is a set of canonical forms for S. Let $w \equiv a_1 a_2 \ldots a_k$ be an arbitrary word from A^+, with $a_i \in A_{\alpha_i}$, $\alpha_i \in Y$, $i = 1, \ldots, k$. If we note that, for $a \in A_\alpha$, the relations

$$a 1_\alpha = 1_\alpha a = a,$$

are consequences of $\mathfrak{P}$, since 1_α represents the identity of S_α and that the defining relations $\mathfrak{R}_\alpha$ for S_α are included in $\mathfrak{P}$, we have

$$\begin{aligned} w &\equiv a_1 a_2 \ldots a_k \\ &= (1_{\alpha_1} a_1)(1_{\alpha_2} a_2) \ldots (1_{\alpha_k} a_k) \\ &= (1_{\alpha_1} 1_{\alpha_2} \ldots 1_{\alpha_k})(a_1 a_2 \ldots a_k) && \text{(relations } 1_\delta a = a 1_\delta) \\ &= 1_{\alpha_1 \alpha_2 \ldots \alpha_k} a_1 a_2 \ldots a_k && \text{(relations } 1_\alpha 1_\beta = 1_{\alpha\beta}) \\ &= 1_\beta^k a_1 a_2 \ldots a_k && \text{(relations } \mathfrak{R}_\beta,\ \beta = \alpha_1 \ldots \alpha_k) \\ &= (1_\beta a_1)(1_\beta a_2) \ldots (1_\beta a_k) && \text{(relations } 1_\delta a = a 1_\delta) \\ &= (a_1 \phi_{\alpha_1,\beta})(a_2 \phi_{\alpha_2,\beta}) \ldots (a_k \phi_{\alpha_k,\beta}). && \text{(relations } 1_\delta a = a\phi_{\gamma,\delta}) \end{aligned}$$

Since the last word is from A_β^*, it can be reduced to a canonical form from $W_\beta \subseteq W$ by applying relations $\mathfrak{R}_\beta \subseteq \mathfrak{R}$.

Finally, from the fact that S is a disjoint union of monoids S_α, $\alpha \in Y$, and that W_α is a set of canonical forms for S_α, it follows that different elements of W represent different elements of S. The theorem now follows by Proposition 5.2.1. ■

5. Rees matrix semigroups

We have already encountered Rees matrix semigroups over a group in Chapter 4. Basic facts about these semigroups can be found in Appendix A. Here we consider a more general situation of a Rees matrix semigroup over a monoid, with the Rees matrix not necessarily regular. First we recall the relevant definitions.

Let S be a monoid, let 0 be an element not belonging to S, let I and Λ be two index sets, and let $P = (p_{\lambda i})_{\lambda\in\Lambda, i\in I}$ be a $|\Lambda| \times |I|$ matrix with entries from $S \cup \{0\}$. The *Rees matrix semigroup* $\mathcal{M}^0[S; I, \Lambda; P]$ is the set $(I \times S \times \Lambda) \cup \{0\}$ with multiplication

$$(i_1, s_1, \lambda_1)(i_2, s_2, \lambda_2) = \begin{cases} (i_1, s_1 p_{\lambda_1 i_2} s_2, \lambda_2) & \text{if } p_{\lambda_1 i_2} \neq 0 \\ 0 & \text{otherwise,} \end{cases}$$
$$0(i, s, \lambda) = (i, s, \lambda)0 = 00 = 0.$$

If all the entries of P are equal to 0, $\mathcal{M}^0[S; I, \Lambda; P]$ is the semigroup of order $|I||S||\Lambda| + 1$ with zero multiplication; the simplest presentation for such a semigroup is its multiplication table. In the rest of this section we will consider the non-trivial case where P contains at least one non-zero entry. Furthermore, we will assume that P contains at least one unit (i.e. invertible element) of S; when this condition is not satisfied generating sets for $\mathcal{M}^0[S; I, \Lambda; P]$ strongly depend on the semigroup S and the particular choice of the matrix P, and finding a reasonable general generating set seems to be a difficult problem. The assumption that P contains at least one unit is equivalent to assuming that P has an entry equal to 1_S. For if P has an invertible entry, which without loss of generality we may denote by p_{11}, then the map $(i, s, \lambda) \mapsto (i, sp_{11}, \lambda)$ is an isomorphism between $\mathcal{M}^0[S; I, \Lambda; P]$ and $\mathcal{M}^0[S; I, \Lambda; P']$, where

$$P = (p'_{\lambda i})_{\lambda\in\Lambda, i\in I} = (p_{11}^{-1} p_{\lambda i})_{\lambda\in\Lambda, i\in I},$$

and evidently $p'_{11} = 1_S$. Rees matrix semigroups over groups (such as completely 0-simple semigroups; see Appendix A) trivially satisfy this additional restriction.

Lemma 5.1. *Let S be a monoid, let A be a set of semigroup generators for S, let $P = (p_{\lambda i})_{\lambda\in\Lambda, i\in I}$ be a $|\Lambda| \times |I|$ matrix with entries from $S \cup \{0\}$ ($0 \notin S$), and with $p_{11} = 1_S$. The Rees matrix semigroup is then generated by the set*

$$\{(1, a, 1) \mid a \in A\} \cup \{(i, 1_S, 1) \mid 1 \neq i \in I\} \cup \{(1, 1_S, \lambda) \mid 1 \neq \lambda \in \Lambda\}.$$

PROOF. Let $i \in I$, $\lambda \in \Lambda$, $s \in S$. Since A is a generating set for S we have $s = a_1 a_2 \dots a_k$, $a_1, \dots, a_k \in A$. Therefore,

$$\begin{aligned}(i, s, \lambda) &= (i, 1_S p_{11} a_1 p_{11} a_2 \dots p_{11} a_k p_{11} 1_S, \lambda) \\ &= (i, 1_S, 1)(1, a_1, 1)(1, a_2, 1) \dots (1, a_k, 1)(1, 1_S, \lambda),\end{aligned}$$

and hence the lemma. ∎

Remark 5.2. The generating set in Lemma 5.1 is a straightforward generalisation of the generating set given in Theorem 4.2.1 for the completely 0-simple semigroups.

Remark 5.3. Although this is again the best possible choice of generators in general (e.g. consider the case where all entries of P except p_{11} are equal to 0), in most cases, including the completely 0-simple case, it is far from being optimal; see Section 3 in Chapter 4. ∎

Theorem 5.4. *Let $T = \mathcal{M}^0[S; I, \Lambda; P]$ be a Rees matrix semigroup, where S is a monoid, $P = (p_{\lambda i})_{\lambda \in \Lambda, i \in I}$ is a $|\Lambda| \times |I|$ matrix with entries from $S \cup \{0\}$ ($0 \notin S$) and with $p_{11} = 1_S$. Let $\langle A \mid \mathfrak{R} \rangle$ be a semigroup presentation for S, let $e \in A^+$ be a non-empty word representing the identity 1_S of S, and let*

$$B = A \cup \{b_i \mid 1 \neq i \in I\} \cup \{c_\lambda \mid 1 \neq \lambda \in \Lambda\}$$

be a new alphabet. Then the presentation

$$\mathfrak{P} = \langle B \mid \mathfrak{R},\ b_i e = b_i,\ e b_i = p_{1i},\ c_\lambda e = p_{\lambda 1},\ e c_\lambda = c_\lambda,\ c_\lambda b_i = p_{\lambda i} \\ (1 \neq i \in I,\ 1 \neq \lambda \in \Lambda) \rangle$$

defines T as a semigroup with zero.

Remark 5.5. This time we opt for presentations of semigroups with zero, as T does have a zero, but does not necessarily have an identity. The presentation $\mathfrak{P}$ has no relations of the form $w = 0$ if and only if the matrix P has no entries equal to 0, i.e. if and only if 0 is indecomposable in T. In this case, by considering $\mathfrak{P}$ as a semigroup presentation, we see that it defines the semigroup $T - \{0\}$. In this way, in particular, we obtain presentations for completely simple semigroups.

PROOF OF THEOREM 5.4. The set $\{(1, a, 1) \mid a \in A\}$ generates the subsemigroup $\{(1, s, 1) \mid s \in S\} \cong S$, and hence satisfies the defining relations $\mathfrak{R}$ of S. Also, we see that a word w from A^+ represents the element $(1, w, 1)$ of T, and, in particular, e represents $(1, 1_S, 1)$. Now we check that the remaining relations from $\mathfrak{P}$ also hold in T:

$$(i, 1_S, 1)(1, 1_S, 1) = (i, 1_S p_{11} 1_S, 1) = (i, 1_S, 1),$$
$$(1, 1_S, 1)(i, 1_S, 1) = \begin{cases} (1, 1_S p_{1i} 1_S, 1) = (1, p_{1i}, 1) & \text{if } p_{1i} \neq 0 \\ 0 & \text{if } p_{1i} = 0, \end{cases}$$
$$(1, 1_S, \lambda)(1, 1_S, 1) = \begin{cases} (1, p_{\lambda 1}, 1) & \text{if } p_{\lambda 1} \neq 0 \\ 0 & \text{if } p_{\lambda 1} = 0, \end{cases}$$
$$(1, 1_S, 1)(1, 1_S, \lambda) = (1, 1_S, \lambda),$$
$$(1, 1_S, \lambda)(i, 1_S, 1) = \begin{cases} (1, p_{\lambda i}, 1) & \text{if } p_{\lambda i} \neq 0 \\ 0 & \text{if } p_{\lambda i} = 0. \end{cases}$$

Let W_1 be a set of canonical forms for S. We shall show that the set

$$W = \{b_i w_1 c_\lambda \mid i \in I,\ \lambda \in \Lambda,\ w_1 \in W_1\} \cup \{0\},$$

where b_1 and c_1 are defined to be equal to the empty word, is a set of canonical forms for T.

Let $w \in B^+$ be an arbitrary word. By induction on $|w|$ we prove that w can be transformed into a word from W by using relations from $\mathfrak{P}$. The case $|w| = 1$ is obvious; so let us suppose that $|w| > 1$. Then either $w \equiv w'a$, $a \in A$, or $w \equiv w'b_i$, $1 \neq i \in I$, or $w \equiv w'c_\lambda$, $1 \neq \lambda \in \Lambda$, where $w' \in B^+$ is a word of shorter length than w. By the inductive hypothesis w' can be transformed into a word $b_j w_1' c_\mu$ from W by using relations from $\mathfrak{P}$. If $w \equiv w'a$ we have

$$\begin{aligned} w &= b_j w_1' c_\mu a \\ &= b_j w_1' c_\mu e a \quad \text{(relations } \mathfrak{R}\text{)} \\ &= b_j w_1' p_{\mu 1} a. \quad \text{(relation } c_\mu e = p_{\mu 1}\text{)} \end{aligned}$$

If $p_{\mu 1} = 0$ then the last word is equal to 0; otherwise, $w_1' p_{\mu 1} a$ is a word from A^+, which can be transformed into a canonical form $w_1 \in W_1$ by using relations $\mathfrak{R}$, thus transforming the word w into the word $b_j w_1 c_1 \in W$. If $w \equiv w'b_i$, we distinguish the case $\mu = 1$, in which

$$w = b_j w_1' b_i = b_j w_1' e b_i = b_j w_1' p_{1i} = \begin{cases} b_j w_1 c_1 & \text{if } p_{1i} \neq 0 \\ 0 & \text{if } p_{1i} = 0, \end{cases}$$

where $w_1 \in W_1$ is the canonical form for $w_1' p_{1i} \in A^+$, and the case $\mu \neq 1$, where

$$w = b_j w_1' c_\mu b_i = b_j w_1' p_{\mu i} = \begin{cases} b_j w_1 c_1 & \text{if } p_{\mu i} \neq 0 \\ 0 & \text{if } p_{\mu i} = 0, \end{cases}$$

where $w_1 \in W_1$ is the canonical form of $w_1' p_{\mu i} \in A^+$. Finally, if $w \equiv w'c_\lambda$, we have

$$w = b_j w_1' c_\mu c_\lambda = b_j w_1' c_\mu e c_\lambda = b_j w_1' p_{\mu 1} c_\lambda = \begin{cases} b_j w_1 c_\lambda & \text{if } p_{\mu 1} \neq 0 \\ 0 & \text{if } p_{\mu 1} = 0, \end{cases}$$

where $w_1 \in W_1$ is the canonical form for $w_1' p_{\mu 1} \in A^+$.

In order to prove that different elements from W represent different elements from T it is enough to note that a typical element $b_i w_1 c_\lambda$ of W represents the element (i, w_1, λ) of T. The theorem now follows from Proposition 5.2.1. ■

If we note that the presentation $\mathfrak{P}$ has $|A| + |I| + |\Lambda| - 2$ generators and $|\mathfrak{R}| + 2|I| + 2|\Lambda| + (|I| - 1)(|\Lambda| - 1)$ relations we obtain the following

Corollary 5.6. *Let S be a finitely presented monoid, and let I and Λ be both finite. Then the Rees matrix semigroup $\mathcal{M}^0[S; I, \Lambda; P]$, with matrix P containing at least one invertible entry, is finitely presented.*

Later, in Section 2 of Chapter 10, we will see that in the completely 0-simple case the converse of this corollary is true as well.

6. Ideal extensions

Let S be a semigroup with zero, and let I and T be semigroups. We say that T is an *ideal extension* of I by S if T contains an ideal J isomorphic to I such that the Rees quotient (see Appendix A) T/J is isomorphic to S. Ideal extensions are not constructions in the strictest sense: not only is there no 'recipe' for obtaining T given S and I, but T may not even exist or it may not be unique. For a detailed introduction to ideal extensions and examples illustrating the above assertions see Petrich (1973).

Here we just prove the following result which we will need in Chapter 8.

Theorem 6.1. *Let S be a semigroup and let I be an ideal of S. If both I and S/I are finitely presented then S is finitely presented as well.*

PROOF. Suppose that I and S/I are defined by (semigroup) presentations $\langle X | \mathfrak{R} \rangle$ and $\langle Y \mid \mathfrak{S} \rangle$ respectively. Suppose for the moment that no generator from Y represents the zero of the semigroup S/I. In this case S is obviously generated by the set $X \cup Y$. Each word $w \in Y^+$ which represents the zero of S/I represents an element of I in S, and so there exists a word $\alpha_w \in X^+$ such that $w = \alpha_w$ in S. Also, for each $x \in X$ and each $y \in Y$ the words xy and yx represent elements from I in S; let β_{xy}, β_{yx} be words from X^+ such that $xy = \beta_{xy}$ and $yx = \beta_{yx}$ hold in S. Let us fix a word $z \in Y^+$ representing the zero of S/I. We define the following new sets of relations:

$$\begin{aligned}
\mathfrak{S}_1 &= \{(u = v) \in \mathfrak{S} \ : \ u \neq z \text{ in } S/I\}, \\
\mathfrak{S}_2 &= \{(u = v) \in \mathfrak{S} \ : \ u = z \text{ in } S/I\}, \\
\overline{\mathfrak{S}}_2 &= \{u = \alpha_u, v = \alpha_v \ : \ (u = v) \in \mathfrak{S}_2\} \cup \{z = \alpha_z\}, \\
\mathfrak{S}_3 &= \{xy = \beta_{xy}, yx = \beta_{yx} \ : \ x \in X, y \in Y\},
\end{aligned}$$

and let T denote the semigroup defined by the presentation

$$\langle X \cup Y \,|\, \mathfrak{R} \cup \mathfrak{S}_1 \cup \overline{\mathfrak{S}}_2 \cup \mathfrak{S}_3 \rangle. \tag{16}$$

We claim that $S \cong T$. It is obvious that S satisfies all the relations from $\mathfrak{R} \cup \mathfrak{S}_1 \cup \overline{\mathfrak{S}}_2 \cup \mathfrak{S}_3$, and hence there is a natural epimorphism $\phi : T \longrightarrow S$.

Assume now that two words $w_1, w_2 \in (X \cup Y)^+$ represent the same element of S. If both w_1 and w_2 contain a letter from X then we can use relations from $\mathfrak{S}_3$ to find words $\overline{w}_1, \overline{w}_2 \in X^+$ such that $w_1 = \overline{w}_1$ and $w_2 = \overline{w}_2$ in T. Since the relation $\overline{w}_1 = \overline{w}_2$ holds in I, it can be deduced from $\mathfrak{R}$, thus giving $w_1 = w_2$ in T. If $w_1 \in Y^+$ and w_1 represents an element of I in S then $w_1 = z$ holds in S/I, and so z can be obtained from w_1 by applying relations from $\mathfrak{S}$. If no relations from $\mathfrak{S}_2$ are needed in this deduction then $w_1 = z = \alpha_z$ holds in T. Otherwise, instead of the first application of a relation from $\mathfrak{S}_2$ we can use the corresponding relation from $\overline{\mathfrak{S}}_2$ and thus obtain a word $\overline{w}_1$ containing a letter from X such that $w_1 = \overline{w}_1$ holds in T. Applying a similar argument to w_2, if necessary, we reduce this case to the case where both w_1 and w_2 contain a letter from X. Finally, if w_1 does not represent an element of I in S then $w_1 = w_2$ can be deduced by just using relations from $\mathfrak{S}_1$, and again $w_1 = w_2$ holds in T. Therefore, ϕ is an isomorphism and $T \cong S$ as required.

Let us now suppose that Y contains generators representing the zero in S/I, and let Y_0 be the set of all non-zero generators from Y. If the zero of S/I is a product of two non-zero elements from S/I then Y_0 generates S/I and the argument above can be repeated. Otherwise a similar argument would show that

$$\langle X \cup Y_0 \,|\, \mathfrak{R} \cup \mathfrak{S}_1 \cup \mathfrak{S}_3 \rangle \tag{17}$$

is a presentation for S.

Finally, note that if both presentations $\langle X \,|\, \mathfrak{R} \rangle$ and $\langle Y \,|\, \mathfrak{S} \rangle$ are finite so are the presentations (16) and (17).■

7. Presentations for subsemigroups—a general rewriting theorem

The problem of finding presentations for subsemigroups can be formulated as follows: *given a semigroup S defined by a presentation $\langle A \,|\, R \rangle$, and given a set of words $X \subseteq A^+$, find a presentation for the subsemigroup T of S generated by X.*

The analogous problems for groups and their subgroups is solved by theorems of Reidemeister and Schreier; see Magnus, Karrass and Solitar (1966). The main idea there is that of *rewriting*: the relations defining the group are *rewritten* under certain rules to give a presentation for the subgroup.

In this section we develop the idea of rewriting for semigroups, and give a general solution to the above problem. As in the case of groups, with this degree of generality, one may not expect a very satisfactory solution. For example, the presentation obtained will have to be infinite, even when the semigroup is finitely presented and the subsemigroup is finitely generated, since a finitely generated subsemigroup of a finitely presented semigroup may be not finitely presented; see Example 8.1.2. The real significance of the presentation obtained is that it gives rise to a 'recipe' for obtaining presentations for subsemigroups in various special cases. We shall use this recipe in Chapters 7 and 9 to find presentations for ideals of finite index and maximal subgroups of (0-)minimal ideals of a semigroup.

Let S be the semigroup defined by a presentation $\langle A \mid \mathfrak{R}\rangle$, and let T be the subsemigroup of S generated by a set $X = \{\xi_i \mid i \in I\}$, where each ξ_i is a word from A^+. We seek a presentation for T in terms of the generators X.

Although so far we have been identifying words in an alphabet with elements that these words represent in a particular semigroup, in the case of the semigroup T we may not do so—generators X of T are already words (not letters) from the alphabet A. So we choose a new alphabet $B = \{b_i \mid i \in I\}$ in one-one correspondence with X. Intuitively, b_i is an abstract image of of the generator ξ_i. In other words, a word $b_{i_1}b_{i_2}\dots b_{i_k} \in B^+$ *represents* the element $\xi_{i_1}\xi_{i_2}\dots\xi_{i_k}$ of T. This can be made more formal by introducing the mapping $\psi : B^+ \longrightarrow A^+$ defined by

$$(b_{i_1}b_{i_2}\dots b_{i_k})\psi = \xi_{i_1}\xi_{i_2}\dots\xi_{i_k}, \tag{18}$$

which we call the *representation mapping.* Note that ψ is defined constructively (if A, $\mathfrak{R}$ and X are defined constructively), and that it is a homomorphism.

On the other hand, each word from A^+ which represents an element of the subsemigroup T is equal in S to a product of elements of X, which is in turn represented by the corresponding word from B^+. Thus we have a mapping

$$\phi : L(A,T) \longrightarrow B^+,$$

where $L(A,T)$ denotes the set of all words from A^+ which represent elements of T, satisfying

$$(w\phi)\psi = w \text{ in } S \tag{19}$$

for all $w \in L(A,T)$. Since ϕ 'rewrites' each word representing an element of T into a corresponding product of generators of T, we call ϕ a *rewriting mapping.*

Note that the existence of a rewriting mapping is a consequence of the Axiom of Choice. However, unlike the representation mapping, a rewriting mapping is not necessarily unique, it is not necessarily a homomorphism and it is not defined constructively by (19).

The main result of this section is

Theorem 7.1. *Let S be the semigroup defined by a presentation $\langle A\,|\,\mathfrak{R}\rangle$, and let T be the subsemigroup of S generated by $X = \{\xi_i\,|\,i \in I\} \subseteq A^+$. Introduce a new* *alphabet* $B = \{b_i \mid i \in I\}$, *and let* ψ *and* ϕ *be the representation mapping and a rewriting mapping. Then* T *is defined by generators* B *and relations*

$$b_i = \xi_i\phi,\ i \in I, \tag{20}$$
$$(w_1w_2)\phi = w_1\phi \cdot w_2\phi, \tag{21}$$
$$(w_3uw_4)\phi = (w_3vw_4)\phi, \tag{22}$$

where $w_1, w_2 \in L(A,T)$, $u = v$ *is any relation from* $\mathfrak{R}$, *and* $w_3, w_4 \in A^*$ *are any words such that* $w_3uw_4 \in L(A,T)$.

We prove the theorem by using Proposition 3.2.3: we prove that relations (20), (21), (22) hold in T and that any other relation which holds in T is a consequence of (20), (21), (22). We do this in three lemmas.

Lemma 7.2. *If* $\alpha = \beta$ *is any of relations (20), (21), (22), then* $\alpha\psi = \beta\psi$ *holds in* S.

Proof. If we use definition (19) of a rewriting mapping and the fact that ψ is a homomorphism we obtain

$$\begin{aligned} &b_i\psi \equiv \xi_i = (\xi_i\phi)\psi, \\ &((w_1w_2)\phi)\psi = w_1w_2 = (w_1\phi)\psi \cdot (w_2\phi)\psi \equiv (w_1\phi \cdot w_2\phi)\psi, \\ &((w_3uw_4)\phi)\psi = w_3uw_4 = w_3vw_4 = ((w_3vw_4)\phi)\psi, \end{aligned}$$

as required. ■

Lemma 7.3. *If* $\alpha, \beta \in L(A,T)$ *are any two words such that the relation* $\alpha = \beta$ *holds in* S, *then the relation* $\alpha\phi = \beta\phi$ *is a consequence of (20), (21), (22).*

Proof. Since $\langle A \mid \mathfrak{R}\rangle$ is a presentation for S, there exists a sequence

$$\alpha \equiv \gamma_1, \gamma_2, \ldots, \gamma_k \equiv \beta,$$

of words from A^+ such that γ_{i+1} can be obtained from γ_i by one application of one relation from $\mathfrak{R}$, for all $i = 1, \ldots, k-1$. Therefore, relations (22) contain all the relations $\gamma_i\phi = \gamma_{i+1}\phi$, $i = 1, \ldots, k-1$, and we have the sequence

$$\alpha\phi \equiv \gamma_1\phi, \gamma_2\phi, \ldots \gamma_k\phi \equiv \beta\phi,$$

in which every term can be obtained from the previous one by one application of one relation (22). ■

Lemma 7.4. *For any word* $w \in B^+$ *the relation*

$$w = (w\psi)\phi$$

is a consequence of (20), (21), (22).

PROOF. If $w \equiv b_{i_1} b_{i_2} \dots b_{i_k}$, with $b_{i_1}, \dots, b_{i_k} \in B$, then

$$w\psi \equiv \xi_{i_1} \xi_{i_2} \cdots \xi_{i_k} \tag{23}$$

by (18). Note that $\xi_{i_1}, \dots, \xi_{i_k} \in L(A,T)$, so that a multiple application of (21) gives

$$(\xi_{i_1} \xi_{i_2} \cdots \xi_{i_k})\phi = (\xi_{i_1}\phi)(\xi_{i_2}\phi) \dots (\xi_{i_k}\phi). \tag{24}$$

Finally, by (20), we have

$$\xi_{i_1}\phi = b_{i_1},\ \xi_{i_2}\phi = b_{i_2}, \dots, \xi_{i_k}\phi = b_{i_k}, \tag{25}$$

and from (23), (24), (25) we obtain the desired result. ■

PROOF OF THEOREM 7.1. Relations (20), (21), (22) hold in T by Lemma 7.2. Let $\alpha, \beta \in B^+$ be any two words such that the relation $\alpha = \beta$ holds in T. This means that $\alpha\psi = \beta\psi$ holds in S, and, by Lemma 7.3, we deduce that $(\alpha\psi)\phi = (\beta\psi)\phi$ is a consequence of (20), (21), (22). Finally, by Lemma 7.4, both $\alpha = (\alpha\psi)\phi$ and $\beta = (\beta\psi)\phi$ are consequences of (20), (21), (22), so that we conclude that $\alpha = \beta$ is a consequence of (20), (21), (22), as required. ■

Remark 7.5. Theorem 7.1 is a semigroup analogue of Theorem 2.6 in Magnus, Karrass and Solitar (1966). Relations (20), (21), (22) correspond respectively to relations (5), (7), (8) of that theorem. In the group case one more family of relations was necessary: relations of the type $\phi(w_1) = \phi(w_2)$, where w_1 and w_2 are freely equal words representing the same element of the subgroup. For semigroups such a family of relations contains only trivial relations because of the nature of the free semigroup. ■

Remark 7.6. The main disadvantages of the presentation from Theorem 7.1 are that it is always infinite and that neither the rewriting mapping ϕ nor the set $L(A,T)$ have been defined constructively. However, Theorem 7.1 yields a uniform method for finding presentations for subsemigroups in various special cases. Given a semigroup S defined by a presentation $\langle A \,|\, \mathfrak{R} \rangle$ and a subsemigroup T of S, this method consists of the following three steps:

- find a generating set X for T, and introduce a new alphabet B in one-one correspondence with X;
- find a (specific) rewriting mapping;
- find a set $\mathfrak{S} \subseteq B^+ \times B^+$ of relations which hold on T and imply all the relations (20), (21), (22).

It is easy to prove that $\langle B \mid \mathfrak{S} \rangle$ is indeed a presentation for T by using Tietze transformations. First, since (20), (21), (22) are defining relations for T, they imply relations $\mathfrak{S}$ which hold in T, so that $\mathfrak{S}$ can be added to the presentation (20), (21), (22), and then relations (20), (21), (22) can be eliminated from the presentation, as they are implied by $\mathfrak{S}$ by assumption. ■

Chapter 5

Reidemeister—Schreier type theorems for ideals

In this chapter we give presentations for an ideal and a right ideal of a semigroup defined by a presentation. The main feature of the obtained presentations is that they are finite if the semigroup is finitely presented and the ideal in question has finite index. This can be considered as an analogue of the Reidemeister—Schreier theorem for groups which asserts that a subgroup of finite index in a finitely presented group is finitely presented.

Of course, the obtained presentation for right ideals is also a presentation for two-sided ideals. There are two main reasons for giving a separate presentation for two-sided ideals. The first reason is that the presentation for two-sided ideals is less complicated and in general contains many fewer relations, and therefore is more hopeful for further applications. The second reason is that the proofs in the two cases are rather different, but both share the common general approach outlined at the end of of the previous chapter; we hope that by giving both presentations we will illustrate this general approach more successfully.

The results of Sections 1 and 2 will appear in Campbell, Robertson, Ruškuc and Thomas (1995b), while the results of Section 3 will appear in Campbell, Robertson, Ruškuc and Thomas (1995d).

1. A generating set and a rewriting mapping for subsemigroups

If I is a (left, right or two-sided) ideal of a semigroup S, the index of I in S is defined to be the number of equivalence classes of the corresponding (left, right, two-sided) Rees congruence η_I; see Section 1 of Appendix A. In effect, the index of I in S is equal to $|S-I|+1$. If I is a two-sided ideal this is equal to the order of the Rees quotient S/I.

This closely parallels the notion of the index of a subgroup in a group. Let G be a group, and let H be a subgroup of G. The index of H in G is the number of (left or right) cosets of H in G, which is equal to the number of equivalence classes of the (left or right) congruence associated with H. If H is a normal subgroup of G, then the index of H in G is equal to the order of the factor group G/H.

Therefore, the notion of the index of a substructure in a structure is closely related to a (one-sided) congruence of the structure associated with the substructure, or, to put it differently, to an action of the structure associated with the substructure. In the cases of subgroups of a group and ideals of a semigroup, the associated actions exist because of the fact that the substructure is an equivalence class of a one-sided congruence; for the connection between actions and one-sided congruences see Appendix A.

In the case of subsemigroups of a semigroup, there is no such natural associated congruence. A subsemigroup may even fail to be a congruence class of a one-sided congruence; an example of this is when S is the free group on generators A, and T is the subsemigroup of S generated by A. Therefore, there seems to be no natural definition for the index of a subsemigroup in a semigroup. Nevertheless, just for the needs of this thesis, we define *the index* of a subsemigroup T of a semigroup S to be $|S-T|+1$—the definition first introduced by Jura (1978). Except for the fact that this agrees with the earlier definition of index for ideals, the main justification for this definition lies in Jura's result that a subsemigroup of finite index in a finitely generated semigroup is itself finitely generated; see Jura (1978). This is very similar to Schreier's result for groups: a subgroup of finite index in a finitely generated group is itself finitely generated; see Magnus, Karrass and Solitar (1966).

In this section we give an alternative proof of Jura's result. This proof has two main advantages over the original proof: our generating set is smaller (actually, it is best possible in general), and its definition is explicit enough to enable us to find a corresponding rewriting mapping, which we will then use in the following two sections.

Let S be a semigroup generated by a set A, and let T be a subsemigroup of S. A *set of representatives of* $S-T$ is a set Ω of words from A^* such that

(1) Ω contains the empty word ϵ;

(2) each non-empty word in Ω represents an element of $S-T$;

(3) each element of $S-T$ is represented by one, and only one, word in Ω.

The empty word ϵ is included in Ω for technical reasons; since we are dealing with semigroups ϵ does not represent an element of S. The *representative function* $w \mapsto \overline{w}$ associates with every element $w \in S-T$ its representative $\overline{w} \in \Omega$ (and, formally, associates the empty word to the empty word). This definition has the following immediate consequences which we will frequently use:

Lemma 1.1. *The representative function has the following properties:*

(i) $\overline{w} \equiv \overline{\overline{w}}$ *for each word* w *representing an element of* $S - T$*;*

(ii) if $w_1, w_2 \in \Omega$ *represent the same element of* S *then* $w_1 \equiv w_2$*;*

(iii) $w = \overline{w}$ *holds in* S *for any word* w *representing an element of* $S - T$*.*

The notions of the set of representatives and representative function parallel closely these notions in the group context. There is, however, a somewhat hidden but important difference. If we are given a finitely presented group G and a finite set of generators for a subgroup of finite index H, the Todd—Coxeter enumeration procedure for groups (see Todd and Coxeter (1936) and Neubüser (1981)) becomes an *algorithm* for finding coset representatives for G modulo H. At present no such algorithm is known for semigroups.

Open Problem 7. Find a procedure which will take as its input a finite semigroup presentation $\langle A \mid \mathfrak{R} \rangle$ (defining a semigroup S) and a finite set of words $X \subseteq A^+$, and which would terminate if and only if the subsemigroup T of S generated by X has finite index in S, and would return in this case a set of representatives of $S - T$.

However, when T is a two-sided ideal then the Todd—Coxeter enumeration procedure for semigroups (see Chapter 14) can be used for finding representatives of $S - T$. If S is defined by a finite presentation $\langle A \mid \mathfrak{R} \rangle$, and if T is given by a finite set of words $X = \{\xi_1, \ldots, \xi_k\} \subseteq A^+$ which generates T as an ideal, then the presentation

$$\langle A \mid \mathfrak{R},\ \xi_1 = \ldots = \xi_k = 0 \rangle$$

defines the Rees quotient S/T. If T has finite index in S, then the above presentation defines a finite semigroup, so that the Todd—Coxeter enumeration procedure would terminate after a finite number of steps, yielding a set of representatives for $S - T$. This can be generalised to right ideals; see Section 3 of Chapter 14.

Now we find a generating set for a subsemigroup in terms of a set of representatives. Let us recall that for a semigroup generated by A and a subset T of S, $L(A,T)$ denotes the set of all words from A^+ which represent elements from T.

Theorem 1.2. *Let* S *be a semigroup generated by a set* A*, and let* T *be a subsemigroup of* S*. If* Ω *is a set of representatives of* $S - T$ *then the set*

$$X = \{\rho_1 a \rho_2 \mid \rho_1, \rho_2 \in \Omega,\ a \in A,\ \ \rho_1 a, \rho_1 a \rho_2 \in L(A,T)\}$$

generates T*.*

PROOF. We prove that each word $w \in L(A,T)$ is equal in S to a product of elements from X. We do this by induction on the length of w. If $|w| = 1$ then $w \equiv a \in A$ and w represents an element of T so that $w \equiv \epsilon a \epsilon \in X$. For $|w| > 1$ let $w \equiv w_1 a w_2$ ($w_1, w_2 \in A^*$, $a \in A$), where $w_1 a$ is the shortest initial segment of w which is in $L(A,T)$. If $w_2 \notin L(A,T)$ (in particular, if $w_2 \equiv \epsilon$) then $w = \overline{w}_1 a \overline{w}_2 \in X$. Otherwise $w = (\overline{w}_1 a) w_2$ and the assertion follows by induction because $\overline{w}_1 a \in X$ and $|w_2| < |w|$. ■

This theorem has a certain similarity to Schreier's theorem in groups (see Magnus, Karrass and Solitar (1966)); for example it gives a bound for the rank (i.e. the minimal cardinality of a generating set) of a subsemigroup.

Corollary 1.3. *Let S be any semigroup and let T be a subsemigroup of S. Then*

$$\mathrm{rank}(T) \leq (|S - T| + 1)^2 \mathrm{rank}(S).$$

In particular, if S is finitely generated and T has finite index, then T is finitely generated. ■

The following example shows that the set X given in Theorem 1.2 is, in general, the best possible generating set for a subsemigroup.

Example 1.4. Let $S = A^+$ be the free semigroup on an alphabet A, let $k > 1$, and let

$$T = \{w \in A^+ \mid |w| \geq k\}.$$

T is obviously a subsemigroup of S. Since each element of S can be written in a unique way as a product of elements of A, the set of representatives of $S - T$ has to be

$$\Omega = \{w \in A^* \mid |w| < k\}.$$

The generating set X given in Theorem 1.2 is

$$\begin{aligned} X &= \{w_1 a w_2 \mid w_1, w_2 \in A^*,\ a \in A,\ |w_1|, |w_2| < k,\ |w_1 a|, |w_1 a w_2| \geq k\} \\ &= \{w \in A^+ \mid k \leq |w| \leq 2k - 1\}. \end{aligned}$$

On the other hand, being a subsemigroup of a free semigroup, T has a unique minimal generating set

$$\begin{aligned} T - T^2 &= \{w \in A^+ \mid |w| \geq k\} - \{w \in A^+ \mid |w| \geq 2k\} \\ &= \{w \in A^+ \mid k \leq |w| \leq 2k - 1\}; \end{aligned}$$

see Lothaire (1983). Hence, X is the unique minimal generating set of T. ■

Our next task is to find a rewriting mapping associated with the generating set given in Theorem 1.2. (For the definition of rewriting mapping see Section 6 of Chapter 6.) First we introduce a new alphabet

$$B = \{b_{\rho,a,\sigma} \mid \rho, \sigma \in \Omega,\ a \in A,\ \rho a, \rho a \sigma \in L(A,T)\} \tag{1}$$

in one-one correspondence with X. The representation mapping is now the unique homomorphism $\psi : B^+ \longrightarrow A^+$ extending the mapping

$$\psi : b_{\rho,a,\sigma} \mapsto \rho a \sigma \ (\rho, \sigma \in \Omega,\ a \in A,\ \rho a, \rho a \sigma \in L(A,T)). \tag{2}$$

For a word $w \in L(A,T)$ let $w'a$ ($w' \in A^*$, $a \in A$) be the shortest initial segment of w in $L(A,T)$, and let w'' be the rest of w. Define

$$\phi \, : \, L(A,T) \longrightarrow B^+$$

by

$$w\phi = \begin{cases} b_{\overline{w'},a,\overline{w''}} & \text{if } w'' \notin L(A,T) \\ b_{\overline{w'},a,\epsilon}(w''\phi) & \text{if } w'' \in L(A,T). \end{cases} \tag{3}$$

Lemma 1.5. *ϕ is a rewriting mapping.*

PROOF. We have to prove that

$$(w\phi)\psi = w$$

holds in S for any $w \in L(A,T)$. We do this by induction on $|w|$. If $|w| = 1$ then $w \equiv a \in A$, and $a \in L(A,T)$. By (3) we have

$$w\phi \equiv a\phi \equiv b_{\epsilon,a,\epsilon},$$

so that

$$(w\phi)\psi \equiv (b_{\epsilon,a,\epsilon})\psi \equiv \epsilon a \epsilon \equiv a \equiv w.$$

Let $|w| > 1$, and, as before, let $w \equiv w'aw''$, where $w'a$ is the shortest initial segment of w which belongs to $L(A,T)$. If $w'' \notin L(A,T)$ then

$$(w\phi)\psi \equiv b_{\overline{w'},a,\overline{w''}}\psi \equiv \overline{w'}a\overline{w''} = w'aw''$$

by Lemma 1.1(iii). If $w'' \notin L(A,T)$, then

$$\begin{array}{rcll} (w\phi)\psi & \equiv & (b_{\overline{w'},a,\epsilon}(w''\phi))\psi & \text{(by (3))} \\ & \equiv & b_{\overline{w'},a,\epsilon}\psi \cdot (w''\phi)\psi & (\psi \text{ is a homomorphism}) \\ & \equiv & \overline{w'}a \cdot (w''\phi)\psi & \text{(definition of } \psi) \\ & = & w'a \cdot (w''\phi)\psi & \text{(Lemma 1.1(iii))} \\ & = & w'aw'', & \text{(inductive hypothesis)} \end{array}$$

which completes the proof. ■

In the following two sections we shall use the generating set X given in Theorem 1.2 and the above rewriting mapping to find presentations for T, when T is a two-sided or one-sided ideal. We finish this section by proving the following technical lemma, which gives certain rules for applying ϕ in the case when T is a right ideal.

Lemma 1.6. *Let T be a right ideal in S, with all the other notation from this section. Each word $\beta \in L(A,T)$ can be written as*

$$\beta \equiv \beta_1 a_1 \dots \beta_{k-1} a_{k-1} \beta_k a_k \beta_{k+1}, \tag{4}$$

where $k \geq 1$, $\beta_1, \dots, \beta_{k+1} \in A^*$, $a_1, \dots, a_k \in A$, $\beta_1, \dots, \beta_{k+1} \notin L(A,T)$, $\beta_1 a_1, \dots, \beta_k a_k \in L(A,T)$, and then

$$\beta\phi \equiv (\beta_1 a_1)\phi \cdot \dots \cdot (\beta_{k-1} a_{k-1})\phi \cdot (\beta_k a_k \beta_{k+1})\phi. \tag{5}$$

If $\beta_i' \in A^+$ is such that $\beta_i = \beta_i'$ in S then

$$\begin{aligned} \beta\phi &\equiv (\beta_1 a_1 \dots \beta_i' a_i \dots \beta_{k-1} a_{k-1} \beta_k a_k \beta_{k+1})\phi \\ &\equiv (\beta_1 a_1)\phi \cdot \dots \cdot (\beta_i' a_i)\phi \cdot \dots \cdot (\beta_{k-1} a_{k-1})\phi \cdot (\beta_k a_k \beta_{k+1})\phi. \end{aligned} \tag{6}$$

In particular,

$$\begin{aligned} \beta\phi &\equiv (\beta_1 a_1 \dots \overline{\beta_i} a_i \dots \beta_{k-1} a_{k-1} \beta_k a_k \beta_{k+1})\phi \\ &\equiv (\beta_1 a_1)\phi \cdot \dots \cdot (\overline{\beta_i} a_i)\phi \cdot \dots \cdot (\beta_{k-1} a_{k-1})\phi \cdot (\beta_k a_k \beta_{k+1})\phi. \end{aligned} \tag{7}$$

Proof. The decomposition (4) can be proved by an easy induction, where $\beta_1 a_1$ should be chosen to be the shortest initial segment of β representing an element of T. Equality (5) then follows directly from the definition (3) of ϕ. To prove (6) it is enough to note that no initial segment of β_i' can represent an element of T since T is a right ideal and $\beta_i' = \beta_i$ does not represent an element of T. Finally, (7) follows from (6) by Lemma 1.1(iii). ■

2. Two sided ideals of finite index

In this section we find a presentation for a two-sided ideal I of a semigroup S defined by a presentation $\langle A \mid \mathfrak{R} \rangle$. The presentation obtained is in terms of the generating set obtained in the previous section. We use the general approach for finding presentations for subsemigroups outlined in Section 7 of Chapter 6. This approach, as well as the final result (Corollary 1.3), is similar to the Reidemeister—Schreier theorems for groups; see Magnus, Karrass and Solitar (1976).

The Reidemeister—Schreier presentation for a subgroup H of a group G depends crucially on a set of coset representatives of H and the action of G on this set. If G is a finitely presented group, and if H is a subgroup of finite index given by a finite generating set, then a set of coset representatives of H and the action of G on this set can be constructed effectively by using the Todd—Coxeter enumeration procedure for groups.

In what follows we have a similar setting: the set of representatives Ω of $S - I$ plays the role of the set of coset representatives, and S acts on the set $\Omega \cup \{0\}$

(where 0 replaces all the elements of I simultaneously), due to the fact that I is an ideal; see Appendix A for details. As we mentioned in Section 1, the semigroup version of the Todd—Coxeter enumeration procedure can be used to find both Ω and the action of S on $\Omega \cup \{0\}$ if I has finite index in S.

On the other hand, the actual presentation that we obtain, as well as the proofs of technical details are rather different from the group case. This is primarily due to the difference in the nature of the corresponding actions: the action of S in $\Omega \cup \{0\}$ in general does not have any symmetry, and, in particular, is not transitive.

We now state the main result of this section.

Theorem 2.1. *Let S be the semigroup defined by a presentation $\langle A \,|\, \mathfrak{R}\rangle$, and let I be a two sided ideal of S. If Ω is a set of representatives of $S - I$, and if B and ϕ are defined by (1) and (3) respectively, then I is defined by generators B and relations*

$$(w_1 w_2)\phi = w_1\phi \cdot w_2\phi, \tag{8}$$

where

$$w_1 \in \{\rho_1 a \rho_2 \mid \rho_1, \rho_2 \in \Omega,\ a \in A,\ \rho_1 a \in L(A, I)\},$$
$$w_2 \in \{\rho_1 a_1 \ldots \rho_k a_k \rho_{k+1} \mid 1 \leq k \leq 3,\ \rho_i \in \Omega,\ a_j \in A,\ \rho_1 a_1 \ldots \rho_k a_k \in L(A, I)\},$$

and

$$(w_3 u w_4)\phi = (w_3 v w_4)\phi, \tag{9}$$

where

$$(u = v) \in \mathfrak{R},\ w_3 \in \Omega,\ w_3 u w_4 \in L(A, I),$$
$$w_4 \in \{\rho_1 a_1 \ldots \rho_k a_k \rho_{k+1} \mid k = 0, 1, 2,\ \rho_i \in \Omega,\ a_j \in A,$$
$$\rho_1 a_1 \ldots \rho_k a_k \rho_{k+1} \notin L(A, I)\}.$$

Remark 2.2. Note that, when I is an ideal, the generating set X and the corresponding alphabet B can be written as

$$X = \{\rho a \sigma \mid \rho, \sigma \in \Omega,\ a \in A,\ \rho a \in L(A, I)\},$$
$$B = \{b_{\rho,a,\sigma} \mid \rho, \sigma \in \Omega,\ a \in A,\ \rho a \in L(A, I)\},$$

since $\rho a \in L(A, I)$ implies $\rho a \sigma \in L(A, I)$. For the same reason we have $w_1, w_2 \in L(A, I)$ in the statement of Theorem 2.1. It is also important to note that for $k = 0$, the word w_4 takes all the values in Ω. ■

If all the sets A, $\mathfrak{R}$ and Ω are finite, then so are the sets of all relations (8) and (9). Therefore we have

Corollary 2.3 (Reidemeister—Schreier Theorem for ideals) *If S is a finitely presented semigroup, and if I is a two-sided ideal of finite index in S, then I is finitely presented.* ■

We prove the theorem along the lines set out at the end of Section 7 of Chapter 6. Actually, we have already gone a certain way in doing this: in Section 1 we have found a generating set for I and an associated rewriting mapping. Therefore, we are left with proving that all relations (8) and (9) hold in I, and that they imply all the relations

$$(\rho a\sigma)\phi = b_{\rho,a,\sigma}, \tag{10}$$
$$(w_1w_2)\phi = w_1\phi \cdot w_2\phi, \tag{11}$$
$$(w_3uw_4)\phi = (w_3vw_4)\phi, \tag{12}$$

where $\rho,\sigma \in \Omega$, $a \in A$, $\rho a \in L(A,I)$, $w_1, w_2 \in L(A,I)$, $w_3, w_4 \in A^*$, $(u = v) \in \mathfrak{R}$, $w_3uw_4 \in L(A,I)$. We do this in a series of lemmas. In all these lemmas we assume the notation from Theorem 2.1 and Section 1.

The first of the above assertions is easy to prove.

Lemma 2.4. *Relations (8) and (9) hold in I.*

PROOF. In the proof of Theorem 6.7.1 we proved that relations (10), (11), (12) hold in I. Relations (8) and (9) are special cases of relations (11) and (12) respectively, and the lemma follows. ■

It is also easy to show that relations (10) are all implied by (8) and (9); actually, with our particular choice of ϕ relations (10) become trivial:

Lemma 2.5. $b_{\rho,a,\sigma} \equiv (\rho a\sigma)\phi$, *for all $\rho,\sigma \in \Omega$, and all $a \in A$ such that $\rho a \in L(A,I)$.*

PROOF. Since I is an ideal, and since $\rho \notin L(A,I)$, it follows that no initial segment of ρ belongs to $L(A,I)$. Therefore, ρa is the shortest initial segment of $\rho a\sigma$ which belongs to $L(A,I)$, and the lemma follows from the definition (3) of ϕ. ■

Relations (11) are much harder to deduce, and we do this in several steps.

Lemma 2.6. *Let $\alpha,\beta,\gamma,\delta \in A^+$ and $a_1, a_2 \in A$ be such that $\alpha,\beta,\gamma,\delta \notin L(A,I)$, $\alpha a_1, \gamma a_2 \in L(A,I)$; then the relation*

$$(\alpha a_1\beta\gamma a_2\delta)\phi = (\alpha a_1\beta)\phi \cdot (\gamma a_2\delta)\phi$$

is a consequence of (8) and (9).

PROOF. Assume first that $\beta\gamma \notin L(A,I)$. Since $\alpha \notin L(A,I)$, and since I is an ideal, no initial segment of α belongs to $L(A,I)$. Similarly, no initial segment of either $\beta\gamma$ or δ belongs to $L(A,I)$. On the other hand $\alpha a_1 \in L(A,I)$ by a condition of the lemma, and $\beta\gamma a_2 \in L(A,I)$ since $\gamma a_2 \in L(A,I)$ and I is an ideal. Therefore, the decomposition of $\alpha a_1\beta\gamma a_2\delta$ in accord with Lemma 1.6 (4) is $\beta_1 a_1'\beta_2 a_2'\beta_3$, with $\beta_1 \equiv \alpha$, $a_1' = a_1$, $\beta_2 \equiv \beta\gamma$, $a_2' = a_2$, $\beta_3 \equiv \delta$. By Lemma 1.6 (6) and (7) we have

$$(\alpha a_1\beta\gamma a_2\delta)\phi \equiv (\overline{\alpha}a_1\overline{\beta}\overline{\gamma}a_2\overline{\delta})\phi. \tag{13}$$

Now note that $\overline{\alpha},\overline{\beta},\overline{\gamma},\overline{\delta} \in \Omega$ by the definition of the representative function, so that the relation

$$(\overline{\alpha}a_1\overline{\beta}\overline{\gamma}a_2\overline{\delta})\phi = (\overline{\alpha}a_1\overline{\beta})\phi \cdot (\overline{\gamma}a_2\overline{\delta})\phi \tag{14}$$

is one of relations (8). Again by Lemma 1.6 (7) we have

$$(\overline{\alpha}a_1\overline{\beta})\phi \equiv (\alpha a_1\beta)\phi,\ (\overline{\gamma}a_2\overline{\delta})\phi \equiv (\gamma a_2\delta)\phi, \tag{15}$$

and from (13), (14) and (15) we conclude that

$$(\alpha a_1\beta\gamma a_2\delta)\phi = (\alpha a_1\beta)\phi \cdot (\gamma a_2\delta)\phi$$

is a consequence of (8), (9) when $\beta\gamma \notin L(A,I)$.

Let us now assume that $\beta\gamma \in L(A,I)$. Then γ can be written as $\gamma \equiv \gamma_1 a_3\gamma_2$, where $\beta\gamma_1 \notin L(A,I)$ and $\beta\gamma a_2 \in L(A,I)$. Obviously, $\gamma_2 \notin L(A,I)$ because $\gamma \notin L(A,I)$ and I is an ideal. If $\gamma_2 a_2\delta \notin L(A,I)$, then similarly as above we have

$$\begin{aligned}
(\alpha a_1\beta\gamma a_2\delta)\phi &\equiv (\alpha a_1\beta\gamma_1 a_3\gamma_2 a_2\delta)\phi \\
&\equiv (\overline{\alpha}a_1\overline{\beta}\overline{\gamma}_1 a_3\overline{\gamma}_2 a_2\overline{\delta})\phi && \text{(Lemma 1.6 (6), (7))} \\
&= (\overline{\alpha}a_1\overline{\beta})\phi \cdot (\overline{\gamma}_1 a_3\overline{\gamma}_2 a_2\overline{\delta})\phi && \text{(relation (8))} \\
&\equiv (\alpha a_1\beta)\phi \cdot (\gamma_1 a_3\gamma_2 a_2\delta)\phi && \text{(Lemma 1.6 (6), (7))} \\
&\equiv (\alpha a_1\beta)\phi \cdot (\gamma a_2\delta)\phi.
\end{aligned}$$

Finally, if $\gamma_2 a_2\delta \in L(A,I)$, then δ can be written as $\delta \equiv \delta_1 a_4\delta_2$, where $\gamma_2 a_2\delta_1 \notin L(A,I)$ and $\gamma_2 a_2\delta_1 a_4 \in L(A,I)$. This time $\delta \notin L(A,I)$ implies $\delta_2 \notin L(A,I)$, and we have

$$\begin{aligned}
(\alpha a_1\beta\gamma a_2\delta)\phi &\equiv (\alpha a_1\beta\gamma_1 a_3\gamma_2 a_2\delta_1 a_4\delta_2)\phi \\
&\equiv (\overline{\alpha}a_1\overline{\beta}\overline{\gamma}_1 a_3\overline{\gamma}_2 a_2\overline{\delta}_1 a_4\overline{\delta}_2)\phi && \text{(Lemma 1.6)} \\
&= (\overline{\alpha}a_1\overline{\beta})\phi \cdot (\overline{\gamma}_1 a_3\overline{\gamma}_2 a_2\overline{\delta}_1 a_4\overline{\delta}_2)\phi && \text{(relation (8))} \\
&\equiv (\alpha a_1\beta)\phi \cdot (\gamma_1 a_3\gamma_2 a_2\delta_1 a_4\delta_2)\phi && \text{(Lemma 1.6)} \\
&\equiv (\alpha a_1\beta)\phi \cdot (\gamma a_2\delta)\phi,
\end{aligned}$$

which completes the proof. ■

Lemma 2.7. *Let $w_1 \in L(A,I)$ be an arbitrary word, and let $\gamma,\delta \in A^+$ and $a_2 \in A$ be such that $\gamma,\delta \notin L(A,I)$ and $\gamma a_2 \in L(A,I)$. The relation*

$$(w_1\gamma a_2\delta)\phi = w_1\phi \cdot (\gamma a_2\delta)\phi$$

is a consequence of (8) and (9).

PROOF. Let

$$w_1 \equiv \beta_1 a_1' \ldots \beta_k a_k' \beta_{k+1}$$

be the decomposition of w_1 in accord with Lemma 1.6 (4). We prove the lemma by induction on k, the case $k = 1$ being Lemma 2.6. For $k > 1$ let us denote β_1 by w_1' and $\beta_2 a_2' \ldots \beta_k a_k' \beta_{k+1}$ by w_1''. Then $w_1'' \in L(A, I)$, and we have

$$\begin{array}{rcll} (w_1 \gamma a_2 \delta)\phi & \equiv & (w_1' a_1 w_1'' \gamma a_2 \delta)\phi & \\ & \equiv & (w_1' a_1)\phi \cdot (w_1'' \gamma a_2 \delta)\phi & \text{(Lemma 1.6)} \\ & = & (w_1' a_1)\phi \cdot w_1''\phi \cdot (\gamma a_2 \delta)\phi & \text{(induction)} \\ & \equiv & (w_1' a_1 w_1'')\phi \cdot (\gamma a_2 \delta)\phi & \text{(Lemma 1.6)} \\ & \equiv & w_1\phi \cdot (\gamma a_2 \delta)\phi, & \end{array}$$

as required. ∎

Lemma 2.8. *For any two words $w_1, w_2 \in L(A, I)$, the relation*

$$(w_1 w_2)\phi = w_1\phi \cdot w_2\phi$$

is a consequence of (8) and (9).

PROOF. Let

$$w_2 \equiv \beta_1 a_1 \ldots \beta_k a_k \beta_{k+1}$$

be the decomposition of w_2 in accord with Lemma 1.6 (4). We prove the lemma by induction on k, the case $k = 1$ being Lemma 2.7. For $k > 1$ we have

$$w_1 \beta_1 a_1 \ldots \beta_{k-1} a_{k-1} \in L(A, I),$$

so that

$$\begin{array}{rcl} (w_1 w_2)\phi & \equiv & ((w_1 \beta_1 a_1 \ldots \beta_{k-1} a_{k-1})(\beta_k a_k \beta k + 1))\phi \\ & = & (w_1 \beta_1 a_1 \ldots \beta_{k-1} a_{k-1})\phi \cdot (\beta_k a_k \beta_{k+1})\phi \end{array} \tag{16}$$

by Lemma 2.7. Now, by induction we have

$$(w_1 \beta_1 a_1 \ldots \beta_{k-1} a_{k-1})\phi = w_1\phi \cdot (\beta_1 a_1 \ldots \beta_{k-1} a_{k-1})\phi. \tag{17}$$

Finally, from (16) and (17) we have

$$\begin{array}{rcl} (w_1 w_2)\phi & = & w_1\phi \cdot (\beta_1 a_1 \ldots \beta_{k-1} a_{k-1})\phi \cdot (\beta_k a_k \beta_{k+1})\phi \\ & \equiv & w_1\phi \cdot (\beta_1 a_1 \ldots \beta_k a_k \beta_{k+1})\phi \\ & \equiv & w_1\phi \cdot w_2\phi \end{array}$$

by Lemma 1.6 (5). ∎

Now we embark on deducing relations (12), again in several steps.

Lemma 2.9. *Let $\alpha, \beta \in A^+$ and $(u = v) \in \mathfrak{R}$. If $\alpha \notin L(A, I)$ and $\alpha u \beta \in L(A, I)$, then the relation*

$$(\alpha u \beta)\phi = (\alpha v \beta)\phi$$

is a consequence of (8) and (9).

Proof. If $\alpha u \notin L(A, I)$ then no initial segment of αu belongs to $L(A, I)$ since I is an ideal. In other words, if $\beta_1 a_1 \ldots \beta_k a_k \beta_{k+1}$ is the decomposition of $\alpha u \beta$ in accord with Lemma 1.6 (4), then αu is an initial segment of β_1, and by Lemma 1.6 (6) we have

$$(\alpha u \beta)\phi \equiv (\alpha v \beta)\phi.$$

Therefore we may assume that $\alpha u \in L(A, I)$.

