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ORIGINAL RESEARCH

Wittgenstein on Weyl: the law of the excluded middle and the natural numbers

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Abstract
In one of his meetings with members of the Vienna Circle, Wittgenstein discusses Hermann Weyl's brief conversion to intuitionism and criticizes his arguments against applying the law of the excluded middle to generalizations over the natural numbers. Like Weyl, however, Wittgenstein rejects the classical model theoretic conception of generality when it comes to infinite domains. Nonetheless, he disagrees with him about the reasons for doing so. This paper provides an account of Wittgenstein's criticism of Weyl that is based on his differing understanding of what a general statement over infinite domains consists in. This difference in their conception of generality is argued to be central to the middle Wittgenstein's overall stance on intuitionism as well. While Weyl (and other intuitionists) reject the law of the excluded middle on grounds of constructivity, Wittgenstein argues that general statements over infinite domains do not express propositions in the first place. The origin of this position as well as its consequences for contemporary debates on generality are further assessed.

Keywords Wittgenstein · Weyl · Intuitionism · Generality

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1 Introduction

In one of his meetings with members of the Vienna Circle, Wittgenstein discusses Hermann Weyl's brief conversion to intuitionism and criticizes his arguments against

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applying the law of the excluded middle (LEM) to generalizations over the natural numbers. For Weyl, a universal statement about the natural numbers is not a genuine statement (or judgement as he calls it) but rather a "general instruction for judgements [Anweisung für Urteile]" (Weyl, 1998a, p. 98), whereas an existential judgement is only genuine if it is accompanied by the construction of a witness. Since the negation of a universal judgement does not entail the construction of a counterexample, the law of the exclude middle looses its assertability. While Wittgenstein seems to agree with this, he gives a different reason for doing so. He is recorded to criticise Weyl for "conflating things", and "lumping two things together" (Wittgenstein, 1979, p. 82). The actual argument against the duality of the quantifiers and the law of the excluded middle, according to Wittgenstein, is that in generalisations over infinite domains—both universal and existential—we are "not dealing with sentences in the first place" (Wittgenstein, 1979, p. 82). This point is later repeated as a broader critique of intuitionism:

> I need hardly say that where the law of the excluded middle doesn't apply, no other law applies either, because in that case we aren't dealing with propositions of mathematics. (Against Weyl and Brouwer). (Wittgenstein, 1975, p. 176)

Instead, Wittgenstein suggests that the respective generality is exhibited by the rules of the symbolism that we use to express when reasoning with the natural numbers.

The purpose of this paper is to understand what exactly Wittgenstein meant by this criticism and how his alternative conception fares against Weyl's own. The origin of Wittgenstein's criticism, I aim to show, lies in the tractarian conception of generality that still looms large in the background of Wittgenstein's comments: General expressions over infinite domains, when understood appropriately, do not fulfil the tractarian standard of sentence-hood. The assessment of Wittgenstein's alternative proposal will then lead to an investigation of mathematical induction as well as a reconsideration of the meaning of existential statements on Wittgenstein's part.

Understanding these points clarifies the middle Wittgenstein's obscure relationship to intuitionism in general. It explains how he appears to be in agreement with the intuitionistic position all the while objecting to it. Wittgenstein's engagement with intuitionism has been discussed in various studies [(cf. Fogelin, 1968; Gonzalez, 1991; Marion, 2003; Rumfitt, 2014)], but apart from Marthieu Marion's excellent paper "Wittgenstein and Finitism", his remarks on Weyl and the relevance of his conception of generality to the whole issue have not received much attention.[1]

[1] When assessing Wittgenstein's relationship to intuitionism, Brouwer's work is usually the main point of reference [(cf. Wittgenstein, 1975, p. 210, 1998, Part V, esp. Sect. 11, Sect. 12, Sect. 27)]. As Rumfitt (2014) has pointed out, Wittgenstein seems to think that Brouwer's arguments against LEM rest on knowability constraints (cf. Wittgenstein, 1979, p. 73), which, given Brouwer's claim that mathematical entities are a creation of the mind [cf. Brouwer (1928, 1981)], directly transform into a lack of determinacy regarding the truth conditions of the matter. Since Wittgenstein is diametrically opposed to this conception of mathematical entities (cf. e.g. Wittgenstein, 1994, p. 6.233) he seeks to replace Brouwer's considerations regarding truth conditions with considerations regarding meaning. His remarks on Weyl are motivated by the same principle. While I agree with Marion that the tractarian analysis of generality is central for this approach, there are some notable differences between our interpretations. At the end of the paper I will compare our two approaches and highlight what I take to be the advantages of my own.

Furthermore, clarifying the relationship between Weyl and Wittgenstein is not only of historical interest. Their positions illustrate two different conceptions of generality with respect to indeterminate domains. Weyl's work and the conception of generality that it introduces suggests possible applications to the question of absolute generality [(cf. Rayo & Uzquiano, 2006; Linnebo, 2022)] and to the philosophical foundations of constructive mathematics (cf. Crosilla, 2022). This paper, however, shows that Wittgenstein's diagnosis of the relevant phenomena provides a more appropriate basis for these endeavours. Yet, the solution that he suggests yields some unfavourable consequences regarding our understanding of generality expressions. This suggests a renewed consideration of these matters.

The structure of this paper is as follows: After considering Weyl's argument against the duality of quantifiers (Sect. 2), I will collect the parts from the *Tractatus* that are relevant for understanding Wittgenstein's comments (Sect. 3). But, as the tractarian account of generality crumbles at the end of the 1920s, it is important to distinguish which parts of it remain (at least relatively) intact such that they can be understood as the proper basis of this criticism (Sect. 4.1). This will then be applied to the remarks he puts forth about Weyl's understanding of universal generalisations (Sect. 4.2) and existential generalisations (Sect. 4.3).

2 Weyl against the duality of the quantifiers

The textual background to the discussion with the Vienna Circle is Weyl's *Symposion* article from 1927 (Weyl, 1998b), but the remarks they discuss refer to his previous article "On the New Crisis in the Foundations of Mathematics" [Weyl (1921), trans. Weyl (1998a)], which marks his brief conversion to intuitionism.[2] Weyl discusses the judgements $\forall \alpha A(\alpha) \vee \exists \alpha \neg A(\alpha)$ for choice sequences α and $\forall n A(n) \vee \exists n \neg A(n)$ for natural numbers n. This paper focuses only on his discussion of the natural numbers. Weyl calls into question that $\forall x P(x) \vee \exists x \neg P(x)$ can be a logical truth when the quantifiers are understood to range over the natural numbers. The crucial point for this is whether the inference from $\neg \forall x P(x)$ to $\exists x \neg P(x)$ is always acceptable. There are two ways in which concerns can be raised about this: (i) it can be objected that this inference simply doesn't hold in all the cases, and, more radically, (ii) it can be called into question whether the negation of a universal statement over the natural numbers yields a meaningful statement at all. The latter option, of course, restricts the possibilities of forming meaningful statements considerably. Surprisingly, however, both, Weyl and Wittgenstein, for their own particular reasons and with differing consequences, will turn out to venture down this road.

