THOUGHT AND PREDICATION IN FREGE AND RUSSELL

Jose Manuel Pereira Mestre Da Conceicao

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Thought and Predication in Frege and Russell

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University of St Andrews

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One is unable to notice something—because it is always before one's eyes.

L. Wittgenstein, *Philosophical Investigations* §129

Abstract

This dissertation offers a reappraisal of how Russell's views about thought and predication around the time of his *Principles of Mathematics* relate to Frege's own theorizing about those topics. It does so by telling a story about the encounter of Russell's world with Frege's logic. The main protagonist in that story is Russell. Briefly, the story is as follows.

Russell inherited from Moore the elements of the largely atomistic worldview that he upheld around 1903. Underlying that worldview was a model of term combination that we may call *the building blocks model*. That model was primarily targeted at the composition of atomic propositions (chapter 3). In one respect, the model proved advantageous, in that it prevented Russell from mistaking propositional functions (or what they stand for) for properties in the traditional sense (chapters 1 and 4). In other respects, it had a deleterious effect on Russell's theorizing. In fact, it would break down even in the case of atomic propositions themselves (chapter 4). However, it was as a model for the kinds of complexity introduced by propositional functions (chapter 5) and that-clauses (chapter 6) that it proved seriously inadequate. By contrast, Frege's model of complexity derived entirely from his account of generality, and was therefore perfectly suited to functions (chapter 2). Yet, Frege's relative indifference towards ontological questions, or at any rate his lack of a developed picture of the world comparable to Russell's, meant that he could avoid any deep commitments with regard to the other two cases (chapters 2 and 7).

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Introduction

This dissertation offers a reappraisal of how Russell's views about thought and predication around the time of his *Principles of Mathematics* relate to Frege's own theorizing about those topics. It does so by telling a story about the encounter of Russell's world with Frege's logic. The main protagonist in that story is Russell. Briefly, the story is as follows.

Russell inherited from Moore the elements of the largely atomistic worldview that he upheld around 1903. Underlying that worldview was a model of term combination that we may call *the building blocks model*. That model was primarily targeted at the composition of atomic propositions (chapter 3). In one respect, the model proved advantageous, in that it prevented Russell from mistaking propositional functions (or what they stand for) for properties in the traditional sense (chapters 1 and 4). In other respects, it had a deleterious effect on Russell's theorizing. In fact, it would break down even in the case of atomic propositions themselves (chapter 4). However, it was as a model for the kinds of complexity introduced by propositional functions (chapter 5) and that-clauses (chapter 6) that it proved seriously inadequate. By contrast, Frege's model of complexity derived entirely from his account of generality, and was therefore perfectly suited to functions (chapter 2). Yet, Frege's relative indifference towards ontological questions, or at any rate his lack of a developed picture of the world comparable to Russell's, meant that he could avoid any deep commitments with regard to the other two cases (chapters 2 and 7).

More specifically, the thesis makes two central claims.

Regarding thought, it argues that Frege's thoughts and Russell's propositions are related neither as species and genus, nor as two species of a single genus. Rather, it tries to show that Russell's worldview from at least 1903 and presumably until 1918 simply made no room, at a fundamental level, for the sort of thing that Frege would conceive a thought to be.

Hence it rejects the representation of Russell's propositions as coarse-grained thoughts, or alternatively of Frege's thoughts as fine-grained propositions, that is often found in the philosophy of language, in particular in the literature on propositional attitudes. And it rejects views in more specialized literature according to which thoughts and propositions are essentially the same kind of thing. Gideon Makin, to take one prominent example, claimed in *The Metaphysicians of Meaning* that both Frege and Russell were committed to *propositionalism* throughout relevant periods in their philosophical careers. Makin took propositionalism to be the view that propositions are the abstract and mind-independent complex entities that serve simultaneously as the meanings of sentences, the bearers of truth, and the objects of propositional attitudes. I aim to show that Makin, and the many others who share this central claim with him, are mistaken.

Regarding predication, the thesis argues that Russell never conceived propositional functions as properties, and that the ground for attributing a corresponding view even to Frege is much weaker than often realized.

Hence the thesis rejects the increasingly widespread view according to which the originators of higher-order logic would have regarded it as a useful tool in ontological theorizing. Ever since Quine it has been common to frame ontological questions about objects in terms of first-level quantification, and more recently some have found it attractive to frame ontological questions about properties in terms of second-level quantification. One example of this approach is found in Robert Trueman's recent book *Properties and Propositions. The Metaphysics of Higher-Order Logic*. I aim to show that an accurate reading of Russell's and Frege's contributions offers no support for this trend.

The thesis is divided into two parts, each concerning one of our main topics.

Part I concerns predication, and includes chapters 1 to 4.

Chapters 1 to 3 are in a certain sense introductory. Chapter 1 gives the Aristotelian background of the traditional notion of a property. There I argue that Aristotle's syllogistic is best interpreted as excluding what Aristotle called 'definite predications', or atomic propositions. An account of how general statements relate to their instances was therefore left for Frege to give. Chapter 2 offers a brief historical perspective on Frege's logic in relation to its predecessors. Then, drawing on Michael Dummett's and Peter Sullivan's interpretations of Frege, I argue, in effect against Dummett himself, that Frege's functions are *not* properties in the relevant sense. Chapter 3 introduces the picture of the world that Russell shared in 1903 and presumably kept in essence until 1918.

Chapter 4 concerns Russell's propositional functions. There I argue against James Levine's view that the conception of analysis that Dummett attributed to Frege in fact applies rather to Russell. I try to identify the features of Russell's ontology that precluded him from having that conception. Still, and still owing to Russell's ontological concerns, there is an extent of agreement with Frege, in so far as it is possible to compare their views. For Russell, propositional functions are not constituents of atomic propositions.

Part II concerns thought, and includes chapters 5 to 7.

Chapter 5 marks the transition between our two main topics. Makin claimed that in 'On denoting' Russell abandoned *sensism*, the view that at least some thought-components are aboutness-shifters. I argue that the view that Russell targeted then was essentially tied to his theory of the variable, and that it is misleading to regard Frege's senses as aboutness-shifters in Makin's sense.

Chapter 6 finally concerns propositions. There I describe Russell's journey from the dual-relation to the multiple-relation theory of judgement, and argue that neither theory concerns judgement as a cognitive act, at least on a certain understanding of what cognition is. Chapter 7 concerns Frege's thoughts, which I represent as essentially involved with the understanding of cognition that Russell's worldview fundamentally excludes.

1 Aristotle's categories

In this chapter, I argue that Aristotle's notion of a predicate is ambiguous between what may be said of a subject, and what may occupy the predicate position in a complex relation. At the same time, I claim that, although Aristotle may never have confused these two senses, he likewise never succeeded in explaining how they are related.

I introduce Aristotle's theory of predication in 1.1, his theories of opposition and conversion in 1.2 and his theory of inference in 1.3. In 1.4 and 1.5, I argue that Aristotle gave us no account of the implication of particular and atomic statements by universal ones respectively. In 1.6 I represent Medieval theories of *suppositio* as failed attempts to overcome those difficulties.

1.1 Predication

In chapter 2 of the *Categories*, Aristotle introduced a four-fold classification of 'things themselves' according to whether they are *predicable of* or (*present*) *in* a subject.

> Of things themselves, some are predicable of a subject, and are never present in a subject. […] Some things, again, are present in a subject, but are never predicable of a subject. […] Other things, again, are both predicable of a subject and present in a subject. […] There is, lastly, a class of things which are neither present in a subject nor predicable of a subject. (*Cat* 2 1a15–b10)

Things of the latter sort, those that are neither present in nor predicable of subjects, are Aristotle's *primary* (or *first*) *substances* (*Cat* 5 2a10). These are the things by reference to which the world, or at any rate Aristotle's world, is to be understood. We might call them *particular* or *individual* things. Ordinary objects encountered in everyday experience will do as examples, though Aristotle may have had in mind the more restricted class of particular living organisms. The example he gives is 'this man'. A first substance is no doubt something that can be demonstrated, and will typically be the kind of thing that may be given a proper name: though not everything that can demonstrated is first substance.

Next (in the order of explanation, but first in Aristotle's presentation) come *secondary* (or *second*) *substances*, which are predicable of, but not present in, other subjects. They are the kinds of thing first substances are: what defines them, or says what they are, or what their nature consists in (*Cat* 5 2a15–25). These kinds come in different degrees of generality: a kind of lower generality is a species, of higher generality a genus. For instance, an individual man, say Socrates, is a man; and man is a rational animal. So Socrates is a rational animal; man being his species, animal his genus, rationality what distinguishes his species from others in the same genus. As Aristotle observes, both the name and the definition of a second substance apply to what it defines.

By contrast, some things describe individuals, but do not quite say what they are. In this case, Aristotle writes, only their names, not their definitions, apply to what they describe. For instance, Socrates may be called 'white' if he is indeed white, but is not defined by it: the definition of 'white' does not apply to him. These are the things that are predicable of, and also present in, subjects. Were Socrates the only white thing, we might say that being a man really defines him, but being white only does so nominally.

What we have so far is a contrast precisely inherited from Aristotle between (substantial) individuals, their substantial or essential properties, and their non-substantial or accidental properties. The remaining category is that of things that are present in subjects, but are not predicable of them. These resemble first substance in being individual, but also accidental properties in being present in something else, that is, in being incapable of existence apart from what they are present in (*Cat* 2 1a20). They are aptly called *non-substantial* or *accidental individuals*.[1] What they are not, is parts of substantial individuals: Aristotle explicitly distinguishes 'being present in' from 'being a part of'. His example is: a particular bit of grammatical knowledge; another would be: this white (pointing to a white surface).

We are used to applying the essence-accident distinction to properties, and so to understanding one notion in terms of the other, and both in relation to substantial individuals: a property is accidental to an individual if he might not have had it, essential otherwise.[2] This is of course unavailable as the substance-accident distinction applies to individuals, which may perhaps explain the controversy over the category of non-substantial individuals. But we can say that, just as substantial individuals are defined by their essential properties, the individual things that are present in them are defined by their

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[1] I follow the traditional interpretation of accidental individuals defended in our time by Ackrill (1963) and challenged by Owen (1965).

[2] In fact, this gloss would do as a characterization of a property that an individual has necessarily, but not essentially. The notion of essence is the more restricted: arguably, Socrates necessarily belongs to the singleton Socrates, but not essentially. But the rough characterization should suffice for present purposes.

accidents. For instance, Socrates is not defined by being white, but being white defines this white (pointing to the surface of his skin). We might indeed call accidental individuals 'first accidents' and accidental properties 'second accidents', by analogy with first and second substances. It follows that accidental individuals are doubly parasitic upon their hosts: they depend not only on their existence, but also on that of their accidents. That is, accidental individuals depend on first substance, and on its being a certain way. It also follows that the essence-accident distinction is a relative one: the same property may be accidental to a first substance, but essential to an accidental individual.[3]

And so we have substantial individuals (first substance), the properties that define them (second substance), those that do not (second accidents), and the individuals they define (first accidents). We should note that there is one respect in which this classification is merely formal, in that it does not seem to determine which individuals are present in which, that is, which fall under the category of first substance. Of course, for Aristotle, it is Socrates, not this white (pointing to him), that is first substance: but with some ingenuity one might argue that this white, while defined by being white, only happens to be man, which is in turn what defines Socrates. Our special interest in Socrates may well be prior to, or at any rate independent of, Aristotle's classificatory scheme. One might reply that this white, pointing to Socrates, is essentially his white. But that would simply beg the question as to which is first substance: we might as well say that Socrates is essentially this white's man. If this is right, then the substance-accident distinction more generally, just like the essence-accident distinction as it applies to properties, is also a relative one.

In *On Interpretation*, as Aristotle writes, predications may 'sometimes concern a universal subject, sometimes an individual' (*DeInt* 7 17b1). Things that are universal are those that may be 'predicated of many subjects' (*DeInt* 7 17a35). One might infer from this that some things can be predicated of some one thing and not more, but the classification from the *Categories* leaves no room for such things. One assumes at any rate that Aristotle's universal subjects include second substance and accidents. Indeed, second substance defines second substance, just as it defines first substance. Things that are predicable of others, then, may themselves be subjects of predication. Hence even the subject-predicate distinction itself is a relative one, as the same thing may be now a subject, now a predicate.

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[3] Accidental individuals are sometimes characterized as tropes, and tropes as particularized properties. But this is misleading. In so far as something is not predicable for Aristotle, it is not a property. Accidental individuals are rather instances of properties.

Now, while everything is a possible subject of predication, not everything is a possible predicate. Some things, as we saw, are never predicable of others. The individual-property distinction, that is, is *absolute*. There is therefore a peculiar class of things singled out with respect to predication. So, although Aristotle goes on to discuss different kinds of predicates in the *Categories*, what he introduces at the outset are in effect two kinds of subjects: those that may *also* be predicates, and those that may not. As Geach points out, the Aristotle of the *Categories* follows the Plato of the *Sophist* in distinguishing sharply between names and predicates (1972: 45). Propositions of the simplest form must have two heterogeneous parts: two names or two verbs together give no proposition, only a name and a verb do.

It is therefore natural to suppose, as the first-second substance distinction already suggested, that the paradigm of a predication for Aristotle consists in the immediate combination of a property and an individual, or, in linguistic terms, in the application of a predicate to a proper name. The model extends intelligibly to the combination of two properties, or to the application of a predicate to a subject that need not be a proper name, in so far as a universal subject may be taken to be, in some sense, a single thing.

We have been led to distinguishing two formulations of Aristotle's view of predication, one in linguistic terms, one in worldly terms. In fact, the notions of subject and predicate belong to grammar, and commentators have long been puzzled by Aristotle's application of primarily linguistic categories to worldly items (cf. Kneale and Kneale 1962: II.2). They have rightly enquired whether he was mainly concerned with the classification of expressions or of what they signify. The likely answer is that he was concerned with both, and with expressions only in so far as they could function as a reliable guide to the world. Aristotle need indeed not be charged with anything like use-mention confusion: metonymy is a common enough phenomenon to account for the seeming ambiguity. Yet the justification for proceeding as Aristotle did must be deeper.

That linguistic distinctions may have so much as heuristic value to real ones calls for explanation, especially in light of the fact that Aristotle sometimes indicates that real distinctions are independent of language. For instance, in *On Interpretation* Aristotle implies that it is *because* some things are universal that propositions may concern universal subjects. However, we need not suppose that real distinctions and their linguistic counterparts are anything but conceptually coeval. For instance, we can try to define a property as what characterizes things, rather than as what a predicate stands for, but of course the notion of 'characterization' may in turn be understood in terms of predication and vice-versa. Given how basic the subject-predicate and individual-property distinctions are, as they seem to be for Aristotle, it is perhaps unsurprising that they can only be understood in terms of each other. But we need not worry about the circularity.

The Latin term 'predicate' translates the Greek 'category'. Literally, then, a categorical proposition is a predicative one, that is, a proposition that expresses a predication. Whenever a proposition may be said to concern, or be about something, what stands for that thing is its subject, and what it says about it the predicate. As Geach put it, a predicate is just 'an expression that gives an assertion about something if we attach it to another expression [i.e., the subject] that stands for what we are making the assertion about' (1950: 461). The two notions are thus complementary.

Like assertion, indeed as a species of it, predication is subject to act-object ambiguity. Etymology suggests that the act sense is prior. The term 'category' acquired a technical meaning in Aristotle, but originally it meant 'accusation' in the legal context. However, one may well come to think that 'accusation' is just as ambiguous in the same respect. Either way, typically, if it is expressed in the conventional form of words, an act of predication expresses a predication in the object sense, while one may presumably express a predication without in fact predicating anything. Geach suggests keeping 'predication' for the latter and 'application' for the former (1950: 462), but context should suffice for distinguishing the two.

1.2 Opposition and conversion

In *On Interpretation* Aristotle identifies categorical propositions as the simplest propositions, out of which more complex ones are built, elsewhere called 'hypothetical'. He thus conceives them as atomic.

Categorical propositions may take the form of affirmations (or positive assertions) or denials (or negative assertions) and concern either individual or universal things (*DeInt* 5 17a5). This gives four basic types of predication, exemplified by 'Socrates is (not) mortal' and 'Man is (not) mortal'. The first are *individual* predications, the second *indefinite*. Indefinite predications have the form of what are nowadays called *generic* propositions.

Now, it is one thing for a subject to be universal, another for a predication itself to be universal. Only predications about universals may be universal, though they need not be. In fact, the indefinite predications from the *Topics* bifurcate in *On Interpretation* into universal and particular ones, such as 'Every man is mortal' and 'Some man is mortal'. As Aristotle observes, 'the word "every" does not make the subject a universal, but rather gives the proposition a universal character' (*DeInt* 7 17b10).

Indefinite or generic predications thus seem to give way to *general* or *quantified* ones. But it is unclear whether Aristotle simply came to think that indefinite predications were ambiguous between particular and universal propositions. What we know is that once the distinction is made, he inclines to equate indefinite with particular predications, as he claims them to behave alike with respect to negation.

That equation may of course be challenged. Aristotle seems to think that, just like 'Some man is white' and 'Some man is not white', 'Man is white' and 'Man is not white' may both be true. This sounds odd, at least in English. In this respect, generic propositions are rather like universal propositions. But of course that, in so far as they are not falsified by particular counter-examples, generic propositions are like particular ones.

Aristotle is at any rate right to observe that definite predications form a peculiar class relative to negation (or denial). The denial of a definite predication necessarily contradicts it, but the denials of indefinite ones do not, whether they are identified with particular or universal propositions: 'Socrates is mortal' is true if and only if 'Socrates is not mortal' is false.

It is indeed with respect to general or quantified predications that Aristotle introduces the four-fold classification according to quantity and quality familiar from *On Interpretation*. For the sake of terminological simplicity, we will follow the tradition in referring to Aristotle's four types of general statement as 'categorical', and from now on leave the term 'predication' to definite predications.

According to Aristotle, in a universal affirmative categorical, the predicate is affirmed universally of the subject; in a universal negative, the predicate is denied universally of the subject; in a particular affirmative, the predicate is affirmed partially of the subject; in a particular negative, the predicate is denied partially of the subject. Using *S* and *P* as schematic for terms in subject and predicate position respectively, and the vowels *a*, *e*, *i* and *o* to label the four kinds of categoricals in accordance with later Medieval tradition, we have the traditional square of opposition:

		Affirmative		Negative
Universal	(*a*)	Every *S* is *P*.	(*e*)	No *S* is *P*.
Particular	(*i*)	Some *S* is *P*.	(*o*)	Not every *S* is *P*.

Each of the schemas *a* to *o* can be understood as a frame for a relation holding between two terms standing for universals. Following Aristotle's preferred formulation in the *Prior Analytics*, *a* is '… belongs to every ---'; *e* is '… belongs to no ---'; *i* is '… belongs to some ---'; *o* is '… does not belong to every ---'. An apt notation has developed in the tradition to represent each of these, respectively '*PaS*', '*PeS*', '*PiS*', '*PoS*'. We can call these relations 'categorical'.

Two sentences one from each column may be opposed as contradictories or as contraries. Sentences of types *a* and *o* and sentences of types *e* and *i* are said to be contradictory, and sentences of types *a* and *e* are said to be contrary.

It is worth noting that Aristotle defines these relations in *syntactic* terms, and only then observes that contradictory statements cannot have the same truth-value, while contrary ones cannot both be true (cf. *De Int* 7 17b15–25). Contraries of contradictories, or subcontraries, as sentences of types *i* and *o* would later be called, cannot both be false. Sentences of types *a* and *e* imply sentences of type *i* and *o* respectively, later called their subalternate.