If $\beta \in L(A, I)$, then we have

$$(\alpha u \beta)\phi = (\alpha u)\phi \cdot \beta\phi, \tag{18}$$

$$(\alpha v \beta)\phi = (\alpha v)\phi \cdot \beta\phi, \tag{19}$$

by Lemma 2.8. Also, since $\alpha \notin L(A, I)$, by Lemma 1.6 (6) we have

$$(\alpha u)\phi \equiv (\overline{\alpha} u)\phi, \ (\alpha v)\phi \equiv (\overline{\alpha} v)\phi. \tag{20}$$

The relation

$$(\overline{\alpha} u)\phi = (\overline{\alpha} v)\phi \tag{21}$$

is one of the relations (9), for $w_1 \equiv \overline{\alpha} \in \Omega$ and $w_2 \equiv \epsilon \in \Omega$. From (18), (19), (20) and (21) we obtain

$$(\alpha u \beta)\phi = (\alpha v \beta)\phi.$$

So we may assume that $\beta \notin L(A, I)$.

Now let

$$\alpha u \equiv \alpha \gamma_1 a_1 \gamma_2 a_2 \ldots \gamma_k a_k \gamma_{k+1}, \quad \alpha v \equiv \alpha \delta_1 a_1' \delta_2 a_2' \ldots \delta_l a_l' \delta_{l+1},$$

where

$$\alpha\gamma_1, \gamma_2, \ldots, \gamma_{k+1}, \alpha\delta_1, \delta_2, \ldots, \delta_{l+1} \notin L(A, I),$$
$$\alpha\gamma_1 a_1, \gamma_2 a_2, \ldots, \gamma_k a_k, \alpha\delta_1 a_1', \delta_2 a_2', \ldots, \delta_l a_l' \in L(A, I).$$

We distinguish the following four cases.

Case 1. $\gamma_{k+1}\beta \notin L(A, I)$ and $\delta_{l+1}\beta \notin L(A, I)$. Since $\beta \notin L(A, I)$, we have

$$\begin{aligned}
(\alpha u \beta)\phi &\equiv (\alpha\gamma_1 a_1 \ldots \gamma_k a_k \gamma_{k+1}\beta)\phi \\
&\equiv (\overline{\alpha}\gamma_1 a_1 \ldots \gamma_k a_k \gamma_{k+1}\overline{\beta})\phi \quad \text{(Lemma 1.6 (6), (7))} \\
&\equiv (\overline{\alpha} u \overline{\beta})\phi \\
&= (\overline{\alpha} v \overline{\beta})\phi \quad \text{(relation (9))} \\
&\equiv (\overline{\alpha}\delta_1 a_1' \ldots \delta_l a_l' \delta_{l+1}\overline{\beta})\phi \\
&\equiv (\alpha\delta_1 a_1' \ldots \delta_l a_l' \delta_{l+1}\beta)\phi \quad \text{(Lemma 1.6 (6), (7))} \\
&\equiv (\alpha v \beta)\phi.
\end{aligned}$$

Case 2. $\gamma_{k+1}\beta \in L(A,I)$ and $\delta_{l+1}\beta \notin L(A,I)$. We can write β as $\beta \equiv \beta_1 a \beta_2$, where $\gamma_{k+1}\beta_1 a$ is the shortest initial segment of $\gamma_{k+1}\beta$ which is in $L(A,I)$. By applying Lemma 1.6 and relation (9), similarly as in the previous case, we obtain

$$(\alpha u\beta)\phi \equiv (\overline{\alpha}u\overline{\beta}_1 a\overline{\beta}_2)\phi = (\overline{\alpha}v\overline{\beta}_1 a\overline{\beta}_2)\phi \equiv (\overline{\alpha}v\overline{\beta})\phi \equiv (\alpha v\beta)\phi.$$

Case 3. $\gamma_{k+1}\beta \notin L(A,I)$ and $\delta_{l+1}\beta \in L(A,I)$. This is dual to the previous case.

Case 4. $\gamma_{k+1}\beta \in L(A,I)$ and $\delta_{l+1}\beta \in L(A,I)$. Let $\gamma_{k+1}\beta_1 a$ and $\delta_{l+1}\beta_1' a'$ be the shortest initial segments of $\gamma_{k+1}\beta$ and $\delta_{l+1}\beta$, respectively, which are in $L(A,I)$, and let $\beta \equiv \beta_1 a\beta_2 \equiv \beta_1' a' \beta_2'$. If $\beta_1 \equiv \beta_1'$, then $a \equiv a'$ and $\beta_2 \equiv \beta_2'$, so that

$$(\alpha u\beta)\phi \equiv (\alpha u\beta_1 a\beta_2)\phi \equiv (\overline{\alpha}u\overline{\beta}_1 a\overline{\beta}_2)\phi = (\overline{\alpha}v\overline{\beta}_1 a\overline{\beta}_2)\phi \equiv (\alpha v\beta)\phi$$

by Lemma 1.6 and relation (9). If $|\beta_1| < |\beta_1'|$ then β can be written as $\beta \equiv \beta_1 a\beta_2 a'\beta_3$, and we have

$$(\alpha u\beta)\phi \equiv (\alpha u\beta_1 a\beta_2 a'\beta_3)\phi \equiv (\overline{\alpha}u\overline{\beta}_1 a\overline{\beta}_2 a'\overline{\beta}_3)\phi = (\overline{\alpha}v\overline{\beta}_1 a\overline{\beta}_2 a'\overline{\beta}_3)\phi \equiv (\alpha v\beta)\phi.$$

The case when $|\beta_1| > |\beta_1'|$ is dealt with analogously.

This completes the proof of the lemma. ■

Lemma 2.10. *For any two words $w_3, w_4 \in A^*$ and any $(u = v) \in \mathfrak{R}$ such that $w_3uw_4 \in L(A,I)$ the relation $(w_3uw_4)\phi = (w_3vw_4)\phi$ is a consequence of (8) and (9).*

PROOF. If $w_3 \notin L(A,I)$ this is Lemma 2.9. Suppose that $w_3 \in L(A,I)$ and write $w_3 \equiv \alpha_1 a_1 \ldots \alpha_k a_k \alpha_{k+1}$, where $\alpha_1, \ldots, \alpha_{k+1} \notin L(A,I)$ and $\alpha_1 a_1, \ldots, \alpha_k a_k \in L(A,I)$. If $\alpha_{k+1}u \notin L(A,I)$ then $(w_3uw_4)\phi \equiv (w_3vw_4)\phi$ by Lemma 1.6, while, if $\alpha_{k+1}u \in L(A,I)$, we have

$$\begin{aligned}(w_3uw_4)\phi &\equiv (\alpha_1 a_1 \ldots \alpha_k a_k \alpha_{k+1} u w_4)\phi \equiv (\alpha_1 a_1)\phi \cdot \ldots \cdot (\alpha_k a_k)\phi \cdot (\alpha_{k+1} u w_4)\phi \\ &= (\alpha_1 a_1)\phi \cdot \ldots \cdot (\alpha_k a_k)\phi \cdot (\alpha_{k+1} v w_4)\phi \equiv (w_3vw_4)\phi\end{aligned}$$

by Lemma 2.9. ■

We can now prove Theorem 2.1.

PROOF OF THEOREM 2.1. By Theorem 1.2 the set

$$X = \{\rho a\sigma \mid \rho, \sigma \in \Omega,\ a \in A,\ \rho a \in L(A,I)\}$$

generates I, and the mapping ϕ, defined by (3), is a corresponding rewriting mapping. Relations (8) and (9) hold in T by Lemma 2.4, and imply relations (10), (11), (12) by Lemmas 2.5, 2.8 and 2.10. Therefore the theorem follows from Theorem 6.7.1 and Remark 6.7.6. ■

As we have already mentioned, defining relations (8), (9) for an ideal are more complicated than the Reidemeister—Schreier relations for a subgroup of a group. In particular, they seem to be rather hard to use for any practical purposes. This prompts us to pose two different types of open problems. First of all, it would be interesting to see if there is a simpler presentation for ideals in general, or, if not, to find simpler presentations in some special cases.

Open Problem 8. Find a presentation for an ideal I of a finitely presented semigroup S, which would be finite whenever I has finite index in S, and which is simpler than the presentation given in Theorem 7.2.1.

Open Problem 9. Find a finite presentation for an ideal I of finite index in a finitely presented semigroup S if S/I is known to be of some special type. In particular, find such presentation if S/I is

(i) a group with a zero adjoined;
(ii) a completely simple semigroup with a zero adjoined;
(iii) a completely 0-simple semigroup;
(iv) an inverse semigroup.

It would also be useful to have computational methods for handling large presentations. In computational group theory such methods are available in the form of so called *Tietze transformations* programs; see Havas et al. (1984).

Open Problem 10. Develop a semigroup version of the Tietze transformation program for simplifying presentations.

Apart from the number and nature of defining relations, there are other, more striking, differences between ideals of finite index and subgroups. We finish this section by pointing out some of them.

Example 2.11. The action of a group on the cosets of a subgroup can be considered as an automaton (see Sims (1994)), giving that the set of all words representing elements of a subgroup of finite index is a regular language. (For the definitions and basic results on finite state automata and regular languages see Howie (1991).) Moreover, the converse also holds: if the set of all words representing elements of a subgroup is a regular language, then the subgroup has finite index; see Sims(1994), Corollary 2.2.

In one direction this also holds for ideals of semigroups. Let S be a semigroup generated by a set A, let I be an ideal of S, and let Ω be a set of representatives of $S-I$. Consider the deterministic automaton $\mathcal{M} = (\Omega \cup \{0\}, A, t, \epsilon, \{0\})$, where

$$t : (\Omega \cup \{0\}) \times A \longrightarrow \Omega \cup \{0\}$$

is the action of the generators A on $\Omega \cup \{0\}$. Then t can be extended naturally to a mapping

$$t : (\Omega \cup \{0\}) \times A^+ \longrightarrow \Omega \cup \{0\},$$

and it is obvious that $t(\epsilon, \alpha) = 0$ if and only if $\alpha \in L(A, I)$. The automaton $\mathcal{M}$ is finite if and only if I has finite index in S, and, in this case, the language $L(A, I)$ is regular.

However, the converse does not hold. For instance, let $F = \langle a, b \mid \ \rangle$ be the free semigroup on $\{a, b\}$, and let I be the ideal consisting of all words containing both a and b. Obviously, the language $L(A, I)$ $(= I)$ is regular, but I does not have finite index in S. ■

Example 2.12. Given a subgroup H of a group G the left (right) congruence η defined by

$$x\eta y \Leftrightarrow x^{-1}y \in H \ \ (xy^{-1} \in H)$$

is a unique left (right) congruence having H as an equivalence class. Therefore, H is a subgroup of finite index in G if and only if H is an equivalence class of a congruence of finite index.

For an ideal I of a semigroup S the Rees congruence η_I does have I as an equivalence class. However, in general, it is not a unique, but only the smallest, congruence with this property. If we change our definition of finite index to mean that I is an equivalence class of a congruence of finite index, then the analogue of Corollary 2.3 fails to hold. To show this, consider the free semigroup $F = \langle a, b \mid \ \rangle$ on two generators a, b and its three-element homomorphic image

$$S = \langle a, b \mid a^2 = a, \ b^2 = b, \ ab = ba \rangle,$$

and let $\phi : F \longrightarrow S$ be the natural epimorphism. The congruence $\ker\phi$ has three congruence classes:

$$K_1 = \{a^i \mid i \geq 1\}, \ K_2 = \{b^i \mid i \geq 1\}, \ K_3 = \{w \mid w \text{ contains both } a \text{ and } b\}.$$

K_3 is an ideal of F, and it has 'finite index' (by our alternative definition). However, K_3 is not finitely generated, let alone finitely presented—elements ab^i, $i \geq 1$ are all indecomposable. ■

Example 2.13. Given a finitely generated group G and a number $k \geq 1$, there are only finitely many subgroups of G with index at most k. If G is finitely presented, then each of these subgroups is also finitely presented by the Reidemeister—Schreier Theorem.

The analogous results are true for ideals: if S is a finitely generated semigroup and $k \geq 1$, then there are only finitely many ideals of S having index at most k (see Jura (1978)); if S is finitely presented each of these ideals is finitely presented by Corollary 2.3.

For groups there even exists an algorithm (usually called the *Low Index Subgroup Algorithm*) which as its input takes a presentation for G and a number k, and gives as its output a list of generating sets for all subgroups of G of index at most k; see Dietze and Schaps (1974). Rather surprisingly, an analogous algorithm does not exists for ideals—its existence would imply a solution to the finiteness problem; for details see Jura (1980). ■

3. Right ideals

In this section we generalise results of the previous section to right ideals. The set of defining relations we obtain is both larger and more complicated than in the case of two-sided ideals.

When the parallelism between groups and semigroups is concerned, it is the right (and left) ideals that really correspond to subgroups, as both of them give rise to one-sided congruences; properties of two sided ideals are closer to normal subgroups. However, all the comments about this parallelism made in the previous section remain valid if the word 'ideal' is replaced by 'one-sided ideal'.

In this section we work in the same setting as in the previous one. Hence, we consider a semigroup S defined by a presentation $\langle A \mid \mathfrak{R}\rangle$, and we let R denote a right ideal of S. A set of representatives of $S - R$ will be denoted by Ω. Thus Ω has properties (1), (2), (3) from Section 1. However, here we need to assume that Ω has an additional property:

(4) each non-empty word of Ω is a word of minimal length representing the corresponding element of S.

If, as usual, $x \mapsto \overline{x}$ denotes the representative function, then the above condition means that

$$|\overline{x}| \leq |x|,$$

for all $x \in A^+ - L(A, R)$. By Theorem 1.2 the set

$$X = \{\rho a \sigma \mid \rho, \sigma \in \Omega,\ a \in A,\ \rho a \in L(A, R)\}$$

generates R. (Here, as in the case of two-sided ideals, the condition $\rho a \in L(A, T)$ automatically implies $\rho a \sigma \in L(A, I)$.) An abstract image of X is the alphabet

$$B = \{b_{\rho,a,\sigma} \mid \rho, \sigma \in \Omega,\ a \in A,\ \rho a \in L(A, R)\},$$

and associated representation and rewriting mappings are defined by (2) and (3) in Section 1. We also define the following six sets of relations:

$$\begin{aligned}
\mathfrak{R}_1 &= \{(\rho_1 a_1 \rho_2)\phi \cdot (\rho_3 a_2 \rho_4)\phi = (\rho_1 a_1)\phi \cdot (\rho_2 \rho_3 a_2 \rho_4)\phi \mid a_i \in A,\ \rho_i \in \Omega,\\
&\qquad \rho_1 a_1, \rho_2 \rho_3 a_2, \rho_3 a_2 \in L(A, R)\};\\
\mathfrak{R}_2 &= \{(\rho_1 a_1 \rho_2)\phi \cdot (\rho_3 a_2 \rho_4)\phi = (\rho_1 a_1 \overline{\rho_2 \rho_3 a_2 \rho_4})\phi \mid a_i \in A,\ \rho_i \in \Omega,
\end{aligned}$$

$$\begin{aligned}
& \rho_1 a_1, \rho_3 a_2 \in L(A,R), \rho_2\rho_3 a_2 \rho_4 \notin L(A,R)\}; \\
\mathfrak{R}_3 &= \{(\rho_1\rho_2)\phi \cdot (\rho_3 a \rho_4 \rho_5)\phi = (\rho_1 \overline{\rho_2 \rho_3 a \rho_4} \rho_5)\phi \mid a \in A,\ \rho_i \in \Omega, \\
& \quad \rho_1\rho_2, \rho_3 a \in L(A,R), \rho_2\rho_3 a \rho_4 \notin L(A,R)\}; \\
\mathfrak{R}_4 &= \{(\rho_1\rho_2)\phi \cdot (\rho_3\rho_4)\phi = (\rho_1 \overline{\rho_2\rho_3}\rho_4)\phi \mid \rho_i \in \Omega, \\
& \quad \rho_1\rho_2, \rho_3\rho_4 \in L(A,R), \rho_2\rho_3 \notin L(A,R)\}; \\
\mathfrak{R}_5 &= \{(\rho_1\rho_2 a \rho_3\rho_4)\phi = (\rho_1 \overline{\rho_2 a \rho_3}\rho_4)\phi \mid a \in A,\ \rho_i \in \Omega, \\
& \quad \rho_1\rho_2 a \in L(A,R), \rho_2 a \rho_3 \notin L(A,R)\}; \\
\mathfrak{R}_6 &= \{(\rho_1 u \rho_2)\phi = (\rho_1 v \rho_2)\phi \mid (u=v) \in \mathfrak{R},\ \rho_i \in \Omega, \rho_1 u \rho_2 \in L(A,R)\}.
\end{aligned}$$

With this notation we have:

Theorem 3.1. *If S is the semigroup defined by a presentation $\langle A \mid \mathfrak{R}\rangle$, and if R is a right ideal of S, then R is defined by the presentation $\langle B \mid \mathfrak{R}_1, \dots, \mathfrak{R}_6\rangle$.*

Theorem 3.1 has an obvious left-right dual. Hence, if we note that when all A, $\mathfrak{R}$ and Ω are finite then all $\mathfrak{R}_1, \dots, \mathfrak{R}_6$ are finite as well, we have:

Corollary 3.2 (Reidemeister—Schreier Theorem for one-sided ideals) *A one-sided ideal of finite index in a finitely presented semigroup is itself finitely presented.* ■

The relations $\mathfrak{R}_1, \dots, \mathfrak{R}_6$ are very complicated. In the case when S is finitely presented and R has finite index, we can find a larger but more natural set of defining relations.

Corollary 3.3. *Let S be a semigroup with finite presentation $\langle A \mid \mathfrak{R}\rangle$, let R be a right ideal of finite index, let Ω be a set of representatives of $S - R$, and let ϕ be the corresponding rewriting mapping. If*

$$m = \max\{|\rho|, |u|, |v| \mid \rho \in \Omega, (u=v) \in \mathfrak{R}\}$$

then R is defined by the presentation

$$\begin{gathered}
\mathfrak{P} = \langle B \mid (w_1 w_2)\phi = w_1\phi \cdot w_2\phi,\ w_3\phi = w_4\phi \\
(w_1, w_2, w_3, w_4 \in L(A,T),\ |w_1 w_2|, |w_3|, |w_4| \leq 5m+2,\ w_3 = w_4 \text{ in } S)\rangle
\end{gathered}$$

Proof. All the relations from the given presentation hold in R, since ϕ is a rewriting mapping, and they include all the relations from the presentation given in Theorem 3.1. ■

We now embark on the proof of Theorem 3.1. Our general strategy is the same as in the previous section: we want to apply Theorem 6.7.1.

Lemma 3.4. *All the relations $\mathfrak{R}_1, \dots, \mathfrak{R}_6$ hold in R.*

PROOF. The lemma follows from the fact that ϕ is a rewriting mapping similarly as in Lemma 2.4. For an example we prove that a general relation

$$(\rho_1 a_1 \rho_2)\phi \cdot (\rho_3 a_2 \rho_4)\phi = (\rho_1 a_1 \overline{\rho_2 \rho_3 a_2 \rho_4})\phi$$

from $\mathfrak{R}_2$ holds in R. This equivalent to proving that

$$((\rho_1 a_1 \rho_2)\phi \cdot (\rho_3 a_2 \rho_4)\phi)\psi = ((\rho_1 a_1 \overline{\rho_2 \rho_3 a_2 \rho_4})\phi)\psi$$

holds in S. Since ψ is a homomorphism and ϕ is a rewriting mapping we have

$$\begin{aligned} ((\rho_1 a_1 \rho_2)\phi \cdot (\rho_3 a_2 \rho_4)\phi)\psi &= \rho_1 a_1 \rho_2 \rho_3 a_2 \rho_4, \\ ((\rho_1 a_1 \overline{\rho_2 \rho_3 a_2 \rho_4})\phi)\psi &= \rho_1 a_1 \overline{\rho_2 \rho_3 a_2 \rho_4}, \end{aligned}$$

and the assertion follows from Lemma 1.1. ■

Lemma 3.5. $(b_{\rho,a,\sigma}\psi)\phi \equiv b_{\rho,a,\sigma}$, *for all* $\rho, \sigma \in \Omega$, $a \in A$ *such that* $\rho a \in L(A,T)$.

PROOF. The proof is exactly the same as the proof of Lemma 2.5. ■

Lemma 3.6. *The following relations are consequences of* $\mathfrak{P}$*:*

(i) $(w_1 w_2)\phi = w_1\phi \cdot w_2\phi$, *for all* $w_1, w_2 \in L(A,R)$*;*
(ii) $(\alpha\beta\gamma)\phi = (\alpha\overline{\beta}\gamma)\phi$, *for all* $\alpha, \gamma \in A^*$, $\beta \in A^+$, $\alpha\beta\gamma \in L(A,R)$, $\beta \notin L(A,R)$.

PROOF. We proceed by simultaneous induction on (i) and (ii). First we prove

(a) $(w_1 w_2)\phi = w_1\phi \cdot w_2\phi$ if $|w_1 w_2| = n$, assuming that $(u_1 u_2)\phi = u_1\phi \cdot u_2\phi$ if $|u_1 u_2| < n$, and that $(\alpha\beta\gamma)\phi = (\alpha\overline{\beta}\gamma)\phi$ if $|\alpha\beta\gamma| < n$;

and then we prove

(b) $(\alpha\beta\gamma)\phi = (\alpha\overline{\beta}\gamma)\phi$ if $|\alpha\beta\gamma| = n$ assuming that $(u_1 u_2)\phi = u_1\phi \cdot u_2\phi$ if $|u_1 u_2| \leq n$, and that $(\delta\zeta\theta)\phi = (\delta\overline{\zeta}\theta)\phi$ if $|\delta\zeta\theta| < n$.

For $n = 1$ there is nothing to prove in (a), while in (b) we have $\alpha \equiv \gamma \equiv \epsilon$ (the empty word), and $\beta \in A$, and the relation $(\alpha\beta\gamma)\phi = (\alpha\overline{\beta}\gamma)\phi$ belongs to $\mathfrak{R}_5$.

Now we prove (a) for a general n. Let $w_1 \equiv \alpha_1 a_1 \alpha_2 a_2 \ldots \alpha_k a_k \alpha_{k+1}$ and $w_2 \equiv \beta_1 b_1 \beta_2 b_2 \ldots \beta_l b_l \beta_{l+1}$ as in Lemma 1.6 (4). If $k > 1$ then

$$\begin{aligned} (w_1 w_2)\phi &\equiv (\alpha_1 a_1 \alpha_2 a_2 \ldots \alpha_k a_k \alpha_{k+1} w_2)\phi \\ &\equiv (\alpha_1 a_1)\phi \cdot (\alpha_2 a_2 \ldots \alpha_k a_k \alpha_{k+1} w_2)\phi && \text{(by (3))} \\ &= (\alpha_1 a_1)\phi \cdot (\alpha_2 a_2 \ldots \alpha_k a_k \alpha_{k+1})\phi \cdot w_2\phi && \text{(induction)} \\ &= (\alpha_1 a_1 \alpha_2 a_2 \ldots \alpha_k a_k \alpha_{k+1})\phi \cdot w_2\phi && \text{(by (3))} \\ &\equiv w_1\phi \cdot w_2\phi. \end{aligned}$$

So we may assume that $k = 1$, i.e. that $w_1 \equiv \alpha_1 a_1 \alpha_2$.

Suppose that $\alpha_2 w_2 \in L(A,R)$, so that

$$(w_1 w_2)\phi \equiv (\alpha_1 a_1)\phi \cdot (\alpha_2 w_2)\phi$$

by the definition of ϕ. If $\alpha_2\beta_1 b_1 \ldots \beta_{l-1}b_{l-1} \in L(A,R)$, then

$$\begin{aligned}
(w_1 w_2)\phi &\equiv (\alpha_1 a_1)\phi \cdot (\alpha_2\beta_1 b_1 \ldots \beta_{l-1}b_{l-1}\beta_l b_l \beta_{l+1})\phi \\
&= (\alpha_1 a_1)\phi \cdot (\alpha_2\beta_1 b_1 \ldots \beta_{l-1}b_{l-1})\phi \cdot (\beta_l b_l \beta_{l+1})\phi && \text{(induction)} \\
&\equiv (\alpha_1 a_1 \alpha_2\beta_1 b_1 \ldots \beta_{l-1}b_{l-1})\phi \cdot (\beta_l b_l \beta_{l+1})\phi && \text{(by (3))} \\
&= (\alpha_1 a_1 \alpha_2)\phi \cdot (\beta_1 b_1 \ldots \beta_{l-1}b_{l-1})\phi \cdot (\beta_l b_l \beta_{l+1})\phi && \text{(induction)} \\
&\equiv (\alpha_1 a_1 \alpha_2)\phi \cdot (\beta_1 b_1 \ldots \beta_{l-1}b_{l-1}\beta_l b_l \beta_{l+1})\phi && \text{(induction)} \\
&\equiv w_1\phi \cdot w_2\phi.
\end{aligned}$$

On the other hand, if $\alpha_2\beta_1 b_1 \ldots \beta_{l-1}b_{l-1} \notin L(A,R)$, then

$$\begin{aligned}
(w_1 w_2)\phi &\equiv (\alpha_1 a_1)\phi \cdot (\alpha_2\beta_1 b_1 \ldots \beta_{l-1}b_{l-1}\beta_l b_l \beta_{l+1})\phi \\
&= (\overline{\alpha_1} a_1)\phi \cdot (\overline{\alpha_2\beta_1 b_1 \ldots \beta_{l-1}b_{l-1}}\,\overline{\beta_l} b_l \overline{\beta_{l+1}})\phi && \text{(induction)} \\
&= (\overline{\alpha_1} a_1 \overline{\alpha_2\beta_1 b_1 \ldots \beta_{l-1}b_{l-1}})\phi \cdot (\overline{\beta_l} b_l \overline{\beta_{l+1}})\phi && \text{(relations } \mathfrak{R}_1\text{)} \\
&\equiv (\overline{\alpha_1} a_1 \overline{\alpha_2\beta_1 b_1 \ldots \beta_{l-1}b_{l-1}})\phi \cdot (\beta_l b_l \beta_{l+1})\phi && \text{(Lemma 1.6 (7))} \\
&= (\alpha_1 a_1 \alpha_2\beta_1 b_1 \ldots \beta_{l-1}b_{l-1})\phi \cdot (\beta_l b_l \beta_{l+1})\phi. && \text{(induction)}
\end{aligned}$$

If $l = 1$, we have finished; otherwise

$$\begin{aligned}
(w_1 w_2)\phi &\equiv (\alpha_1 a_1 \alpha_2)\phi \cdot (\beta_1 b_1 \ldots \beta_{l-1}b_{l-1})\phi \cdot (\beta_l b_l \beta_{l+1})\phi \\
&\equiv (\alpha_1 a_1 \alpha_2)\phi \cdot (\beta_1 b_1 \ldots \beta_{l-1}b_{l-1}\beta_l b_l \beta_{l+1})\phi \equiv w_1\phi \cdot w_2\phi
\end{aligned}$$

by induction. So we may assume that $\alpha_2 w_2 \notin L(A,R)$, and hence that no initial segment of $\alpha_2 w_2$ is in $L(A,R)$. Now

$$(w_1 w_2)\phi \equiv (\alpha_1 a_1 \alpha_2\beta_1 b_1 \ldots \beta_l b_l \beta_{l+1})\phi \equiv (\alpha_1 a_1 \overline{\alpha_2\beta_1 b_1 \ldots \beta_l b_l \beta_{l+1}})\phi,$$

by Lemma 1.6 (7), and

$$\begin{aligned}
w_1\phi \cdot w_2\phi &\equiv (\alpha_1 a_1 \alpha_2)\phi \cdot (\beta_1 b_1 \ldots \beta_l b_l \beta_{l+1})\phi \\
&\equiv (\overline{\alpha_1} a_1 \overline{\alpha_2})\phi \cdot (\overline{\beta_1} b_1)\phi \cdot \ldots \cdot (\overline{\beta_{l-1}} b_{l-1})\phi \cdot (\overline{\beta_l} b_l \overline{\beta_{l+1}})\phi \\
& && \text{(Lemma 1.6 (5), (7))} \\
&= (\overline{\alpha_1} a_1 \overline{\alpha_2\beta_1 b_1})\phi \cdot \ldots \cdot (\overline{\beta_{l-1}} b_{l-1})\phi \cdot (\overline{\beta_l} b_l \overline{\beta_{l+1}})\phi \\
& && \text{(relations } \mathfrak{R}_2\text{)} \\
&= \ldots \\
&= (\overline{\alpha_1} a_1 \overline{\alpha_2\beta_1 b_1 \ldots \beta_{l-1}b_{l-1}})\phi \cdot (\overline{\beta_l} b_l \overline{\beta_{l+1}})\phi && \text{(relations } \mathfrak{R}_2\text{)} \\
&= (\overline{\alpha_1} a_1 \overline{\alpha_2\beta_1 b_1 \ldots \beta_{l-1}b_{l-1}\beta_l b_l \beta_{l+1}})\phi && \text{(relations } \mathfrak{R}_2\text{)} \\
&\equiv (\alpha_1 a_1 \overline{\alpha_2\beta_1 b_1 \ldots \beta_{l-1}b_{l-1}\beta_l b_l \beta_{l+1}})\phi && \text{(Lemma 1.6 (7))} \\
&\equiv (w_1 w_2)\phi.
\end{aligned}$$

This completes the proof of (a).

We now start on the proof of (b). First suppose that $\alpha \in L(A,R)$, so that $\alpha \equiv \alpha_1 a_1 \ldots \alpha_k a_k \alpha_{k+1}$ as in Lemma 1.6 (4). If $\alpha_{k+1}\beta\gamma \notin L(A,R)$, then

$$(\alpha\beta\gamma)\phi \equiv (\alpha_1 a_1 \ldots \alpha_k a_k \alpha_{k+1}\beta\gamma)\phi \equiv (\alpha_1 a_1 \ldots \alpha_k a_k \alpha_{k+1}\overline{\beta}\gamma)\phi \equiv (\alpha\overline{\beta}\gamma)\phi$$

by Lemma 1.6 (6), and we are done. If $\alpha_{k+1}\beta\gamma \in L(A,R)$, then

$$\begin{aligned}(\alpha\beta\gamma)\phi &\equiv (\alpha_1 a_1 \ldots \alpha_k a_k)\phi \cdot (\alpha_{k+1}\beta\gamma)\phi = (\alpha_1 a_1 \ldots \alpha_k a_k)\phi \cdot (\alpha_{k+1}\overline{\beta}\gamma)\phi \\ &\equiv (\alpha_1 a_1 \ldots \alpha_k a_k \alpha_{k+1}\overline{\beta}\gamma)\phi \equiv (\alpha\overline{\beta}\gamma)\phi\end{aligned}$$

by Lemma 1.6 and induction. Therefore we may suppose that $\alpha \notin L(A,R)$.

If $\alpha\beta \notin L(A,R)$, the result follows immediately from Lemma 1.6; so suppose that $\alpha\beta \in L(A,R)$. If $\gamma \in L(A,R)$ (in particular, $\gamma \neq \epsilon$), then

$$(\alpha\beta\gamma)\phi = (\alpha\beta)\phi \cdot \gamma\phi = (\alpha\overline{\beta})\phi \cdot \gamma\phi = (\alpha\overline{\beta}\gamma)\phi$$

by induction; so we may assume that $\gamma \notin L(A,R)$. Since $\alpha \notin L(A,R)$, we may write

$$\alpha\beta \equiv \alpha\beta_1 b_1 \beta_2 b_2 \ldots \beta_k b_k \beta_{k+1},$$

where $\alpha\beta_1 b_1$ is the shortest prefix of $\alpha\beta$ in $L(A,R)$. If $k > 1$, then

$$\begin{aligned}(\alpha\beta\gamma)\phi &\equiv (\alpha\beta_1 b_1 \beta_2 b_2 \ldots \beta_k b_k \beta_{k+1}\gamma)\phi \\ &\equiv (\alpha\beta_1 b_1 \beta_2 b_2 \ldots \beta_{k-1} b_{k-1})\phi \cdot (\beta_k b_k \beta_{k+1}\gamma)\phi && \text{(Lemma 1.6)} \\ &\equiv (\overline{\alpha}\beta_1 b_1 \beta_2 b_2 \ldots \beta_{k-1} b_{k-1})\phi \cdot (\overline{\beta_k} b_k \overline{\beta_{k+1}}\overline{\gamma})\phi && \text{(Lemma 1.6)} \\ &= (\overline{\alpha}\overline{\beta_1 b_1 \beta_2 b_2 \ldots \beta_{k-1} b_{k-1}})\phi \cdot (\overline{\beta_k} b_k \overline{\beta_{k+1}}\overline{\gamma})\phi && \text{(induction)} \\ &= (\overline{\alpha}\overline{\beta_1 b_1 \beta_2 b_2 \ldots \beta_{k-1} b_{k-1} \beta_k b_k \beta_{k+1}}\overline{\gamma})\phi && \text{(relations } \mathfrak{R}_3\text{)} \\ &\equiv (\overline{\alpha}\overline{\beta}\overline{\gamma})\phi.\end{aligned}$$

Since $|\overline{\beta}| \leq |\beta|$, a similar argument shows that $(\alpha\overline{\beta}\gamma)\phi = (\overline{\alpha}\overline{\overline{\beta}}\overline{\gamma})\phi$. But $\overline{\beta} \equiv \overline{\overline{\beta}}$ by Lemma 1.1, and thus $(\alpha\beta\gamma)\phi = (\alpha\overline{\beta}\gamma)\phi$. Hence we may assume that $k = 1$, so that $\alpha\beta \equiv \alpha\beta_1 b_1 \beta_2$. If $\beta_2\gamma \in L(A,R)$, then the definition of ϕ gives that

$$(\alpha\beta\gamma)\phi = (\alpha\beta_1 b_1 \beta_2\gamma)\phi \equiv (\alpha\beta_1 b_1)\phi \cdot (\beta_2\gamma)\phi \equiv (\overline{\alpha}\beta_1 b_1)\phi \cdot (\overline{\beta_2}\gamma)\phi.$$

Since $\beta_1 b_1, \gamma \notin L(A,R)$, we may use induction and relations $\mathfrak{R}_4$ to get

$$(\alpha\beta\gamma)\phi = (\overline{\alpha}\overline{\beta_1 b_1})\phi \cdot (\overline{\beta_2}\overline{\gamma})\phi = (\overline{\alpha}\overline{\beta_1 b_1 \beta_2}\overline{\gamma})\phi = (\overline{\alpha}\overline{\beta}\overline{\gamma})\phi.$$

As above we obtain $(\alpha\beta\gamma)\phi = (\alpha\overline{\beta}\gamma)\phi$. So we may assume that $\beta_2\gamma \notin L(A,R)$. Given this, Lemma 1.6 gives that

$$(\alpha\beta\gamma)\phi \equiv (\alpha\beta_1 b_1 \beta_2\gamma)\phi \equiv (\overline{\alpha}\overline{\beta_1} b_1 \overline{\beta_2}\overline{\gamma})\phi,$$

and relations $\mathfrak{R}_5$ then give that

$$(\alpha\beta\gamma)\phi = (\overline{\alpha}\overline{\beta_1 b_1 \beta_2}\overline{\gamma})\phi = (\overline{\alpha}\overline{\beta}\overline{\gamma})\phi.$$

Again we have that

$$(\alpha\overline{\beta}\gamma)\phi = (\overline{\alpha}\overline{\overline{\beta}}\overline{\gamma})\phi \equiv (\overline{\alpha}\overline{\beta}\overline{\gamma})\phi = (\alpha\beta\gamma)\phi,$$

and this concludes the proof. ∎

Lemma 3.7. *The relations*

$$(w_3uw_4)\phi = (w_3vw_4)\phi,$$

where $(u = v) \in \mathcal{R}$, $w_3, w_4 \in A^*$, $w_3uw_4 \in L(A, R)$, *are consequences of* $\mathfrak{P}$.

PROOF. If $w_3u \notin L(A, R)$, then $w_3v \notin L(A, R)$, and

$$(w_3uw_4)\phi \equiv (\overline{w_3u}w_4)\phi \equiv (\overline{w_3v}w_4)\phi \equiv (w_3vw_4)\phi$$

by Lemma 1.6; so we may assume that $w_3u \in L(A, R)$.

If $w_4 \in L(A, R)$ (in particular, $w_4 \neq \epsilon$), then

$$(w_3uw_4)\phi = (w_3u)\phi \cdot w_4\phi = (w_3v)\phi \cdot w_4\phi = (w_3vw_4)\phi$$

by Proposition 3.6 (i) and induction, so we may assume that $w_4 \notin L(A, R)$.

Assume that $w_3 \in L(A, R)$; then $w_3 \equiv \alpha_1 a_1 \ldots \alpha_k a_k \alpha_{k+1}$ as in Lemma 1.6. If $\alpha_{k+1}uw_4 \in L(A, R)$ then

$$\begin{aligned}(w_3uw_4)\phi &\equiv (\alpha_1 a_1 \ldots \alpha_k a_k \alpha_{k+1}uw_4)\phi = (\alpha_1 a_1 \ldots \alpha_k a_k)\phi \cdot (\alpha_{k+1}uw_4)\phi \\ &= (\alpha_1 a_1 \ldots \alpha_k a_k)\phi \cdot (\alpha_{k+1}vw_4)\phi \equiv (w_3vw_4)\phi\end{aligned}$$

by Lemma 1.6, Proposition 3.6 and induction, while if $\alpha_{k+1}uw_4 \notin L(A, R)$ we have

$$(w_3uw_4)\phi \equiv (\alpha_1 a_1 \ldots \alpha_k a_k \alpha_{k+1}uw_4)\phi \equiv (\alpha_1 a_1 \ldots \alpha_k a_k \alpha_{k+1}vw_4)\phi \equiv (w_3vw_4)\phi$$

by Lemma 1.6.

Given that $w_3 \notin L(A, R)$ and that $w_4 \notin L(A, R)$, we have

$$(w_3uw_4)\phi = (\overline{w}_3u\overline{w}_4)\phi = (\overline{w}_3v\overline{w}_4)\phi = (w_3vw_4)\phi$$

by Proposition 3.6 (ii) and relations $\mathfrak{R}_6$ as required. ∎

PROOF OF THEOREM 3.1. The set X generates T by Theorem 1.2, and ϕ is a corresponding rewriting mapping by Lemma 1.5. All the relations from $\mathfrak{P}$ hold in T by Lemma 3.4 and imply all relations (20), (21), (22) of Theorem 6.7.1 by Lemmas 3.5, 3.6 and 3.7. Therefore, the theorem follows by Theorem 6.7.1 and Remark 6.7.6. ∎

As we have seen, ideals of finite index in a finitely presented semigroup are always finitely presented. Despite numerous attempts the author has not succeeded in proving the analogous result for semigroups, so that we have

Open Problem 11. Is a subsemigroup of finite index in a finitely presented semigroup necessarily finitely presented?

It is not even clear what should be conjectured for an answer to the above question. On one hand, the definition of index of a subsemigroup is very artificial, in the sense that it does not correspond to any one-sided congruence (see Section 1), and this might be a reason for a negative answer. On the other hand, all the attempts to generalise the results for ideals to subsemigroups have broken down on technical details, giving no clear indication for construction of a possible counterexample. Also, in the following chapter we give a piece of positive evidence for the above problem in the case of subsemigroups of free semigroups.

Chapter 6

Subsemigroups of free semigroups

One of the best known applications of the Reidemeister—Schreier theorem for groups is a proof of the fact that a subgroup of a free group is itself free—the result known as the Nielsen—Schreier theorem; see Magnus, Karrass and Solitar (1966). Of course, there is no hope for such a nice result for free semigroups, since it is well known that a subsemigroup of a free semigroup need not be free. Nevertheless, one might expect that the Reidemeister—Schreier type results of the previous chapter can be used to prove some other, weaker, properties of subsemigroups of free semigroups, and this is the theme of this chapter. More specifically we will consider the following

Main Problem. Let F be a (finitely generated) free semigroup.

(i) Is every subsemigroup (ideal, one-sided ideal) of F free?

(ii) Is every subsemigroup (ideal, one-sided ideal) of F finitely generated?

(iii) Is every subsemigroup (ideal, one-sided ideal) of F which is finitely generated finitely presented?

(iv) Is every subsemigroup (ideal, one-sided ideal) of finite index in F finitely presented?

The answers that we obtain can be summarised as in the following table

		subsemigroups	ideals	one-sided ideals
(i)	free	−	−	−
(ii)	f.g.	−	−	−
(iii)	f.g. $\Rightarrow$ f.p	−	+	+
(iv)	f.i. $\Rightarrow$ f.p.	+	+	+

The results of this chapter will appear in Campbell, Robertson, Ruškuc and Thomas (1995d).

1. Subsemigroups

It is well known that a subsemigroup of a free semigroup is not necessarily free; see, for example, Lothaire (1983). It can also fail to be finitely generated, as the following example shows.

Example 1.1. Let $F = \{a, b\}^+$ be the free semigroup on two generators, and consider the subsemigroup S of F generated by the set

$$X = \{ab^i \mid i \geq 1\}.$$

Obviously, no element of X is a product of other elements of X, and therefore all the elements of X are indecomposable in S, so that S is not finitely generated. Actually, it is easy to show that S is a free semigroup on generators X. Since a free semigroup on more that two generators contains F as a subsemigroup, it also contains a non-finitely generated subsemigroup.

However, any subsemigroup S of the free monogenic semigroup $\mathbb{N}$ is finitely generated. To see this, first note that S is isomorphic to the semigroup

$$\{s/d \mid s \in S\},$$

where d is the greatest common divisor of all elements of S, and hence we may assume that $d = 1$. This means that there are numbers $s_1, \ldots, s_k \in S$ such that g.c.d.$(s_1, \ldots, s_k) = 1$. It is well known that if $a, b \in \mathbb{N}$ are coprime then every big enough positive integer can be written as a positive linear combination $\alpha a + \beta b$, $\alpha > 0$, $\beta > 0$, of a and b; see Rose (1988). A straightforward generalisation of this gives that every big enough positive integer can be written as a positive linear combination of $s_1, \ldots, s_k$. In other words, all but finitely many elements of S belong to the subsemigroup of S generated by $\{s_1, \ldots, s_k\}$, and hence S is finitely generated. ■

Also, a finitely generated subsemigroup of a free semigroup may be not finitely presented.

Example 1.2. Let F be the free semigroup on three generators a, b and c, and let S be the subsemigroup generated by $v = ba$, $w = ba^2$, $x = a^3$, $y = ac$ and $z = a^2c$. Clearly, S is finitely generated, and we claim that it is not finitely presented.

First recall that, if a semigroup is finitely presented with respect to one generating set, then it is finitely presented with respect to any finite generating set; see Proposition 3.3.1. So it is sufficient to show that S is not finitely presented with respect to the particular generating set $\{v, w, x, y, z\}$.

Decomposing the word $\alpha = ba^{3(n+1)}c$ in two different ways, we see that the relations $vx^nz = wx^ny$ $(n \geq 0)$ hold in S, and so any set of defining relations must imply these. Since any proper subword of α representing an element of S can be expressed as an element of $\{v, w, x, y, z\}^+$ in only one way, there is no non-trivial relation holding in S which we can apply to a proper subword of vx^nz or wx^ny. So any set of defining relations for S must include all the relations $vx^nz = wx^ny$, and so S is not finitely presented.

Since a free group on two generators contains a free subsemigroup on countably many generators (Example 1.1), it contains a free subsemigroup on three generators, and hence it contains a finitely generated subsemigroup which is not finitely presented.

On the other hand, all subsemigroups of the free monogenic semigroup $\mathbb{N}$ are finitely presented. This follows from the fact that every finitely generated commutative semigroup is finitely presented (see Rédei (1965)), and the fact that each subsemigroup of $\mathbb{N}$ is finitely generated (Example 1.1). ∎

Remark 1.3. If, instead of ordinary semigroup presentations, we consider so called Malcev presentations (i.e. presentations of semigroups embeddable into groups), then we have a completely different situation—every finitely generated subsemigroup of a free semigroup can be defined by a finite Malcev presentation; see Spehner (1989).

The main result of this section gives an affirmative answer to part (iv) of Main Problem in the case of subsemigroups. It also gives some positive evidence for Open Problem 29 from Chapter 7.

Theorem 1.4. *If $F = A^+$ is a finitely generated free semigroup and S is a subsemigroup of F of finite index, then S is finitely presented.*

Let us recall that the index of S in F is defined to be $|F - S| + 1$. The crucial fact for the proof of this theorem is the following

Lemma 1.5. *If $F = A^+$ is a finitely generated free semigroup and S is a subsemigroup of F of finite index, then there is an ideal I of finite index in F with $I \subseteq S$.*

PROOF. Let $F - S = \{\alpha_1, \ldots, \alpha_k\}$, and let

$$p = \max\{|\alpha_i| \mid 1 \leq i \leq k\}.$$

Let I be the set of all words from F of length at least $p+1$; then certainly $I \subseteq S$. Also, it is easy to see that I is an ideal in F, and since

$$F - I = \{w \in F \mid |w| \leq p\},$$

I has finite index in F. ∎

PROOF OF THEOREM 1.4. Let I be an ideal of F of finite index, such that $S \subseteq I$ (Lemma 1.5). By Corollary 7.2.3 I is finitely presented. On the other hand, I has finite index in S as well; in other words, S is an ideal extension of I by a finite semigroup; see Section 6 of Chapter 6. Therefore, S is finitely presented by Theorem 6.6.1. ∎

2. Ideals

In this section we consider the Main Problem for two-sided ideals of free semigroups. We show that the first two questions have negative answers, and that the third question has a positive answer. The fourth question has a positive answer by Corollary 7.2.3.

In what follows there is a possibility of confusion over the use of the word 'generate': an ideal can be generated by a set *as a semigroup* or *as an ideal.* We introduce the convention that 'generate' will always mean 'generate as a semigroup' unless stated otherwise.

Theorem 2.1. *Let $F = A^+$ be a free semigroup and let $I \neq F$ be a proper two-sided ideal of F. Then I is not free.*

PROOF. Let $a \in A - I$, and let w be an element of I of minimal length. Both words aw and wa belong to I since I is a two-sided ideal, and neither of them is a product of two elements of I; hence each generating set for I contains both these words. But then $w(aw) = (wa)w$ is a non-trivial relation holding in I, and I is not free. ∎

Example 2.2. Let $F = \{a, b\}^+$ be the free semigroup on two generators, and let I be the ideal generated (as an ideal) by a, i.e.

$$I = \{w_1 a w_2 \mid w_1, w_2 \in \{a, b\}^*\}.$$

Each word ab^i, $i \geq 1$, belongs to I, and is indecomposable in I. Therefore, I is not finitely generated. ∎

Now we show that finitely generated ideals of a free semigroup are finitely presented. Actually, we can prove a more general statement:

Theorem 2.3. *If I is a finitely generated ideal in a free semigroup F, then I has finite index in F.*

PROOF. We show that if I has infinite index in F, then I is not finitely generated.

Suppose that I has infinite index in F, and let $w_1, w_2, \ldots$ be distinct elements of $F - I$. Let x be an element of I of minimal length, so that $xw_1, xw_2, \ldots$ are elements of I. If $xw_i = uv$ with $u, v \in I$, then x is a prefix of u (by the minimality of $|x|$), so that v is a suffix of w_i; since $v \in I$, we have that $w_i \in I$, a contradiction. So xw_i cannot be expressed as a non-trivial product of elements of I, and hence any generating set for I must contain all $xw_1, xw_2, \ldots$. So I is not finitely generated. ∎

Corollary 2.4. *If I is a finitely generated ideal in a finitely generated free semigroup F, then I is finitely presented.*

PROOF. I has finite index in F by Theorem 2.3, and is therefore finitely presented by Corollary 7.2.3. ∎

3. One-sided ideals

In this section we consider the Main Problem for one-sided ideals of free semigroups. We give negative answers to questions (i) and (ii), and a positive answer to question (iii). However, the proof of this last fact is completely different from the proof of the corresponding result for two-sided ideals, since a finitely generated one-sided ideal of a free semigroup does not necessarily have finite index. Question (iv) has a positive answer as a direct consequence of Corollary 7.3.2.

Theorem 3.1. *Let $F = A^+$ be a free semigroup and let $R \neq F$ be a proper right ideal. If R is finitely generated as a semigroup then it is not free.*

PROOF. Since $F \neq R$ there exists $a \in A$ such that $a \notin R$. Suppose that $a^i \notin R$ for all $i \geq 1$. Let w be an element of R of minimal length. Then $wa^i \in R$, $i \geq 1$, since R is a right ideal, but wa^i is not a product of two elements of R. Therefore, each generating set of R contains all the words wa^i, $i \geq 1$, and R is not finitely generated, a contradiction. Thus R contains some power of a. Let a^i be the minimal such power; obviously $i > 1$. The word a^{i+1} belongs to R since R is a right ideal, but a^{i+1} is not a product of two elements of R since $i > 1$; hence each generating set for R contains both a^i and a^{i+1}. Since a^i and a^{i+1} satisfy the non-trivial relation $a^i a^{i+1} = a^{i+1} a^i$, R cannot be free. ■

Example 3.2. Let F be the free semigroup on two generators $\{a, b\}$, and let $R = aF^1$ be the principal right ideal generated (as a right ideal) by a. The set $\{ab^i \mid i \geq 0\}$ is a unique minimal generating set for R, and it is easy to see that R is free on this generating set. ■

The previous example also shows that a right ideal of a finitely generated free semigroup is not necessarily finitely generated.

We devote the rest of this section to proving that a finitely generated right ideal of a free semigroup is always finitely presented. We prove this directly, by finding an explicit presentation, but first we prove a necessary and sufficient condition for an ideal of a free semigroup to be finitely generated.

Theorem 3.3. *Let $F = A^+$ be a free semigroup, let $\alpha_1, \dots, \alpha_m \in F$, and let*

$$R = \alpha_1 F^1 \cup \ldots \cup \alpha_m F^1 \tag{1}$$

be the right ideal generated (as a right ideal) by $\{\alpha_1, \dots, \alpha_m\}$. Then R is finitely generated (as a semigroup) if and only if there exists a constant N such that each word of F of length at least $N + 1$ contains some α_i as a subword.

PROOF. ($\Rightarrow$) Assume that R is finitely generated. Without loss of generality we may assume that no α_i is a prefix of an α_j for $i \neq j$; for, otherwise, $\alpha_j F^1 \subseteq \alpha_i F^1$, and we could omit $\alpha_j F^1$ from (1). For any word $\eta \in F$, the word $\alpha_i\eta$, $1 \le i \le m$, belongs to R; conversely, every word from R has the form $\alpha_i\eta$ for some i and some η. Since R is finitely generated, there is an upper bound on the length of generators in a finite generating set for R. In particular, for some constant N, R is generated by the set

$$\Sigma = \{\alpha_i\eta \mid 1 \le i \le m,\ \eta \in F^1,\ |\eta| \le N\}. \tag{2}$$

Let ζ be any element of F, and consider the element $\alpha_1\zeta$ of R. Since Σ is a generating set for R, we may write

$$\alpha_1\zeta = \alpha_{i_1}\eta_1\alpha_{i_2}\eta_2\ldots\alpha_{i_k}\eta_k,$$

for some $i_1,\ldots,i_k$ and some $\eta_1,\ldots,\eta_k$ with $\eta_j \le N$, $j = 1,\ldots,k$. (Note that, in a free semigroup both symbols $=$ and $\equiv$ have the same meaning, and we prefer using the former.) Since α_1 is not a proper prefix of α_{i_1} and vice versa, we have $\alpha_1 = \alpha_{i_1}$, so that

$$\zeta = \eta_1\alpha_{i_2}\eta_2\ldots\alpha_{i_k}\eta_k.$$

If $|\zeta| > N$, then, since $|\eta_1| \le N$, we have $k \ge 2$, and hence α_{i_2} is a subword of ζ, thus proving this half of the theorem.

($\Leftarrow$) Suppose that $R = \alpha_1F^1 \cup \ldots \cup \alpha_mF^1$, and that there exists a constant N such that every word of F of length at least $N+1$ contains some α_i as a subword. We show that Σ, as defined by (2), is a (finite) generating set for R.

Let β be an arbitrary element of R, say $\beta = \alpha_{i_1}\gamma_1$, with $1 \le i_1 \le m$, $\gamma_1 \in F^1$. If $|\gamma_1| \le N$ then $\beta \in \Sigma$ and we have finished. Otherwise, let δ_1 be the prefix of γ_1 consisting of the first $N+1$ letters of γ_1. By the hypothesis, δ_1 contains some α_{i_2} as a substring, and so we have $\beta = \alpha_{i_1}\eta_1\alpha_{i_2}\gamma_2$ for some η_1 and γ_2 in F^1 with $|\eta_1| \le N$. Continuing in this way yields

$$\beta = \alpha_{i_1}\eta_1\alpha_{i_2}\eta_2\ldots\alpha_{i_k}\eta_k,$$

expressing β as a product of elements of Σ, as required. ■

Remark 3.4. Since every right ideal which is finitely generated as a semigroup is also finitely generated as a right ideal (with the same generating set), Theorem 3.3 in effect gives a necessary and sufficient condition for a right ideal of a free semigroup to be finitely generated as a semigroup.

Of course, Theorem 3.3 has a dual for left ideals. The two put together give the following, rather surprising, connection between finitely generated left and right ideals of a free semigroup.

Corollary 3.5. *If F is a free semigroup and $\alpha_1, \alpha_2, \ldots, \alpha_m \in F$, then the right ideal $\alpha_1F^1 \cup \alpha_2F^1 \cup \ldots \cup \alpha_mF^1$ is finitely generated as a semigroup if and only if the left ideal $F^1\alpha_1 \cup F^1\alpha_2 \cup \ldots \cup F^1\alpha_m$ is finitely generated as a semigroup.* ■

Also as a consequence of Theorem 3.3, we can see that finitely generated right ideals of a free semigroup do not necessarily have finite index (compare with Theorem 2.3).

Example 3.6. Let $F = \{a, b\}^+$ be the free semigroup on a and b, and let

$$R = a^2F^1 \cup abF^1 \cup b^2F^1.$$

Since any word of length at least 3 must contain a^2, ab or b^2 as a subword, R is a finitely generated right ideal of F by Theorem 3.3. However, the set $F - R = \{a, b\} \cup baF^1$ is not finite; in other words R does not have finite index in F. ■

Finally, we prove the main result of this section.

Theorem 3.7. *If R is a right ideal of a free semigroup F, and if R is finitely generated as a semigroup, then R is finitely presented.*

PROOF. Let $F = A^+$ be the free semigroup on A. Since R is finitely generated and each generator of R involves only finitely many elements from A, and since $Ra \subseteq R$ for each $a \in A$, we see that A is finite. Let $A = \{a_1, a_2, \ldots, a_n\}$.

Let R be generated by $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$, where $\alpha_i \in A^+$ for each i, and let B be the alphabet $\{b_1, b_2, \ldots, b_m\}$, where b_i represents the generator α_i of R. Let $M = \max\{|\alpha_i| \mid 1 \leq i \leq m\}$. We have a relation

$$\gamma(b_1, b_2, \ldots, b_m) = \delta(b_1, b_2, \ldots, b_m)$$

in R if and only if

$$\gamma(\alpha_1, \alpha_2, \ldots, \alpha_m) \equiv \delta(\alpha_1, \alpha_2, \ldots, \alpha_m)$$

in A^+. We define the *weight* of the relation $\gamma(b_1, b_2, \ldots, b_m) = \delta(b_1, b_2, \ldots, b_m)$ to be the length of the word $\gamma(\alpha_1, \alpha_2, \ldots, \alpha_m)$ in A^+. Let $\mathfrak{R}$ denote the set of all relations $\gamma(b_1, b_2, \ldots, b_m) = \delta(b_1, b_2, \ldots, b_m)$ of weight at most $3M$ that hold in R. We claim that $\langle B \mid \mathfrak{R} \rangle$ is a presentation for R (and the proof will then be completed, since $\mathfrak{R}$ is finite).