Weyl starts his argument by considering (i). Can, he asks, the judgement that $\exists x \neg P(x)$ be understood as the negation of the universal judgement? The intuition behind a positive answer to this question is the idea that one might imagine a procedure of running through the natural numbers and checking for each number n if it

[2] In this article Weyl proclaims "So I now abandon my own attempt and join Brouwer" (Weyl, 1998a, p. 98) and subsequently adopts his own version of Brouwer's theses. Even thought their general direction is the same, there are some differences in detail between Weyl and Brouwer, which, however, mostly concern the technical part of the project. For a discussion, see van Dalen (1995).

holds that $P(n)$. Under this procedure, either it is true that all n have P or at some point a counterexample will have to turn up. According to Weyl, considerations like this appeal to the idea that such a procedure of going through all the numbers is completable. But therein lies a mistake:

> [T]his point of view of a completed run through an infinite sequence is nonsensical. I cannot get general judgements about numbers by looking at the individual numbers but only by looking at the essence of number [*das Wesen der Zahl*]. (Weyl, 1998a, p. 95)

Thus, instead of taking it as a claim about all numbers, we are to understand the content expressed in $\forall x P(x)$ as appealing, without any mentioning of instances, to the "essence of number". This means that for an arbitrary natural number, we can infer that it has property P *qua* being a natural number, but we can no longer explicate the truth of $\forall x P(x)$ by reference to all the instances falling under the concept of natural number. This type of generality is nowadays called generic generality.[3] Weyl considers these expressions no longer to be judgements, but to contain merely a "general instruction for judgements" (Weyl, 1998a, p. 98) (about concrete numbers).

Yet, the above quote is somewhat inconclusive. Weyl only objects to the idea of *running* through an infinite series, but that in itself does not exclude the possibility that statements over an infinite domain may nonetheless have determinate truth conditions—we might just not be able to epistemically attain them. If the domain is determinately circumscribed and if it is also determinate for each n whether it has P, what should keep the corresponding general statements from being equally determinate as well? Incidentally, the matter is not helped by the fact that Weyl indeed considers the domain of natural numbers (contrary to the domain of choice sequences) to be "extensionally-definite" (Weyl, 1998a, p. 96). A possible response to this is to strengthen the epistemic demands for the respective judgements and thus to require a (finite) verification of the universal claim or the construction of a counterexample. If a constructivist stance is adopted outright in this way, then it is enough to acknowledge that one cannot run through an infinite series in order to dismiss the inference from $\neg\forall x P(x)$ to $\exists x \neg P(x)$. Considering Weyl's remark, it seems like this is indeed the case for him and the indeterminacy in question is a consequence of the constructivist standpoint.

But there is also evidence to the contrary. In continuation of the quote above, Weyl claims that "God himself cannot make use of a different reason for deciding" (Weyl, 1998a, p. 97). He repeats the emphasis on God the second time he treats the topic, which is the one Wittgenstein and the Vienna Circle were discussing (cf. Weyl, 1998b, p. 133), and the point will also come up in Wittgenstein's own discussion of the infinite. God, however, surely would be able to run through the natural numbers, for God is conceived as a being that is not under the restrictions of what Russell aptly termed 'medical finitude'. The only conceivable way in which it can be upheld that

[3] Of course, this also means that under this conception, we no longer *need* any reference to instances to express such a general statement. In this respect, generic generality has, by its contrast to an instance based one, its use in the debate on absolute generality (cf. Rayo & Uzquiano, 2006). Since instance based conceptions often struggle with the possibility of diagonalization, such a generic conception suggests itself as an alternative (cf. Linnebo, 2018).

God is not able to run through all the natural numbers is to consider their domain to be indeterminate in the first place. Then one cannot say that God would be able to run through them simply because there is no clear conception of what this amounts to. If there were such a conception, God as an omnipotent being would by definition be capable of doing so. An argument based on this conception of the indeterminacy of the natural numbers would no longer depend on a constructivist premise. Instead, the constructivist requirement would follow from the conceived indeterminacy of the domain. An argument of this sort will be considered when discussing Wittgenstein.[4] Weyl, however, does not mention anything along these lines, and it would also contradict his claim regarding the extensional determinacy of the natural numbers. Thus, it seems more plausible to attribute to Weyl the strategy mentioned in the previous paragraph and thereby an overall constructivist premise.

This reading is also supported by his treatment of existential generalisations. In addition to his remarks about infinitary generalisations, Weyl also criticises the received understanding of the existential quantifier.

> An existential statement—say, "there exists an even number"—is not at all a judgement in the strict sense, which claims a state of affairs. Existential states of affairs are empty inventions of logicians. "2 is an even number": This is an actual judgement expressing a state of affairs; "there is an even number" is merely a judgement abstract gained from this judgement. (Weyl, 1998a, p. 95)

An existential statement can only convey a proper judgement if it contains the construction of a witness. In this case, the existential judgement abstracts from this particular number and only states, in covertly modal terms, the possibility of a construction. But Weyl is quick to emphasize that "[o]nly the successful construction can provide justification for this; the mere possibility is out of the question." (Weyl, 1998a, p. 96) Even though Weyl's remark appears in the discussion of claims regarding the natural numbers, he seems to introduce this criticism as a general idea that is not necessarily connected to the infinite. Concerns about the infinite provide major support for it, since the availability of instances is usually considered to be guaranteed on finite domains, but that should not cover up the fact that it is the construction of a witness and not finitude which makes $\exists x P(x)$ a determinate judgement.

Thus, Weyl introduces epistemic demands for the legitimacy of existential and (so I have argued) universal judgements from an overall constructive standpoint. These demands have the consequence that $\exists x \neg P(x)$ can no longer be taken to be the negation of $\forall x P(x)$. That something does not follow from the essence of the concept of natural number does not imply that there is a counterexample readily constructible. So much for the duality of the quantifiers. But Weyl even goes one step further. Regarding (ii), the question how the negations $\neg\forall x P(x)$ and $\neg\exists x P(x)$ are then to be understood, Weyl's verdict is unreservedly negative: "Neither the negation of the one nor of the other makes any comprehensible sense." (Weyl, 1998a, p. 97). In consequence, "[t]his means that it becomes quite impossible even to formulate an 'axiom of the exclude middle' for them"

[4] Further arguments like this have been put forward, for instance, in Dummett (1991) and Feferman (2014). As argued in Sect. 4.2, Wittgenstein seems to have in common with them that his rejection of standard quantification over infinite domains does not depend on a constructivist premise of the sort that Weyl is relying on.

(Weyl, 1998a, p. 99). Thus, as far as judgement abstractions and general instructions for judgements are concerned, neither of their negations are said to yield any such abstractions or general instructions themselves, let alone any determinate judgements *simpliciter*. Weyl's reasons for this might be put in the following way: How are we to understand that something does not follow from the concept of natural number? A statement like that denies a logical subordination between two concepts, but it does not give any positive determination. Similarly, the negation of some possibility of construction does not yield any claim regarding the possibility of any judgement itself. Neither of the two is thus sufficiently determined.

