

Subgroup sum graphs of finite abelian groups

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Abstract Let G be a finite abelian group, written additively, and H a subgroup of G . The *subgroup sum graph* $\Gamma_{G,H}$ is the graph with vertex set G , in which two distinct vertices x and y are joined if $x + y \in H \setminus \{0\}$. These graphs form a fairly large class of Cayley sum graphs. Among cases which have been considered previously are the *prime sum graphs*, in the case where $H = pG$ for some prime number p . In this paper we present their structure and a detailed analysis of their properties. We also consider the simpler graph $\Gamma_{G,H}^+$, which we refer to as the *extended subgroup sum graph*, in which x and y are joined if $x + y \in H$: the subgroup sum is obtained by removing from this graph the partial matching of edges having the form $\{x, -x\}$ when $2x \neq 0$. We study perfectness, clique number and independence number, connectedness, diameter, spectrum, and domination number of these graphs and their complements. We interpret our general results in detail in the prime sum graphs.

Keywords Subgroup sum graph · Finite abelian group · clique number · chromatic number · Cayley graphs · Spectrum

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1 Introduction

Cayley graphs are excellent models for interconnection networks. Hence, there are many investigations in connection with parallel processing and distributed computing. The definition of the Cayley graph was introduced by Arthur Cayley in 1878 to explain the concept of abstract groups which are described by a set of generators. Cayley graphs of finite cyclic groups are studied in the name of circulant graphs [2, 15, 16, 34, 33] and Cayley graphs of finite groups are considered in [1, 11, 20, 21, 30–32, 37]. Other graphs from finite groups are also studied in [6, 8, 18, 36]. Several authors studied Cayley graphs of finite abelian groups in [10, 17, 35, 38]. The generalized Cayley graphs of finite rings with respect to subsets are studied in [28, 29]. A variant of Cayley graphs are the *Cayley sum graphs* which have been studied by a number of authors [3, 9, 14, 19].

The square element graph of rings was studied by Biswas, Sen Gupta and Sen [4, 25, 26], while the power graph of semigroups was studied in [5]. The power graph of groups are studied through the orders of elements in a group in [6, 8, 18, 36]. Raveendra Prathap and Tamizh Chelvam [23, 24] defined and studied the square graph and cubic power graph of finite abelian groups. Let G be a finite abelian group with identity element 0. The square graph of G denoted $\Gamma_{sq}(G)$ is an undirected simple graph with vertex set G and two distinct vertices a and b are adjacent in $\Gamma_{sq}(G)$ if $a + b = 2t$ for some $t \in G$ and $2t \neq 0$. Having defined the square graph $\Gamma_{sq}(G)$ of G , authors studied various properties of the complement of square graph in [23]. Subsequently another graph called the cubic power graph $\Gamma_{cpg}(G)$ is introduced and studied in [24]. These graphs can be generalized, in the context of Cayley graphs of finite abelian groups, in parallel with the generalizations made in the case of finite rings [28, 29]. For a fixed positive integer n , the generalized sum graph $\Gamma_{gsg}(G)$ is the simple undirected graph with vertex set G and two distinct vertices x and y are adjacent if $x + y \in S = \{nt \mid nt \neq 0, t \in G\}$. One can see that when $n = 2$, $\Gamma_{gsg}(G) = \Gamma_{sq}(G)$ and when $n = 3$, $\Gamma_{gsg}(G) = \Gamma_{cpg}(G)$. Further note that when $n = 1$ and S is a generating set for G , $\Gamma_{gsg}(G)$ is the Cayley graph $\text{Cay}(G, S)$ [20]. When n is a prime number p , we call the generalized sum graph as the prime sum graph $\Gamma_{psg}(G)$.

In this paper, we extend the process of generalization, by defining the *subgroup sum graph* $\Gamma_{G,H}$, where G is a finite abelian group and H a subgroup of G : the vertices are the elements of G , and x and y are joined if $x+y \in H \setminus \{0\}$. A closely related graph, which we call the *extended subgroup sum graph* $\Gamma_{G,H}^+$ is defined similarly, but without the restriction $x + y \neq 0$ for adjacency; it turns out to be easier to work with.