We need to show that any relation $\gamma(b_1, b_2, \ldots, b_m) = \delta(b_1, b_2, \ldots, b_m)$ which holds in R is a consequence of $\mathfrak{R}$. So suppose we have a relation

$$b_{i_1} b_{i_2} b_{i_3} \ldots b_{i_r} = b_{j_1} b_{j_2} b_{j_3} \ldots b_{j_s} \tag{3}$$

holding in R, so that

$$\alpha_{i_1} \alpha_{i_2} \alpha_{i_3} \ldots \alpha_{i_r} \equiv \alpha_{j_1} \alpha_{j_2} \alpha_{j_3} \ldots \alpha_{j_s}$$

in A^+. We argue by induction on the weight of relation (3), the case of weight 1 being a relation in $\mathfrak{R}$. If $\alpha_{i_1} \equiv \alpha_{j_1}$, we have $b_{i_2}b_{i_3}\dots b_{i_r} = b_{j_2}b_{j_3}\dots b_{j_s}$, and the result follows by the inductive hypothesis. So suppose that $|\alpha_{i_1}| < |\alpha_{j_1}|$, so that α_{i_1} is a proper prefix of α_{j_1}, say $\alpha_{j_1} \equiv \alpha_{i_1}\zeta$. We now have

$$\alpha_{i_1}\alpha_{i_2}\alpha_{i_3}\dots\alpha_{i_r} \equiv \alpha_{i_1}\zeta\alpha_{j_2}\alpha_{j_3}\dots\alpha_{j_s},$$

so that

$$\alpha_{i_2}\alpha_{i_3}\dots\alpha_{i_r} \equiv \zeta\alpha_{j_2}\alpha_{j_3}\dots\alpha_{j_s}.$$

If α_{i_2} is a prefix of ζ then $\zeta \in R$, and so

$$\zeta \equiv \alpha_{k_1}\alpha_{k_2}\dots\alpha_{k_t}.$$

Both relations

$$b_{i_2}b_{i_3}\dots b_{i_r} = b_{k_1}\dots b_{k_t}b_{j_2}\dots b_{j_s}$$

and

$$b_{j_1} = b_{i_1}b_{k_1}\dots b_{k_t}$$

are of lower weight than (3), and hence are consequences of $\mathfrak{R}$ by the inductive hypothesis. Now we have

$$b_{i_1}b_{i_2}\dots b_{i_r} = b_{i_1}b_{k_2}\dots b_{k_t}b_{j_2}\dots b_{j_s} = b_{j_1}b_{j_2}\dots b_{j_s},$$

which shows that (3) is a consequence of $\mathfrak{R}$ as well. So let us assume now that ζ is a proper prefix of α_{i_2}. Then we have

$$\alpha_{i_2} \equiv \zeta\alpha_{j_2}\alpha_{j_3}\dots\alpha_{j_{q-1}}\xi$$

for some q and ξ, where ξ is a prefix of α_{j_q}. Since $\zeta\alpha_{j_2}\alpha_{j_3}\dots\alpha_{j_{q-1}}\xi$ represents an element of R and $\zeta\alpha_{j_2}\alpha_{j_3}\dots\alpha_{j_{q-1}}\xi$ is a prefix of $\zeta\alpha_{j_2}\alpha_{j_3}\dots\alpha_{j_q}$, we see, since R is a right ideal, that $\zeta\alpha_{j_2}\alpha_{j_3}\dots\alpha_{j_q}$ represents an element of R. So

$$\zeta\alpha_{j_2}\alpha_{j_3}\dots\alpha_{j_q} \equiv \alpha_{k_1}\alpha_{k_2}\dots\alpha_{k_t}$$

for some $\alpha_{k_1}, \alpha_{k_2}, \dots, \alpha_{k_t}$. Now $|\zeta\alpha_{j_2}\alpha_{j_3}\dots\alpha_{j_{q-1}}\xi| = |\alpha_2| \le M$, and so

$$\begin{aligned} |\alpha_{j_1}\alpha_{j_2}\alpha_{j_3}\dots\alpha_{j_{q-1}}\alpha_{j_q}| &= |\alpha_{i_1}\zeta\alpha_{j_2}\alpha_{j_3}\dots\alpha_{j_{q-1}}\alpha_{j_q}| \\ &\le |\alpha_{i_1}| + |\zeta\alpha_{j_2}\alpha_{j_3}\dots\alpha_{j_q-1}\xi| + |\alpha_{j_q}| \\ &\le 3M. \end{aligned}$$

Now

$$\alpha_{j_1}\alpha_{j_2}\alpha_{j_3}\dots\alpha_{j_{q-1}}\alpha_{j_q} \equiv \alpha_{i_1}\zeta\alpha_{j_2}\alpha_{j_3}\dots\alpha_{j_{q-1}}\alpha_{j_q} \equiv \alpha_{i_1}\alpha_{k_1}\alpha_{k_2}\dots\alpha_{k_t},$$

and the corresponding relation $b_{j_1}b_{j_2}\dots b_{j_{q-1}}b_{j_q} = b_{i_1}b_{k_1}b_{k_2}\dots b_{k_t}$ is a relation of weight at most $3M$, and so is in $\mathfrak{R}$ by definition. So we may use this relation from $\mathfrak{R}$ to transform the relation

$$b_{i_1}b_{i_2}b_{i_3}\dots b_{i_r} = b_{j_1}b_{j_2}b_{j_3}\dots b_{j_s}$$

to $b_{i_1}b_{i_2}b_{i_3}\dots b_{i_r} = b_{i_1}b_{k_1}b_{k_2}\dots b_{k_t}b_{j_{q+1}}b_{j_{q+2}}\dots b_{j_s}$. So we only need to deduce the relation

$$b_{i_2}b_{i_3}\dots b_{i_r} = b_{k_1}b_{k_2}\dots b_{k_t}b_{j_{q+1}}b_{j_{q+2}}\dots b_{j_s},$$

which is of lower weight than our original relation, and the result follows by induction. ■

Chapter 7

Semigroup and group presentations

For a little while we are going to depart from our theme of finding presentations for subsemigroups of semigroups, in order to consider the following, seemingly completely different, question: *given a semigroup presentation* $\mathfrak{P} = \langle A \mid \mathfrak{R} \rangle$ *what is the connection between the semigroup* $S(\mathfrak{P})$ *defined by* $\mathfrak{P}$ *and the group* $G(\mathfrak{P})$ *defined by* $\mathfrak{P}$*?* In Section 1 we review the elementary facts related to the above question, and we also mention the existing results about conditions for $G(\mathfrak{P})$ to contain $S(\mathfrak{P})$ as a subsemigroup. In Section 2 we prove a necessary and sufficient condition for the minimal two-sided ideal of $S(\mathfrak{P})$ to be a disjoint union of copies of $G(\mathfrak{P})$—a phenomenon first suggested by studying the computer evidence. This main result is comparatively easy to prove, but it has far reaching consequences. First of all, it will prove very useful in studying the structure of semigroups defined by presentations; first easy examples of this type of investigation are in Section 3, but this theme will be developed fully in Chapters 11, 12 and 13. The main result will also serve as a motivation for another Reidemeister—Schreier type rewriting theorem, which we give in Chapter 10.

Section 1 has an introductory character. The results of Section 2, including the main result (Theorem 2.1), will appear in Campbell, Robertson, Ruškuc and Thomas (1995). The examples in Section 3 are special cases of examples given in the same paper, but are explained here in more detail.

1. Some known connections

Let $\mathfrak{P} = \langle A \mid \mathfrak{R} \rangle$ be a semigroup presentation, and let $S = S(\mathfrak{P})$ be the semigroup defined by $\mathfrak{P}$. Obviously, $\mathfrak{P}$ can be considered as a group presentation as well, and we denote by $G = G(\mathfrak{P})$ the group defined by $\mathfrak{P}$.

At the first glance the question about connections between $S(\mathfrak{P})$ and $G(\mathfrak{P})$ does not seem to be particularly promising; connections, if there are any at all, seem to be very loose. First of all, different presentations defining the same group can be defining different semigroups. For example, take

$$\mathfrak{P}_i = \langle a \mid a^{i+1} = a^i \rangle,\ i \geq 1.$$

Then $G(\mathfrak{P}_i)$ is the trivial group for all i, while $S(\mathfrak{P}_i)$ is the aperiodic monogenic semigroup of order i. Moreover, the presentation

$$\langle a, b \mid a^2 = a,\ b^2 = b \rangle,$$

which also defines the trivial group, defines an infinite semigroup in which all the words $(ab)^i$, $i = 1, 2, \ldots$, are distinct. As noted in Campbell, Robertson and Thomas (1993a), a 'dual' example does not exist:

Proposition 1.1. *Let $\mathfrak{P}$ be a semigroup presentation. If $G(\mathfrak{P})$ is infinite then so is $S(\mathfrak{P})$.* ■

The reason for this is that the natural homomorphism

$$\phi : S(\mathfrak{P}) \longrightarrow G(\mathfrak{P}),$$

which extends the identity mapping $\mathrm{id} : A \longrightarrow A$, and which exists by Proposition 3.2.1 and the definition of $G(\mathfrak{P})$, is an epimorphism when $S(\mathfrak{P})$ is finite. It is wrong, however, to assume that ϕ is *always* an epimorphism; the most that we can say is that the image of ϕ is the *subsemigroup* of $G(\mathfrak{P})$ generated by A, and this is not necessarily equal to $G(\mathfrak{P})$. For example, take $\mathfrak{P}$ to be $\langle a \mid \rangle$. Then $S(\mathfrak{P})$ is the free monogenic group $\mathbb{N}$, while $G(\mathfrak{P})$ is the free cyclic group $\mathbb{Z}$, and it is clear that ϕ is not an epimorphism. A more general example is provided by the presentation $\langle a_1, \ldots, a_n \mid \rangle$, which defines the free semigroup F_n on n generators and the free group FG_n on n generators respectively.

However, F_n and FG_n are strongly related in a different way: FG_n contains a subsemigroup isomorphic to F_n, or, in other words, F_n can be *embedded* into FG_n. The problem of determining whether a given semigroup is *embeddable* into a group is very old, and has received much attention over the years; see Adian (1966), Kashintsev (1992) and Guba (1994). This problem is also strongly related to our main question of this chapter, because of the following, easily proved result.

Proposition 1.2. *Let $\mathfrak{P}$ be a semigroup presentation. The semigroup $S(\mathfrak{P})$ is embeddable into a group if and only if the natural homomorphism $\phi : S(\mathfrak{P}) \longrightarrow G(\mathfrak{P})$ is an embedding.* ■

Probably the best known sufficient condition for this to happen is what is usually called Adian's embeddability theorem. For a given semigroup presentation $\langle A \mid \mathfrak{R} \rangle$, define the graph $L(\mathfrak{P})$ (usually called the *left Adian's graph* of $\mathfrak{P}$) as follows. The set of vertices of $L(\mathfrak{P})$ is A. Two vertices $a_1, a_2 \in A$ are adjacent in $L(\mathfrak{P})$ if and only if $\mathfrak{R}$ contains a relation of the form $a_1 u = a_2 v$ or a relation of the form $a_2 u = a_1 v$, where $u, v \in A^*$. The *right Adian graph* $R(\mathfrak{P})$ of $\mathfrak{P}$ is defined dually, by considering last letters of relations from $\mathfrak{R}$. Then we have

Proposition 1.3. *Let $\mathfrak{P}$ be a semigroup presentation. If neither of the graphs $L(\mathfrak{P})$ nor $R(\mathfrak{P})$ contains a loop, then the natural homomorphism $\phi : S(\mathfrak{P}) \longrightarrow G(\mathfrak{P})$ is an embedding.* ■

This theorem was proved by Adian (1966). An immediate consequence of it is that the semigroup $S(\mathfrak{P})$ defined by the presentation

$$\mathfrak{P} = \langle A \mid aub = cvd \rangle,$$

where $u, v \in A^*$, $a, b, c, d \in A$, $a \neq c$, $b \neq d$, has a soluble word problem, since it can be embedded into the one-relator group $G(\mathfrak{P})$, and since every one-relator group has a soluble word problem by the theorem of Magnus; see Magnus, Karrass and Solitar (1966).

There are many generalisations of Proposition 1.3. One direction of generalisation is to presentations satisfying so called *small overlap conditions*; see Higgins (1992). This theory has a certain similarity to small cancellation theory for groups; see Lyndon and Schupp (1977). Another generalisation is to so called *relative presentations*—a way of combining semigroup and group presentations; see Pride (1995).

Of course, the theory of embeddable semigroups has certain limitations, since embeddable semigroups are very special in the class of all semigroups. In particular, it does not say anything interesting for finite semigroups, since a finite semigroup is embeddable into a group if and only if it is a group.

In a separate and much more recent development, a group of mathematicians around E.F. Robertson used an implementation of the Todd—Coxeter enumeration procedure for semigroups to investigate semigroups defined by various well known group presentations. One of the most striking observations made during the course of this research was that in many cases the semigroup in question contained a number of copies of the corresponding group. A closer examination showed that these groups were the Schützenberger groups of the minimal two-sided ideal. The following result by Robertson and Ünlü (1993) explains the phenomenon in the case where there is only one copy of the group.

Proposition 1.4. *Let $\mathfrak{P} = \langle A \mid \mathfrak{R} \rangle$ be a semigroup presentation. If $S(\mathfrak{P})$ has a minimal ideal M which is a group then M is isomorphic to $G(\mathfrak{P})$.* ■

The phenomenon was also explained in various particular cases in Campbell, Robertson and Thomas (1993) and Walker (1992). In the following section we give a general explanation, i.e. we give a necessary and sufficient condition for the Schützenberger group of the minimal ideal of a semigroup defined by a presentation to be isomorphic to the group defined by the same presentation. In Section 3, as well as in Chapters 11, 12, 13 we have examples showing how this result can be used in describing the structure of semigroups defined by presentations, and in solving the word and finiteness problems.

2. An isomorphism theorem

In this section we are going to consider the semigroup $S(\mathfrak{P})$ defined by a presentation $\mathfrak{P} = \langle A \mid \mathfrak{R} \rangle$, and, in addition, we are going to assume that $S(\mathfrak{P})$ has both minimal left ideals and minimal right ideals. Some basic facts about such semigroups can be found in Section 3 of Appendix A. In particular, $S(\mathfrak{P})$ has a (unique) minimal two-sided ideal M, which is the disjoint union of all minimal left ideals, as well as the disjoint union of all minimal right ideals, and is a completely simple semigroup. For basic facts about completely simple semigroups the reader is referred to Section 2 of Appendix A. In particular, the intersection of any minimal left ideal and any minimal right ideal is a group (called the Schützenberger group of M), and all these groups are isomorphic. Here we investigate when the Schützenberger group of M is isomorphic to the group $G(\mathfrak{P})$ defined by $\mathfrak{P}$.

Theorem 2.1. *Let $\mathfrak{P} = \langle A \mid \mathfrak{R} \rangle$ be a semigroup presentation, and assume that the semigroup $S(\mathfrak{P})$ defined by $\mathfrak{P}$ has both minimal left ideals and minimal right ideals. Let L be an arbitrary minimal left ideal, and let R be an arbitrary minimal right ideal of $S(\mathfrak{P})$, so that $H = R \cap L$ is the Schützenberger group of the minimal two sided ideal M of $S(\mathfrak{P})$. Finally, let $G(\mathfrak{P})$ be the group defined by $\mathfrak{P}$, and let $\phi : S(\mathfrak{P}) \longrightarrow G(\mathfrak{P})$ be the natural homomorphism. Then:*

(i) *ϕ is an epimorphism.*

(ii) *$\phi\restriction_H : H \longrightarrow G(\mathfrak{P})$ is an epimorphism. In other words, the group defined by $\mathfrak{P}$ is a homomorphic image of the Schützenberger group of M.*

(iii) *$\phi\restriction_H$ is an isomorphism if and only if the idempotents of M are closed under multiplication. In this case, M is the disjoint union of $|I||\Lambda|$ copies of the group $G(\mathfrak{P})$, where $|I|$ is the number of minimal right ideals of $S(\mathfrak{P})$, and $|\Lambda|$ is the number of minimal left ideals of $S(\mathfrak{P})$.*

PROOF. To simplify notation we will write S for $S(\mathfrak{P})$ and G for $G(\mathfrak{P})$. Also, to avoid confusion, we assume that $A = \{a_j \mid j \in J\}$, and then denote generators of G by $B = \{b_j \mid j \in J\}$. After this the definition of ϕ becomes

$$a_j\phi = b_j \ (j \in J). \tag{1}$$

Also, since M is a completely simple semigroup (Proposition A.3.3) we adopt the notation for these semigroups introduced in Section 2 of Appendix A. Thus, L_λ, $\lambda \in \Lambda$, is the family of all minimal left ideals of S (and hence of M as well; see Proposition A.3.3), and R_i is the family of all minimal right ideals of S. For $i \in I$ and $\lambda \in \Lambda$ the group $R_i \cap L_\lambda$ is denoted by $H_{i\lambda}$, its identity is denoted by $e_{i\lambda}$, and we have

$$H_{i\lambda} = e_{i\lambda} S e_{i\lambda}. \tag{2}$$

Finally, without loss of generality we may assume

$$R = R_1, \ L = L_1, \ H = H_{11}, \ e = e_{11}.$$

(i) As we noted before, ϕ is well defined by (1), and its image is the subsemigroup of G generated by B. So, to prove the first part of the theorem, we have to show that every element of G is represented by a word from B^+. Obviously, it is enough to show this for the inverse b_j^{-1} of an arbitrary element b_j of B. Consider the element ea_je of S; by (2) it belongs to the group H, and hence has an inverse $w_j \in H$:

$$ea_jew_j = w_jea_je = e.$$

Since e is an idempotent of S we have $e\phi = 1_G$, so that

$$1_G = (ea_jew)\phi = (e\phi)(a_j\phi)(e\phi)(w\phi) = b_j(w\phi),$$

and the inverse of b_j in G is represented by the positive word $w\phi \in B^+$, as required.

(ii) For any $g \in G$ there exists $s \in S$ such that $s\phi = g$ by (i). The element ese belongs to H by (2), and we have

$$(ese)\phi = (e\phi)(s\phi)(e\phi) = s\phi = g,$$

showing that $\phi\restriction_H$ is onto.

In order to prove the third (and most important) part of the theorem, we need the following technical lemma:

Lemma 2.2. *If the idempotents of M are closed under multiplication then:*

(i) $e_{i_1\lambda_1}e_{i_2\lambda_2}e_{i_3\lambda_3} = e_{i_1\lambda_1}e_{i_3\lambda_3}$, *for all* $i_1, i_2, i_3 \in I$ *and all* $\lambda_1, \lambda_2, \lambda_3 \in \Lambda$.

(ii) For any $i \in I$, $\lambda \in \Lambda$, *the mapping* $\eta : x \mapsto e_{i\lambda}xe_{i\lambda}$ *is an epimorphism from* S *onto* $H_{i\lambda}$.

PROOF. (i) By Proposition A.2.2 (iv) we have $e_{i_1\lambda_1}e_{i_2\lambda_2} \in H_{i_1\lambda_2}$, and, since idempotents are closed, it must be the identity $e_{i_1\lambda_2}$ of $H_{i_1\lambda_2}$. Similarly,

$$e_{i_1\lambda_2}e_{i_3\lambda_3} = e_{i_1\lambda_3} \text{ and } e_{i_1\lambda_1}e_{i_3\lambda_3} = e_{i_1\lambda_3},$$

and the result follows.

(ii) Let $x, y \in S$ be arbitrary. Since $e_{i\lambda} \in M$, and M is an ideal, we have $e_{i\lambda}x \in M$, so that

$$e_{i\lambda}x \in H_{i_1\lambda_1} = e_{i_1\lambda_1}Se_{i_1\lambda_1},$$

for some $i_1 \in I$ and some $\lambda_1 \in \Lambda$. Hence

$$e_{i\lambda}x = e_{i_1\lambda_1}s_1e_{i_1\lambda_1}$$

for some $s_1 \in S$, and, similarly,

$$ye_{i\lambda} = e_{i_2\lambda_2}s_2e_{i_2\lambda_2}$$

for some $i_2 \in I$, $\lambda_2 \in \Lambda$ and some $s_2 \in S$. Now, if we use (i), we have

$$\begin{aligned}(x\eta)(y\eta) &= e_{i\lambda}xe_{i\lambda}e_{i\lambda}ye_{i\lambda} = e_{i\lambda}xe_{i\lambda}ye_{i\lambda} = e_{i_1\lambda_1}s_1e_{i_1\lambda_1}e_{i\lambda}e_{i_2\lambda_2}s_2e_{i_2\lambda_2} \\ &= e_{i_1\lambda_1}s_1e_{i_1\lambda_1}e_{i_2\lambda_2}s_2e_{i_2\lambda_2} = e_{i\lambda}xye_{i\lambda} = (xy)\eta,\end{aligned}$$

and hence η is a homomorphism. That η is onto follows directly from (2). ■

Now we can prove the last part of Theorem 2.1.

(iii) ($\Leftarrow$) If the set of idempotents of M is closed under multiplication, then, by Lemma 2.2 (ii), the set $\{ea_je \mid j \in J\}$ generates the group $H = eSe$, and these generators satisfy all the relations from $\mathfrak{R}$. Since G is the group defined by $\mathfrak{P}$, there exists an epimorphism $\psi : G \longrightarrow H$ such that

$$b_j\psi = ea_je, \; j \in J.$$

Now we have

$$\begin{aligned}&(b_j\psi)\phi = (ea_je)\phi = (e\phi)(a_j\phi)(e\phi) = 1_Gb_j1_G = b_j, \\ &((ea_je)\phi)\psi = b_j\psi = ea_je,\end{aligned}$$

so that $\phi\restriction_H \circ\psi$ and $\psi \circ \phi\restriction_H$ are identity mappings, and hence $\phi\restriction_H$ is an isomorphism.

($\Rightarrow$) Let us suppose that $\phi\restriction_H$ is an isomorphism, but that $e_{i_1\lambda_1}e_{i_2\lambda_2}$ is not an idempotent for some $i_1, i_2 \in I$ and some $\lambda_1, \lambda_2 \in \Lambda$. Certainly,

$$e_{i_1\lambda_1}e_{i_2\lambda_2} \in H_{i_1\lambda_2}$$

by Proposition A.2.2 (iv). By Proposition A.2.2 (vii) there exist

$$s \in H_{i_11}, \; s' \in H_{1\lambda_2},$$

such that

$$s's = e,$$

and the mapping

$$\zeta : H_{i_1\lambda_2} \longrightarrow H_{11}, \; x\zeta = s'xs,$$

is an isomorphism; see Figure 5.

In particular, we have

$$s'e_{i_1\lambda_1}e_{i_2\lambda_2}s \neq e.$$

On the other hand

$$\begin{aligned}(s'e_{i_1\lambda_1}e_{i_2\lambda_2}s)\phi &= (s'\phi)(e_{i_1\lambda_1}\phi)(e_{i_2\lambda_2}\phi)(s\phi) = (s'\phi)(s\phi) \\ &= (ss')\phi = e\phi = 1_G,\end{aligned}$$

which contradicts the assumption that $\phi\restriction_H$ is an isomorphism. ■

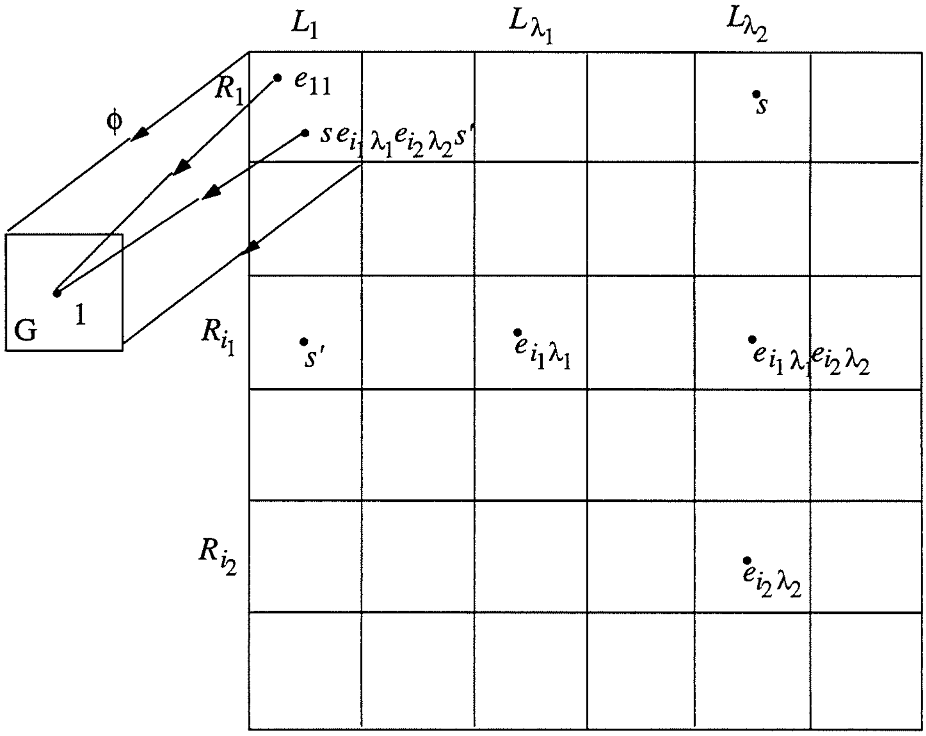


Figure 5.

Remark 2.3. If the set of idempotents E of a completely simple semigroup T is closed, then, by Proposition A.2.4 (ii), T is isomorphic to the Rees matrix semigroup $\mathcal{M}[H; I, \Lambda; P]$, where H is the Schützenberger group of T, and all entries of P are equal to 1_H. This Rees matrix semigroup is, in turn, easily seen to be isomorphic to the direct product $RZ_I \times H \times LZ_\Lambda$, where RZ_I is the semigroup of right zeros (i.e. a semigroup satisfying $xy = x$ for all x and all y) of order $|I|$, and LZ_Λ is the semigroup of left zeros of order $|\Lambda|$. Therefore, in the notation of Theorem 2.1, if the idempotents of M are closed under multiplication, then $M \cong RZ_I \times G(\mathfrak{P}) \times LZ_\Lambda$.

In order to use Theorem 2.1 for investigating the structure of the minimal ideal M of a semigroup S defined by a presentation $\mathfrak{P}$, first we have to find all the minimal left and right ideals of S, then to find $|I||\Lambda|$ idempotents of M, and then to check that these idempotents are closed under multiplication; in the following section we shall illustrate all three stages on concrete examples. However, the third stage tends to be rather tedious, since there are roughly $(|I||\Lambda|)^2$ products to be examined. Fortunately, there is a special case, which we will encounter frequently, in which this step can be omitted:

Corollary 2.4. *Let $\mathfrak{P}$ be a semigroup presentation, and assume that the semigroup $S(\mathfrak{P})$ defined by $\mathfrak{P}$ has minimal left ideals and a unique minimal right ideal (or, dually, minimal right ideals and a unique minimal left ideal). Then every minimal left ideal (every minimal right ideal) is a group isomorphic to the group $G(\mathfrak{P})$ defined by $\mathfrak{P}$.*

PROOF. Let R $(= R_1)$ be the unique minimal right ideal of $S(\mathfrak{P})$. Then R is the minimal two-sided ideal as well, since the minimal two sided ideal is the union of all minimal right ideals; see Propositions A.3.3 and A.2.2 (ii). If L_λ, $\lambda \in \Lambda$, are the minimal left ideals of $S(\mathfrak{P})$, then, by the same theorems, $R = R_1 = \bigcup_{\lambda\in\Lambda} L_\lambda$, so that $H_{1\lambda} = R_1 \cap L_\lambda = L_\lambda$. Finally, the idempotents of a single minimal right ideal form a semigroup of right zeros by Proposition A.2.2 (viii), and, in particular, are closed under multiplication. Hence, each $H_{1\lambda} = L_\lambda$ is isomorphic to $G(\mathfrak{P})$ by Theorem 2.1 (iii). ■

Remark 2.5. It is easy to see that the minimal ideal M of a semigroup S is a group if and only if S has a unique minimal left ideal and a unique minimal right ideal. Therefore, Proposition 1.4 is a special case of Corollary 2.4.

3. Examples: presentations defining $\mathcal{A}_5$

In this section we illustrate how the results of the previous section can be used to investigate the structure of semigroups defined by presentations. We shall consider several presentations which, when considered as group presentations, all define the alternating group $\mathcal{A}_5$ of degree 5, and we shall describe the semigroups defined by these presentations. In Chapters 11, 12 and 13 we will have further examples of using Theorem 2.1 in describing the structure of various semigroups. The considerations there will be more complicated, but will proceed roughly along the same lines as the considerations in this section.

It is well known that $\mathcal{A}_5$ is defined by the following (group) presentation:

$$\mathfrak{P}_1 = \mathfrak{P}_1(2,3,5) = \langle a, b \mid a^2 = b^3 = (ab)^5 = 1\rangle.$$

Of course, $\mathfrak{P}_1$ is not a semigroup presentation. (It is, however, a monoid presentation, but it is easy to see that the monoid defined by this presentation is again $\mathcal{A}_5$.) Let us consider the following semigroup modification of $\mathfrak{P}_1$:

$$\mathfrak{P}_2 = \mathfrak{P}_2(2,3,5) = \langle a, b \mid a^3 = a,\ b^4 = b,\ (ab)^5 = a^2\rangle.$$

The first step towards applying Theorem 2.1 is to identify minimal left ideals and minimal right ideals. This is a task that we shall be often faced with. In the following proposition we give some useful criteria for the (one-sided) ideal generated by an element to be minimal.

Proposition 3.1. *Let S be a semigroup, and let $s \in S$.*

(i) s generates a minimal left (right) ideal if and only if for each $s_1 \in S$ there exists $s_2 \in S$ such that $s_2 s_1 s = s$ ($s s_1 s_2 = s$).

(ii) s generates the unique minimal left (right) ideal if and only if for each $s_1 \in S$ there exists $s_2 \in S$ such that $s_2 s_1 = s$ ($s_1 s_2 = s$).

PROOF. (i) The left ideal generated by s is $S^1 s$; see Section 1 of Appendix A. For an arbitrary $s_1 \in S$ consider the left ideal $S s_1 s$. Certainly, $S s_1 s \subseteq S^1 s$, and $S^1 s$ is a minimal ideal if and only if this inclusion is never proper, i.e. if and only if for each $s_1 \in S$ there exists $s_2 \in S$ such that $s_2 s_1 s = s$.

(ii) ($\Rightarrow$) Assume that s generates a unique minimal left ideal L, and let $s_1 \in S$ be arbitrary. By Proposition A.3.2 (i), L is the minimal two-sided ideal of S, so that $s s_1 \in L$, i.e. $s s_1 = s_1' s$ for some $s_1' \in S^1$. By (i) there exists $s_2' \in S^1$ such that $s_2' s_1' s = s$, so that, for $s_2 = s_2' s \in S$, we have

$$s_2 s_1 = s_2' s s_1 = s_2' s_1' s = s,$$

as required.

($\Leftarrow$) Since for any $s_1 \in S$ there exists $s_2 \in S$ such that $s_2(s_1 s) = s$, s generates a minimal left ideal $L = S^1 s$ by (i). Assume that S possesses another minimal left ideal $L_1 = S^1 s_1$. Then $L \cap L_1 = \emptyset$ by Proposition A.3.2 (i). On the other hand, since there exists $s_2 \in S$ such that $s_2 s_1 = s$, we see that $s \in L \cap L_1$, which is a contradiction. Therefore, L is a unique minimal left ideal of S.

Proofs for right ideals are dual. ∎

Now we return to our presentation $\mathfrak{P}_2$.

Lemma 3.2. *The element a generates the unique minimal left ideal L in the semigroup $S(\mathfrak{P}_2)$.*

PROOF. By Proposition 3.1 (ii) it is enough to prove that for each word $w \in \{a, b\}^+$ there exists a word $w_1 \in \{a, b\}^+$ such that $w_1 w = a$ holds in $S(\mathfrak{P}_2)$. We prove this by induction on the length of w. If $|w| = 1$ then either $w \equiv a$, and we have

$$a^2 \cdot w \equiv a^3 = a,$$

or $w \equiv b$, in which case

$$a(ab)^4 a \cdot w \equiv a(ab)^5 = aa^2 = a.$$

For $|w| \geq 2$, we can write $w \equiv w' w''$, where $|w'| = 2$ and $w'' \in \{a, b\}^*$. Note that $|aw''| < |w|$ and $|bw''| < |w|$, so that by the inductive hypothesis there exist $w_1, w_2 \in \{a, b\}^+$ such that the relations

$$w_1 a w'' = a, \; w_2 b w'' = a$$

hold in $S(\mathfrak{P}_2)$. There are four possibilities for w': $w' \equiv a^2$, $w' \equiv ab$, $w' \equiv ba$ and $w' \equiv b^2$. Now we have

$$\begin{aligned}
&w_1 a \cdot a^2 w'' \equiv w_1 a^3 w'' = w_1 a w'' = a,\\
&w_1 a(ab)^4 \cdot abw'' \equiv w_1 a(ab)^5 w'' = w_1 aa^2 w'' = w_1 a w'' = a,\\
&w_1 (ab)^4 a \cdot baw'' \equiv w_1 (ab)^5 aw'' = w_1 a^2 aw'' = w_1 aw'' = a,\\
&w_2 b^2 \cdot b^2 w'' \equiv w_2 b^4 w'' = w_2 bw'' = a,
\end{aligned}$$

which completes the inductive argument. ■

Lemma 3.3. *L is the unique minimal two sided ideal of $S(\mathfrak{P}_2)$. A word w represents an element of L if and only if w contains the letter a.*

PROOF. The first assertion follows from the fact that the minimal two-sided ideal is the union of all minimal left ideals and the fact that L is the unique minimal left ideal. Since $a \in L$, and since L is a two-sided ideal, it is clear that every word which contains a represents an element of L. For the converse, note that for any two words w_1 and w_2 which represent the same element of $S(\mathfrak{P}_2)$, w_1 contains a if and only if w_2 contains a. On the other hand, if $b^i \in L$ for some $i = 1, 2, 3$, then the minimality of L would imply the existence of a word w such that $wab^i = b^i$, which is impossible. ■

The fact that $S(\mathfrak{P}_2)$ has a unique minimal left ideal does not automatically allow us to apply the results from the previous section. We need to show that $S(\mathfrak{P}_2)$ has minimal right ideals as well.

Lemma 3.4. *For each word $w_1 \in \{a, b\}^+$ there exists a word $w_2 \in \{a, b\}^+$ such that the relation*

$$aw_1 w_2 = a$$

holds in $S(\mathfrak{P}_2)$.

PROOF. We prove the lemma by induction on the length of w_1. If $|w_1| = 1$ then either $w_1 \equiv a$, and we have

$$aa \cdot a \equiv a^3 = a,$$

or $w_1 \equiv b$, in which case

$$ab \cdot (ab)^4 a \equiv (ab)^5 a = a^2 a = a.$$

If $|w_1| \geq 2$ then w_1 can be written as $w_1 \equiv w_1' w_1''$, where $w_1' \in \{a, b\}^*$ and $|w_1''| = 2$. By the inductive hypothesis there exist $w_2', w_2'' \in \{a, b\}^+$ such that the relations

$$aw_1' aw_2' = a, \; aw_1' bw_2'' = a \tag{3}$$

hold in $S(\mathfrak{P}_2)$. There are four possibilities for w_1'': $w_1'' \equiv a^2$, $w_1'' \equiv b^2$, $w_1'' \equiv ab$ and $w_1'' \equiv ba$. In the first three of them we have

$$aw_1'a^2 \cdot aw_2' \equiv aw_1'a^3w_2' = aw_1'aw_2' = a,$$
$$aw_1'b^2 \cdot b^2w_2'' \equiv aw_1'b^4w_2'' = aw_1'bw_2'' = a,$$
$$aw_1'ab \cdot (ab)^4aw_2' \equiv aw_1'(ab)^5aw_2' = aw_1'a^2aw_2' = aw_1'aw_2' = a.$$

So there remains only the case $w_1 \equiv w_1'ba$ to be considered. First note that using

$$aba^2 = (ab)(ab)^5 \equiv (ab)^5(ab) = a^2ab = ab,$$

it is easy to prove

$$ab^ia^2 = ab^i \tag{4}$$

for any $i \geq 1$. Now, if w_1' is a power of b, say $w_1' \equiv b^i$, $1 \leq i \leq 3$, then by using (4) we obtain

$$\begin{aligned} aw_1'ba \cdot ab^{3-i}(ab)^4a &\equiv ab^{i+1}a^2b^{3-i}(ab)^4a = ab^4(ab)^4a \\ &= ab(ab)^4a \equiv (ab)^5a = a^2a = a. \end{aligned}$$

Otherwise w_1' can be written as $w_1' \equiv w_1'''ab^i$, where $0 \leq i \leq 3$, and, similarly as above, we have

$$\begin{aligned} aw_1'ba \cdot aw_2'' &\equiv aw_1'''ab^{i+1}a^2w_2'' = aw_1'''ab^{i+1}w_2'' \\ &\equiv aw_1'bw_2'' = a, \end{aligned}$$

by using (3) and (4). ∎

Lemma 3.5. *Let R_i be the right ideal of $S(\mathfrak{P}_2)$ generated by b^ia for $i = 0, 1, 2, 3$. These ideals are the only minimal right ideals of $S(\mathfrak{P}_2)$ and are all distinct.*

PROOF. Each R_i is a minimal right ideal by Proposition 3.1 (i) and Lemma 3.4. Since each word containing b is equal to a word ending with b^ia for some $i = 0, 1, 2, 3$, we have $L = R_0 \cup R_1 \cup R_2 \cup R_3$ by Lemma 3.3, so that $S(\mathfrak{P}_2)$ has no minimal right ideals other than R_i, $i = 0, 1, 2, 3$. Finally, note that $S(\mathfrak{P}_2)$ has the following property:

$$b^jaw_1 = b^kaw_2 \Longrightarrow j = k = 0 \text{ or } j, k > 0 \text{ and } j \equiv k \pmod 3.$$

This implies that all R_i, $i = 0, 1, 2, 3$, are distinct. ∎

If we combine Lemmas 3.2, 3.3, 3.4, 3.5 and Corollary 2.4 we obtain

Theorem 3.6. *The semigroup $S(\mathfrak{P}_2)$ defined by the presentation*

$$\mathfrak{P}_2 = \mathfrak{P}_2(2,3,5) = \langle a, b \mid a^3 = a,\ b^4 = b,\ (ab)^5 = a^2 \rangle$$

has a unique minimal left ideal L, which is a disjoint union of four minimal right ideals, each of which is isomorphic to the alternating group $\mathcal{A}_5$ of degree 5. The set $S(\mathfrak{P}_2) - L$ is a cyclic group of order 3. The semigroup $S(\mathfrak{P}_2)$ is finite of order 243 and is a union of groups. ∎

The presentation $\mathfrak{P}_2(2,3,5)$ is a special case of the following presentation

$$\mathfrak{P}_2(l,m,n) = \langle a, b \mid a^{l+1} = a,\ b^{m+1} = b,\ (ab)^n = a^l \rangle.$$

The group defined by this presentation is usually denoted by (l,m,n); for details on these groups see Coxeter and Moser (1980). Theorem 3.6 does not depend on the particular choice of the parameters $l = 2$, $m = 3$, $n = 5$. In other words we have

Theorem 3.7. *The semigroup $S(\mathfrak{P}_2)$ defined by the presentation*

$$\mathfrak{P}_2 = \mathfrak{P}_2(l,m,n) = \langle a, b \mid a^{l+1} = a,\ b^{m+1} = b,\ (ab)^n = a^l \rangle,$$

where $l, m, n \geq 1$, has a unique minimal left ideal L which is a disjoint union of $m+1$ minimal right ideals, each of which is isomorphic to the group (l,m,n). The set $S(\mathfrak{P}_2) - L$ is the cyclic group of order m. The semigroup $S(\mathfrak{P}_2)$ is a union of groups and is finite if and only if the group (l,m,n) is finite. ∎

For a sketch of a proof of the above theorem the reader is referred to Campbell, Robertson, Ruškuc and Thomas (1995). This theorem has an obvious left right dual:

Theorem 3.8. *The semigroup $S(\mathfrak{P}_3)$ defined by the presentation*

$$\mathfrak{P}_3 = \mathfrak{P}_3(l,m,n) = \langle a, b \mid a^{l+1} = a,\ b^{m+1} = b,\ (ab)^n = b^m \rangle,$$

where $l, m, n \geq 1$, has a unique minimal right ideal R which is a disjoint union of $l+1$ minimal right ideals each of which is isomorphic to the group (l,m,n). The set $S(\mathfrak{P}_3) - L$ is the cyclic group of order l. The semigroup $S(\mathfrak{P}_2)$ is a union of groups and is finite if and only if the group (l,m,n) is finite. ∎

In particular, the presentation

$$\mathfrak{P}_3(2,3,5) = \langle a, b \mid a^3 = a,\ b^4 = b,\ (ab)^5 = b^3 \rangle$$

defines a semigroup of order 182 consisting of three copies of $\mathcal{A}_5$ and one copy of C_2.

We finish off this section by considering one more modification of the presentation $\mathfrak{P}_1$:

$$\mathfrak{P}_4 = \langle a, b \mid a^3 = a,\ b^4 = b,\ ab^3a = a^2,\ ba^2b = b^2,\ a(ab)^5a = a^2 \rangle.$$

It is clear that the group $G(\mathfrak{P}_4)$ defined by $\mathfrak{P}_4$ is again isomorphic to the alternating group $\mathcal{A}_5$. We are going to determine the structure of the semigroup $S(\mathfrak{P}_4)$ defined by $\mathfrak{P}_4$. An easy inductive argument based on the relations

$$\begin{aligned}
&a \cdot a^2 = a,\ b^2 \cdot b^2 = b,\\
&b^3a \cdot ab = b^2ba^2b = b^2b^2 = b,\\
&a^2b^2 \cdot ba = aab^3a = aa^2 = a,
\end{aligned}$$

proves that both a and b generate minimal left ideals in this semigroup; we denote these ideals by L_a and L_b respectively. Next we note that the last letter of a word is an invariant of the presentation $\mathfrak{P}_4$, so that $L_a \neq L_b$. Each word from $\{a,b\}^+$ ends either with a or with b, so that $S(\mathfrak{P}_4) = L_a \cup L_b$. In particular, $S(\mathfrak{P}_4)$ has no proper two-sided ideals by Proposition A.3.2, so that $S(\mathfrak{P}_4)$ is simple. A dual argument shows that a and b generate minimal right ideals R_a and R_b respectively, that $R_a \neq R_b$ and that $S(\mathfrak{P}_4) = R_a \cup R_b$. Therefore, $S(\mathfrak{P}_4)$ is a completely simple semigroup with two minimal left ideals and two minimal right ideals.

Let $H_{a,a}$, $H_{a,b}$, $H_{b,a}$, $H_{b,b}$ denote the groups $R_a \cap L_a$, $R_a \cap L_b$, $R_b \cap L_a$, $R_b \cap L_b$ respectively. Now we need to find the four idempotents which are the identities of these groups. Two of them obviously are $a^2 \in H_{a,a}$ and $b^3 \in H_{b,b}$. The other two are $a^2b^3 \in H_{a,b}$ and $b^3a^2 \in H_{b,a}$ because of

$$a^2b^3a^2b^3 \equiv a(ab^3a)ab^3 = aa^2ab^3 = a^2b^3,$$
$$b^3a^2b^3a^2 \equiv b^2(ba^2b)b^2a^2 = b^2b^2b^2a^2 = b^3a^2.$$

Finally, similar easy calculations prove that these four idempotents are closed under multiplication, so that by Theorem 2.1 we have

$$H_{a,a} \cong H_{a,b} \cong H_{b,a} \cong H_{b,b} \cong G(\mathfrak{P}_4) \cong \mathcal{A}_5.$$

We have proved the following

Theorem 3.9. *The semigroup $S(\mathfrak{P}_4)$ defined by the presentation*

$$\mathfrak{P}_4 = \langle a, b \mid a^3 = a,\ b^4 = b,\ ab^3a = a^2,\ ba^2b = b^2,\ a(ab)^5a = a^2 \rangle$$

is a completely simple semigroup with two minimal left ideals and two minimal right ideals, and is a disjoint union of four copies of the alternating group $\mathcal{A}_5$. ■

Chapter 8

Minimal ideals

In Section 3 of the previous chapter we have seen that understanding the structure of the minimal two-sided ideal M of a semigroup S defined by a presentation $\mathfrak{P}$ can be a big step towards understanding the structure of the whole semigroup. The crucial piece of information about M concerns the nature of its Schützenberger group H, since M is a disjoint union of copies of H. Theorem 9.2.1 asserts that if the idempotents of M are closed under multiplication, then H is isomorphic to the group G defined by the presentation $\mathfrak{P}$. However, this theorem gives very little information about H when the idempotents of M are not closed. In this chapter we develop another Reidemeister—Schreier type rewriting method, giving a presentation for H regardless of whether or not the idempotents are closed.

Actually, we shall consider a more general situation where M is a 0-minimal ideal which is a completely 0-simple semigroup. This is indeed a generalisation of the case where M is a minimal two-sided ideal with minimal left and right ideals; for if M is a minimal two sided ideal of S with minimal left and right ideals, and if we adjoin a zero to the semigroup S, then $M \cup \{0\}$ is a (unique) 0-minimal two-sided ideal of $S \cup \{0\}$, and is a completely 0-simple semigroup.

The reason for this generalisation is that it widens possibilities for applications in the investigation of the structure of semigroups defined by presentations. For suppose that M is the minimal ideal of S. After we have described the structure of M, the natural way to proceed in investigating S would be to form the Rees quotient $S_1 = S/M$, and to investigate S_1. However, the semigroup S_1 has a zero, which is a trivial minimal ideal, so that a theory of minimal ideals alone will not give any further information on S_1. On the other hand, with a theory of 0-minimal ideals one has a hope of continuing: if S_1 has a 0-minimal ideal M_1 we would investigate M_1 and then form the quotient $S_2 = S_1/M_1$ and so on.

The results of Sections 1 and 2 will appear in Campbell, Robertson, Ruškuc and Thomas (1995b). Section 3 contains a new result (Theorem 3.2), and a sketch of an alternative rewriting theorem which will appear in Campbell, Robertson, Ruškuc and Thomas (1995a). The example in Section 4 appears for the first time in this thesis.

1. A generating set for the Schützenberger group

Let S be a semigroup with zero and let $\mathfrak{P} = \langle A \,|\, \mathfrak{R} \rangle$ be a semigroup presentation for S. Suppose that S has a 0-minimal two-sided ideal M. By Proposition A.3.1 M is either a semigroup with zero multiplication or it is a 0-simple semigroup. Here we are interested in the latter case. Moreover, we assume that M is a completely 0-simple semigroup.

We adopt our standard notation for completely 0-simple semigroups; see Section 2 of Appendix A. Thus, L'_λ, $\lambda \in \Lambda$, are all 0-minimal left ideals of M, and R'_i, $i \in I$, are all 0-minimal right ideals of M. By Proposition A.3.4 these ideals are 0-minimal in S as well. We also let

$$R_i = R'_i - \{0\},\ L_\lambda = L'_\lambda - \{0\},\ H_{i\lambda} = R_i \cap L_\lambda\ \ (i \in I,\ \lambda \in \Lambda).$$

For any $i \in I$ and any $\lambda \in \Lambda$ either $H_{i\lambda}$ is a group, in which case we denote by $e_{i\lambda}$ a word representing the identity of this group, or $H_{i\lambda} \cup \{0\}$ is a semigroup with zero multiplication. Each R_i contains at least one group $H_{i\mu}$ and each L_λ contains at least one group $H_{j\lambda}$. In particular M contains at least one group, and there is no loss of generality if we assume that this group is H_{11}. We write

$$H_{11} = H,\ e \equiv e_{11},\ R_1 = R,\ L_1 = L.$$

If $H_{i\lambda}$ is a group, then it is isomorphic to H. We call H the *Schützenberger group* of M. In this section we find a generating set for H, and in the following section we find a presentation for H in terms of this generating set.

Let

$$\Lambda_0 = \Lambda \cup \{0\},$$

(where we assume $0 \notin \Lambda$), and define

$$L_0 = \{0\}.$$

Then S acts by postmultiplication on the set $\{L_\lambda \,|\, \lambda \in \Lambda_0\}$; see Proposition A.3.4. This means that there exists a mapping

$$\rho_l : \Lambda_0 \times A^+ \longrightarrow \Lambda_0$$

such that

$$L_\lambda w = L_{(\lambda,w)\rho_l}, \tag{1}$$

for all $\lambda \in \Lambda_0$ and all $w \in A^+$. To simplify notation, we shall write just λw instead of $(\lambda, w)\rho_l$, so that (1) becomes

$$L_\lambda w = L_{\lambda w}. \tag{2}$$

Since ρ_l is an action, it satisfies

$$\lambda(w_1 w_2) = (\lambda w_1) w_2 \tag{3}$$

for any $w_1, w_2 \in A^+$, and hence it is completely determined by the values λa ($\lambda \in \Lambda_0$, $a \in A$). Also, since $\Lambda_0 = \{0\}$, we have

$$0w = 0, \tag{4}$$

for all $w \in A^+$.

For each $\lambda \in \Lambda$, let $x_\lambda \in A^+$ be a word representing an element of $H_{1\lambda} = R \cap L_\lambda$; we choose x_1 to be identical to e. Since $H = H_{11}$ is a group, we have

$$L_1 x_\lambda = L_\lambda, \ \lambda \in \Lambda,$$

by Proposition A.2.1 (x), or, in other words,

$$1x_\lambda = \lambda, \ \lambda \in \Lambda. \tag{5}$$

Each L_λ contains at least one group $H_{i\lambda}$. Therefore, there exists a function

$$\Lambda \longrightarrow I, \ \lambda \mapsto i_\lambda,$$

such that $H_{i_\lambda \lambda}$ is a group for every $\lambda \in \Lambda$. Since $H_{11} \subseteq L_1$ is a group we can define

$$i_1 = 1. \tag{6}$$

Finally, since both H_{11} and $H_{i_\lambda \lambda}$ are groups, $H_{i_\lambda 1}$ contains a unique (semigroup) inverse of x_λ by Proposition A.2.1 (xi). This means that there exists a word $x'_\lambda \in A^+$ which represents an element of $H_{i_\lambda 1}$, and such that the relations

$$x_\lambda x'_\lambda = e, \ x'_\lambda x_\lambda = e_{i_\lambda \lambda} \tag{7}$$

hold in S; see Figure 6.

Obviously, since $x_1 \equiv e$ and $i_1 = 1$, we can define

$$x'_1 \equiv e. \tag{8}$$

With this notation we have the following description of H:

Lemma 1.1. $H = \{x_\lambda w x'_{\lambda w} \mid \lambda \in \Lambda, \ w \in A^+, \ \lambda w \neq 0\}$.

Proof. Let us denote the set

$$\{x_\lambda w x'_{\lambda w} \mid \lambda \in \Lambda, \ w \in A^+, \ \lambda w \neq 0\}$$

by K, and consider a typical element $x_\lambda w x'_{\lambda w}$ of K. Since $\lambda w \neq 0$, we have

$$L_\lambda w = L_{\lambda w} \neq L_0 = \{0\},$$

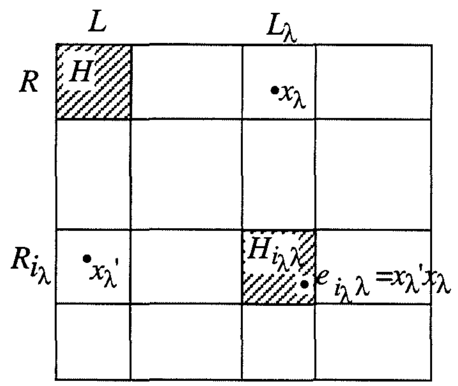


Figure 6.

and since $x_\lambda \in H_{1\lambda} = R \cap L_\lambda$, we have

$$x_\lambda w \in L_{\lambda w} \not\ni 0.$$

On the other hand,

$$x'_{\lambda w} \in H_{i_{\lambda w}1} = R_{i_{\lambda w}} \cap L,$$

and since $H_{i_{\lambda w},\lambda w}$ is a group, we have

$$x_\lambda w x'_{\lambda w} \in H_{11} = H$$

by Proposition A.2.1 (ix). Hence $K \subseteq H$.

Now let $w \in A^+$ be any word representing an element of H ($w \in L(A, H)$ in our usual notation). Then obviously $L_1 w = L_1$, so that $1w = 1$, and since e represents the identity of H, we have

$$w = ewe \equiv x_1 w x'_1 \equiv x_1 w x'_{1w} \in K.$$

Hence we have $H \subseteq K$, and the lemma follows. ■

Now we can prove the main result of this section.

Theorem 1.2. *The set*

$$X = \{x_\lambda a x'_{\lambda a} \mid \lambda \in \Lambda,\ a \in A,\ \lambda a \neq 0\}$$

generates H as a semigroup.

PROOF. By Lemma 1.1 we have $X \subseteq H$, so that we only have to prove that each element of H can be written as a product of elements of X. Again by Lemma 1.1, any element of of H is equal to a word of the form $x_\lambda w x'_{\lambda w}$, where $\lambda \in \Lambda$, $w \in A^+$ and $\lambda w \neq 0$, and to prove the lemma it is sufficient to prove that this word is equal in S to a product of elements of X. We do this by induction on the length of w.

If $|w| = 1$ then $w \equiv a \in A$, so that $x_\lambda w x'_{\lambda w} \in X$. Assume that $|w| > 1$, so that $w \equiv aw_1$, where $a \in A$, $w_1 \in A^+$. Also let $\lambda \in \Lambda$ be such that $\lambda w \neq 0$; in particular we have $\lambda a \neq 0$, because of (3) and (4). This means that

$$x_\lambda a \in L_{\lambda a} \not\ni 0. \tag{9}$$

Since $H_{i_{\lambda a}, \lambda a}$ is a group, its identity $e_{i_{\lambda a}, \lambda a}$ is a right identity for $L_{\lambda a}$ (see Proposition A.2.1 (xii)), and hence

$$x_\lambda a e_{i_{\lambda a}, \lambda a} = x_\lambda a. \tag{10}$$

Now note that by (7) we have

$$x'_{\lambda a} x_{\lambda a} = e_{i_{\lambda a}, \lambda a}, \tag{11}$$

and from (10) and (11) we obtain

$$x_\lambda w x'_{\lambda w} \equiv x_\lambda a w_1 x'_{\lambda(aw_1)} = (x_\lambda a x'_{\lambda a})(x_{\lambda a} w_1 x'_{(\lambda a) w_1}). \tag{12}$$

Since

$$(\lambda a) w_1 = \lambda(aw_1) = \lambda w \neq 0,$$

the word $x_{\lambda a} w_1 x'_{(\lambda a) w_1}$ is equal in S to a product of elements of X by the inductive hypothesis, and since $x_\lambda a x'_{\lambda a} \in X$, (12) yields a decomposition of $x_\lambda w x'_{\lambda w}$ into a product of elements of X. ∎

Obviously, Theorem 1.2 has a left-right dual. In order to formulate it we need some more notation. Let $I_0 = I \cup \{0\}$, where $0 \notin I$, and let $R_0 = \{0\}$. Then the action of S on the set $\{R_i \mid i \in I_0\}$ (see Proposition A.3.4) can be described by means of a mapping

$$\rho_r : A^+ \times I_0 \longrightarrow I_0,\ (w, i) \mapsto wi,$$

such that

$$wR_i = R_{wi}.$$

For each $i \in I$ let $y_i \in A^+$ be a word representing an element of $H_{i1} = R_i \cap L$, let $\lambda_i \in \Lambda$ be such that $H_{i\lambda_i}$ is a group, and let y'_i be a word representing a unique inverse of y_i in $H_{1\lambda_i}$. In addition, let

$$y_1 \equiv e,\ \lambda_1 = 1,\ y'_1 \equiv e.$$

Then we have

Theorem 1.3. *The set*

$$X = \{y'_{ai} a y_i \mid a \in A,\ i \in I,\ ai \neq 0\}$$

generates H as a semigroup. ∎

Although at the beginning of this section we assumed that S is defined by the presentation $\mathfrak{P} = \langle A \mid \mathfrak{R} \rangle$, so far we have only used the assumption that A generates S. Therefore, as a consequence of Theorems 1.2 and 1.3 we have the following

Corollary 1.4. *Let S be a semigroup with zero having a 0-minimal ideal M which is a completely 0-simple semigroup. If M has $|I|$ 0-minimal right ideals and $|\Lambda|$ 0-minimal left ideals, and if H is the Schützenberger group of M then*

$$\operatorname{rank}(H) \leq \operatorname{rank}(S) \cdot \min(|I|, |\Lambda|).$$

In particular, if S is finitely generated, and M has finitely many 0-minimal right ideals or finitely many 0-minimal left ideals then H is finitely generated as well.
■

Now we want to find a rewriting mapping associated with the generating set X given in Theorem 1.2. First we introduce a new alphabet

$$B = \{b_{\lambda,a} \mid \lambda \in \Lambda,\ a \in A,\ \lambda a \neq 0\}$$

in one-one correspondence with X. The representation mapping is the unique homomorphism

$$\psi : B^+ \longrightarrow A^+$$

such that

$$b_{\lambda,a}\psi = x_\lambda a x'_{\lambda a}. \tag{13}$$

Next we define a mapping

$$\overline{\phi} : \{(\lambda, w) \mid \lambda \in \Lambda,\ w \in A^+,\ \lambda w \neq 0\} \longrightarrow B^+$$

inductively by

$$\begin{aligned} (\lambda, a)\overline{\phi} &= b_{\lambda,a}, \\ (\lambda, aw)\overline{\phi} &= b_{\lambda,a} \cdot (\lambda a, w)\overline{\phi}. \end{aligned} \tag{14}$$

If $w \in L(A, H)$, then $1w = 1 \neq 0$, so that we can define

$$\phi : L(A, H) \longrightarrow B^+$$

by

$$w\phi = (1, w)\overline{\phi}. \tag{15}$$

Lemma 1.5. *Let $\lambda \in \Lambda$ and $w \in A^+$ be arbitrary. If $\lambda w \neq 0$ then the relation*

$$((\lambda, w)\overline{\phi})\psi = x_\lambda w x'_\lambda$$

holds in S. In particular, ϕ is a rewriting mapping.