To the theory of opposition Aristotle adds a theory of conversion in the *Prior Analytics* as a preamble to the syllogistic. The converse of a categorical proposition is a categorical implied by the first, which has for subject the former's predicate and for predicate its subject. This implies that subject and predicate terms be interchangeable (*salva congruitate*, as it were). Only propositions of types *a*, *e* and *i* have converses in this sense. Thus '*PeS*' and '*PiS*' convert *simply* to '*SeP*' and '*SiP*' respectively, and '*PaS*' converts *per accidens* to '*SiP*'. Presupposing terms not to be empty, Aristotle's account of the logical relations between categoricals is entirely correct.[4]

1.3 Inference

The theory of inference that Aristotle put forward in the *Prior Analytics* belongs to a subject on which he 'had nothing else of an earlier date to speak of at all' (*SE* 24 184b2). The syllogistic, as it came to be known, was the first systematic formal account of inference grounded in an analysis of the structure of the types of sentences it concerned. It is arguments built out of the four types of sentences from the square of opposition that form

[4] The age-old controversy over existential import need not concern us here.

the subject matter of Aristotle's logic. For our purposes, we need only consider the assertoric (or non-modal) syllogistic.[5]

The terms of the conclusion of a categorical syllogism are called 'extremes'. Owing to the theory of conversion, the extremes are connected by the premises via a third term, called 'middle'. Thus each syllogism has exactly two premises, each of which is of type *a*, *e*, *i* or *o*, and has exactly one of the extremes and the middle term. An example would be: 'Every man is an animal; every animal is mortal; therefore, every man is mortal.'[6]

Now, individuals, we saw, cannot be said of anything. They are not possible predicates, and so cannot occupy (at any rate by themselves) the predicate position in any predication. The syllogistic thus turns out to be a calculus of possible predicates or properties.[7] Yet, an (equally valid) argument such as 'Socrates is an animal; every animal is mortal; therefore, Socrates is mortal' can be easily obtained from the one mentioned above just by substituting 'Socrates' for 'every man'. Accordingly, later logicians found it natural to reinterpret singular propositions as universally quantified ones, and to replace proper names with suitable predicates. For instance, 'Socrates' might be replaced with 'is (identical with) Socrates'. Terms thus defined are necessarily complex. Hence the strategy is consistent with the doctrine of the *Categories*, as it does not require individuals or proper names to be said of anything by themselves.

Aristotle arranged the types of two-premise combinations that satisfy these conditions into three figures (literally, schemas), according to whether the middle term occupies the subject position in exactly one, both, or neither premise. These correspond to the first, second and third syllogistic figures respectively.[8] In the first figure, the middle term serves alternatively as subject and as predicate in the premises; in the second and third figures, it is one of the extremes that plays one of the two roles in one of the premises and the other in the conclusion.

In an argument such as the above, terms are ordered after the relative size of their extensions. This probably suggested to Aristotle calling 'major' both the term that is the predicate of the conclusion and the premise that includes it, and calling 'minor' both the

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[5] In this section, we draw mostly on Kneale and Kneale 1962, but also on Smith 2020 and Bobzien 2020.

[6] The formulation in argument form is due to Theophrastus. Aristotle actually stated syllogisms in conditional form (e.g., 'If every man is an animal, and every animal is mortal, then every man is mortal') (cf. Kneale and Kneale 1962: 111).

[7] Examples with individuals in subject position are too rare to be of any great significance.

[8] A fourth figure was recognized later in the tradition, following Theophrastus's systematization of five 'indirect' moods in the first figure, plus one subaltern mood (cf. Kneale and Kneale 1962: 100).

term that is the subject of the conclusion and the premise that includes it. But for that reason, the usage really fits only the first figure. Still, using 'M' for the middle, 'P' for the major and 'S' for the minor term, the three figures may be represented as follows:

First	*Second*	*Third*
S-M	M-S	S-M
M-P	M-P	P-M

Now, Aristotle defined a syllogism as a certain kind of (deductively) valid argument. This means that an 'invalid syllogism' is a contradiction in terms. He rather chose to say that, where a conclusion does not follow from a set of premises, there is no syllogism:

> A syllogism is discourse in which, certain things being stated, something other than what is stated follows of necessity from their being so. I mean by the last phrase that they produce the consequence, and by this, that no further term is required from without in order to make the consequence necessary. (*APr* I 1, 24b18)

It has often been noted that Aristotle's definition of validity appears to differ from the classical conception in three respects. First, the phrase 'certain things being stated' suggests that a valid argument must have at least two premises. Second, 'other than what is stated' suggests that the conclusion of a valid argument must differ from the premises. Third, and perhaps more importantly, the qualification 'from their being so' has been taken to imply that an argument is valid only if its conclusion is related to the premises in virtue of their content. By contrast, classical logic allows arguments with any number of premises, arguments valid by repetition, and arguments with so-called 'irrelevant' conclusions. However, it is not clear what to make of these differences. As we just saw, Aristotle was concerned with a notably narrow class of two-premise arguments, and did not discuss systematically inferences that do not depend on the internal structure of sentences. It may well be that the more restrictive aspects of his definition should be discounted in light of how circumscribed his interests were.

A more interesting feature of Aristotle's account of validity is that, like the theory of opposition, it is not primarily given in straightforwardly semantic terms. On the contrary, truth, for one, is not mentioned, and the definition has, if anything, a syntactic character.

The theory of reduction is the pinnacle of the syllogistic. It is Aristotle's proof theory.[9] It was in the course of expounding it that he premiered the use of letters as schematic variables. Aristotle aimed to establish which of the two-premise combinations in each figure yielded valid conclusions (i.e., 'syllogized'). He showed successfully that there are four (valid) forms of syllogism in the first and second figures, and six in the third. These were later called moods. The difference between the first and second figures on the one hand, and the third on the other, is due to the fact that moods in the third figure all have particular conclusions. By contrast, the first and second figures each have two moods with universal conclusions. By the square of opposition, these imply corresponding particular propositions. Accordingly, two moods with particular conclusions, identified as 'redundant' or 'subaltern', were later added to each figure. The number of moods in the first and second figures was thus raised to six, equalling the number of moods in the third figure.

Aristotle's strategy was, first, to assume syllogisms in one of the figures to be *complete*, or not in need of proof, and then to transform or reduce *incomplete* syllogisms in other figures to syllogisms in that figure. He chose the first figure as the basis of the reduction, since the validity of arguments in that figure is evident, or at any rate easier to grasp. He then took to reduce incomplete to complete syllogisms directly or indirectly. Indirect proofs are by *reductio ad impossibile* and by *ekthesis*, or exposition.[10] Direct proofs are proofs by conversion. Invalidity, Aristotle proved by counterexample.

In the Middle Ages, a mnemonic was invented to codify the theory of reduction. Each mood was assigned a name that describes its method of reduction. Each name has three out of the four vowels used to identify categoricals (*a*, *e*, *i*, *o*), which give the form of each premise of the mood. Consonants identify the method of reduction: 's' means simple conversion, 'p' conversion *per accidens*, 'c' *reductio*, 'm' transposition of the premises. The first letter is the same as that of the name of the complete mood to which the mood is reduced. The oldest version of the mnemonic is given by Sherwood in his *Introduction to Logic*:

Barbara celarent darii ferio baralipton
Celantes dabitis fapesmo frisesomorum
Cesare camestres festino baroco

[9] Cf. Kneale and Kneale 1962: 67 and Smith 2020.

[10] There is a controversy over the interpretation of *ekthesis*, but this particular method is not essential. All but two incomplete syllogisms can be proved directly, and all can be proved by *reductio* (cf. Smith 2020).

Darapti felapton disamis datisi bocardo ferison.

Barbara, *celarent*, *darii* and *ferio* are Aristotle's complete syllogisms. Aristotle actually went on to prove that *darii* and *ferio* can be reduced to *barbara* and *celarent*, thus reducing all of the 18 valid forms to just two. And with the help of obversion, invented by medieval logicians to convert particular negative categoricals, even *celarent*, and with it every syllogism, can in effect be reduced to *barbara*. It is unclear how Aristotle would have regarded obversion, but he might have welcomed the result.

We can give a direct proof of the second-figure syllogism *camestres* as a simple example of Aristotle's procedure. *Camestres* is '*MaP*; *MeS*; therefore, *PeS*'. The object is to show that the conclusion follows using only conversion and complete syllogisms. The minor premise converts simply to '*SeM*'. Together with the major premise, this implies '*SeP*' by *celarent* (that is, '*SeM*; *MaP*; therefore, *SeP*'). The latter again converts simply to '*PeS*', which is what we wanted.

The example is instructive in a further respect. The validity of *celarent* is not more evident than that of *camestres*. This goes to show that Aristotle's concern was distinctively axiomatic. In fact, he seems to have been aware that the choice of first-figure syllogisms as complete was indeed arbitrary.

The syllogistic, and the theory of reduction in particular, is as elegant as it was original. Aristotle's chief achievement as a formal logician thus arose with a peculiar interest in proof. This may be unsurprising in light of his apparent tendency to think in syntactic terms even outside the theory of reduction. As we have noted in passing, his account of validity and other logical notions is not straightforwardly semantic. It would of course be anachronistic, if not altogether spurious, to characterize Aristotle as an inferentialist of sorts. But neither this nor the restricted applicability of the syllogistic by themselves detract from his achievement. It was the style rather the scope of his approach that defined logic as a discipline.

1.4 Particular statements

That said, the interpretation of categoricals has long been thought problematic. We will now look into one of its puzzling features.

The square of opposition displays two fundamental logical relations of opposition among categoricals, in terms of which other relations can be defined: contradictoriness and contrariety. For instance, the subordinate of a categorical can be defined as the contradictory of its contrary; its superordinate (as it were), as the contrary of its contradictory; its subcontrary, as the contradictory of the contrary of its contradictory.

Now, we can interpret contradictoriness in terms of *propositional* negation. But we are given no account of contrariety. In particular, it is not supposed to be understood in terms of *term* negation, since contraries are not obverses: a categorical of type *e* is not 'Every *S* is not-*P*', but rather 'Every *S* is-not *P*'. Instead of a *negative* (or *infinite*) *term*, it involves, as it were, a *negative copula*.

At the same time, this negative copula is obviously not the negation of the type *a* copula 'Every … is *P*', but rather the negation of the type *i* one 'Some … is *P*'.

Alternatively, we could define contradictoriness as the single relation of opposition. But then we would still need a *second* copula: if not a negative one, then a *particular* one.

Hence there is a misleading appearance of uniformity in the square of opposition. We can interpret it as involving either a single (well understood) relation of opposition, or a single affirmative copula, but not both. In the first case, we need a further particular copula; in the second, a further negative one. Either way, there is no single relation corresponding to 'is' or 'belongs to', but two: either a negative and an affirmative copula, or a universal and a particular one.

This is best seen from the interpretation of categoricals within an intuitive set theory that has become standard, in which terms stand for classes rather than universals (which can be thought of as their intensions), and relations between them are analysed as relations of inclusion and exclusion between classes, and their negations. Inclusion works as an affirmative or positive copula, exclusion as a negative one:

PaS: $S \subset P$

PeS: $S \cap P = \varnothing$

PiS: $\sim(S \cap P = \varnothing)$

PoS: $\sim(S \subset P)$

Here exclusion is represented as a complex relation. But since it is defined in terms of intersection, identity and the null set, we still lack an account of how it relates to inclusion.

We could alternatively define '*PeS*' in terms of inclusion and relative complement:

PeS: $\mathrm{S} \subset \mathrm{P}^{\mathrm{C}}$

But while this would enable us to treat inclusion as the single (affirmative) copula, we would then lack an account of how negation relates to relative complement, or how propositional negation relates to term negation. Note that on the new interpretation '*PeS*' is in fact given as 'Every *S* is not-*P*'.

It thus seems that we must lack an account either of how the universal copula relates to the particular one, or of how propositional negation relates to the negative copula or to term negation. But what is missing from Aristotle's theory is precisely an explanation of how 'some', 'every' and 'not' are related. In particular, there is no genuine explanation of the implication of 'some' by 'every' beyond the intuitive (truth-functional) characterization of subordination. Aristotle's bifurcation of indefinite predications in effect left him with no (formal) account of how universal categoricals entail particular ones.

1.5 Atomic statements

The possibility of analysing '*PeS*' as '$S \subset P^C$', and the possibility of reducing all syllogistic moods to *barbara* with the help of obversion, independently suggest that the universal copula 'Every … is *P*' is in some sense the *fundamental* categorical relation, whether or not it be understood in terms of inclusion (the choice of which is of course arbitrary).

In light of Aristotle's overall presentation, it would be entirely natural to regard this fundamental copula as corresponding to, or as identical with, at least one of the notions of predication that he introduced in the *Categories*. Aristotle's exposition in the *Topics* and *On Interpretation*, especially, suggests that definite, indefinite and quantified predications (or categoricals) differ only with regard to their subjects. If so, such phrases as 'said of', 'predicated of', 'belongs to', etc, are mere notational variants, and can be used to express, as it were, the 'is' of predication.

Now, categorical statements are naturally interpreted as expressing relations between universals (or, as we also saw, classes). And admittedly Aristotle nowhere spoke of predication as a relation in the *Categories*. Rather, and perhaps especially in the paradigmatic case of the connection between an individual and a universal, he implied predication to be a matter of immediate combination. But let us suppose for the sake of argument that predication *can* be construed as a relation even in the paradigmatic case of definite predications.

Still, even the fundamental categorical relation from the *Prior Analytics* cannot be identified with predication in the sense of the *Categories*. The basic thought here is that the theory of conversion is antithetical to the latter.

Let us say that a relation is *symmetric* if and only if its terms can be interchanged *salva veritate*, and that it is *even* (for lack of a better term) if and only if its terms can be interchanged *salva congruitate*, that is, if they are of the same grammatical type. Obviously, every symmetric relation is even, but not conversely. Less obviously, every even relation is 'equal-level' in Frege's *Grundgesetze* sense, though not conversely.

Owing to the theory of conversion, Aristotle's four types of categorical statement express even relations, and two express symmetric relations. They are even because conversion demands precisely that terms in subject and predicate position can be interchanged *salva congruitate*. Types *e* and *i* express symmetric relations because they convert simply, which means that their terms can be interchanged *salva veritate*.

This contrasts sharply with the theory of the *Categories*. Definite predications, even if we suppose them to be relational, express *uneven* relations. Individuals cannot be said of anything, and so cannot be interchanged with what is said of them. Proper names and predicates (now in a strictly linguistic sense) form different types. Hence the relation that according to the *Categories* holds between an individual and a universal, if indeed it is a relation, is simply excluded from the syllogistic.

Note that this is not *just* to say that individuals are excluded from the syllogistic; *that* we already knew. But it is to realize that the *type* distinction that Aristotle introduced in the *Categories* is completely absent from the syllogistic.

The same reasoning applies to indefinite predications. It would be natural to think otherwise, as indefinite predications, just like categoricals, may be said to express relations between universals. To recap, in an indefinite predication a universal is *said of* another universal. But a universal is something that *may* be said of something else. Hence in an indefinite predication something is said of something that may itself be said of something else. Likewise, the terms of categoricals, again owing to conversion, can, indeed must, occupy subject and predicate positions indifferently. *A fortiori* the same term may occupy now the subject, now the predicate position. Hence in both cases possible predicates are possible subjects.

However, this is not to say that an indefinite predication is an *even* relation in the sense defined above. The subject and the predicate of an indefinite predication still belong to different grammatical categories. In definite and indefinite predications alike, then, subject and predicate have different forms. In particular, only the predicate has, so to speak, a *predicative* form.

By contrast, in categorical statements either both subject and predicate have a predicative form, or neither does. There 'subject' and 'predicate' do *not* indicate different forms, or grammatical *types*. Rather, they only indicate different *cases*. In effect, in the syllogistic, 'subject' and 'predicate' are just labels for different positions in a relation.

The intuitive set-theoretic interpretation of categoricals is again useful in this connection. Exclusion is a symmetric relation, inclusion is not; but both are even relations. All terms have the same type. In this case, none has a predicative form. In the sense of the *Categories*, '*S*' and '*P*' are not predicates but *names* (of classes). (If the class-theoretical relations were further interpreted within the predicate calculus, *both* would be predicates.)

In fact, in categoricals there is no more reason to say of one term that it is (not) said of the other than vice-versa. One way to spell out this point is to say that the terms of categoricals are on a par with respect to what they are *about*, if indeed they are about anything. On the class-theoretic interpretation, they are about *both* subject and predicate terms.[11] By contrast, in the sense of the *Categories*, only the subject can be said to be what the predication is about. Even indefinite predications are about their universal subject.

Now one may certainly regard with suspicion the idea (which we will also find in Russell) that something, namely a universal, may belong to different types, as indefinite predications seem to require. In fact, Aristotle himself may have come to so regard them. At any rate, he may have eliminated indefinite predications altogether once he distinguished universal and particular propositions.

Still, this leaves definite predications untouched. That is, it remains the case that none of the relations expressed by categoricals is uneven, unlike the relation (if it is a relation) that obtains between individuals and universals. Hence the syllogistic is simply inapplicable to definite predications.

1.6 Theories of *suppositio*

An alternative interpretation of categoricals, according to which they concern individuals, developed in the tradition. That interpretation also enables a uniform account of the copula. However, even if that interpretation were coherent, categorical statements would still not be predications in the sense of the *Categories*.

[11] Frege makes a very similar point in 1880/1: 33.

Theories of *suppositio* gained shape as medieval logicians tried to account for the meanings of expressions like ‘man’, ‘every man’ and ‘some man’ so as to combine the theory of conversion with the possibility that terms refer to individuals. Since no account of such theories ever became standard, I shall again refer briefly to Sherwood’s *Introduction to Logic*, which contains the oldest surviving complete survey of the subject, and occasionally to Ockham.

Suppositio (or supposition) is the fourth of Sherwood’s *proprietates terminorum* (or properties of terms) alongside signification, appellation and copulation. Signification is the basic property. It is ‘a presentation of the form of something to the understanding’ (Kretzmann 1966: 105): the form, or universal, associated with a term; e.g., the signification of ‘man’ is the form of man. Appellation is defined in terms of signification as the (set or plurality of) objects to which the latter is correctly applied at the time of use: those objects, that is, which have the form associated with the signification of the term, when the term is used. It is therefore the *actual* and *present*, or *contemporary*, extension of a term. The appellation of ‘man’ are the presently existing men: the things that have the form of man. Roughly, then, if signification is the intention of a term, appellation is its extension. Finally, supposition is what a term stands for in some range of contexts. It has accordingly been dubbed ‘contextual reference’ (Potter 2020: 12). As Ockham notes, in a broad sense appellation is one kind of supposition (Boehner 1957: 69). The notion of copulation is not very clear in Sherwood. Its point appears to be to extend the idea of supposition to terms in predicate position, and in that respect it is redundant. It acquires an altogether different meaning in later writers. I shall accordingly mostly refer simply to supposition in what follows.[12]

Supposition comes in three varieties: material, formal, and personal. A term has material supposition if it supposits for itself, as ‘man’ does in ‘man has three letters’. It has formal supposition, as the name suggests, if it supposits for the form which is its signification, as ‘man’ does in ‘man is a species’.[13] And it has personal supposition if it supposits for some part or subset of its appellation or extension. Further subdivisions within material and formal supposition need not concern us.

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[12] Sherwood does not yet speak of the ampliation and restriction of a term, according to whether the use of a term broadens or restricts its appellation. These serve to explain tense and modality.

[13] The reader of Carnap, for whom, in the *formal* mode, words have, in Sherwood’s terms, *material* supposition, will find this terminology confusing.

Personal supposition divides into *determinate* and *confused.* It is determinate when a term supposits for a single thing, confused when it supposits for many. 'Man' has determinate supposition in 'a man runs' since it supposits for one man, though, as Russell might put it, it does not matter which. We might say that, in its occurrence in phrases such as 'some/a man', 'man' supposits indeterminately for one determinate man.

Confused supposition may or may not be distributive; thus we have *merely confused*, and *distributive confused* supposition. A term has distributive supposition when it 'supposits for many in such a way as to supposit for any' (Kretzmann 1966: 108). For instance, 'man' does so in 'every man is an animal', the idea being that here 'man' stands for each man individually. On merely confused supposition, however, Sherwood has remarkably little to say. We are told only that 'animal' supposits merely confusedly in 'every man is an animal', and are left to guess what it may be for a term to supposit for several things in a non-distributive way, and why it should do so there.