2 Preliminaries

In this section, we recollect certain basic definitions and properties of graphs which are essential for further reference. Throughout this paper, $\Gamma = (V, E)$

is a finite simple graph with vertex set V and edge set E . A graph Γ is said to be *connected* if there exists a path between every pair of distinct vertices in Γ . A graph Γ is said to be *complete* if every pair of distinct vertices are adjacent through an edge and the complete graph on n vertices is denoted by K_n . The *degree* of a vertex v is the number of the edges in Γ which are incident with v . Note that degree of each vertex v in K_n is $n - 1$. The distance $d(u, v)$ between the vertices u and v in Γ is the length of the shortest path between u and v . If no path exists between u and v in Γ , then $d(u, v) = \infty$. For a vertex $v \in V(\Gamma)$, the *eccentricity* $e(v)$ of v is the maximum distance from v to any other vertex in $V(\Gamma)$. That is, $e(v) = \max\{d(v, w) : w \in V(\Gamma)\}$. The *radius* of Γ is the minimum eccentricity among the vertices of Γ and is denoted by $rad(\Gamma)$. i.e., $rad(\Gamma) = \min\{e(v) : v \in V(\Gamma)\}$. The *diameter* of Γ is the maximum eccentricity among the vertices of Γ and is denoted by $diam(\Gamma)$. i.e., $diam(\Gamma) = \max\{e(v) : v \in V(\Gamma)\}$. The *girth* of Γ is the length of a shortest cycle in Γ and is denoted by $girth(\Gamma)$. A graph is said to be *self-centered* graph if radius and diameter of the graph are equal. A *clique* of Γ is a maximal complete subgraph of Γ and the number of vertices in the largest clique of Γ is called the *clique number* of Γ and is denoted by $\omega(\Gamma)$.

An *independent set* is a set of vertices in a graph Γ , in which no two vertices are adjacent. The cardinality of a maximal independent set is called the *independence number* and is denoted by $\beta(\Gamma)$. A (vertex) *proper colouring* of Γ is an assignment of colours from a set C such that no two adjacent vertices receive same colour. If $|C| = k$, we say that the corresponding colouring is a proper k -colouring. A graph is *k -colourable* if it has a proper k -colouring. The *chromatic number* of a graph Γ is the least k such that Γ is k -colourable and is denoted by $\chi(\Gamma)$. The *clique cover number* $\theta(\Gamma)$ is the smallest number of complete subgraphs required to cover all the vertices of Γ . Note that the independence number and clique cover number of Γ are just the clique number and chromatic number of the complementary graph $\bar{\Gamma}$. A graph Γ is *perfect* if every induced subgraph of Γ has clique number equal to chromatic number. The *Weak Perfect Graph Theorem* of Lovász [22] asserts that the complement of a perfect graph is also perfect; so every induced subgraph of a perfect graph has independence number equal to clique cover number. We also make use of the theorem of Dilworth [12] asserting that the comparability graph (or incomparability graph) of a partial order is perfect. The *open neighbourhood* $N_\Gamma(v)$ of the vertex v in Γ is the set of vertices adjacent to v , while the *closed neighbourhood* of v is $\{v\} \cup N_\Gamma(v)$. The *domination number* of a graph is the least cardinality of a set of vertices for which the union of their closed neighbourhoods is the whole vertex set.

3 Definition and Basic Properties

In this section, we give formal definitions of our graphs, describe their structure in terms of the structure of G and H , examine connectedness, diameter, girth, and self-centredness, and show that these graphs are perfect. Let G be a finite

abelian group, and H a subgroup of G . We define the *extended subgroup sum graph* $\Gamma_{G,H}^+$ to have vertex set G and edges $\{x, y\}$ whenever $x + y \in H$; and the *subgroup sum graph* $\Gamma_{G,H}$ to have the same vertex set and edges $\{x, y\}$ whenever $x + y \in H \setminus \{0\}$. We see that the generalized sum graph $\Gamma_{gsg}(G)$ previously mentioned is the subgroup sum graph $\Gamma_{G,nG}$, while for any prime p , the prime sum graph $\Gamma_{psg}(G)$ is the subgroup sum graph $\Gamma_{G,pG}$.

The next result deals with the case where the subgroup H is trivial (either $\{0\}$ or G).

Theorem 1 *Let G be a finite abelian group.*

- (a) *If $H = \{0\}$, then the subgroup sum graph $\Gamma_{G,H}$ is a null graph on the vertex set G , while the extended subgroup sum graph $\Gamma_{G,H}^+$ is a partial matching where elements other than the identity and involutions are joined to their inverses.*
- (b) *If $H = G$, then the extended subgroup sum graph $\Gamma_{G,H}^+$ is complete, and the subgroup sum graph $\Gamma_{G,H}$ is obtained by deleting a matching covering all vertices except the identity and involutions.*

Proof If $H = \{0\}$, then the only edges in $\Gamma_{G,H}^+$ are those of the form $\{a, -a\}$ where $2a \neq 0$; there are no edges in $\Gamma_{G,H}$.