PROOF. We prove the first part of the lemma by induction on the length of w. If $|w| = 1$ then $w \equiv a \in A$, and we have

$$((\lambda, w)\overline{\phi})\psi \equiv ((\lambda, a)\overline{\phi})\psi \equiv b_{\lambda,a}\psi \equiv x_\lambda a x'_{\lambda a} \equiv x_\lambda w x'_{\lambda w}.$$

For $|w| > 1$ we can write $w \equiv aw_1$, where $a \in A$, $w_1 \in A^+$, and then

$$\begin{aligned}
((\lambda, w)\overline{\phi})\psi &\equiv ((\lambda, aw_1)\overline{\phi})\psi \\
&\equiv (b_{\lambda,a} \cdot (\lambda a, w_1)\overline{\phi})\psi && \text{(by (14))} \\
&\equiv b_{\lambda,a}\psi \cdot ((\lambda a, w_1)\overline{\phi})\psi && (\psi \text{ is a homomorphism}) \\
&\equiv x_\lambda a x'_{\lambda a}((\lambda a, w_1)\overline{\phi})\psi && \text{(by (13))} \\
&= x_\lambda a x'_{\lambda a} x_{\lambda a} w_1 x'_{(\lambda a)w_1} && \text{(induction)} \\
&= x_\lambda a e_{i_{\lambda a}, \lambda a} w_1 x'_{(\lambda a)w_1} && \text{(by (7))} \\
&= x_\lambda a w_1 x'_{(\lambda a)w_1} && \text{(Proposition A.2.1 (xii))} \\
&\equiv x_\lambda w x'_{\lambda w}, && \text{(by (3))}
\end{aligned}$$

as required. To prove that ϕ is a rewriting mapping we have to prove that

$$(w\phi)\psi = w$$

holds in S for any $w \in L(A, H)$; see Section 7 of Chapter 6. Note that $w \in L(A, H)$ implies $1w = 1$, so that by the first part of the lemma we have

$$(w\phi)\psi \equiv ((1, w)\overline{\phi})\psi = x_1 w x'_{1w} \equiv x_1 w x'_1 \equiv ewe = w,$$

since e represents the identity of H. ∎

2. A Reidemeister—Schreier type theorem for the Schützenberger group

In this section we use all the notation introduced in Section 1. In particular, S denotes a semigroup with zero having a 0-minimal ideal M which is a completely 0-simple semigroup, and $\mathfrak{P} = \langle A \,|\, \mathfrak{R} \rangle$ is a presentation for S. We are going to find a presentation for the Schützenberger group H of M in terms of the generating set X given in Theorem 1.2. In doing this we make use of the mapping $\overline{\phi}$ defined by (14) and the rewriting mapping ϕ defined by (15).

Our main result is the following

Theorem 2.1. *The presentation*

$$\begin{gathered}
\mathfrak{Q} = \langle B \mid (x_\lambda a x'_{\lambda a})\phi = b_{\lambda,a},\ (\mu, u)\overline{\phi} = (\mu, v)\overline{\phi} \\
(\lambda, \mu \in \Lambda,\ a \in A,\ (u = v) \in \mathfrak{R},\ \lambda a \neq 0,\ \mu u \neq 0)\rangle
\end{gathered}$$

defines H (as a semigroup) in terms of generators X.

Before proving the above result we give some corollaries and remarks.

First of all we remark that Theorem 2.1 has a left-right dual, which gives a presentation for H in terms of generators Y from Theorem 1.3. In order to formulate this result, we need to define objects dual to B, ψ, $\overline{\phi}$ and ϕ. The definitions are as follows:

$$\begin{aligned}
&C = \{c_{a,i} \mid a \in A,\ i \in I,\ ai \neq 0\},\\
&\psi_r : C^+ \longrightarrow A^+,\ c_{a,i}\psi_r = y'_{ai}ay_i,\\
&\overline{\phi}_r : \{(w,i) \mid w \in A^+,\ i \in I,\ wi \neq 0\} \longrightarrow B^+,\\
&(a,i)\overline{\phi}_r = c_{a,i},\ (wa,i)\overline{\phi}_r = (w,ai)\overline{\phi}_r \cdot c_{a,i},\\
&\phi_r : L(A,H) \longrightarrow B^+,\\
&w\phi_r = (w,1)\overline{\phi}_r.
\end{aligned}$$

Now we have

Theorem 2.2. *The presentation*

$$\begin{aligned}
\mathfrak{Q}_r = \langle C \mid (y'_{ai}ay_i)\phi_r = c_{a,i},\ (u,j)\overline{\phi}_r = (v,j)\overline{\phi}_r\\
(i,j \in I,\ a \in A,\ (u = v) \in \mathfrak{R},\ ai \neq 0,\ uj \neq 0)\rangle
\end{aligned}$$

defines H (as a semigroup) in terms of the generators Y. ■

Both $\mathfrak{Q}$ and $\mathfrak{Q}_r$ are semigroup presentations for H. However, a semigroup presentation for an arbitrary group is a group presentation for that group, so that $\mathfrak{Q}$ and $\mathfrak{Q}_r$ are group presentations for H as well.

Next we note that if the sets A, $\mathfrak{R}$ and Λ are all finite, then the presentation $\mathfrak{Q}$ is finite as well; alternatively, if the sets A, $\mathfrak{R}$ and I are finite, then the presentation $\mathfrak{Q}_r$ is finite. Therefore we have

Corollary 2.3. *Let S be a semigroup with zero, let M be a 0-minimal ideal of S which is a completely 0-simple semigroup, and let H be the Schützenberger group of M. If S is finitely presented and if M has finitely many 0-minimal left ideals or finitely many 0-minimal right ideals, then H is finitely presented as well.* ■

We obtain another consequence of our main result if we combine Corollary 2.3 with Corollary 6.5.6.

Corollary 2.4. *Let S be a semigroup with zero, and let M be a 0-minimal ideal of S which is a completely 0-simple semigroup. If S is finitely presented and if M has finitely many 0-minimal left ideals and finitely many 0-minimal right ideals then M is finitely presented as well.*

PROOF. Being a completely 0-simple semigroup, M is isomorphic to a Rees matrix semigroup $\mathcal{M}^0[H; I, \Lambda; P]$, where H is the Schützenberger group of M, $|I|$ is the number of 0-minimal right ideals of M, $|\Lambda|$ is the number of 0-minimal left ideal of M, and P is a regular $|\Lambda| \times |I|$ matrix with entries from $H \cup \{0\}$; see Proposition A.2.4. Under the conditions of the corollary, the group H is finitely presented by Corollary 2.3, so that M is finitely presented by Corollary 6.5.6. ■

Actually, we can do even better: we can find a representation for M as a Rees matrix semigroup, provided that we know representatives of all $H_{1\lambda}$, $\lambda \in \Lambda$, and all H_{i1}, $i \in I$.

Corollary 2.5. *For each $\lambda \in \Lambda$ and each $i \in I$ let $\overline{x}_\lambda$ be a word representing an element of $H_{1\lambda}$, and let $\overline{y}_i$ be a word representing an element of H_{i1}. The 0-minimal ideal M is isomorphic to the Rees matrix semigroup $\mathcal{M}^0[H; I, \Lambda; P]$, where H is the group defined by the presentation $\mathfrak{Q}$, and*

$$P = ((\overline{x}_\lambda \overline{y}_i)\phi)_{\lambda \in \Lambda, i \in I},$$

where 0ϕ is defined to be 0.

PROOF. The corollary follows directly from Theorem 2.1 and Proposition A.2.4. ■

(Note that the representative $\overline{x}_\lambda$, $\lambda \in \Lambda$, is arbitrary, and is not necessarily equal to x_λ.)

Now we are also in a position to fulfill the promise given at the end of Section 5 in Chapter 6, and prove the converse of Corollary 6.5.6 for completely 0-simple semigroups.

Corollary 2.6. *Let S be a completely 0-simple semigroup with 0-minimal left ideals L'_λ, $\lambda \in \Lambda$, and 0-minimal right ideals R'_i, $i \in I$, and let H be the Schützenberger group of S. Then S is finitely presented if and only if H is finitely presented and both sets I and Λ are finite.*

PROOF. ($\Rightarrow$) Let S be finitely presented. In particular, S is finitely generated, so that from

$$\mathrm{rank}(S) \geq \max(|I|, |\Lambda|)$$

(Corollary 4.2.3) we conclude that both I and Λ are finite, and then it follows that H is finitely presented by Corollary 2.3.

($\Leftarrow$) By Proposition A.2.3, the semigroup S is isomorphic to a Rees matrix semigroup $\mathcal{M}^0[H; I, \Lambda; P]$, with a regular matrix P. In particular P contains at least one entry from H, and hence this implication follows from Corollary 6.5.6. ■

Finally, we would like to point out the similarity between Theorems 2.1 and 2.2 and the Reidemeister—Schreier theorem for groups; see Magnus, Karrass and

Solitar (1966). Unlike the case of ideals of finite index (Chapter 7), this time even the actual presentation is more like the usual Reidemeister—Schreier presentation for a subgroup of a group. This is primarily due to the fact that H itself is a group, as well as to the fact that the action of a semigroup on the 0-minimal left ideals of a 0-minimal two-sided ideal is much more symmetrical, and in particular is transitive.

Now we embark on the proof of Theorem 2.1. The main idea is again to apply the general method for finding presentations for subsemigroups described in Remark 6.7.6. We have already found a generating set (Theorem 1.2), and a corresponding rewriting mapping (Lemma 1.5). So there remains to prove that all the relations from the presentation $\mathfrak{Q}$ hold in H, and that they imply the relations

$$b_{\lambda,a} = (x_\lambda a x'_{\lambda a})\phi, \tag{16}$$
$$(w_1 w_2)\phi = w_1\phi \cdot w_2\phi, \tag{17}$$
$$(w_3 u w_4)\phi = (w_3 v w_4)\phi, \tag{18}$$

where $\lambda \in \Lambda$, $a \in A$, $\lambda a \neq 0$, $w_1, w_2 \in L(A,H)$, $w_3, w_4 \in A^+$, $(u = v) \in \mathfrak{R}$, $w_3 u w_4 \in L(A,H)$. We do this in several steps.

Lemma 2.7. *All the relations from $\mathfrak{Q}$ hold in S.*

PROOF. We have to prove that for any relation $\alpha = \beta$ from $\mathfrak{Q}$, the relation $\alpha\psi = \beta\psi$ holds in S. Let $\lambda \in \Lambda$, $a \in A$, with $\lambda a \neq 0$. Since $x_\lambda a x'_{\lambda a} \in L(A,H)$, we have $1 x_\lambda a x'_{\lambda a} = 1$, and by Lemma 1.5 we have

$$((x_\lambda a x'_{\lambda a})\phi)\psi \equiv ((1, x_\lambda a x'_{\lambda a})\overline{\phi})\psi = x_1 x_\lambda a x'_{\lambda a} x'_1 \equiv e x_\lambda a x'_{\lambda a} e = x_\lambda a x'_{\lambda a} \equiv b_{\lambda,a}\psi.$$

Similarly, for $(u = v) \in \mathfrak{R}$ and $\mu \in \Lambda$ with $\mu u \neq 0$, we have

$$((\mu, u)\overline{\phi})\psi = x_\mu u x'_{\mu u} = x_\mu v x'_{\mu v} = ((\mu, v)\overline{\phi})\psi,$$

as required. ■

Lemma 2.8. *Let $\lambda \in \Lambda$, $w_1, w_2 \in A^+$, be such that $\lambda w_1 w_2 \neq 0$. Then*

$$(\lambda, w_1 w_2)\overline{\phi} \equiv (\lambda, w_1)\overline{\phi} \cdot (\lambda w_1, w_2)\overline{\phi}.$$

PROOF. We prove the lemma by induction on $|w_1|$, the case $|w_1| = 1$ being the definition (14) of $\overline{\phi}$. For $|w_1| > 1$ let $w_1 \equiv a w'_1$, so that

$$\begin{aligned}
(\lambda, w_1 w_2)\overline{\phi} &\equiv (\lambda, a w'_1 w_2)\overline{\phi} \\
&\equiv (\lambda, a)\overline{\phi} \cdot (\lambda a, w'_1 w_2)\overline{\phi} && \text{(by (14))} \\
&\equiv (\lambda, a)\overline{\phi} \cdot (\lambda a, w'_1)\overline{\phi} \cdot (\lambda a w'_1, w_2)\overline{\phi} && \text{(induction)} \\
&\equiv (\lambda, a w'_1)\overline{\phi} \cdot (\lambda a w'_1, w_2)\overline{\phi} && \text{(by (14))} \\
&\equiv (\lambda, w_1)\overline{\phi} \cdot (\lambda w_1, w_2)\overline{\phi},
\end{aligned}$$

as required. ■

Lemma 2.9. *If* $w_1, w_2 \in L(A, H)$ *then*

$$(w_1w_2)\phi \equiv w_1\phi \cdot w_2\phi.$$

PROOF. Since $w_1 \in L(A, H)$, we have $1w_1 = 1$, so that

$$\begin{aligned} (w_1w_2)\phi &\equiv (1, w_1w_2)\overline{\phi} \equiv (1, w_1)\overline{\phi} \cdot (1w_1, w_2)\overline{\phi} \equiv (1, w_1)\overline{\phi} \cdot (1, w_2)\overline{\phi} \\ &\equiv w_1\phi \cdot w_2\phi \end{aligned}$$

by Lemma 2.8. ■

Lemma 2.10. *Let* $w_3, w_4 \in A^+$ *and* $(u = v) \in \mathfrak{R}$. *If* $w_3uw_4 \in L(A, H)$ *then the relation*

$$(w_3uw_4)\phi = (w_3vw_4)\phi$$

is a consequence of $\mathfrak{Q}$.

PROOF. Since $u = v$ holds in S, we have $\lambda u = \lambda v$, and hence

$$\begin{aligned} (w_3uw_4)\phi &\equiv (1, w_3uw_4)\overline{\phi} \\ &\equiv (1, w_3)\overline{\phi} \cdot (1w_3, u)\overline{\phi} \cdot (1w_3u, w_4)\overline{\phi} && \text{(Lemma 2.8)} \\ &= (1, w_3)\overline{\phi} \cdot (1w_3, v)\overline{\phi} \cdot (1w_3v, w_4)\overline{\phi} && \text{(relation } (\mu, u)\overline{\phi} = (\mu, v)\overline{\phi}) \\ &\equiv (1, w_3vw_4)\overline{\phi} && \text{(Lemma 2.8)} \\ &\equiv (w_3vw_4)\phi, \end{aligned}$$

as required. ■

PROOF OF THEOREM 2.1. The set X generates H by Theorem 1.2, and ϕ is a corresponding rewriting mapping by Lemma 1.5. All the relations from $\mathfrak{Q}$ hold in H by Lemma 2.7. Relations (16) are included in the presentation $\mathfrak{Q}$, the relations (17) are identically true by Lemma 2.9, and the relations (18) are consequences of $\mathfrak{Q}$ by Lemma 2.10. Therefore, $\mathfrak{Q}$ is a presentation for H by Theorem 6.7.1 and Remark 6.7.6. ■

In the following section we are going to see how Theorem 2.1 can be modified to give a presentation for the Schützenberger group of a minimal two-sided ideal, and in Section 4, as well as in Chapters 11 and 13, we will use one of these modifications to determine the structure of various semigroups. Unfortunately, we do not have any 'nice and natural' examples of calculating the Schützenberger group of a particular 0-minimal ideal (although it would not be difficult to construct artificial examples). Nevertheless, as we mentioned in the introduction to this chapter, we believe that Theorem 2.1 can be a powerful tool for investigating the structure of more complicated semigroups, just as the Reidemeister—Schreier theorem is in groups. However, applying the latter has been made much easier by implementing it on computers and making it a part of all major group theory systems; see Schönert et al. (1993). This, once again, highlights the need for the development of computational methods for semigroups. We make some initial steps in this direction in Chapter 14, where we will also pose certain computational problems related to possible implementations of Theorem 2.1.

3. Minimal two-sided ideal

Let us now consider the situation where our semigroup S does not necessarily have a zero, but has a (unique) minimal two-sided ideal M which is a completely simple semigroup. This is equivalent to assuming that S has both minimal left ideals and minimal right ideals. Adjoining a zero to S yields a semigroup S^0 with zero, in which the set $M^0 = M \cup \{0\}$ is a (unique) 0-minimal ideal. It is obvious that M^0 is a completely 0-simple semigroup, and we can use for M^0 all the notation introduced in the previous two sections. Since M^0 is obtained from M by adjoining a zero, we see that L_λ, $\lambda \in \Lambda$, are minimal left ideals of M, and hence are the minimal left ideals of S by Proposition A.3.3. Similarly, R_i, $i \in I$, are the minimal right ideals of S. Also, since M is completely simple, each $H_{i\lambda} = R_i \cap L_\lambda$ is a group.

If $\mathfrak{P} = \langle A \mid \mathfrak{R} \rangle$ is a presentation for S, then S^0 can be defined by the presentation

$$\mathfrak{P}^0 = \langle A, 0 \mid \mathfrak{R},\ 00 = 0,\ a0 = 0a = 0\ \ (a \in A) \rangle.$$

The Schützenberger groups for M and M^0 are identical, and the rewriting described in the previous section can be applied to the presentation $\mathfrak{P}^0$ to obtain a presentation for this group, which we again denote by H. For any word w and any $\lambda \in \Lambda$ we have $\lambda w = 0$ if and only if w contains 0 since 0 is indecomposable in S^0. Therefore, Theorem 2.1 takes the following form in the case of a minimal ideal:

Theorem 3.1. *The presentation*

$$\mathfrak{Q} = \langle B \mid (x_\lambda a x'_{\lambda a})\phi = b_{\lambda,a},\ (\mu, u)\overline{\phi} = (\mu, v)\overline{\phi} \\ (\lambda, \mu \in \Lambda,\ a \in A,\ (u = v) \in \mathfrak{R}) \rangle$$

defines H (as a semigroup) in terms of generators X. ■

However, by a careful choice of representatives x_λ, $\lambda \in \Lambda$, and their inverses x'_λ, $\lambda \in \Lambda$, we can obtain a simpler presentation for H. First of all, since all the sets $H_{i\lambda}$ are groups, it is natural to choose $x_\lambda \in H_{1\lambda}$ to be the identity of the group $H_{1\lambda}$:

$$x_\lambda \equiv e_{1\lambda}.$$

Now we invoke Proposition A.2.2 (iii), which asserts that the idempotents $e_{1\lambda}$, $\lambda \in \Lambda$, of R form a semigroup of right zeros, so that we have

$$e_{11}e_{1\lambda} = e_{1\lambda},\ e_{1\lambda}e_{11} = e_{11}.$$

Therefore, for the inverse x'_λ of x_λ we can choose

$$x'_\lambda \equiv e_{11} \equiv e.$$

With this choice of representatives we have

Theorem 3.2. *The presentation*

$$\mathfrak{T} = \langle B \mid (\mu, u)\overline{\phi} = (\mu, v)\overline{\phi},\ (\lambda, e)\overline{\phi} = 1\ (\lambda, \mu \in \Lambda,\ (u = v) \in \mathfrak{R})\rangle$$

defines H as a monoid.

PROOF. It is easy to check that the given relations hold in H. For the relations

$$(\mu, u)\overline{\phi} = (\mu, v)\overline{\phi}$$

this has been proved in Lemma 2.7, while for the other group of relations we have

$$((\lambda, e)\overline{\phi})\psi = x_\lambda e x'_{\lambda e} \equiv e_{1\lambda}e_{11}e_{11} = e_{11} \equiv e,$$

by Lemma 1.5. Now we want to show that the given relations imply

$$(x_\lambda a x'_{\lambda a})\phi = b_{\lambda,a},$$

for all $\lambda \in \Lambda$ and all $a \in A$, and the theorem will follow from Theorem 3.1.

First note that Lemma 2.10 has been proved by using only the relations $(\mu, u)\overline{\phi} = (\mu, v)\overline{\phi}$. Therefore, the relations

$$(w_3 u w_4)\phi = (w_3 v w_4)\phi,$$

$w_3, w_4 \in A^+$, $(u = v) \in \mathfrak{R}$, are consequences of the presentation $\mathfrak{T}$ as well. This, in turn, implies that if $w_5, w_6 \in L(A, H)$ represent the same element of H then the relation

$$w_5\phi = w_6\phi$$

is a consequence of $\mathfrak{T}$. In particular, since

$$e_{1\lambda}e_{11} = e_{11}$$

holds in S, we have

$$\begin{aligned} 1 &= (1, e)\overline{\phi} \equiv e_{11}\phi = (e_{1\lambda}e_{11})\phi \equiv (1, e_{1\lambda}e_{11})\overline{\phi} \\ &\equiv (1, e_{1\lambda})\overline{\phi} \cdot (1e_{1\lambda}, e_{11})\overline{\phi} \equiv (1, e_{1\lambda})\overline{\phi} \cdot (\lambda, e_{11})\overline{\phi} \equiv (1, e_{1\lambda})\overline{\phi}, \end{aligned}$$

where the relations $(\lambda, e)\overline{\phi} = 1$ and Lemma 2.8 have been used. Finally we have

$$\begin{aligned} (x_\lambda a x'_{\lambda a})\phi &\equiv (1, e_{1\lambda} a e_{11})\overline{\phi} \equiv (1, e_{1\lambda})\overline{\phi} \cdot (1e_{1\lambda}, a)\overline{\phi} \cdot (1e_{1\lambda}a, e_{11})\overline{\phi} \\ &= 1 \cdot (\lambda, a)\overline{\phi} \cdot 1 \equiv b_{\lambda,a}, \end{aligned}$$

as required. ∎

The presentation $\mathfrak{T}$ has some clear advantages over the presentation $\mathfrak{Q}$ from Theorem 3.1: not only does it have fewer relations ($|\Lambda|(|\mathfrak{R}| + 1)$, as compared to $|\Lambda|(|\mathfrak{R}| + |A|)$), but it also does not require us to know exact expressions for the representatives x_λ, $\lambda \in \Lambda$. The only information it uses is the action of S on its minimal left ideals (which is built into the definition of $\overline{\phi}$), and a word representing an idempotent of the minimal two-sided ideal of S. This makes it easier to apply, so that, whenever we are faced with the task of finding a presentation for the Schützenberger group of a minimal two-sided ideal, we will use the presentation $\mathfrak{T}$.

We finish off this section by mentioning another rewriting theorem for the Schützenberger group of the minimal two sided ideal introduced in Campbell, Robertson, Ruškuc and Thomas (1995a). It gives a presentation for H in terms of generators

$$\{eae \mid a \in A\} \cup \{e_{1\lambda}e_{i1} \mid i \in I, \lambda \in \Lambda\}.$$

As always, first we introduce a new alphabet in one-one correspondence with the given generating set:

$$B_1 = \{b_a \mid a \in A\} \cup \{c_{\lambda i} \mid \lambda \in \Lambda, i \in I\}.$$

Next we define a mapping

$$\phi_1 : A^+ \longrightarrow B_1^+$$

inductively by

$$\begin{aligned} a\phi_1 &= b_a, \\ (aw)\phi_1 &= b_a c_{1a,w1}(w\phi), \end{aligned}$$

$a \in A$, $w \in A^+$. It is important to note that this rewriting mapping depends both on the action of S on its minimal left ideals and on the action of S on its minimal right ideals. We also need words $g_{\lambda i} \in A^+$, $\lambda \in \Lambda$, $i \in I$, such that

$$g_{\lambda i} = e_{1\lambda}e_{i1}$$

holds in S, and which satisfy

$$g_{\lambda 1} \equiv g_{1i} \equiv e.$$

With this notation Campbell, Robertson, Ruškuc and Thomas (1995a) proved the following

Theorem 3.3. *The presentation*

$$\begin{aligned} \mathfrak{U} = \langle B_1 \mid (ue_{i1})\phi_1 = (ve_{i1})\phi,\ (ee_{i1})\phi = e_{i1}\phi,\ (g_{\lambda i}e_{j1})\phi = c_{\lambda i}(e_{j1}\phi) \\ ((u = v) \in \mathfrak{R},\ i, j \in I,\ \lambda \in \Lambda)\rangle \end{aligned}$$

defines H as a group. ■

This theorem yields a finite presentation for H if and only if S is finitely presented and has finitely many minimal left ideals as well as finitely many minimal right ideals, which is a weaker result than Corollary 2.3. This was the main reason for the author to consider this presentation 'worse' than the presentation $\mathfrak{Q}$ from Theorem 3.1. A closer analysis, however, shows that the sizes of two presentations are not comparable in general. The presentation $\mathfrak{Q}$ has $|A||\Lambda|$ generators and $|\Lambda|(|\mathfrak{R}| + |A|)$ relations, while the presentation $\mathfrak{U}$ has $|A| + |I||\Lambda|$ generators and $|I|(|\mathfrak{R}| + |I||\Lambda| + 1)$ relations. The presentation $\mathfrak{T}$ from Theorem 3.2 has the same number of generators as $\mathfrak{Q}$, but it has fewer relations than either $\mathfrak{Q}$ or $\mathfrak{U}$.

Nevertheless, the presentation $\mathfrak{U}$ does have certain disadvantages when compared with both $\mathfrak{Q}$ and $\mathfrak{T}$. First of all, it is less general, as it does not generalise naturally to 0-minimal ideals. It also requires more 'input data', since we have to know two actions of S and all the idempotents e_{i1}, $e_{\lambda 1}$, $i \in I$, $\lambda \in \Lambda$. However, it should be pointed out that the particular examples that we give in this thesis can be solved with more or less the same degree of difficulty by using any of the given presentations. The reason why we opt for the presentation $\mathfrak{T}$ in all these examples is that we wish to emphasise its generality, rather than to develop a separate rewriting procedure for each particular example.

4. Example: another presentation for $\mathcal{A}_5$

In Section 3 of Chapter 9 we showed how Theorem 9.2.1 can be used for investigating the structure of semigroups defined by presentations. All the presentations considered there were, when considered as group presentations, presentations for the alternating group $\mathcal{A}_5$ of degree 5. In this section we illustrate the general theory developed in the previous three sections on a further presentation defining $\mathcal{A}_5$:

$$\mathfrak{P}_5 = \mathfrak{P}_5(2,3,5) = \langle a, b \mid a^3 = a,\ b^4 = b,\ b(ab)^5 = b,\ ba^2b = bab^3ab \rangle.$$

(Notice that the last relation is redundant in the group case.)

Our aim is to determine the structure of the semigroup $S = S(\mathfrak{P}_5)$ defined by the above presentation. Similarly as in Section 3 of Chapter 9 we begin by determining minimal left ideals and minimal right ideals of S.

Lemma 4.1. *For any word $w_1 \in \{a,b\}^+$ there exists a word $w_2 \in \{a,b\}^+$ such that*

$$w_2 w_1 b = b \ \ (\textit{respectively } b w_1 w_2 = b)$$

holds in S. In particular each of the words b, ba, ba^2 (respectively b, ab, a^2b) generates a minimal left (respectively right) ideal. All these minimal left (right) ideals are distinct, and they are the only minimal left (right) ideals of S.

Proof. It is easy to see that, by applying relations $a^3 = a$ and $b^4 = b$, w_1 can be written as

$$w_1 = a^{i_{k+1}} b^{j_k} a^{i_k} b^{j_{k-1}} a^{i_{k-1}} \dots b^{j_1} a^{i_1},$$

where

$$k \geq 0,\ 0 \leq i_1, i_{k+1} \leq 2,\ 1 \leq i_2, \ldots, i_k \leq 2,\ 1 \leq j_1, \ldots, j_k \leq 3.$$

(If w_1 is a power of a then we ought to take $k = 0$, while if w_1 is a power of b we ought to take $k = 1$, $i_1 = i_2 = 0$.) We prove the first part of the lemma by induction on k.

For $k = 0$, we have $w_1 = a^{i_1}$, and hence

$$(ba)^4 ba^{3-i_1} \cdot w_1 b = (ba)^4 ba^3 b = (ba)^5 b = b.$$

Now let $k \geq 1$. By the inductive hypothesis we have

$$w_2' b^{j_{k-1}} a^{i_{k-1}} \ldots b^{j_1} a^{i_1} b = b,$$

for some $w_2' \in A^+$. Now, if $i_k = 1$, then we have

$$\begin{aligned}
& w_2'(ba)^4 b^{4-j_k} b(ab)^4 a^{3-i_{k+1}} \cdot w_1 b \\
= \ & w_2'(ba)^4 b^{4-j_k} b(ab)^4 \underline{a^3} b^{j_k} a b^{j_{k-1}} a^{i_{k-1}} \ldots b^{j_1} a^{i_1} b \\
= \ & w_2'(ba)^4 b^{4-j_k} \underline{b(ab)^5} b^{j_k - 1} a b^{j_{k-1}} a^{i_{k-1}} \ldots b^{j_1} a^{i_1} b \\
= \ & w_2'(ba)^4 \underline{b^4} a b^{j_{k-1}} a^{i_{k-1}} \ldots b^{j_1} a^{i_1} b \\
= \ & w_2' \underline{b(ab)^5} b^{j_{k-1}-1} a^{i_{k-1}} \ldots b^{j_1} a^{i_1} b \\
= \ & w_2' b^{j_{k-1}} a^{i_{k-1}} \ldots b^{j_1} a^{i_1} b \\
= \ & b,
\end{aligned}$$

while if $i_k = 2$ we have

$$\begin{aligned}
& w_2'(ba)^4 b(ba)^4 b^{4-j_k} b(ab)^4 a^{3-i_{k+1}} \cdot w_1 b \\
= \ & w_2'(ba)^4 b(ba)^4 b^{4-j_k} b(ab)^4 \underline{a^3} b^{j_k} a^2 b^{j_{k-1}} a^{i_{k-1}} \ldots b^{j_1} a^{i_1} b \\
= \ & w_2'(ba)^4 b(ba)^4 b^{4-j_k} \underline{b(ab)^5} b^{j_k - 1} a^2 b^{j_{k-1}} a^{i_{k-1}} \ldots b^{j_1} a^{i_1} b \\
= \ & w_2'(ba)^4 b(ba)^4 \underline{b^4} a^2 b^{j_{k-1}} a^{i_{k-1}} \ldots b^{j_1} a^{i_1} b \\
= \ & w_2'(ba)^4 b(ba)^4 \underline{ba^2 b^{j_{k-1}}} a^{i_{k-1}} \ldots b^{j_1} a^{i_1} b \\
= \ & w_2'(ba)^4 b\underline{(ba)^4 bab^3} a b^{j_{k-1}} a^{i_{k-1}} \ldots b^{j_1} a^{i_1} b \\
= \ & w_2'(ba)^4 \underline{bb^3} a b^{j_{k-1}} a^{i_{k-1}} \ldots b^{j_1} a^{i_1} b \\
= \ & w_2' \underline{(ba)^5 b^{j_{k-1}}} a^{i_{k-1}} \ldots b^{j_1} a^{i_1} b \\
= \ & w_2' b^{j_{k-1}} a^{i_{k-1}} \ldots b^{j_1} a^{i_1} b \\
= \ & b,
\end{aligned}$$

thus proving the first part of our lemma.

It now follows from Proposition 9.3.1 that the left ideals generated by b, ba and ba^2 are all minimal; we denote these ideals by L_1, L_2 and L_3 respectively.

In order to see that L_1, L_2 and L_3 are the only minimal left ideals of S, let $w \in A^+$ be an arbitrary word. Then the word bw contains a letter b, and is therefore equal to a word of the form w_1ba^i, $0 \le i \le 2$. By the first part of the lemma, there exists a word w_2 such that

$$w_2bw = w_2w_1ba^i = ba^i \in L_{i+1} \cap S^1w.$$

Hence, if w generates a minimal left ideal it must be L_{i+1}, since the minimal left ideals of a semigroup are disjoint.

To prove that L_1, L_2, L_3 are all distinct, first we note that the last letter of a word is an invariant of the presentation $\mathfrak{P}_5$, and hence

$$L_1 \neq L_2,\ L_1 \neq L_3.$$

Also, the last power of a modulo 2 is also an invariant of $\mathfrak{P}_5$ (i.e. if $w_1ba^i = w_2ba^j$ then $i \equiv j \pmod 2$), so that

$$L_2 \neq L_3.$$

The assertions about minimal right ideals can be proved in a similar way. ∎

As in the above proof, let L_λ, $\lambda \in \Lambda = \{1, 2, 3\}$, denote the (minimal) left ideals of S generated by b, ba, ba^2 respectively, and let R_i, $i \in I = \{1, 2, 3\}$, denote the (minimal) right ideals generated by b, ab, a^2b respectively. As in the previous sections, we let

$$L = L_1,\ R = R_1,\ H = H_{11} = R_1 \cap L_1.$$

We are now going to find a presentation for H by applying Theorem 3.2. Since S has two generators and three minimal left ideals, the presentation $\mathfrak{T}$ will have 6 generators, which we denote by

$$t_{1,a}, t_{2,a}, t_{3,a}, t_{1,b}, t_{2,b}, t_{3,b}.$$

(Notice that we use the letter t rather than b in order to avoid having, say, $b_{1,b}$ as a generator.) We also need to describe the action of S on its minimal left ideals. As we already noted in Section 1, this action is determined by the actions of thegenerators, which are easily seen to be

$$\begin{aligned}
L_1a &= S^1ba = L_2,\\
L_2a &= S^1ba^2 = L_3,\\
L_3a &= S^1ba^3 = S^1ba = L_2,\\
L_1b &= S^1b^2 = L_1,\\
L_2b &= S^1bab = L_1,\\
L_3b &= S^1ba^2b = L_1.
\end{aligned}$$

The above information can be given more compactly in the form of the following table:

	a	b
1	2	1
2	3	1
3	2	1

Finally, we need a word e representing the identity of H, and it is easy to see that b^3 satisfies these conditions.

A presentation for H is obtained by rewriting the defining relations for S and the word b^3. The definition (14) of the rewriting mapping

$$\overline{\phi} : \Lambda \times A^+ \longrightarrow B^+$$

in the particular case of the semigroup S becomes

$$\begin{aligned} &(\lambda, a)\overline{\phi} = t_{\lambda,a}, \ (\lambda, b)\overline{\phi} = t_{\lambda,b}, \\ &(\lambda, aw)\overline{\phi} = t_{\lambda,a} \cdot (\lambda a, w)\overline{\phi} \\ &(\lambda, bw)\overline{\phi} = t_{\lambda,b} \cdot (\lambda b, w)\overline{\phi}, \end{aligned}$$

where $\lambda \in \Lambda$, $w \in \{a, b\}^+$. Therefore, if $w \equiv \alpha_1\alpha_2 \ldots \alpha_k \in A^+$, where $\alpha_1, \ldots, \alpha_k \in A$, we can calculate $(\lambda, w)\overline{\phi}$ by means of the following table

	α_1	α_2	$\ldots$	α_k
λ	t_{λ_1,α_1}	t_{λ_2,α_2}	$\ldots$	t_{λ_k,α_k}

where $\lambda_1, \ldots, \lambda_k$ are obtained recursively by

$$\lambda_1 = \lambda, \ \lambda_{j+1} = \lambda_j\alpha_j \ (j = 1, \ldots, k-1).$$

First we rewrite the relations from $\mathfrak{P}_5$:

	a	a	a	$=$		a
1	$t_{1,a}$	$t_{2,a}$	$t_{3,a}$	$=$	1	$t_{1,a}$
2	$t_{2,a}$	$t_{3,a}$	$t_{2,a}$	$=$	2	$t_{2,a}$
3	$t_{3,a}$	$t_{2,a}$	$t_{3,a}$	$=$	3	$t_{3,a}$

	b	b	b	b	$=$		b
1	$t_{1,b}$	$t_{1,b}$	$t_{1,b}$	$t_{1,b}$	$=$	1	$t_{1,b}$
2	$t_{2,b}$	$t_{1,b}$	$t_{1,b}$	$t_{1,b}$	$=$	2	$t_{2,b}$
3	$t_{3,b}$	$t_{1,b}$	$t_{1,b}$	$t_{1,b}$	$=$	3	$t_{3,b}$

	b	a	b	a	b	a	b	a	b	a	b	$=$		b
1	$t_{1,b}$	$t_{1,a}$	$t_{2,b}$	$t_{1,a}$	$t_{2,b}$	$t_{1,a}$	$t_{2,b}$	$t_{1,a}$	$t_{2,b}$	$t_{1,a}$	$t_{2,b}$	$=$	1	$t_{1,b}$
2	$t_{2,b}$	$t_{1,a}$	$t_{2,b}$	$t_{1,a}$	$t_{2,b}$	$t_{1,a}$	$t_{2,b}$	$t_{1,a}$	$t_{2,b}$	$t_{1,a}$	$t_{2,b}$	$=$	2	$t_{2,b}$
3	$t_{3,b}$	$t_{1,a}$	$t_{2,b}$	$t_{1,a}$	$t_{2,b}$	$t_{1,a}$	$t_{2,b}$	$t_{1,a}$	$t_{2,b}$	$t_{1,a}$	$t_{2,b}$	$=$	3	$t_{3,b}$

	b	a	a	b	$=$		b	a	b	b	b	a	b
1	$t_{1,b}$	$t_{1,a}$	$t_{2,a}$	$t_{3,b}$	$=$	1	$t_{1,b}$	$t_{1,a}$	$t_{2,b}$	$t_{1,b}$	$t_{1,b}$	$t_{1,a}$	$t_{2,b}$
2	$t_{2,b}$	$t_{1,a}$	$t_{2,a}$	$t_{3,b}$	$=$	2	$t_{2,b}$	$t_{1,a}$	$t_{2,b}$	$t_{1,b}$	$t_{1,b}$	$t_{1,a}$	$t_{2,b}$
3	$t_{3,b}$	$t_{1,a}$	$t_{2,a}$	$t_{3,b}$	$=$	3	$t_{3,b}$	$t_{1,a}$	$t_{2,b}$	$t_{1,b}$	$t_{1,b}$	$t_{1,a}$	$t_{2,b}$

and then we rewrite the word b^3:

	b	b	b		
1	$t_{1,b}$	$t_{1,b}$	$t_{1,b}$	$=$	1
2	$t_{2,b}$	$t_{1,b}$	$t_{1,b}$	$=$	1
3	$t_{3,b}$	$t_{1,b}$	$t_{1,b}$	$=$	1

The resulting presentation

$$\begin{gathered}\langle t_{1,a}, t_{2,a}, t_{3,a}, t_{1,b}, t_{2,b}, t_{3,b} \mid t_{1,a}t_{2,a}t_{3,a} = t_{1,a},\ t_{2,a}t_{3,a}t_{2,a} = t_{2,a},\\ t_{3,a}t_{2,a}t_{3,a} = t_{3,a},\ t_{1,b}^4 = t_{1,b},\ t_{2,b}t_{1,b}^3 = t_{2,b},\ t_{3,b}t_{1,b}^3 = t_{3,b},\ t_{1,b}(t_{1,a}t_{2,b})^5 = t_{1,b},\\ t_{2,b}(t_{1,a}t_{2,b})^5 = t_{2,b},\ t_{3,b}(t_{1,a}t_{2,b})^5 = t_{3,b},\ t_{1,b}t_{1,a}t_{2,a}t_{3,b} = t_{1,b}t_{1,a}t_{2,b}t_{1,b}^2t_{1,a}t_{2,b},\\ t_{2,b}t_{1,a}t_{2,a}t_{3,b} = t_{2,b}t_{1,a}t_{2,b}t_{1,b}^2t_{1,a}t_{2,b},\ t_{3,b}t_{1,a}t_{2,a}t_{3,b} = t_{3,b}t_{1,a}t_{2,b}t_{1,b}^2t_{1,a}t_{2,b},\\ t_{1,b}^3 = 1,\ t_{2,b}t_{1,b}^2 = 1,\ t_{3,b}t_{1,b}^2 = 1\rangle\end{gathered}$$

is a monoid presentation for H by Theorem 3.2. However, as we noted before, this is also a group presentation for H, and in what follows we treat it as such. Now the three relations

$$t_{1,a}t_{2,a}t_{3,a} = t_{1,a},\ t_{2,a}t_{3,a}t_{2,a} = t_{2,a},\ t_{3,a}t_{2,a}t_{3,a} = t_{3,a}$$

obtained by rewriting $a^3 = a$, are equivalent to the single relation

$$t_{2,a}t_{3,a} = 1.$$

We can easily see that rewriting any other relation from $\mathfrak{P}_5$ gives only one relation, while rewriting b^3 gives two more relations, so that we obtain the following presentation for H

$$\begin{gathered}\langle t_{1,a}, t_{2,a}, t_{3,a}, t_{1,b}, t_{2,b}, t_{3,b} \mid t_{2,a}t_{3,a} = 1,\ t_{1,b}^3 = 1,\ (t_{1,a}t_{2,b})^5 = 1,\\ t_{2,a}t_{3,b} = t_{2,b}t_{1,b}^2t_{1,a}t_{2,b},\ t_{2,b}t_{1,b}^2 = 1,\ t_{3,b}t_{1,b}^2 = 1\rangle.\end{gathered}$$

Now by using the first, fifth and sixth relations we can eliminate $t_{3,a}$, $t_{3,b}$ and $t_{2,b}$ as

$$t_{3,a} = t_{2,a}^{-1},\ t_{2,b} = t_{3,b} = t_{1,b}, \tag{19}$$

and we obtain

$$\langle t_{1,a}, t_{2,a}, t_{1,b} \mid t_{1,b}^3 = 1,\ (t_{1,a}t_{1,b})^5 = 1,\ t_{2,a} = t_{1,a}\rangle.$$

Next, by using the last relation

$$t_{2,a} = t_{1,a}, \tag{20}$$

we eliminate $t_{2,a}$ and obtain

$$\langle t_{1,a}, t_{1,b} \mid t_{1,b}^3 = 1,\ (t_{1,a}t_{1,b})^5 = 1\rangle.$$

Finally, if we introduce new generators x and y by

$$x = t_{1,b},\ y = t_{1,a}t_{1,b},$$

and then eliminate $t_{1,a}$ and $t_{1,b}$ by using

$$t_{1,b} = x,\ t_{1,a} = yx^{-1} = yx^2, \tag{21}$$

we end up with the presentation

$$\langle x, y \mid x^3 = 1,\ y^5 = 1\rangle, \tag{22}$$

which defines the free product $C_3 * C_5$ of a cyclic group of order 3 and a cyclic group of order 5.

Therefore, the Schützenberger group H of the minimal two-sided ideal M of S is isomorphic to $C_3 * C_5$, while the minimal ideal itself consists of nine copies of this group.

We can be more precise and express M as a Rees matrix semigroup by using Corollary 2.5. To this end we need representatives $\overline{x}_\lambda$ of $H_{1\lambda}$, $\lambda = 1, 2, 3$, as well as representatives $\overline{y}_i$ of H_{i1}, $i = 1, 2, 3$. One choice of these representatives is

$$\overline{x}_1 \equiv b,\ \overline{x}_2 \equiv ba,\ \overline{x}_3 \equiv ba^2,\ \overline{y}_1 \equiv b,\ \overline{y}_2 \equiv ab,\ \overline{y}_3 \equiv a^2b.$$

The entries of a Rees matrix

$$P = (p_{\lambda i})_{1\le\lambda\le 3,\, 1\le i\le 3}$$

are obtained by rewriting the words $\overline{x}_\lambda\overline{y}_i$. For example, rewriting $\overline{x}_1\overline{y}_1 \equiv b^2$ yields

	b	b
1	$t_{1,b}$	$t_{1,b}$

so that $p_{11} = t_{1,b}^2$. Similarly

$$\begin{aligned}
p_{12} &= (\overline{x}_1\overline{y}_2)\phi \equiv (bab)\phi \equiv t_{1,b}t_{1,a}t_{2,b},\\
p_{13} &= (\overline{x}_1\overline{y}_3)\phi \equiv (ba^2b)\phi \equiv t_{1,b}t_{1,a}t_{2,a}t_{3,b},\\
p_{21} &= (\overline{x}_2\overline{y}_1)\phi \equiv (bab)\phi \equiv t_{1,b}t_{1,a}t_{2,b},\\
p_{22} &= (\overline{x}_2\overline{y}_2)\phi \equiv (ba^2b)\phi \equiv t_{1,b}t_{1,a}t_{2,a}t_{3,b},\\
p_{23} &= (\overline{x}_2\overline{y}_3)\phi \equiv (ba^3b)\phi \equiv t_{1,b}t_{1,a}t_{2,a}t_{3,a}t_{2,b},\\
p_{31} &= (\overline{x}_3\overline{y}_1)\phi \equiv (ba^2b)\phi \equiv t_{1,b}t_{1,a}t_{2,a}t_{3,b},\\
p_{32} &= (\overline{x}_3\overline{y}_2)\phi \equiv (ba^3b)\phi \equiv t_{1,b}t_{1,a}t_{2,a}t_{3,a}t_{2,b},\\
p_{33} &= (\overline{x}_3\overline{y}_3)\phi \equiv (ba^4b)\phi \equiv t_{1,b}t_{1,a}t_{2,a}t_{3,a}t_{2,a}t_{3,b}.
\end{aligned}$$

By using (19), (20), (21) we can express these entries in terms of generators x and y:

$$p_{11} = t_{1,b}^2 = x^2,$$
$$p_{12} = p_{21} = t_{1,b}t_{1,a}t_{2,b}xyx^2x = xy,$$
$$p_{13} = p_{22} = p_{31} = t_{1,b}t_{1,a}t_{2,a}t_{3,b} = t_{1,b}t_{1,a}^2t_{1,b} = xyx^2yx^2x = xyx^2y,$$
$$p_{23} = p_{32} = t_{1,b}t_{1,a}t_{2,a}t_{3,a}t_{2,b} = t_{1,b}t_{1,a}t_{1,a}t_{1,a}^{-1}t_{1,b} = xyx^2x = xy,$$
$$p_{33} = t_{1,b}t_{1,a}t_{2,a}t_{3,a}t_{2,a}t_{3,b} = t_{1,b}t_{1,a}t_{1,a}t_{1,a}^{-1}t_{1,a}t_{1,b} = xyx^2y.$$

Therefore

$$P = \begin{pmatrix} x^2 & xy & xyx^2y \\ xy & xyx^2y & xy \\ xyx^2y & xy & xyx^2y \end{pmatrix}.$$

Moreover, P can be transformed into the normal form by Propositions A.2.5 and A.2.6. To do this we first premultiply the rows by x, y^4x^2 and $y^4x^2y^4x^2$ respectively, and then we postmultiply the columns by 1, y^4x and y^4xy^4x, and we obtain

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & y^4xy^4x \\ 1 & y^4xy^4x & y^4xy^4x \end{pmatrix}.$$

Finally, we determine $S - M$. First we note that $b \in M$, so that every word which contains a letter b represents an element of M. Conversely, if a word w represents an element of M then w contains b, for the letter b is an invariant of the presentation $\mathfrak{P}_5$. Therefore, a and a^2 are the only elements of $S - M$, and $S - M$ is a cyclic group of order two.

We have proved

Theorem 4.2. *The semigroup S defined by the presentation*

$$\mathfrak{P}_5 = \langle a, b \mid a^3 = a,\ b^4 = b,\ b(ab)^5 = b,\ ba^2b = bab^3ab \rangle$$

has a minimal two-sided ideal M, which is isomorphic to the Rees matrix semigroup

$$\mathcal{M}[H; \{1,2,3\}, \{1,2,3\}; P],$$

where

$$H = \langle x, y \mid x^3 = 1,\ y^5 = 1 \rangle$$

is the free product $C_3 * C_5$ and

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & y^4xy^4x \\ 1 & y^4xy^4x & y^4xy^4x \end{pmatrix}.$$

$S - M$ is a cyclic group of order 2. The semigroup S is infinite, and is a union of groups. ∎

Further applications of our rewriting theorem to structural investigations can be found in Chapters 11 and 13.

Chapter 9

Fibonacci semigroups

In this chapter we apply methods developed in the previous two chapters to investigate the structure of semigroups defined by presentations for the Fibonacci groups and generalised Fibonacci groups.

Section 1 contains definitions and some historical remarks. The results of Sections 2 and 4 will appear in Campbell, Robertson, Ruškuc and Thomas (1995a), and the results of Section 3 have appeared in Campbell, Robertson, Ruškuc and Thomas (1994). However, the proofs of many theorems have been altered in order to bring them in line with our general theory.

1. Definitions and history

In this chapter we consider semigroups $S(r,n)$ defined by presentations

$$\mathfrak{P}(r,n) = \langle a_1,\ldots,a_n \mid a_i a_{i+1}\ldots a_{i+r-1} = a_{i+r} \ \ (i=1,\ldots,n)\rangle,$$

and, more generally, semigroups $S(r,n,k)$ defined by presentations

$$\mathfrak{P}(r,n,k) = \langle a_1,\ldots,a_n \mid a_i a_{i+1}\ldots a_{i+r-1} = a_{i+r+k-1} \ \ (i=1,\ldots,n)\rangle,$$

where $r,n,k \in \mathbb{N}$, and all subscripts are reduced modulo n. It is obvious that the presentation $\mathfrak{P}(r,n)$ is identical to the presentation $\mathfrak{P}(r,n,1)$.

The groups $F(r,n)$ defined by $\mathfrak{P}(r,n)$ are called *Fibonacci groups.* These groups have been a subject of extensive investigation over the years. This investigation began by a question of Conway (1965) as to whether or not the group $F(2,5)$ defined by the presentation

$$\mathfrak{P}(2,5) = \langle a_1,a_2,a_3,a_4,a_5 \mid a_1a_2=a_3,\ a_2a_3=a_4,\ a_3a_4=a_5,\\ a_4a_5=a_1,\ a_5a_1=a_2\rangle$$

is cyclic of order 11. It was quickly determined that this was indeed the case; see Conway et al. (1967). At the same time it was observed that this can be proved without using inverses or cancellation laws, so that the semigroup

$S(2,5)$ is also isomorphic to C_{11}. Thus the connection between Fibonacci groups and semigroups $S(r,n)$, which we shall call *Fibonacci semigroups*, has attracted attention at the earliest stage of the research.

Much of the research on the Fibonacci groups has been motivated by the finiteness problem: *for which $r,n \in \mathbb{N}$ is the group $F(r,n)$ finite?* For example, $F(2,1)$ and $F(2,2)$ are trivial groups, $F(2,3)$ is the quaternion group, $F(2,4)$ is cyclic of order 5, $F(2,5)$ is cyclic of order 11, $F(2,6)$ is infinite, $F(2,7)$ is cyclic of order 29 and $F(2,n)$ is infinite for $n \geq 8$. For a survey and extensive bibliography on the Fibonacci groups the reader is referred to Thomas (1991). In particular, this paper contains the information known at that time on orders of the groups $F(r,n)$, for $2 \leq r,n \leq 10$; see Table 1.

$r\backslash n$	2	3	4	5	6	7	8	9	10
2	1	8	5	11	∞	29	∞	∞	∞
3	8	2	∞	22	1512	?	∞	?	∞
4	3	63	3	∞	?	?	?	?	∞
5	24	∞	624	4	∞	?	∞	∞	?
6	5	5	125	7775	5	∞	?	?	∞
7	48	342	∞	?	7^6-1	6	∞	?	∞
8	7	∞	7	?	∞	8^7-1	7	∞	?
9	80	8	6560	∞	?	∞	9^8-1	8	∞
10	9	999	4905	9	?	?	∞	10^9-1	9

Table 1.

The question of connections between Fibonacci groups and Fibonacci semigroups arose again with the development of computational methods for finitely presented semigroups, since presentations $\mathfrak{P}(r,n)$ provided natural and interesting examples of semigroup presentations. A record of this research can be found in Campbell, Robertson and Thomas (1992), where the authors produced a table with orders of semigroups $S(r,n)$, $2 \leq r,n \leq 10$; see Table 2. This prompted a conjecture made in Campbell, Robertson and Thomas (1993a) that the order of a Fibonacci semigroup $S(r,n)$ is equal to g.c.d.(n,r) times the order of the corresponding Fibonacci group $F(r,n)$. This conjecture was proved in Campbell, Robertson, Ruškuc and Thomas (1994), by showing that $S(r,n)$ is a completely simple semigroup which is a disjoint union of g.c.d.(n,r) copies of $F(r,n)$. In this chapter we prove this result by using Theorem 9.2.1 and Corollary 9.2.4. However, it should be noted that the actual result is older than Theorem 9.2.1, and that it has actually motivated Theorem 9.2.1. This is an example of how computing methods can be used as tools for experimenting in mathematics, and how the results of such experiments can point to more general and purely theoretical theorems.

$r \backslash n$	2	3	4	5	6	7	8	9	10
2	2	8	10	11	∞	29	∞	∞	∞
3	8	6	∞	22	4536	?	∞	?	∞
4	6	63	12	∞	?	?	?	?	∞
5	25	∞	624	20	∞	?	∞	∞	?
6	10	15	250	7775	30	∞	?	?	∞
7	48	342	∞	?	$7^6 - 1$	42	∞	?	∞
8	14	∞	28	?	∞	?	56	∞	?
9	80	24	6560	∞	?	∞	?	72	∞
10	18	999	9810	45	?	?	∞	?	90

Table 2.

Rather than considering Fibonacci semigroups $S(r,n)$, for most of this chapter we will be considering their generalisations $S(r,n,k)$, which we call *generalised Fibonacci semigroups.* The corresponding groups $F(r,n,k)$, usually called *generalised Fibonacci groups*, were introduced in Campbell and Robertson (1974). A survey of results and bibliography for these groups can again be found in Thomas (1991). We do not consider this generalisation just for the sake of generalisation, but because of the fact that the generalised Fibonacci semigroups show a greater variety of behaviour. Apart from the already mentioned situation where $S(r,n,k)$ is a finite union of disjoint copies of $F(r,n,k)$, we will see that there are two more different situations: $S(r,n,k)$ can be a finite union of a number of disjoint copies of some group which is not necessarily isomorphic to $F(r,n,k)$, and which is infinite regardless of whether or not $F(r,n,k)$ is infinite, or $S(r,n,k)$ can be a semigroup without any minimal (left, right or two-sided) ideals.

In Section 2 we find for which values of r,n,k the semigroup $S(r,n,k)$ has minimal ideals, and prove that for these values $S(r,n,k)$ is a completely simple semigroup. A straightforward application of Corollary 9.2.4 gives then a structure theorem for the Fibonacci semigroups $S(r,n)$ in Section 3. Finally, in Section 4 we apply the Reidemeister—Schreier type rewriting technique developed in Chapter 10 to find a presentation for the Schützenberger group of $S(r,n,k)$ in the case when $S(r,n,k)$ is a completely simple semigroup, and prove that this group is infinite regardless of whether or not the group $F(r,n,k)$ is infinite.

2. Minimal ideals of generalised Fibonacci semigroups

In this section we begin describing the structure of the generalised Fibonacci semigroups $S(r,n,k)$. Let us recall that $S(r,n,k)$ is defined by the presentation

$$\mathfrak{P}(r,n,k) = \langle a_1,\ldots,a_n \mid a_i a_{i+1} \ldots a_{i+r-1} = a_{i+r+k-1} \ \ (i = 1,\ldots,n)\rangle,$$

where all the subscripts are reduced modulo n. Let us also define

$$\begin{aligned} d_1 &= \text{g.c.d.}(n,k), \\ d_2 &= \text{g.c.d.}(n, r+k-1), \\ d &= \text{g.c.d.}(d_1,d_2) = \text{g.c.d.}(n,k,r+k-1). \end{aligned}$$

Our strategy is to use the general theory developed in Chapters 9 and 10, whichever is appropriate. Both these methods have the same beginning—detection of minimal left and right ideals, and this is the main theme of this section.

We begin by considering the case where there are no minimal ideals at all, and, within this case, we start by considering the case $r = 1$, which is somewhat exceptional.

Theorem 2.1. *Let $n, k \in \mathbb{N}$ be arbitrary, and let $d_1 = \text{g.c.d.}(n,k)$. Then $S(1,n,k)$ is the free semigroup of rank d_1, and $F(1,n,k)$ is the free group of rank d_1.*

PROOF. For $r = 1$ the presentation $\mathfrak{P}(r,n,k)$ becomes

$$\mathfrak{P}(1,n,k) = \langle a_1,\ldots,a_n \mid a_i = a_{i+k} \ \ (i = 1,\ldots,n)\rangle.$$

Therefore we have

$$a_i = a_{i+pk},$$

for any $i = 1,\ldots,n$ and any $p \geq 0$. Since the subscripts are reduced modulo n, and since $d_1 = \text{g.c.d.}(n,k)$, we have

$$a_i = a_{i+d_1}. \tag{1}$$

On the other hand, all the relations $a_i = a_{i+k}$ are consequences of (1), so that $\mathfrak{P}(1,n,k)$ is equivalent to

$$\langle a_1,\ldots,a_n \mid a_i = a_{i+d_1} \ \ (i = 1,\ldots,n)\rangle.$$

The relations of the above presentation can be used to eliminate generators $a_{d_1+1}, a_{d_1+2},\ldots,a_n$, so that we are left with the presentation

$$\langle a_1,\ldots,a_n \mid \ \rangle,$$

which defines the free semigroup of rank d_1 and the free group of rank d_1 respectively. ■

Next we deal with the general case where there are no minimal ideals.

Theorem 2.2. *Let $r,n,k \in \mathbb{N}$, $r > 1$, be arbitrary, and let $d = \text{g.c.d.}(n,k,r+k-1)$. If $d > 1$ then the semigroup $S(r,n,k)$ has no minimal (left, right or two-sided) ideals.*

PROOF. The existence of minimal left ideals or minimal right ideals implies the existence of a minimal two-sided ideal by Proposition A.3.2. Therefore, to prove the theorem, it is sufficient to prove that $S(r,n,k)$ has no minimal two-sided ideals.