Note, however, that this also entails that the inference from $\neg\exists x\varphi(x)$ to $\forall x\neg\varphi(x)$, which is intuitionistically acceptable, can no longer be asserted. In this respect, Weyl's remark is somewhat surprising. Intuitionists usually understand the negation of such statements to mean that their assertions can be shown to lead to an absurdity. The negation of an existential statement is then implicitly universal since not a particular construction is denied, but the possibility of any such construction. Furthermore, understanding negation in this way is systematically compatible with Weyl's overall position, and there even is some textual evidence for it in Weyl's paper as well.[5]

In summary, Weyl argues against the duality of the quantifiers and, on top of that, he claims that $\neg\forall x P(x)$ and $\neg\exists x P(x)$ don't serve to express meaningful statements. Where in this regard does Wittgenstein object? While he seems to agree with Weyl's verdict on the negation of universal and existential generalisations, his misgivings regard two points: a) the asymmetry of calling a statement meaningful while denying the same for its negation (even though this somewhat brushes over the distinction between concrete judgements and general instructions for judgements that Weyl clearly draws) and b) Weyl's use of the notion of existence and his overall reliance on constructivity. To develop these points, some theoretical background in Wittgenstein's thinking is required. As the meetings with the Vienna Circle were still revolving around tractarian themes, the *Tractatus*'s conception of generality is the place to start.

3 The conceptual background to Wittgenstein's response

Wittgenstein repeatedly discusses matters regarding the notion of generality with the Vienna Circle. In an astounding number of times he offers criticisms and refinements to the tractarian conception of generality (cf. Wittgenstein, 1979, pp. 38, 44, 51, 187), but his response to Weyl is still largely shaped by his old account. The challenge for the interpreter is thus to separate the (momentarily) surviving remains of this conception for a basis of engagement with Weyl from what fell victim to his own critical retrospective remarks. I start by laying out the original tractarian account of generality.

In the *Tractatus*, Wittgenstein distinguished between the way generality is to be expressed and what the truth of a generality statement consists in:

5 Cf. in this respect Linnebo (2022), who develops Weyl's notion of generic generality and combines it with intuitionistic negation, and see (Weyl, 1998a, p. 98), where he discusses the respective argument sufficient to establish a judgement of the form $\forall x\neg\varphi(x)$.

> I dissociate the concept *all* from truth-functions. Frege and Russell introduced generality in association with logical product or logical sum. This made it difficult to understand the propositions '$(\exists x).fx$' and '$(x).fx$', in which both ideas are embedded. (Wittgenstein, 1994, p. 5.521)

The two ideas separate as follows.[6] An *expression* of generality, i.e., "the concept all", is given by an expression like 'P(x)'. The generality that it incorporates is contained in the variable. The variable characterizes a logico-linguistic form which certain propositions share. Understanding the variable is tantamount to understanding the logical form it embodies and thereby tantamount to understanding the range of propositions that share it. To obtain a proposition which can be used to assert the *truth* of a generalisation one needs to add a quantifier that binds the variable, like '$\forall x P(x)$'. The resulting expression is a conjunction or disjunction of the propositions corresponding to each of the instances that are captured by the variable. In this way, the two ideas are combined in quantified statements like

$$\forall x P(x) \equiv P(a) \wedge P(b) \wedge \cdots \wedge P(n)$$
$$\exists x P(x) \equiv P(a) \vee P(b) \vee \cdots \vee P(n)$$

To dive a bit deeper into these two components observe that an expression of generality is intimately connected to the logical symbolism chosen. This is the case even for a non-perfect symbolism. To highlight this point, Wittgenstein remarks that generality "makes constants prominent" (Wittgenstein, 1994, p. 5.522). A fuller explanation of this idea is given by Ramsey:

> It is not 'aRb' but '(x).xRb' which makes Rb prominent. In writing (x).xRb we use the expression Rb to collect together the set of propositions xRb which we want to assert to be true; and it is here that the expression Rb is really essential because it is this which is common to this set of propositions. ((Ramsey, 1990, p. 19)

Thus the expression of generality consists in picking out a class of propositions that can be formed *with respect to a certain symbolic structure*. In this way, the generality expression itself is parasitic on the structure chosen beforehand; it gives us a class of propositions, selected as those that share a common form in this particular symbolic structure. In this way it also differs slightly from the standard model theoretic account of generality in which the variable ranges over the objects of a given domain. On Wittgenstein's conception it is the propositions themselves that serve as inputs into a truth function in a (conceptually distinct) subsequent step (cf. also Potter, 2009). Finally, regardless of the symbolism chosen, any expression of generality ultimately serves to pick out a class of elementary propositions to which, lastly, the truth functions will be applied. This ensures the well-foundedness of the procedure.

[6] I want to thank Peter Sullivan for helping me understand this passage; any misunderstandings in this respect are of course my own.

Next is the application of truth functions to a collection of propositions.[7] Wittgenstein points out three ways to specify such a collection of propositions:

> 1. direct enumeration, in which case we can simply substitute for the variable the constants that are its values; 2. giving a function fx whose values for all values of x are the propositions to be described; 3. giving a formal law that governs the construction of the propositions, in which case the bracketed expression has as its members all the terms of a series of forms. (Wittgenstein, 1994, p. 5.501)

We can see that only option two refers to the mechanism of generality that was discussed before. Option one need not require any shared logico-linguistic form between the propositions. Option three specifies the collection yet in a different way by giving a recursive rule according to which expressions of a certain type can be constructed.

In his work following up on the *Tractatus* Wittgenstein will put great effort in separating method two from method three, but in the *Tractatus* itself he holds that "how the description of the terms of the expression takes place is unessential" (Wittgenstein 1994, p. 5.501). The way of description is unessential from the standpoint that these three methods may yield truth-functionally equivalent results. This indicates a primacy of the notion of truth-function over the expression of generality. This impression is further sustained by considering again Ramsey's analysis from above. Ramsey's remark implies that regarding method two, different systems of notation can be used that may yield different expressions of generality, and thus different ways of collecting propositions. Yet these differences, too, move to the background when the truth of these generalisations is in question.

Next, consider negation. Negation is understood in a classical sense according to which the negation of a statement is the *affirmation of its opposite*. This, of course, requires that there is such an opposite in the first place. Thus, Wittgenstein writes "The propositions 'p' and '$\neg p$' have opposite sense, but there corresponds to them one and the same reality." (Wittgenstein, 1994, p. 4.0621) This point gets further illustrated in his analogy to illuminate the tractarian conception of truth (cf. Wittgenstein, 1994, p. 4.063).

> [I]magine a black spot on white paper: you can describe the shape of the spot by saying, for each point on the sheet, whether it is black or white. To the fact that a point is black there corresponds a positive fact, and to the fact that a point is white (not black), a negative fact. (Wittgenstein, 1994, p. 4.063)

This analysis has many facets, but only two of them are important for present concerns: (i) An assertion of falsehood corresponds to a point on the paper just as an assertion of truth does.[8] (ii) The black and white parts make up the whole paper. Thereby, this analogy suggests that the notion of negation is intrinsically connected to the idea that logical space is *total*. As far as negation is affirmation of the opposite, there must be

[7] Of course, Wittgenstein does not use the two quantifiers in the logical notation of the *Tracataus* itself, but incorporates them by the use of the N-operator. This may have implications for the logical system of the *Tractatus* further down the line, but it is not important for the investigation of his conception of generality.