If $H = G$, then every pair $\{a, b\}$ with $a \neq b$ is an edge of $\Gamma_{G,G}^+$, and all those with $b \neq -a$ are edges of $\Gamma_{G,H}$. \square

The graphs considered in Theorem 1 are not very interesting, so where necessary below we assume that $1 < |H| < |G|$. For a prime p and an abelian group G , we have the following for the prime sum graph $\Gamma_{psg}(G)$.

- (a) $pG = \{0\}$ if and only if G is elementary abelian (a direct sum of cyclic groups C_p of order p).
- (b) $pG = G$ if and only if p does not divide $|G|$.

Let $S(G)$ denote the set of solutions of $2x = 0$ in a group G , and $s(G) = |S(G)|$. If $G = C_{2^{k_1}} \times \cdots \times C_{2^{k_r}} \times A$, where $|A|$ is odd and $k_1, \dots, k_r > 0$, then $s(G) = 2^r$.

The basic structure of these graphs $\Gamma_{G,H}^+$ and $\Gamma_{G,H}$ are given in the next result.

Theorem 2 *Let H be a subgroup of the abelian group G , with $|H| = k$ and $|G/H| = m$.*

- (a) *The extended subgroup sum graph $\Gamma_{G,H}^+$ has $(m + s(G/H))/2$ connected components, of which $s(G/H)$ are complete graphs K_k (whose vertex sets are the cosets of H having order 1 or 2 in G/H), and $(m - s(G/H))/2$ are complete bipartite graphs $K_{k,k}$ (whose vertex sets are the union of two cosets $H + a$ and $H - a$ for some $a \in G$ with $2a \notin H$).*
- (b) *The subgroup sum graph $\Gamma_{G,H}$ is obtained from the extended subgroup sum graph $\Gamma_{G,H}^+$ by deleting a perfect matching from every component which is complete bipartite, and deleting a matching from a complete component on a coset $H + a$ covering all elements other than elements of $S(G)$ lying in this coset (if any).*

Proof (a) The neighbours of a in $\Gamma_{G,H}^+$ are the elements of the coset $H - a$ (possibly excluding a). If $H - a \neq H + a$, then we have a complete bipartite graph on these two cosets. If $H - a = H + a$, so that this coset has order 2 in G/H , then we have a complete graph on this coset.

(b) To obtain the subgroup sum graph, we must delete edges of the form $\{a, -a\}$ for which $a \neq -a$ (that is, a is not the identity or an involution). \square

For an abelian group G , Theorem 2 gives a complete description of graphs $\Gamma_{G,H}^+$ and $\Gamma_{G,H}$, and enables us to determine their properties and parameters, as we do in the rest of the paper. First, though, we describe the components in a little more detail, and introduce three parameters we will use throughout the paper, counting three different types of cosets of H in G :

- *Type 1*, cosets $H + a$ for which $2a \notin H$ (that is, cosets distinct from their inverses in G/H). For such cosets, $(H + a) \cup (H - a)$ is a connected component of $\Gamma_{G,H}^+$, and is complete bipartite $K_{k,k}$; in $\Gamma_{G,H}$, it is $K_{k,k}$ minus a matching, which is connected if $k > 2$ and two disjoint edges if $k = 2$.
- *Type 2*, cosets $H + a$ for which $2a \in H$ but $H + a$ does not contain a solution of $2x = 0$. For these, $H + a$ is a connected component of both graphs, and is complete in the first and complete minus a perfect matching in the second. (This type can only occur if $|H|$ is even. Again we have to exclude $k = 2$ here since in that case K_2 minus a matching consists of two isolated vertices.)
- *Type 3*, cosets $H + a$ containing a solution of $2x = 0$. For these, $H + a$ is a connected component, and is complete in the first graph and complete minus a matching of size $(k - s(H))/2$ in the second. (This case always occurs, since the coset H has this form. See below for the proof that every such coset contains $s(H)$ solutions of $2x = 0$.)

For $i = 1, 2, 3$, we let m_i be the number of cosets of Type i , so that $m_1 + m_2 + m_3 = m = |G/H|$, and $m_2 + m_3 = s(G/H)$.

We claim that the number of solutions of $2x = 0$ in a coset is equal to $s(H)$ if the coset has Type 3 and 0 otherwise. This is clear for the coset H , so let $H + a$ be another coset. Clearly there are no solutions of $2x = 0$ in the coset unless it has Type 3, so suppose this is the case. Then $K = H \cup (H + a)$ is a subgroup of G , and $S(K)$ is a subgroup of G containing $S(H)$ as a subgroup of index 2. Thus $s(K) = 2s(H)$, so there are $s(H)$ elements in $H + a$ satisfying $2x = 0$. In particular, we see that $m_3 s(H) = s(G)$.