Now, it is relatively straightforward how Sherwood's distinctions map onto Aristotle's square of opposition. Terms in subject position have distributive supposition when predications are universal (types *a* and *e*), determinate when they are particular (types *i* and *o*). Terms in predicate position have confused supposition in universal affirmative predications (type *a*). That there is meant to be *some* connection between universal quantification and distributive supposition on the one hand, and existential (or particular) quantification and determinate supposition on the other, is clear enough. But what exactly it consists in, we cannot yet say. In fact, so far the theory comes dangerously close to a disquotational account of the meaning of quantifier phrases, according to which 'man' stands for every man in 'every man' and for a man in 'a man', which is of course worthless as an analysis of their complexity.

Geach claims in *Reference and Generality* that the later doctrine of distribution was born out of a confusion between confused and distributive supposition. But there is an obvious connection between the two theories. First, as Kretzmann notes, a distributed term seems to be, already in Sherwood and according to him even more clearly in Peter of Spain, just a term that has distributive supposition (1966: 120).[14] And second, both theories

[14] Incidentally, *contra* Kretzmann, this does *not* contradict Geach's claim, since it does not exhaust the content of the doctrine of distribution.

probably derive their applicability to terms in predicate position from the theory of conversion.[15]

As Geach notes, while the doctrine of distribution may seem intuitively plausible for terms in subject position, it is harder to maintain for terms in predicate position. Terms are said to be distributed in subject position in universal categoricals and in predicate position in negative ones. They are undistributed in all other cases. Now, type *e* and *i* propositions convert simply, and so their predicates have the same kind of supposition as their subjects: distributive and determinate respectively.[16] It follows, as traditional logic had it, that subject terms of universal categoricals and predicate terms of universal negatives are distributed, and that subject terms of particular ones are not. But it does not follow, as traditional logic also had it, that predicate terms of particular negatives are distributed, since type *o* propositions do not convert. Predicates of universal affirmatives are again peculiar. They are meant to supposit non-distributively for many things. Yet by parity of reasoning they ought to have determinate supposition, since type *a* propositions convert *per accidens* to type *i* propositions. They would seem to supposit for at least two things of a certain class, though not all; and again in such a way that it may not matter which. But this rather suggests that the only stated difference between determinate and merely confused supposition—namely, a numerical one—is immaterial.

The point of the distinction becomes apparent only from the role it plays in Sherwood's account of 'logical descent', that is, the transition from a general term to its 'inferiors', or corresponding individual terms. Sherwood gives a list of five rules about such transitions. They are not quite rules of inference. For instance, his fifth rule regarding confused and determinate supposition reads: 'an argument from distributive confused supposition to determinate supposition does follow, but not from merely confused supposition' (Kretzmann 1966: 119). The transition from distributive to determinate supposition corresponds to the implication of particular by universal categoricals, as in 'Every man is an animal; therefore, some man is an animal'. But the transition from 'Every man is an animal' to 'Every man is some animal' is invalid, since there is no one animal

[15] The Kneales remark that the extension of supposition to predicates may have in fact been motivated by conversion (cf. 1962: 248–9).

[16] At one point, Geach suggests that the doctrine of distribution already breaks down for subjects of type *e* propositions, but this is uncharitable. One might be led to suppose that 'No man' supposited for no man, which is indeed absurd; but of course type *e* propositions can be recast in the form 'Every *S* is not *P*'.

which every man is.[17] As Ockham notes, 'it is not open to us to make the logical descent from 'animal' to its inferior terms; for this does not follow: 'Every man is an animal; therefore every man is this animal; or every man is that animal (and so on for every animal)' (Boehner 1957: 77).

What the difference between determinate and confused supposition is getting at is (what we should like to describe as) the scope ambiguity that a sentence of type *a* must have if it is understood as involving not just a universal quantifier attached to the term in subject position, but also a particular quantifier attached to the term in predicate position. On Sherwood's view, that is, every quantified predication is in fact a relational proposition involving two quantifiers. His theory therefore requires an account not only of simple, but also of multiple generality. This seems to be entirely an imposition of the theory of conversion, and a particularly odd one, since there are no articles in Latin, definite or indefinite, and so no temptation as there might be in English to treat 'Every man is *an* animal' as 'Every man is *some* animal'.

Further distinctions and rules allowed Sherwood and his successors to deal with cases of multiple generality with a complexity well beyond those allowed by the limited forms of categorical propositions. But the example of confused and determinate supposition suffices to illustrate the general problem with the theory of supposition. The theory promised to ground an explanation of the validity of inferences involving generality upon an account of the meaning of expressions involving quantifiers. But it rather looks as though the opposite is the case.

The circularity becomes explicit in Ockham's case by his deliberate choice to *define* each kind of supposition directly in terms of logical descent, rather than the other way around. While he achieves greater clarity than Sherwood, it comes at the cost of a complete reversal of explanatory priority. Thus he writes that '[t]here is determinate *suppositio* when it is possible to make the logical descent to singulars by a disjunctive proposition'; merely confused *suppositio* when the descent is possible only 'by way of a proposition with a disjunctive predicate' and 'the original proposition can be inferred from any singular'; and distributive *suppositio* when 'it is licit to make a logical descent in some way to a copulative proposition if the term has many inferiors, but a formal inference cannot be made to the original proposition from one of the instances' (Boehner 1957: 76–8).

As a theory of multiple generality, then, the theory of supposition is descriptive at best, and can only provide a series of useful rules to detect certain fallacies.

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[17] Sherwood gives a different example.

As a theory of predication, however, it does not fare much better either. In Aristotle, 'every' and 'some' 'go with the predicate' (cf. Geach 1962 §41). On the theory of supposition, they indicate the relata of a simple relation of predication. But we should now enquire what relation that is.

Following Ockham, Geach interprets the several kinds of supposition in terms of conjunctions and disjunctions of names and sentences. In particular, he defines determinate supposition (contextually) in terms of propositional disjunction thus: if *a* and *b* are all of the *T*'s, a predication '*F*(some *T*)' is equivalent to '*Fa* or *Fb*'.[18]

Take a sentence of type *i*, 'Some *S* is *P*'. Both terms have determinate supposition. The sentence is equivalent to 'Some *S* is some *P*'. Now, suppose that the appellation of '*S*' is s_1 and s_2 and that the appellation of '*P*' is p_1 and p_2. Applying Geach's truth-condition to the original sentence once gives 's_1 is some *P*, or s_2 is some *P*' (or 'Some *S* is p_1, or some *S* is p_2'; here scope is irrelevant). Applying it twice gives 's_1 is p_1, or s_1 is p_2, or s_2 is p_1, or s_2 is p_2'. The problem is, as should have been obvious from the start, all of s_1, s_2, p_1, and p_2 are *individuals*. But the only relation that holds between individuals in the *Categories* is *inherence* (or *being in* something). Hence the 'is' here cannot be the 'is' of predication in any of the senses of the *Categories*.

However, it could be the 'is' of identity. And, for Ockham, it is. As he writes, 'for the truth of the proposition 'This is an angel' […] it is sufficient and necessary that subject and predicate should stand for the same thing' (Boehner 1957: 83). It is therefore a short leap from Sherwood's account of *suppositio* to Ockham's view of predication as identity. And conversely, while Ockham's nominalism may have been independently motivated, it is especially suited to the theory of supposition. As Geach notes, the two-name view of predication was the 'predominant logical theory of the Middle Ages' (1972: 51).[19] But whatever its merits, the view is not Aristotle's, as it just eradicates predication altogether. For a sentence such as 'Socrates is (a) man' to express a predication in Aristotle's sense, 'man' must have *formal*, not *personal* supposition.

In Sherwood's terms, then, Aristotle's incapacity to explain how quantified propositions relate to their atomic instances translates into the incapacity to relate personal to formal supposition. For all their ingenuity and acute sensitivity to scope ambiguities in natural language, then, medieval logicians were unable advance beyond Aristotle in

[18] We will return Geach's account in our discussion of Russell's theory of denoting in chapter 5.

[19] For an account of Ockham's view and its refutation by Burleigh see Kneale and Kneale 1962: 266ff.

explaining how general statements relate to their instances. The following chapter will consider how, much later, Frege effected this advance.

2 Frege's logic

In this chapter, I argue that Frege's replacement of Aristotle's notion of a predicate by his own notion of a function does not thereby eliminate it.

When Frege wrote, five obstacles to the development of logic were still outstanding: two problems inherited from Aristotle concerning the implication of atomic and particular statements by universal ones, two problems inherited from the Stoics and Boole regarding the relationship between term and propositional logic, and the problem of multiple generality inherited from the medieval logicians. As we now know, all of these had to be solved before Frege could realize his early ambition to formulate a *lingua characteristica* that was also a *calculus ratiocinator* for mathematics. Frege's breakthrough came with his introduction of the notion of a function, which underlies his invention of the notation of quantifiers and bound variables in *Begriffsschrift*.

Roughly, my strategy is to show, first, that the way in which Frege solved these issues, or at any rate the way in which they can be solved within Frege's framework, supports central features of Dummett's interpretation of his account of generality; second, that it is Aristotle's notion of a term that Frege's notion of a function can be said to eliminate; and third, that Dummett's notion of a simple predicate can be equated with Aristotle's notion of a predicate.

In 2.1 I introduce two problems concerning the relationship between propositional and term logic. In 2.2 I introduce Frege's notion of a function and its role in accounting for the implication of atomic and particular statements by universally quantified ones. In 2.3 I argue that Frege's account of generality entails Dummett's distinction of constituents and components. In 2.4 I argue that Frege's account of multiple generality entails Dummett's distinction of analysis and decomposition, and that extant objections to the uniqueness of analysis are misconceived. In 2.5 I acknowledge that, although Frege recognized a category of simple predicates, it is unclear whether they are simple in Dummett's sense. In 2.6 I argue *contra* Dummett that Frege could have consistently held that simple predicates stand for objects. In 2.7, I argue that simple predicates can be equated with Aristotle's predicates.

2.1 Leibniz's ideal

Frege set out early in his career to formulate a *lingua characteristica* for mathematics that was also a *calculus ratiocinator* in the sense of Leibniz: that is, a language powerful enough to express every mathematical thought, mere adherence to the grammar of which might guarantee correctness in reasoning (cf. Frege 1882: 84–5). This he achieved, by and large, with *Begriffsschrift*.

As Dummett observed, Frege's success was so overwhelming in all major respects that it is now hard for us to imagine being in the position of someone trying to do what he did in his time. In fact, without Frege's work, we might not have a clear idea of what such an endeavour might require. With the benefit of hindsight, we can say that, in a certain sense, what Frege needed to do was to bring together into a single system the logic of generality (including multiple generality) and propositional logic.

Before Frege, only Leibniz and Boole made serious attempts at such a synthesis, but neither came close to a solution. However, analogies between Aristotelian categorical syllogisms and Stoic hypothetical syllogisms had been known since Antiquity.

To the Stoics' credit, we do not need to say half as much about their logic as we did about the syllogistic. It is recognizably a version of propositional logic. The Stoics were in fact responsible for remarkable innovations in the field, including the use of propositional variables. Chrysippus, who was ranked above Aristotle as a formal logician, produced a system of inference schemata comparable to modern expositions of the propositional calculus, using (a strict) conditional, negation and exclusive disjunction. Philo defined for the first time a *truth-functional* or *material* conditional. Apart from direct contributions to propositional logic, the Stoics also studied a number of paradoxes (including the liar and a version of Frege's puzzle) and developed a theory of meaning and truth based on the notion of a *lekton*, a sort of propositional content invariant with respect to time.

Yet the Stoic logical tradition was progressively neglected in late Antiquity, with the result that no primary sources survived. Its influence upon subsequent developments was therefore greatly diminished. The few principles of propositional logic that reached Leibniz, Boole and Frege are likely to have derived from Modern expositions of Medieval theories of *consequentiae*, which were probably rediscovered independently of the Stoics, but were also less systematic. Our own knowledge of that tradition is second-hand and reconstructive.[20]

One major cause for such neglect may have been the fact that Stoic and Peripatetic logical systems were for a long time perceived as rivals, together with Aristotle's stature as a philosopher. But it is odd that they should have even looked like alternative, rather than complementary systems, even to Ancient eyes. First, Aristotle's own formulation of

[20] See the account of Stoic logic given by Kneale and Kneale 1962: III and Bobzien 2020.

syllogisms in conditional form simply uses the kind of statement that the Stoics were interested in. Second, Stoic hypothetical syllogisms may themselves have been inspired by work by Aristotle's pupil Theophrastus.

Theophrastus is credited with the invention of a system of syllogisms that he also called 'hypothetical', derived from the consideration of the syllogistic figures themselves. A Theophrastian hypothetical syllogism derived from *barbara* would be: 'If *S* exists, *P* exists; if *P* exists, *Q* exists; therefore, if *S* exists, *Q* exists'. As in categorical syllogisms, here variables are *term* variables. In turn, a Stoic hypothetical syllogism would be given in the form: 'if the first, then the second; if the second, then the third; therefore, if the first, then the third'. Here, ordinal number terms function as *propositional* variables. Still, the relationship between the two sorts of hypothetical syllogisms is straightforward. Substitution instances of the Theophrastian pattern are substitution instances of the Stoic pattern, but not conversely.

Hence if the rivalry between Stoic and Peripatetic logic had any ground at all, it could only have been the alternative account of universal *categorical* propositions proposed by the Stoics. A type *a* sentence such as 'Every *S* is *P*' would be analysed by the Stoics roughly as 'If something is *S*, it is *P*'. Now, this analysis had to seem peculiar not only to the Peripatetics but also to the Stoics. On the one hand, it contained term variables like Theophrastian hypotheticals; on the other, it resembled a conditional statement without really being one.

By the nineteenth century, term and propositional logic had long come to be seen as complementary rather than rival fields, studying two different, unrelated kinds of inference. But it would be left to Boole to exploit the analogy between categorical and hypothetical syllogisms further.

Boole's 'Algebra of logic' dates from 1848. It consisted in a system of uninterpreted formulas designed after the model of arithmetical equations that could be interpreted either as expressing relations between extensions of concepts (or *primary propositions*), or as expressing relations between propositions (or *secondary propositions*). Boole's aim was to show how problems from term and propositional logic could be solved using similar algorithms, similar also to those already in use to solve arithmetical equations. Not only did he achieve this, but his 'algebraic' approach to logic would become the predominant style of logical theorizing in the following century.

Up until this point, much of Boole's innovations had been largely anticipated by Leibniz almost two centuries earlier. Yet, as Frege was to recognize, Boole advanced beyond Leibniz in 'one point of fundamental importance' (Frege 1880/1: 10). Boole's *reduction* of secondary to primary propositions, which had eluded even Leibniz, finally promised to close the gap between term and propositional logic.

In rather Stoic fashion, Boole began by relativizing truth to time, and then analysed relations between propositions as relations between classes of times at which they were true. He could thus interpret, e.g., 'if *p*, then *q*' as 'the class of instances at which *p* is true is included in the class of instants at which *q* is true', or in other words, 'Every instant at which *p* is true is an instant at which *q* is true'; '*p* and *q*' was interpreted as 'the intersection between the class of instants at which *p* is true and the class of instants at which *q* is true is not empty'; in other words, 'Some instants at which *p* is true are instants at which *q* is true'; and so on.

In spite of the artificiality that Frege noted (in particular with respect to 'eternal truths', which could be relativized to time only by courtesy), Boole's technique was ingenious and to some extent perfectly admissible.

Yet a moment's reflection suffices to see that Boole did not in fact succeed in fulfilling his promise. Once his equations were interpreted, they could only express either relations of concepts or relations of propositions, *but not both*. The analogy between term and propositional logic is therefore exploited, but not really explained. Hence Boole in no way provided a *synthesis* of the two fields.

Furthermore, while Boole provided a means to reinterpret conditional statements as categorical statements about classes of times, each of those categorical statements can be re-expressed in the Stoic alternative form. To take one example, 'Every instant at which *p* is true is an instant at which *q* is true' can be re-expressed as 'If something is a class of instants at which *p* is true, then it is a class of instants at which *q* is true'. But that form is again *not* an instance of 'if *p*, then *q*'. Hence the relationship between Peripatetic and Stoic formulations of categorical statements also remained unexplained.

Frege repeatedly complained about the inadequacy of Boole's system for his own purpose of designing an expressively complete formal language for mathematics. He charitably attributed that inadequacy to their different aims (cf. Frege 1880/1: 12). Some of Frege's criticisms were relatively superficial, in the sense that they could have easily been met without essential changes to Boole's system, such as his use of notation borrowed from mathematics or his choice of primitive signs. But others were definitive.

Frege's objection to the expressive completeness of Boole's system is straightforward. Since Boole reduced hypothetical to categorical syllogisms, the resulting system could only be as expressively powerful as Aristotle's syllogistic. It was thus unable to express not only multiple generality, but more generally any difference of scope involving at least one quantifier. Boole's algebra would therefore fail miserably as a *lingua characteristica* for mathematics.

Interestingly, Frege also objected to the formal adequacy of Boole's system independently of the range of its applicability. Frege's idea seems to have been that Boole's algorithms were not based on a perspicuous syntax, and did not therefore have a semantic foundation. In other words, although Boole's formulae, once interpreted, could be *associated with* certain thoughts, they did not actually *express* those thoughts, since they were not based on their analysis. Hence Boole's system left something to be desired even as a mere *calculus ratiocinator*. In that sense, for Frege a *calculus* would only be a logic if it was *also* a *lingua*.

Frege's approach to the problem, as he himself observed (1880/1: 17), may be regarded as a reversal of Boole's strategy. Rather than reducing relations of propositions to relations of concepts, in a certain sense he did the opposite.

2.2 Functions

Frege introduced the notion of a function in *Begriffsschrift* §9 as a preamble to his account quantification in §11. In his early account, a function is an expression, not what a functional expression might be thought to stand for. Specifically, a function is the part of a complex expression that remains constant when some of its other parts are substituted by other expressions. In Frege's own words,

> Suppose that a simple or complex symbol occurs in one or more places in an expression (whose content need not be a possible content of judgement). If we imagine this symbol as replaceable by another (the same one each time) at one or more of its occurrences, then the part of the expression that shows itself invariant under such replacement is called the function; and the replaceable part, the argument of the function. (Frege 1879 §9)

For example, the name 'Cato' may be imagined as variable in either one of its occurrences in 'Cato killed Cato'. If it is the first occurrence that is imagined as variable, the function is '…killed Cato'; if it is the second, the function is 'Cato killed…'; if it is both, the function is '…killed…' (cf. 1879 §9). Hence, 'It easy to see how regarding a content as a function of an argument leads to the formation of concepts' (1879 Preface).

Someone might protest that Frege's exposition involves use-mention confusion. The function is not '…killed…', say, but what that expression stands for. To be sure, Frege himself later condemned his earlier practice of speaking of signs rather than what they symbolize (e.g., 1891: 138) and adopted precisely that view of functions. But here he is introducing a notation for the first time, basically describing the formation rules for his newly invented language. It is natural that his explanations are primarily given in syntactic terms.

More to the point, it is at any rate one thing to charge Frege with use-mention confusion, quite another to accuse him of incoherence. In fact, the way Frege explains functions in *Begriffsschrift* is perfectly coherent. As Geach first observed, Frege explains them as *linguistic* functions (cf., e.g., 1961: 143), but linguistic functions are functions in the perfectly ordinary mathematical sense of correlations of arguments and values.

A typical numerical function is a correlation between numbers. To take a simple example, the function x^2 is a function that takes numbers as arguments and delivers numbers as values. It is, as we say, a function from numbers to numbers. For instance, it gives the value 4 for the argument 2.

Analogously, a linguistic function is simply a function that takes expressions as arguments and delivers expressions as values. Some linguistic functions in this sense are actually studied in mathematics. And even the very *expressions* for numerical functions can be regarded as linguistic functions. For instance, the linguistic function 'x^2' gives the value '2^2' for the argument '2'. A function in the sense of *Begriffsschrift* is a linguistic function in just this sense.

But not conversely. Not every linguistic function is a function in the sense of *Begriffsschrift*. There is a significant restriction on the acceptable *values* of a linguistic function if it is to be counted as a function in the sense of *Begriffsschrift*. The argument of a function in the sense of *Begriffsschrift* must figure as a *part* of its value, which is of course *not* required in the general case.