[8] This also entails that negation is used to characterise the content of an assertion (i.e. that something is not the case) and is hence applied on the level of propositions and not on the level of judgements/assertions (i.e. that one cannot make this or that judgement).

an underlying space to assure that it is sufficiently determined what the opposite is supposed to be. This is also connected to Wittgenstein's remarks that a proposition "reaches through the whole of logical space" (Wittgenstein, 1994, p. 3.42). Negation would not be sufficiently (positively) determined for a proposition that does not reach through the whole of logical space.[9] Connected with this idea is the thought that whatever can be expressed by a statement with a negation can in principle also be expressed without it. The negation of a proper statement like "The wall is not blue" can be understood as saying that the wall must *have any other colour*. The negation of an improper statement like "(The sound) Eb is not blue" can only be understood as saying that the concept of colour cannot be applied to the sounds. The reference to logical space shows that by a meaningful proposition Wittgenstein has only the first one in mind.

The duality of the quantifiers follows from combining the previously discussed account of generality with this understanding of negation. Since quantification can be expressed via elementary truth functions, applying negation to a quantified proposition is then tantamount to applying negation to its instances, i.e. to the propositions serving as the conjuncts and disjuncts, respectively (cf. also Wittgenstein, 1994, p. 5.52):

$$\neg\forall x\, P(x) \equiv \neg(P(a) \wedge \cdots \wedge P(n)) \equiv \neg P(a) \vee \cdots \vee \neg P(n)$$
$$\neg\exists x\, P(x) \equiv \neg(P(a) \vee \cdots \vee P(n)) \equiv \neg P(a) \wedge \cdots \wedge \neg P(n)$$

By the determinacy of negation the resulting instances on the right side are all proper propositions. In summary, then, it is only possible to understand the negation of a quantified proposition as affirming the opposite because (a) the quantified proposition is connected to its instances in this way *and* (b) negation is defined on them by presupposing that their opposite is sufficiently circumscribed.

So much for the tractarian account of generality. Wittgenstein diversifies this account extensively in retrospective discussions. Amongst other things, he takes back his remark that the differences between the three methods of specifying collections of propositions are unessential. He discovers that enumeration can only be seen as an expression of generality in a limited sense, and that specifying expressions via recursion is a completely different endeavour. However, the central point for understanding his remarks on Weyl, and thus his overall stance on intuitionism, is that for quantification over the natural numbers (and other infinite domains), the interaction between quantification and negation no longer works as described above.

4 Wittgenstein on the tractarian account of generality and his remarks about Weyl

Assessing Wittgenstein's response to Weyl entails two tasks: the need to differentiate Wittgenstein's retrospective criticism of his own account from those parts of the account which he still makes use of; and the need to come to an understanding of

[9] This is of course not the case with intuitionistic negation, where φ, $\neg\varphi$, and $\neg\neg\varphi$ are propositions that differ in their inherent complexity. In contrast to (Wittgenstein, 1994, p. 4.0621) $\neg\varphi$ is thus not to be understood as the opposite of φ such that $\neg\neg\varphi$ does not entail φ.

how much the extent of his agreement with Weyl influences his criticism. In this main section of the paper, I will first look at Wittgenstein's new discussion of generality and its consequences regarding infinite domains (Sect. 4.1). Afterwards, I will apply those findings to Wittgenstein's take on universal generalisations over the natural numbers (Sect. 4.2) and on existential statements (Sect. 4.3).

4.1 Loosening the tractarian account of generality

Upon his return to philosophy, Wittgenstein comes to realize that generality has too many facets to be treated uniformly.[10] In earlier conversations with the Circle he distinguished between expressions consisting of plain enumerative statements like "Everyone in this room wears pants", generality with respect to grammatical forms, for which he gives the example "all (primary) colors", and, finally, expressions like "all numbers" (cf. Wittgenstein, 1979, p. 44). Regarding a difference between the first two expressions, he notes that a conjunction like $\varphi(a) \wedge \varphi(b) \wedge \cdots \wedge \varphi(n)$ is only a proper expression of generality if it also contains another expression ψ saying that the individuals a to n are all the individuals under consideration (Wittgenstein, 1979, pp. 38–39). This particular piece of information is not captured by the truth functional understanding. Should the general statement be true, the expressions $\varphi(a) \wedge \varphi(b) \wedge \cdots \wedge \varphi(n)$ and $\varphi(a) \wedge \varphi(b) \wedge \cdots \wedge \varphi(n) \wedge \psi$ will be truth-functionally equivalent. With generalisations over grammatical categories like "all colors" such an additional completeness claim ψ is not required. Through our understanding of the grammatical category itself it is conveyed that there are only such and such primary colors. This is a first step in the broadification of the notion of generality. In addition to that, Wittgenstein now also seems to allow for generality to enter into elementary propositions by giving an incomplete picture of a state of affairs (cf. Wittgenstein, 1979, p. 41).

Yet, even though Wittgenstein notes that generality can be conveyed in different ways, assertions of general propositions like those mentioned in the previous paragraph are still thought of in terms of truth-functionality (and this should also include the incomplete elementary propositions). The matter changes more drastically, however, when we consider the notion of "all numbers" in contrast to the examples above. Wittgenstein says about generalisation over all numbers:

> [W]e know that such a proposition has been misunderstood and that induction has nothing to do with the totality of numbers. (Wittgenstein, 1979, p. 45)

This quote contains two aspects. It contains the negative claim that there is no such thing as a *totality* of all numbers, and the subsequent positive one that because of it, the proper expression of generality proceeds via induction. Starting with the negative observation, why does Wittgenstein think that there is no totality of natural numbers?

The reason lies in the infinity of the domain of natural numbers. To this we find an extensive discussion in the *Philosophical Remarks* which were compiled around the same time that the conversations with the Circle took place. Interestingly, Wittgenstein

[10] G.E. Moore, for instance, notes him saying in a lecture that generality is "very difficult, because many different kinds—This doesn't mean a genuine concept, with species of it.—What's meant is that we express 'all, some, every' in entirely different ways" (Wittgenstein, 2016, p. 4:45).

claims (contrary to Brouwer and seemingly also against Weyl) that speaking of a totality of all numbers is not just confined by epistemic demands, i.e. that judgements about infinitely many instances are constrained by requirements of constructivity, but that it is *all in all meaningless*.

> It isn't just impossible 'for us men' to run through the natural numbers one by one; it's *impossible*, it means nothing.
>
> Nor can you say, 'A proposition cannot deal with all the numbers one by one, so it has to deal with them by means of the concept of number', as if this were a *pis aller*: 'Because we can't do it *like this*, we have to do it another way'. But it's not like that: of course it's possible to deal with the numbers one by one, but that doesn't lead to the totality. For the totality is only given as a concept. (Wittgenstein, 1975, Sect. 124)

Moreover, contrary to Weyl, this is also sufficient for Wittgenstein to reject the claim that the natural numbers are "extensionally-definite". In fact, he was just quoted to reject the whole idea that the natural numbers have any extension at all. Wittgenstein also evokes the notion of God to illustrate his point, and almost mockingly writes: "'Can god know all the places of the expansion of π?' would have been a good question for the schoolmen to ask." (Wittgenstein, 1975, p. 149) Unlike Weyl, however, he manages to be more consequential on this part. As noted before, these remarks on God would not be understandable if the infinite is merely thought of as epistemically inaccessible to us. God surely would not have *that* problem. But even God cannot do something that is conceptually meaningless—at least we could not conceive of it. Understood this way, "going through the extension of all the natural numbers" is like "colouring a concept" or "finding the difference between a hedgehog". It is a meaningless task. In following along with this idea, Wittgenstein thus rejects standard quantification over infinite domains without any reliance on a constructivist premise.