Thus, in $\Gamma_{G,H}^+$ with $k > 2$, there are $m_1/2$ components which are complete bipartite $K_{k,k}$, and $m_2 + m_3$ components which are complete graphs K_k . In $\Gamma_{G,H}$, there are $m_1/2$ components which are a complete bipartite graph minus a perfect matching, m_2 components which are complete graphs K_k minus a perfect matching, and m_3 components which are K_k minus a matching of size $(k - s(H))/2$.

In particular, we see that the numbers m , $s(G)$, $s(H)$ and $s(G/H)$ determine the number of cosets of each type. This is also true if $k = 2$.

4 Connectedness, diameter, girth and perfectness

We consider first a list of graph properties and parameters. Recall that hereafter H is a non-trivial subgroup of a finite abelian group G , $k = |H|$ and $m = |G : H|$.

Proposition 1 *Let G be an abelian group and H be a subgroup of G . Suppose that $|G| > 2$. Then the following are equivalent:*

- (a) $\Gamma_{G,H}^+$ is connected;
- (b) $\Gamma_{G,H}$ is connected;
- (c) $H = G$.

Proof The number of components of $\Gamma_{G,H}^+$ is $(m + s(G/H))/2$. Since each of m and $s(G/H)$ is at least 1, the graph is connected if and only if both are 1, which implies that $H = G$. The converse is clear from Theorem 1.

Since $\Gamma_{G,H}$ has at least as many components as $\Gamma_{G,H}^+$, we see that if it is connected then $H = G$. Conversely, if $H = G$, then we could only disconnect the graph by deleting a matching if $|G| = 2$. \square

Corollary 1 *Suppose that $|G| > 2$ and $H \neq G$. Then the complements of $\Gamma_{G,H}^+$ and $\Gamma_{G,H}$ are connected and have diameter at most 2, with equality except for $\bar{\Gamma}_{G,\{0\}}$ (which is complete). If $k > 2$, then these complements have radius equal to diameter, and so are self-centred.*

Proof It is clear that the diameter of complements of $\Gamma_{G,H}^+$ and $\Gamma_{G,H}$ is 2. If a vertex g has eccentricity 1, then it is joined to all other vertices, that is, it is isolated in $\Gamma_{G,H}^+$ or $\Gamma_{G,H}$ as appropriate. This can only occur if $k = |H| = 2$. \square

Corollary 2 (a) *The extended subgroup sum graph $\Gamma_{G,H}^+$ has girth 3 if $|H| > 2$, 4 if $|H| = 2$ and G/H is not an elementary abelian 2-group and ∞ otherwise.*
 (b) *The subgroup sum graph $\Gamma(G, H)$ has girth 3 if $|H| > 3$, 6 if $|H| = 3$ and ∞ if $|H| = 2$.*

Proof (a) There is always at least one component K_k , since $s(G/H) \geq 1$; if $k > 2$, this component contains a triangle. Otherwise, there is a component $K_{k,k}$ unless $s(G/H) = m$.

(b) If we delete a non-perfect matching from a complete graph on at least four vertices, what is left contains a triangle. If $k = 3$, then $s(G) = s(G/H)$ and so there are no cycles of length 3, but removing a perfect matching from $K_{3,3}$ produces a 6-cycle. If $k = 2$, then the graph consists of isolated vertices and edges. \square

Corollary 3 *The subgroup sum graph and extended subgroup sum graph are perfect graphs.*

Proof It suffices to show that each connected component is perfect. For the extended graph, these components are complete or complete bipartite, which are well known to be perfect. In the other case, we have to deal with the complement or bipartite complement of a matching. In the first case, the result holds since a matching is clearly perfect. The graph obtained by removing a perfect matching from $K_{k,k}$ is the comparability graph of the partial order on $\{a_1, \dots, a_k, b_1, \dots, b_k\}$ in which $\{a_1, \dots, a_k\}$ and $\{b_1, \dots, b_k\}$ are antichains and $a_i < b_j$ if and only if $i \neq j$; and the comparability graph of a poset is perfect, by Dilworth's Theorem. \square

From the above theorem, we have the following corollary for the cubic power graph of a finite abelian group.

Corollary 4 [24, Theorem 4.11] *Let G be a finite abelian group. Then the cubic power graph $\Gamma_{cpg}(G)$ is perfect.*

Also, we have the following corollary which is applicable for the complement of the square graph of a finite abelian group.

Corollary 5 [23, Theorem 3.2] *Let G be a finite abelian group. Then the complement $\bar{\Gamma}_{sq}(G)$ is perfect.*

5 Cliques and cocliques

We now compute the clique number and independence number of subgroup sum graphs $\Gamma_{G,H}$ where H is a proper subgroup of G .