Actually, we consider the homomorphic image T of $S(r,n,k)$ obtained by adding the relations

$$a_i = a_{i+d},\ i = 1,\ldots,n,$$

to $\mathfrak{P}(r,n,k)$. After eliminating generators $a_{d+1}, a_{d+2}, \ldots, a_n$ we obtain the following presentation for T:

$$\langle a_1,\ldots,a_d \mid (a_i a_{i+1} \ldots a_d a_1 \ldots a_{i-2} a_{i-1})^l a_i = a_i \ \ (i = 1,\ldots,d)\rangle, \qquad (2)$$

where $l = (r-1)/d$ and the subscripts are reduced modulo d. We prove that T has no minimal two-sided ideals. Since a homomorphic image of a minimal ideal is again a minimal ideal, it will then follow that S has no minimal two-sided ideals either.

For a word $w \equiv a_{i_1} a_{i_2} \ldots a_{i_p}$, we denote by $\gamma(w)$ the number of pairs (i_j, i_{j+1}), $1 \le j < p$, such that

$$i_j + 1 \not\equiv i_{j+1} \pmod{d}.$$

Since

$$\gamma((a_i a_{i+1} \ldots a_{i-2} a_{i-1})^l a_i) = 0 = \gamma(a_i),$$

and since the first and the last letters of a word are invariants of the presentation (2), we conclude that $\gamma(w)$ is also an invariant, i.e. if two words w_1 and w_2 represent the same element of T then $\gamma(w_1) = \gamma(w_2)$. We also note that

$$\gamma(w_1 w_2) \ge \gamma(w_1) + \gamma(w_2),$$

for any two words w_1, w_2.

Assume now that T has a minimal two-sided ideal M, and let w be a word representing an element of M. Consider the word $a_1^2 w$. It also represents an element of M, and, because of minimality, there exist words w_3, w_4 such that

$$w = w_3 a_1^2 w w_4$$

holds in T. Note that

$$\gamma(a_1^2) = 1,$$

and so we have

$$\gamma(w) = \gamma(w_3 a_1^2 w w_4) \ge \gamma(w_3) + \gamma(a_1^2) + \gamma(w) + \gamma(w_4) > \gamma(w),$$

which is a contradiction. ∎

Now we come to the main result of this section.

Theorem 2.3. *Let $r, n, k \in \mathfrak{N}$, $r > 1$, be arbitrary, let $d_1 = \text{g.c.d.}(n,k)$, $d_2 = \text{g.c.d.}(n, r+k-1)$ and assume that $d = \text{g.c.d.}(n,k,r+k-1) = 1$.*

(i) Each generator a_i of $S(r,n,k)$ generates a minimal left ideal, as well as a minimal right ideal.

(ii) Two generators a_i and a_j generate the same minimal left (respectively right) ideal if and only if $i \equiv j \pmod{d_1}$ ($i \equiv j \pmod{d_2}$).

(iii) $S(r,n,k)$ is a completely simple semigroup with d_1 minimal left ideals and d_2 minimal right ideals.

In order to prove the theorem we need some technical lemmas. In all of them we assume the conditions given in the theorem.

Lemma 2.4. *For each i, $1 \le i \le n$, there exist words $\alpha_1(i)$ and $\alpha_2(i)$ such that the relations*

$$\alpha_1(i) \cdot a_i = a_{i+k}, \quad a_i \cdot \alpha_2(i) = a_{i+r+k-1} \tag{3}$$

hold in $S(r,n,k)$.

PROOF. It is enough to take

$$\alpha_1(i) \equiv a_{i-r+1} a_{i-r+2} \dots a_{i-1}, \; \alpha_2(i) \equiv a_{i+1} a_{i+2} \dots a_{i+r-1},$$

as both relations (3) then become defining relations of $S(r,n,k)$. ∎

Lemma 2.5. *For each i, $1 \le i \le n$, and for each $l \ge 1$, there exist words $\beta_1(i,l)$ and $\beta_2(i,l)$ such that the relations*

$$\beta_1(i,l) \cdot a_i = a_{i+ld_1}, \quad a_i \cdot \beta_2(i,l) = a_{i+ld_2}$$

hold in $S(r,n,k)$.

PROOF. Since $d_1 = \text{g.c.d.}(n,k)$ there exist $r, s \in \mathbb{N}$ such that

$$rk = ld_1 + sn.$$

Now let

$$\beta_1(i,l) \equiv \alpha_1(i + (r-1)k) \cdot \ldots \cdot \alpha_1(i+k) \cdot \alpha_1(i).$$

By Lemma 2.4 we have

$$\begin{aligned}
\beta_1(i,l) \cdot a_i &\equiv \alpha_1(i + (r-1)k) \cdot \ldots \cdot \alpha_1(i+k) \cdot \alpha_1(i) \cdot a_i \\
&= \alpha_1(i + (r-1)k) \cdot \ldots \cdot \alpha_1(i+k) \cdot a_{i+k} \\
&= \dots \\
&= \alpha_1(i + (r-1)k) \cdot a_{i+(r-1)k} \\
&= a_{i+rk} = a_{i+ld_1+sn} = a_{i+ld_1},
\end{aligned}$$

since the subscripts are reduced modulo n.

The construction for $\beta_2(i,l)$ is dual. ∎

Lemma 2.6. *For any i, j with $1 \leq i, j \leq n$, there exist words $\gamma_1(i,j)$ and $\gamma_2(i,j)$ such that the relations*

$$\gamma_1(i,j) \cdot a_i a_j = a_j, \; a_i a_j \cdot \gamma_2(i,j) = a_i$$

hold in $S(r,n,k)$.

PROOF. Since g.c.d.$(d_1, d_2) = 1$, there exist $s, t \in \mathbb{N}$ such that

$$i + sd_1 \equiv j + td_2 - 1 \pmod{n}. \tag{4}$$

Now, by Lemma 2.5, we have

$$a_{j+td_2} \cdot \beta_2(j + td_2, n - t) = a_{j+td_2+(n-t)d_2} = a_{j+nd_2} = a_j, \tag{5}$$

so that

$$a_i a_j = a_i a_{j+td_2} \beta_2(j + td_2, n - t), \tag{6}$$

and therefore

$$\beta_1(i,s) \cdot a_i a_j = a_{i+sd_1} a_{j+td_2} \cdot \beta_2(j+td_2, n-t) = a_{j+td_2-1} a_{j+td_2} \cdot \beta_2(j+td_2, n-t), \tag{7}$$

by (6), Lemma 2.5 and (4). Next note that

$$\begin{aligned} & a_{j+td_2-r+1} a_{j+td_2-r+2} \ldots a_{j+td_2-2} \cdot a_{j+td_2-1} a_{j+td_2} \beta_2(j + td_2, n - t) \\ = \; & a_{j+td_2+k} \beta_2(j + td_2, n - t) \end{aligned} \tag{8}$$

by a relation from $\mathfrak{P}(r,n,k)$, and also that

$$\begin{aligned} & \beta_1(j + td_2 + k, n - \frac{k}{d_1}) \cdot a_{j+td_2+k} \cdot \beta_2(j + td_2, n - t) \\ = \; & a_{j+td_2+k+nd_1-k} \cdot \beta_2(j + td_2, n - t) = a_{j+td_2} \cdot \beta_2(j + td_2, n - t) = a_j \end{aligned} \tag{9}$$

by Lemma 2.5 and (5). From (4)-(9) it follows that for

$$\gamma_1(i,j) \equiv \beta_1(j + td_2 + k, n - \frac{k}{d_1}) \cdot a_{j+td_2-r+1} \cdot \ldots \cdot a_{j+td_2-2} \cdot \beta_1(i,s)$$

we have

$$\gamma_1(i,j) \cdot a_i a_j = a_j,$$

exactly as required.

The construction for $\gamma_2(i,j)$ is dual. ■

PROOF OF THEOREM 2.3. We prove all statements for left ideals; the proofs for right ideals are dual.

(i) In order to prove that a_i generates a minimal left ideal it is enough to prove that for any word w_1 there exists a word w_2 such that the relation

$$w_2 w_1 a_i = a_i$$

holds in $S(r,n,k)$; see Proposition 9.3.1. This, however, follows easily from Lemma 2.6 by induction on the length of w_1.

(ii) Let

$$i \equiv j \pmod{d_1}.$$

Without loss of generality we may assume that $i < j$, so that

$$i + rd_1 = j$$

for some $r \in \mathbb{N}$. By Lemma 2.5 we have

$$\begin{aligned} \beta_1(i,r) \cdot a_i &= a_{i+rd_1} = a_j, \\ \beta_1(j,n-r) \cdot a_j &= a_{j+nd_1-rd_1} = a_{j-rd_1} = a_i, \end{aligned}$$

and we conclude that a_i and a_j generate the same (minimal) left ideal.

For the converse assume that a_i and a_j generate the same (minimal) left ideal, so that

$$a_i = w_1 a_j, \ a_j = w_2 a_i, \tag{10}$$

for some words w_1 and w_2. Note that the subscript of the last letter modulo d_1 is an invariant of $\mathfrak{P}(r,n,k)$, in the sense that

$$w_3 a_{i_1} = w_4 a_{i_2} \Longrightarrow i_1 \equiv i_2 \pmod{d_1}.$$

In particular, from (10), we have

$$i \equiv j \pmod{d_1},$$

as required.

(iii) This part of the lemma is a direct consequence of parts (i) and (ii). ∎

3. The structure of Fibonacci semigroups

The results of this section are easy consequences of the main result of the previous section and Corollary 9.3.4. However, we include them in a separate section, mainly because we consider them as 'nice', but also because they are the earliest of all the results presented in this thesis.

Let us recall that the Fibonacci semigroup $S(r,n)$ is defined by the presentation

$$\mathfrak{P}(r,n) = \langle a_1, \ldots, a_n \mid a_i a_{i+1} \ldots a_{i+r-1} = a_{i+r} \ \ (i = 1, \ldots, n) \rangle,$$

and that the group defined by $\mathfrak{P}(r,n)$ is denoted by $F(r,n)$.

Theorem 3.1. *Let $r, n \in \mathbb{N}$, $r > 1$ be arbitrary, and let $d_2 = \text{g.c.d.}(n, r)$. The Fibonacci semigroup $S(r, n)$ is a completely simple semigroup with one minimal left ideal ($S(r, n)$ itself) and d_2 minimal right ideals, each of which is a group isomorphic to the Fibonacci group $F(r, n)$. In particular, $S(r, n)$ is a disjoint union of d_2 copies of $F(r, n)$.*

PROOF. Note that $\mathfrak{P}(r, n) = \mathfrak{P}(r, n, 1)$, and since

$$d = \text{g.c.d.}(n, 1, r) = 1,$$

$S(r, n, k)$ is a completely simple semigroup by Theorem 2.3. By the same theorem, the number of minimal left ideals is $d_1 = \text{g.c.d.}(n, 1) = 1$, while the number of minimal right ideals is d_2. Finally, each of these minimal right ideals is a group isomorphic to $F(r, n)$ by Corollary 9.2.4. ∎

An immediate corollary of this theorem is the reduction of the finiteness problem for Fibonacci semigroups to the same problem for Fibonacci groups.

Corollary 3.2. *A Fibonacci semigroup $S(r, n)$ is finite if and only if the corresponding Fibonacci group $F(r, n)$ is finite.*

PROOF. If $r > 1$, the corollary follows from Theorem 3.1. For $r = 1$, $S(r, n, k)$ and $F(r, n, k)$ are the free monogenic semigroup and free cyclic group respectively, and are both infinite. ∎

Similarly, we have

Corollary 3.3. *A Fibonacci semigroup $S(r, n)$ has a soluble word problem if and only if the corresponding Fibonacci group $F(r, n)$ has a soluble word problem.* ∎

Here we have a surprising connection with Adian's results mentioned in Section 1 of Chapter 9. There we had conditions for a semigroup to be embeddable into the group with the same presentation, thus reducing the word problem for the semigroup to the word problem for the group. Here, again, we have the word problem for a semigroup reduced to a word problem for the corresponding group, but for the opposite reason: the semigroup consists of a finite number of copies of the group.

Another connection with Adian's results is that Theorem 3.1 generalises to presentations whose Adian graphs (see Section 1 of Chapter 9) satisfy certain technical conditions.

Theorem 3.4. *Let*

$$\mathfrak{P} = \langle a_1, \ldots, a_n \mid a_1 = \alpha_1, \ldots, a_n = \alpha_n \rangle$$

be a presentation in which each α_i is a word of length at least two and each a_i occurs as the first, second and last letter of three (not necessarily distinct) α_j.

Assume further that the right Adian graph of $\mathfrak{P}$ is connected, while the left Adian graph has d components. Then the semigroup S defined by $\mathfrak{P}$ is a completely simple semigroup with a unique minimal left ideal and d minimal right ideals, each of which is a group isomorphic to the group defined by $\mathfrak{P}$. In particular, S is finite if and only if G is finite, and S has a soluble word problem if and only if G has a soluble word problem. ∎

4. Generalised Fibonacci semigroups

Now we return to investigating the generalised Fibonacci semigroups $S(r, n, k)$, i.e. the semigroups defined by the presentations

$$\mathfrak{P}(r, n, k) = \langle a_1, \ldots, a_n \mid a_i a_{i+1} \ldots a_{i+r-1} = a_{i+r+k-1} \ \ (i = 1, \ldots, n) \rangle.$$

As before we let

$$d_1 = \text{g.c.d.}(n, k),\ d_2 = \text{g.c.d.}(n, r + k - 1).$$

We already know that if $r = 1$ or $\text{g.c.d.}(n, k, r + k - 1) > 1$ the semigroup $S(r, n, k)$ is infinite and does not have any minimal ideals; see Theorems 2.1 and 2.2. Hence, in this section we concentrate on the case where $r > 1$ and

$$\text{g.c.d.}(n, k, r + k - 1) = \text{g.c.d.}(d_1, d_2) = 1.$$

In this case we know that $S(r, n, k)$ is a completely simple semigroup with d_1 minimal left ideals and d_2 minimal right ideals; see Theorem 2.3. In particular, $S(r, n, k)$ is a disjoint union of $d_1 d_2$ copies of its Schützenberger group $H(r, n, k)$. Here we apply the general Theory from Chapters 9 and 10 to find the group $H(r, n, k)$.

The case $d_1 = 1$ or $d_2 = 1$ is easy.

Theorem 4.1. *Let $r, n, k \in \mathbb{N}$, $r > 1$, be arbitrary, and let $d_1 = \text{g.c.d.}(n, k)$ and $d_2 = \text{g.c.d.}(n, r + k - 1)$. If $d_1 = 1$ (respectively $d_2 = 1$) then the generalised Fibonacci semigroup $S(r, n, k)$ has a unique minimal left (right) ideal and d_2 (d_1) minimal right (left) ideals, each of which is isomorphic to the generalised Fibonacci group $F(r, n, k)$. In particular, $S(r, n, k)$ is finite if and only if $F(r, n, k)$ is finite.*

PROOF. The proof is exactly the same as the proof of Theorem 3.1. ∎

In the rest of this section we consider the case $d_1 > 1$ and $d_2 > 1$. First we give an example which shows that the Schützenberger group $H(r, n, k)$ of $S(r, n, k)$ may be not too closely related to $F(r, n, k)$.

Example 4.2. Let us consider the semigroup $S(2,6,2)$. It is defined by the presentation

$$\mathfrak{P}(2,6,2) = \langle a_1, a_2, a_3, a_4, a_5, a_6 \mid a_1a_2 = a_4,\ a_2a_3 = a_5,\ a_3a_4 = a_6,\\ a_4a_5 = a_1,\ a_5a_6 = a_2,\ a_6a_1 = a_3 \rangle.$$

Note that here $d_1 = 2$ and $d_2 = 3$. Also note that the group $F(2,6,2)$ is easily seen to be cyclic of order 7.

By Theorem 2.3, $S(2,6,2)$ is a completely simple semigroup with two minimal left ideals and three minimal right ideals (see Figure 7).

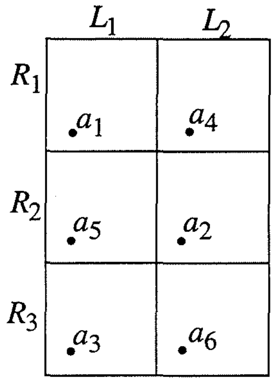


Figure 7.

We are going to apply Theorem 10.3.2 in order to find a presentation for the Schützenberger group $H(2,6,2)$ of $S(2,6,2)$.

Since $S(2,6,2)$ has six generators and two minimal left ideals, the obtained presentation will have twelve generators which we denote by $b_{\lambda,a_i} = b_{\lambda,i}$, $\lambda = 1,2$, $i = 1,\dots,6$. By Theorem 2.3 we have that the action of $S(2,6,2)$ on its minimal left ideals L_λ, $\lambda = 1,2$, has the following simple form:

	a_1	a_2	a_3	a_4	a_5	a_6
1	1	2	1	2	1	2
2	1	2	1	2	1	2

Now we rewrite the defining relations for $S(2,6,2)$ using the same method as in Section 4 of Chapter 10.

	a_1	a_2	$=$		a_4
1	b_{11}	b_{12}	$=$	1	b_{14}
2	b_{21}	b_{12}	$=$	2	b_{24}

	a_2	a_3	$=$		a_5
1	b_{12}	b_{23}	$=$	1	b_{15}
2	b_{22}	b_{23}	$=$	2	b_{25}

	a_3	a_4	$=$		a_6
1	b_{13}	b_{14}	$=$	1	b_{16}
2	b_{23}	b_{14}	$=$	2	b_{26}

	a_4	a_5	$=$		a_1
1	b_{14}	b_{25}	$=$	1	b_{11}
2	b_{24}	b_{25}	$=$	2	b_{21}

	a_5	a_6	$=$		a_2
1	b_{15}	b_{16}	$=$	1	b_{12}
2	b_{25}	b_{16}	$=$	2	b_{22}

	a_6	a_1	$=$		a_3
1	b_{16}	b_{21}	$=$	1	b_{13}
2	b_{26}	b_{21}	$=$	2	b_{23}

Next we note that

$$\begin{aligned} a_4a_1a_4a_1 &= a_4a_4a_5a_4a_1 = a_4a_4a_2a_3a_4a_1 = a_4a_4a_2a_6a_1 \\ &= a_4a_4a_2a_3 = a_4a_4a_5 = a_4a_1, \end{aligned}$$

so that a_4a_1 represents an idempotent of $S(2,6,2)$. We now rewrite this idempotent:

	a_4	a_1		
1	b_{14}	b_{21}	$=$	1
2	b_{24}	b_{21}	$=$	1

By Theorem 10.3.2 the presentation

$$\begin{gathered} \langle b_{11}, b_{12}, b_{13}, b_{14}, b_{15}, b_{16}, b_{21}, b_{22}, b_{23}, b_{24}, b_{25}, b_{26} \mid b_{11}b_{12} = b_{14},\ b_{21}b_{12} = b_{24}, \\ b_{12}b_{23} = b_{15},\ b_{22}b_{23} = b_{25},\ b_{13}b_{14} = b_{16},\ b_{23}b_{14} = b_{26},\ b_{14}b_{25} = b_{11},\ b_{24}b_{25} = b_{21}, \\ b_{15}b_{16} = b_{12},\ b_{25}b_{16} = b_{22},\ b_{16}b_{21} = b_{13},\ b_{26}b_{21} = b_{23},\ b_{14}b_{21} = 1,\ b_{24}b_{21} = 1\rangle \end{gathered}$$

is a monoid presentation for $H(2,6,2)$, and, again, we consider it as a group presentation for $H(2,6,2)$.

Let us first eliminate $b_{15}, b_{16}, b_{25}, b_{26}$ by using

$$b_{15} = b_{12}b_{23},\ b_{25} = b_{22}b_{23},\ b_{16} = b_{13}b_{14},\ b_{26} = b_{23}b_{14}, \tag{11}$$

so that we obtain the presentation

$$\begin{gathered} \langle b_{11}, b_{12}, b_{13}, b_{14}, b_{21}, b_{22}, b_{23}, b_{24} \mid b_{11}b_{12} = b_{14},\ b_{21}b_{12} = b_{24},\ b_{14}b_{22}b_{23} = b_{11}, \\ b_{24}b_{22}b_{23} = b_{21},\ b_{23}b_{13}b_{14} = 1,\ b_{14}b_{21} = 1,\ b_{24}b_{21} = 1\rangle. \end{gathered}$$

Next we eliminate b_{14} and b_{24} by

$$b_{14} = b_{24} = b_{21}^{-1}, \tag{12}$$

and we obtain

$$\begin{gathered} \langle b_{11}, b_{12}, b_{13}, b_{21}, b_{22}, b_{23} \mid b_{11}b_{12}b_{21} = 1,\ b_{21}b_{12}b_{21} = 1, \\ b_{22}b_{23} = b_{21}b_{11},\ b_{22}b_{23} = b_{21}^2,\ b_{23}b_{13} = b_{21}\rangle. \end{gathered}$$

Now we can eliminate b_{21} by

$$b_{21} = b_{11}, \tag{13}$$

and obtain

$$\langle b_{11}, b_{12}, b_{13}, b_{22}b_{23} \mid b_{11}b_{12}b_{11} = 1,\ b_{22}b_{23} = b_{11}^2,\ b_{23}b_{13} = b_{11}\rangle.$$

The remaining three relations can be used to eliminate b_{23}, b_{13} and b_{12} as

$$b_{23} = b_{22}^{-1}b_{11}^{2},\ b_{13} = b_{23}^{-1}b_{11} = b_{11}^{-2}b_{22}b_{11},\ b_{12} = b_{11}^{-2}, \tag{14}$$

leaving the presentation

$$\langle x, y \mid \ \rangle,$$

where

$$x = b_{11},\ y = b_{22}. \tag{15}$$

Therefore, the Schützenberger group $H(2,6,2)$ of $S(2,6,2)$ is isomorphic to the free group of rank 2, and $S(2,6,2)$ is a disjoint union of six copies of this group.

It is possible to use Corollary 10.2.5 to obtain a representation of $S(2,6,2)$ as a Rees matrix semigroup over the free group of rank two. One choice of representatives is

$$\overline{x}_1 \equiv a_1,\ \overline{x}_2 \equiv a_4,\ \overline{y}_1 \equiv a_1,\ \overline{y}_2 \equiv a_5,\ \overline{y}_3 \equiv a_3.$$

The entries of a Rees matrix $P = (p_{\lambda i})_{1\leq\lambda\leq 2,\,1\leq i\leq 3}$ are obtained by rewriting the words $\overline{x}_\lambda\overline{y}_i$ and writing the result in terms of x and y, by using (11)-(15). The obtained matrix is

$$P = \begin{pmatrix} x^2 & x^{-1}y^{-1}x^2 & x^{-1}yx \\ 1 & x & x^{-1}y^{-1}x^2 \end{pmatrix},$$

and $S(2,6,2)$ is isomorphic to the Rees matrix semigroup

$$\mathcal{M}[FG_2; \{1,2,3\},\ \{1,2\}; P],$$

where FG_2 is the free group on x and y. ■

Let us now return to the general case where we have $r, n, k \in \mathbb{N}$, $r > 1$, $d_1 > 1$, $d_2 > 1$ and $d = 1$. By Theorem 2.3, $S(r,n,k)$ has d_1 minimal left ideals; we denote them by L_λ, $\lambda \in \Lambda = \{1, \ldots, d_1\}$, so that

$$a_i \in L_\lambda \iff i \equiv \lambda \pmod{d_1}.$$

Similarly as in Example 4.2, the action of $S(r,n,k)$ on its minimal left ideals is given by

$$\lambda a_i \equiv i \pmod{d_1}.$$

Along the lines of Section 1 of Chapter 10, we introduce a new alphabet

$$B = \{b_{\lambda i} \mid \lambda \in \Lambda,\ 1 \leq i \leq n\},$$

where $b_{\lambda,i}$ stands for b_{λ,a_i}. The rewriting mapping $\overline{\phi}$ is defined by

$$\begin{aligned}(\lambda, a_i)\overline{\phi} &= b_{\lambda,i},\\ (\lambda, a_i w)\overline{\phi} &= b_{\lambda,i} \cdot (\lambda a_i, w)\overline{\phi},\end{aligned}$$

where $1 \leq i \leq n$, $\lambda \in \Lambda$, $w \in \{a_1, \dots, a_n\}^+$. If we introduce the following convention

$$\begin{aligned}b_{\lambda,i} &= b_{\lambda \text{ (mod } d_1), \, i \text{ (mod } n)},\\ (\lambda, w)\overline{\phi} &= (\lambda \text{ (mod } d_1), w)\overline{\phi},\end{aligned}$$

where $\lambda, i \in \mathbb{N}$, the definition of the rewriting mapping $\overline{\phi}$ becomes

$$\begin{aligned}(\lambda, a_i)\overline{\phi} &= b_{\lambda,i},\\ (\lambda, a_i w)\overline{\phi} &= b_{\lambda,i} \cdot (i, w)\overline{\phi}.\end{aligned} \tag{16}$$

Also we have

$$(\lambda, w_1 w_2)\overline{\phi} \equiv (\lambda, w_1)\overline{\phi} \cdot (\lambda w_1, w_2)\overline{\phi}; \tag{17}$$

see Lemma 10.2.8.

Let $H = H(r, n, k)$ denote the Schützenberger group of $S(r, n, k)$. By Theorem 2.3, $S(r, n, k)$ is a disjoint union of $d_1 d_2$ copies of H. By Theorem 10.3.2, the presentation

$$\begin{aligned}\mathfrak{T} = \langle B \mid (\lambda, a_i a_{i+1} \dots a_{i+r-1})\overline{\phi} = (\lambda, a_{i+r+k-1})\overline{\phi},\ (\lambda, e)\overline{\phi} = 1\\ (\lambda \in \Lambda,\ i = 1, \dots, n)\rangle,\end{aligned}$$

where e is a word representing an idempotent of $S(r, n, k)$, defines H as a monoid and also as a group. Note that $\mathfrak{T}$ has nd_1 generators and $nd_1 + d_1$ relations.

The author has not succeeded in his attempt to transform the presentation $\mathfrak{T}$ into a presentation from which it would be possible to describe H. Therefore we pose the following

Open Problem 12. Describe the group defined by the presentation $\mathfrak{T}$. For which $r, n, k \in \mathbb{N}$ is this group:

(a) free?
(b) isomorphic to $F(r, n, k)$?

Nonetheless, it is possible to reduce the number of relations in $\mathfrak{T}$ enough to prove that H is always infinite.

Theorem 4.3. *If $r, n, k \in \mathbb{N}$, with $r > 1$, $d_1 > 1$, $d_2 > 1$ and $d = 1$, then the Schützenberger group of $S(r, n, k)$ is infinite.*

PROOF. First we introduce some notation:

$$s = \frac{n}{d_2}, \tag{18}$$

$$u_i \equiv a_{i+1}a_{i+2}\dots a_{i+r-1},\ i = 1, \dots, n. \tag{19}$$

With this notation a general Fibonacci relation $a_i a_{i+1} \dots a_{i+r-1} = a_{i+r+k-1}$ becomes

$$a_i u_i = a_{i+r+k-1}. \tag{20}$$

With (16) in mind, relations

$$(\lambda, a_i a_{i+1} \dots a_{i+r-1})\overline{\phi} = (\lambda, a_{i+r+k-1})\overline{\phi},$$

become

$$b_{\lambda,i} \cdot (i, u_i)\overline{\phi} = b_{\lambda,i+r+k-1}, \tag{21}$$

$\lambda \in \Lambda$, $i = 1, \dots, n$.

For $i \in \mathbb{N}$, $1 \le i \le n$, let us consider the following subset of relations (21):

$$\begin{gathered} b_{\lambda,i+l(r+k-1)} \cdot (i + l(r+k-1), u_{i+l(r+k-1)})\overline{\phi} = b_{\lambda,i+(l+1)(r+k-1)} \\ (\lambda \in \Lambda,\ l = 0, \dots, s-1). \end{gathered} \tag{22}$$

Since $d_2 = \text{g.c.d.}(n, r+k-1)$, it is easy to see that there are exactly d_2 such subsets. We split relations (22) into two further groups:

$$\begin{gathered} b_{\lambda,i+l(r+k-1)} \cdot (i + l(r+k-1), u_{i+l(r+k-1)})\overline{\phi} = b_{\lambda,i+(l+1)(r+k-1)} \\ (\lambda \in \Lambda,\ l = 0, \dots, s-2), \end{gathered} \tag{23}$$

and

$$\begin{gathered} b_{\lambda,i+(s-1)(r+k-1)} \cdot (i + (s-1)(r+k-1), u_{i+(s-1)(r+k-1)})\overline{\phi} = b_{\lambda,i+s(r+k-1)} \\ (\lambda \in \Lambda). \end{gathered} \tag{24}$$

Since $s = n/d_2$, and since $r+k-1$ is divisible by d_2, relations (24) can be written as

$$b_{\lambda,i+(s-1)(r+k-1)} \cdot (i + (s-1)(r+k-1), u_{i+(s-1)(r+k-1)})\overline{\phi} = b_{\lambda,i}\ (\lambda \in \Lambda). \tag{25}$$

We are now going to show that in the set of relations (22) relations (25) can be replaced by a single relation

$$(i, u_i u_{i+(r+k-1)} \dots u_{i+(s-1)(r+k-1)})\overline{\phi} = 1. \tag{26}$$

Note that the last letter of u_i is a_{i+r-1}, and also note that

$$r - 1 \equiv r + k - 1 \pmod{d_1},$$

since $d_1 = \text{g.c.d.}(n,k)$, so that (26) can be written as

$$(i,u_i)\overline{\phi}\cdot(i+r+k-1,u_{i+r+k-1})\overline{\phi}\cdot\ldots\cdot(i+(s-1)(r+k-1),u_{i+(s-1)(r+k-1)})\overline{\phi}=1. \quad (27)$$

First we have

$$\begin{aligned} & b_{\lambda,i}\cdot(i,u_i)\overline{\phi}\cdot(i+r+k-1,u_{i+r+k-1})\overline{\phi}\cdot\ldots \\ & \cdot(i+(s-1)(r+k-1),u_{i+(s-1)(r+k-1)})\overline{\phi} \\ = \; & b_{\lambda,i+r+k-1}(i+r+k-1,u_{i+r+k-1})\overline{\phi}\cdot\ldots \\ & \cdot(i+(s-1)(r+k-1),u_{i+(s-1)(r+k-1)})\overline{\phi} \quad \text{(by (23))} \\ = \; & \ldots \\ = \; & b_{\lambda,i+(s-1)(r+k-1)}\cdot(i+(s-1)(r+k-1),u_{i+(s-1)(r+k-1)})\overline{\phi} \\ = \; & b_{\lambda,i}, \qquad\qquad\qquad\qquad \text{(by (25))} \end{aligned}$$

and hence (27) is a consequence of (23) and (25), since we are dealing with a group presentation. Now we prove that (27) and (23) imply (25). First note that, by (23),

$$b_{\lambda,i+(s-1)(r+k-1)} = b_{\lambda,i+(s-2)(r+k-1)}\cdot(i+(s-2)(r+k-1),u_{i+(s-2)(r+k-1)})\overline{\phi}.$$

By iterating this we obtain

$$b_{\lambda,i+(s-1)(r+k-1)} = b_{\lambda,i}\cdot(i,u_i)\overline{\phi}\cdot\ldots\cdot(i+(s-2)(r+k-1),u_{i+(s-2)(r+k-1)})\overline{\phi}. \quad (28)$$

On the other hand,

$$\begin{aligned} & (i,u_i)\overline{\phi}\cdot\ldots\cdot(i+(s-2)(r+k-1),u_{i+(s-2)(r+k-1)})\overline{\phi} \\ = \; & [(i+(s-1)(r+k-1),u_{i+(s-1)(r+k-1)})\overline{\phi}]^{-1} \end{aligned} \quad (29)$$

by (27), so that from (28) and (29) we obtain

$$b_{\lambda,i+(s-1)(r+k-1)}\cdot(i+(s-1)(r+k-1),u_{i+(s-1)(r+k-1)})\overline{\phi} = b_{\lambda,i},$$

as required.

To summarise what we have done, a set of d_1 relations (24) can be replaced by a single relation. There are d_2 such sets, and after each has been replaced by a single relation, $\mathfrak{T}$ becomes a presentation on nd_1 generators and $nd_1+d_1+d_2-d_1d_2$ relations. Since $d_1 > 1$, $d_2 > 1$ and $d = \text{g.c.d.}(d_1,d_2) = 1$, we conclude that at least one of d_1 and d_2 is greater than 2, so that $d_1d_2 > d_1+d_2$. In other words, H can be presented by a presentation having fewer relations than generators, and hence H is infinite. ■

Remark 4.4. In Campbell, Robertson, Ruškuc and Thomas (1995a) the authors prove the above result by the same method, but they use the rewriting defined at the end of Section 3 in Chapter 10. In this way they obtain a presentation on $n+(d_1-1)(d_2-1)$ generators and n relations

$$(a_ia_{i+1}\ldots a_{i+r-1})\phi_1 = (a_{i+r+k-1})\phi_1.$$

This is certainly a nicer presentation than the one we presented here. This adds some evidence to our claim that the two rewriting theorems are not comparable in strength.

We end this chapter by summarising all the information about generalised Fibonacci semigroups we obtained.

Theorem 4.5. *Let $r, n, k \in \mathbb{N}$, and let $d_1 = \text{g.c.d.}(n, k)$, $d_2 = \text{g.c.d.}(n, r+k-1)$, $d = \text{g.c.d.}(n, k, r+k-1)$.*

(i) If $r = 1$ then $S(r, n, k)$ is the free semigroup of rank d_1.

(ii) If $r > 1$ and $d > 1$ then $S(r, n, k)$ does not have any minimal (left, right or two-sided) ideals.

(iii) If $r > 1$ and $d_1 = 1$ then $S(r, n, k)$ is a completely simple semigroup with a unique minimal left ideal and d_2 minimal right ideals, each of which is a group isomorphic to the generalised Fibonacci group $F(r, n, k)$.

(iv) If $r > 1$ and $d_2 = 1$ then $S(r, n, k)$ is a completely simple semigroup with a unique minimal right ideal and d_1 minimal left ideals, each of which is isomorphic to the generalised Fibonacci group $F(r, n, k)$.

(iv) If $r > 1$, $d_1 > 1$, $d_2 > 1$ and $d = 1$ then $S(r, n, k)$ is a completely simple semigroup with d_1 minimal left ideals and d_2 minimal right ideals, and is a union of $d_1 d_2$ copies of an infinite group.

(v) If $d_1 = 1$ or $d_2 = 1$, then $S(r, n, k)$ is finite if and only if the corresponding group $F(r, n, k)$ is finite; otherwise $S(r, n, k)$ is infinite regardless of whether $F(r, n, k)$ is finite or infinite. ∎

Chapter 10

Semigroups defined by Coxeter type presentations

In this chapter we investigate the structure of semigroups defined by a semigroup variant of Coxeter type presentations for groups. The results of this chapter will appear in Campbell, Robertson, Ruškuc and Thomas (1995c).

1. Coxeter graphs and presentations

Let Γ be a digraph with a finite set of vertices

$$A = A(\Gamma) = \{a_i \mid i \in I\},$$

which satisfies the following three conditions:

(Γ1) Γ has no directed circuits;

(Γ2) every vertex a_i of Γ is labelled by a natural number $p_i > 0$;

(Γ3) every edge (a_i, a_j) is labelled by a natural number $p_{ij} > 0$.

(Note that condition (Γ1) implies that for any $a_i, a_j \in A$ at least one of (a_i, a_j) or (a_j, a_i) is not an edge in Γ.)

With every such graph we associate a presentation

$$\mathfrak{P}(\Gamma) = \langle A \mid \mathfrak{R} \rangle, \tag{1}$$

where

$$\mathfrak{R} = \{a_i^{p_i+1} = a_i \mid i \in I\} \cup \{\alpha(i,j) \mid i,j \in I\},$$

with

$$\alpha(i,j) = \begin{cases} (a_i a_j)^{p_{ij}} = a_i^{p_i} & \text{if } (a_i, a_j) \text{ is an edge} \\ (a_j a_i)^{p_{ji}} = a_j^{p_j} & \text{if } (a_j, a_i) \text{ is an edge} \\ a_i a_j = a_j a_i & \text{otherwise.} \end{cases}$$

If $I = \{1, \ldots, n\}$, $p_i = 2$ for all $i \in I$, and if we consider $\mathfrak{P}(\Gamma)$ as a group presentation, then it is equivalent to the presentation

$$\langle a_1, \ldots, a_n \mid a_i^2 = (a_j a_k)^{q_{jk}} = 1 \ (1 \leq i \leq n,\ 1 \leq j < k \leq n) \rangle,$$

where

$$q_{ij} = \begin{cases} p_{ij} & \text{if } (a_i, a_j) \text{ is an edge} \\ p_{ji} & \text{if } (a_j, a_i) \text{ is an edge} \\ 2 & \text{otherwise.} \end{cases}$$

These presentations are usually called *Coxeter presentations*, and the groups defined are called *Coxeter groups*. They were first introduced by Coxeter (1936), and it soon turned out that many important classes of groups can be presented as Coxeter groups, with possibly a few additional relations.

Here we will be interested in semigroups $S(\Gamma)$ defined by (1). The semigroup defined by (1) for a particular graph Γ will be denoted by $S(\Gamma)$, while the corresponding group will be denoted by $G(\Gamma)$. We often refer to $G(\Gamma)$ as a *Coxeter type group*.

After introducing some more definitions and examples in this section, in Section 2 we go on to prove that $S(\Gamma)$ has a minimal two-sided ideal which is a disjoint union of copies of $G(\Gamma)$. In Section 3 we prove that $S(\Gamma)$ is a union of groups and find a necessary and sufficient condition for $S(\Gamma)$ to be finite. Finally, in Sections 4 and 5 we consider two classes of presentations (1) which, when considered as group presentations, define alternating and symmetric groups respectively. We determine both the structure and the orders of the semigroups defined by these presentations.

First, however, we give some examples of graphs and associated presentations.

Example 1.1. Let $I = \{1, \ldots, n\}$, and let Γ be the empty graph on $A = \{a_1, \ldots, a_n\}$. The associated presentation is

$$\mathfrak{P}(\Gamma) = \langle a_1, \ldots, a_n \mid a_i^{p_i+1} = a_i,\ a_j a_k = a_k a_j \ \ (1 \leq i, j, k \leq n) \rangle.$$

The semigroup $S(\Gamma)$ and the group $G(\Gamma)$ are both finite and abelian. Actually, $S(\Gamma)$ has a minimal two-sided ideal which is isomorphic to $G(\Gamma)$; see Campbell, Robertson, Ruškuc and Thomas (1995c). ■

Example 1.2. Let $I = \{1, \ldots, n\}$, and let $\Gamma = \Gamma_n^S$ be the oriented simple path on $a_1, \ldots, a_n$; in other words, the set of edges of Γ is

$$\{(a_1, a_2), (a_2, a_3), \ldots, (a_{n-1}, a_n)\}.$$

Let the labels of Γ be

$$p_i = 2,\ p_{j,j+1} = 3 \ \ (1 \leq i \leq n,\ 1 \leq j < n).$$

The presentation $\mathfrak{P}(\Gamma)$ is

$$\mathfrak{P}(\Gamma) = \langle a_1, \dots, a_n \mid a_i^3 = a_i,\ (a_j a_{j+1})^3 = a_j^2,\ a_k a_l = a_l a_k \\ (1 \le i \le n,\ 1 \le j < n,\ 1 \le k < l - 1 < n) \rangle.$$

The group $G(\Gamma)$ defined by Γ is the symmetric group $\mathcal{S}_{n+1}$ of degree $n+1$; see Moore (1897) or Coxeter and Moser (1980). The structure of the semigroup $S(\Gamma)$ will be described in Section 5. ∎

Example 1.3. Let $I = \{1, \dots, n\}$, and let the graph $\Gamma = \Gamma_n^A$ have vertices $A = \{a_1, \dots, a_n\}$, edges

$$\{(a_i, a_j) \mid 1 \le i < j \le n\},$$

and labels

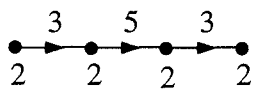
$$p_i = 3,\ p_{jk} = 2\ \ (1 \le i \le n,\ 1 \le j < k \le n).$$

The presentation $\mathfrak{P}(\Gamma)$ is

$$\mathfrak{P}(\Gamma) = \langle a_1, \dots, a_n \mid a_i^4 = a_i,\ (a_j a_k)^2 = a_j^3\ \ (1 \le i \le n,\ 1 \le j < k \le n) \rangle.$$

The group defined by this presentation is the alternating group $\mathcal{A}_{n+2}$ of degree $n+2$; see again Moore (1897) or Coxeter and Moser (1980). The semigroup defined by $\mathfrak{P}(\Gamma)$ will be described in Section 4. ∎

Example 1.4. Let $I = \{1, 2, 3, 4\}$, and let Γ be the following graph:



The group $G(\Gamma)$ is not finite; see Coxeter (1936). However, the homomorphic image of $G(\Gamma)$ obtained by adding the relation

$$(a_1 a_2 a_3)^5 = 1$$

is the finite simple group $\mathrm{PSL}(2, 11)$; see Soicher (1987). ∎

For any digraph Γ satisfying (Γ1), (Γ2), (Γ3), we denote by $A_1 = A_1(\Gamma)$ the set of initial vertices of Γ, i.e.

$$A_1(\Gamma) = \{a_i \in A \mid (a_j, a_i) \text{ is not an edge in } \Gamma \text{ for any } j \in I\}.$$

Since Γ has no directed circuits, and since $A(\Gamma)$ is finite, it follows that

$$A_1(\Gamma) \ne \emptyset.$$

Next we define recursively a sequence of sets $A_n = A_n(\Gamma)$, $n \in \mathbb{N}$, as follows:

$$A_{n+1}(\Gamma) = \{a_j \in A \mid (a_i, a_j) \text{ is an edge in } \Gamma \text{ for some } a_i \in A_n(\Gamma)\}.$$

Again, since Γ does not have directed circuits, there exists $n_0 \in \mathbb{N}$ such that

$$A_n(\Gamma) = \emptyset$$

for all $n \geq n_0$; we also have

$$A = \bigcup_{n \in \mathbb{N}} A_n.$$

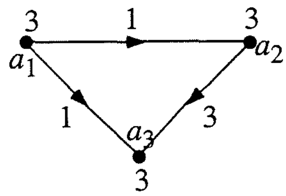
A graph is said to be *finitely related* if the group $G(\Gamma)$ is finite. It is obvious that all the graphs from Examples 1.1, 1.2 and 1.3 are finitely related, while the graph from Example 1.4 is not. We define *strongly finitely related* graphs recursively as follows:

(SFR1) every graph with one vertex is strongly finitely related;

(SFR2) a graph Γ with $n+1$ vertices is strongly finitely related if and only if it is finitely related and every subgraph $\Gamma - \{a_i\}$, $a_i \in A_1(\Gamma)$, of Γ is strongly finitely related.

The following example shows that there are finitely related graphs which are not strongly finitely related.

Example 1.5. Let Γ be the graph

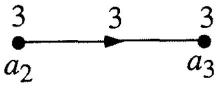


The corresponding presentation is

$$\mathfrak{P}(\Gamma) = \langle a_1, a_2, a_3 \mid a_1^4 = a_1,\ a_2^4 = a_2,\ a_3^4 = a_3,$$
$$a_1 a_2 = a_1^3,\ a_1 a_3 = a_1^3,\ (a_2 a_3)^3 = a_2^3\rangle.$$

The group $G(\Gamma)$ defined by $\mathfrak{P}(\Gamma)$ is obviously cyclic of order 3, and hence Γ is finitely related.

The set of initial vertices of Γ is $A_1(\Gamma) = \{a_1\}$. The graph $\Gamma - \{a_1\}$ is



and the corresponding presentation

$$\langle a_2, a_3 \mid a_2^4 = a_2,\ a_3^4 = a_3,\ (a_2a_3)^3 = a_2^3\rangle$$

defines (3,3,3) group, which is infinite; see Coxeter and Moser (1980). Thus $\Gamma - \{a_1\}$ is not finitely related, and Γ is not strongly finitely related. ∎

However, all the graphs from Examples 1.1, 1.2 and 1.3 are strongly finitely related.

Theorem 1.6. *Each of the following graphs is strongly finitely related:*

(i) every finitely related graph with two vertices;
(ii) every graph without edges;
(iii) Γ_n^S, *for all* $n \in \mathbb{N}$;
(iv) Γ_n^A, *for all* $n \in \mathbb{N}$.

PROOF. (i) and (ii) are obvious. For (iii) and (iv) note that

$$A_1(\Gamma_n^S) = A_1(\Gamma_n^A) = \{a_1\},$$

and that

$$\Gamma_n^S - \{a_1\} \cong \Gamma_{n-1}^S,\ \Gamma_n^A - \{a_1\} \cong \Gamma_{n-1}^A,$$

for all $n > 1$, and the assertions follow by induction. ∎

2. Minimal ideals of $S(\Gamma)$

Throughout this section we assume the notation introduced in the previous section. Thus Γ is a digraph satisfying (Γ1), (Γ2), (Γ3), $\mathfrak{P} = \mathfrak{P}(\Gamma)$ is the associated presentation defined by (1), $S = S(\Gamma)$ is the semigroup defined by $\mathfrak{P}(\Gamma)$, and that $G = G(\Gamma)$ is the group defined by $\mathfrak{P}(\Gamma)$. We also assume that the set of initial vertices of Γ is

$$A_1 = A_1(\Gamma) = \{a_{i_1}, a_{i_2}, \ldots, a_{i_t}\}.$$

Our aim is to prove, via a sequence of lemmas, that $S(\Gamma)$ possesses a minimal two-sided ideal which is a union of copies of $G(\Gamma)$. More precisely, we show that $S(\Gamma)$ has minimal right ideals and a unique minimal left ideal, and then we apply Corollary 9.2.4.

Lemma 2.1. *If* $a_i, a_j \in A_1$, *then the relation*

$$a_ia_j = a_ja_i$$

holds in S.

PROOF. By the definition of A_1, neither (a_i, a_j) nor (a_j, a_i) is an edge in Γ, and therefore the relation $\alpha(i,j)$ of $\mathfrak{P}(\Gamma)$ is $a_i a_j = a_j a_i$. ∎

Lemma 2.2. *For any word $w \in A_1^*$ there exists a word $w_1 \in A_1^+$ such that the relation*

$$w_1 w = w w_1 = a_{i_1} a_{i_2} \dots a_{i_t}$$

holds in S.

PROOF. The subsemigroup of S generated by A_1 is commutative by Lemma 2.1, and each generator $a_i \in A_1$ generates a cyclic subgroup of S of order p_i, so that the lemma follows. ∎

Lemma 2.3. *If (a_i, a_j) is an edge in Γ, then the relation*

$$(a_i a_j)^{p_{ij}+1} = a_i a_j$$

holds in S.

PROOF. Since (a_i, a_j) is an edge of Γ, the relation $\alpha(i,j)$ of $\mathfrak{P}$ is $(a_i a_j)^{p_{ij}} = a_i^{p_i}$, and we have

$$(a_i a_j)^{p_{ij}+1} = (a_i a_j)^{p_{ij}} a_i a_j = a_i^{p_i} a_i a_j = a_i a_j,$$

as required. ∎

Lemma 2.4. *If $a_i, a_j \in A$ are arbitrary then the relation*

$$a_i a_j a_i^{p_i} = a_i a_j$$

holds in S.

PROOF. If neither (a_i, a_j) nor (a_j, a_i) is an edge, then $\alpha(i,j)$ is $a_i a_j = a_j a_i$ and we have

$$a_i a_j a_i^{p_i} = a_j a_i a_i^{p_i} = a_j a_i = a_i a_j.$$

If (a_i, a_j) is an edge, in which case $\alpha(i,j)$ is $(a_i a_j)^{p_{ij}} = a_i^{p_i}$, by Lemma 2.3 we have

$$a_i a_j a_i^{p_i} = a_i a_j (a_i a_j)^{p_{ij}} = a_i a_j.$$

Finally, if (a_j, a_i) is an edge, so that $\alpha(i,j)$ is $(a_j a_i)^{p_{ji}} = a_j^{p_j}$, we have

$$\begin{aligned} a_i a_j a_i^{p_i} &= a_i a_j a_j^{p_j} a_i^{p_i} = a_i a_j (a_j a_i)^{p_{ji}} a_i^{p_i} = a_i a_j (a_j a_i)^{p_{ji}-1} a_j a_i a_i^{p_i} \\ &= a_i a_j (a_j a_i)^{p_{ji}} = a_i a_j a_j^{p_j} = a_i a_j, \end{aligned}$$

thus completing the proof. ∎

Lemma 2.5. *Let $a_i \in A$ be arbitrary, and let $w \in A^+$ be a word which contains letter a_i. Then the relation*

$$wa_i^{p_i} = w$$

holds in S.

PROOF. Since w contains a_i, it can be written as

$$w \equiv w_1 a_i a_{j_1} a_{j_2} \dots a_{j_s}.$$

By Lemma 2.4 we have

$$a_i a_{j_k} a_i^{p_i} = a_i a_{j_k} \quad (k = 1, \dots, s),$$

and hence

$$\begin{aligned} w &\equiv w_1 a_i a_{j_1} a_{j_2} \dots a_{j_s} = w_1 a_i a_{j_1} a_i^{p_i} a_{j_2} \dots a_{j_s} = w_1 a_i a_{j_1} a_i^{p_i} a_{j_2} a_i^{p_i} \dots a_{j_s} \\ &= \dots = w_1 a_i a_{j_1} a_i^{p_i} a_{j_2} a_i^{p_i} \dots a_{j_s} a_i^{p_i} = w_1 a_i a_{j_1} a_{j_2} \dots a_{j_s} a_i^{p_i}, \end{aligned}$$

thus proving the lemma. ■

Lemma 2.6. *For any $a_i, a_j \in A$ there exists a word $w \in A^+$ such that the relation*

$$wa_i = a_i a_j$$

holds in S.

PROOF. By Lemma 2.4 it is enough to take $w \equiv a_i a_j a_i^{p_i - 1}$. ■

Lemma 2.7. *For any $a_i \in A$ and any word $v \in A^+$ there exists a word $w \in A^+$ such that the relation*

$$wa_i = a_i v$$

holds in S.

PROOF. We prove the lemma by induction on the length of v, the case of length 1 being Lemma 2.6. Let $|v| > 1$, so that $v \equiv v_1 a_j$ for some $v_1 \in A^+$, $a_j \in A$. By the inductive hypothesis there exists a word $w_1 \in A^+$ such that the relation

$$w_1 a_i = a_i v_1 \tag{2}$$

holds in S, while by Lemma 2.6 we have

$$w_2 a_i = a_i a_j, \tag{3}$$

for some $w_2 \in A^+$. From (2) and (3), for $w \equiv w_1 w_2$, we obtain

$$wa_i \equiv w_1 w_2 a_i = w_1 a_i a_j = a_i v_1 a_j \equiv a_i v,$$

and hence the lemma. ■

Lemma 2.8. *For any two words $u, v \in A^+$ there exists a word $w \in A^+$ such that the relation*

$$wu = uv$$

holds in S.

PROOF. We prove the lemma by induction on the length of u, the case $|u| = 1$ being Lemma 2.7. If $|u| > 1$, then we can write $u \equiv u_1 a_i$, with $u_1 \in A^+$ and $a_i \in A$. By Lemma 2.7 there exists a word w_1 such that

$$w_1 a_i = a_i v. \tag{4}$$

By the inductive hypothesis there exists a word w such that

$$wu_1 = u_1 w_1. \tag{5}$$

From (4) and (5) we obtain

$$wu \equiv wu_1 a_i = u_1 w_1 a_i = u_1 a_i v \equiv uv,$$

which completes the proof. ∎

Let us recall that we have assumed that $A_1 = \{a_{i_1}, \ldots, a_{i_t}\}$.

Lemma 2.9. *For any word $u \in A^+$ there exists a word $v \in A^+$ such that the relation*

$$vu = a_{i_1} a_{i_2} \ldots a_{i_t}$$

holds in S.

PROOF. Consider the set $T \subseteq A^+$ of all words w such that the relation $w = vu$ holds in S for some $v \in A^+$. (Notice that we do not require that w be of the form vu, but only that it be equal in S to such a word.) Let w be a word from T with the minimal number of letters from $A - A_1$. We want to show that $w \in A_1^+$, and the lemma will then follow from Lemma 2.2.

Suppose that w contains a letter from $A - A_1$. Then w can be written as

$$w \equiv \beta a_j w_1,$$

where $\beta \in A_1^*$, $a_j \in A - A_1$, $w_1 \in A^*$. Since

$$A = \bigcup_{i \in \mathbb{N}} A_i,$$

we have $a_j \in A_k$ for some $k \in \mathbb{N}$, $k > 1$. This means that that there exists $a_{j_1} \in A_{k-1}$ such that (a_{j_1}, a_j) is an edge in Γ, and the relation $\alpha(i, j)$ is

$$(a_{j_1} a_j)^{p_{j_1 j}} = a_{j_1}^{p_{j_1}}.$$

By Lemma 2.8 there exists $\beta_1 \in A^+$ such that

$$\beta_1\beta = \beta a_{j_1}(a_{j_1}a_j)^{p_{j_1j}-1}a_{j_1},$$

and so we have

$$\beta_1 w \equiv \beta_1\beta a_j w_1 = \beta a_{j_1}(a_{j_1}a_j)^{p_{j_1j}-1}a_{j_1}a_j w_1 = \beta a_{j_1}(a_{j_1}a_j)^{p_{j_1j}}w_1 = \beta a_{j_1}w_1.$$

Continuing in this way we obtain words $\beta_1,\dots,\beta_{k-1} \in A^+$ such that

$$\beta_{k-1}\dots\beta_1 w = \beta a_{j_{k-1}}w_1,$$

where $a_{j_{k-1}} \in A_1$. The word $\beta a_{j_{k-1}}w_1$ obviously contains fewer letters from $A - A_1$ than $w \equiv \beta a_j w_1$ does. On the other hand if $w = vu$ then

$$\beta a_{j_{k-1}}w_1 = \beta_{k-1}\dots\beta_1 w = (\beta_{k-1}\dots\beta_1 v)u,$$

and hence $\beta a_{j_{k-1}}w_1 \in T$, which is a contradiction with the choice of w. ∎

Lemma 2.10. *The word $a_{i_1}\dots a_{i_t}$ generates a unique minimal left ideal in S.*

PROOF. The lemma follows directly from Lemma 2.9 and Proposition 9.3.1. ∎

Lemma 2.11. *If $w \in A^+$ is a word containing all the letters from A_1, and if $a_j \in A$ is an arbitrary letter, then there exists a word $u \in A^+$ such that the relation*

$$w = ua_j$$

holds in S.

PROOF. Since

$$A = \bigcup_{n\in\mathbb{N}} A_n,$$

there exists $n \in \mathbb{N}$ such that $a_j \in A_n$. We prove the lemma by induction on n.

If $a_j \in A_1$, then w contains a_j, and hence

$$w = wa_j^{p_j} \equiv wa_j^{p_j-1}\cdot a_j,$$

by Lemma 2.5. Suppose that the statement is true for all the letters from A_n, and let $a_j \in A_{n+1}$. Then there exists $a_k \in A_n$ such that (a_k, a_j) is an edge in Γ. By the inductive hypothesis there exists a word u_1 such that

$$w = u_1 a_k.$$

Now if we choose

$$u \equiv u_1 a_k (a_k a_j)^{p_{kj}-1} a_k,$$

we obtain

$$ua_j \equiv u_1 a_k (a_k a_j)^{p_{kj}} = u_1 a_k a_k^{p_k} = u_1 a_k = w,$$

as required. ∎

Lemma 2.12. *Let $w \in A^+$ be any word containing all the letters from A_1. Then for any word $w_1 \in A^+$ there exists a word $w_2 \in A^+$ such that the relation*

$$ww_1w_2 = w,$$

holds in S.