However, being more consequential at this point also entails being more radical. For how can it be explained that Wittgenstein holds that there is no such thing as a totality of all numbers? Furthermore, in light of the fact that we have a flourishing practice of making judgements about what we call 'all natural numbers', how else are we to interpret such talk if not by reference to such a totality? Wittgenstein's reasoning on this matter can be reconstructed in two steps. First, he notes that the tractarian account of generality fails for infinite domains, and second, he comes up with an alternative conception of generality that rules out any conception of the natural numbers forming a totality.

In retrospective, Wittgenstein claims that the tractarian account of generality only works if the truth functions are finite. He argues that there cannot be an infinite combination of truth-functions, because there cannot be an infinite conjunction of propositions.

> What is the meaning of such a mathematical proposition as "$(\exists n).4 + n = 7$"? Might it be a disjunction - $4 + 0 = 7. \vee .4 + 1 = 7. \vee .4 + 2 = 7. \vee$. etc. *ad inf.* But what does that mean? I can understand a proposition with a beginning and an end. But can one also understand a proposition with no end?

> If no finite product makes a proposition true, that means *no* product makes it true. (Wittgenstein, 1975, Sect. 127)

As this remark suggests, the finitude of the truth functions is coupled with the finitude of the conjunction of propositions, which, again, is connected to the conception that the elements of a generalisation are not objects, but propositions.[11]

A consequence of this is that the variable based way of expressing generality (i.e. method two from Wittgenstein (1994), p. 5.501) becomes untenable for infinite domains, as well. It would lead straight away to such an illicit attempt to form an infinite logical product or sum. This is thus a main retrospective criticism of his own position, and it also serves to explain the mysteriously sounding remark directed at Weyl:

> The generality does not show itself in the letters. They do not have anything to do with generality. Generality shows itself in that it goes on in this way. (Wittgenstein, 1979, p. 82)

The first two sentences can be understood as referring to (and rejecting) the 'old way' of understanding general propositions that was so clearly put by Ramsey. With this, Wittgenstein seems to warn that any proposition that uses variables to express generality over the natural numbers does so only in superficial similarity to the tractarian conception of generality.

But the account is not yet complete. Wittgenstein has rejected infinite products, but he has not given an argument as to why, and, furthermore, it is not yet clear why the natural numbers should cease to be a totality as a consequence of it. The first point is difficult to answer, because Wittgenstein himself goes straight into presenting the alternative that results form such a rejection. As has been argued in Methven (2016), one argument can be found by considering a remark by Ramsey, according to which a theory of infinitary conjunctions or disjunctions is something "which we cannot express for lack of symbolic power" (Ramsey, 1929, p. 146). Methven argues that if we were to allow for infinite conjunctions and disjunctions we would not be able to account for their understanding other than by going beyond our grasp of the rules of the symbolism, because that would essentially involve completing an infinite process. We would thus have to attribute ourselves the understanding of a meaning that we cannot exhibit.

This looks similar to Weyl's argument in that Ramsey's explanation also appeals to epistemic demands. However, whereas Weyl is concerned with the legitimacy of individual universal or existential judgements (and argues that they are only legitimate when accompanied by a proof/construction), Ramsey's point is about the legitimacy of a *theory* of infinite conjunctions and disjunctions in general. In this respect, the objection that we cannot exhibit their meaning holds for any infinite disjunction and

[11] This point echoes well with his retrospective criticism of his tractarian position in his Cambridge Lectures from 1930: "In the *Tractatus* I thought '$(\exists x) fx$ is a definite logical sum, only I can't at the moment tell you which' (Wittgenstein, 2016, p. 7:39). And further: "My mistake was to think that the [logical] product, though we couldn't find it now, was contained in it [i.e. the general proposition] (Wittgenstein, 2016, p. 7:40). Wittgenstein passes over the more complicated interplay of generality expression and truth functionality in this retrospective, but all in all it is clear that he now thinks that the tractarian conception is no longer sufficient because infinite logical products cannot exist.

conjunction, regardless of whether it is accompanied by a proof or not. This is taken to be sufficient reason to reject the whole understanding of such generalisations as infinite products, instead of introducing, like Weyl, restrictions on the legitimacy of asserting *individual* judgements of such a type.

His own reasons for rejecting infinite conjunctions notwithstanding, Wittgenstein introduces an alternative conception of generality, from which it follows that there is no totality of all numbers. According to Wittgenstein in 1929, the infinity of the natural numbers is not something that is to be understood in terms of the size of a collection of objects. Instead, infinity is a feature of the rules for the symbolism that is used to express the concept.

> The rules for a number-system–say, the decimal system–contain everything that is infinite about the numbers. That, e.g., these rules set no limits on the left or the right hand to the numerals; this is what contains the expression of infinity. (Wittgenstein, 1975, Sect. 141)

Adding any arbitrary finite number to any other arbitrary finite number also produces a finite number. The absence of any internal limitation to this is the sense in which the natural numbers are infinite. Furthermore, following Methven, the possibility of increasing finite numbers arbitrarily often (but finitely many times) is to be distinguished from the notion of possibility that features in the distinction between the potential and the actual infinite. In Wittgenstein's eyes it is a category mistake to infer the potential or actual existence of a totality of instances falling under the concept from the fact that there is no restriction to the application of the successor function (cf. Wittgenstein, 1975, Sect. 142, Sect. 143). This explains why Wittgenstein claims that induction, which is taken as the expression of such generality, "has nothing to do with the totality of numbers". In this respect, Wittgenstein's conception of generality differs starkly from Weyl's. Weyl's epistemic concerns are to be contrasted with Wittgenstein's attempt to understand the infinite not in terms of any extension, but as a feature of the rules of the symbolism, most notably of the rules governing induction.[12] But how is this new type of generality to be understood, and how does it align with the common mathematical understanding and justification of induction?

4.2 Universal generalisations and induction

Induction is a notorious issue for Wittgenstein (and in Wittgenstein scholarship), and it makes up a topic on its own. Due to some (initially at the very least) disturbing remarks, it is questionable if his overall position on the matter is tenable with respect to current mathematical practice (or any practice of mathematics beyond elementary arithmetic and geometry). But before this becomes relevant, the immediate concerns for understanding the remarks on Weyl are:

1. Since there is no variable based generality any more, what kind of generality is contained in the expression of induction?

[12] For further discussion of this point, especially regarding differences between Wittgenstein and Ramsey in dealing with this new understanding of infinity, cf. Methven (2020).