5.1 General results

We compute the clique number and independence number of the graphs $\Gamma_{G,H}$ where H is a proper subgroup of G . By corollary 3, the chromatic number is equal to the clique number, and the clique cover number is equal to the independence number, so we get these two further parameters for free.

Theorem 3 *Let G be a finite abelian group, and H a non-trivial subgroup of G . Suppose that $|H| = k$, $|G/H| = m$, and let m_1, m_2, m_3 be as above.*

- (a) *The clique number of the extended subgroup sum graph $\Gamma_{G,H}^+$ is equal to k , and the independence number is equal to $km_1/2 + m_2 + m_3$.*
- (b) *The clique number of the subgroup sum graph $\Gamma_{G,H}$ is $(k + s(H))/2$.*
- (c) *The independence number of the subgroup sum graph $\Gamma_{G,H}$ is equal to $km_1/2 + 2(m_2 + m_3)$ if $s(H) < |H|$, and $km_1/2 + 2m_2 + m_3$ if $s(H) = |H|$.*

Proof (a) Clearly H is a maximal clique, while a maximal independent set takes one bipartite block from each complete bipartite component and one vertex from each complete component. (By assumption, $k \geq 2$, so the cliques of size 2 in the complete bipartite graphs are not larger than k .)

(b) When deleting edges from the extended subgroup sum graph, the bipartite components remain bipartite, and so have clique number 2.

In a complete component, corresponding to a coset of Type 2, we delete a perfect matching, giving a graph with clique number $k/2$. If there is a coset of Type 3, it contains a clique of size $(k + s(H))/2$. The second quantity is larger, and there is always a Type 3 coset, namely H itself. Now since $k \geq 2$, we have $(k + s(H))/2 \geq 2$.

(c) It is easy to see that if we delete a perfect matching from $K_{k,k}$ with $k \geq 2$, the resulting graph still has independence number k .

A Type 2 coset corresponds to a complete graph with a perfect matching removed, so contains an independent set of size 2. Similarly, a Type 3 coset contains an independent set of size 2 unless all its elements satisfy $2x = 0$, in which case it is complete and has independence number 1 (this occurs if and only if $s(H) = |H|$).

Summing over all cosets gives the result. \square

Using the above Theorem 3, one can obtain the clique number of the cubic power graph obtained in [24, Theorem 3.8]. In fact, in the language of the cubic power graph $\Gamma_{cpg}(G)$ of an abelian group G , $H = 3G$ and $|H| = k = \frac{|G|}{3^r}$. Hence, we have the following corollary.

Corollary 6 [24, Theorem 3.8] *Let G be a finite abelian group. Then the chromatic number $\omega(\Gamma_{cpg}(G)) = s(H) + \left\lceil \frac{|G|/3^r - s(H)}{2} \right\rceil = (k + s(H))/2$.*

5.2 Prime sum graphs

Now we calculate these numbers for the prime sum graph of a finite abelian group. It is clear from our analysis that the prime 2 behaves very differently from odd primes. We will deal with odd primes first. Note that, by our earlier assumptions, we can assume that $|G|$ is divisible by the prime p in question, otherwise $pG = G$.

5.2.1 Odd prime p

Let G be a finite abelian group. Then $H = pG$ is a subgroup of G whose index is p^r , where r is the number of cyclic summands of p -power order when G is expressed as a direct sum of cyclic groups; the quotient G/pG is an elementary abelian p -group. So $s(G/H) = 1$. Also, all 2-elements of G are contained in pG , so (in the notation used above) we have $m_2 = 0$, $m_3 = 1$, and $m_1 = p^r - 1$. Also, if G has q cyclic summands of 2-power order, then $s(H) = 2^q$, and $s(H) = |H|$ if and only if $G \cong C_2^q \times C_p^r$.

From our earlier results, we find that $\Gamma_{G,pG}^+$ has clique number $k = |G|/p^r$ and independence number $k(p^r - 1)/2 + 1$, while $\Gamma_{G,pG}$ has clique number $k/2 + 2^{q-1}$ and independence number $k(p^r - 1)/2 + 2$ unless $G \cong C_2^q \times C_p^r$, in which case the independence number is $k(p^r - 1)/2 + 1$. (If $q = 0$, then $|G|$ is odd, so k is odd and the clique number is $(k + 1)/2$.)