PROOF. We prove the lemma by induction on the length of w_1. First we consider the case $|w_1| = 1$ i.e. $w_1 \equiv a_j \in A$. If $a_j \in A_1$ then w contains a_j, so that

$$ww_1a_j^{p_j-1} \equiv wa_j^{p_j} = w,$$

by Lemma 2.5. If $a_j \notin A_1$ then there exists $a_k \in A$ such that (a_k, a_j) is an edge. By Lemma 2.11 we have

$$w = ua_k,$$

for some $u \in A^+$, so that

$$ww_1a_k(a_ja_k)^{p_{kj}-1} = ua_k(a_ja_k)^{p_{kj}} = u(a_ka_j)^{p_{kj}}a_k = ua_k^{p_k}a_k = ua_k = w.$$

If $|w_1| > 1$ then we can write $w_1 \equiv w_1'a_k$, with $w_1' \in A^+$ and $a_k \in A$. The word ww_1' contains all the letters from A_1, so that the above argument shows that there exists $w_2' \in A^+$ such that

$$ww_1'a_kw_2' = ww_1'. \tag{6}$$

By the inductive hypothesis there exists $w_2'' \in A^+$ such that

$$ww_1'w_2'' = w. \tag{7}$$

From (6) and (7) we obtain

$$ww_1w_2'w_2'' \equiv ww_1'a_kw_2'w_2'' = ww_1'w_2'' = w,$$

as required. ■

Lemma 2.13. *Any word $w \in A^+$ which contains all the letters from A_1 generates a minimal right ideal in S.*

PROOF. The lemma is an immediate consequence of Lemma 2.12 and Proposition 9.3.1. ■

By combining Lemmas 2.10 and 2.13 with Corollary 9.2.4 we obtain the following:

Theorem 2.14. *Let Γ be a digraph satisfying (Γ1), (Γ2), (Γ3), and let $\mathfrak{P}(\Gamma)$ be the associated presentation defined by (1). The semigroup $S(\Gamma)$ defined by $\mathfrak{P}(\Gamma)$ has a unique minimal left ideal (which is also a unique minimal two-sided ideal). This minimal left ideal is a disjoint union of minimal right ideals, each of which is a group isomorphic to the group $G(\Gamma)$ defined by $\mathfrak{P}(\Gamma)$.* ■

Example 2.15. The semigroup $S(\Gamma)$ defined by the graph Γ from Example 1.4 is infinite, since the group $G(\Gamma)$ is infinite. Nevertheless, by Theorem 2.14, $S(\Gamma)$ has a unique minimal left ideal, which is a disjoint union of minimal right ideals, each of which is a group isomorphic to $G(\Gamma)$. Let T be the homomorphic image of $S(\Gamma)$ obtained by adding the relation $(a_1a_2a_3)^5 = a_1^2$ to the presentation $\mathfrak{P}(\Gamma)$. Being a homomorphic image of $S(\Gamma)$, T has minimal right ideals and a unique minimal left ideal. Hence, by Corollary 9.2.4, the minimal left ideal of T is a disjoint union of copies of the group defined by the same presentation; in this case the group in question is $\mathrm{PSL}(2, 11)$. Computer evidence shows that T is a finite semigroup of order 133880 containing 201 copies of $\mathrm{PSL}(2, 11)$. ■

3. Structure and finiteness of $S(\Gamma)$

As in the previous section, Γ denotes a digraph satisfying (Γ1), (Γ2), (Γ3), $\mathfrak{P}(\Gamma)$ denotes the corresponding presentation (1), $S(\Gamma)$ denotes the semigroup defined by $\mathfrak{P}(\Gamma)$ and $G(\Gamma)$ denotes the group defined by $\mathfrak{P}(\Gamma)$. By Theorem 2.14, $S(\Gamma)$ has a unique minimal left ideal, which we denote by $L(\Gamma)$. This ideal is also a minimal two-sided ideal of $S(\Gamma)$, and is a union of copies of $G(\Gamma)$. Theorem 2.14, however, does not give any information as to the number of copies of $G(\Gamma)$ in $L(\Gamma)$. A natural way for further investigation of $S(\Gamma)$ is to consider the following two questions.

(i) How many copies of $G(\Gamma)$ does $L(\Gamma)$ contain?
(ii) What is the structure of the Rees quotient $S(\Gamma)/L(\Gamma)$?

In this section we prove some initial general results on the previous two questions, and then in the following two sections we look at two special cases, where we obtain the full description.

The following is the crucial technical result that we shall need.

Lemma 3.1. *Let $a_i \in A_1$, and let $u, v \in A^+$ be such that the relation $u = v$ holds in $S(\Gamma)$. Then u contains a_i if and only if v contains a_i.*

Proof. Since $a_i \in A_1$, every relation of $\mathfrak{P}(\Gamma)$ which contains a_i has one of the forms $a_i^{p_i+1} = a_i$ or $(a_ia_j)^{p_{ij}} = a_i^{p_i}$ or $a_ia_j = a_ja_i$, and hence contains a_i on both its sides. The result now follows from the fact that $S(\Gamma)$ is defined by $\mathfrak{P}$. ■

As before we assume that $A_1 = \{a_{i_1}, \ldots, a_{i_t}\}$.

Lemma 3.2. *A word $w \in A^+$ represents an element of the minimal left ideal $L(\Gamma)$ (or, equivalently, an element of a minimal right ideal) if and only if w contains all the letters of A_1.*

PROOF. First note that the assertion for $L(\Gamma)$ and the assertion for minimal right ideals are indeed equivalent, since $L(\Gamma)$ is a disjoint union of all minimal right ideals of $S(\Gamma)$.

Let us now assume that w represents an element of $L(\Gamma)$. Then so does the word $a_{i_1} \dots a_{i_t} w$, and since $L(\Gamma)$ is minimal we have

$$w_1 a_{i_1} \dots a_{i_t} w = w,$$

for some word w_1. The word $w_1 a_{i_1} \dots a_{i_t} w$ contains all the letters from A_1, and hence so does w by Lemma 3.1. Therefore, we have proved the direct part of our lemma. The converse part is an immediate consequence of Lemma 2.13. ■

Lemma 3.3. *Let $a_i \in A_1$. The subsemigroup $S_i(\Gamma)$ of $S(\Gamma)$ generated by the set $A - \{a_i\}$ is isomorphic to the semigroup $S(\Gamma - \{a_i\})$.*

PROOF. The presentation $\mathfrak{P}(\Gamma - \{a_i\})$ can be obtained from $\mathfrak{P}(\Gamma)$ by removing the generating symbol a_i, as well as all the relations containing this generating symbol. Hence $S_i(\Gamma)$ satisfies all the relations from the presentation $\mathfrak{P}(\Gamma - \{a_i\})$. Assume now that $u, v \in A^+$ are arbitrary words representing the same element of $S_i(\Gamma)$. This means that v can be obtained from u by applying relations from $\mathfrak{P}(\Gamma)$. By Lemma 3.1, neither u nor v contains a_i, and hence none of the applied relations contains a_i either. Therefore v can be obtained from u by applying relations from $\mathfrak{P}(\Gamma - \{a_i\})$. The lemma now follows from Proposition 3.2.3. ■

Lemma 3.4. $L(\Gamma) - S(\Gamma) = \bigcup_{j=1}^{t} S_{i_j}$.

PROOF. This follows from Lemma 3.2. ■

The information about $S(\Gamma)$ that we have gathered so far is sufficient to prove the following:

Theorem 3.5. *$S(\Gamma)$ is a union of (Coxeter type) groups.*

PROOF. We prove the theorem by induction on the number of vertices of Γ. If Γ has only one vertex a_1, then

$$\mathfrak{P}(\Gamma) = \langle a_1 \mid a_1^{p_1+1} = a_1 \rangle,$$

and, obviously, both $S(\Gamma)$ and $G(\Gamma)$ are isomorphic to the cyclic group of order p_1.

Assume now that Γ has at least two vertices. By Theorem 2.14, $S(\Gamma)$ has a unique minimal left ideal $L(\Gamma)$, which is a union of copies of (the Coxeter type group) $G(\Gamma)$. The set $S(\Gamma) - L(\Gamma)$ is a union of semigroups S_{i_j}, $1 \leq j \leq t$, by Lemma 3.4. Each S_{i_j} is isomorphic to $S(\Gamma - \{a_{i_j}\})$ by Lemma 3.3, and hence is a union of (Coxeter type) groups by the inductive hypothesis. Therefore, $S(\Gamma)$ itself is a union of (Coxeter type) groups. ■

Now we concentrate on minimal right ideals, with the aim of finding an upper bound for the number of these ideals.

Lemma 3.6. *Any minimal right ideal R of $S(\Gamma)$ is generated by a word of the form ua_i, where $a_i \in A_1$, $u \in A^*$, and u does not contain a_i, but it contains all the other letters from A_1.*

PROOF. Let w be a word representing an element of R. Then w contains all the letters from A_1 by Lemma 3.2. Let ua_i be the shortest initial segment of w containing all the letters from A_1. Then it is clear that $a_i \in A_1$, and that u contains all the letters from A_1 except a_i. The word ua_i represents an element of a minimal right ideal by Lemma 3.2, and it is obvious that this minimal right ideal has to be R. Because of its minimality, R is generated by any of its elements, and, in particular, is generated by ua_i. ■

Theorem 3.7. *The number of minimal right ideals of $S(\Gamma)$ is not greater than*

$$(\sum_{j=1}^{t} |S(\Gamma - \{a_{i_j}\})|) + 1.$$

PROOF. The theorem is a direct consequence of Lemmas 3.3 and 3.6. The additional term 1 has to be added to cover the case where $|A_1| = 1$, in which case the only element of A_1 also generates a minimal right ideal. ■

As a consequence we have an upper bound for the order of $S(\Gamma)$, as well as a necessary and sufficient condition for $S(\Gamma)$ to be finite.

Theorem 3.8. *(i)* $|S(\Gamma)| \leq ((\sum_{j=1}^{t} |S(\Gamma - \{a_{i_j}\})|) + 1) \cdot |G(\Gamma)| + \sum_{j=1}^{t} |S(\Gamma - \{a_{i_j}\}|.$

(ii) $S(\Gamma)$ is finite if and only if Γ is strongly finitely related.

PROOF. (i) This inequality is an immediate consequence of Theorems 2.14 and 3.7 and Lemmas 3.3 and 3.4.

(ii) We prove this part of the theorem by induction on the number of vertices of Γ. If Γ has one vertex, then it is strongly finitely related and the semigroup $S(\Gamma)$ is finite. Hence the assertion holds in this case.

Let us now consider the case where Γ has at least two vertices, and let us assume first that Γ is strongly finitely related. This means that that Γ is finitely related, i.e. that the group $G(\Gamma)$ is finite, and that each of the graphs $\Gamma - \{a_{i_j}\}$, $1 \leq j \leq t$, is strongly finitely related. By the inductive hypothesis, the latter condition implies that each semigroup $S(\Gamma - \{a_{i_j}\})$, $1 \leq j \leq t$, is finite, and hence $S(\Gamma)$ is finite by (i).

For the converse, assume that Γ is not strongly finitely related. Then either $G(\Gamma)$ is infinite, or $\mathfrak{P}(\Gamma - \{a_{i_j}\})$ is not strongly finitely related for some j, $1 \leq j \leq t$. In the former case, $S(\Gamma)$ is infinite since it contains a copy of $G(\Gamma)$ by Theorem 2.14. In the latter case the semigroup $S(\Gamma - \{a_{i_j}\})$ is infinite by the inductive hypothesis, and hence $S(\Gamma)$ is infinite by Lemmas 3.3 and 3.4. ∎

Remark 3.9. In the following section we shall see that both bounds from Theorems 3.7 and 3.8 can be achieved, and in Section 5 we shall see that there are cases when they are not achieved.

4. Semigroups defined by Coxeter type presentations for alternating groups

Let us recall from Example 1.3 that the graph Γ_n^A, $n \geq 1$, has vertices $\{a_1, \dots, a_n\}$, edges (a_i, a_j), $1 \leq i < j \leq n$, and labels

$$p_i = 3, \ p_{jk} = 2 \ \ (1 \leq i \leq n, \ 1 \leq j < k \leq n).$$

The graph Γ_4^A is shown in Figure 8.

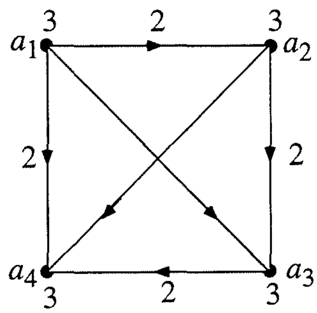


Figure 8.

The presentation associated with Γ_n^A is

$$\mathfrak{P}(\Gamma_n^A) = \langle a_1, \dots, a_n \mid a_i^4 = a_i, \ (a_j a_k)^2 = a_j^3 \ \ (1 \leq i \leq n, \ 1 \leq j < k \leq n)\rangle.$$

The group defined by this presentation is the alternating group $\mathcal{A}_{n+2}$ of degree $n+2$. We denote the semigroup defined by this presentation by $S_A(n)$. Note that this semigroup is finite by Theorems 1.6 and 3.8. In this section we shall determine the exact order and the structure of $S_A(n)$.

By Theorem 2.14, $S_A(n)$ has a unique minimal left ideal, which we denote by $L_A(n)$, and this ideal is a disjoint union of minimal right ideals each of which is isomorphic to $\mathcal{A}_{n+2}$. As outlined at the beginning of the previous section, we describe $S_A(n)$ by describing $S_A(n)/L_A(n)$ and determining the number of minimal right ideals in $L_A(n)$. The first of these tasks is easy to achieve. If we note that a_1 is the unique initial vertex of Γ_n^A, by Lemmas 3.3 and 3.4 we have

Lemma 4.1. *$S_A(n) - L_A(n) \cong S_A(n-1)$ for all $n \geq 2$.* ∎

In order to determine the number of minimal right ideals in $L_A(n)$, we first define two mappings on words:

$$\begin{gathered}\phi : \{a_1, a_2, a_3, \ldots\}^+ \longrightarrow \{a_2, a_3, a_4, \ldots\}^+, \\ (a_{i_1} a_{i_2} \ldots a_{i_s})\phi = a_{i_1+1} a_{i_2+1} \ldots a_{i_s+1}, \\ \psi = \phi^{-1}.\end{gathered} \tag{8}$$

These mappings will be used in the following section as well.

Lemma 4.2. *(i) If $u, v \in \{a_1, \ldots, a_n\}^+$ are any two words such that the relation $u = v$ holds in $S_A(n)$, then the relation $u\phi = v\phi$ holds in $S_A(n+1)$.*

(ii) If $u, v \in \{a_2, \ldots, a_n\}^+$ are any two words such that the relation $u = v$ holds in $S_A(n)$ then the relation $u\psi = v\psi$ holds in $S_A(n-1)$.

PROOF. (i) This part of the lemma follows from the fact that applying ϕ to both sides of a relation from $\mathfrak{P}(\Gamma_n^A)$ yields a relation from $\mathfrak{P}(\Gamma_{n+1}^A)$.

(ii) Since $u = v$ holds in $S_A(n)$, v can be obtained from u by applying relations from $\mathfrak{P}(\Gamma_n^A)$. None of these relations contains a_1 by Lemma 3.1. Applying ψ to both sides of a relation not containing a_1 yields a relation from $\mathfrak{P}(\Gamma_{n-1}^A)$, and hence the result. ∎

Lemma 4.3. *Let $w_1, w_2 \in \{a_2, \ldots, a_n\}^*$. The words $w_1 a_1$ and $w_2 a_1$ generate the same minimal right ideal in $S_A(n)$ if and only if $w_1 \equiv w_2 \equiv \epsilon$ (the empty word), or $w_1\psi = w_2\psi$ holds in $S_A(n-1)$.*

PROOF. Both $w_1 a_1$ and $w_2 a_1$ certainly generate minimal right ideals by Lemma 3.2. Also, by Lemma 4.2, if $w_1\psi = w_2\psi$ holds in $S_A(n-1)$, then $w_1\psi\phi = w_2\psi\phi$ holds in $S_A(n)$. Since $\psi = \phi^{-1}$, we have $w_1 = w_2$ in $S_A(n)$, and $w_1 a_1$ and $w_2 a_1$ indeed generate the same minimal right ideal.

For the converse, assume that $w_1 a_1$ and $w_2 a_1$ generate the same minimal right ideal. Then

$$w_1 a_1 w_3 = w_2 a_1,$$

for some $w_3 \in \{a_1, \ldots, a_n\}^*$. This means that $w_2 a_1$ can be obtained from $w_1 a_1 w_3$ by applying relations from $\mathfrak{P}(\Gamma_n^A)$. If a relation is applied to a subword of $w_1 a_1 w_3$ which contains a letter from w_1, then this relation does not start with a_1, and hence is $a_i^4 = a_i$ for some i, $2 \leq i \leq n$, or is $(a_j a_k)^2 = a_j^3$ for some j, k, $2 \leq j < k \leq n$. In either case the applied relation does not contain a_1, and is hence applied solely to w_1. Therefore, we obtain that either $w_1 \equiv w_2 \equiv \epsilon$, or $w_1 = w_2$ holds in $S_A(n)$, in which case $w_1\psi = w_2\psi$ holds in $S_A(n-1)$ by Lemma 4.3. ∎

Now we have the following theorem describing the structure of $S_A(n)$.

Theorem 4.4. *The semigroup $S_A(n)$, $n \geq 2$, possesses a unique minimal left ideal $L_A(n)$ which is a disjoint union of $|S_A(n-1)|+1$ minimal right ideals, each of which is isomorphic to the alternating group $\mathcal{A}_{n+2}$. The set $S_A(n) - L_A(n)$ is a semigroup isomorphic to $S_A(n-1)$. The semigroup $S_A(n)$ is a union of alternating groups $\mathcal{A}_3, \ldots, \mathcal{A}_{n+2}$. It is finite, and its order is given recursively by*

$$|S_A(1)| = 3,$$
$$|S_A(n+1)| = (|S_A(n)|+1)\frac{(n+3)!}{2} + |S_A(n)|.$$

PROOF. The theorem follows from Theorem 2.14 and Lemmas 3.6, 4.1 and 4.3. ■

The semigroup $S_A(4)$ is shown in Figure 9. It has 4 $\mathcal{D}$-classes, each of which has exactly one $\mathcal{L}$-class. Note that in this figure, contrary to the convention, $\mathcal{L}$-classes are represented by rows, and $\mathcal{R}$-classes are represented by columns. The order of this semigroup is 1145091.

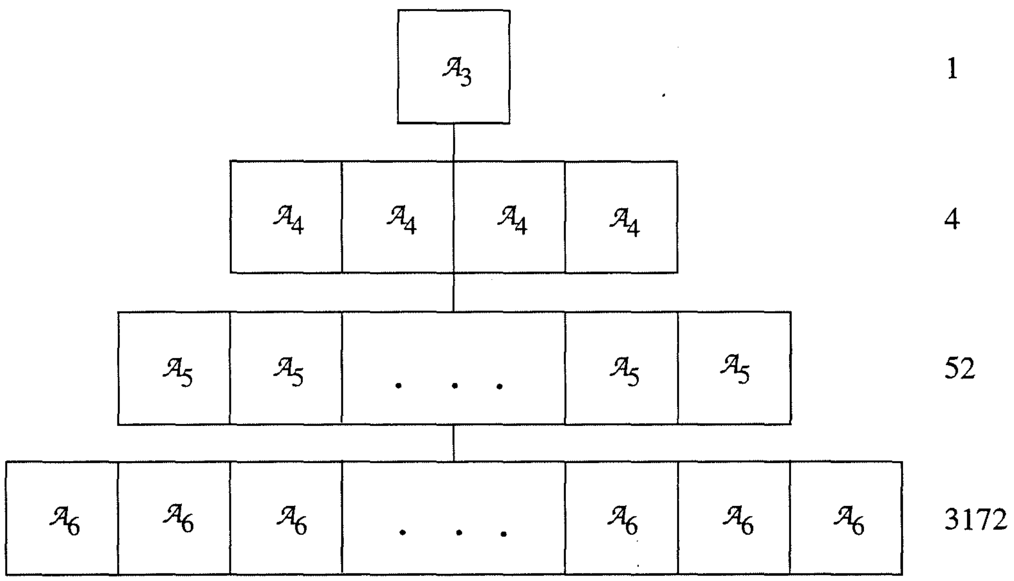


Figure 9.

5. Semigroups defined by Coxeter type presentations for symmetric groups

Now we investigate the structure of the semigroups $S(\Gamma_n^S)$, $n \geq 1$; see Example 1.2. The graph Γ_n^S has vertices $\{a_1, \ldots, a_n\}$, edges (a_i, a_{i+1}), $1 \leq i < n$, and labels

$$p_i = 2,\ p_{j,j+1} = 3 \ \ (1 \le i \le n,\ 1 \le j < n).$$

The graph Γ_4^S is shown in Figure 10.

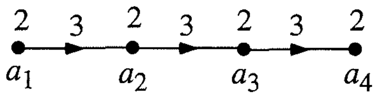


Figure 10.

The presentation associated with Γ_n^S is

$$\mathfrak{P}(\Gamma_n^S) = \langle a_1, \dots, a_n \mid a_i^3 = a_i,\ (a_j a_{j+1})^3 = a_j^2,\ a_k a_l = a_l a_k,$$
$$(1 \le i \le n,\ 1 \le j < n-1,\ 1 \le k < l-1 \le n-1)\rangle.$$

The group $G(\Gamma_n^S)$ defined by this presentation is the symmetric group $\mathcal{S}_{n+1}$ of degree $n+1$. We denote by $S_S(n)$ the semigroup defined by this presentation.

We investigate the semigroup $S_S(n)$ along the same lines as we did for $S_A(n)$ in the previous section: we find the number of minimal right ideals in the unique minimal left ideal $L_S(n)$, and we describe $S_S(n) - L_S(n)$. However, the technical side of the argument is more complicated than in the previous section. In it we make use of the mappings ϕ and ψ defined by (8) in the previous section, as well as of the following recursively defined sequence of sets of words:

$$W_1 = \{a_1\},$$
$$W_{n+1} = \{(w\phi)a_i a_{i-1} \dots a_1 \mid w \in W_n,\ 1 \le i \le n+1\} \cup \{a_1\}.$$

Clearly,

$$W_n \subseteq \{a_1, \dots, a_n\}^+,$$

so that W_n can be considered as a subset of $S_S(n)$.

Again, a_1 is the only initial vertex of Γ_n^S. In particular, the minimal left ideal $L_S(n)$ consists of all words containing a_1; see Lemma 3.2. Hence every word from W_n represents an element of $L_S(n)$. We aim to prove that each minimal right ideal of $S_S(n)$ contains one and only one element from W_n.

We begin with some technical results.

Lemma 5.1. *If $i \le j \le n$ then there exists a word $w \in \{a_1, \dots, a_n\}^+$ such that the relation $a_i = wa_j$ holds in $S_S(n)$.*

PROOF. If $i = j$, we can choose $w_1 \equiv a_i^2$, because of the relation $a_i^3 = a_i$. So assume that $i \ne j$. Let w_k, $1 \le k < n$, denote the word $a_k(a_k a_{k+1})^2 a_k$. Then we have

$$w_k a_{k+1} \equiv a_k(a_k a_{k+1})^3 = a_k a_k^2 = a_k,$$

and hence, for $w \equiv w_i \dots w_{j-2} w_{j-1}$, we have

$$wa_j \equiv w_i \dots w_{j-2} w_{j-1} a_j = w_i \dots w_{j-2} a_{j-1} = \dots = w_i a_{i+1} = a_i,$$

as required. ∎

Lemma 5.2. *The relation*

$$a_k^2 a_{k+1} a_k a_{k+1} = a_k a_{k+1} a_k$$

holds in $S_S(n)$ for any k, $1 \leq k < n$.

PROOF. Note that by Lemma 2.4 we have

$$a_{k+1} a_k a_{k+1}^2 = a_{k+1} a_k, \; a_k a_{k+1} a_k^2 = a_k a_{k+1},$$

and hence

$$\begin{aligned} a_k^2 a_{k+1} a_k a_{k+1} &= (a_k a_{k+1})^3 a_{k+1} a_k a_{k+1} = a_k (a_{k+1} a_k)^2 a_{k+1}^2 a_k a_{k+1} \\ &= a_k (a_{k+1} a_k)^2 a_k a_{k+1} = (a_k a_{k+1})^2 a_k^2 a_{k+1} \\ &= (a_k a_{k+1})^2 a_{k+1} = a_k a_{k+1} a_k a_{k+1}^2 = a_k a_{k+1} a_k, \end{aligned}$$

as required. ∎

Lemma 5.3. *(i) If $u, v \in \{a_1, \dots, a_n\}^+$ are any two words such that the relation $u = v$ holds in $S_S(n)$, then the relation $u\phi = v\phi$ holds in $S_S(n+1)$.*

(ii) If $u, v \in \{a_2, \dots, a_n\}^+$ are any two words such that the relation $u = v$ holds in $S_S(n)$ then the relation $u\psi = v\psi$ holds in $S_S(n-1)$.

PROOF. The proof is exactly the same as the proof of Lemma 4.2. ∎

Lemma 5.4. *For any three words $u, w_1, w_2 \in \{a_1, \dots, a_n\}^*$ the words ua_1w_1 and ua_1w_2 generate the same minimal right ideal in $S_S(n)$.*

PROOF. As a direct consequence of Lemma 3.2, both ua_1w_1 and ua_1w_2 generate the same minimal right ideal as the word ua_1. ∎

Lemma 5.5. *Let $w_1, w_2 \in W_n$, with*

$$w_1 \equiv (\overline{w}_1\phi) a_i \dots a_1, \; w_2 \equiv (\overline{w}_2\phi) a_j \dots a_1,$$

$\overline{w}_1, \overline{w}_2 \in W_{n-1}$, $1 \leq i, j \leq n$. If w_1 and w_2 generate the same minimal right ideal in $S_S(n)$ then $\overline{w}_1$ and $\overline{w}_2$ generate the same minimal right ideal in $S_S(n-1)$.

PROOF. If w_1 and w_2 generate the same minimal right ideal in $S_S(n)$, then there exists a word $v \in \{a_1, \ldots, a_n\}^*$ such that the relation

$$w_1 = w_2 v$$

holds in $S_S(n)$. Therefore, $w_2 v$ can be obtained from w_1 by applying relations from $\mathfrak{P}(\Gamma_n^S)$. Since w_1 contains a_1, so does any word representing the same element in $S_S(n)$ by Lemma 3.1. Thus we have a sequence of words

$$w_1 \equiv u_1 a_1 v_1,\ u_2 a_1 v_2,\ u_3 a_1 v_3,\ \ldots, u_s a_1 v_s \equiv w_1 v,$$

in which every word $u_{k+1} a_1 v_{k+1}$, $1 \leq k < s$, is obtained from $u_k a_1 v_k$ by one application of one relation from $\mathfrak{P}(\Gamma_n^S)$, and in which the words $u_1, \ldots, u_s$ do not contain a_1.

Since $u_1 \equiv (\overline{w}_1 \phi) a_i \ldots a_2$, we have

$$u_1 \psi \equiv \overline{w}_1 a_{i-1} \ldots a_1,$$

so that $u_1 \psi$ contains a_1 and hence generates a minimal right ideal in $S_S(n-1)$. We now prove by induction that for every k, $1 \leq k \leq s$, the words $u_k \psi$ and $u_1 \psi$ generate the same minimal right ideal in $S_S(n-1)$. Note that there is nothing to prove for $k = 1$, and assume that the statement is true for some $k \geq 1$. We distinguish the following four cases, depending on which relation is applied to $u_k a_1 v_k$ to obtain $u_{k+1} a_1 v_{k+1}$, and to which part of $u_k a_1 v_k$ this relation is applied.

Case 1. The relation is applied to a subword of u_k. This means that $u_k = u_{k+1}$ in $S_S(n)$, and hence $u_k \psi = u_{k+1} \psi$ in $S_S(n-1)$ by Lemma 5.3. It now follows by the inductive hypothesis that $u_{k+1} \psi$ and $u_1 \psi$ generate the same minimal right ideal in $S_S(n-1)$.

Case 2. The relation is applied to a subword of $u_k a_1 v_k$ which contains a_1 and at least one letter from u_k. Since u_k does not contain a_1, it must be of the form $u_k \equiv u'_k a_i$ with $i > 2$, and the applied relation must be $a_1 a_i = a_i a_1$, so that $u_{k+1} \equiv u'_k$. By the inductive hypothesis $u_k \psi$ and $u_1 \psi$ generate the same minimal right ideal in $S_S(n-1)$. In particular, $u_k \psi$ contains a_1 by Lemma 3.2. On the other hand

$$u_k \psi \equiv (u'_k \psi) a_{i-1}.$$

Since $i > 2$, $u'_k \psi$ also contains a_1, and hence generates a minimal right ideal in $S_S(n-1)$. This minimal right ideal is the same as the minimal right ideal generated by $u_k \psi$ by Lemma 5.4, and the assertion follows by induction.

Case 3. v_k has the form $v_k \equiv a_i v'_k$, $i > 2$, and the applied relation is $a_1 a_i = a_i a_1$. Then $u_{k+1} \equiv u_k a_i$, so that $u_{k+1} \psi \equiv (u_k \psi) a_{i-1}$, and $u_{k+1} \psi$ generates the same minimal right ideal as $u_k \psi$ by Lemma 5.4.

Case 4. One of the relations $a_i^3 = a_i$ or $(a_j a_{j+1})^3 = a_j^2$ is applied to a subword of $a_1 v_k$. In this case $u_k \equiv u_{k+1}$, and the assertion follows by induction.

What we have just proved implies, in particular, that $u_1\psi$ and $u_s\psi$ generate the same minimal right ideal in $S_S(n-1)$. On the other hand, we have

$$u_1\psi \equiv ((\overline{w}_1\phi)a_i \dots a_2)\psi \equiv \overline{w}_1 a_{i-1} \dots a_1.$$

Since $\overline{w}_1 \in W_{n-1}$, $\overline{w}_1$ contains a_1, and hence $u_1\psi$ and $\overline{w}_1\psi$ generate the same minimal right ideal by Lemma 5.4. Similarly, $u_s\psi$ and $\overline{w}_2$ generate the same minimal right ideal, which finally implies that $\overline{w}_1$ and $\overline{w}_2$ generate the same minimal right ideal, exactly as required. ■

Let

$$\eta : A^* \longrightarrow \mathcal{S}_{n+1}$$

be the unique homomorphism such that

$$a_i\eta = (i\; i+1),\; i = 1, \dots, n.$$

Since $\mathcal{S}_{n+1}$ is the group defined by the presentation $\mathfrak{P}(\Gamma_n^S)$, it follows that η induces an epimorphism from $S_S(n)$ onto $\mathcal{S}_{n+1}$; we denote this epimorphism also by η. For a word $w \in \{a_1, \dots, a_n\}^+$ which contains letter a_1, by $\iota(w)$ we denote the longest initial segment of w not containing a_1, and by $\tau(w)$ we denote the terminal segment of w starting immediately after the first a_1. Therefore, we have

$$w \equiv \iota(w)a_1\tau(w).$$

Lemma 5.6. *Let $u, v \in \{a_1, \dots, a_n\}^+$ be two words which both contain a_1 and which represent the same element of $S_S(n)$. Then*

$$(\iota(u)\tau(u))\eta = (\iota(v)\tau(v))\eta$$

holds in $\mathcal{S}_{n+1}$.

Proof. Since $S_S(n)$ is defined by $\mathfrak{P}(\Gamma_n^S)$, it follows that v can be obtained from u by applying relations from $\mathfrak{P}(\Gamma_n^S)$. The letter a_1 is an invariant of $\mathfrak{P}(\Gamma_n^S)$ by Lemma 3.1, so that all the words obtained in this process contain a_1, and hence it is enough to prove the lemma in the case where u and v differ by one application of one relation from $\mathfrak{P}(\Gamma_n^S)$.

So we suppose that v is obtained from u by applying a relation $u' = v'$ from $\mathfrak{P}(\Gamma_n^S)$. Notice that we have

$$u'\eta = v'\eta,$$

since $\mathcal{S}_{n+1}$ satisfies all the relations from $\mathfrak{P}(\Gamma_n^S)$. Therefore, if $u' = v'$ is applied to a subword of $\tau(u)$, then we have $\iota(u) \equiv \iota(v)$ and $(\tau(u))\eta = (\tau(v))\eta$, so that

$$(\iota(u)\tau(u))\eta = (\iota(u))\eta \cdot (\tau(u))\eta = (\iota(v))\eta \cdot (\tau(v))\eta = (\iota(v)\tau(v))\eta.$$

Next assume that $u' = v'$ is applied to a subword of $\iota(u)$. Since $\iota(u)$ does not contain a_1, it follows that the relation $u' = v'$ does not contain a_1 either, and hence applying $u' = v'$ to $\iota(u)$ yields $\iota(v)$; we also have $\tau(u) \equiv \tau(v)$. Similarly as in the previous case we now obtain $(\iota(u)\tau(u))\eta = (\iota(v)\tau(v))\eta$.

Now consider the case where $u' = v'$ has been applied to a subword of u containing the first appearance of a_1. If the relation $u' = v'$ is $a_1^3 = a_1$, then either $\iota(u)\tau(u) \equiv \iota(v)a_1^2\tau(v)$ or $\iota(u)a_1^2\tau(u) \equiv \iota(v)\tau(v)$, and in any case the assertion follows from $a_1^2\eta = (1\ 2)(1\ 2) = (1)$. If $u' = v'$ is $a_1a_j = a_ja_1$, $j > 2$, then $\iota(u)\tau(u) \equiv \iota(v)\tau(v)$, and the result follows trivially. Finally, if $u' = v'$ is $(a_1a_2)^3 = a_1^2$ then $\iota(u) \equiv \iota(v)$ and either

$$a_1\tau(u) \equiv a_1^2u_1,\ a_1\tau(v) \equiv (a_1a_2)^3v_1,$$

or

$$a_1\tau(u) \equiv (a_1a_2)^3u_1,\ a_1\tau(v) \equiv a_1^2v_1,$$

for some words u_1, v_1. In both cases the result follows from $(a_2(a_1a_2)^2)\eta = (12) = a_1\eta$. This completes the proof of the lemma. ■

Lemma 5.7. *Each minimal right ideal of $S_S(n)$ contains exactly one element of W_n.*

PROOF. We prove the lemma by induction on n. For $n = 1$, $S_S(n)$ is the cyclic group of order 2, and is its own unique minimal right ideal, so that that the assertion holds in this case.

Let us now assume that $n > 1$, and let R be any minimal right ideal of $S_S(n)$. First we prove that R contains an element from W_n. By Lemma 3.6, R is generated by a word of the form w_1a_1 with $w_1 \in \{a_2, \ldots, a_n\}^*$. Let $w \equiv w_1a_1$ be such a word of the minimal possible length. If w_1 is the empty word then $w \equiv a_1 \in W_n$. So we may assume that w_1 is non-empty, and we can write

$$w_1 \equiv w_2a_i,$$

where $i \geq 2$.

If $i > 2$ then

$$w \equiv w_2a_ia_1 = w_2a_1a_i.$$

By Lemma 5.4, w_2a_1 belongs to R, which is a contradiction to our choice of w. Therefore we may assume that $i = 2$, i.e. that

$$w \equiv w_2a_2a_1.$$

The word w_2a_2 does not contain a_1 and hence the word $(w_2a_2)\psi$ represents an element of $S_S(n-1)$. Moreover, since

$$(w_2a_2)\psi \equiv w_2\psi \cdot a_1,$$

$(w_2a_2)\psi$ generates a minimal right ideal $\overline{R}$ in $S_S(n-1)$. By the inductive hypothesis $\overline{R}$ contains a word $u \in W_{n-1}$. The minimality of $\overline{R}$ implies that there exists a word $\overline{v}$ such that

$$(w_2a_2)\psi = u\overline{v}$$

holds in $S_S(n-1)$. Then

$$w_2a_2 \equiv ((w_2a_2)\psi)\phi = (u\overline{v})\phi \equiv u\phi \cdot \overline{v}\phi$$

by Lemma 5.3, and thus

$$u\phi \cdot v\phi \cdot a_1 = w \in R.$$

Let $v \in \{a_2, \ldots, a_n\}^*$ be a word of minimal length such that $(u\phi)va_1 \in R$. We shall now prove that $va_1 \equiv a_ia_{i-1}\ldots a_1$ for some i, $1 \leq i \leq n$. Suppose not. Let

$$va_1 \equiv a_{i_t}a_{i_{t-1}}\ldots a_{i_1},$$

and let k be the least positive integer such that

$$j = i_{k+1} \neq k+1.$$

Certainly we have $k \neq 0$, as $a_{i_1} = a_1$. Let also

$$v_1 \equiv a_{i_t}a_{i_{t-1}}\ldots a_{i_{k+2}},$$

so that

$$va_1 \equiv v_1a_ja_ka_{k_1}\ldots a_1.$$

We now distinguish the following three cases.

Case 1. $j > k+1$. In this case we have

$$(u\phi)va_1 \equiv (u\phi)v_1a_ja_ka_{k-1}\ldots a_1 = (u\phi)v_1a_ka_{k-1}\ldots a_1a_j.$$

By Lemma 5.4, it follows that $(u\phi)v_1a_ka_{k-1}\ldots a_1 \in R$, which is a contradiction with the choice of v.

Case 2. $j < k$. Now we have

$$\begin{aligned} R \ni (u\phi)va_1a_{j+1} &\equiv (u\phi)v_1a_ja_ka_{k-1}\ldots a_1a_{j+1} \\ &= (u\phi)v_1a_ka_{k-1}\ldots a_{j+2}a_ja_{j+1}a_ja_{j+1}a_{j-1}\ldots a_1 \\ &= (u\phi)v_1a_ka_{k-1}\ldots a_{j+2}a_2^2a_ja_{j+1}a_ja_{j+1}a_{j-1}\ldots a_1, \end{aligned}$$

by Lemma 2.5, since $u\phi$ contains a_2. By Lemma 5.1 there exists a word β such that

$$a_2 = \beta a_j,$$

so that

$$\begin{aligned}
& (u\phi)v_1a_ka_{k-1}\dots a_{j+2}a_2^2a_ja_{j+1}a_ja_{j+1}a_{j-1}\dots a_1 \\
= \;& (u\phi)v_1a_ka_{k-1}\dots a_{j+2}a_2\beta a_j^2a_{j+1}a_ja_{j+1}a_{j-1}\dots a_1 \\
= \;& (u\phi)v_1a_ka_{k-1}\dots a_{j+2}a_2\beta a_ja_{j+1}a_ja_{j-1}\dots a_1 && \text{(Lemma 5.2)} \\
= \;& (u\phi)v_1a_ka_{k-1}\dots a_{j+2}a_2^2a_{j+1}a_ja_{j-1}\dots a_1 \\
= \;& (u\phi)v_1a_ka_{k-1}\dots a_1, && \text{(Lemma 2.5)}
\end{aligned}$$

where Lemma 5.2 has been used. We have proved that

$$(u\phi)v_1a_k\dots a_1 \in R,$$

which is a contradiction since

$$|v_1a_k\dots a_1| < |v_1a_ja_k\dots a_1| = |va_1|.$$

Case 3. $j = k$. Similarly as in the previous case we have

$$\begin{aligned}
R \ni (u\phi)v_1a_ja_ka_{k-1}\dots a_1 \;&\equiv\; (u\phi)v_1a_k^2a_{k-1}\dots a_1 \\
&= (u\phi)v_1a_2^2a_k^2a_{k-1}\dots a_1 && \text{(Lemma 2.5)} \\
&= (u\phi)v_1a_2\beta a_ka_k^2a_{k-1}\dots a_1 && \text{(Lemma 5.1)} \\
&= (u\phi)v_1a_2\beta a_ka_{k-1}\dots a_1 \\
&= (u\phi)v_1a_2^2a_{k-1}\dots a_1 \\
&= (u\phi)v_1a_{k-1}\dots a_1, && \text{(Lemma 2.5)}
\end{aligned}$$

which is again a contradiction to the choice of v.

We have proved that

$$va_1 \equiv a_i\dots a_1,$$

for some i, and therefore $(u\phi)va_1$ belongs to both W_n and R, as required.

Now we prove uniqueness. Assume that R contains two distinct words w_1 and w_2 from W_n. Neither of w_1 or w_2 is identical to a_1. To prove this it is enough to note that application of any relation from $\mathfrak{P}(\Gamma_n^S)$ to a word of the form ua_1v, where u is a word which does not contain a_1 but contains a_2 yields another word of the same form, and that all the words of W_n, except a_1, have this form.

Therefore we may assume that

$$w_1 \equiv (\overline{w}_1\phi)a_i\dots a_1,\; w_2 \equiv (\overline{w}_2\phi)a_j\dots a_1,$$

for some $\overline{w}_1, \overline{w}_2 \in W_{n-1}$ and some i, j, $1 \le i, j \le n$. By Lemma 5.5, $\overline{w}_1$ and $\overline{w}_2$ generate the same minimal right ideal in $S_S(n-1)$, and hence we have

$$\overline{w}_1 \equiv \overline{w}_2$$

by the inductive hypothesis.

Now we have to prove that $i = j$. Since w_1 and w_2 both belong to R the relation

$$(\overline{w}_1\phi)a_i \dots a_1 v = (\overline{w}_2\phi)a_j \dots a_1$$

holds in $S_S(n-1)$ for some word v. If, as before, η denotes the unique epimorphism $\{a_1, \dots, a_n\}^* \to \mathcal{S}_{n+1}$ extending the mapping $a_i \mapsto (i\; i+1)$, then we have

$$((\overline{w}_1\phi)a_i \dots a_1 v)\eta = ((\overline{w}_2\phi)a_j \dots a_1)\eta,$$

from which it follows easily that

$$v\eta = (1\,2)(2\,3)\dots(j\; j+1)(i\; i+1)\dots(2\,3)(1\,2) = \sigma, \tag{9}$$

since $\overline{w}_1$ and $\overline{w}_2$ are identical. On the other hand, Lemma 5.6 gives

$$((\overline{w}_1\phi)a_i \dots a_2 v)\eta = ((\overline{w}\phi)a_j \dots a_2)\eta,$$

so that

$$v\eta = (2\,3)(3\,4)\dots(j\; j+1)(i\; i+1)\dots(3\,4)(2\,3) = \tau. \tag{10}$$

From (9) and (10) we have

$$\sigma = \tau.$$

Now we can see that it is impossible to have $i < j$ because then we would have

$$1\sigma = j + 1 \neq 1 = 1\tau.$$

Similarly, $i > j$ would imply

$$(i+1)\sigma = 1 \neq 2 = (i+1)\tau.$$

Therefore we have $i = j$, so that $w_1 \equiv w_2$, thus completing the proof. ∎

Lemma 5.8. *$S_S(n)$ has c_n minimal right ideals, where the sequence $(c_n)_{n=1}^{\infty}$ is defined recursively by*

$$c_1 = 1,\ c_{n+1} = (n+1)c_n + 1.$$

Proof. $S_S(n)$ has exactly $|W_n|$ elements by Lemma 5.7, and this is easily seen to be equal to c_n. ∎

Finally, as in Section 4, the fact that a_1 is the only initial vertex of Γ_n^S together with Lemmas 3.3 and 3.4 gives the following

Lemma 5.9. $S_S(n) - L_S(n) \cong S_S(n-1)$ *for all* $n > 1$. ∎

By combining Lemmas 5.8 and 5.9 with Theorem 2.14 we obtain

Theorem 5.10. *The semigroup $S_S(n)$, $n \geq 1$, has a unique minimal left ideal $L_S(n)$ which is a disjoint union of c_n minimal right ideals, each of which is isomorphic to the symmetric group $\mathcal{S}_{n+1}$. For $n \geq 2$, the set $S_S(n) - L_S(n)$ is a semigroup isomorphic to $S_S(n-1)$. The semigroup $S_S(n)$ is a union of copies of the symmetric groups $\mathcal{S}_2, \mathcal{S}_3, \ldots, \mathcal{S}_{n+1}$. It is finite and its order is*

$$|S_S(n)| = \sum_{i=1}^{n} c_i(i+1)!. \blacksquare$$

The egg-box picture (again with $\mathcal{L}$-classes shown as rows and $\mathcal{R}$-classes shown as columns) of the semigroup $S_S(4)$ is shown in Figure 11. The order of this semigroup is 5180.

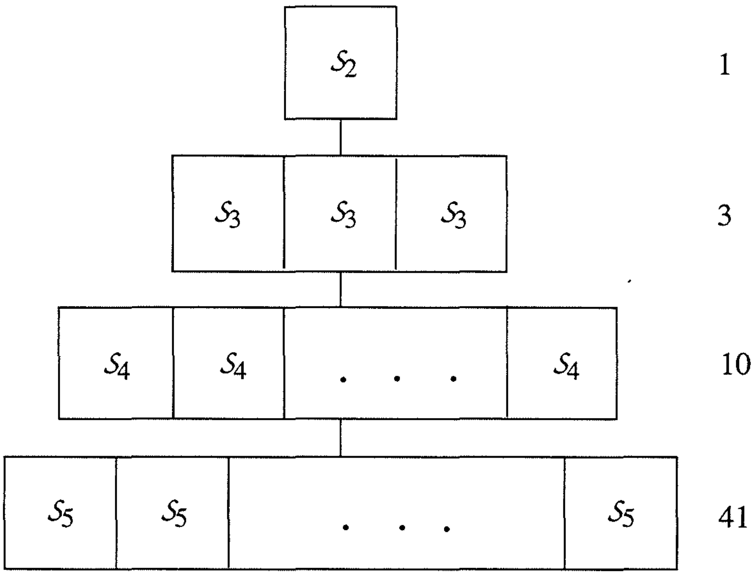


Figure 11.

Chapter 11

Certain one-relator products of cyclic groups

In this chapter we consider various special cases of semigroups defined by presentations of the form
$$\langle a, b \mid a^{p+1} = a,\ b^{q+1} = b,\ \alpha = \beta \rangle, \tag{1}$$
where $p, q \in \mathbb{N}$, $\alpha, \beta \in \{a, b\}^+$. The semigroup S defined by (1) can be viewed as the semigroup free product $C_p * C_q$ of a cyclic group of order p and a cyclic group of order q factored by the smallest congruence containing (α, β). We classify the semigroups S we consider with respect to the following three properties:

(F1) S has a minimal two-sided ideal which is a disjoint union of copies of the group defined by (1);

(F2) S has a minimal two-sided ideal which is a disjoint union of copies of a group which is not isomorphic to the group defined by (1).

(F3) S has no minimal (left, right or two-sided) ideals.

The results of Section 1 have appeared in Campbell, Robertson, Ruškuc and Thomas (1994a) and the results of Section 2 will appear in Campbell, Robertson, Ruškuc, Thomas and Ünlü (1995).

1. Semigroups defined by presentations $\langle a, b \mid a^3 = a,\ b^{q+1} = b,\ ab^r ab^2 = b^2 a \rangle$

In this Section we consider the semigroup S defined by the presentation

$$\langle a, b \mid a^3 = a,\ b^{q+1} = b,\ ab^r ab^2 = b^2 a \rangle. \tag{2}$$

We assume that q is even, that r is odd and that $1 \leq r \leq q$.

We are going to apply the general theory developed in Chapter 10 to determine the structure of S. It will turn out that S has two minimal left ideals and three minimal right ideals. After rewriting presentation (2) into a presentation for the

Schützenberger group H of the minimal two-sided ideal M of S we will see that this group is finite metabelian, thus enabling us to find the order of S. A closer analysis will show that H is isomorphic to the group G defined by (2) if and only if $q = 2r$.

Semigroups defined by (2) were one of the first to be investigated by using computational methods, and these investigations made a significant influence on the general theory from Chapters 9 and 10. This is the primary reason for including these, rather special, semigroups in this thesis.

As usual, we begin by determining minimal left and right ideals of S. This turns out to be a rather tedious job, and we need several technical results.

Lemma 1.1. *The following relations hold in S:*

(i) $bab^q = ba$;
(ii) $ba^2b^q = ba^2$;
(iii) $a^2b^{2i}a = b^{2i}a,\ i \in \mathbb{N}$;
(iv) $bab^{2i}a^2 = bab^{2i},\ i \in \mathbb{N}$;
(v) $ba^2b^{2i}a^2 = ba^2b^{2i},\ i \in \mathbb{N}$;
(vi) $(b^qa^2)^2 = b^qa^2$;
(vii) $b^2 \cdot ba^2 = ba^2 \cdot b^2$;
(viii) $(ab^ra)^2 = b^qa^2$.

Proof. (i) We have

$$bab^q = b^{q+1}ab^q = b^{q-1}b^2ab^q = b^{q-1}ab^rab^2b^q = b^{q-1}ab^rab^2 = b^{q-1}b^2a = ba.$$

(ii) Using (i) we have

$$ba^2b^q = bab^qab^q = bab^qa = ba^2.$$

(iii) We prove (iii) by induction on i. For $i = 1$ we have

$$a^2b^2a = a^2ab^rab^2 = ab^rab^2 = b^2a.$$

Now if

$$a^2b^{2i}a = b^{2i}a,$$

then

$$a^2b^{2i+2}a = a^2b^{2i}b^2a = a^2b^{2i}ab^rab^2 = b^{2i}ab^rab^2 = b^{2i+2}a,$$

thus completing the inductive argument.

(iv) Let $s = q - 2$. We prove by induction that

$$bab^{is}a^2 = bab^{is} \tag{3}$$

for all $i \in \mathbb{N}$. Since g.c.d.$(q, s) = 2$ and since $b^{q+1} = b$, (iv) will then follow.

For $i = 1$ we have

$$\begin{aligned} bab^s a^2 &= b^{q-1} b^2 ab^{q-2} a^2 = b^{q-1} ab^r ab^q a^2 = b^{q-1} ab^r a^3 \\ &= b^{q-1} ab^r a = b^{q-1} ab^r ab^q = b^{q-1} b^2 ab^{q-2} = bab^s, \end{aligned}$$

where (ii) has been used. Now assume that (3) holds for some i. Then we have

$$\begin{aligned} bab^{(i+1)s} a^2 &= bab^{is} b^s a^2 = bab^{is} a^2 b^s a^2 = bab^{is-2} b^2 aab^s a^2 = bab^{is-2} ab^r ab^2 ab^s a^2 \\ &= bab^{is-2} ab^r ab^2 ab^s = bab^{is-2} b^2 a^2 b^s = bab^{is} a^2 b^s = bab^{(i+1)s}, \end{aligned}$$

thus completing the inductive argument.

(v) Using (i) and (iv) we have

$$ba^2 b^{2i} a^2 = bab^q ab^{2i} a^2 = bab^q ab^{2i} = ba^2 b^{2i}.$$

(vi) By (ii) we have

$$b^q a^2 b^q a^2 = b^q a^4 = b^q a^2.$$

(vii) By (iii) and (v) we have

$$ba^2 b^2 = ba^2 b^2 a^2 = b^3 a^2..$$

(viii) By (i) and (vii) we have

$$\begin{aligned} (ab^r a)^2 &= (ab^r ab^q)^2 = (ab^r ab^2 b^{q-2})^2 = (b^2 ab^{q-2})^2 \\ &= b^2 ab^{q-2} b^2 ab^{q-2} = b^2 ab^q ab^{q-2} = b^2 a^2 b^{q-2} = b^q a^2, \end{aligned}$$

since $q - 2$ is even. ■

Lemma 1.2. *For any word $w_1 \in \{a, b\}^+$ there exists a word $w_2 \in \{a, b\}^+$ such that the relation*

$$w_2 w_1 ba = ba$$

holds in S.

PROOF. We prove the lemma by induction on the length of $w_1 ba$. However, in this case it is convenient to define the length of a word to be its length in the free product $\{a\}^+ * \{b\}^+$. Thus, for example, both words aba and $b^3 a^2 b^5$ have lengths 3, while the word $ababa$ has length 5.

If the length of $w_1 ba$ is 2, then $w_1 ba$ has the form $b^i a$ for some $i \in \mathbb{N}$, so that premultiplying by a suitable power of b yields ba. If the length of $w_1 ba$ is greater than 2, then $w_1 ba$ can be written as

$$w_1 ba = b^i a^j b^k a w_1',$$

where $0 \leq i \leq q$, $1 \leq j \leq 2$, $1 \leq k \leq q$, $w_1' \in \{a,b\}^*$, and w_1' is either empty or ends with ba. We shall show that by premultiplying the word $b^ia^jb^ka$ we can obtain the word ba. The lemma will then follow by induction, since baw_1' is shorter than $b^ia^jb^kaw_1'$.

First of all, by premultiplying $b^ia^jb^ka$ by b^{q+1-i} we obtain the word ba^jb^ka. Next by premultiplying by $b^{q-1}ab^rab$ we obtain

$$b^{q-1}ab^rab \cdot ba^jb^ka = b^{q-1}b^2aa^jb^ka = ba^{j+1}b^ka.$$

By repeating this we obtain the word $bab^ka = bab^{q+k}a$. Now premultiplying by $b^{q-1}ab^{r-1}$ yields

$$b^{q-1}ab^{r-1} \cdot bab^{q+k}a = b^{q-1}b^2ab^{q+k-2}a = bab^{q+k-2}a.$$

By repeating this we obtain the word bab^2a if k is even or bab^ra if k is odd. We now consider these two cases separately.

First we consider the word bab^2a, and we premultiply it by $b^{q-1}ab^{r-1}$ to obtain

$$b^{q-1}ab^{r-1} \cdot bab^2a = b^{q-1}b^2a^{l+1} = ba^2,$$

and then we premultiply ba^2 by $b^{q-1}ab^rab$:

$$b^{q-1}ab^rab \cdot ba^2 = b^{q-1}b^2aa^2 = ba.$$

Now we consider the word bab^ra, and we premultiply it by $b^{q-1}ab^rab$ to obtain

$$b^{q-1}ab^rab \cdot bab^ra = b^{q-1}b^2a^2b^ra = ba^2b^ra,$$

and then we premultiply it by bab^{r-1}:

$$bab^{r-1} \cdot ba^2b^ra = b(ab^ra)^2 = b^{q+1}a^2 = ba^2.$$

Again premultiplying by $b^{q-1}ab^rab$ yields ba. ■

Lemma 1.3. *Let L_1 and L_2 be the left ideals of S generated by ba and bab respectively. Then L_1 and L_2 are the only two minimal left ideals of S, and the action of S on $\{L_1, L_2\}$ is given by*

	a	b
1	1	2
2	1	1

PROOF. That L_1 and L_2 are minimal left ideals follows from Lemma 1.2 and Proposition 9.3.1. To prove that $L_1 \neq L_2$ note that for any $i, j \geq 0$ and any $w_1, w_2 \in \{a,b\}^*$ we have

$$w_1ab^i = w_2ab^j \implies i \equiv j \pmod 2,$$

so that we can never have $ba = wbab$.

Finally, to prove that L_1 and L_2 are the only minimal left ideals of S, we show that for any word $w_1 \in \{a,b\}^+$ there exists a word $w_2 \in \{a,b\}^+$ such that $w_2w_1 \in L_1 \cup L_2$. First we premultiply w_1, to obtain a word of length at least three. This word has the form $w_1'ba^ib^j$. By Lemma 1.2 this word can be premultiplied to give ba^ib^j. Next, from

$$b^{q-1}ab^rab \cdot ba^ib^j = b^{q-1}b^2aa^ib^j = ba^{i+1}b^j,$$

it follows that we may assume that $i = 1$. Now if we premultiply the word bab^j by $b^{q-1}ab^{r-1}$ we obtain

$$b^{q-1}ab^{r-1} \cdot bab^2b^{j-2} = bab^{j-2}.$$

By continuing in this way we obtain the word $ba \in L_1$ if j is even or $bab \in L_2$ if j is odd.

The given action of S on $\{L_1, L_2\}$ follows from the above argument. ■

Lemma 1.4. *S has a unique minimal two-sided ideal M. A word $w \in \{a,b\}^+$ represents an element of M if and only if w contains ba as a subword. The set $S - M$ has exactly $3q+2$ elements, and these elements are represented by the non-empty words of the form a^ib^j, $0 \le i \le 2$, $0 \le j \le q$.*

PROOF. The existence of M is a consequence of the fact that S has minimal left ideals; see Lemma 1.4 and Proposition A.3.2. By the same theorem we have $M = L_1 \cup L_2$. In particular, M is generated by ba, and hence every word containing ba as a subword represents an element of M. For the converse note that we have the following invariant of presentation (2): if $w_1, w_2 \in \{a,b\}^+$ are such that $w_1 = w_2$ holds in S then w_1 contains ba as a subword if and only if w_2 contains ba as subword. Therefore, a word not containing ba as a subword does not represent an element of M. There are exactly $3q+2$ such words, and they are a^ib^j, $0 \le i \le 2$, $0 \le j \le q$, $i \ne 0$ or $j \ne 0$. All these words represent distinct elements of S since the only relations that can be applied to a^ib^j are $a^3 = a$ and $b^{q+1} = b$. ■

Lemma 1.5. *For any word $w_1 \in \{a,b\}^+$ there exists a word $w_2 \in \{a,b\}^+$ such that the relation*

$$baw_1w_2 = ba$$

holds in S.

PROOF. We prove the lemma by induction on the length of baw_1, where the length is again taken in the free product sense. In the case of length 2 we have $baw_1 \equiv ba^i$, so that postmultiplying by a suitable power of a yields ba.

If the length of baw_1 is greater than 2, then baw_1 can be written as $w_1'ba^ib^ja^k$, where $1 \leq i \leq q$, $1 \leq j \leq 2$, $0 \leq k \leq q$, $w_1' \in \{a,b\}^*$ and w_1' is either empty or ends with ba. Similarly as in Lemma 1.2, we show that $ba^ib^ja^k$ can be reduced by postmultiplying to ba, and the lemma will follow by induction.