2. Since there is no truth-functionality any more, what does the truth or just: legitimacy of such a generalisation via induction consists in?

Wittgenstein's answer to these questions will turn out to differ quite extensively from Weyl's.

In general, an inductive definition is a way of specifying a set of elements by specifying one or more initial elements and a collection of rules which can be applied to them to obtain further elements. The inductively defined set is then obtained by taking the closure under these rules. That the elements of such a set all share a certain property P can then be demonstrated by showing that the initial elements have it and that it is inherited by succession among the specified rules. However, according to Wittgenstein, this conception of definition (and consequently also of proof) by induction is not to be understood in a way that suggests that there is a predetermined totality along which the operations so defined move; the generality lies solely in the recognition of the procedure itself:

> The rule for infinity can be expressed symbolically as follows: $[f(1), f(\xi), f(\xi+1)]$. Note that we have to go step by step, starting from $f(1)$. This is not the kind of generality represented by $(x)\varphi(x)$. (Wittgenstein, 1980, p. 14)

And G.E Moore notes from the same lecture:

> Though the rule has some sort of generality, it's not comparable to $(x)\varphi(x)$, because we have to go step by step: we never get to anything general. (Wittgenstein, 2016, p. 4:39)

In particular, the "rule for infinity" is observable as a property of the symbolism given any *finite and particular* instance of it. It is *not* a different means to attain the same totality that is else implicated by using a variable (i.e. method two from the *Tractatus*). While use of a variable serves to pick out a class of propositions by reference to a shared logico-linguistic form, induction does not give us such a class. It is merely a property of the rules that govern how the symbolism works.[13] This is the case when a set is defined via induction but also when we conceive of any general property that the 'elements' of such an inductively defined set may have.

Therefore, Wittgenstein even objects to the inductive generality being expressed by $\forall x P(x)$ precisely because it suggests a similarity to the variable based expression of generality. Moreover, as far as induction characterizes the form by which generality regarding the natural numbers is expressed, Wittgenstein thinks that it simply does not make sense to negate it. The expression of generality that it contains is not sentence-like and hence there is no sentential negation of it. In short, we do not have $\neg\forall x P(x)$, because we don't even have $\forall x P(x)$. The similarity to a "normal" generalization, which of course permits such a negation, is, as it was pointed out, only superficial.

[13] It is actually quite difficult to separate the two methods by using such a description, and Wittgenstein's own comments can also be understood to apply to the other as well. (After all, also in case of $(x)\varphi(x)$ we understand the generality in a finite piece of symbolism by understanding what kind of expressions we can construct with it.) One notable difference is that with induction the elements form part of a series. There is also a certain incongruence compared with method 2 in that induction requires the starting point, the number 1, to be given. Moore, in his notes of the same lecture, has picked up on this point: "ξ seems here to denote a sort of generality comparable to the generality of x in $(x)\varphi(x)$. But this is not so. ξ has no meaning whatever, unless I have 1."(Wittgenstein, 2016, 4:39) See also (Wittgenstein, 1974, p. 443).

With that, there is enough material to explain why Wittgenstein writes in partial confirmation of Weyl:

> Therefore, where there is a statement it can be negated; and where a certain structure cannot be negated, there is no statement either. The principle of excluded middle doesn't hold in this case for the simple reason that we are not dealing here with sentences. (Wittgenstein, 1979, p. 82)

The second sentence is intended as a clarification against Weyl: induction is a non-truth functional expression of generality, and thereby it does not lead in Wittgenstein's eyes to a proposition. It is simply a wholly different kind of expression. The quote suggests that Wittgenstein thinks that Weyl would consider an expression of generic generality to be a genuine statement, against which he emphasises the logical difference between the two. Yet, Weyl also acknowledges that such a generalisation is of a different quality and accounts for this by introducing his notion of a "general instruction for judgements". Nonetheless, Weyl does not go as far as Wittgenstein in his revision of the relevant conception of generality. For him, general instructions of judgements can still be expressed via variables—and that is a decisive advantage when it comes to addressing the second question.

Given Wittgenstein's answer to question 1, it becomes quite challenging to find an adequate answer to question 2. If induction is indeed an *expression* of generality, i.e. a way to understand what a general claim consists in, then how can it serve to *justify* a general claim? If we cannot conceive of its generality without having a 'proof' by induction, then such a 'proof' does not establish the truth of the relevant statement but its *meaning* (cf. Wittgenstein, 1974, p. 406). This leads to a number of questions regarding Wittgenstein's account: When he later on says that certain statements that are proven by induction—like the fundamental theorem of algebra or Fermat's last theorem—are not mathematical propositions (cf. Wittgenstein, 1974, Sect. 168, Sect. 189), what notion of proof and what notion of proposition is at play here? Furthermore, are all instances of induction (even beyond the natural numbers, for example along polynomial degree) equally legitimate (as expressions of generality)?[14] Finally, what is the relationship between a generally stated schema or principle of induction and the particular inductive proofs that correspond to it (cf. Frascolla, 1994, p. 80f.)? I raise these questions here, not in order to answer them (if that is indeed possible), but to point out the repercussions that this account has when it comes to expressing and justifying generalisations—and to highlight a difference to Weyl, who largely avoids these issues by staying closer to orthodoxy.

Weyl simply thinks of induction as given by the properties of the natural numbers and considers it as a way of establishing what lies in the essence of the respective concepts (cf. Weyl, 1998a, p. 100).[15]

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[14] The main question here is whether Wittgenstein is actually being as restrictive as one might think or whether this is just a mere dispute about words. Cf. Marion (1998), Potter (2011), Ramharter (2014) and the references on p.200.

[15] A reviewer has noted that Weyl's notion of 'laying in the essence of a concept' has a connection to Edmund Husserl's notion of "laws that are grounded in the concept of concept" (Husserl, 2001, p. 241). Husserl's mentioning of "operations which are grounded in the Idea of the cardinal number" (Husserl, 2003, p. 414) can be seen as an exemplification of this. Just like associativity and commutativity are grounded in

> [C]omplete induction cannot, nor need it be, further explained, for it is nothing but the mathematical basic intuition [*Urintuition*] of the "always one more". The proper judgements that can be gained form these universal judgements come into being by substituting the *arbitrary* number that they are about, with a *determined* number. (Weyl, 1998a, p. 100)

In his characterisation of induction, Weyl makes use of the notion of an arbitrary number, which is nothing else than the variable based way of expressing generality. He is entitled to do so, since he does not, like Wittgenstein, doubt that there is a totality of natural numbers over which it ranges. This is crucial in that it allows for a justification of induction (even though Weyl claims that induction cannot be further explained). The other component that is needed for this is the assumption that the natural numbers are defined inductively. That these two lead to a justification of the induction principle is established in a brief argument by Dummett (cf. Dummett, 2000, p. 9). The key idea is to make use of the notion of a free variable proof, i.e. a proof that seeks to establish a universal generalisation for a certain domain by reasoning with an arbitrary exemplary member. The following explanation then suffices: Given that we have a free variable proof of $P(x) \rightarrow P(x+1)$ and a proof of $P(0)$, one can construct for any n a proof of $P(n)$ by applying the conditional n times. This procedure itself is a free variable proof of $P(x)$ and hence establishes a claim of generic generality.