A *Begriffsschrift* function from names to sentences is the paradigm of a Fregean incomplete expression. A sentence, Frege writes, 'may always be regarded as a function of one of the symbols that occur in it' (1879 §11). Once a function is obtained from a sentence, it can be appended to a 'concavity' in Frege's 'content-stroke'. The concavity is just Frege's sign for what came to be known as the universal quantifier, after Peirce introduced the phrase.

Suppose, then, that the sentence 'Cato killed Cato' is regarded as a function of the second occurrence of the name 'Cato'. If we now replace it by a variable (a Gothic letter in Frege's symbolism) and let it be preceded by '$\forall x$' (or by a concavity in the content-stroke with a Gothic letter standing over it) we obtain an expression such as '$\forall x$ (Cato killed x)'. 'This signifies [the judgement] that the function is a fact whatever we take its argument to be' (1879 §11), that is, that Cato killed everyone (assuming for convenience the relevant restriction of the domain).

Now from the new judgement, or sentence, 'we can always deduce any number we like *of judgements with less general content*, by substituting something different each time for the Gothic letter' (1879 §11). Those judgements can of course be conjoined. Frege could thus straightforwardly account for the implication, by a universally quantified (or universal) statement, not just of its (possibly atomic) instances, but also of their conjunction. An account of the implication of existentially quantified (or particular) statements also immediately follows, though indirectly, as these are entailed by the individual instances of universally quantified statements.

2.3 Constituents and components

On Frege's account of generality, then, a function is at once what a quantifier attaches to, as well as the pattern instantiated by any one of the instances of the quantified sentence formed by attaching the quantifier to that function. If, say, '$\forall x$' attaches to 'Cato killed x', 'Cato killed Brutus' can be recognized as one of its instances precisely because it instantiates the pattern 'Cato killed…', which is to say that the function 'Cato killed x' can be extracted from it.

It is this double role of incomplete expressions that explains the relationship between a quantified sentence and its instances. It corresponds to the distinction that Dummett draws between the *constituents* and the *components* of a sentence. The function 'Cato killed x' occurs both in the quantified sentence '$\forall x$ (Cato killed x)' and in the atomic sentence 'Cato killed Brutus', and so may be said to be a component of both. But, Dummett insists, it does not do so in quite the same way. Since the function is formed from 'Cato killed Brutus', it cannot be one of the items from which the atomic sentence itself is formed in the first place. By contrast, it *is* one of the items from which the quantified sentence is formed: it is precisely that one to which the quantifier attaches. The same function is therefore a *genuine constituent* of the quantified sentence, but a *mere component* of any of its instances.

The contrast becomes clearer as logical complexity increases. In the atomic case, there might be a temptation to regard the function-argument analysis of a sentence as somehow still revealing its constituents (cf. 2.5 below). That temptation disappears in more complex cases. Consider a quantified sentence of the form '$\forall x\,(Fx \to Gx)$'. Such a sentence is formed by attaching the quantifier '$\forall x$' to the function '$Fx \to Gx$'. In turn, that function can be obtained from any one of the instances of the quantified sentence, say, '$Fa \to Ga$', by letting '*a*' go variable. But now it is obvious that the function '$Fx \to Gx$' is *not* in turn a constituent of '$Fa \to Ga$', which is naturally thought to be formed from two atomic sentences by means of a propositional connective.

Now, although '$Fa \to Ga$' would not be regarded as formed from '$Fx \to Gx$', there is a question whether the instances of the quantified sentence need to be mentioned in an account of its formation at all. To be sure, '$\forall x\,(Fx \to Gx)$' may be regarded as having been formed either according to (1) or according to (2):

(1)

$\forall x(Fx \to Gx)$
|
$Fx \to Gx$
|
$Fa \to Ga$
/ \
Fa Ga

(2)

$\forall x(Fx \to Gx)$
|
$Fx \to Gx$
/ \
Fx Gx
| |
Fa Ga

On the account represented by (2), the function '$Fx \to Gx$' is a constituent of the quantified sentence, but is not represented as a component of its instances. Or rather, that account does not involve its instances at all, which means that it does not provide an explanation of how the formation of the quantified sentence relates to its instances.

Yet, as far as the interpretation of Frege is concerned, there can be no doubt that Frege would have regarded (1) as embodying the correct account of the formation of the quantified sentence. Note that on the alternative account implication is not a *propositional* connective, as it links functions rather than sentences. This means that it would have to be redefined as a propositional connective in a language in which non-quantified but logically complex sentences could be expressed. Frege, however, always chose to define connectives as propositional once and for all, which is the only way to avoid ambiguity in a language such as the one that he invented.

Hence, for Frege the logic of generality is in a way an *extension* of propositional logic via function-argument analysis. That extension is enabled by the fact sentences of any complexity may be analysed in terms of argument and function (provided only that they are *complete* expressions). It is in this sense that Frege's claim to have reduced relations of concepts to relations of propositions, in direct opposition to Boole, should be interpreted. Frege could thus both avoid the problem of accounting for their relationship entirely, as he never really regarded them as separate in the first place, and provide an explanation of how the Stoic construal of universal categoricals related to conditional statements.

2.4 Analysis and decomposition

A quantified sentence may itself be regarded as an instance of a generalization. For instance, '$\forall x$ (Cato killed x)' may be regarded as an instance of '$\exists y \forall x$ (y killed x)', that is, 'Someone killed everyone'. (Frege himself would have expressed existential quantification by means of the universal quantifier and negation.) We can again imagine 'Cato' as variable in '$\forall x$ (Cato killed x)' in order to obtain the function '$\forall x$ (…killed x)', proceed to replace it with a variable, and then attach '$\exists y$' to it so as to form the doubly general sentence.

Frege's account of multiple generality is therefore already given with his account of simple generality. Or rather, generality *simpliciter* is fully explained in terms of the function-argument analysis of sentences, which again may of course be complex, and the attachment of quantifiers to functions with single arguments. Hence the problem of multiple generality simply reduces to the problem of distinguishing the relative scopes of quantifiers. Indeed, Frege's account obviates the need, not just for theories of *suppositio*, but also for independent theories of relations.[21]

As Dummett observes, 'an incomplete expression may never be considered as derived from another incomplete expression by the removal of some constituent expression: we have always to start with a complete expression, and form whatever incomplete expressions we want to consider from that' (1973: 40). That is, the notion of a function with two arguments does not have to receive special consideration in Frege's account of multiple generality involving only universal and particular quantification. (It might have to, if there were primitive double-binding operators.)

Incidentally, this appears to contradict what Frege writes about functions with more than one argument:

> Suppose that a symbol occurring in a function has so far been imagined as not replaceable; if we now imagine it as replaceable at some or all of the positions where it occurs, this way of looking at it gives us a function with a further argument besides the previous one. In this way we get functions of two or more arguments. (Frege 1879 §9)

It would therefore seem as if functions with two arguments could be gotten by further extracting a function from a function. For instance, '*x* killed *y*' could be obtained from '*x* killed Cato' by varying 'Cato'. This would imply that two-argument functions are, conversely, functions from names to one-argument functions.

However, Frege's preference for (1) above, along with other examples (cf. Dummett 1973: ch. 3), definitively settles, in Dummett's favour, the question whether he would have accepted the principle of the completeness of the values of functions.

So just as a single function may be regarded as a constituent of a (quantified) sentence and a mere component of another one (one of its instances), so can a single (quantified) sentence be regarded both as having been formed from a function, and as the basis for the formation of another function. Dummett systematizes these two ways of regarding or analysing a sentence as a difference between analysis proper and decomposition (cf. 1981: 271).

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[21] Of the sort Boolean logicians such as DeMorgan and Peirce had been working out before Frege. Peirce is however to be credited with developing a system essentially equivalent to Frege's, shortly after the publication of *Begriffsschrift*, and developed independently from it.

To decompose a sentence, according to Dummett, is to represent it in terms of argument and function after the manner explicitly introduced by Frege, in order to exhibit it as a possible instance of a generalization. Decomposition is thus a means by which to represent sentences as instantiating some pattern with a view to the explanation of inferences involving generality.

Analysis proper consists instead in the attempt to reconstitute all of the steps by which a sentence may be regarded as having been formed. The underlying assumption is that a sentence is constructed stepwise. Unlike decomposition, analysis essentially proceeds in stages, revealing, at each stage, the last step in the construction of the sentence. The final stage is reached once no constituent is further analysable (in *this* sense of analysis; whether or not those constituents are definable is a further question). Analysis thus aims at uncovering the *ultimate* constituents of the sentence, consisting, in effect, in an account of its constructional history.

A general characterization of the components of a sentence follows. An expression is a mere component of a sentence if it may be extracted from it by decomposition. But it is one of its actual constituents if it occurs at any step of its analysis (or, conversely, of its construction). The last step in the construction of any quantified sentence consists precisely in attaching a quantifier to a function (or 'closing' an open sentence), which are thus always among their constituents. But since they are obtained from their instances only by decomposition, they are their mere components.

While decomposition does not reveal the constituents of a sentence, it presupposes not only that the sentence has been formed, but also that it has been formed in a particular way. In fact, decomposition presupposes analysis in the sense that any meaningful decomposition of a sentence is circumscribed by its analytical structure and ultimate constituents.

This is again especially clear in the case of complex quantified sentences. As Frege writes, one may be misled by a superficial analogy into thinking that 'The number 20 can be represented as the sum of four squares' and 'Every positive integer can be represented as the sum of four squares' may be decomposed in exactly the same ways. In particular, one may be inclined to think that, just as the first sentence may be decomposed into the argument 'The number 20' and the function '…can be represented as the sum of four squares', the second sentence may be decomposed into the argument 'Every positive integer' and the function '…can be represented as the sum of four squares'. However,

> We may see that this view is mistaken if we observe that 'the number 20' and 'every positive integer' are not concepts of the same rank. What is asserted of the number 20 cannot be asserted in the same sense of [the concept] 'every positive integer'; of course it may in certain circumstances be assertible of every positive integer. The expression 'every positive integer' just by itself, unlike 'the number 20', gives no complete idea; it gets a sense only through the context of the sentence. (Frege 1879 §9)

It is the *analysis* of the sentence 'Every positive integer can be represented as the sum of four squares' as '$\forall x$ (x is a positive integer $\rightarrow$ x can be represented as the sum of four squares)' that shows the function '…can be represented as the sum of four squares' *not* to be a *mere* component of that sentence, i.e., *not* to be extractable from it by decomposition. But the same function *can* be extracted from the consequent of any of its instances, such as 'The number 20 can be represented as the sum of four squares'. (The same analysis provides for the decomposition 'Every … can be represented as the sum of four squares', but that is of course a different function.)

The fact that analysis circumscribes decomposition allowed Dummett to resolve a seeming tension within Frege's theorizing. Although according to Frege a single sentence may be multiply decomposed, Dummett argued, it may yet have a unique analysis.

Many of Dummett's critics have focused on this feature of his interpretation. There are, according to them, several counter-examples to the uniqueness of analysis. These include instances of logical abstraction, simple propositional tautologies, and sentences that express converse but non-symmetrical relations. In each case, we have two sentences with essentially different structures that nevertheless express the same thought. But for Dummett, to analyse a thought is simply to analyse the sentence that expresses it. Hence, the argument goes, thoughts cannot have unique intrinsic structures as Dummett contends.

It is unclear whether sentences that express non-symmetrical converse relations really have different structures. Frege did often claim that sentences that can be formalized as 'aRb' and '$bR'a$', where R is not symmetrical and R' is its converse, express the same thought and differ only in 'illumination' or 'tone'. A standard example is the difference between the active and the passive voice. Now, it is unclear whether, for Frege, converse non-symmetrical relations really are distinct. But more to the point, even if they are distinct, there is a sense in which the sentences in question have the *same* structure.

At any rate, all three putative counter-examples the uniqueness of analysis are bound to miss their mark. What we have in each case is, at best, a single thought being expressed by sentences with a different structure, and hence a different analysis. But the questions that this raises concern, not the uniqueness of analysis, but the individuation of thoughts and their expression. It would be misleading even to speak of thoughts' having alternative analyses, in the relevant sense of analysis, on pain of equivocation. If a thought is expressed by sentences with different analyses, those would not be alternative analysis of the same sentence. Perhaps Frege allowed different sentences of the concept-script to express the same thought, whether or not he was right to do so, or could do so consistently. Yet each sentence from the concept-script is parsed in a unique way. Arguably, Dummett meant nothing more than this by the uniqueness of analysis; at any rate, we need mean nothing more by it.

2.5 Simple and complex predicates

So far, we have only considered quantified sentences. However, Dummett argued that the category of functions is insufficient for the purposes of analysis, if analysis is meant to include not only an account of the formation of quantified sentences, but also of atomic ones.

The basic reason is relatively simple. To recap, functions are genuine constituents of quantified sentences but not of their instances. What this means is that they are formed from the latter by decomposition, and hence cannot be among the items from which they themselves are formed. This, we saw, is especially clear when the sentence is complex. But if we now suppose that the instance in question is an atomic sentence, it should suffice to observe that the exact same thing is true in that case as well.

In order to explain how atomic sentences are formed, Dummett claims, we need to draw a further distinction. A function, in Frege's sense, Dummett calls a *complex predicate* (cf. Dummett 1973: 28). But complex predicates must be sharply distinguished from *simple predicates*. Just as complex predicates were just what decomposition required, simple predicates are just what analysis proper ultimately requires. In particular, a simple predicate is the kind of expression that *can* serve the purpose of explaining the formation or composition of an atomic sentence, rather than one that *may* be formed from it by decomposition. In Sullivan's words, simple predicates are those predicates that would be found among 'the primitive vocabulary that defines the expressive resources of a language' (2010: 108).

Note that simple and complex predicates can be, and often are, typographically indistinguishable. The difference lies, not in relative complexity, but in explanatory role. In fact, there will be an overlap between the two categories whenever an atomic sentence is regarded as a possible instance of a generalization. But this overlap ought not impress us, for it is precisely at the atomic level that there is any need for such distinction. It is the case of a complex predicate that is extracted from an atomic sentence that should be treated as 'degenerate' (cf. 1973: 30). As Dummett writes,

> we do need to recognize the separate existence of the simple predicate '…snores' as well: for precisely because the complex predicate '*x* snores' has to be regarded as formed from such as sentence as 'Herbert snores', it cannot itself be one of the ingredients from which 'Herbert snores' was formed […] Rather, if there is to be economy, it is the degenerate 'complex' predicate that should be dispensed with, with the sign of generality, in 'Everyone snores', being regarded as in this special case attached directly to the simple predicate. (Dummett 1973: 30–1)

Now, one might think that there is no real need to recognize a category of simple predicates. After all, decomposition must have a converse operation, which we might call 'saturation'. Can we not, contrary to what Dummett says, think of the atomic sentence 'Herbert snores' precisely as having been formed simply by substituting the name 'Herbert' for the variable in the complex predicate '*x* snores'?

Well, no. Or rather, not in the relevant sense. There is, in fact, a kind of equivocation here. We can indeed think of the atomic sentence as in some sense 'formed' by putting together a name and a complex predicate. But that will still not constitute an account of its analysis, just as it would not if the sentence were complex. The point about decomposition having a converse holds equally well in the latter case.

Dummett has an argument to show that the analysis of an atomic sentence does not reduce to one of its decompositions. Suppose we were trying to give an account of the meaning or truth-conditions of a sentence of the form '$\forall x\, Fx$', but had not yet recognized a category of simple predicates. We would say that the sentence is true iff '*Fx*' is true of an arbitrary object. But now consider one of its instances, say, '*Fa*'. The following equivalence certainly holds: '*Fx*' is true of *a* iff '*Fa*' is true. If '*Fa*' is not atomic, its meaning will in turn be explained by its further analysis. However, if it is atomic,

> we shall go round in a circle. '*Fx*' will be true of *a* just in case '*Fa*' is true: but this takes us back to where we started. We cannot explain what it is to grasp the condition for '*Fa*' to be true in terms of grasping the conditions for '*Fx*' to be true of an arbitrary object, and then

> explain what it is to grasp *that* condition in terms of grasping the condition for '*Fa*', '*Fb*', '*Fc*', etc., to be true. (Dummett 1981: 293; example replaced by a schema.)

We should therefore distinguish the complex predicate '*Fx*' from the simple predicate '*F*…'. Armed with this distinction, we can then proceed as follows. '*Fa*' is true iff '*F*…' is true of *a*, and '*F*…' is true of *a* iff *Fa*.

Now unlike the *relative* distinction between constituent and mere components, it could be doubted whether Frege would have recognized the *absolute* distinction between simple and complex predicates. Dummett himself acknowledged that Frege 'tacitly assimilated simple predicates to complex ones' (1973: 30). For Dummett, this assimilation is understandable in so far as Frege's object was to explain generality rather than the composition of atomic sentences. But the question is not just whether Frege's conception of analysis could have included an account of the formation of atomic sentences. The question is rather, if it did, whether it would look like Dummett's.

It is arguable that it would not. In 1891 and 1892b, Frege modified his early account of functions in two important respects. In 'Function and concept', he identified concepts with (total) functions (1891: 20) from objects to truth-values (1891: 15), and functions with something 'incomplete, in need of supplementation, or "unsaturated"' (1891: 6). Functions were no longer the sort of expressions that Frege had previously identified them with, but what those expressions stand for (1891: 2–3); but neither were they sets of ordered pairs (1891: 9 fn.), which, according to Frege, are complete, self-standing objects.

In 'On concept and object' Frege further clarified that this unsaturated character of functions was essential to them. Frege's function-argument distinction is a relative one, in the sense that any function may itself be an argument for another function. But the concept-object distinction he now emphasised is absolute; hence, on no account can a concept become an object.

At the same time, it was to the notion of unsaturation that Frege assigned the role of combining the elements of thought into a whole:

> not all the parts of a thought can be complete; at least one must be 'unsaturated', or predicative; otherwise they would not hold together. For example, the sense of the phrase 'the number 2' does not hold together with that of the expression 'the concept *prime number*' without a link. We apply such a link in the sentence 'the number 2 falls under the concept *prime number*'; it is contained in the words 'falls under', which need to be completed in two

> ways; and only because their sense is thus 'unsaturated' are they capable of serving as a link. Only when they have been supplemented in this twofold respect do we get a complete sense, a thought. (1892b: 205)

Frege thus seemingly ascribed to functions a role in the composition of atomic sentences, namely, the role of enabling thought components to combine together. It might therefore seem that, in Dummett's terms, Frege would have treated ('degenerate' cases of) complex predicates as constituents of atomic sentences after all.

However, on Dummett's behalf, it should be noted that, *because* Frege adhered to the context principle, he had no real need to account for the unity of thought at all (cf. 2.6 below). *A fortiori*, he had no need to do it in terms of the unsaturated character of functions.

At the same time, Frege's use of 'unsaturatedness' here qualifies the *sense* rather than the *meaning* of functional expressions. But functions are meanings, not senses; that is, they are not, properly speaking, constituents of thoughts at all. Hence, even if their unsaturated character played any role in accounting for the unity of thought, it would have to be an indirect one at best. In fact, the unsaturatedness of the *sense* of functional expressions can be interpreted as an expression, albeit an unclear one, of the context principle.

2.6 The context principle

Dummett finds support for his claim that Frege implicitly recognized a category of simple predicates in the following passage from Frege's review of Boole.

> […] instead of putting a judgement together out of an individual as subject and an already previously formed concept as predicate, we do the opposite and *arrive at a concept by splitting up the content* of possible judgement. Of course, if the expression of the content of possible judgement is to be analysable in this way, *it must already be itself articulated*. We may infer from this that at least the properties and relations which *are not further analysable* must have their own simple designations. (Frege 1880/1: 17, emphasis added.)

Here Frege makes three interconnected points. In the first sentence, he expresses the priority of judgement in concept-formation. This can be interpreted in Dummett's terms as concerning decomposition. In fact, Dummett provides a precise sense in which sentences (or judgements) are prior to the functions (or concepts) into which they are decomposed. As we saw, functions are *mere* components of the sentences from which they are formed.