5.2.2 The prime 2

We can write $G = A \times B$, where A has odd order and B has order a power of 2. Suppose that B is the product of r cyclic groups, of which q have order 2. Then $2G = A \times 2B$, where B is the product of $r - q$ cyclic groups. We have $G/2G$ elementary abelian of order 2^r ; so every coset $2G + x$ satisfies $2x \in 2G$, and there are 2^q cosets containing involutions. Thus $m_1 = 0$, $m_2 = 2^r - 2^q$, and $m_3 = 2^q$. Moreover, $S(2G)$ is an elementary abelian of order 2^{r-q} , so $S(2G) = 2G$ if and only if G is the product of cyclic groups of orders 2 and 4 only.

So $\Gamma^+(G, 2G)$ has clique number $k = |2G| = |G|/2^r$, and independence number 2^r ; and $\Gamma(G, 2G)$ has clique number $(k + 2^{r-q})/2$, and independence number 2^{r+1} unless G is a product of cyclic groups of orders 2 and 4, in which case the independence number is $2^{r+1} - 2^q$.

6 Spectrum

Since the graphs $\Gamma_{G,H}^+$ and $\Gamma_{G,H}$ are disconnected if $H < G$, we can compute the spectrum by considering the components separately. Theorem 2 gives the number of components of each type. As before we let $|H| = k$ and $|G : H| = m$.

For $\Gamma_{G,H}^+$, the situation is simple: the components are either complete bipartite $K_{k,k}$ (with eigenvalues k and $-k$ each with multiplicity 1, and 0 with multiplicity $2k - 2$) or complete K_k (with eigenvalues $k - 1$ with multiplicity 1 and -1 with multiplicity $k - 1$).

For $\Gamma_{G,H}$ things are a little more complicated. There are three types of components:

- (a) $K_{k,k}$ minus a perfect matching. This is a distance-regular graph; its eigenvalues are $k - 1$ and $-(k - 1)$, each with multiplicity 1, and 1 and -1 , each with multiplicity $k - 1$.
- (b) K_k minus a perfect matching.
- (c) K_k minus a partial matching covering $k - s(H)$ vertices.

Consider the graph K_{a+b} , with a even, minus a matching covering a vertices. (We assume that $a > 0$ and $a + b \geq 4$ to avoid trivial cases.) This graph occurs in various contexts. For example, it is a Turán graph, maximizing the number of edges in a graph containing no complete subgraph of order $(a/2) + b + 1$. More relevant here, it is a *generalized line graph* in the sense of Hoffman; if $a > 2$, its smallest eigenvalue is -2 , with multiplicity $(a/2) - 1$ (see [7]). For $a, b > 0$, its eigenvalues are as follows:

- (a) the roots of the quadratic $x^2 - (a + b - 3)x - (a + 2b - 2)$, each with multiplicity 1;
- (b) 0, with multiplicity $a/2$;
- (c) -2 , with multiplicity $(a/2) - 1$;
- (d) -1 , with multiplicity $b - 1$.

These (and the corresponding eigenvectors) can be calculated, using the fact that the partition into the vertices covered and uncovered by the matching is equitable, in the sense of Godsil and Royle [13]. In detail: let the adjacency matrix of a graph Γ have the form $A(\Gamma) = \begin{pmatrix} A & J \\ J & B \end{pmatrix}$, where A is the adjacency matrix of K_a minus a matching, B the adjacency matrix of K_b , and J an all -1 matrix of appropriate size. Then the subspace of \mathbb{R}^{a+b} spanned by $(1^a, 0^b)$ and $(0^a, 1^b)$ is $A(\Gamma)$ -invariant, and the restriction of (Γ) to this subspace is $\begin{pmatrix} a-2 & b \\ a & b-1 \end{pmatrix}$, with eigenvalues as in (a). The orthogonal subspace is spanned by three types of vectors: those with zero entries in the last b coordinates, and in which entries in positions corresponding to the ends of (deleted) matching edges are negatives of each other; those with zero entries in the last b coordinates, and with entries in positions corresponding to the ends of matching edges are equal and all entries sum to 0; and those with zero entries in the first a coordinates, and with the remaining entries summing to zero. These are seen to be eigenvectors with the eigenvalues and multiplicities given in (b)–(d).

7 Domination

The *domination number* of a graph is the least cardinality of a set of vertices for which the union of their closed neighbourhoods is the whole vertex set. The domination number of K_k is clearly 1, and that of $K_{k,k}$ is 2. So the domination number of $\Gamma_{G,H}^+$ is $m = |G : H|$. Similarly the domination number of $K_{k,k}$ minus a perfect matching is 2 (two vertices on one of the deleted edges form a dominating set), while the domination number of K_k minus a matching is 1 if the matching is not a perfect matching and 2 if it is. So the domination number of $\Gamma_{G,H}$ is $m_1 + 2m_2 + m_3 = |G : H| + m_2$.