The contrast to Wittgenstein is thus the continued reliance on the variable based notion of generality, which enables reference to an arbitrary instance, and with that, use of the concept of a free variable proof. This contrast is to Weyl's favour: When generality is thus conceived independently of induction, induction can be justified along the lines of Dummett's argument as a principle of establishing the *truth* of general claims.[16] It can therefore retain its usual role as a principle of proofs rather than a conception of generality. Furthermore, even though Weyl denied this possibility, intuitionistic negation is applicable to his notion of generality, for precisely the reason that there is an independent conception of generality whose expressions *can* be negated. Since this is not the case for Wittgenstein's approach, there is little hope that the same will work for him. Wittgenstein may simply have been too radical in this respect.

However, Weyl's constructive leanings come to the fore even more drastically when considering existential statements. And this is where Wittgenstein's criticism has its main focus.

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the essence of addition, the principle of induction can be seen as grounded in the essence of number. Weyl was well acquainted with Husserl's work and had even referenced him in his Habilitationsvortrag (cf. Weyl, 1910). I thank one of the reviewers for pointing out this connection—which is certainly worth investigating in itself—and for providing these references and translations.

[16] It should be noted that such a justification wouldn't work for Wittgenstein on two levels: For one, there is the appeal to the notion of a free variable proof *within* the justification, and secondly, the justification, in line with its constructive aims, is itself a template for a free variable proof of $P(x)$, and therefore it does not have the appropriate conception of generality. If induction were indeed an *expression* of generality then it cannot be justified by showing that it is a method yielding the desired property for any element of the domain, because the conception of 'any element' is vacuous as an independent notion.

4.3 Existential generalisations and constructions

Existential generalisations over infinite domains are, for Wittgenstein, just like universal generalisations, devoid of any meaning. Again, the reason is the infinitude of the disjunction that would express such a proposition. Wittgenstein's main point regarding existential generalisations can be illustrated by the following example: Letting $G(x)$ mean that x is the sum of two primes, we can express that the first ten natural number candidates (or any finite series of numbers) contain a counterexample to Goldbach's conjecture via the finite disjunction

$$\neg G(4) \vee \neg G(6) \vee \cdots \vee \neg G(24).$$

This *does* constitute a meaningful proposition, which can be true or false, even if we weren't able to verify for each of the disjuncts whether it is true or false.

Contrary to Weyl, then, Wittgenstein considers existential statements as meaningful, not necessarily because they are made true by a construction, but because their domain is regarded as sufficiently determined:

> If you answer the question whether the figure 7 occurs in the expansion of π by saying: Yes, it occurs at the 25th place, you have answered only the question whether 7 occurs at the 25th place but not the question whether 7 occurs at all. If the question has sense then the answer has sense too, no matter if it turns out positive or negative. (Wittgenstein, 1979, p. 82)

Usually, one would expect that "7 occurs" is a logical consequence of "7 occurs at the 25th place". Yet, according to Wittgenstein, the former is not a genuine statement, since it cannot be expressed via a finite conjunction or disjunction. *A fortiori*, then, "7 occurs" cannot be thought of as a logical consequence of "7 occurs at the 25th place". Wittgenstein's understanding of meaningfulness and truth of an existential statement is thus not *primarily* connected to any claim regarding constructivity. This explains why, even if a mathematical statement refers to a concrete number, Wittgenstein is keen to point out that "this has nothing at all to do with existence" (Wittgenstein, 1979, p. 81). It is not existence but determinacy that ensures meaning and truth-aptness. Furthermore, once determinacy is ensured, it is not required that we in fact have identified one particular instance as a witness to the claim.

The difference between Wittgenstein's demand for determinacy and Weyl's requirement that any legitimate existential statement requires a construction can best be illustrated in the finite case. Consider the following proof that there exist two irrational numbers x and y such that x^y is rational: We know that $\sqrt{2}$ is irrational, thus if $\sqrt{2}^{\sqrt{2}}$ is rational, we would choose $x = y = \sqrt{2}$. If $\sqrt{2}^{\sqrt{2}}$ is not rational, choose $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$. With this information, can one say that $\exists x \in \{\sqrt{2}, \sqrt{2}^{\sqrt{2}}\}$ such that at least one of the substitution instances for $x^{\sqrt{2}}$ is rational? If one requires a proof of at least one $x^{\sqrt{2}}$ being rational, the answer seems to be 'no'. If one requires only that the domain of quantification is determinate, then the answer should be 'yes'. Thus, constructivity and determinacy are not the same in Wittgenstein's eyes: determinacy can be given without constructivity and constructivity does not ensure determinacy that has not been established beforehand.[17]

So, how does all this explain Wittgenstein's remark with which this paper started: "where the law of the excluded middle doesn't apply, no other law of logic applies either, because we aren't dealing with propositions of mathematics" (Wittgenstein, 1975, Sect. 151)? As the previous discussion shows, this remark rests on the distinction between finite and infinite domains. On finite domains it is not only meaningful to assert the negation of a universal statement, but doing so also yields an existential statement (and vice versa). This establishes the duality of the quantifiers and negation to behave classically.[18] In case of infinite domains, neither the universal nor the existential generalisation can be asserted as genuine statements, such that it is flat out denied that there is any meaningful dichotomy. Strictly speaking, therefore, LEM retains ubiquitous assertability. It holds for all statements and should it not appear to hold, Wittgenstein thinks that we are not dealing with propositions in the first place.

Now that my interpretation has been laid out, it will be helpful to highlight some points in which I differ from Mathieu Marion's previous understanding of these passages (cf. Marion (1995, 2003)). While I agree with Marion's emphasis on the importance of the tractarian background to Wittgenstein's comments, my suggestion differs in two central respects on how this background is applied to Weyl. Marion takes Wittgenstein to have misunderstood Weyl in the following way:

> [Wittgenstein] took it to be that a general proposition such as $\forall x\, F(x)$ cannot have a negation because such a negation would be equivalent to a purely existential proposition, and the latter can only be judgement abstracts. But Weyl was not exactly "lumping" universal and existential propositions together: he thought that negations of general propositions with unrestricted quantifications are not umfangs-definit, and this is why these can't be negated. (Marion, 1995, p. 150)

As my interpretation goes, Wittgenstein does not accuse Weyl of lumping together universal and existential propositions, but a notion of construction (or existence) with a notion of determinacy. Wittgenstein denies that a construction yields a determinate claim for an infinite domain and he considers questions regarding existence and questions regarding well-determinateness to be *prima facie* distinct. A constructive statement cannot bring determinacy into an otherwise indeterminate domain (even though a construction does yield a determinate proposition and a way to conceive a bounded domain). This interpretation has the advantage that it takes into account

[17] Still, even though a construction does not yield a full existential statement over an infinite domain, to every construction there corresponds a bounded existential statement, and every bounded existential statement seems to allow for a construction in the eyes of the intuitionist—insofar as the property in question is decidable. Given decidability, one can thus recover each conception from the other for the case of the existential statements.

[18] It also entails the validity of the finitary de Morgan law, which the intuitionist usually denies.