In the second sentence, Frege claims that the possibility of analysing a sentence, or 'the expression of the content of possible judgement', in terms of argument and function depends upon its prior 'articulation'. This can in turn be interpreted as the circumscription of decomposition by analysis in the sense explained in 2.4 above.

It is in the third sentence that Frege claims that the prior articulation of judgement relative to concept-formation is ultimately grounded in 'properties and relations which are not further analysable', i.e., in simple properties and relations. 'But', Frege then adds,

> it doesn't follow from this that the ideas of these properties and relations are formed apart from objects: on the contrary they arise with the first judgement in which they are ascribed to things. Hence in the concept-script their designations never occur on their own […] (Frege 1880/1: 17)

Here Frege effectively prefigures his statement of the context principle in the *Grundlagen*, according to which it is only in the context of a sentence that a word has meaning. The implication is that the context principle concerns, *not* the priority of judgement in concept-formation, but rather simple properties, relations and objects.

Frege thus recognizes a significant asymmetry between objects and simple properties (for short) on the one hand, and complex properties on the other. Judgements are prior to complex properties only. But this is not to say that simple properties are in turn prior to judgements, since 'not prior to' does not entail 'subsequent to'. Rather, from an explanatory point of view, simple properties are *coeval* both with the judgements in which they occur, and with the objects with which they combine to form those judgements.

There can therefore be no doubt that Frege recognized *some* notion of simplicity. The question now is how to characterize it. There are two alternatives to consider here.

One is that Frege could have meant by simplicity just relative logical simplicity. In that case, Frege's *simple* predicates would be *complex* in *Dummett*'s sense. In particular, they would be genuinely incomplete expressions, and stand for functions.

On this interpretation, the question whether or not Frege envisaged an account of the composition of atomic sentences loses some of its significance. If he did, then his account was simply largely unsatisfactory, basically for the kind of reasons that lead Dummett to introduce a category of simple predicates (in *his* sense, cf. 2.5 above). If he did not, then it would have been reasonable for him to simply presuppose some such account might be independently available.

What anyhow remains significant is that Frege recognized *any* asymmetry between simple and complex predicates *at all*. For that asymmetry still translates into an asymmetry with respect to predicate formation. Again, complex predicates are formed from judgements, which are in turn formed from the simple predicates and objects with which they are coeval. Hence, even if Frege's simple predicates are not simple in Dummett's sense, they are simple in the sense that 'degenerate cases' of complex predicates are simple. And so they still form a class upon which the class of 'non-degenerate' complex predicates is ultimately grounded.

Another way to characterize Frege's notion of simplicity would be Dummett's. On Dummett's reading, Frege's simple properties are simple in just the sense in which simple predicates are simple. Simple and complex predicates have, for Dummett, contrasting explanatory roles. Complex predicates serve the explanation of quantification, and are subject to the principle of the extraction of functions. Simple predicates concern the composition of atomic sentences, and are subject to the context principle, which merely constrains their combinatorial potential. As Dummett might phrase it, according to the context principle, the meaning of a word encompasses its systematic contribution to the meanings of the sentences in which it occurs.

Dummett's separation of the context principle from the principle of the extraction of functions has the following consequence. An incomplete expression is an expression whose argument-places are internal to it (cf. Dummett 1973: 32). But that is tantamount to its being subject to the principle of the extraction of functions. Complex predicates are such that they *must* be formed by decomposition. But then, by implication, simple predicates are *not* incomplete expressions. They are incomplete only in so far as they are not sentences; 'but,' Dummett writes, 'in *that* sense, proper names are equally incomplete' (1973: 32). However, this prevents them neither from being able to combine with other expressions, nor from being able to combine with expressions of certain kinds *only*. In accordance with the context principle, they will still have what Dummett calls their 'valencies'. So while the argument-places of complex predicates represent their dependence upon the sentences from which they derive, the argument-places of simple predicates are merely an indication of their grammar.

The context principle thus obviates the need for an *explanation* of how simple predicates (or properties) *may* combine with names (or objects) to form atomic sentences (or judgments). For that possibility is internal to them. On Dummett's view, simple predicates are just what analysis presupposes, and analysis just what decomposition presupposes. In Frege's terms, simple properties are what is presupposed by the prior articulation of judgements relative to concept-formation. Either way, neither have any existence apart from the possibility of their occurrence in judgements or their expression. It is in this sense that Frege's explanation of the composition of atomic sentences above in terms of the 'unsaturatedness' of the *sense* of predicative expressions can be interpreted as concerning the context principle rather than functional extraction. His explanation would nevertheless be at least poorly phrased, since, on this picture, names too would be 'unsaturated'.

Now for Frege complex predicates being incomplete expressions means not just that they have their argument-places internally, but also that they stand for functions. A second implication, then, is that simple predicates do *not* stand for functions, since functions are just what incomplete expressions stand for, and simple predicates are *not* incomplete expressions.

Dummett's position on this point is equivocal, or at least elusive. Concerning the *senses* of simple predicates, his view was always that they were *not* functions, even though he changed his mind on whether they were objects. (He defended the claim that they were objects against Geach in 1981: ch. 13, but by 2007 he had come to think that they were *sui generis*, cf. 2007: 122.) However, concerning their *meaning*, his final view seems to have remained that simple predicates stood for functions.

Be that as it may, what is interesting for our concerns is that *Frege*'s own stance on this issue is harder to determine than one might think. One might expect it to be straightforwardly that *simple* predicates stand for functions. However, Frege's discussion of the so-called paradox of the concept *horse*, of all things, suggests contrary evidence.

Kerry had taken issue with Frege's *absolute* distinction of concept and object by drawing attention to such putative counter-examples as 'The concept *horse* is easily attained', in which the concept *horse* is treated as an object. Such a sentence is not only meaningful but true, but according to Frege it can be neither. In his reply, Frege made two points.

First, Frege conceded that not everything that he wished to express can in fact be expressed. In *Frege*'s sense of a concept, i.e., a function, Kerry's sentence is indeed ill-formed. More generally, there can be no generalizations across different levels of Frege's logical hierarchy. For instance, precisely because no concept-word is replaceable by a proper name in any context in a Fregean language, it is impossible to express in such a language that no concept is an object.

It is Frege's second reply that we should look to. Ultimately Frege's strategy is to distinguish two interpretations of Kerry's sentence. Frege quite simply *agrees* with Kerry that there is an interpretation on which 'The concept *horse* is easily attained' comes out true. But that interpretation concerns *Kerry*'s sense—and everyone else's, for that matter—of 'concept'. In the usual, psychological sense of 'concept', which Frege does not propose to eliminate, the concept *horse* is indeed easily attained. Frege asks his reader to allow him to use the word 'concept' in a *new* way, just as he himself would allow his reader to keep using the word 'concept' in its old sense. But in the old sense of 'concept', a concept is indeed an object in *Frege*'s sense of the word, i.e., not a function. As he paradoxically put it, in a sentence such as 'the concept "horse" is a concept easily attained', 'the three words "the concept 'horse' " do designate an object, but on that very account they do not designate a concept, as I am using the word' (1892b: 195).

Before the 1890s, Frege's terminology was still developing. In *Grundlagen*, he used the word 'concept' for the content of functional expressions, but had not yet distinguished their sense from their meaning. In *Begriffsschrift*, he used the word 'function', not for their content, but for those expressions themselves (cf. 2.2 above). For their content, Frege then chose the word 'property' (cf. 1879 §10).

Now, it would be easy enough to restate Kerry's paradox as a paradox about the *property horse*. And there is absolutely no reason to suppose that Frege's reply to such paradox would be any different than the one he gave Kerry. In *his* sense of 'property', he might have said, it is impossible to express that no property is an object. But in different, perhaps traditional senses, properties *are* objects in *Frege*'s sense of the word, i.e., not functions.

A surprisingly strong case can therefore be made for Dummett's claim that Frege implicitly recognized a category of simple predicates. Complex predicates stand for functions. In Frege's new sense, concepts and properties are functions. But there remain senses of concepts and properties according to which they are *not* functions. And in so far as they are not functions, they are apt to constitute thoughts or whatever sentences express. It would therefore be entirely consistent with what Frege actually wrote to say that simple predicates stand for properties in the traditional sense of a property.

To recap, I have argued that Dummett's overall interpretation of Frege's conception of analysis is largely correct. In particular, his account of complex predicates as constituents of quantified sentences alone is unexceptionable, and carries with it a contrast between analysis and decomposition. Alleged counter-examples to the uniqueness of analysis are off-target and concern, not the relationship between analysis and decomposition, but the relationship between thoughts and their expression. Although I do not wish to minimize these questions, they are orthogonal to our immediate concerns.

As regards simple predicates, things are less straightforward. Frege did recognize some such a category, but although Dummett's account of simplicity is consistent with what Frege wrote, it does not seem to be implied by it. Either way, it is undeniable that complex predicates depend *asymmetrically* on judgements, which in turn depend *symmetrically* on simple predicates. Now Frege may or may not have envisaged an account of the composition of atomic sentences. Whether or not he did, it is nevertheless surprisingly consistent with what he actually wrote not only that predicates can be simple in Dummett's sense, but also that they can stand for objects.

2.7 Simple properties

Frege had prophesised in the preface to *Begriffsschrift* that 'the replacement of the concepts of *subject* and *predicate* by *argument* and *function* will prove itself in the long run' (1879 Preface). Indeed, we have seen how that replacement helped solve the five problems mentioned at the beginning. It now remains to bring fully to light the relationship of Frege's logic not just to Aristotle's syllogistic, but also to his ontology.

To recap, Frege is with the Stoics regarding the formulation of universally quantified statements, but is further able to explain how they relate to conditional statements, again owing to function-argument analysis. To use Russell's term (cf. 4.2 below), a formal or variable implication '$\forall x\ (Sx \rightarrow Px)$' has material implications as its instances, say '$Sa \rightarrow Pa$', a mere component of which is a constituent of the variable implication, namely the complex predicate '$Sx \rightarrow Px$'.

For Frege, '$\forall x\ (Sx \rightarrow Px)$' defines the relation of subordination of the concept S to the concept P, in terms of which the inclusion of the extension of S in the extension of P may be defined. So we have the following definitions:

PaS: $S \subset P$: $\forall x\ (x \in S \rightarrow x \in P)$: $\forall x\ (Sx \rightarrow Px)$

Here the notation is slightly misleading, as '*S*' and '*P*' first appear as terms for classes and then as predicates. But the relationship between term, general and propositional logic finally becomes clear. In particular, it is *terms* (for classes) that can be said to be defined away, or eliminated, by Frege's general logic.

Bobzien writes that '[f]rom a modern perspective, Aristotle's system can be understood as a sequent logic in the style of natural deduction and as a fragment of first-order logic' (2020). While it may be illuminating to note a resemblance between Aristotle's theory of reduction and modern systems of natural deduction, representing the syllogistic as a fragment of first-order logic is misleading. The relations studied by the syllogistic belong to the *second* level of Frege's hierarchy, their arguments being (first-level) functions.

To Frege's eyes, Aristotle's tendency to treat '*PaS*' as a predication betrayed a conflation between the subordination of concepts and an object's falling under a concept. As we saw in chapter 1, it is unclear whether Aristotle himself was guilty of such equivocation. Still, it was Frege who could provide the means to re-establish the connection between the term logic of the syllogistic and the ontology of the *Categories*.

Once the analysis of '*PaS*' as '$\forall x\ (Sx \rightarrow Px)$' is reached, it is a small step to arrive at the atomic statements that form its instances. And it is these atomic statements, if anything, that can be regarded as the simple predications from the *Categories*. But if those predications express, as it were, the immediate combination of an individual and a universal, they have to be analyzed, as Dummett recommends, in terms, not of complex predicates, but of simple ones.

3 Russell's world

In this chapter, we give an overview of the picture of the world that was operative in Russell's mind around the time that he wrote the *Principles of Mathematics* of 1903, as he was working towards a version of Frege's logic. We introduce simple terms in 3.1, atomic propositions in 3.2, concepts in 3.3 and complex propositions in 3.4, and discuss Russell's three problems involving relations in the remaining sections: logical assertion in 3.5, propositional unity in 3.6, and abstract relations in 3.7.

We should be able to draw two general lessons from this overview. First, Russell's world was broadly atomistic, and contained only crude forms of complexity, partly mereological, and roughly consisting in some form of juxtaposition of simple elements, their separation, and their reconfiguration. Second, the features of his ontology that Russell recognized as problematic not only derived from Moore, but also seemed to him to be misplaced in a work on the principles of mathematics.

3.1 Terms

Russell's world is a world of *terms*. Terms are simple, and divide into *things* and *concepts*. Things combine with concepts to form simple propositions, of which they are the simple subjects. Propositions combine with other terms, including other propositions, to form complex propositions: complex terms some proper part of which are propositions. This form of combination can be reiterated to form ever more complex propositions. Nothing falls outside of the scope of this scheme. Anything that the world contains is either a simple term, or a complex term ultimately constituted by simple terms.

This, in rough outline, was the simple worldview that Russell would successively alter and modify, but never completely abandon, at least until 1918. He arrived at it via the method that he called 'philosophical grammar', which was in effect the form that his ontological theorizing assumed.

Philosophical grammar consists in assigning 'the meaning of each word in the sentence expressing the proposition' (1903a §46). The underlying assumption is that 'every word occurring in a sentence must have *some* meaning', and its aim is 'a classification, not of words, but of ideas' (1903a §46). It is the sort of enquiry in which, Russell says, 'grammar, though not our master, will yet be taken as our guide' (1903a §46).

Terms are objects. They are what can be counted, and they form a single, absolutely general category. As Russell writes, the word 'term'

> is the widest word in the philosophical vocabulary. I shall use it synonymously with the words unit, individual and entity. The first two emphasize the fact that every term is *one*, while the third is derived from the fact that every term has being, i.e., is in some sense. A man, a moment, a number, a class, a relation, a chimaera, or anything else that can be mentioned, is sure to be a term; and to deny that such and such a thing is a term must always be false. (1903a §47)

The possibility of being mentioned, or referred to, is therefore sufficient for something to be a term. In turn, the connection with counting and the self-contradictoriness involved in denying singular claims of existence suggest a Quinean criterion of objecthood. To be, one might say, is to be a term, and to be a term, is to be the value of a variable bound by the existential quantifier. Russell did not at this stage draw an explicit connection with quantification, but he might have done so.

Likewise, Russell sometimes speaks of 'things that do not exist':

> Points, instants, bits of matter, particular states of mind, and particular existents generally, are things in the above sense, and so are many terms which do not exist, for example, the points in a non-Euclidean space, and the pseudo-existents of a novel. (1903a §48)

But these are simply things that do not exist *in time* or *space*. They are *abstract*, as we might say. Their recognition did not for a moment suggest to Russell that reference (or quantification) could not be merely apparent even before 1905, as sometimes seems to be assumed.

Terms are simple. However, it is not always clear what Russell means by simplicity. Sometimes he seems to imply that terms are simple in that they have no proper parts that are terms. This goes with a logical or grammatical criterion of simplicity, according to which simple terms are just what simple words mean. As we shall see, this is the criterion that guides Russell's thinking at least intuitively, again in accordance with philosophical grammar.

However, Russell also speaks in a way that suggests an alternative, ontological criterion of simplicity. 'Again', he writes, 'every term is immutable and indestructible. What a term is, it is, and no change can be conceived in it which would not destroy its identity and make it another term' (1903a §47). It would now seem as if a term were simple in so far as it had *no* parts. If so, Russell conceived of a thing's corruption as the separation of its parts, and its change as their reconfiguration. And if so, few of his examples of simple terms are simple in this sense: in general, for instance, no ordinary object is. Here, however, and at any rate for expository purposes, I will adopt for the most part the first criterion of simplicity.

Terms divide into *things* and *concepts*. Paradigmatically, things are what proper names stand for. As Russell writes, a wider notion of a proper name than the usual one would be needed in order to achieve complete generality, since he means to include 'all particular points and instants' (1903a §48). One might indeed wonder if points and instants could really have proper names in any normal sense. In any case, perhaps the category of demonstratives might serve as a better guide to Russell's meaning.

Concepts are what 'all other words' stand for. By 'all other words' Russell mostly means adjectives and verbs. Adjectives and verbs stand respectively for *predicates* and *relations*, into which concepts divide. Of intransitive verbs, Russell carelessly remarks that they express relations to indeterminate relata. The view is especially odd in light of the example he chooses to illustrate it, 'Smith breathes' (1903a §48).

3.2 Atomic propositions

Things combine with concepts to form simple propositions. A proposition is simple, or atomic, if none of its proper parts is a proposition. Russell does not at this stage use the word 'atomic', but in fact atomic propositions form his paradigm case.

Propositions are a kind of complex term. Like fusions, complex terms have parts; unlike fusions they have structure: the terms that constitute them remain their determinate parts.

Now simple terms are indeed '[w]hatever may be an object of thought, or may occur in any true or false proposition' (1903a §47). Famously, for Russell, a proposition is not just a declarative sentence in the indicative mood. Rather, 'a proposition, unless it happens to be linguistic, does not itself contain words: it contains the entities indicated by words' (1903a §51).

There are two implications here. Any possible constituent of a proposition is a term and a term is just an entity. Conversely, anything is a possible constituent of a proposition. Hence there is no gap between the realm of propositions and the realm of terms.

Russell introduces the notion of 'the *terms* of a proposition as those terms, however numerous, which occur in a proposition and may be regarded as subjects about which the proposition is' (1903a §48). This allows him to analyse propositions in terms of what they are *about* and what they *say* about what they are about. What a proposition is about, Russell calls its *logical subject*; what it says about it, an *assertion*:

> every proposition may be divided … into a term (a subject) and something which is said about the subject, which something I shall call an *assertion*. Thus 'Socrates is a man may be divided into *Socrates* and *is a man*.' (1903a §43)

The qualification 'logical' is important because, as we shall see in chapter 5, a proposition is not always about the grammatical subject of the sentence that expresses it.

Russell singles out subject-predicate propositions as those among atomic propositions that have a unique analysis in terms of subject and assertion. By contrast, relational propositions may have several subject-assertion analyses; exactly two if the relation has two places.

> In a large class of propositions […] it is possible, in one or more ways, to distinguish a subject and an assertion about the subject. […] In a relational proposition, say '*A* is greater than *B*', we may regard *A* as the subject, and 'is greater than *B*' as the assertion, or *B* as the subject and '*A* is greater than' as the assertion. There are thus, in the case proposed, two ways of analyzing the proposition into subject and assertion. Where a relation has more than two terms […] there will be more than two ways of making the analysis. But in some propositions, there is only a single way: these are subject-predicate propositions, such as 'Socrates is human'. (1903a §48)

At this stage, it is not altogether clear why Russell does not consider a third analysis, in which *A* and *B* might be regarded as subjects, and 'is greater than' as the assertion. After all, he does define the 'terms of a proposition' in such a way that would clearly allow for it ('however numerous'). Moreover, in that case, there would be no very good reason *not* to regard this third possibility as in some sense fundamental. We shall consider Russell's reasons in chapter 4.

The same notion of the 'terms of a proposition' also allowed Russell to define a criterion for the replaceability of terms in propositions *salva congruitate*:

> It is a characteristic of the terms of a proposition that any one of them may be replaced by any other entity without our ceasing to have a proposition. (1903a §48)

Roughly, then, the logical subjects of a proposition may be replaced at will by just any term. Conversely, any term is the possible subject of a proposition: that is, any term is a possible term of a proposition.

3.3 Concepts

According to Russell, such sentences as 'Socrates is human' and 'Humanity belongs to Socrates' are logically equivalent, but do not express the same proposition. The proposition that Socrates is human is *only* about Socrates, but the proposition that humanity belongs to Socrates is *also* about humanity. This 'indicates that humanity is a concept, not a thing' (1903a §48). The implication is that things always are, or may be regarded as being, the logical subjects of the propositions in which they occur, which can in turn always be said to be about them.