First of all, by postmultiplying $ba^ib^ja^k$ by an appropriate power of a, we can obtain the word ba^ib^ja. Next we postmultiply ba^ib^ja by b^rab^2 and obtain

$$ba^ib^ja \cdot b^rab^2 = ba^ib^{j+2}a.$$

Continuing in this way yields ba^ib^qa if j is even or ba^ib^ra if j is odd. Note that $ba^ib^qa = ba^{i+1}$ by Lemma 1.1 (i), and postmultiplying by a^{2-i} yields ba. Let us now consider the word ba^ib^ra, and let us postmultiply it by b^2; we obtain

$$ba^ib^ra \cdot b^2 = ba^{i-1}b^2a.$$

If $i = 1$, the obtained word is b^3a, and this word can be transformed into $b^{q+1}a = ba$ by repeatedly postmultiplying by b^rab^2. If $i = 2$, then we have the word bab^2a, which is of the form $ba^ib^ja^k$ with j even, and we have already shown that such a word can be reduced by postmultiplication to ba. ∎

Lemma 1.6. *Let R_1, R_2, R_3 be the right ideals of S generated by ba, aba, a^2ba respectively. Then R_1, R_2, R_3 are the only three minimal right ideals of S.*

PROOF. That R_1, R_2, R_3 are minimal follows from Lemma 1.5 and Proposition 9.3.1. To prove that they are distinct first note that for any $i, j \geq 0$, and any $w_1, w_2 \in \{a,b\}^*$, we have

$$b^iaw_1 = b^jaw_2 \Longrightarrow i \equiv j \pmod 2.$$

Therefore, $R_1 \neq R_2$ and $R_1 \neq R_3$. In order to prove that $R_2 \neq R_3$ we have to note a less apparent invariant of $\mathfrak{P}$: for any $i, k \geq 0$, any $j, l \geq 1$, and any $w_1, w_2 \in \{a,b\}^*$, where w_1 and w_2 do not start with b, we have

$$a^ib^jw_1 = a^kb^lw_2 \Longrightarrow i + j \equiv k + l \pmod 2.$$

Finally, every word containing ba as a subword has the form a^ib^jaw, where $i \geq 0$, $j \geq 1$, and hence $M = R_1 \cup R_2 \cup R_3$, so that R_1, R_2 and R_3 are the only minimal right ideals of S. ∎

Lemma 1.7. *The minimal two-sided ideal M of S is a completely simple semigroup, and is a union of six copies of a group.*

PROOF. The lemma follows directly from Lemmas 1.3 and 1.6 and Proposition 9.3.1. ∎

Lemma 1.8. *The Schützenberger group H of the minimal two-sided ideal M of S is defined by the presentation*

$$\langle x, y, z \mid x^2 = 1,\ y^{q/2} = 1,\ xyxy^{-1}(xy^{-1}xy)^2 = 1\rangle.$$

It is a metabelian group of order $q(2^{q/2} - 1)$.

Proof. We find a presentation for H by rewriting the presentation (2) for S in accord with Theorem 10.3.2. Since S has two generators and two minimal left ideals we need four new generating symbols, which we denote by $t_{1,a}$, $t_{2,a}$, $t_{1,b}$, $t_{2,b}$. The action of S on its minimal left ideals is given in Lemma 1.3.

We rewrite the defining relations for S as follows:

	a	a	a	$=$		a
1	$t_{1,a}$	$t_{1,a}$	$t_{1,a}$	$=$	1	$t_{1,a}$
2	$t_{2,a}$	$t_{1,a}$	$t_{1,a}$	$=$	2	$t_{2,a}$

	b	b	$\dots$	b	$=$		b
1	$t_{1,b}$	$t_{2,b}$	$\dots$	$t_{1,b}$	$=$	1	$t_{1,b}$
2	$t_{2,b}$	$t_{1,b}$	$\dots$	$t_{2,b}$	$=$	2	$t_{2,b}$

	a	b	b	$\dots$	b	a	b	b	$=$		b	b	a
1	$t_{1,a}$	$t_{1,b}$	$t_{2,b}$	$\dots$	$t_{1,b}$	$t_{2,a}$	$t_{1,b}$	$t_{2,b}$	$=$	1	$t_{1,b}$	$t_{2,b}$	$t_{1,a}$
2	$t_{2,a}$	$t_{1,b}$	$t_{2,b}$	$\dots$	$t_{1,b}$	$t_{2,a}$	$t_{1,b}$	$t_{2,b}$	$=$	2	$t_{2,b}$	$t_{1,b}$	$t_{2,a}$

The word $b^q a^2$ represents an idempotent by Lemma 1.1 (vi), and it is obvious that this idempotent belongs to M. Rewriting $b^q a^2$ gives

	b	b	$\dots$	b	a	a		
1	$t_{1,b}$	$t_{2,b}$	$\dots$	$t_{2,b}$	$t_{1,a}$	$t_{1,a}$	$=$	1
2	$t_{2,b}$	$t_{1,b}$	$\dots$	$t_{1,b}$	$t_{2,a}$	$t_{1,a}$	$=$	1

Thus we obtain the following presentation for H:

$$\begin{aligned}\langle t_{1,a}, t_{2,a}, t_{1,b}, t_{2,b} \mid t_{1,a}^2 = 1,\ (t_{1,b}t_{2,b})^{q/2} = 1,\\ t_{1,a}(t_{1,b}t_{2,b})^{(r-1)/2}t_{1,b}t_{2,a}t_{1,b}t_{2,b} = t_{1,b}t_{2,b}t_{1,a},\\ t_{2,a}(t_{1,b}t_{2,b})^{(r-1)/2}t_{1,b}t_{2,a}t_{1,b}t_{2,b} = t_{2,b}t_{1,b}t_{2,a},\ t_{2,a}t_{1,a} = 1\rangle.\end{aligned}$$

If we use

$$t_{2,a} = t_{1,a}^{-1} = t_{1,a}$$

to eliminate $t_{2,a}$, and if we denote $t_{1,a}$, $t_{1,b}$, $t_{2,b}$ by x, z, t respectively, we obtain the presentation

$$\langle x, z, t \mid x^2 = 1,\ (zt)^{q/2} = 1,\ x(zt)^{(r-1)/2}zxzt = ztx,\ x(zt)^{(r-1)/2}zxzt = tzx\rangle.$$

From the last two relations we obtain

$$zt = tz.$$

Now we introduce a new generator y by

$$zt = y,$$

and then eliminate t, so that we obtain

$$\langle x, y, z \mid x^2 = 1,\ y^{q/2} = 1,\ xy^{(r-1)/2}zxy = yx,\ yz = zy\rangle.$$

Next we introduce a new generator

$$u = y^{(r-1)/2}z,$$

eliminate z by using this generator, and obtain

$$\langle x, y, u \mid x^2 = 1,\ y^{q/2} = 1,\ xuxy = yx,\ yu = uy\rangle.$$

Finally, by eliminating u, we obtain the desired presentation:

$$\langle x, y \mid x^2 = 1,\ y^{q/2} = 1,\ xyxy^{-1}(xy^{-1}xy)^2 = 1\rangle.$$

To see that H is metabelian (i.e. that the derived subgroup H' is abelian) we introduce a new generator

$$v = xy^{-1}xy,$$

after which our presentation becomes

$$\langle x, y, v \mid x^2 = 1,\ y^{q/2} = 1,\ v = xy^{-1}xy,\ xvx = v^{-1},\ zvz^{-1} = v^2\rangle,$$

so that H' is the cyclic group generated by v. From the second and the last relations it follows that the order of v divides $2^{q/2} - 1$, and it is relatively easy (using the Reidemeister—Schreier Theorem, say) to prove that v indeed has order $2^{q/2} - 1$. Since the group H/H' is clearly isomorphic to $C_2 \times C_{q/2}$, it follows that $|H| = q(2^{q/2} - 1)$, as required. ∎

If we combine Lemmas 1.3, 1.4, 1.6 and 1.8 we obtain the following:

Theorem 1.9. *Let $q, r \in \mathbb{N}$, and assume that q is even and r is odd. The semigroup S defined by the presentation*

$$\langle a, b \mid a^3 = a,\ b^{q+1} = b,\ ab^rab^2 = b^2a\rangle$$

has a unique minimal two-sided ideal M, which is a completely simple semigroup with two minimal left ideals and three minimal right ideals. The Schützenberger group H of M is the metabelian group of order $q(2^{q/2} - 1)$ defined by the presentation

$$\langle x, y \mid x^2 = 1,\ y^{q/2} = 1,\ xyxy^{-1}(xy^{-1}xy)^2 = 1\rangle. \tag{4}$$

S is a finite semigroup of order $q2^{q/2} + 2q + 2$. The egg-box picture of S is shown in Figure 12.

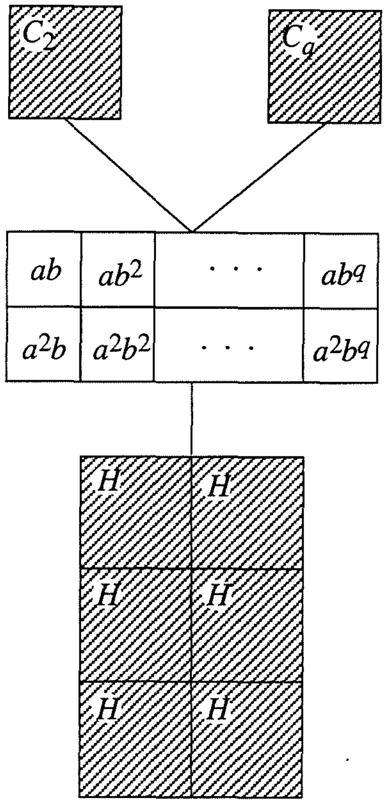


Figure 12.

As we have already mentioned, presentation (2) is rather special when compared to the other presentations considered in this thesis. Nevertheless it had a big influence on the general theory from Chapters 9 and 10. There are several reasons for this. First of all, every presentation (2) (with q even and r odd) defines a finite semigroup of relatively small order; see Theorem 1.9, and compare with Theorems 11.4.5, 12.4.4 and 12.5.10. Hence these semigroups are very convenient for a computational investigation using the Todd—Coxeter enumeration procedure.

Next, unlike Coxeter type semigroups and (non-generalised) Fibonacci semigroups which all have a unique minimal left ideal or a unique minimal right ideal, a semigroup S defined by (2) has two minimal left ideals and three minimal right ideals. This opens up a possibility for the Schützenberger group H of the minimal two-sided ideal M not to be isomorphic to the group G defined by (2). Actually, it was quickly discovered by E.F. Robertson that both possibilities $H \cong G$ and $H \not\cong G$ occur for various values of q and r, and it was conjectured that $H \cong G$ if and only if $q = 2r$. Walker (1992) proved this conjecture in one direction: if $q = 2r$ then $H \cong G$.

Using Theorem 1.9 we can prove the whole conjecture. If G is the group defined by

$$\langle a, b \mid a^2 = 1,\ b^q = 1,\ ab^r ab^2 = b^2 a \rangle,$$

from the last relation we see that b^r is conjugate to a, so that $b^{2r} = 1$. If we let $s = \text{g.c.d.}(q/2, r)$, then we see that $b^{2s} = 1$. Since r is odd we have $r \equiv s \pmod{2s}$, and so we have the following presentation for H:

$$\langle a, b \mid a^2 = 1,\ b^{2s} = 1,\ ab^s ab^2 = b^2 a \rangle. \tag{5}$$

The group defined by the above presentation is $H^{s,2,-2}$ in the notation from Campbell, Coxeter and Robertson (1977). Theorem 8.2 from the same paper implies that this group is metabelian of order $2s(2^s - 1)$. Now, if $q \neq 2r$, then $s < q/2$, so that $|H| > |G|$, so that H and G are not isomorphic. On the other hand, it is a routine matter to check that presentations (4) and (5) indeed define isomorphic metabelian groups for $q = 2r$.

Early computational evidence enabled E.F. Robertson to establish the following multiplication table for the products of idempotents of the minimal two-sided ideal M of S:

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	e_1	e_1	e_1	e_4	e_4	$e_4 b^{q-2r}$
e_2	e_2	e_2	e_2	e_5	e_5	$e_5 b^{q-2r}$
e_3	e_3	e_3	e_3	$e_6 b^{2r}$	$e_6 b^{2r}$	e_6
e_4	e_1	e_1	$e_1 b^{2r}$	e_4	e_4	e_4
e_5	e_2	e_2	$e_2 b^{2r}$	e_5	e_5	e_5
e_6	$e_3 b^{q-2r}$	$e_3 b^{q-2r}$	e_3	e_6	e_6	e_6

where

$$e_1 = b^q a^2,\ e_2 = ab^q a,\ e_3 = b^{2r-1}a^2ba^2,$$
$$e_4 = b^{2r-2}a^2ba^2b,\ e_5 = a^2b^{2r-1}a^2b,\ e_6 = b^{q-1}a^2b.$$

With Lemma 1.1 in mind it is easy to see that these idempotents are closed under the multiplication if and only if $q = 2r$. In this way the computer evidence suggested Theorem 9.2.1.

Considering the non-isomorphic case $q \neq 2r$ was the first step towards a general rewriting theory for the Schützenberger group of the minimal two-sided ideal developed in Chapter 10. (Note that the generalised Fibonacci semigroup $S(2, 6, 2)$ from Section 3 of Chapter 11, although 'nice', was not a suitable working example, since it is infinite and does not admit a straightforward investigation by using Todd—Coxeter enumeration procedure.) Note that, for $q = 2r$, the idempotent $e = e_1$ has the property

$$ew_1w_2e = ew_1ew_2e,$$

for any words w_1, w_2; see the proof of Theorem 9.2.1. If $q \neq 2r$, e does not have this property (as it would imply $H \cong G$). However, it was noted that e has the following weaker property:

$$ea^{i_1}b^{j_1}a^{i_2}b^{j_2}\dots a^{i_k}b^{j_k}a^{i_{k+1}}e = ea^{i_1}eb^{j_1}ea^{i_2}eb^{j_2}e\dots ea^{i_k}eb^{j_k}ea^{i_{k+1}}e,$$

giving rise to an obvious rewriting mapping, and later to a presentation for H. A generalisation of this approach yielded the following more general, but very technical, rewriting theorem:

Theorem 1.10. *Let S be the semigroup defined by the presentation*

$$\mathfrak{P} = \langle a_1, a_2 \mid a_1^{n_1+1} = a_1,\ a_2^{n_2+1} = a_2,\ \alpha_1 = \beta_1,\ \dots, \alpha_m = \beta_m\rangle,$$

where $n_1, n_2, m > 0$, and each α_i and each β_i is a non-empty word in a_1 and a_2. Let S have both minimal left ideals and minimal right ideals, let M be the minimal two-sided ideal of S and let H be the Schützenberger group of M. Suppose that there exists an idempotent e from M satisfying the following two conditions:

(E1) $ea_1^i a_2 = ea_1^i ea_2$, *for* $1 \leq i \leq n_1$;
(E2) $ea_2^i a_1 = ea_2^i ea_1$, *for* $1 \leq i \leq n_2$.

Let $e = a_t^p\gamma$, where $t \in \{1,2\}$, and γ is a word which does not start with a_t. Let B be the alphabet $\{x_{i,j} \mid 1 \leq i \leq 2,\ j \in \mathbb{N}\}$, with the convention that $x_{i,j} = x_{i,k}$ if $j \equiv k \pmod{n_i}$. We define a mapping $\phi\{a_1, a_2\}^+ \longrightarrow B^+$ by

$$a_i^j\phi = x_{i,j},\ (\delta_1 a_i a_j \delta_2)\phi = (\delta_1 a_i)\phi \cdot (a_j\delta_2)\phi,\ \textit{if } i \neq j.$$

Then H is defined by the presentation $\langle B \mid \mathfrak{R}\rangle$, where $\mathfrak{R}$ is the following set of relations:

(i) $(a_{j_1}^{k_1}\alpha_i a_{j_2}^{k_2})\phi = (a_{j_1}^{k_1}\beta_i a_{j_2}^{k_2})\phi$, *for* $1 \leq j_p \leq 2$, $1 \leq k_p \leq n_{j_p}$, $1 \leq i \leq m$;
(ii) $(a_j^k e)\phi = a_j^k\phi$, *for* $1 \leq j \leq 2$, $1 \leq k \leq n_j$;
(iii) $(ea_j^k)\phi = a_j^k\phi$, *for* $1 \leq j \leq 2$, $1 \leq k \leq n_j$;
(iv) $x_{t,q+p} = x_{t,q}x_{t,p}$ *if e is of the form $a_t^p\delta a_t^q$ (otherwise there is no relation here).* ■

(For details on the above theorem and for a proof see Campbell, Robertson, Ruškuc and Thomas (1994a).)

Due to the lack of feasible examples, the authors hoped for a while that it is always possible to find an idempotent satisfying conditions (E1) and (E2) (or their left-right duals). However, as the following example shows this is not the case.

Example 1.11. Let S be the semigroup defined by the presentation

$$\langle a, b \mid a^3 = a,\ b^3 = b,\ (ab)^3 = (ab)^2,\ ab^2aba^2b = ab^2a^2b,$$
$$bab^2aba = (ba)^2,\ ba^2b^2a^2ba = ba^2ba,\ ab^2bab = ab^2ab,\ abab^2ab = abab,$$
$$(b^2a^2)^3 = b^2a^2,\ (ab^2)^3 = (ab^2)^2,\ (ba^2)^2(ba)^2(ab)^2a^2b^2 = (ba^2)^2b^2\rangle.$$

Let $w_1 \equiv x_1^{i_1}y_1^{j_1}w_1'z_1^{k_1}u_1^{l_1}$ and $w_2 \equiv x_2^{i_2}y_2^{j_2}w_2'z_2^{k_2}u_2^{l_2}$ be two words, where $\{x_1, y_1\} = \{x_2, y_2\} = \{z_1, u_1\} = \{z_2, u_2\} = \{a, b\}$, w_1' does not begin with y_1 and does not end with z_1, and w_2' does not begin with y_2 and does not end with z_2. From the nature of our presentation we see that if $w_1 = w_2$ holds in S then we have

$$x_1 = x_2,\ y_1 = y_2,\ z_1 = z_2,\ u_1 = u_2,$$
$$i_1 \equiv i_2 \pmod 2,\ j_1 \equiv j_2 \pmod 2,\ k_1 \equiv k_2 \pmod 2,\ l_1 \equiv l_2 \pmod 2.$$

By using the Todd—Coxeter enumeration procedure we can see that S is finite of order 224. In particular, S has minimal left ideal and minimal right ideals. Let e be a word representing an idempotent of the minimal two-sided ideal M of S. It is clear that e must contain both a and b. Also, it is clear that e can be chosen to have the length (in the free product sense) at least three. Let us assume that $e \equiv e'a_2a_1^ia_2^j$, where $1 \leq i, j \leq 2$. Then for $k \neq i$ we have

$$ea_1^ka_2 \equiv e'a_2a_1^ia_2^ja_1^ka_2 \neq e'e_2a_1^ia_2^ja_1^ke'a_2a_1^ia_2^{j+1} \equiv ea_1^kea_2,$$

and e does not satisfy condition (E1). Similarly, if $e \equiv e'a_1a_2^ia_1^j$, the e does not satisfy (E2). ■

The above example, as well as the fact that Theorem 1.10 is technically very complicated, prompted a search for a 'better' presentation for the Schützenberger group of a minimal two-sided ideal. The first such result is Theorem 10.3.3, where the idempotents of the minimal two-sided ideal and their interaction still play an important role. However, work on Theorem 10.3.3 suggested that the notion of rewriting mapping is at least as important as the idempotents, thus paving the path for the general rewriting theory described in Section 7 of Chapter 6, which, in turn, lead to Reidemeister—Schreier type results in Chapters 7 and 10.

2. Semigroups defined by presentations $\langle a, b \mid a^3 = a, b^{q+1} = b, \alpha = b^r\rangle$

In this section we consider semigroups defined by presentations

$$\langle a, b \mid a^3 = a,\ b^{q+1} = b,\ \alpha = b^r\rangle, \tag{6}$$

where $q, r \in \mathbb{N}$, $\alpha \in \{a, b\}^+$, and we classify them with respect to the properties (F1), (F2), (F3); see the introduction to this chapter.

A typical procedure for determining the type of a semigroup S defined by a presentation (6) consists of the following steps:

1. checking if S has minimal left and right ideals;
2. determining the number of minimal left ideals and the number of minimal right ideals;
3. if there is more than one minimal left ideal and more than one minimal right ideal then rewriting (6) into a presentation for the Schützenberger group of the minimal two sided ideal.

Except for the case $\alpha \equiv a^2$, the type of a semigroup S defined by (6) depends only on α and not on q and r. More precisely, the type of S crucially depends on the beginning and ending of α. Therefore, we consider the following cases

$$\alpha \equiv a,\ \alpha \equiv a^2,\ \alpha \equiv ab\beta ba,\ \alpha \equiv ab\beta b,\ \alpha \equiv b\beta ba,\ \alpha \equiv ab\beta ba^2$$
$$\alpha \equiv a^2b\beta ba,\ \alpha \equiv b\beta ba^2,\ \alpha \equiv a^2b\beta b,\ \alpha \equiv b\beta b,\ \alpha \equiv a^2b\beta ba^2,$$

where $\beta \in \{a,b\}^*$. Note that because of the relations $a^3 = a$ and $b^{q+1} = b$ the above cases cover all the possibilities for α.

The number of cases can be reduced further by using the following obvious

Lemma 2.1. *Let $q, r \in \mathbb{N}$, let $\alpha \in \{a,b\}^+$, and let α^R be the reverse of α (i.e., if $\alpha \equiv x_1x_2\ldots x_k$, with $x_i \in \{a,b\}$, then $\alpha^R \equiv x_k\ldots x_2x_1$). The semigroup S defined by the presentation*

$$\langle a,b \mid a^3 = a,\ b^{q+1} = b,\ \alpha = b^r\rangle$$

is anti-isomorphic to the semigroup S^R defined by the presentation

$$\langle a,b \mid a^3 = a,\ b^{q+1} = b,\ \alpha^R = b^r\rangle. \blacksquare$$

Hence, for example, the case $\alpha \equiv ab\beta b$ is dual to $\alpha \equiv b\beta ba$, and we shall consider only the former. In what follows S and G will always denote the semigroup and the group defined by the considered presentation. We will be omitting most technical details, for which the reader is referred to Campbell, Robertson, Ruškuc, Thomas and Ünlü (1995).

The case $\alpha \equiv a$

In this case we consider the semigroup S defined by the presentation

$$\langle a,b \mid a^3 = a,\ b^{q+1} = b,\ a = b^r\rangle, \tag{7}$$

which, after eliminating a, becomes

$$\langle b \mid b^{3r} = b,\ b^{q+1} = b\rangle.$$

Therefore, S is isomorphic to the cyclic group of order g.c.d.$(q, 2r)$. In particular S has a unique minimal left ideal and a unique minimal right ideal and is isomorphic to the group defined by (7).

The case $\alpha \equiv a^2$

Now we have the presentation

$$\langle a, b \mid a^3 = a,\ b^{q+1} = b,\ a^2 = b^r \rangle. \tag{8}$$

Let us first assume that g.c.d.$(q, r) = 1$. Choose k such that $kr \equiv 1 \pmod{q}$, and introduce a new generator $c = b^r$. Then we may delete generator $b = c^k$ to obtain

$$\langle a, c \mid a^3 = a,\ c^{k(q+1)} = c^k,\ c = a^2 \rangle.$$

Next we delete c, and obtain

$$\langle a \mid a^3 = a,\ a^{2k(q+1)} = a^{2k} \rangle,$$

so that S is the cyclic group of order 2. In particular, S has a unique minimal left ideal and a unique minimal right ideal and is isomorphic to the group defined by (8).

Now we consider the case where g.c.d.$(q, r) = s \neq 1$. Let T be the homomorphic image of S obtained by adding the relation $b^{s+1} = b$ to (8). Therefore, T is defined by

$$\langle a, b \mid a^3 = a,\ b^{s+1} = b,\ a^2 = b^s \rangle,$$

which is equivalent to

$$\langle a, b \mid a^2 = b^s = 1 \rangle,$$

so that T is the monoid free product of the cyclic group C_2 of order 2 and the cyclic group C_s of order s. In particular, T has no minimal (left, right or two-sided) ideals, which implies that S does not have minimal (left, right or two-sided) ideals.

The case $\alpha \equiv ab\beta ba$

In this case S is defined by the presentation

$$\langle a, b \mid a^3 = a,\ b^{q+1} = b,\ ab\beta ba = b^r \rangle.$$

The element b generates a minimal left ideal L and a minimal right ideal R. Also, we have $L = R$, and this is the minimal two-sided ideal of S. This minimal two-sided ideal is isomorphic to G by Corollary 9.2.4. $S - L$ has exactly two elements a and a^2, so that $|S| = |G| + 2$, and S is finite if and only if G is finite.

The case $\alpha \equiv ab\beta b$

The presentation (6) in this case becomes

$$\langle a, b \mid a^3 = a,\ b^{q+1} = b,\ ab\beta b = b^r \rangle.$$

Element b again generates a unique minimal right ideal R. However, in this case S has three minimal left ideals L_1, L_2 and L_3 generated by b, ba and ba^2 respectively. By Corollary 9.2.4, each of L_1, L_2, L_3 is a group isomorphic to G. $S - R$ again has two elements a and a^2, and $|S| = 3|G| + 2$. S is finite if and only if G is finite.

The case $\alpha \equiv ab\beta ba^2$

In this case S is defined by

$$\langle a, b \mid a^3 = a,\ b^{q+1} = b,\ ab\beta ba^2 = b^r \rangle.$$

As in the previous two cases b generates a unique minimal right ideal R, but now S has two minimal left ideals L_1 and L_2 generated by b and ba respectively. Each of L_1, L_2 is a group isomorphic to G. The order of S is given by $|S| = 2|G| + 2$, and S is finite if and only if G is finite.

The case $\alpha \equiv b\beta ba^2$

Now S is defined by

$$\langle a, b \mid a^3 = a,\ b^{q+1} = b,\ b\beta ba^2 = b^r \rangle.$$

Assume that

$$b\beta b \equiv b^{i_1} a^{j_1} b^{i_2} a^{j_2} \dots b^{i_k} a^{j_k} b^{i_{k+1}},$$

with $k \geq 0$, $i_1, \dots, i_{k+1} \in \{1, \dots, q\}$, $j_1, \dots, j_k \in \{1, 2\}$.

First we consider the case $k = 0$ or $k > 0$ and $j_1 = j_2 = \dots = j_k = 2$. Let T be the homomorphic image of S defined by the presentation

$$\langle a, b \mid a^3 = a,\ b^{q+1} = b,\ b\beta ba^2 = b^r,\ ba^2 = a^2 b = b^2 = b \rangle.$$

This presentation is clearly equivalent to

$$\langle a, b \mid a^2 = 1,\ b^2 = b \rangle.$$

Therefore T is the monoid free product of the cyclic group C_2 and the two-element semilattice (b with 1 adjoined to it), and hence does not have minimal (left, right or two-sided) ideals. Therefore, if $k = 0$ or $j_1 = \ldots = j_k = 2$, then S has no minimal (left, right or two-sided) ideals.

Now we consider the case where $k > 0$ and $j_l = 1$ for some l. It is possible to show that S satisfies the relation $ba^2 = b$, so that S can be defined by a presentation

$$\langle a, b \mid a^3 = a,\ b^{q+1},\ ba^2 = b,\ b^{m_1}ab^{m_2}a \ldots b^{m_l}ab^{m_{l+1}} = b^r \rangle,$$

where $l \geq 1$, $m_1, \ldots, m_l \in \{1, \ldots, q\}$. S has two minimal left ideals L_1 and L_2 generated by b and ba respectively; it also has three minimal right ideals R_1, R_2 and R_3 generated by b, ab and a^2b respectively. The action of S on L_1 and L_2 is given by

	a	b
1	2	1
2	1	1

Rewriting the relations $a^3 = a$ and $b^{q+1} = b$ in accord with Theorem 10.3.2 gives

$$t_{1,a}t_{2,a} = 1,\ t_{1,b}^q = 1, \tag{9}$$

while rewriting $ba^2 = b$ does not give any new relations. The word b^q represents the idempotent of $L_1 \cap R_1$; rewriting this word gives

$$t_{1,b} = t_{2,b}. \tag{10}$$

With (9) and (10) in mind, we see that rewriting $ba^2 = b$, $b^{m_1}ab^{m_2}a \ldots b^{m_l}ab^{m_{l+1}} = b^r$ gives

$$t_{1,b}^{m_1}t_{1,a}t_{1,b}^{m_2}t_{1,a} \ldots t_{1,a}t_{1,b}^{m_{l+1}} = t_{1,b}^r.$$

Therefore, H is defined by the presentation

$$\langle x, y \mid y^q = 1,\ y^{m_1}xy^{m_2}x \ldots xy^{m_{l+1}} = y^r \rangle.$$

It is easy to see that H can be both finite and infinite, depending on l and $m_1, \ldots, m_{l+1}$. S is finite if and only if H is finite and $|S| = 6|H| + 2$.

The case $\boldsymbol{\alpha \equiv b\beta b}$

In this case S is defined by the presentation

$$\langle a, b \mid a^3 = a,\ b^{q+1} = b,\ b\beta b = b^r \rangle.$$

Let us assume that

$$b\beta b \equiv b^{i_1}a^{j_1}b^{i_2}a^{j_2} \ldots b^{i_k}a^{j_k}b^{i_{k+1}},$$

where $k \geq 0$, $i_1, \ldots, i_{k+1} \in \{1, \ldots, q\}$, $j_1, \ldots, j_k \in \{1, 2\}$. If $k = 0$ then S is the semigroup free product of two cyclic groups, and hence has no minimal (left, right or two-sided) ideals. When $k > 0$, all the words $(ba^{3-j_1})^l b$, $l \geq 1$, generate distinct right ideals, and all the words $b(a^{3-j_k}b)^l$, $l \geq 1$, generate distinct left ideals. Hence S has no minimal left ideals and no minimal right ideals. However, S might have minimal two-sided ideals. For instance, if S is defined by

$$\langle a, b \mid a^3 = a,\ b^2 = b,\ baba^2b = b \rangle,$$

then b generates a minimal two-sided ideal.

The case $\alpha \equiv a^2 b\beta ba^2$

Now we have the presentation

$$\langle a, b \mid a^3 = a,\ b^{q+1} = b,\ a^2 b\beta ba^2 = b^r \rangle. \tag{11}$$

This case is very similar to the case $\alpha \equiv b\beta ba^2$. Let us assume that

$$b\beta b \equiv b^{i_1} a^{j_1} b^{i_2} b^{j_2} \ldots b^{i_k} a^{j_k} b^{i_{k+1}}.$$

If $k = 0$ or if $j_1 = j_2 = \ldots = j_k = 2$, then S has no minimal (left, right or two-sided) ideals. If $k > 0$ and at least one j_s is 1, then (11) is equivalent to

$$\langle a, b \mid a^3 = a,\ b^{q+1} = b,\ a^2 b = ba^2 = b,\ a^2 b^{m_1} ab^{m_2} a \ldots b^{m_l} ab^{m_{l+1}} = b^r \rangle.$$

S has two minimal left ideals L_1 and L_2 which are generated by b and ba respectively, and it has two minimal right ideals R_1 and R_2 which are generated by b and ab respectively. The Schützenberger group H of the minimal two-sided ideal M of S has a presentation

$$\langle x, y \mid y^q,\ y^{m_1} xy^{m_2} x \ldots y^{m_l} xy^{m_{l+1}} = y^r \rangle.$$

S is finite if an only if H is finite, and we have $|S| = 4|H| + 2$.

The results from this section are summarised in Table 3.

α	S	comments	size of m.i.
a	(F1)		1×1
a^2	(F1) (F3)	if g.c.d.$(q, r) = 1$ if g.c.d.$(q, r) > 1$	1×1
$ab\beta ba$	(F1)		1×1
$ab\beta b$	(F1)		1×3
$b\beta ba$	(F1)		3×1
$ab\beta ba^2$	(F1)		1×2
$a^2b\beta ba$	(F1)		2×1
$b\beta ba^2$	(F2) (F3)	if $b\beta b$ contains bab otherwise	3×2
$a^2b\beta b$	(F2) (F3)	if $b\beta b$ contains bab otherwise	2×3
$b\beta b$	(F3)	may have minimal two-sided ideal	
$a^2b\beta ba^2$	(F2) (F3)	if $b\beta b$ contains bab otherwise	2×2

Table 3.

Chapter 12

Computational methods

Many results of the previous chapters have been influenced by computational evidence obtained from an implementation of the Todd—Coxeter enumeration procedure running on computers at the University of St Andrews. By commenting on this influence we hope to have underlined the usefulness of computational methods in the study of semigroup presentations and semigroups in general. However, the formulation and the proofs of the results did not depend on any computational evidence, and so the over-simplified abstraction of the Todd—Coxeter enumeration procedure shown in Figure 4 sufficed for the needs of the first eleven chapters. In this chapter we give due credit to the Todd—Coxeter enumeration procedure by describing it in detail, and by giving three modifications of this procedure. These modifications enumerate Rees quotients by one-sided ideals, minimal one-sided ideals and the idempotents of minimal two-sided ideals. These procedures can be combined with the Reidemeister—Schreier type results of Chapters 7 and 10, giving these results a more computational flavour. We also pose several open problems which we consider important for the further development of computational methods for semigroup presentations.

In Section 1 we give a brief summary of the development of computational methods in semigroup theory. In Section 2 we establish the notation that we need to describe various Todd—Coxeter type procedures, and we describe the standard Todd—Coxeter enumeration procedure. In Sections 3, 4 and 5 we give the mentioned three modifications of this standard procedure. The notation from 2 and the results from 4 and 5 will appear in Campbell, Robertson, Ruškuc and Thomas (1995a). The result from Section 3 appears here for the first time.

1. A short history

As we mentioned in Chapter 3, semigroup theory draws a significant amount of motivation and ideas from group theory. Therefore it is natural to look at computational group theory for hints on the development of computational semigroup theory.

Computational group theory is a relatively young mathematical discipline; Neubüser (1995) dates the beginning of its history to 1953, which is the year of the first implementation of the Todd—Coxeter enumeration procedure for groups. However, the same author points out that the 'prehistory' of computational group theory began at the same time as group theory itself, thus implying that the delay in development was not due to lack of interest. The main reason for the delay is that computational group theory requires well developed theoretical group theory, as well as advanced computers, and it was only in the second half of this century that both these requirements were met. The result has been rather dramatic: the number of results has been growing so rapidly, that an intention of writing a book on the subject in 1970 resulted in 1994 in the book of Sims (1994) dealing exclusively with group presentations. This book is the best reference available for computational methods for group presentations. For a general discussion on computational group theory and its development the reader is referred to Neubüser (1995).

The first applications of computational techniques to semigroups seem to have been the determination of semigroups of small orders; see Forsythe (1955), Tetsuya et al. (1955) and Plemmons (1967), (1970); see also Chapter 15 in Jürgensen et al. (1991). In these papers the authors develop computational methods for handling semigroups defined by their multiplication tables. However, as we mentioned in Chapter 3, multiplication tables do not seem to be the best way to approach semigroups in general. The other two approaches—transformation semigroups and semigroups defined by presentations—although harder to develop, seem more promising in the long term.

Lallement and McFadden (1990) give a collection of algorithms for investigating finite transformation semigroups, as well as a bibliography containing earlier partial attempts in this field. The algorithms of Lallement and McFadden give the structure of a transformation semigroup S (given by its generators) in terms of Green's relations: they calculate all $\mathcal{D}$-classes of S, and for each $\mathcal{D}$-class D they determine whether or not it is regular, find the number of $\mathcal{L}$-classes and $\mathcal{R}$-classes of D and find a generating set for the Schützenberger group (as a permutation group) of D. A computer implementation of these algorithms is described in Champarnaud and Hansel (1991). Konieczny (1994) generalised these results to semigroups of binary relations.

The development of computational methods for semigroups defined by presentations has been concentrated around the two main procedures: the Knuth—Bendix rewriting procedure (introduced in Knuth and Bendix (1970); see also Sims (1994)) and the Todd—Coxeter enumeration procedure, which is the main theme of this chapter.

The origins of the Todd—Coxeter enumeration procedure are in Todd and Coxeter (1936), where they described the procedure for groups. Neumann (1967) showed how this procedure can be used for semigroups, but he did not prove the validity of the modification. This gap was filled by Jura (1978). The first computer implementation of the procedure is attributed to J.M. Parrington by Cannon (1969). The first FORTRAN version of the Todd—Coxeter enumeration procedure running in St Andrews was by E.F. Robertson and Y. Ünlü; see Robertson and Ünlü (1993). An improved C version is due to T.G. Walker; see Walker (1992). Two modifications of the Todd-Coxeter enumeration procedure for enumerating minimal one-sided ideals and idempotents of the minimal two-sided ideal are described in Campbell, Robertson, Ruškuc and Thomas (1995a). Also in Campbell, Robertson, Ruškuc and Thomas (1994a), (1995a) and (1995b) foundations are laid for a Reidemeister—Schreier type theory for semigroups, which has also been a topic of a detailed discussion in this thesis.

2. Todd—Coxeter enumeration procedure

We begin this section by describing certain data structures and actions on these data structures, in terms of which we will describe the Todd—Coxeter enumeration procedure and various modifications of this procedure.

Let $A = \{a_1, \ldots, a_n\}$ be a finite alphabet, and let ∞ be a symbol. We define the following three data structures.

D1 A finite set C of non-negative integers containing 0 and 1.

D2 A mapping
$$\tau : (C \cup \{\infty\}) \times A \longrightarrow C \cup \{\infty\},$$
such that
$$(0, a)\tau = 0, \ (\infty, a)\tau = \infty,$$
for all $a \in A$.

D3 A set K which is a subset of $C \times C$.

The elements of C are called *cosets*. In the standard version of the Todd—Coxeter enumeration procedure they are simply names for elements of the enumerated semigroup S. The cosets 0 and 1 *do not* represent elements of S. They are included in C for technical reasons, and can be conveniently thought of as representing an adjoined zero and an adjoined identity. In the variants of the procedure described in Section 3 cosets are names for the elements of a Rees quotient S/L, where L is a left ideal of S, while in Section 4 cosets are names for minimal left ideals of S.

The mapping τ is called the *coset table*. This mapping extends to a mapping
$$\tau : (C \cup \{\infty\}) \times A^+ \longrightarrow C \cup \{\infty\}$$
by
$$(i, w_1w_2)\tau = ((i, w_1)\tau, w_2)\tau.$$

An equality $(i, w)\tau = j$ means that the coset i multiplied by the word w gives the coset j. An equality $(i, w)\tau = \infty$ means that the result of multiplication of i by w has not yet been defined. We say that row i of the coset table is *complete* if $(i, a)\tau \neq \infty$ for all $a \in A$.

The set K is called the *coincidence set*. Intuitively, it consists of pairs (i, j), $i, j \in C$, which have been found to be equal during the course of the procedure.

The terms 'coset' and 'coset table' originate from computational group theory, as the Todd—Coxeter enumeration procedure for groups enumerates cosets of a subgroup in a group. We have retained them here for historical reasons. The terms 'element' and 'ideal' instead of 'coset' may suit better the particular procedures described here.

Now we introduce five actions on the above data structures.

E1 Make a new definition. Suppose that the coset table has a row which is not complete. Let i be the first such a row, and let j be the least suffix such that $ia_j = \infty$. If k is the greatest natural number which has belonged to C during the procedure, add $k+1$ to C, redefine $(i, a_j)\tau = k+1$ in the definition of the coset table, and define $(k+1, a)\tau = \infty$, for all $a \in A$.

E2 Push a row of a relation. For i in C and a relation $u = v$, with $u, v \in A^+$, we compare $(i, u)\tau$ and $(i, v)\tau$ and make them equal as follows. Let $u \equiv u_1 a$ and $v \equiv v_1 b$, with $a, b \in A$.

- **(i)** If $(i, u)\tau = \infty$, $(i, v)\tau = k \neq \infty$ and $(i, u_1)\tau = l \neq \infty$ then redefine $(l, a)\tau = k$ in the definition of the coset table.
- **(ii)** If $(i, u)\tau = k \neq \infty$, $(i, v)\tau = \infty$ and $(i, v_1)\tau = l \neq \infty$, then redefine $(l, b)\tau = k$ in the definition of the coset table.
- **(iii)** If $(i, u)\tau = k \neq \infty$, $(i, v)\tau = l \neq \infty$ and $k \neq l$, then add the pair (k, l) to the coincidence set K.

E3 Push a row of a 1-word. Let $f \in A^+$ and $i \in C$. We force $(i, f)\tau$ to be equal to 1 as follows. Let $f \equiv f_1 a$, $a \in A$.

- **(i)** If $(i, f_1)\tau = k \neq \infty$ and $(i, f)\tau = \infty$, then redefine $(k, a)\tau = 1$.
- **(ii)** If $(i, f)\tau = k \notin \{1, \infty\}$ then add the pair $(1, k)$ to the coincidence set K.

E4 Push row 1 of a 0-word. Let $f \in A^+$. We force $(1, f)\tau$ to be equal to 0 as follows. Let $f \equiv f_1 a$, $a \in A$.

- **(i)** If $(1, f_1)\tau = k \neq \infty$ and $(i, f)\tau = \infty$ then redefine $(k, a)\tau = 0$.
- **(ii)** If $(1, f)\tau = k \notin \{0, \infty\}$ then add the pair $(0, k)$ to the coincidence set K.

E5 Process a coincidence. Let (i, j) be an element of the coincidence set K. We examine the consequences of identifying i and j as follows. If $i = j$

then remove (i,j) from K and end the action. If $i > j$ then replace (i,j) by (j,i) and continue. If $i < j$ then do the following three steps.

- **(i)** For each $a \in A$ with $(j,a)\tau = k \neq \infty$ do the following two steps:
  - **(a)** if $(i,a)\tau = \infty$ then redefine $(i,a)\tau = k$;
  - **(b)** if $(i,a)\tau = l \notin \{\infty, k\}$ add the pair (l,k) to K.
- **(ii)** Replace j by i in the definition of the coset table and in the remaining elements of K.
- **(iii)** Remove j from C and (i,j) from K.

Let

$$\mathfrak{P} = \langle a_1, \ldots, a_m \mid u_1 = v_1, \ \ldots, u_n = v_n \rangle$$

be a finite semigroup presentation. The standard *Todd-Coxeter enumeration procedure* (shortly, TC) is defined as follows.

Start with

$$C = \{0,1\}, \ K = \emptyset, \ (0,a)\tau = 0, \ (1,a)\tau = \infty \ \ (a \in A).$$

A single pass of the procedure consists of the following three steps:

- **(1)** E1;
- **(2)** for every $i \in C$ and every $u_j = v_j$ from $\mathfrak{R}$ do E2;
- **(3)** repeat E5 until $K = \emptyset$.

The passes are repeated until the data structures D1, D2, D3 all stabilise (i.e. until a pass not changing any of them is performed).

The following proposition asserts that the TC procedure *does* enumerate elements of the semigroup defined by $\mathfrak{P}$.

Proposition 2.1. *Let*

$$\mathfrak{P} = \langle a_1, \ldots, a_n \mid u_1 = v_1, \ \ldots, u_n = v_n \rangle$$

be a finite semigroup presentation, and let S be the semigroup defined by $\mathfrak{P}$. The Todd—Coxeter enumeration procedure terminates after a finite number of passes if and only if S is finite. When the procedure stops, the final coset table C gives the right regular representation of S acting on S with a zero and an identity adjoined, and $|C| - 2$ is the order of S. ■

A proof of the above theorem can be found in Jura (1978a). One may note that the coset 0 is superfluous in the TC procedure: at any stage of the procedure we have $ia = 0$ if and only if $i = 0$, because the procedure does not involve action E4. The TC procedure, as described above, has straightforward modifications for enumerating monoids (in which case relations of the form $u = 1$ are allowed), semigroups with zero (in which case relations of the form $u = 0$ are allowed) and monoids with zero (where both above types of relations are allowed). Finally, we note that the TC procedure *is not* an algorithm, since it does not necessarily terminate.

3. Enumerating one-sided Rees quotients

As before, let us have a finite semigroup presentation

$$\mathfrak{P} = \langle A \mid \mathfrak{R} \rangle = \langle a_1, \ldots, a_m \mid u_1 = v_1, \ldots, u_n = v_n \rangle,$$

and let S be the semigroup defined by $\mathfrak{P}$. In addition, let us have a finite set of words

$$X = \{\xi_1, \ldots, \xi_r\} \subseteq A^+,$$

and let R be the right ideal of S generated by X. Let S^* be the semigroup S with an identity adjoined to it (regardless of whether S has an identity or not). Obviously, R is a right ideal of S^*, and there is an action

$$\sigma : S^*/R \times S \longrightarrow S^*/R$$

of S on the set $S^*/R = (S^* - R) \cup \{0\}$, given by

$$(x, s)\sigma = \begin{cases} xs & \text{if } x \in S^* - R \text{ and } xs \in S - R \\ 0 & \text{otherwise} \end{cases}$$

(see Appendix A).

In this section we describe a modification of the Todd—Coxeter enumeration procedure, which we call the RQ (Rees quotient) procedure, and which terminates if and only if R has finite index in S, and gives the index of R in S, the action σ and a set of representatives of $S - R$; see Chapter 7 for definitions of the above concepts. Of course the described procedure has a left-right dual, for enumerating the Rees quotient of a semigroup by a left ideal. These procedures can be used in conjunction with the Reidemeister—Schreier type results from Section 3 of Chapter 7, thus giving effective finite presentations for one-sided ideals of finite index. Finally, we note that to obtain an effective finite presentation for the minimal two-sided ideal I of S generated by X, we have to enumerate the two-sided Rees-quotient R/I, which is equivalent to enumerating (by using the TC procedure) the semigroup defined by the presentation

$$\langle a_1, \ldots, a_m \mid \mathfrak{R}, \ \xi_1 = 0, \ \ldots, \ \xi_r = 0 \rangle.$$

The RQ procedure is based on the data structures and actions introduced in the previous section. The initialisation of the procedure is the same as for the TC procedure:

$$C = \{0, 1\}, \ K = \emptyset, \ (0, a)\tau = 0, \ (1, a)\tau = \infty \ \ (a \in A). \tag{1}$$

One pass of the procedure consists of the following four steps:

(1) E1;
(2) for all $i \in C$ and all $u_j = v_j$ in $\mathfrak{R}$ do E2;
(3) for every $\xi_i \in X$ do E4;
(4) repeat E5 until $K = \emptyset$.

The passes are repeated until all the data structures D1, D2, D3 stabilise.

During the procedure to each $i \in C$ we assign a word z_i by the following rule: z_1 is the empty word ϵ; $z_0 \equiv \xi_1$; if i has been defined as $i = (j, a_k)\tau$ then $z_i \equiv z_j a_k$. It is easy to check that

$$(1, z_i)\tau = i,$$

for all $i \in C - \{0\}$.

Now we start working towards our main result (Theorem 3.10), which asserts that RQ does indeed enumerate the Rees quotient S/R.

Lemma 3.1. *For every $k \geq 0$ there exists $n_0 \in \mathbb{N}$ such that one of the following two statements is true:*

(i) for every $n \geq n_0$ we have $k \notin C$ after the nth pass of the procedure; or
(ii) the nth pass of the procedure, $n \geq n_0$, does not change any of the values $(k, a_j)\tau$, $1 \leq j \leq m$.

PROOF. The only action which adds a new element to C is E1, but the added element is always greater than any other previously used element of C. Therefore, if k is removed from C at some stage of the procedure, it will never be added to C again, and (i) holds.

So assume that at some stage we add k to C, and that we never remove it after it. For a fixed j, $1 \leq j \leq m$, the values $(k, a_j)\tau$ can change during the procedure, but they always decrease. Therefore, $(k, a_j)\tau$ will take only finitely many values, and will eventually stabilise. Consequently, any of $(k, a_j)\tau$, $j = 1, \ldots, m$, will take only finitely many values and they will all eventually stabilise, so that (ii) holds. ∎

When case (ii) occurs we say that row k has stabilised. Lemma 3.1 in effect says that each row of the coset table will eventually disappear or stabilise. However, as noted in Neubüser (1982), Lemma 3.1 does not imply that we can *predict* when this is going to happen, as this would yield a solution to the finiteness problem.

Lemma 3.1 enables us to 'take the limit of the procedure' in the following sense. First we define $D \subseteq \mathbb{N} \cup \{0\}$ to be the set of all eventually stable rows, i.e.

$$i \in D \iff \text{case (ii) of Lemma 3.1 holds for } i.$$

Next we define a mapping

$$\sigma : D \times A^+ \longrightarrow D$$

by

$$(i, w)\sigma = (i, w)\tau, \text{ after row } i \text{ has stabilised.}$$

It is clear that σ is an action of A^+ on D. In the following lemma we prove that σ induces an action of S on D.

Lemma 3.2. *Let $w_1, w_2 \in A^+$. If $w_1 = w_2$ holds in S then*

$$(i, w_1)\sigma = (i, w_2)\sigma,$$

for all $i \in D$. In other words, σ defines an action of S on D.

PROOF. The lemma is a direct consequence of the fact that the action E2 is a part of the RQ procedure. ∎

Our main task is to prove that the action σ is actually equivalent to the action ρ; for the definition of equivalent actions see Appendix A. To do this we need several technical results.

Lemma 3.3. *If at some stage of the* RQ *procedure the coset table satisfies the property*

$$i = (j, a_k)\tau \Longrightarrow (1, z_i)\rho = (1, z_j a_k)\rho, \tag{2}$$

for all $i, j \in C$ and all $k = 1, \ldots, m$, then it satisfies

$$i = (j, w)\tau \Longrightarrow (1, z_i)\rho = (1, z_j w)\rho,$$

for all $i, j \in C$ and all $w \in A^+$.

PROOF. We prove the lemma by induction on the length of w, the case $|w| = 1$ being (2). Let us assume that $|w| > 1$, so that $w \equiv a_k w_1$, with $a_k \in A$, and let us have

$$i = (j, w)\tau. \tag{3}$$

Let us denote

$$(j, a_k)\tau = l, \tag{4}$$

so that (2) gives

$$(1, z_l)\rho = (1, z_j a_k)\rho. \tag{5}$$

Also, from (3) we have

$$i = (j, a_k w_1)\tau = ((j, a_k)\tau, w_1)\tau = (l, w_1)\tau,$$

and by the inductive hypothesis we have

$$(1, z_i)\rho = (1, z_l w_1)\rho. \tag{6}$$

Finally, from (5) and (6) we deduce

$$\begin{aligned}(1, z_j w)\rho &= (1, z_j a_k w_1)\rho = ((1, z_j a_k)\rho, w_1)\rho = ((1, z_l)\rho, w_1)\rho \\ &= (1, z_l w_1)\rho = (1, z_i)\rho,\end{aligned}$$

exactly as required. ■

Lemma 3.4. *After any pass of the* RQ *procedure the coset table satisfies the following property*

$$i = (j, w)\tau \Longrightarrow (1, z_i)\rho = (1, z_j w)\rho, \tag{7}$$

for all $i, j \in C$ and all $w \in A^+$.

PROOF. We prove the lemma by induction on the number of passes. The coset table before the first pass is defined by (1), and obviously satisfies (7). We prove that applying one pass of the procedure to a coset table satisfying (7) yields another coset table also satisfying (7). In fact, by Lemma 3.3, it is enough to prove that the new table satisfies (2). From the definition of the RQ procedure it is clear that the coset table is changed only in steps E1, E2(i), E2(ii), E4(i), E5. We consider each of these steps separately, showing that they do not affect property (2).

E1. Let

$$i = (j, a_k)\tau$$

be a definition. Then z_i is defined to be

$$z_i \equiv z_j a_k,$$

and we have

$$(1, z_i)\rho = (1, z_j a_k)\rho.$$

Since the new coset table is identical to the previous one, except for the entry $(j, a_k)\tau$, we see that the new table satisfies (2).

E2(i). Suppose that

$$i = (j, a_k)\tau$$

has been obtained by pushing row l of the relation $u_p = v_p$. This means that

$$u_p \equiv \overline{u}_p a_k,\ (l, \overline{u}_p)\tau = j,\ (j, a_k)\tau = \infty,\ (l, v_p)\tau = i.$$

By the inductive hypothesis we have

$$(1, z_l \overline{u}_p)\rho = (1, z_j)\rho,\ (1, z_l v_p)\rho = (1, z_i)\rho.$$

Next note that

$$(1, z_l u_p)\rho = (1, z_l v_p)\rho,$$

since $u_p = v_p$ holds in S, so that we have

$$\begin{aligned}(1, z_i)\rho &= (1, z_l v_p)\rho = (1, z_l u_p)\rho = (1, z_l \overline{u}_p a_k)\rho = ((1, z_l \overline{u}_p)\rho, a_k)\rho \\ &= ((1, z_j)\rho, a_k)\rho = (1, z_j a_k)\rho,\end{aligned}$$

as required.

E2(ii). This action is analogous to the previous one.

E4(i). Suppose that

$$i = (j, a_k)\tau$$

has been obtained by pushing the first row of the zero word $\xi_p \in X$. This means that

$$\xi_p \equiv \overline{\xi}_p a_k,\ (1, \overline{\xi}_p)\tau = j,\ (j, a_k)\tau = \infty,\ i = 0.$$

Also, since ξ_p represents an element of R, we have $(1, \xi_p)\rho = 0$, so that

$$\begin{aligned}(1, z_i)\rho &= (1, \xi_1)\rho = 0 = (1, \xi_p)\rho = (1, \overline{\xi}_p a_k)\rho = ((1, \overline{\xi}_p)\rho, a_k)\rho \\ &= ((1, z_j)\rho, a_k)\rho = (1, z_j a_k)\rho,\end{aligned}$$

by induction.

E5. Now we consider processing a coincidence (i_1, i_2). If this coincidence has been obtained by pushing a row of a relation or by pushing row 1 of a zero word, it is easy to repeat the argument above to prove that

$$(1, z_{i_1})\rho = (1, z_{i_2})\rho. \tag{8}$$

If (i_1, i_2) is a consequence of another coincidence $j_1 = j_2$ satisfying

$$(1, z_{j_1})\rho = (1, z_{j_2})\rho,$$

then we have

$$i_1 = (j_1, a_k)\tau,\ i_2 = (j_2, a_k)\tau,$$

for some k, $1 \le k \le m$, so that

$$(1, z_i)\rho = (1, z_{j_1} a_k)\rho = ((1, z_{j_1})\rho, a_k)\rho = ((1, z_{j_2})\rho, a_k)\rho = (1, z_{j_2} a_k)\rho = (1, z_{i_2})\rho,$$

by induction. Therefore, in any case, (8) holds.

Now, if $i_1 < i_2$, then i_2 is removed from C (possibly producing more coincidences), and every occurrence of i_2 is replaced by i_1. For example, $i_2 = (i, a_k)\tau$ becomes $i_1 = (i, a_k)\tau$. Now we have

$$(1, z_i a_k)\rho = (1, z_{i_2})\rho = (1, z_{i_1})\rho,$$

which completes the inductive step. ■

Lemma 3.5. *Action σ satisfies the property*

$$i = (j, w)\sigma \Longrightarrow (1, z_i)\rho = (1, z_j w)\rho,$$

for all $i, j \in D$ and all $w \in A^+$.

PROOF. The lemma is an immediate consequence of the definitions of D and σ and Lemma 3.4. ■

Lemma 3.6. *If $i \in D$, $i \neq 0$, then $(1, z_i)\rho \neq 0$ in S^*/R.*

PROOF. If $(1, z_i)\rho = 0$ in S^*/R, then z_i represents an element of R in S, so that $z_i = \xi_j w$ for some j, $1 \leq j \leq r$, and some $w \in A^+$. By Lemma 3.2 we have

$$i = (1, z_i)\sigma = (1, \xi_j w)\sigma = ((1, \xi_j)\sigma, w)\sigma = (0, \xi_j)\sigma = 0,$$

which is a contradiction. ■

Lemma 3.7. *If $w \in A^+$ is such that $(1, w)\rho \neq 0$ in S^*/R then $(1, w)\sigma \neq 0$.*

PROOF. If $(1, w)\sigma = 0$ then by Lemma 3.5 we have

$$0 = (1, z_0)\rho = (1, z_1 w)\rho = (1, w)\rho,$$

which is a contradiction. ■

Lemma 3.8. *If $w \in A^+$ does not represent an element of R then the relation $w = z_{(1,w)\sigma}$ holds in S.*

PROOF. By Lemma 3.7 we have $(1, w)\sigma \neq 0$. Now note that

$$(1, z_{(1,w)\sigma})\sigma = (1, w)\sigma,$$

by the definition of words z_j, so that, By Lemma 3.5, we have

$$(1, z_1 z_{(1,w)\sigma})\rho = (1, z_1 w)\rho.$$

Since $w \notin R$, we obtain $z_{(1,w)\sigma} = w$ in S. ■

Lemma 3.9. *Actions ρ and σ are equivalent.*

PROOF. Let us define mappings

$$\phi_1 : S^*/R \longrightarrow D, \quad \phi_2 : D \longrightarrow S^*/R$$

by

$$x\phi_1 = \begin{cases} (1, x)\sigma & \text{if } x \neq 0 \\ 0 & \text{otherwise} \end{cases}, \quad x\phi_2 = \begin{cases} z_x & \text{if } x \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

These mappings are well defined by Lemmas 3.2 and 3.6. If $x \in S^*/R$, $x \neq 0$, then $(1,x)\sigma \neq 0$ by Lemma 3.7, and we have

$$(x\phi_1)\phi_2 = ((1,x)\sigma)\phi_2 = z_{(1,x)\sigma} = x,$$

by Lemma 3.8. Also, if $x \in D$, $x \neq 0$, then $z_x \notin R$ by Lemma 3.6, and we have

$$(x\phi_2)\phi_1 = z_x\phi_1 = (1, z_x)\sigma = x.$$

It is clear that

$$(0\phi_1)\phi_2 = 0,\ (0\phi_2)\phi_1 = 0,$$

and so ϕ_1 and ϕ_2 are mutually inverse.