Wittgenstein's explicit comments on the notion of existence in Weyl, which Marion does not discuss.[19]

This brings me to the second point, which is the explanation why LEM does not apply in such cases. For Marion, the reason is of a more general sort. He writes:

> Wittgenstein's claim [...] is that the lack of validity of the law of the excluded middle in mathematics is a distinguishing feature of all mathematical propositions as opposed to empirical propositions. It would be a mistake, according to him, to base the rejection of this law on some logic-chopping. He could not have meant, however that one cannot recognize any substantial differences of "grammar" in mathematics. [...] Indeed, most of Wittgenstein's later philosophy of mathematics is built on the grammatical distinction between finite sequences and infinite series. (Marion, 1995, p. 160)

While I agree with the second part of the quote, my interpretation does not align with the first one. There is indeed a sense in which the propositions of mathematics are different from empirical propositions. This also has its origin in the *Tractatus*, where mathematical propositions are treated as "pseudo-propositions" (Wittgenstein, 1994, p. 6.2). However, insistence on the special status that mathematical propositions have (in the *Tractatus* and later on) blurs the distinction between the finite and the infinite that does all the heavy lifting here. Were it not for this difference, we could not explain why Wittgenstein uses the expression "mathematical proposition" in a quite distinctive way in his middle period, in particular, when he claims that a generalisation proved by induction is not a mathematical proposition (cf. Wittgenstein, 1975, Sect. 168, Sect. 189). For this implies that there are mathematical expressions that *are* genuine mathematical propositions and some that are not—and the difference between the two kinds is best understood as the difference between the finite and the infinite.

Thus, I agree with Marion that there is a "substantial difference in grammar" between the finite and the infinite, but I would like to add that this difference is elucidated by Wittgenstein in distinguishing between genuine mathematical propositions and expressions that fail to obtain even that status. In this way I propose to understand Wittgenstein's remark that

> There can be no such question as, Do the figures 0,1,2...9 occur in π? I can only ask if they occur at one particular point, or if they occur among the first 10.000 figures. No expansion, however far it may go, can refute the statement "They do occur"—therefore this statement cannot be verified either. What is verified is an entirely different assertion, namely that this sequence occurs at this or that point, Hence you cannot affirm or deny such a statement, and therefore you cannot apply the law of the excluded middle to it. (Wittgenstein, 1979, p. 71)

[19] It also connects nicely to Wittgenstein's leniency towards applying differing understandings of the notion of existence in mathematics which is embodied in remarks such as: "Really, existence is what is proved by the procedures we call 'existence proofs'. When the intuitionists and others talk about this they say: 'This state of affairs, existence, can be proved only thus and not thus.' And they don't see that by saying that they have simply defined what *they* call existence." (Wittgenstein, 1974, p. 374) Thus, existence, as the multi-faceted concept that Wittgenstein takes it to be, cannot serve as a criterion of determinacy.

That a certain sequence of figures occurs in a bounded and hence finite interval expresses a mathematical proposition to which the law of the excluded middle applies, whereas "there occur these figures" or "there do not occur these figures" without a bounded specification does not amount to a mathematical proposition at all. When Wittgenstein denies the applicability of LEM and all the other logical laws, he does not mean the bounded expressions from before (and thereby all of mathematics) but only statements regarding the infinite.

Finally, it should be said that my criticism of Marion applies to the local case of his reading Wittgenstein on Weyl. In his paper, Marion concludes that Wittgenstein's conception of quantification and generality was much closer to the finitism of Skolem and Goodstein than it was to the intuitionists. I don't want to argue this point. In fact, given what I take to be Wittgenstein's understanding of existential generalisations and of negation, it looks like my interpretation would line up nicely with his conclusion.

5 Conclusion

Wittgenstein's criticism focuses on the way Weyl argues for something that they both agree on, namely that for infinite domains, $\forall x P(x) \vee \exists x \neg P(x)$ is not a logical truth and $\neg\forall x P(x)$ and $\neg\exists x P(x)$ are not even meaningful expressions. To this effect, Weyl proposes an understanding of general statements over indeterminate domains via conceptual entailment (called generic generality). But the notion of indeterminacy that he employed is entangled with his constructivist leanings. The natural numbers are extensionally definite, but epistemically indeterminate, because one can't *run* through all of them. Wittgenstein is more consequential in this respect. He considers infinite domains to be indeterminate *simpliciter* (and not to be extensionally-definite). Weyl's reliance on constructivity is perhaps most clearly felt in his interpretation of the existential statement. Here, Wittgenstein points out that there are two factors at play: determinacy and verifiability. A domain can be determinate and thus permit a determinate existential generalisation even without explicit knowledge of a witness to it.

The basis for this interpretation was Wittgenstein's retrospective discussion of the *tractarian* conception of generality and his realisation that it fails with respect to infinite domains. The reason for this failure was located in his observation that there cannot be an infinite logical product or sum, such that no truth function can be applied to such a generalisation. As a consequence, generality could no longer be expressed via variables, like the *Tractatus* suggested, because these variables are understood as collecting the propositions to which the truth functions are to be applied. The alternative, expressing generality via induction, however, does not retain the full status of a statement for precisely the reason that it lacks truth functionality. Thus, even though Wittgenstein broadened his account of generality in the late 1920 s, it seemed that at least initially he stayed with the truth-functional account as characteristic of genuine assertive statements. This explains why Wittgenstein, even though he shared scepticism towards the infinite with the intuitionists, decidedly differs from their position when it comes to an assessment of the validity of the logical laws.

From a systematic point of view, Wittgenstein's diagnosis of the notion of indeterminacy seems to be preferable for the simple reason that he does not assume a constructivist perspective. But the lessons that he draws from it bear some unfavourable consequences for his treatment of generality and, in connection to that, for his understanding of induction. Weyl's notion of generic generality, on the other hand, does not have unfavourable consequences regarding the status of induction, and it is compatible with intuitionistic negation. Because of this, it seems to have promising applications to modern debates on absolute generality [(cf. Rayo & Uzquiano, 2006; Linnebo, 2018, 2022)] and the legitimacy of generalised inductive definitions (cf. Crosilla, 2022). A combination of both of their accounts thus seems to be a golden middle way. The key question that this leads to is: Can there be a conception of the indeterminacy of the relevant domains that still retains the variable based account of generality, but that does not rely on constructivity beforehand? This might result in a rejection of standard quantification and classical logic (without, as done by Weyl) presupposing constructivity to begin with.[20] Furthermore, the logic of this treatment of generality need not be exactly the one that is standard for intuitionists, i.e. it need not follow all principles that are connected to the intuitionistic interpretation of the logical constants [the so-called BHK-interpretation (cf. Troelstra & van Dalen, 1988, Ch.1.3)]. For instance, since it is motivated by the demand of tracking provability conditions, and therefore by matters that are not of concern here, insistence on the disjunction property should no longer be required. A similar argument could be made regarding the rejection of the de Morgan laws, or the determinacy of atomic propositions. The resulting approach would thus be novel and a promising suggestion for current debates on generality,[21]

Declarations

Competing interests The author has no competing interests to declare.

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[20] Cf. in this respect Dummett (1991), ch. 25, and Feferman (2014).

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