Things, then, can only occur as what something is said *about*: in 'Socrates is human', 'we no longer have a proposition at all if we replace *human* by something other than a predicate' (1903a §48). One is here reminded of Aristotle's class of individuals, which could not be *said of* anything. Hence even if relational propositions had in some sense unique analyses, subject-predicate propositions could still be defined as those having a unique logical subject.

Concepts, by contrast, may be the logical subjects of some propositions (those that are about them), but will be assertions in others (those that are not). One is now reminded of Aristotle's universals, which could both be *said of* other things and have themselves things *said of* them. Russell characterizes this dual role as alternative *modes of occurrence*. Concepts may occur in propositions *as things* or *as concepts*. As he put it, 'Socrates is a thing, because Socrates cannot occur otherwise than as term in a proposition: Socrates is not capable of that curious twofold use which is involved in *human* and *humanity*' (1903a §48).

Russell considers but ultimately rejects the possibility of *not* identifying *human* and *humanity*. He concedes that there is a grammatical difference between the two words, but insists that that difference is *not* also logical. He argues that it would be self-contradictory to suppose that a concept occurring as concept (say, *human*) differed from the concept occurring as term (*humanity*), since that very supposition requires that the concept occurring as concept (*human*) be treated as term, as it is one of the logical subjects of the proposition that frames the supposition.

Now, it might seem as if Russell's argument simply begs the question against an opponent who would distinguish *human* and *humanity*. On the assumption that no concepts occur as terms, if there were such a proposition as that *human* differs from *humanity*, then the first term of the relation would simply not be a concept, since there *is* no proposition the terms of which are concepts, by Russell's own criterion of substitution.

This charge neglects that concepts are, at bottom, terms themselves: it is as it were their mode of occurrence as concepts that is peculiar. As Russell puts it, the difference 'lies solely in external relations, and not in the intrinsic nature of the terms' (1903a §49).

Likewise, when Russell says that his argument proves that 'we were right in saying that terms embrace everything that can occur in a proposition' (1903a §49), it is true that that only follows on the assumption that concepts *may* occur as things, which is what his argument was trying to prove. But although there is a kind of circularity, the reason why the argument proves nothing is not that it is simply question begging. Rather, on Russell's conception, it is again simply intrinsic to concepts that they may occur as terms: it is definitional of terms, *any* terms, that they *may* be the *terms of* a proposition.

Concepts, then, are terms that may occur as concepts in propositions. When they do so, they are assertions, or the part of the proposition that says something about its logical subject. Strictly speaking, a concept is an assertion if it is a predicate. If it is a relation, we saw, the assertion is the concept plus all but one of its relata. But relations too may occur as subjects, it again being 'plain that the difference must be one in external relations' (1903a §52).

With modes of occurrence of concepts in place, Russell gives a 'logical genealogy' or derivation of relational from subject-predicate propositions. 'The simplest propositions are those in which one predicate occurs otherwise than as a term, and there is only one term of which the predicate in question is asserted' (1903a §57): again, 'Socrates is human' is an example. Next, if we turn the adjective into the corresponding abstract noun, we can form the relational proposition which asserts a relation between the same thing and the concept now occurring as a term: 'Socrates has humanity'. These propositions, we saw, are equivalent but not identical.

Russell now considers a third case, arrived at when the abstract noun is turned into the corresponding count noun: 'Socrates is a man'. Here 'man' is not a predicate but what

Russell calls a *class-concept*, although they 'differ little' from one another: a class being 'the sum or conjunction of all the terms which have the given predicate' (cf. 1903a §57).

But the new sentence, Russell claims, is ambiguous between 'Socrates is-a man' and 'Socrates is a-man'. Both express relational propositions, but the second one is no longer equivalent to 'Socrates is human' and 'Socrates has humanity', as it expresses 'the identity of Socrates with an ambiguous individual' (1903a §57 fn.). It is in conjunction with the latter that Russell introduces *denoting concepts*, which will occupy us in chapter 5.

3.4 Complex propositions

Just as simple terms combine into propositions, propositions combine into more complex propositions. More generally, a proposition may be complex (i.e., have propositions as parts) and yet have parts which are not propositions. One important class of this special case are propositions that express propositional attitudes. In chapter 6 we will take a look at Russell's own theory of judgement.

Among the complex propositions all parts of which are propositions are those expressed by logically compound sentences, i.e., sentences formed from other sentences with the help of (binary) logical connectives. Russell would later call these 'molecular complexes'.

One instance of a molecular complex is implication. An implication, for Russell, asserts a relation between two propositions. That relation, in his usage, is expressed by 'implies'. But according to Russell, consideration of inference by *modus ponendo ponens* (*modus ponens*, for short), the principle 'that if the hypothesis in an implication is true, it may be dropped and the consequent asserted' (1903a §38), brings out the necessity of distinguishing implication from a corresponding relation expressed by 'therefore'.

Russell makes his case by reference to Lewis Carroll's 'What the tortoise said to Achilles'. The puzzle that Carroll presented in that paper pointed towards the need to distinguish rules from statements, or inference rules from formation rules. If only formation rules are taken into account, starting with '$p\rightarrow q$' and 'p', we can form '$((p\rightarrow q)\wedge p)\rightarrow q)$' and '$(((p\rightarrow q)\wedge p)\wedge(((p\rightarrow q)\wedge p)\rightarrow q)\rightarrow q)$', and so on endlessly. But without inference rules, we shall be unable to describe the inference from '$p\rightarrow q$' and 'p' to 'q'.

Russell tries to draw a similar lesson. He claims that the puzzle points to 'the distinction between a proposition actually asserted, and a proposition considered merely as a complex concept' (1903a §38). If *implies* and *therefore* are not distinguished, we shall be unable to argue (or rather describe the inference) from '*p*' and '*p* implies *q*' to '*q*'. For 'the proposition "*p* implies *q*" asserts an implication, though it does not *assert p* or *q*'—whereas the inference in question should 'enable us to assert *q* provided *p* is true and implies *q*' (1903a §38). He concludes that, while *implies* holds between *unasserted propositions*, *therefore* must hold between *asserted* ones. Yet 'these are psychological terms, whereas the difference which I desire to express is genuinely logical' (1903a §38).

Now, in the psychological sense of assertion, two of Russell's points are perfectly sensible. If one asserts '$p \rightarrow q$', one thereby asserts *neither* '*p*' *nor* '*q*'. Likewise, if one asserts a sentence, one does *not* thereby change its meaning. Otherwise 'no proposition which can possibly in any context be unasserted could be true, since when asserted it would become a different proposition' (1903a §38). Together, these two points make up roughly what Geach later called 'the Frege point' (Geach 1972: 255).

However, Russell adds that, in his peculiar *logical* sense of assertion, assertion is a quality which 'true propositions have […] not belonging to false ones'. And yet at the same time, 'in "*p* implies *q*", *p* and *q* are not asserted, and yet they may be true' (1903a §38). It is not at all clear, then, what role he intends logical assertion to play. Characteristically, however, Russell is content with '[l]eaving this puzzle to logic' while insisting that 'there is a difference of some kind between an asserted and an unasserted proposition' (1903a §38).

3.5 Logical assertion

We thus arrive at the first of three interrelated problems that Russell discusses primarily in connection with relations. But in fact the problems are general. For although subject-predicate propositions are not strictly relational, Russell takes them to nevertheless imply a relation between subject and predicate (cf. 1903a §53).

The first problem concerns the difference between *logically asserted* and *logically unasserted* propositions, or as Russell also puts it, between *propositions* and *propositional concepts* (cf. 1903a §55). The second is the problem of *propositional unity*. The third is the problem of *abstract relations*. We shall now briefly consider each of these in turn.

Russell begins by considering the difference between a sentence and its nominalization, for instance 'Caesar died' and 'the death of Caesar'. The first, he says, expresses a proposition, while the second refers to a propositional concept. A propositional concept is just what a proposition *becomes* if it is turned into a *logical subject* by nominalizing its verb. The difference therefore involves Russell's former distinction of alternative modes of occurrence for concepts.

Now a problem arises out of Russell's account of how propositions and propositional concepts relate to their own truth or falsehood. On the one hand, he writes, 'neither truth nor falsity belongs to a mere logical subject': 'the death of Caesar has an external relation to truth or falsehood (as the case may be), whereas "Caesar died" in some way or other contains its own truth or falsehood as an element' (1903a §52). On the other hand, 'if this is the correct analysis, it is difficult to see how "Caesar died" differs from "the truth of Caesar's death" in the case where it is true, or "the falsehood of Caesar's death" in the other case. Yet it is quite plain that the latter, at any rate, is never equivalent to "Caesar died"' (1903a §52).

Russell concludes that there 'appears to be an ultimate notion of assertion, given by the verb, which is lost as soon as we substitute a verbal noun, and is lost when the proposition in question is made the subject of some other proposition'. This is again the logical sense of assertion, which is 'very difficult to bring before the mind', and in which only true propositions are asserted. Unlike a concept, then, a proposition seems to be 'an entity which cannot be made a logical subject' (1903a §52).

Needless to say, these passages are both confusing and confused. For instance, when Russell claims that propositions *contain* their truth or falsehood, it is unclear whether he means simply that they are necessarily true or false, or whether he means they are necessarily true or necessarily false, the latter being suggested by his claim that only true propositions are asserted. As we shall see in chapter 6, he probably means both. On the one hand only propositions are truth-evaluable, but on the other they bear an intrinsic relation to their truth-value.

Leaving that to one side for the moment, there is something right about what Russell says here, but also something wrong.

What is right is his insistence that, in so far as they are logical subjects, propositional concepts, have *no* relation to truth and falsehood, let alone an 'external' one. Again, they are not evaluable with respect to truth. Such phrases as 'the so-and-so' do not *assert* anything, they are *names* of something (or else are 'incomplete symbols', if one adopts Russell's later theory of descriptions). In this sense, 'the death of Caesar' is simply a name of an object which we might take to be an event, following Ramsey (1927: 141). But in so far as an event is a complex *object*, there is no point in calling it a *propositional* complex, supposing a proposition to be the sort of thing 'which it takes a sentence to express' (Ramsey 1991: 8), that is, something with a verb.

At the same time, as Ramsey also writes, we may indeed use the phrase 'the death of Caesar' as in 'the death of Caesar implies the death of someone'. However, this is just an alternative way to say 'that Caesar died implies that someone did', which is in turn just an alternative way to say 'if Caesar died, then someone did'. But then in this case there is no point in calling 'the death of Caesar' a propositional *complex*, since it is simply a proposition. If this is right, then Russell's logical notion of assertion can simply be regarded as the result of a confused attempt to solve an issue that should not have been there in the first place.

It is important to realize that it is Russell's view of concepts that allows him to maintain the identity between propositions and propositional concepts. But it is just as important to realize why he would have wanted to maintain it. On Russell's view, propositions are complexes. This by itself suggests that propositions and propositional concepts have the same kind of complexity. One might say on Russell's behalf that, in general, complexes come in two kinds: those the terms of which are combined by a 'relating' relation, and those the terms of which are combined in some other way. The problem is that Russell seemed to lack a model for any other sort of combination but the propositional one, in which two or more things are linked by a relation. Hence complexes of the first kind are propositions, but complexes of the second kind really have to be propositions as well if indeed they are anything at all.

3.6 Propositional unity

One might of course argue the other way around. If terms *only* combine into propositions so that nothing can form propositional complexes, but propositions are propositional complexes, then terms cannot really combine into propositions. This, roughly, is Russell's problem of propositional unity.

Russell introduces it as a problem about analysis. 'The twofold nature of the verb, as actual verb and as verbal noun', he writes, 'may be expressed […] as the difference between a relation in itself and a relation actually relating' (1903a §54). Now, consider a proposition such as the one expressed by '*A* differs from *B*'. Its analysis after Russell's philosophical grammar consists in giving the meanings of the parts of the sentence: '*A*' means *A*, 'differs' means difference, '*B*' means *B*.

Now, the problem is that *A*, difference and *B*, 'thus placed side by side, do not reconstitute the proposition': the relation as it 'occurs in the proposition actually relates *A* and *B*', but after analysis it 'is a notion which has no connection with *A* and *B*' (1903a §54). And it would be no good to equate the original proposition with '*A* is referent and *B* relatum with respect to difference', because '*A*, referent, difference, relatum, *B*' is 'still merely a list of terms, not a proposition' (1903a §54). Hence the 'verb, when used as a verb, embodies the unity of the proposition' (1903a §54), which is destroyed when the verb is considered as a term.

It might seem as if Russell was once again ignoring an obvious possibility here. Could he not just say that 'differs' means *differs*, rather than difference? Well, no. Again, concepts are terms. What that partly implies is that it is only in propositions that concepts may occur *as* concepts, because it is only in propositions that concepts may *occur* at all. In other words, the mode of occurrence of a concept is essentially its mode of occurrence *in a proposition*. Outside propositions there just are no modes of occurrence. But outside propositions, concepts are, just like anything else, terms.

The fact that Russell formulated the problem of propositional unity as a problem about analysis thus has the advantage of showing that there is not just one problem here, but two.

The problem of propositional concepts already generates a problem about unity. Propositions and propositional concepts are in some sense identical, an identity which it would be the business of a faithful analysis to preserve. But Russell's conception of complexity fits only propositional formation, as he had no alternative model of 'combination into a whole'. Hence propositions are identical with something that can really have no place in Russell's picture. That is certainly enough to generate a problem of unity.

However, even if Russell had a *second* model of complexity with which to account for the formation of propositional concepts, the things to which the *first* model had to be applied (i.e., terms) were not suited to meet the requirements of the model. Hence it turns out that it was no accident that Russell thought of propositions as propositional concepts after all, for propositional concepts are all that terms are suited to make up.

An analogy with children's building blocks may be apposite here. We can think of Russell's terms as stacking blocks, and of his propositions as constructions built out of blocks that are not simply stacked upon each other. For a construction to be a proposition, one of the blocks, which we may call a *relating relation*, must be an *interlocking* block, or at any rate such that it is capable of holding the remaining blocks together.

The two problems can now be stated as follows.

For something to have any kind of complexity at all on Russell's view, it must be like a construction with an interlocking block that actually interlocks other blocks. Even a pile of blocks that are not actually interlocked are just blocks, and so *not* a new complex block. Hence the only genuinely complex blocks are propositions, not propositional concepts.

At the same time, Russell's world only contains blocks (terms) none of which is an interlocking block (or relating relation). So the best one can do is to pile them up. But to do so is not yet to form a new complex block (a proposition). The analytical version of the problem of unity thus translates in the analogy into the problem of how to turn a non-interlocking block into an interlocking one prior to the construction. In chapter 4 we will see how even propositional functions fail to fit Russell's simple model of complexity.

3.7 Abstract relations

In spite of Russell's view of the 'twofold nature of the verb', he was still led to consider the question whether a 'relating relation' could in some sense be identical with the corresponding relation 'abstractly considered' (1903a §81). He formulated his third problem as follows: are relational propositions constituted by *abstract relations* themselves, or merely by their *specific instances*?

In the *Principles*, Russell ultimately settled with the first view, that abstract relations themselves enter into relational propositions. Roughly, his reasoning seems to have been the following. Although both views faced versions of the problem of unity and Bradley's regress, only the first view is indispensable.

First Russell observed that if abstract relations entered into relational propositions, their unity would be compromised. He then considered a means to block the problem. Russell proposed that abstract relations might not enter into relational propositions *on their own*. That is, the analysis of a relational proposition would reveal, not only an abstract relation, but further *relating relations* as their constituents, holding between the abstract relation and the terms that it was supposed to relate.

However, Russell concluded that this proposal must fail for two reasons.

The first one is rather obvious. In so far as there was a problem of unity for the first relation, there must also be a problem of unity for the new ones. The new relating relations may of course be considered in abstraction too, and are indeed so considered after analysis. So neither the original constituents nor they together with the new relations are be able to reconstitute the proposition.

One might still try to avoid the problem by recognizing yet further relating relations holding between the original constituents and the relating relations just introduced. Of course the new relations can also be abstractly considered. The problem is effectively only blocked if further relations are introduced at each stage *ad infinitum*.

But this leads immediately to the second problem. For 'in this continually increasing complexity we are supposed to be only analysing the *meaning* of our original propostion': but then '[t]his attempt, in fact, leads to an endless process of the inadmissible kind' (1903a §55).

In light of Russell's conclusion, his disposal of an argument of Bradley's against the reality of relations, sometimes known as *Bradley's regress*, may seem surprising. According to Russell, Bradley's idea is that two terms can only be related by a given relation if they are somehow related to that relation. But then the terms are related to the relation only if a further relation holds between each of them and the first relation, and so on *ad infinitum*.

In this case, however, Russell claimed that 'the process is one of implications, not one of analysis' (1903a §55). *Maybe* Bradley had succeeded in proving that a single relational proposition entailed an infinite series of relational propositions, each of which asserting a relation holding between the terms of the proposition that precedes it immediately in that series. But so long as those propositions are all *different* propositions, that is, so long as they form no part of the meaning of the original proposition, the 'endless process' is of the harmless kind (cf. 1903a §99).

The same distinction would allow Russell to maintain that subject-predicate propositions are genuinely *not* relational, although they *imply* that some relation holds between the subject and the predicate. For instance, the proposition that Socrates is human has a single logical subject, but it implies that Socrates has humanity, which has two (cf. 1903a §53). Incidentally, that not every proposition is relational implies that not every verb means a relation after all, since (the verbal expression of) any proposition contains a verb. The 'is' in 'Socrates is human', Russell says, is only a 'pseudo-relation' (cf. 1903a §94, fn).

(It would also enable his *opponent* to reply to his own 1912 regress argument in the *Problems of Philosophy* against a nominalist attempt to replace universals by relations of similarity between particulars.)

We are bound to conclude that Russell's different attitudes towards Bradley's regress and his own argument against abstract relations entering into relational propositions must be explained by the fact that Bradley's regress is problematic only when the problem of propositional unity is already in view.

Anyhow, since Russell's discussion of abstract relations *does* have the problem of unity in view, he considers the alternative possibility that it is not abstract relations themselves that enter into relational propositions, but only their specific instances.

On this new assumption, rather than having a 'twofold nature', each 'relating relation' would be an instance of the abstract relation revealed by analysis. For instance, it is not the abstract relation of difference that occurs, by itself or at all, in the propositions that *A* differs from *B*, *C* from *D*, etc. Rather, there are in each case 'specific differences' between *A* and *B*, *C* and *D*, etc., intrinsic to the terms, that are able to actually relate them. It is presumably owing to the fact that specific differences are 'intrinsic to the terms' that *A*, *B* and their specific difference would be able to, as it were, reconstitute the proposition that *A* differs from *B*.

However, Russell raises another regress against specific differences. Specific differences between different terms, he claims, will be different amongst themselves. For any two assertions of difference, then, there must be a further one asserting the difference between those differences. And the case is general: if R is an abstract relation, and R_1 and R_2 are two of its specific instances holding between different pairs of terms, so that $R_1 \neq R_2$, then there is *another* (i.e., different) specific relation R_3, holding between R_1 and R_2.

Yet, Russell acknowledges that this 'endless process' is merely one 'of implications', and so of the harmless kind. It is just that any two relational propositions imply *another* (i.e., different) proposition asserting a difference between the relations they involve. Indeed, his definitive argument against 'specific relations' concerned, not a regress of sorts, but the possibility of inference.

'[I]f no two pairs of terms can have the same relation,' Russell writes, 'it follows that no two terms can have anything in common, and hence different differences will not be in any definable sense *instances* of difference', and so the view of specific relations 'fails to solve the difficulty for which it was invented' (1903a §55).

Russell elsewhere makes essentially the same point in connection with assertion, but there it is more clearly spelled out. As he puts it, '[i]n "Socrates is a man", we can plainly distinguish Socrates and something that is asserted about him; we should admit unhesitatingly that the *same* thing may be said about Plato or Aristotle' (1903a §81). Otherwise, if we had not the *same* but only different 'specific assertions', inferences that presuppose identity of predication, including simple generalizations, would be invalid on pain of ambiguity. And the same of course holds for relations generally.