The complement of a disconnected graph Γ with no isolated vertices has domination number 2: two vertices in different components of Γ form a dominating set in $\bar{\Gamma}$. By Proposition 1, provided that $H < G$, the domination numbers of $\bar{\Gamma}_{G,H}^+$ and $\bar{\Gamma}_{G,H}$ are both 2.

8 Reconstructing the group

In studies of graphs defined on groups, one topic which has been considered is the extent to which the graph determines the group. For example, Solomon and Woldar [27] showed that the *commuting graph* (with x joined to y if $xy = yx$) of a finite simple group determines the group.

In our situation, things are a bit different: the graphs only determine certain parameters of the pair (G, H) . We note that, in either case, the number of vertices of the graph is equal to $|G|$. The connected components have sizes k and $2k$, and the number k occurs at least once (for the subgroup H); so we can also determine $|H|$.

Suppose that we are given $\Gamma_{G,H}^+$. Then the number of components of size k is equal to $s(G/H)$. So this is all the information we get: two pairs (G_1, H_1) and (G_2, H_2) with $s(G_1/H_1) = s(G_2/H_2)$ have isomorphic subgroup sum graphs.

For the graph $\Gamma_{G,H}$, we obtain the above information, and also the numbers $s(H)$ and $s(G)$. For the components of size k are either complete minus a perfect matching of size $k/2$ (m_2 of these) or complete minus a matching of size $(k - s(H))/2$ (m_3 of these). Since $s(H) \neq 0$, we can distinguish the two types, and hence find m_2 and m_3 ; and from a Type 3 coset we can recover $s(H)$, and hence $s(G) = m_3 s(H)$.

9 A generalization

Let H and K be subgroups of the abelian group G with $H \neq K$. We can extend our previous definition by defining the *generalized subgroup sum graph* $\Gamma_{G;H,K}$ to be the graph with vertex set G in which x and y are joined if $x + y \in H$ but $x + y \notin K$.

We lose no generality by assuming that $K \leq H$, since with the definition just given, $\Gamma_{G;H,K} = \Gamma_{G;H,H \cap K}$. We shall always make this assumption. Now our previous subgroup sum graph $\Gamma_{G,H}$ is $\Gamma_{G;H,\{0\}}$. Moreover, $\Gamma_{G;G,H}$ is the complement of the extended subgroup sum graph $\Gamma_{G,H}^+$. We have not investigated these graphs further.

Conflict of interest

The authors declare that they have no conflict of interest.

References

1. Afkhami, M., Khashyarmansh, K., Nafar, K.: Generalized Cayley graphs associated to commutative rings. *Linear Algebra Appl.* **437**, 1040–1049 (2012).
2. Alspach, B., Parsons, T.D.: Isomorphism of Circulant Graphs and Digraphs, *Discrete Mathe.* **25**, 97–108 (1979).
3. Biswas, A., Saha, J.P.: Spectrum of twists of Cayley and Cayley sum graphs, *Advances in Applied Mathematics* **132**, article 102272 (2022).
4. Biswas, B., Sen Gupta, R.: On the connectedness of square element graphs over arbitrary rings. *South East Asian Bull. Math.* **43(2)**, 153–164 (2019).
5. Biswas, B., Sen Gupta, R., Sen, M.K., Kar, S.: Some properties of square element graphs over semigroups. *AKCE International Journal of Graphs and Combinatorics.* **17(1)**, 118–130 (2020).
6. Cameron, P. J., Ghosh, S.: The power graph of a finite group. *Discrete Math.* **311**, 1220–1222 (2011).
7. Cameron, P. J., Goethals, J.-M., Seidel, J.J., Shult, E. E.: Line graphs, root systems and elliptic geometry. *J. Algebra.* **43**, 305–327 (1976).
8. Chakrabarty, I., Ghosh, S., Sen, M. K.: Undirected power graphs of semi groups. *Semi-group Forum.* **78**, 410–426 (2009).
9. DeVos, M., Goddyn, L., Mohar, B., Šámal, R.: Cayley sum graphs and eigenvalues of (3,6) fullerenes, *Journal of Combinatorial Theory, B*, 99, 358–369 (2009).