We now want to prove that for each $x \in S^*/R$ and each $s \in S$ we have

$$((x\phi_1, s)\sigma)\phi_2 = (x,s)\rho, \tag{9}$$

which means that σ and ρ are equivalent; see Section 1 of Appendix A.

If $x \neq 0$ and $(x,s)\rho \neq 0$ then

$$\begin{aligned} ((x\phi_1, s)\sigma)\phi_2 &= (((1,x)\sigma, s)\sigma)\phi_2 = ((1,xs)\sigma)\phi_2 \\ &= z_{(1,xs)\sigma} = xs = (x,s)\rho. \end{aligned}$$

If $x \neq 0$ and $(x,s)\rho = 0$ then $(1,xs)\sigma = 0$ by Lemma 3.2, and we have

$$((x\phi_1, s)\sigma)\phi_2 = ((1,xs)\sigma)\phi_2 = 0\phi_2 = 0 = (x,s)\rho.$$

Finally, if $x = 0$, we have

$$((x\phi_1, s)\sigma)\phi_2 = ((0,s)\sigma)\phi_2 = 0\phi_2 = 0 = (x,s)\rho.$$

Therefore, (9) holds in every case, and the lemma follows. ■

Now we can prove our main result.

Theorem 3.10. *Let*

$$\mathfrak{P} = \langle A \mid \mathfrak{R} \rangle = \langle a_1, \ldots, a_m \mid u_1 = v_1, \ldots, u_n = v_n \rangle$$

be a finite semigroup presentation, let S be the semigroup defined by $\mathfrak{P}$, let $X = \{\xi_1, \ldots, \xi_r\}$ be a finite subset of A^+, and let R be the minimal right ideal of S generated by X. The RQ *procedure, with $\mathfrak{P}, X$ as its input, will terminate if and only if R has finite index in S. When the procedure terminates, the final coset table τ gives the action of S on S^*/R, $|C|-1$ is the index of R in S, and $\{z_i \mid 0 \neq i \in C\}$ is a set of representatives of $S - R$.*

PROOF. If R has has infinite index in S, then $S - R$ is infinite, so that the set D is infinite by Lemma 3.9. On the other hand, at any stage of the RQ procedure, the set C is finite. Hence, the procedure does not terminate in this case.

Assume now that R has finite index t in S, and let $\{w_1, w_2, \ldots, w_t\}$, with $w_1 \equiv \epsilon$, be a set of representatives of $S - R$ (see Section 1 of Chapter 7). In addition, we define w_0 to be ξ_1. After finitely many passes of the procedure all the cosets $1w_k$, $0 \leq k \leq t$, will be defined. Of course, all these cosets are distinct (by Lemma 3.9), and will remain distinct in the rest of the procedure. By Lemma 3.1, after finitely many passes all the rows $1w_k$, $0 \leq k \leq t$, will stabilise. Since $|D| = |S^*/R| = t + 1$ by Lemma 3.9, we conclude that

$$D = \{1w_0, 1w_1, \ldots, 1w_t\}.$$

Since S acts on D, it follows that $\{1w_0, 1w_1, \ldots, 1w_t\}$ is closed under this action. Now, for any $i \in C$ we have $1z_i = i$, and since $1 = 1w_1 \in D$, we conclude that $i \in D$ as well. Therefore, at this stage of the procedure we have $C = D$, so that there are no new definitions to be made, and hence the procedure terminates. The remaining assertions follow from Lemma 3.9. ■

As we already mentioned, the TC procedure, the RQ procedure, and the obvious left-right dual of the RQ procedure can be combined with Theorems 7.2.1 and 7.3.1 to give the following:

Theorem 3.11. *Let S be a finitely presented semigroup, and let $\mathfrak{P}$ be a finite presentation for S. A (left, right or two-sided) ideal I of finite index in S is finitely presented. If I is given by a finite set of words generating I as an ideal, then a finite presentation for I can be effectively found.* ■

4. Enumerating minimal left ideals

In this section we describe another modification of the TC enumeration procedure, which we call the Minimal Ideal (MI) enumeration procedure, which enumerates minimal left ideals of a finitely presented semigroup. We again start with a finite presentation

$$\mathfrak{P} = \langle A \mid R \rangle = \langle a_1, \ldots, a_m \mid u_1 = v_1, \ldots, u_n = v_n \rangle,$$

and we let S denote the semigroup defined by $\mathfrak{P}$. We also assume that S has minimal left ideals L_λ, $\lambda \in \Lambda$. We denote L_1 by L, and we assume that a word $f \in A^+$ represents an element of L. S acts on the set $\{L_\lambda \mid \lambda \in \Lambda\}$ by postmultiplication by Proposition A.3.2; we let

$$\rho : \{L_\lambda \mid \lambda \in \Lambda\} \times A^+ \longrightarrow \{L_\lambda \mid \lambda \in \Lambda\}$$

be this action.

The MI procedure starts with the same initial data structures as the TC and RQ procedures:

$$C = \{0,1\},\ K = \emptyset,\ 0a = 0,\ 1a = \infty\ \ (a \in A).$$

One pass of the procedure consists of the following four steps:

(1) E1;
(2) for all $i \in C$ and all $u_j = v_j$ in $\mathfrak{R}$ do E2;
(3) for all $i \in C$ and the word f do E3;
(4) repeat E4 until $K = \emptyset$.

The procedure terminates when all the data structures stabilise.

Similarly as in the TC procedure, the coset 0 is superfluous. To each $i \in C$, $i \neq 0$, we assign a word $z_i \in A^*$, such that $z_1 \equiv \epsilon$ and $z_i \equiv z_j a_k$ if the coset i has been defined by $i = (j, a_k)\tau$. With this notation we have

Theorem 4.1. *Let*

$$\mathfrak{P} = \langle A \mid R \rangle = \langle a_1, \ldots, a_m \mid u_1 = v_1, \ldots, u_n = v_n \rangle,$$

be a finite semigroup presentation, let S be the semigroup defined by $\mathfrak{P}$, and let f be a word representing an element of a minimal left ideal L of S. The MI *procedure terminates if and only if S has finitely many minimal left ideals, and in this case the following statements are true.*

(i) The number of elements of the final coset table C is equal to the number of the minimal left ideals of S.
(ii) The final coset table τ defines an action of S on C, and this action is equivalent to the action of S on its minimal left ideals.
(iii) The set $\{z_i \mid i \in C\}$ is a set of representatives of minimal left ideals of S. ■

This theorem can be proved almost identically as Theorem 3.10. The only difference would be in the proof of Lemma 3.4, where, instead of considering the effect of the action E4, we would have to consider the action E3. A slightly different proof of Theorem 4.1 can be found in Campbell, Robertson, Ruškuc and Thomas (1995a).

Theorem 4.1 (and its obvious left-right dual) can be used in conjunction with the Reidemeister—Schreier type theorems from Section 3 of Chapter 10, thus giving

Theorem 4.2. *Let S be a finitely presented semigroup, let $\mathfrak{P}$ be a finite presentation for S, and assume that S has minimal left ideals and minimal right ideals. If S has finitely many minimal left ideals or finitely many minimal right ideals then the Schützenberger group H of the minimal two-sided ideal of S is finitely presented. If we are given a word representing an element of the minimal two sided ideal, a finite presentation for H can effectively be found.* ■

At the moment we do not have an enumeration procedure which could be used together with the Reidemeister—Schreier type rewriting theorems for the Schützenberger group of a 0-minimal two-sided ideal which is a completely 0-simple semigroup; see Sections 1 and 2 of Chapter 10. Therefore we pose the following

Open Problem 13. Find an enumeration procedure which would enumerate 0-minimal left ideals of a 0-minimal two-sided ideal which is a completely 0-simple semigroup.

It would also be interesting to investigate whether the assumption that we know a word representing an element of L is essential for the MI procedure:

Open Problem 14. Is there a procedure which takes as its input a finite semigroup presentation $\mathfrak{P}$, and terminates if and only if the semigroup S defined by $\mathfrak{P}$ has a minimal left ideal, in which case it returns a word representing an element of this ideal? Does such a procedure exist if S is known to have minimal left ideals?

Another possible way for the further development of TC based computational methods is to develop methods to investigate the global structure of finitely presented semigroups. The most natural way to approach this problem seems to be via Green's relations.

Open Problem 15. Is there a procedure which would enumerate all $\mathcal{D}$-classes of a finitely presented semigroup give by a finite presentation?

Open Problem 16. Find a procedure which would enumerate $\mathcal{L}$-classes (respectively $\mathcal{R}$-classes) of a $\mathcal{D}$-class which is given by a word representing an element in this $\mathcal{D}$-class.

A potentially serious difficulty in Problem 33 is that in general $\mathcal{D}$ is not a one-sided congruence, so that a semigroup S does not act on the set of all its $\mathcal{D}$-classes in a natural way. By way of contrast, $\mathcal{R}$ and $\mathcal{L}$ are one-sided congruences, so that the following problem might be more hopeful:

Open Problem 17. Find a procedure which enumerates all $\mathcal{R}$-classes ($\mathcal{L}$-classes) of a finitely presented semigroup.

We also pose the following problem from Campbell, Robertson, Ruškuc and Thomas (1995a), which might turn out to be related to the above problem:

Open Problem 18. Suppose that we use an arbitrary word f (not necessarily representing an element of a minimal left ideal) as the initial data for the MI procedure. Under which conditions does the procedure terminate after a finite number of passes, and what conclusions about the algebraic properties of S can be drawn from the final coset table?

5. Enumerating the idempotents of a minimal right ideal

We again suppose that S is the semigroup defined by the presentation

$$\mathfrak{P} = \langle A \mid \mathfrak{R} \rangle = \langle a_1, \ldots, a_m \mid u_1 = v_1, \ldots, u_n = v_n \rangle,$$

and we assume that S has a minimal right ideal R and finitely many minimal left ideals L_λ, $\lambda \in \Lambda = \{1, \ldots, q\}$. We also assume that a word $g \in A^+$ representing an element of R is given. We describe an *algorithm*, which we denote by ID, and which gives words representing the idempotents of R (i.e. the identities of the groups $R \cap L_\lambda$, $\lambda \in \Lambda$).

We note that if S has finitely many minimal right ideals R_i, $i \in I = \{1, \ldots, p\}$, then an obvious left-right dual of the ID algorithm can be used to find the idempotents of L_1. This, together with the actions of S on its minimal left ideals and minimal right ideals, transforms the Reidemeister—Schreier type rewriting described in Theorem 10.3.3 into another effective way for obtaining a finite presentation for the Schützenberger group of the minimal two-sided ideal. Also, the ID algorithm can be used to enumerate the idempotents of every R_i, $i \in I$, thus enumerating all the idempotents of the minimal two-sided ideal—a result of some interest on its own.

The ID algorithm consists of the following steps:

(1) Perform MI with $\mathfrak{P}$ and g; let $\rho : \Lambda \times S \to \Lambda$ be the obtained action of S on its minimal left ideals.

(2) Perform TC for $\mathfrak{P}$ together with defining words z_i, $i \in C$, such that $1z_i = i$, until $1f$ has been defined.

(3) Let $X = Y = \emptyset$.

(4) Perform one pass of TC.

(5) Construct the following digraph Γ, depending on the coset table. The set of vertices of Γ is C. Two vertices $i, j \in C$ are joined by an oriented edge $(i \to j)$ if and only if $j = ia_k$ for some $a_k \in A$.

(6) For every $k \in C$ such that there exists an oriented path from $1g$ to k do the next step.

(7) If $kz_k = k$ and if $(1, z_k)\rho_l \notin Y$ then add z_k to X and add $(1, z_k)\rho$ to Y.
(8) Repeat the steps 4, 5, 6, 7 until $|Y| = q$.

Theorem 5.1. *The* ID *algorithm terminates after a finite number of steps. After* ID *has terminated, the set* X *consists of words representing the idempotents idempotents from* $R \cap L_1, \ldots, R \cap L_q$.

We give a sketch of proof of the above theorem.

Lemma 5.2. *If, at any stage of* ID, *we have* $1w_1 = 1w_2$ *for some* $w_1, w_2 \in A^+$, *then* $w_1 = w_2$ *holds in* S.

PROOF. The lemma follows from the fact that the underlying procedure in ID is TC. ■

Lemma 5.3. *At any stage of* ID *the elements of* X *represent distinct idempotents of* R.

PROOF. From $kz_k = k$ it follows that $1z_k^2 = 1z_k$, and hence $z_k^2 = z_k$ holds in S by Lemma 5.2. From the condition that there exists an oriented path from $1g$ to k it follows that $1gw = k$ for some $w \in A^+$, so that $gw = z_k$ by Lemma 5.2. Hence $z_k \in R$. Finally, the condition $(1, z_k)\rho_l \notin Y$ ensures that each element of X belongs to a different minimal left ideal. ■

PROOF OF THEOREM 5.1. Let w_1, gw_2, ..., gw_q be words representing the idempotents from $R \cap L_1$, $R \cap L_2, \ldots, R \cap L_q$ respectively. After a finite number of steps all cosets $1gw_1$, $1gw_2$, ..., $1gw_q$ will be defined. Suppose that these cosets have numbers $k_1, \ldots, k_q$ respectively. Again after finitely many steps the cosets $k_1 z_1, \ldots, k_q z_q$ will be defined. If $z_{k_i} \equiv b_{i1} b_{i2} \ldots b_{it_i}$, where all b_{ij} are from A, let

$$k = \max(\{k_1, \ldots, k_q\} \cup \{k_i b_{i1} \ldots b_{it} \mid 1 \leq i \leq q,\ 1 \leq t \leq t_i\}).$$

After a finite number every row i, $i \leq k$, in the coset table will stabilise. At that stage $1z_{k_i}^2$ must be equal to $1z_{k_i}$ because $z_{k_i}^2 = z_{k_i}$ holds in S. Therefore, at worst, the algorithm will terminate at that stage. It now follows easily from Lemma 5.3 that X contains all q idempotents from R. ■

Appendix A

Semigroups and their ideals

Here we list standard definitions and results that we use in the main text. All the results are well known and can be found in any introductory book on semigroup theory, such as Clifford and Preston (1961) and (1967), Howie (1976) and Lallement (1979).

1. Basics

Let S be a semigroup. A non-empty subset $\emptyset \neq T \subseteq S$ is a *subsemigroup* of S if it is closed under multiplication, i.e. if for any $t_1, t_2 \in T$ we have $t_1 t_2 \in T$. If T_i, $i \in I$, is a family of subsemigroups of S, and if $\bigcap_{i \in I} T_i \neq \emptyset$, then $\bigcap_{i \in I} T_i$ is a subsemigroup of S. For a non-empty subset $X \subseteq S$, the intersection T of all subsemigroups of S which contain X is the *subsemigroup generated by* X; we write $T = \langle X \rangle$. An alternative description of $\langle X \rangle$ is

$$\langle X \rangle = \{x_{i_1} x_{i_2} \dots x_{i_k} \mid k \geq 1,\ x_{i_1}, \dots, x_{i_k} \in X\}.$$

A non-empty subset R of S is a *right ideal* if $rs \in R$ for all $r \in R$ and all $s \in S$. Again, a non-empty intersection of right ideals is a right ideal. The right ideal generated by a set $\emptyset \neq X \subseteq S$ is the intersection of all right ideals of S which contain X. For a semigroup S, by S^1 we mean S if S has an identity, and S with an identity adjoined to it otherwise. With this notation the right ideal generated by X is

$$XS^1 = \{xs \mid x \in X,\ s \in S^1\}.$$

Left ideals are defined dually. The left ideal generated by a set X is S^1X. A set I is a *two sided ideal* (or simply *ideal*) if I is both a left ideal and a right ideal. The two sided ideal generated by a set X is S^1XS^1. Note that any (left, right or two-sided) ideal is automatically a subsemigroup.

Let S_1 and S_2 be two semigroups. A mapping $\phi : S_1 \longrightarrow S_2$ is a *homomorphism* if

$$(xy)\phi = (x\phi)(y\phi)$$

for all $x, y \in S_1$. A homomorphism which is one-one (respectively onto, a bijection) is called a *monomorphism* (respectively *epimorphism, isomorphism*). If $S_1 = S_2$ then any homomorphism is called an *endomorphism*, and any isomorphism is called an *automorphism.*

A *congruence* on a semigroup S is an equivalence relation η, which is *compatible* with the multiplication in S:

$$(s_1, s_2) \in \eta,\ (t_1, t_2) \in \eta \Longrightarrow (s_1 t_1, s_2 t_2) \in \eta.$$

If η is a congruence on S, then the factor set S/η is a semigroup under the multiplication $\overline{s}_1 \overline{s}_2 = \overline{s_1 s_2}$, where $\overline{s}$ denotes the η-class of s. There is the usual connection between congruences and homomorphisms: if η is a congruence, then S/η is a homomorphic image of S under the *natural homomorphism* $\eta^\natural : s \mapsto \overline{s}$; if $\phi : S \longrightarrow T$ is a homomorphism then the *kernel*

$$\ker \phi = \{(s_1, s_2) \mid s_1, s_2 \in S,\ s_1\phi = s_2\phi\}$$

of ϕ is a congruence on S and $S/\ker \phi \cong \mathrm{Im}\phi$; also $\ker \eta^\natural = \eta$ and $(\ker \phi)^\natural = \phi$.

A *left congruence* is an equivalence relation on $\eta \subseteq S \times S$ which is *left compatible*:

$$s \in S,\ (s_1, s_2) \in \eta \Longrightarrow (ss_1, ss_2) \in \eta.$$

A *right congruence* is defined dually. A binary relation is a congruence if and only if it is both a left congruence and a right congruence. The *index* of a (left, right or two-sided) congruence is the number of equivalence classes of that congruence.

Each ideal I of a semigroup S gives rise to a congruence

$$\eta_I = \{(s_1, s_2) \mid s_1, s_2 \in I \text{ or } s_1 = s_2\}, \tag{1}$$

usually called the *Rees congruence.* The *Rees quotient* S/η_I, which is usually denoted by S/I, can be thought of as the set $(S - I) \cup \{0\}$ with the multiplication

$$s_1 s_2 = \begin{cases} s_1 s_2 & \text{if } s_1, s_2, s_1 s_2 \in S - I, \\ 0 & \text{otherwise.} \end{cases}$$

It is important to note that Rees congruences by no means exhaust the class of all congruences of a general semigroup. If I is a left or right ideal, then (1) defines a left or right Rees congruence on S. The *index* of a (left, right, two-sided) ideal is the index of the corresponding Rees congruence.

A *right action* (or simply *action*) of a semigroup S on a set X is a mapping $\rho : X \times S \longrightarrow X$ with the property

$$((x, s_1)\rho, s_2)\rho = (x, s_1 s_2)\rho.$$

When there is no danger of confusion we write just xs for $(x, s)\rho$. A *left action* is defined dually. An action is *faithful* if

$$(\forall x \in X)(xs_1 = xs_2) \Longrightarrow s_1 = s_2.$$

Any semigroup acts on itself by postmultiplication, but this action is not necessarily faithful. Any semigroup acts faithfully by postmultiplication on the set S^1. Also, a semigroup S acts on a factor set S/η, where η is a right congruence, with

$$\overline{s}t = \overline{st}\ (\overline{S} \in S/\eta,\ t \in S). \tag{2}$$

Conversely, for any action $\rho : X \times S \longrightarrow S$ and any $x \in X$, the relation

$$\{(s_1, s_2) \mid xs_1 = xs_2\}. \tag{3}$$

is a right congruence on S.

Actually, there is a one-one correspondence between right congruences on a semigroup S and cyclic actions of S with an indecomposable generator. (An action $\rho : X \times S \longrightarrow X$ is cyclic if $x_0S^1 = X$ for some $x_0 \in X$; x_0 is indecomposable if $x_0 \notin x_0S$.) To see this assume first that η is a right congruence on S. Let S^* denote the semigroup S with an identity 1 adjoined to it (regardless of whether S already has one). Then $\eta \cup \{(1,1)\}$ is a right congruence on S^*. The corresponding action (2) of S on S^*/η is cyclic, with an indecomposable generator $\overline{1}$. Conversely, if $\rho : X \times S \longrightarrow X$ is a cyclic action with an indecomposable generator x_0, then corresponding relation (3) is a right congruence on S, and it is easy to see that this indeed establishes the desired one-one correspondence.

Assume that we have two actions

$$\rho : X \times S \longrightarrow X,\ \sigma : Y \times T \longrightarrow Y.$$

The actions ρ and σ are said to be *equivalent* if there exists an isomorphism $\phi : S \longrightarrow T$ and a bijection $\psi : X \longrightarrow Y$ such that

$$(x, s)\rho = ((x\psi, s\phi)\sigma)\psi^{-1},$$

for all $x \in X$ and all $s \in S$.

2. Simple and 0-simple semigroups

A semigroup is said to be *simple* if it has no proper ideals. A semigroup S with zero is 0-*simple* if 0 and S are the only ideals of S. For technical reasons the trivial semigroup and the two-element semigroup with the zero multiplication are not included among either simple or 0-simple semigroups. Adjoining a zero to a simple semigroup yields a 0-simple semigroup, but there are 0-simple semigroups which cannot be obtained in this way.

It is clear that the set of all left (respectively right, two-sided) ideals of a semigroup S is partially ordered under inclusion. The minimal elements of this poset (if they exist) are called *minimal left* (respectively *right, two-sided*) *ideals.* Minimal non-zero left (respectively right, two-sided) ideals in a semigroup with zero are called 0-*minimal left* (respectively *right, two-sided*) *ideals.*

If a (0-)simple semigroup satisfies the descending chain conditions for both left and right ideals it is said to be *completely* (0)*-simple*. In particular, every finite (0-)simple semigroup is completely (0-)simple. In the following theorem we list the properties of completely 0-simple semigroups that we use in the main text. The theorem is a compilation of results from Rees (1940) and Clifford (1941). Most of the results are the special cases of more general theorems about Green's relations given in Section 4. Nevertheless, we state them separately because a big part of the main text does not require familiarity with Green's relations.

Proposition 2.1. *Let S be a completely 0-simple semigroup. Then*

(i) S possesses both 0-minimal left ideals and 0-minimal right ideals.

Let L'_λ, $\lambda \in \Lambda$, be all 0-minimal left ideals of S, let R'_i, $i \in I$, be all 0-minimal right ideals of S, and let $L_\lambda = L'_\lambda - \{0\}$ and $R_i = R'_i - \{0\}$.

(ii) $S = (\bigcup_{i \in I} R_i) \cup \{0\} = (\bigcup_{\lambda \in \Lambda} L_\lambda) \cup \{0\}$.

(iii) $R_i \cap R_j = L_\lambda \cap L_\mu = \emptyset$, *for* $i \neq j$ *and* $\lambda \neq \mu$.

(iv) Every set $H_{i\lambda} = R_i \cap L_\lambda$ is non-empty.

(v) $|H_{i\lambda}| = |H_{j\mu}|$ *for all* $i, j \in I$ *and all* $\lambda, \mu \in \Lambda$.

(vi) For any $i \in I$ and any $\lambda \in \Lambda$ either $H_{i\lambda}$ is a group (in which case $e_{i\lambda}$ denotes its identity), or $H_{i\lambda} \cup \{0\}$ is a semigroup with zero multiplication.

(vii) If $H_{i\lambda}$ and $H_{j\mu}$ are groups then $H_{i\lambda} \cong H_{j\mu}$.

(viii) Each R_i contains at least one group $H_{i\mu}$. Each L_λ contains at least one group $H_{j\lambda}$.

(ix) For $s_1 \in H_{i\lambda}$ and $s_2 \in H_{j\mu}$ we have $s_1 s_2 \in H_{i\mu}$ if $H_{j\lambda}$ is a group, and $s_1 s_2 = 0$ otherwise.

(x) Let $s \in H_{i\lambda}$. If $H_{j\lambda}$ is a group for some $j \in I$, then $sR_j = R_i$ and $sH_{j\mu} = H_{i\mu}$ for all $\mu \in \Lambda$. If $H_{i\mu}$ is a group then $L_\mu s = L_\lambda$ and $H_{j\mu} s = H_{j\lambda}$ for all $j \in I$.

(xi) If $H_{i\lambda}$ and $H_{j\mu}$ are groups then for any $s \in H_{j\lambda}$ there exists (a unique) $s' \in H_{i\mu}$ such that $ss's = s$ and $s'ss' = s'$; also $ss' = e_{j\mu}$ and $s's = e_{i\lambda}$. s' is uniquely determined by any of the previous four conditions, and the mapping $x \mapsto s'xs$ is an isomorphism from $H_{j\mu}$ onto $H_{i\lambda}$.

(xii) If $H_{i\lambda}$ is a group then $e_{i\lambda}$ is a left identity for R_i and a right identity for L_λ.

(xiii) $H_{i\lambda} \cup \{0\} = e_{i\lambda} S e_{i\lambda}$ *for all* $i \in I$, $\lambda \in \Lambda$. ■

Often a completely 0-simple semigroup is visualised as a so called *egg-box picture*. In Figure 13 we have an example of such a picture. The completely 0-simple semigroup in question has eight 0-minimal right ideals (rows) and ten 0-minimal left ideals (columns). All the intersections $H_{i\lambda} = R_i \cap L_\lambda$ are non-empty, and are of the 'same size'; those of them which are groups are shaded.

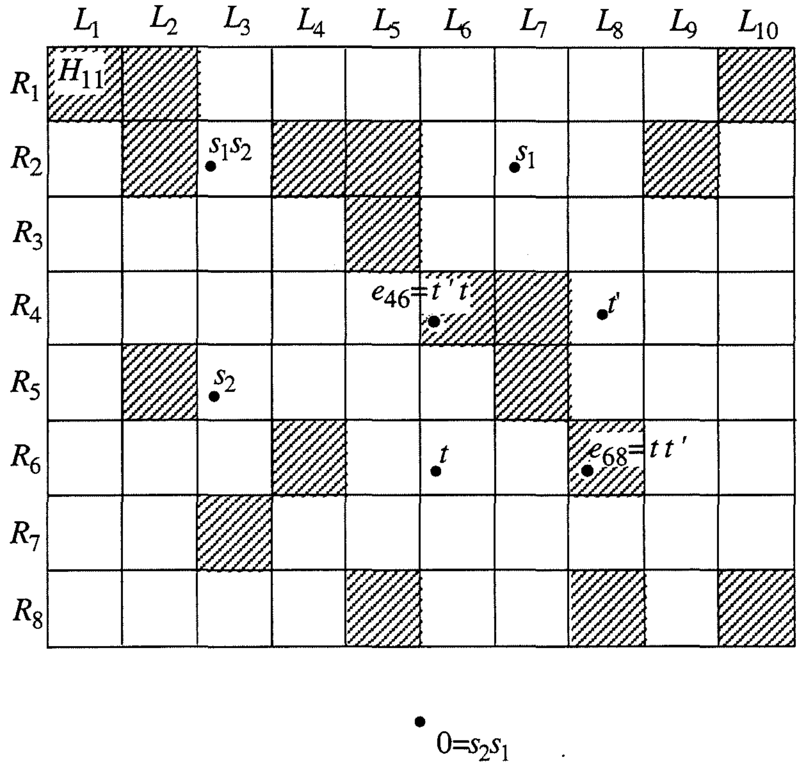


Figure 13.

Each R_i and each L_λ contains at least one group. The product s_1s_2 of $s_1 \in H_{27}$ and $s_2 \in H_{53}$ belongs to H_{23} since H_{57} is a group, but s_2s_1 is zero since H_{23} is not a group. Since both H_{46} and H_{68} are groups, an arbitrary element $t \in H_{66}$ has a unique 'inverse' $t' \in H_{48}$.

Adjoining a zero to a completely simple semigroup yields a completely 0-simple semigroup, so that each statement of Proposition 2.1 has a corollary for completely simple semigroups. We state these corollaries separately, since frequently in the main text we deal with simple semigroups rather than with 0-simple semigroups.

Proposition 2.2. *Let S be a completely simple semigroup. Then*

(i) S has both minimal left ideals and minimal right ideals.

Let L_λ, $\lambda \in \Lambda$, be all minimal left ideals of S, and let R_i, $i \in I$, be all minimal right ideals of S.

(ii) $S = \bigcup_{i\in I} R_i = \bigcup_{\lambda\in\Lambda} L_\lambda$.
(iii) $R_i \cap R_j = L_\lambda \cap L_\mu = \emptyset$ *if* $i \neq j$ *and* $\lambda \neq \mu$.

(iv) *The set* $H_{i\lambda} = R_i \cap L_\lambda = R_i L_\lambda$ *is non-empty and is a group for all* $i \in I$ *and all* $\lambda \in \Lambda$.

We denote by $e_{i\lambda}$ *the identity of the group* $H_{i\lambda}$.

(v) $H_{i\lambda} \cong H_{j\mu}$ *for all* $i, j \in I$ *and all* $\lambda, \mu \in \Lambda$.

(vi) *If* $s \in H_{i\lambda}$ *then* $sR_j = R_i$, $L_\mu s = L_\lambda$, $sH_{j\mu} = H_{i\mu}$, $H_{j\mu}s = H_{j\lambda}$.

(vii) *For any* $i, j \in I$, *any* $\lambda, \mu \in \Lambda$ *and any* $s \in H_{i\mu}$ *there exists (a unique)* $s' \in H_{j\lambda}$ *such that* $ss's = s$, $s'ss' = s'$, $ss' = e_{j\mu}$ *and* $s's = e_{i\lambda}$. s' *is uniquely determined by any of the previous four conditions and the mapping* $x \mapsto s'xs$ *is an isomorphism from* $H_{j\mu}$ *onto* $H_{i\lambda}$.

(viii) $e_{i\lambda}$ *is a left identity for* R_i *and a right identity for* L_λ.

(ix) $H_{i\lambda} = e_{i\lambda} S e_{i\lambda}$ *for all* $i \in I$, $\lambda \in \Lambda$. ∎

There exists a nice structure theory for completely 0-simple semigroups due to Suschkewitsch (1928) and Rees (1940). Let G be a group and let I and Λ be two index sets. Let also $P = (p_{\lambda i})_{\lambda \in \Lambda, i \in I}$ be a $|\Lambda| \times |I|$ matrix (usually called *Rees matrix*) with entries from $G \cup \{0\}$, with the additional property (usually called *regularity*) that each row and each column of P contains at least one non-zero entry. The set $(I \times G \times \Lambda) \cup \{0\}$ can be made into a semigroup by defining

$$(i, g, \lambda)(j, h, \mu) = \begin{cases} (i, g p_{\lambda j} h, \mu) & \text{if } p_{\lambda j} \neq 0 \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

$$(i, g, \lambda)0 = 0(i, g, \lambda) = 00 = 0. \tag{5}$$

This semigroup is denoted by $\mathcal{M}^0[G; I, \Lambda; P]$ and is called a *Rees matrix semigroup*.

Proposition 2.3. *(i) Let* G *be a group, let* I *and* Λ *be two index sets, and let* $P = (p_{\lambda i})_{\lambda \in \Lambda, i \in I}$ *be a regular* $|\Lambda| \times |I|$ *matrix with entries from* $G \cup \{0\}$. *Then the Rees matrix semigroup* $\mathcal{M}^0[G; I, \Lambda; P]$ *is a completely 0-simple semigroup.*

(ii) Let S *be a completely 0-simple semigroup, with all the notation from Proposition 2.1. Assume that* $H_{i_0\lambda_0}$ *is a group. For each* $i \in I$ *and each* $\lambda \in \Lambda$ *choose* $s_i \in H_{i\lambda_0}$ *and* $t_\lambda \in H_{i_0\lambda}$, *and let* $P = (t_\lambda s_i)_{\lambda \in \Lambda, i \in I}$. *Then* P *is regular and* $S \cong \mathcal{M}^0[H_{i_0\lambda_0}; I, \Lambda; P]$. *In particular, every completely 0-simple semigroup is isomorphic to a Rees matrix semigroup.* ∎

If the matrix P does not contain any zeros, then the zero is indecomposable in $\mathcal{M}^0[G; I, \Lambda; P]$. Removing the zero from this semigroup yields another semigroup which we denote by $\mathcal{M}[G; I, \Lambda; P]$.

Proposition 2.4. *(i) Let* G *be a group, let* I *and* Λ *be two index sets, and let* $P = (p_{\lambda i})_{\lambda \in \Lambda, i \in I}$ *be a* $|\Lambda| \times |I|$ *matrix with entries from* G. *Then the Rees matrix semigroup* $\mathcal{M}[G; I, \Lambda; P]$ *is a completely simple semigroup.*

(ii) Let S be a completely simple semigroup, with all the notation from Proposition 2.2, and let $i_0 \in I$ and $\lambda_0 \in \Lambda$ be arbitrary. For each $i \in I$ and each $\lambda \in \Lambda$ choose $s_i \in H_{i\lambda_0}$ and $t_\lambda \in H_{i_0\lambda}$, and let $P = (t_\lambda s_i)_{\lambda\in\Lambda, i\in I}$. Then $S \cong \mathcal{M}[H_{i_0\lambda_0}; I, \Lambda; P]$. In particular, every completely simple semigroup is isomorphic to a Rees matrix semigroup. ■

The following theorem gives a necessary and sufficient condition for two Rees matrix semigroups to be isomorphic.

Proposition 2.5. *Two Rees matrix semigroups $\mathcal{M}^0[G; I, \Lambda; P]$ and $\mathcal{M}^0[K; J, M; Q]$ are isomorphic if and only if there exist an isomorphism $\theta : G \longrightarrow K$, bijections $\psi : I \longrightarrow J$, $\chi : \Lambda \longrightarrow M$ and elements $u_i, v_\lambda \in K$ ($i \in I, \lambda \in \Lambda$) such that*

$$p_{\lambda i}\theta = v_\lambda q_{\lambda\chi, i\psi} u_i$$

for all $\lambda \in \Lambda$ and all $i \in I$. ■

As an immediate consequence we have the following result for simple semigroups.

Proposition 2.6. *Every completely simple semigroup is isomorphic to a Rees matrix semigroup $\mathcal{M}[G; I, \Lambda; P]$ with P in* normal form, *i.e. with $p_{i1} = p_{1\lambda} = 1_G$ for all $i \in I$ and all $\lambda \in \Lambda$.* ■

Finally, we remark that (4) and (5) define an associative binary operation even if G is not a group, but a semigroup, and P is not regular. The obtained semigroup, which we also call Rees matrix semigroup, is no longer 0-simple, but has many similarities with 0-simple semigroups.

3. Semigroups with minimal ideals

As we said before, a (left, right, two-sided) ideal I of a semigroup S is said to be minimal if it contains no other (left, right, two-sided) ideals of S, and is said to be 0-minimal if there are no (left, right, two-sided) ideals of S properly contained between $\{0\}$ and I. A semigroup does not necessarily have minimal or 0-minimal ideals of any kind. Also it can have several minimal left ideals and several minimal right ideals, or several 0-minimal ideals of any kind. However, it can have at most one minimal two-sided ideal, because of the fact that the intersection of two two-sided ideals I_1 and I_2 is always non-empty as it contains $I_1 I_2$. There is a strong link between (0-)minimal ideals and (0-)simple semigroups.

Proposition 3.1. *(i) The minimal two-sided ideal of a semigroup is either trivial or is a simple semigroup.*

(ii) A 0-minimal ideal of a semigroup is either a 0-simple semigroup or is a semigroup with zero multiplication. ■

Of particular interest for this thesis are the semigroups which have minimal one-sided ideals. The following two propositions state the main properties of such semigroups.

Proposition 3.2. *Let S be a semigroup and assume that S has minimal left (right) ideals L_λ, $\lambda \in \Lambda$ (R_i, $i \in I$). Then:*

(i) S has a minimal two-sided ideal which is the disjoint union of all minimal left (right) ideals of S;

(ii) for any $s \in S$ and any $\lambda \in \Lambda$ ($i \in I$) the set $L_\lambda s$ (sR_i) is a minimal left (right) ideal of S; in other words, S acts by postmultiplication (premultiplication) on the set of all minimal left (right) ideals. ■

Proposition 3.3. *Let S be a semigroup and assume that S has both minimal left ideals and minimal right ideals. Let L_λ, $\lambda \in \Lambda$, be all minimal left ideals, and let R_i, $i \in I$, be all minimal right ideals of S.*

(i) The minimal two sided ideal M of S is a completely simple semigroup. Minimal left and right ideals of M coincide with minimal left and right ideals of S.

(ii) If $e_{i\lambda}$ is the identity of the group $H_{i\lambda} = R_i \cap L_\lambda$ then $H_{i\lambda} = e_{i\lambda}Se_{i\lambda}$. ■

Slightly surprisingly, the analogues of Propositions 3.2 and 3.3 for semigroups with 0-minimal one-sided ideals do not hold; see Clifford and Preston (1967). However, we have

Proposition 3.4. *Let M be a 0-minimal ideal of a semigroup S, and assume that M is a completely 0-simple semigroup, with all the notation for these semigroups from Section 2.*

(i) Each L'_λ is a 0-minimal left ideal of S and each R'_i is a 0-minimal right ideal of S.

(ii) For each $\lambda \in \Lambda$ ($i \in I$) and each $s \in S$, the set $L_\lambda s$ (sR_i) is either $\{0\}$ or else is L_μ (R_j) for some $\mu \in \Lambda$ ($i \in I$). In other words, S acts on the set $\{L_\lambda \mid \lambda \in \Lambda\} \cup \{0\}$ ($\{R_i \mid i \in I\} \cup \{0\}$) by postmultiplication (premultiplication). ■

4. Green's relations

Green's relations were introduced in Green (1951), and have since then become a standard tool for investigating the structure of semigroups. Most of the results of the previous two sections can be interpreted in terms of these relations.

Let S be a semigroup. Two elements $a, b \in S$ are *$\mathcal{L}$-equivalent* ($a\mathcal{L}b$) if they generate the same left ideal, i.e. if $S^1a = S^1b$, i.e. if there exist $c, d \in S^1$ such that $ca = b$ and $db = a$. Obviously, $\mathcal{L}$ is an equivalence relation. It is easy to see that it is even a right congruence. In a similar way, two elements $a, b \in S$ are *$\mathcal{R}$-equivalent* ($a\mathcal{R}b$) if $aS^1 = bS^1$, and $\mathcal{R}$ is a left congruence.

The intersection of $\mathcal{L}$ and $\mathcal{R}$ is denoted by $\mathcal{H}$, while the smallest equivalence containing both $\mathcal{L}$ and $\mathcal{R}$ is denoted by $\mathcal{D}$. Since $\mathcal{L}$ and $\mathcal{R}$ commute (i.e. $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$), we have $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$. Finally, two elements $a, b \in S$ are *$\mathcal{J}$-equivalent* if $S^1aS^1 = S^1bS^1$. Relations $\mathcal{H}$, $\mathcal{D}$ and $\mathcal{J}$ are in general only equivalences. For an element $a \in S$, L_a, R_a, H_a, D_a and J_a respectively denote the equivalence classes of a with respect to equivalences $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, $\mathcal{D}$ and $\mathcal{J}$.

The usefulness of Green's relations comes from the fact that we are able to describe the structure of a $\mathcal{D}$-class with some precision. First of all, a $\mathcal{D}$ class is a disjoint union of $\mathcal{R}$-classes, as well as a disjoint union of $\mathcal{L}$ classes. Now, since $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$, we see that the intersection of an $\mathcal{R}$-class and an $\mathcal{L}$-class within the same $\mathcal{D}$-class is non-empty, and is an $\mathcal{H}$ class, so that a $\mathcal{D}$ class can be visualised as an egg box picture, with rows, columns and squares representing respectively $\mathcal{R}$-classes, $\mathcal{L}$-classes and $\mathcal{H}$-classes.

Proposition 4.1 (Green's Lemma) *Let S be a semigroup and let $a, b \in S$.*

(i) If $a\mathcal{R}b$ with

$$as = b \text{ and } bs' = a,$$

where $s, s' \in S^1$, then the mappings

$$\lambda_s : x \mapsto xs \text{ and } \lambda_{s'} : x \mapsto xs'$$

are mutually inverse $\mathcal{R}$-class preserving bijections from L_a onto L_b and vice versa.

(ii) If $a\mathcal{L}b$ with

$$sa = b \text{ and } s'b = a,$$

where $s, s' \in S^1$, then the mappings

$$\rho_s : x \mapsto sx \text{ and } \rho_{s'} : x \mapsto s'x$$

are mutually inverse $\mathcal{L}$-class preserving bijections from R_a onto R_b and vice versa. ■

An immediate consequence of this is:

Proposition 4.2. *If $a\mathcal{D}b$ then $|H_a| = |H_b|$.* ■

Proposition 4.3 (Green's Theorem) *If H is an $\mathcal{H}$-class of a semigroup S, then either $H^2 \cap H = \emptyset$ or H is a subgroup of S.* ■

An element a of a semigroup S is said to be *regular* if there exists an element $b \in S$ such that $aba = a$. An element $a' \in S$ is an *inverse* of a if $aa'a = a$ and $a'aa' = a'$. It is possible to show that an element is regular if and only if it has an inverse. A semigroup is *regular* if all its elements are regular.

Proposition 4.4. *If an element a of a semigroup S is regular, then so are all the elements of its $\mathcal{D}$-class D_a.* ■

If all the elements of a $\mathcal{D}$-class D are regular, we say that D is a regular $\mathcal{D}$-class.

Proposition 4.5. *Let D be a regular $\mathcal{D}$-class of a semigroup S, and let $a, b, c \in D$.*

(i) Each $\mathcal{L}$-class and each $\mathcal{R}$-class in D contains at least one idempotent, and hence at least one group $\mathcal{H}$-class.
(ii) $ab \in D$ if and only if $L_a \cap R_b$ is a group, in which case $ab \in R_a \cap L_b$.
(iii) If a' is an inverse of a then $a' \in D$; also $aa', a'a \in D$, and $\mathcal{H}$-classes $R_a \cap L_{a'}$ and $R_{a'} \cap L_a$ are groups with identities aa' and $a'a$ respectively.
(iv) If $\mathcal{H}$-classes $R_a \cap L_c$ and $R_c \cap L_a$ are groups, then the $\mathcal{H}$-class H_c contains one and only one inverse a' of a. The mapping $x \mapsto axa'$ is an isomorphism between $R_c \cap L_a$ and $R_a \cap L_c$.
(v) All group $\mathcal{H}$-classes of D are isomorphic. ■

Proposition 4.6. *Let S be a completely 0-simple semigroup with all the notation from Proposition 2.1. Then S has exactly two $\mathcal{D}$-classes: $\{0\}$ and $S - \{0\}$; both of them are regular. $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{H}$-classes of $S - \{0\}$ are respectively R_i $(i \in I)$, L_λ $(\lambda \in \Lambda)$ and $H_{i\lambda}$ $(i \in I,\ \lambda \in \Lambda)$. Also $\mathcal{J} = \mathcal{D}$ in S.* ■

Let S be a semigroup and let $s \in S$. Consider the set

$$I_a = \{x \in S \mid S^1xS^1 \subsetneq S^1aS^1\}.$$

I_a is an ideal of S^1aS^1 and the Rees quotient $P_a = S^1aS^1/I_a$ is called a *principal factor* of S. P_a can be thought of as the $\mathcal{J}$-class J_a with a zero adjoined to it, and the multiplication defined by

$$s_1s_2 = \begin{cases} s_1s_2 & \text{if } s_1, s_2, s_1s_2 \in J_a \\ 0 & \text{otherwise.} \end{cases}$$

If a belongs to the minimal ideal M of S then I_a is empty, and P_a is defined to be $J_a = M$.

Proposition 4.7. *Each principal factor of a semigroup is either a simple semigroup, a 0-simple semigroup or a semigroup with zero multiplication.* ■

Appendix B

Open problems

In this appendix we list all the open problems posed in the main text, with references to the pages they appear.

Open Problem 19. (*Page 24*) Find a formula for the rank of a general (finite) completely 0-simple semigroup.

Open Problem 20. (*Page 42*) Find the minimal number k such that there exists a presentation of the form $\langle A, B \mid \mathfrak{R},\ \mathfrak{S}\rangle$ for the full transformation semigroup T_n, where the presentation $\langle A \mid \mathfrak{R}\rangle$ defines the symmetric group $\mathcal{S}_n$ and $|\mathfrak{S}| = k$.

Open Problem 21. (*Page 42*) Find presentations for the semigroup Sing_n of all singular mappings on the set $\{1, \dots, n\}$, and for the semigroup $K(n, r)$ of all mappings on $\{1, \dots, n\}$ of rank at most r.

Open Problem 22. (*Page 43*) Can the symmetric inverse semigroup I_n be defined by a presentation of the form $\langle A, t \,|\, \mathfrak{R}_n, \mathfrak{S}_n\rangle$, where $\langle A \,|\, \mathfrak{R}_n\rangle$ is a presentation for the symmetric group $\mathcal{S}_n$, and $|\mathfrak{S}_n|$ does not depend on n? If yes, what is the minimal possible cardinality for $\mathfrak{S}_n$?

Open Problem 23. (*Page 57*) Find the minimal number k such that there exists a presentation of the form $\langle A, B \mid \mathfrak{R},\ \mathfrak{S}\rangle$ for the special linear semigroup $\mathrm{SLS}(2, p)$ (respectively, for the general linear semigroup $\mathrm{GLS}(2, p)$) such that the presentation $\langle A \mid \mathfrak{R}\rangle$ defines the special linear group $\mathrm{SL}(2, p)$ (respectively, the general linear group $\mathrm{GL}(2, p)$) and $|\mathfrak{S}| = k$.

Open Problem 24. (*Page 57*) Find presentations for the semigroup $\mathrm{GLS}(d, R)$ of all $d \times d$ matrices over a ring R for various d and various R. In particular, find presentations for $\mathrm{GLS}(d, R)$ in the following cases:

(i) $d = 2$, $R = \mathrm{GF}(p^n)$—a general finite field;
(ii) $d = 2$, $R = \mathbb{Z}_m$—the ring of integers modulo m;
(iii) $d > 2$, $R = \mathbb{Z}_p$, p prime.

Open Problem 25. (*Page 84*) Find a procedure which will take as its input a finite semigroup presentation $\langle A \mid \mathfrak{R} \rangle$ (defining a semigroup S) and a finite set of words $X \subseteq A^+$, and which would terminate if and only if the subsemigroup T of S generated by X has finite index in S, and would return in this case a set of representatives of $S - T$.

Open Problem 26. (*Page 94*) Find a presentation for an ideal I of a finitely presented semigroup S, which would be finite whenever I has finite index in S, and which is simpler than the presentation given in Theorem 7.2.1.

Open Problem 27. (*Page 94*) Find a finite presentation for an ideal I of finite index in a finitely presented semigroup S if S/I is known to be of some special type. In particular, find such presentation if S/I is

(i) a group with a zero adjoined;
(ii) a completely simple semigroup with a zero adjoined;
(iii) a completely 0-simple semigroup;
(iv) an inverse semigroup.

Open Problem 28. (*Page 94*) Develop a semigroup version of the Tietze transformation program for simplifying presentations.

Open Problem 29. (*Page 101*) Is a subsemigroup of finite index in a finitely presented semigroup necessarily finitely presented?

Open Problem 30. (*Page 160*) Describe the structure of the Schützenberger group of the generalised Fibonacci semigroup $S(r, n, k)$, where $r, n, k \in \mathbb{N}$, $r > 1$, g.c.d.$(n, k, k + r - 1) = 1$. For which r, n, k is this group:

(a) free?
(b) isomorphic to $F(r, n, k)$?

Open Problem 31. (*Page 221*) Find an enumeration procedure which would enumerate 0-minimal left ideals of a 0-minimal two-sided ideal which is a completely 0-simple semigroup.

Open Problem 32. (*Page 221*) Is there a procedure which takes as its input a finite semigroup presentation $\mathfrak{P}$, and terminates if and only if the semigroup S defined by $\mathfrak{P}$ has a minimal left ideal, in which case it returns a word representing an element of this ideal? Does such a procedure exist if S is known to have minimal left ideals?

Open Problem 33. (*Page 221*) Is there a procedure which would enumerate all $\mathcal{D}$-classes of a finitely presented semigroup given by a finite presentation?

Open Problem 34. (*Page 221*) Find a procedure which would enumerate $\mathcal{L}$-classes (R-classes) of a $\mathcal{D}$-class which is given by a word representing an element of this $\mathcal{D}$-class.

Open Problem 35. (*Page 221*) Find a procedure which enumerates all $\mathcal{R}$-classes ($\mathcal{L}$-classes) of a finitely presented semigroup.

Open Problem 36. (*Page 222*) Suppose that we use an arbitrary word f (not necessarily representing an element of a minimal left ideal) as initial data for the MI procedure. Under which conditions does the procedure terminate after a finite number of passes, and what conclusions about the algebraic properties of S can be drawn from the final coset table?

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Table of notation

Here we list mathematical notation that we have used in the main text. Each entry is followed by a short definition and a reference to the page in the main text where it is defined, or, if it is not explicitly defined, a reference to the first page where it appears.

$\restriction$	restriction of a mapping to a set	23
∞	undefined coset in a coset table	209
1_G	the identity of G	23
A^*	the set of all words over A; the free monoid on A	3
A^+	the set of all non-empty words over A; the free semigroup on A	3
A_0^*	the free monoid with zero on A	3
A_0^+	the free semigroup with zero on A	3
$A_1(\Gamma)$	the set of initial vertices of Γ	166
$A(\Gamma)$	the set of vertices of Γ	164
$\mathcal{A}_n$	the alternating group of degree n	41
$\langle A \mid \mathfrak{R} \rangle$	semigroup presentation with generating symbols A and defining relations $\mathfrak{R}$	4
$BR(S, \theta)$	Bruck-Reilly extension	70
$\mathrm{Aut}(\mathfrak{A})$	the automorphism group of $\mathfrak{A}$	37
B_X	semigroup of all binary relations on X	2
C_n	the cyclic group of order n	123
$C(S)$	the least full self-conjugate subsemigroup of S	21
$\mathcal{D}$	Green's relation	233
$e_{i\lambda}$	the identity of $H_{i\lambda}$	228
$\mathrm{End}(\mathfrak{A})$	the endomorphism semigroup of $\mathfrak{A}$	37
$\eta^\natural$	the natural homomorphism	226
η_I	the Rees congruence	226
ϵ	the empty word	3
$E(S)$	the set of idempotents of S	11
$F(r, n)$	Fibonacci group	147
$F(r, n, k)$	generalised Fibonacci group	149

$F(S)$	the subsemigroup of S generated by its idempotents	11
Γ_n^A	Coxeter graph for $\mathcal{A}_{n+2}$	166
Γ_n^S	Coxeter graph for $\mathcal{S}_{n+1}$	165
$\Gamma(P)$	the graph associated to a Rees matrix	24
$\Gamma(S)$	the graph associated to a completely 0-simple semigroup	11
$\mathrm{GF}(p^n)$	the finite field with p^n elements	36
$G(\Gamma)$	the group defined by $\mathfrak{P}(\Gamma)$	165
$\mathrm{GL}(d,F)$	the general linear group of $d \times d$ matrices over F	33
$\mathrm{GL}(d,p^n)$	$\mathrm{GL}(d,F)$ with $F = \mathrm{GF}(p^n)$	33
$\mathrm{GLS}(d,F)$	the general linear semigroup of $d \times d$ matrices over F	33
$\mathrm{GLS}(d,p^n)$	$\mathrm{GLS}(d,F)$ with $F = \mathrm{GF}(p^n)$	33
$\mathcal{H}$	Green's relation	233
$G(\mathfrak{P})$	the group defined by the presentation $\mathfrak{P}$	112
$H_{i\lambda}$	$R_i \cap L_\lambda$	228
$H(r,n,k)$	the Schützenberger group of $S(r,n,k)$	156
$I(r,d,F)$	the ideal of all matrices from $\mathrm{GLS}(d,F)$ of rank at most r	33
I_n	I_X for $X = \{1,\dots,n\}$	43
I_X	the symmetric inverse semigroup on X	2
$\mathcal{J}$	Green's equivalence	233
$J(r,d,F)$	the set of all matrices from $\mathrm{GLS}(d,F)$ of rank equal to r	33
$J(n,r)$	the set of all mappings from T_n of rank exactly r	31
ker	the kernel	226
$K(n,r)$	the semigroup of all mappings of rank at most r	30
$\mathcal{L}$	Green's relation	232
$L(A,T)$	the set of all words from A^+ representing elements of T	79
$L(\Gamma)$	the unique minimal left ideal of $S(\Gamma)$	174
L_λ	a minimal left ideal; $L'_\lambda - \{0\}$	228
L'_λ	a 0-minimal left ideal	228
$L(\mathfrak{P})$	left Adian's graph of $\mathfrak{P}$	113
$\mathcal{M}[G;I,\lambda;P]$	Rees matrix semigroup over G	230
$\mathcal{M}^0[G;I,\lambda;P]$	Rees matrix semigroup over $G \cup \{0\}$	230
$\mathbb{N}$	the set of natural numbers (positive integers); the free monogenic semigroup	8
$N_{i\lambda}$	the normal subsemigroup of $H_{i\lambda}$ generated by $H_{i\lambda} \cap F(S)$	21
$\binom{n}{r}$	$n!/(r!(n-r)!)$	31
Ω	set of representatives of $S - T$ or $S - I$ or $S - R$	83
$\mathfrak{P}(\Gamma)$	the presentation associated to Γ	164

$\phi(i,\lambda,j,\mu)$	a bijection $H_{i\lambda} \to H_{j\mu}$	12
$\mathfrak{P}(r,n)$	Fibonacci presentation	147
$\mathfrak{P}(r,n,k)$	generalised Fibonacci presentation	147
$\mathrm{PSL}(2,11)$	projective special linear group	166
$\mathcal{P}(S \times T)$	the set of all subsets of $S \times T$	67
PT_n	PT_X for $X = \{1,\ldots,n\}$	43
PT_X	the partial transformation semigroup	2
$\mathcal{R}$	Green's relation	233
$\mathrm{rank}(S)$	the rank of the semigroup S	10
$\mathrm{rank}(S:T)$	the rank of S modulo T	18
$\rho(S)$	the least group congruence on S	21
R_i	a minimal right ideal; $R_i' - \{0\}$	228
R_i'	a 0-minimal right ideal	228
$R(\mathfrak{P})$	right Adian's graph of $\mathfrak{P}$	113
S^*	S with an identity adjoined to it	212
S^1	S if S has an identity, S^* otherwise	225
$S_A(n)$	the semigroup defined by $\mathfrak{P}(\Gamma_n^A)$	177
$S(\Gamma)$	the semigroup defined by $\mathfrak{P}(\Gamma)$	165
$\mathrm{Sing}(d,F)$	the semigroup of singular $d \times d$ matrices over F	33
$\mathrm{Sing}(d,p^n)$	$\mathrm{Sing}(d,F)$ with $F = \mathrm{GF}(p^n)$	33
$\mathrm{Sing}(n)$	the semigroup of all singular mappings	30
$\mathrm{SL}(d,F)$	the special linear group of $d \times d$ matrices over F	33
$\mathrm{SL}(d,p^n)$	$\mathrm{SL}(d,F)$ with $F = \mathrm{GF}(p^n)$	33
$\mathrm{SLS}(d,F)$	the special linear semigroup of $d \times d$ matrices over F	33
$\mathrm{SLS}(d,p^n)$	$\mathrm{SLS}(d,F)$ with $F = \mathrm{GF}(p^n)$	33
$\mathcal{S}_n$	$\mathcal{S}_X$ for $X = \{1,\ldots,n\}$	40
$S(\mathfrak{P})$	the semigroup defined by the presentation $\mathfrak{P}$	112
$S(n,r)$	Stirling number of the second kind	31
$S(r,n)$	Fibonacci semigroup	147
$S(r,n,k)$	generalised Fibonacci semigroup	147
$S_S(n)$	the semigroup defined by $\mathfrak{P}(\Gamma_n^S)$	180
$S^{\times T}$	the cartesian product of $\lvert T\rvert$ copies of S	63
$S^{\oplus T}$	the direct product of $\lvert T\rvert$ copies of S	63
$S \Diamond T$	the Schützenberger product of S and T	67
$S \,\mathrm{Wr}\, T$	the unrestricted wreath product of S by T	63
$S \,\mathrm{wr}\, T$	the restricted wreath product of S by T	63
$\mathcal{S}_X$	the symmetric group on X	2
$S(Y; S_\alpha, \phi_{\alpha,\beta})$	strong semilattice of monoids	72
T_n	T_X for $X = \{1,\ldots,n\}$	1
T_X	the full transformation semigroup on X	1
$V(\pi)$	the value of a path π	24
V_{xy}	the set of values of all paths from x to y	24

$\lvert w\rvert$	the length of a word w	52
$w_1 \equiv w_2$	the word w_1 is identical to the word w_2	4
$w_1 = w_2$	the words w_1 and w_2 represent the same element	4
$\langle X\rangle$	the subsemigroup generated by X	225
$\lceil x\rceil$	the least integer which is not less than x	28
$\xi(p)$	the primitive root of 1 modulo p	41
X^T	the transpose of matrix X	51
$\mathbb{Z}$	integers	42
$\mathbb{Z}_p$	integers modulo p	47

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