Characteristically, Russell concludes his discussion of abstract relations by admitting that '[a]ll these points lead to logical problems, which, in a treatise on logic, would deserve to be fully and thoroughly discussed' (1903a §55). Indeed, he could not come up with any solution to the problem of unity. Likewise, Russell was also happy to leave the problem of logical assertion 'to the logicians with the above brief indication of a difficulty' (1903a §52), even though he did 'not know how to give a clear account of the precise nature of the distinction' between the verb and the verbal noun (1903a §54).

The casual attitude that Russell displays here is indeed quite remarkable. But it does reveal a sound instinct. The twin problems of propositional unity and logical assertion both arise from highly peculiar features of his ontology.

However, Russell's attitude is revealing in a further respect. The *Principles of Mathematics* is as a matter of fact partly a treatise on logic. And yet Russell is in *some* sense right to insist that his discussion of these problems is somewhat out of place in that book. So, what went wrong?

There may be no philosophically interesting answer to this question. From a historical point of view, however, the question is important.

In the Preface to the *Principles*, and in occasional footnotes to the main text, Russell acknowledges Moore's influence on the most philosophical parts of the book. Russell's acknowledgement has two sides. First, he attributes to Moore four distinctive doctrines: the 'non-existential' nature of propositions, their mind-independence, the plurality and mutual independence of terms, and the reality of relations:

> On fundamental questions of philosophy, my position, in all its chief features, is derived from Mr G. E. Moore. I have accepted from him the non-existential nature of propositions (except such as happen to assert existence) and their independence of any knowing mind; also, the pluralism as regards the world, both that of existents and that of entities, as composed of an infinite number of mutually independent entities, with relations with are ultimate, and not reducible to adjectives of their terms or of the whole which these compose. (1903a Preface)

Second, he claims that those doctrines are indispensable for the development of mathematics along logical lines that he is about to undertake:

> Before learning these views from him, I found myself completely unable to construct any philosophy of arithmetic, whereas their acceptance brought about an immediate liberation from a large number of difficulties which I believe to be otherwise insuperable. The doctrines just mentioned are, in my opinion, quite indispensable to any even tolerably satisfactory philosophy of mathematics, as I hope the following pages will show. But I must leave it to my readers to judge how far the reasoning assumes these doctrines, and how far it supports them. Formally, my premisses are simply assumed; but the fact that they allow mathematics to be true, which most currently philosophies do not, is surely a powerful argument in their favour. (1903a Preface)

Russell's indifference towards the problems that his ontology encountered is therefore explained by his relatively provisional commitment to Moore's philosophy. Moreover, of the four doctrines, only the reality of relations might with some plausibility be said to be necessary for mathematics. In fact, far from being necessary for the logical development of mathematics, Russell's ontology was especially ill-suited to the task. Or so we shall argue in the next chapter.

4 Propositional Functions

Russell recognized two ways in which propositions could be analysed: one in terms of simple constituents and another in terms of assertions. At the same time, an assertion was, according to him, 'what Frege calls a function' (1903a §482). A natural assumption would therefore be that Russell's two kinds of analysis correspond to Dummett's distinction of analysis and decomposition. Indeed, Levine (2002) has argued, though on different grounds, that it was in fact Russell who had the conception of analysis that Dummett ascribed, in Levine's view mistakenly, to Frege.

In this chapter, I argue that, although Russell's simple concepts are clearly *not* functions in Frege's sense, he did not in fact have, and could in principle not have had, the same conception of analysis as Frege.

I review Russell's two kinds of analysis in 4.1 and his discussion of formal implication in 4.2. In 4.3 I argue *contra* Russell that Frege's functions are not assertions but propositional functions. In 4.4 I argue that, *pace* Levine, although Russell regarded propositional functions as (in some sense) constituents of quantified propositions, he did not regard them as (in any sense) mere components of their instances. In 4.5 I argue that, in so far as Russell *might* have regarded propositional functions as components of propositions, he would have still regarded any constituent of a proposition as ultimate. In 4.6 I argue that even Russell's concepts are not ultimate constituents in Dummett's sense.

4.1 Concepts and assertions

As we saw in chapter 3, Russell's conception of philosophical grammar consisted in analysing a proposition into its simple constituents by indicating the terms corresponding to each word composing the sentence that expresses it. In addition, Russell recognized that a proposition could also be analysed in terms of what it *says*, that is, into a subject and an assertion. The question that immediately arises, and the one that we shall be concerned with in this section, is why Russell saw any need to distinguish these two forms of analysis.

To recap, if an individual occurs in a proposition, the proposition is necessarily about it, i.e., it is its *logical* subject. But whether a proposition is about a concept depends on its mode of occurrence. A concept is the logical subject of a proposition if, and only if, it occurs there *as a term*.

Hence Russell's two kinds of analysis are apparently related in a straightforward way. The constitution of a proposition determines what it may be regarded as saying of a subject. If a proposition involves either an individual or a concept occurring as a term, it is about it. It can then be divided into that individual or concept as its subject and the remainder of its parts as the assertion, or what it says about that subject.

In reality, this initial characterization of assertions will have to be qualified in at least two crucial respects (cf. 4.3 and 4.6 below). But it is natural to enquire at this stage why Russell took the distinction between concepts and assertions to be at all necessary.

Both individuals and concepts, occurring as terms, combine with concepts occurring as concepts. Concepts occurring as concepts may themselves be regarded as what propositions say about their subjects. Russell even called 'predicates' those concepts that are not relations. It would therefore be entirely natural to regard the category of assertions as redundant at least relative to that of predicates, at least in their predicative mode of occurrence.

To be sure, the first of Russell's reasons for distinguishing assertions from predicates was simply misguided. It concerned the problem of propositional unity. As we saw, again in the previous chapter, in order for simple terms to combine, one of them must act as tying the remaining ones together. It is concepts in their predicative mode of occurrence that are responsible for holding together the constituents of the propositions in which they occur. However, the mode of occurrence of a term as a concept is only a mode of occurrence *within* a proposition. Strictly speaking, terms have *no* modes of occurrence by themselves, that is, outside propositions. But for Russell that is precisely how they appear when they appear as the outcome of analysis. Any relation that is mentioned in the analysis of a proposition is not a 'relating relation', but only a relation 'abstractly considered'. Hence, 'when a proposition is completely analysed into its simple constituents, these constituents taken together do not reconstitute it' (1903a §81).

Things are otherwise, Russell seemed to think, with regard to the analysis of propositions into subject and assertion:

> A less complete analysis of propositions into subject and assertion has also been considered; and this analysis does much less to destroy the proposition. A subject and an assertion, if simply juxtaposed, do not, it is true, constitute a proposition; but as soon as the assertion is actually asserted of the subject, the proposition reappears. The assertion is everything that remains of the proposition when the subject is omitted: the verb remains an asserted verb, and is not turned into a verbal noun; or at any rate the verb retains that

> curious indefinable intricate relation to the other terms of the proposition which distinguishes a relating relation from the same relation abstractly considered. (1903a §81)

Russell claims that this second kind of analysis 'does *much less* to destroy the proposition', thereby implying that the notion of an assertion could be appealed to in order to address the problem of unity. But this raises a couple of issues.

The first is a worry of equivocation. The problem of propositional unity concerned the fact that terms have no modes of occurrence outside propositions, and so cannot acquire their capacity to bind other terms together prior to the constitution of the proposition. In Russell's terms, a 'verbal noun' cannot be turned into an 'asserted verb' prior to a proposition's being 'logically asserted'. But now Russell appears to be maintaining that assertions have that special mode of occurrence even when they are not 'actually asserted' of any subject. Hence the sense in which Russell now claims that 'the proposition reappears' as soon as 'the assertion is actually asserted of the subject' cannot be his peculiar 'logical' sense.

The second is a very simple form of circularity. What analysis in terms of assertions seems to provide is effectively a context for concepts to occur as concepts outside propositions. But this is of course misleading: an assertion is not really a *separable* part of a proposition. Rather, it is what *it* says of *its* subject. Hence the terms that occur in assertions do not *really* occur outside propositions at all. Perhaps assertions may be said to be *abstracted from* propositions. But then it is only in virtue of their being so abstracted that concepts are allowed to retain their modes of occurrence. Hence Russell's subject-assertion analysis does 'much less to destroy' propositional unity simply because it does not 'destroy it' at all. Rather, it simply presupposes it. An appeal to assertions as a means to restore propositional unity would therefore be simply incoherent.

Now as we saw in chapter 3, Russell was committed to a worldview of external relations between simple terms which leads to the problem of unity. Simple terms may happen to constitute propositions, but that possibility is not otherwise built into their nature as terms. However, suffice it to say for now that, in the absence of such a view, there might be no problem in the first place. For instance, as we saw in chapter 2, Frege's context principle, for one, would be enough to diffuse it.

That said, a second reason for distinguishing assertions from predicates was entirely sound. Russell's predicates are simple terms. But as a matter of fact, what a proposition may be held to say of a subject may be complex.

For instance, the analysis of 'Socrates is wise' into the subject Socrates and the assertion 'is wise' might be thought to coincide with its analysis into the simple constituents Socrates and wisdom (ignoring the problem of unity for the sake of argument). But the analysis of 'Socrates is wise and mortal' into the subject Socrates and the assertion 'is wise and mortal' would never be confused with its analysis into the simple constituents Socrates, wisdom and mortality. It is therefore only in the former case that subject and assertion might be thought to coincide in some way with the simple terms out of which the proposition is constructed.

Russell's notion of an assertion might therefore be in line to fulfil the role of a complex predicate. The notion of what may be said of something, or the *general* notion of a predicate, is broader than that of a *simple* predicate. Thus analysis in terms of assertions *need not* coincide with analysis into ultimate constituents (and in fact it will do so only when the proposition is of the subject-predicate form, since Russell also rejected the notion of a 'relational assertion', cf. 4.3 below). This is also why Russell claims that analysis in terms of assertions is 'less complete' than analysis in terms of simple constituents.

Superficially, then, with the caveat that Russell's predicates could not really be the sort of thing for which Dummett's simple predicates stand, as they are not intrinsically predicative, Russell's two kinds of analysis resemble Dummett's distinction of analysis and decomposition. Just as analysis in Dummett's sense aims at uncovering the ultimate constituents of sentences, so does Russell's philosophical grammar reveal the building blocks of propositions. Just as decomposition consists in forming complex predicates from complete sentences, so does Russell's analysis in terms of assertions serve to identify what a proposition says of a subject. And just as analysis circumscribes decomposition, so does the constitution of a proposition determine what it may say about a subject.

In what follows, however, we shall see that this equation could only hold at the atomic level at best, and even then in a highly misleading fashion. Our first step in that direction is to look into Russell's reasons for wanting to explain propositional functions in terms of assertions.

4.2 Formal implication

Russell distinguished material from formal implications. A material implication is what we would call a material conditional, that is, a sentence of the form 'if p, then q' (in symbols, '$p \rightarrow q$'), false only when 'p' is true and 'q' false. While Russell does not quite define material implication in terms of disjunction, he recognizes that it is equivalent to '*q*, or else ~*p*' (cf. 1903a §16). However he typically expresses material implications in the form '*p* implies *q*', thereby signalling his interpretation of implication as a relation between unasserted propositions.

By contrast, a formal implication is a universally quantified proposition of the form '$\forall x(Fx \rightarrow Gx)$', the instances of which are material implications. Here the quantifier '$\forall x$' attaches to the *propositional function* '$Fx \rightarrow Gx$'. A propositional function is in turn a proposition with at least one real (or free) variable, which becomes a 'determinate proposition' once all of them are replaced by terms:

> We may explain (but not define) this notion as follows: *fx* is a propositional function if, for every value of *x*, *fx* is a proposition, determinate when *x* is given. (1903a §22)

A formal implication is therefore the universal closure of a propositional function of a certain kind, whose value is a material conditional. As an instance, Russell gives '*x* is a man implies *x* is mortal for every value of *x*'. Again we might read this as 'for all *x*, if *x* is a man then *x* is mortal', or indeed 'All men are mortal'. Russell took all of these propositions to be equivalent, though not identical (cf. 1903a §40); we need not consider here why not. The quantifier '$\forall x$' is meant to be unrestricted (cf. §41).

Russell's account of inferences from general statements to their instances is slightly different from Frege's. For Russell, a propositional function singles out a class of propositions. In the case of formal implication, the propositional function in the scope of the quantifier singles out a class of material implications. A formal implication is then 'the affirmation of *every* material implication' of that class (cf. 1903a §45).

Still, Russell would agree with Frege that a tree for a formal implication should look like (1) rather than (2) (cf. 2.3 above):

(1) $\forall x(Fx \rightarrow Gx)$

|

$Fx \rightarrow Gx$

|

$Fa \rightarrow Ga$

/ \

Fa Ga

(2) $\forall x(Fx\rightarrow Gx)$

|

$Fx\rightarrow Gx$

/ \

Fx Gx

| |

Fa Ga

As he put it,

> our formal implication asserts a class of implications, not a single implication at all. We do not, in a word, have one implication containing a variable, but rather a variable implication. We have a class of implications, no one of which contains a variable, and we assert that every member of this class is true. (1903a §42)

Incidentally, at this point we find a first wrinkle in Russell's discussion. On the one hand, Russell rightly stresses the fact that the scope of the quantifier in '$\forall x(Fx\rightarrow Gx)$' is '$Fx\rightarrow Gx$', not either '$Fx$' or '$Gx$' separately. Otherwise, the seeming formal implication would after all be a *material* implication of the form '$\forall xFx\rightarrow\forall xGx$', which is of course equivalent to '$\forall x\forall y(Fx\rightarrow Gy)$'. As Russell writes,

> we must not first vary our *x* in '*x* is a man', and then independently vary it in '*x* is a mortal', for this would lead to the proposition that 'everything is a man' implies 'everything is a mortal', which, though true, is not what was meant. This proposition would have to be expressed, if the language of variables were retained, by two variables, '*x* is a man implies *y* is a mortal'. But this formula too is unsatisfactory, for its natural meaning would be: 'If anything is a man, then everything is a mortal'. (1903a §42)

On the other hand, Russell implies that, in order to arrive at '$\forall x(Fx\rightarrow Gx)$', it is *necessary* to form '$Fx\rightarrow Gx$' from a complex complete proposition such as '$Fa\rightarrow Ga$'. As he writes elsewhere, 'the genesis remains essential, for we are not here expressing a relation of two propositional functions' (1903a §89). But as the difference between (1) and (2) shows, it is only *sufficient* to do so. What *is* required is that '$\rightarrow$' be in the scope of '$\forall x$' rather than conversely. Russell therefore seems to conflate (2) with (3):

(3)

$$
\begin{array}{ccc}
 & \forall xFx \rightarrow \forall xGx & \\
 & / \quad \backslash & \\
\forall xFx & & \forall xGx \\
| & & | \\
Fx & & Gx \\
| & & | \\
Fa & & Ga
\end{array}
$$

Note that the bottom halves of (2) and (3) are identical. It is not before the *third* line (from the bottom) is reached that it is determined which formula is constructed. What matters is whether the quantifier is first attached to each simple propositional function as in (3), or only to '$Fx \rightarrow Gx$' as in (2).

Russell however treats the free variables in 'Fx' and 'Gx' as if they were implicitly bound, thus ignoring the possibility of (2). Hence for him it is already determined which sentence is being constructed (in this case, '$\forall xFx \rightarrow \forall xGx$') as soon as the *second* line is reached. This comes out in the following passage:

> The point to be emphasized is, of course, that our x, though variable, must be the same on both sides of the implication, and this requires that we should not obtain our formal implication by first varying first (say) Socrates in 'Socrates is a man', and then in 'Socrates is a mortal', but that we should start from the whole proposition 'Socrates is a man implies Socrates is a mortal', and vary Socrates in this proposition as a whole. (1903a §42)

Hence at this stage Russell seemed to conceive the constructional history of a sentence somehow not to fully determine the scope of a given formula. He thus seemed to draw a distinction where there is in fact none.

More pressing to our immediate concerns is the peculiar, though related, question that Russell raises about '$Fa \rightarrow Ga$', in connection with (1). A proposition of the form '$Fa \rightarrow Ga$' may be regarded as asserting '$F\ldots \rightarrow G\ldots$' of a. But would that be a (complex) relation of two simple assertions, or a single complex assertion?

> It may be said that there is a relation between the two assertions 'is a man' and 'is a mortal', in virtue of which, when the one holds, so does the other. Or again, we may analyse the whole proposition 'Socrates is a man implies Socrates is a mortal' into Socrates and an assertion about him, and say that the assertion in question holds of all terms. Neither of these theories replaces the above analysis of '*x* is a man implies *x* is a mortal' into a class of material implications; but whichever of the two is true carries the analysis one step further. (1903a §44)

One imagines that the contrast that Russell has in mind here is analogous to that between (1) and (2) above. We may think of '*F*…→*G*…' as being formed from two assertions '*F*…' and '*G*…' as in (4), or from '*Fa*→*Ga*' directly as in (5):

(4)

```
        F…→ G…
         /    \
      F…        G…
      |         |
      Fa        Ga
```

(5)

```
        F…→G…
          |
        Fa→Ga
         /    \
      Fa        Ga
```

Now there are two questions to consider here. One concerns the ground for Russell's question. After all, Russell already had an analysis of formal implication in terms of propositional functions. It is therefore unclear why he now wanted to, as he sometimes put it, 'explain propositional functions by means of assertions'.

The other question concerns the ground for Russell's answer. It is also hard to gather what exactly is at stake, so much so that one may be tempted to regard both alternatives as admissible. It may therefore come as a surprise that Russell thought that neither is. We take this second question first.

4.3 Complex assertions

To recap, the idea of an assertion is the idea of a predicate, that is, the idea of what a proposition says of a subject. And what a proposition says of a subject may be said of a different subject in another proposition. For instance, the proposition 'Socrates is a man implies Socrates is a mortal' says the *same thing* about Socrates as the proposition 'Plato is a man implies Plato is a mortal' says about Plato. As Russell put it,

> it seems very hard to deny that the proposition in question tells a fact *about* Socrates, and that the *same* fact is true about Plato or a plum-pudding or the number 2. (1903a §82)

Now the problem is that '[a]n assertion was to be obtained from a proposition by simply omitting one of the terms occurring in the proposition':

> But when we omit Socrates, we obtain '… is a man implies … is a mortal'. In this formula it is essential that, in restoring the proposition, the *same* term should be substituted in the two places where dots indicate the necessity of a term. It does not matter what term we choose, but it must be identical in both places. Of this requisite, however, no trace whatever appears in the would-be assertion, and no trace can appear, since all mention of the term to be inserted is necessarily omitted. (1903a §82)

Russell's 'dots' still suggest that something remains in Socrates' place when he is omitted from the proposition (or in place of his name when *it* is omitted from the sentence). We can rather express the assertion as '(   ) is a man implies (   ) is a mortal', or indeed simply '   is a man implies   is a mortal'.

Now if *this* were what was said first of Socrates, then of Plato, then it would also be what is said of both in, e.g., 'Socrates is a man implies Plato is a mortal'. But, clearly, it is not. Here, at best, something is *now* said of Socrates that before was *only* said of Plato (namely, '   is a man implies Plato is a mortal'), and something is *now* said of Plato that before was *only* said of Socrates (namely, 'Socrates is a man implies   is a mortal').

The notion of a complex assertion is therefore incoherent. An assertion is what a proposition says of a subject. But sometimes what is said of a subject is complex, in the sense that it requires the subject, as it were, to occur more than once in a proposition. But as Russell defined the notion, little sense can be made of the identity or difference of the *argument-places* of an assertion. Hence '$Fa \rightarrow Ga$' cannot be analyzed into the complex assertion '$F\ldots \rightarrow G\ldots$' as in (5) above. As he writes,

> it was our intention, if possible, to explain propositional functions by means of assertions; hence, if our intention can be carried out, the above propositions must be assertions concerning Socrates. There is, however, a very great difficulty in so regarding them. (1903a §82)

For the same reason, though, it is immaterial whether we regard '$F\ldots \rightarrow G\ldots$' as a complex assertion, or as a relation of assertions as in (4). For that view, too,

> suffers from the difficulty that it is essential to the relation of assertions involved that both assertions should be made of the *same* subject, though it is otherwise irrelevant what subject we choose. (1903a §44)

Incidentally, this was in essence Russell's objection to Frege's notion of a function. In fact, he regarded the Fregean notion as a generalized version of his own notion of an assertion, which rendered complex predication unintelligible. As he claimed, 'what Frege calls a function' is:

> What remains of the said unity when one of its terms is simply removed, or, if the term occurs several times, when it is removed from one or more of the places in which it occurs, or, if the unity has more than one term, when two or more of its terms are removed from some or all of the places where they occur. (1903a §482)

The problem is straightforward. Functions may have any number of arguments. But as Russell realised, the notion of something that remains of a certain unity when some of its parts are removed simply makes no sense of the identity and difference of argument-places that the general notion of a function requires.