10. Dobson, E., Spiga, P., Verret, G.: Cayley graphs on abelian groups. *Combinatorica*. **36(4)**, 371–393 (2016).
11. Dejter, I.J., Serra, O.: Efficient dominating sets in Cayley graphs. *Discrete Appl. Math.* **129**, 319–328 (2003).
12. Dilworth, Robert P.: A decomposition theorem for partially ordered sets. *Ann. Math.* **51**, 161–166 (1950).
13. Godsil, C., Royle, G.: *Algebraic Graph Theory*, Springer, New York (2001).
14. Gryniewicz, D.J., Serra, O., Lev, V.F.: Connectivity of addition Cayley graphs, *Journal of Combinatorial Theory Series B* **99(1)**, 202–217 (2017).
15. Huang, Q.: A classification of circulant DCI(CI)- digraphs of 2-power order. *Discrete Math.* **265**, 71–84 (2003).
16. Jixiang, M., Qionxiang, H.: Isomorphisms of circulant digraphs. *Appl. Math. J. Chinese Univ. Ser. A*. **9**, 405–409 (1994).
17. Klotz, W., Sander, T.: Integral Cayley graphs over abelian groups. *Electron. J. Combin.* **17**, # 81 (2010).
18. Kelarev, A.V., Quinn, S.J.: A combinatorial property and power graph of semigroups. *Commentationes Mathematicae Universitatis Carolinae*. **45(1)**, 1–7 (2004).
19. Konyagin, S., Shkredov, I.D.: On subgraphs of random Cayley sum graphs, arXiv 1710.07230.
20. Lakshmivarahan, S., Dhall, S.K.: Ring, torus, hypercube architectures algorithms for parallel computing. *Parallel Computing*. **25**, 1877–1906 (1999).
21. Lee, J.: Independent perfect domination sets in Cayley graphs. *J. Graph Theory*, **37(4)**, 213–219 (2001).
22. Lovsz, László.: Normal hypergraphs and the perfect graph conjecture. *Discrete Math.* **2**, 253–267 (1972).
23. Raveendra Prathap, R., Tamizh Chelvam, T.: Complement graph of the square graph of finite abelian groups. *Houston Journal of Mathematics*. **46(4)**, 845–857 (2020).
24. Raveendra Prathap, R., Tamizh Chelvam, T.: The cubic power graph of finite abelian groups. *AKCE International Journal of Graphs and Combinatorics*. **18(1)**, 16–24 (2021).
25. Sen Gupta, R., Sen, M.K.: The square element graph over a finite commutative ring. *South East Asian Bull. Math.* **39(3)**, 407–428 (2015).
26. Sen Gupta, R., Sen, M.K.: The square element graph over a ring. *Southeast Asian Bull. Math.* **41(5)**, 663–682 (2017).
27. Solomon, R. M., Woldar, A.J.: Simple groups are characterized by their non-commuting graphs. *J. Group Theory*. **16**, 793–824 (2013).
28. Tamizh Chelvam, T., Anukumar Kathirvel, S.: Generalized unit and unitary Cayley graphs of finite rings. *J. Algebra Appl.* **18(1)**, # 1950006 [21 pages] (2019).
29. Tamizh Chelvam, T., Anukumar Kathirvel, S., Balamurugan, M.: Domination in generalized unit and unitary Cayley graphs of finite rings. *Indian J. Pure Appl. Math.* **51(2)**, 533–556 (2020).
30. Tamizh Chelvam, T., Kalaimurugan, G., Chou, Well Y.: Signed star domination number of Cayley graphs. *Discrete Math. Algorithms Appl.* **4(2)**, # 1250017 [10 pages] (2012).
31. Tamizh Chelvam, T., Mutharasu, S.: Subgroups as efficient dominating set in Cayley Graphs. *Discrete Appl. Math.* **161(9)**, 1187–1190 (2013).
32. Tamizh Chelvam, T., Mutharasu, S.: Efficient open domination in Cayley graphs. *Appl. Math. Lett.* **25**, 1560–1564 (2012).
33. Tamizh Chelvam, T., Raja, S., Gutman, I.: Strongly regular integral circulant graphs and their energies. *Bulletin of International Mathematical Virtual Institute*. **2**, 9–16 (2012).
34. Tamizh Chelvam, T., Raja, S.: Integral circulant graphs with four distinct eigenvalues. *Discrete Math. Algorithms Appl.* **10(5)**, # 1850062 [10 pages] (2018).
35. Tamizh Chelvam, T., Rani, I.: Independent Domination number of a Cayley Graph on \mathbb{Z}_n . *J. Combin. Math. Combin. Comput.* **69**, 251–255 (2009).
36. Tamizh Chelvam, T., Sattanathan, M.: Power graph of finite abelian groups. *Algebra and Discrete Mathematics*. **16(1)**, 33–41 (2013).
37. Tamizh Chelvam, T., Sivagami, M.: Structure and substructure connectivity of Cayley graphs. *Arab Journal of Mathematical Science*. **27(1)**, 94–103 (2021).
38. Vatandoost, E., Golkhandypour, Y.: Integral Cayley graph on some of abelian groups. *TJEAS Journal*. **3(6)**, 489–492 (2013).