> Frege wishes to have the empty places where the argument is to be inserted indicated in some way; thus he says that in $2x^3+x$ the function is $2(\ )^3+(\ )$. But here his requirement that the two empty places are to be filled by the same letter cannot be indicated: there is no way of distinguishing what we mean from the function involved in $2x^3+y$. (1903a §482)

It might be thought that the notion of a single-place assertion or function could escape Russell's criticism, and so that the notion of an assertion was simply narrower than the notion of what may be said of a subject, or a predicate. But in fact Russell went further and objected even to Frege's notion of a *one*-place function:

> Frege's general definition of a function, which is intended to cover also functions which are not propositional, may be shown to be inadequate by considering what may be called the identical function, i.e. *x* as a function of *x*. If we follow Frege's advice, and remove *x* in hopes of having the function left, we find that nothing is left at all; yet nothing is not the meaning of the identical function. (1903a §482)

On Russell's understanding of assertions, 'identical functions' would indeed be nothing if functions were assertions. This is an extreme case, but even one-place functions of different levels must have different argument-places, and so, again, cannot be assertions. To borrow Quine's phrase, 'no entity without identity': but none without difference either. Assertions make no sense of the identity and difference of argument-places, then, because they render the very notion of an argument-*place* unintelligible.

What was needed, Russell saw, was the notion of a *variable*. In order to mark the *identity* of argument-places, the use of variables, or some equivalent device, is required. What the two propositions 'Socrates is a man implies Socrates is a mortal' and 'Plato is a man implies Plato is a mortal' say first of Socrates, then of Plato, is therefore not ' is a man implies is a mortal', but '*x* is a man implies *x* is a mortal'.

Once the identity of argument-places is secured, so too is their difference. The propositional function '*x* is a man implies *x* is a mortal' can now be contrasted with '*x* is a man implies *y* is a mortal'. The first is a complex predicate and says something of a single subject. The second is a complex relation and says something of possibly more than one. If only assertions were available, no such distinction could be drawn.

Now, Frege's mature 'ontic' understanding of functions as 'unsaturated entities', along with the role of 'unsaturatedness' in logical combination (cf. 2.5 above), certainly invites Russell's identification of Frege's functions with assertions. However, in the passage form the *Principles* quoted above, Russell misrepresents Frege's account of functions in *Begriffsschrift* at a crucial juncture.

We have to recall, first of all, that Frege originally introduced functions as incomplete *expressions*. By contrast, Russell never understood assertions in linguistic terms. But more importantly, Frege's exposition precisely emphasizes the *replaceability* or *variability* of expressions, rather than their *omission*. To go back to Frege's own words, if we imagine a 'symbol as *replaceable* by another […], then the part of the expression that shows itself *invariant under such replacement* is called the function; and the *replaceable part*, the argument of the function' (1879 §9, italics added; quoted in full at 2.2). An argument-place, for Frege, was never an 'empty' one: it is a *variable* one.

Like Russell, then, Frege contrasted functions with what he later called '*related* argument-places' (or argument-places that *must* be filled by the same argument) with functions with what he later called '*unrelated* argument-places' (or argument-places that *may* be filled by different arguments). He would have therefore agreed with Russell that the 'gaps' that identify argument-places cannot be *mere* gaps. But unlike Russell, Frege never really treated them as such. On the contrary, Frege's use of dots and dashes (or Greek letters) rather than Roman letters to distinguish argument-places was precisely designed to contrast different kinds of *variables* (free or real, and bound or apparent, respectively).

The similarity between Frege's original exposition of functions and Russell's own account of it in the *Principles* is therefore almost entirely superficial. Russell never explained assertions at the level of language. And while Frege would later characterize functions in non-linguistic terms, he nowhere indicated that they should be conceived as what Russell called a 'rump of a proposition' (§482). It would have been rather more charitable for Russell to acknowledge that, for Frege no less than for himself,

> The fact seems to be that we want the notion of any term of a certain class, and that this is what our empty places really stand for. (1903a §482)

It is therefore Russell's notion of a *propositional function*, and not that of an assertion, that best captures the essence of Frege's notion of a function.

This already shows that Russell's subject-assertion analysis cannot quite coincide with Dummettian decomposition. Still, it might be thought that, as soon as the correction is made, representing a proposition as the value of a propositional function for some argument was to Russell what decomposing a sentence into a complex predicate was to Dummett. That was precisely Levine's (2002) suggestion. However, Russell's ground for wanting to analyse '$Fa \rightarrow Ga$' into '$F\ldots \rightarrow G\ldots$' casts serious doubts upon it.

4.4 Predicates as components

To be fair to Levine, it is unclear what that ground really was. Russell claims in §44 that the analysis of '$Fa \rightarrow Ga$' in terms of assertions would *not* replace the analysis of a propositional function in the scope of a formal implication 'into a class of material implications' (cf. above). Russell would therefore *not* have regarded (4) or (5) as in any way alternatives to (1). But by the same token it is unclear why he thought that 'whichever of the two is true carries the analysis one step further' (§44).

A hint is provided by Russell's original characterization of an assertion as what a proposition *says* of a subject. Russell's notion of an assertion is the notion of a *predicate*. (Even his choice of the word 'assertion' was conditioned by his having identified non-relational concepts as predicates, cf. 4.1.)

By implication, for Russell the notion of a propositional function is *not* a notion of a predicate. It is no accident that he nowhere describes propositional functions as what is *said* of anything. In this respect, then, Russell's propositional functions are *not* like Frege's functions, or Dummett's complex predicates for that matter, at least as Russell conceives them. Russell's distinction between assertions and propositional functions was *not*, for him, a distinction between two kinds of predicates.

As a consequence, rightly or wrongly, Russell thought he had found a gap in his own account of quantification. According to him, a formal implication like '$\forall x(Fx \rightarrow Gx)$' *asserts* a class of (material) implications. More specifically, it asserts *that* the propositions that form the class identified by the propositional function '$Fx \rightarrow Gx$' are true. But each of those propositions, say '$Fa \rightarrow Ga$', in turn *asserts* something. However, since propositional functions are *not* predicates, what it asserts is *not* that '$Fx \rightarrow Gx$' holds of a. Thus Russell robbed himself of a unified account of how what quantified statements say relates to what their instances say. What formal implications assert is *not* what their instances assert of their subjects. (Contrast this with Frege, for whom '$\forall x(Fx \rightarrow Gx)$' means that '$Fx \rightarrow Gx$' is true of anything that takes the place of the variable, cf. 2.2.) Hence the need to further analyse '$Fa \rightarrow Ga$' in terms of assertions.

In Dummett's terms, the point can be stated as follows. Although Russell conceived of propositional functions as *constituents* of formal implications, he did not regard them as (even *mere*) *components* of their instances. Hence what Russell seemed to lack was the conception of a constituent that is not also an *ultimate* constituent. But as we saw in 2.3, this distinction was the hallmark of Frege's account of generality.

Now, why did Russell *not* think of propositional functions as predicates? The answer may lie in the roughly mereological model of complexity introduced in chapter 3. For Russell, something is complex if it has parts that are juxtaposed in a certain way. And a predicate is the part of a proposition that says something of its subject. But strictly speaking a propositional function is not a *part* of a proposition: it is an indeterminate proposition that becomes determinate once its variables are replaced by terms. This means that a propositional function and a corresponding proposition overlap partially, but not completely: propositions do not contain (real) variables. What they do have in common is at best an *assertion*: that which remains when terms are omitted from the proposition, or variables from the propositional function.

An obvious reply would be to say that, although propositional functions are not literally parts of the propositions that are their possible values, they may still be said to *occur in* those propositions in the sense displayed by their characterization. For instance, '*Fx*' may be said to *occur in* '*Fa*' precisely because that sentence is one of its substitution instances (cf. Oliver 2010).

Indeed it may. But that would simply amount to a rejection of Russell's model of complexity, and so beg the question against him. And if Frege and Russell had different models of complexity in mind, they would have had alternative conceptions of analysis. This is not to say in turn that Frege completely lacked a model of part-whole complexity. In fact, if Dummett is right about simple predicates, a complete account of analysis requires some such conception, even if it is not exhausted by it.

4.5 Constituents as ultimate constituents

Yet it is important to realize that Russell nevertheless considered an alternative means to recognize propositional functions as in some sense predicates. But the fact that he ultimately rejected the strategy to pursue that means further strengthens the case against his having had a conception of analysis similar to Dummett's.

Suppose that 'Plato is a man implies Plato is a mortal' and 'Socrates is a man implies Socrates is a mortal' are indeed values of the propositional function '*x* is a man implies *x* is a mortal' for Plato and Socrates as arguments respectively. 'The natural interpretation of this statement would be that the one proposition has to Plato the same relation as the other has to Socrates. But', Russell writes,

> this requires that we should regard the propositional function in question as definable by means of its relation to the variable. Such a view, however, requires a propositional function more complicated than the one we are considering. It we represent '*x* is a man implies *x* is a mortal' by ϕx, the view in question maintains that ϕx is the term having to *x* the relation *R*, where *R* is some definite relation. The formal statement of this view is as follows: For all values of *x* and *y*, '*y* is identical with ϕx' is equivalent to '*y* has the relation *R* to *x*'. It is evident that this will not do as an explanation, since it has far greater complexity than what it was to explain. (1903a §82)

There are three questions to consider here. One is what form *R* might take. A related question is what the significance of such definition would be. Finally, the third question is why is it 'evident that this will not do as an explanation'.

Russell proposes to define propositional functions effectively as functions, that is, by means of a relation between their values (that is, propositions) and their arguments (that is, simple terms). In symbols, Russell's definition of ϕx becomes: $\forall x \forall y\ (y = \phi x \leftrightarrow yRx)$. Now, there is certainly a sense in which, as Russell claims, the *definiens* is more complex than the *definiendum*, regardless of how *R* is construed, if only because it is a relation. But there is also a sense in which that is inevitable. So it remains unclear why the definition cannot do as an explanation; at any rate it is certainly not evident why not.

In work that remains unpublished, Peter Sullivan has made the observation that *R* should take the form of a recursive definition of the values of the propositional function derived from the *analysis* (in Dummett's sense) of the propositions that are in fact those values. All that this requires is that Dummett's principle of the completeness of the values of functions be respected. As Sullivan observes, Russell had himself anticipated that principle in the second edition of *Principia Mathematica*, where functions are finally explicitly said to be able to occur only through their values.

This allows us to recognize the significance of Russell's proposal. We saw in 2.2 that although a function in the sense of *Begriffsschrift* is a linguistic function, not every linguistic function is a function in the sense of *Begriffsschrift*. In order for a linguistic function to be function in that sense, it must figure as in some specifiable sense a *part* of its value. In Dummett's terms, it must be a recognizable *component* (albeit a *mere* component) of the sentences in which it is said to occur. If successful, Russell's demand for a definition of a propositional function in terms of a relation between its arguments and its values would have just the effect of restricting its possible values to those in which in might be said to occur as a component. Hence Russell's demand to be shown that a propositional function satisfies certain conditions before he is willing to call it a predicate (or a genuinely *propositional* function) is perfectly sound.

Now it may indeed be questioned whether Russell was in a position to deliver this result in 1903. Although the *Principia* principle certainly plays a role in Russell's discussion of formal implication already in the *Principles*, his grasp of the relationship between the scope of a formula and its constructional history was at best imperfect then (cf. 4.2 above).

Yet, it is not necessary to ascribe to Russell even an excusable degree of logical incompetence. Whether or not Russell could have delivered a recursive definition of the values of (complex) propositional functions, apparently he would *not* have regarded it as an *analysis* of the propositions in question at all.

The reason is roughly the same as the one already given above. Let us assume that Russell had at his disposal a complete account of the construction of a formal implication, again along the following lines:

$$\forall x(Fx \rightarrow Gx)$$
$$|$$
$$Fx \rightarrow Gx$$
$$|$$
$$Fa \rightarrow Ga$$

We can further assume that he would have regarded the following as the analysis of '$Fa \rightarrow Ga$':

$$Fa \rightarrow Ga$$
$$/ \qquad \backslash$$
$$Fa \qquad Ga$$

Finally, we can also assume that he could have analysed the atomic propositions in terms of assertions:

$$Fa \qquad\qquad\qquad Ga$$
$$/ \qquad \backslash \qquad / \qquad \backslash$$
$$F\ldots \quad a \qquad G\ldots \quad a$$

Confusingly, the crucial bit of evidence comes from Russell summary what is wrong with the analysis of '*Fa*→*Ga*' into a *complex* rather than a *simple* assertion:

> The second theory appears objectionable on the ground that the suggested analysis of 'Socrates is a man implies Socrates is a mortal' seems scarcely possible. The proposition in question consists of two terms and a relation, the terms being 'Socrates is a man' and 'Socrates is a mortal'; and it would seem that when a relational proposition is analysed into a subject and an assertion, the subject must be one of the terms of the relation which is asserted. (1903a §44)

Hence a sentence such as '*Fa*→*Ga*' can be regarded *both* as being formed from two atomic propositions, *and* as asserting something of *a*. In Dummett's terms, it can *both* be analysed *and* be decomposed. So far, so good.

But Russell's further observation entails that the analysis of the proposition is *incompatible* with its decomposition (in Dummett's senses). For Russell, analysis must *immediately* deliver the *logical subject* of the proposition, or what it is about. But that is exactly what cannot happen when a proposition is complex. Russell is here once again implicitly rejecting that something may be in some sense a constituent of a proposition, and yet not its *ultimate* constituent. This need not mean that he could not recognize or indeed provide a Fregean or Dummettian account of the construction of a proposition. But it *does* mean that he would not regard it as its *analysis*.

In the end, this conclusion ought not to be surprising. After all, it was Russell himself who gave a very clear account of what he took the analysis of a proposition to be. As we saw in chapter 3, it was the conception of philosophical grammar that he introduced in the *Principles of Mathematics*.

4.6 Simple relations

We saw in 4.1 that Russell distinguished predicates and assertions because he needed the notion of a complex predicate. We then saw in 4.3 that he nevertheless came to regard the notion of a complex assertion as incoherent. Now even though Russell's grounds for rejecting complex assertions were also sufficient for rejecting complex *relational* assertions, and indeed even simple ones, he had independent reasons for rejecting *simple* relational assertions. We will now look into those.

We have noted in 4.3 that there was a tension between what Russell said regarding simple assertions and what he said about Fregean functions with a single argument-place. In short, Russell seemed to argue *only* against the latter while identifying both, which is of course incoherent. In this particular instance, however, the incoherence concealed an insight.

To recap, not only may what may be said of a subject require the subject to, as it were, occur more than once in a proposition as in the case of complex predicates, but also may it be said of two different subjects at once. This is the case of relations. Here we have two cases: simple and complex relations.

For Russell, though, a proposition can be analysed into an assertion about a subject only if it is either a subject-predicate proposition (e.g., '*F*…'), or a simple relational proposition all of whose terms but one are fixed (e.g., '*Ra*…') (cf. 1903a §81). Hence he rejected the notion of a relational assertion right from the outset. Yet, each category of relations gave rise to a different problem.

Russell's argument against complex relational assertions is an instance of his general case against complex assertions, and we need not return to it. In short, the notion of a complex assertion, whether or not it be relational, makes no sense of the notion of an argument place.

By contrast, his argument against treating even simple relations as assertions is rather that doing so destroys the *sense* or *directionality* of the relation. He would therefore allow a simple relational proposition to be alternatively analysed into either one of the assertions obtained from it by omitting any one of its subjects, but not into an assertion obtained by omitting both (cf. chapter 3). As he wrote,

> there is no difficulty in the notion of the class of all propositions of the form *xRy*. […] Yet it is very difficult to regard *xRy* as analyzable into the assertion *R* concerning *x* and *y*, for the very sufficient reason that this view destroys the *sense* of the relation, *i.e.* its direction from *x* to *y*, leaving us with some assertion which is symmetrical with respect to *x* and *y*, such as 'the relation *R* holds between *x* and *y*.' (1903a §82)

This problem is recognizably the same as the one that Russell would later discuss in connection with his multiple-relation theory of judgement (cf. chapter 6). He continues:

> Given a relation and its terms, in fact, two distinct propositions are possible. Thus if we take *R* itself to be an assertion, it becomes an ambiguous assertion: in supplying the terms, if we are to avoid ambiguity, we must decide which is referent and which relatum. (§82)

At first sight, it might seem as if Russell were just expressing the converse problem that he found with complex assertions. Just as complex assertions could not have *identical* argument-places, relational assertions cannot have *different* ones.

But here Russell does *not* in fact dismiss the intelligibility of argument-places in this connection. Rather, he implies that if simple relations were assertions, they would be *symmetrical*. That is, they would be relations the argument-places of which could be filled indifferently. But that the order in which argument-places are filled is indifferent still presupposes that they are *distinct* argument-*places*.

Russell thus seems to have implicitly recognized two kinds of argument-places, appropriate in each case to *simple* and *complex* predicates and relations. Accordingly, he spoke of different kinds of variability:

> We may quite legitimately regard …*Ry* as an assertion, as was explained before; but here *y* has become constant. We may then go on to vary *y*, considering the class of assertions …*Ry* for different values of *y*; but this process does not seem to be identical with that which is indicated by the independent variability of *x* and *y* in the propositional function *xRy*. Moreover, the suggested process requires the variation of an element in an assertion, namely of *y* in …*Ry*, and this is in itself a new and difficult notion. (1903a §82)

The development of Russell's conception of simple relations later realized this distinction in a different way. In 1903, Russell held a 'directionalist' view of relations, which he would abandon for 'positionalism' by 1913 (cf. chapter 6). According to his later positionalist view, relations have determinate *positions*, which are to be occupied by the terms that they relate. But these positions are not properly speaking argument-places, at any rate not the argument-places of a propositional function.

In the *Principles*, Russell conspicuously failed to account for this distinction in any satisfactory way, so much so that he ended up rejecting the idea of a simple relational assertion altogether. And even his account of the positions of a relation in *Theory of Knowledge* was not much better off, as he failed once again to give a satisfactory account of the identity and difference of those positions, as for instance his discussion of identity there reveals.

Yet, the fact that Russell came to see the need to distinguish the positions and the argument-places of simple and complex relations can be regarded as anticipating to some extent Dummett's much later distinction between the 'valencies' of simple predicates, represented by 'gaps' that remain 'external' to them, and the argument-places of complex predicates, which are 'integral to [their] being' (Dummett 1973: 32–3).

However, that is about as far as similarities between Russell and Dummett should go. In fact, the difference between Russell's understanding of simple relations and predicates (that is, concepts) and Dummett's conception of simple predicates should not be underestimated. It is not, as it were, as if analysis in Dummett's sense *included*, perhaps as a last step, philosophical grammar in Russell's sense.

On the contrary, for Dummett, analysis is exactly what decomposition presupposes, and simple predicates exactly what analysis presupposes (cf. chapter 2). That is also why Dummett's simple predicates are as it were essentially predicative unlike Russell's concepts, why neither Dummett nor Frege had to face a corresponding problem of propositional unity, and why neither would have a need for an intermediate category of assertions such as Russell's. That, indeed, is one of the lessons of the context principle.

5 Denoting

This chapter marks the transition from our first main topic (predication) to the second one (thoughts). On one end, it closes off our discussion of functions by identifying further points of contrast between Frege's and Russell's understanding of quantification. On the other end, it lays the groundwork for our rejection of Makin's characterization of Frege's thoughts and Russell's propositions. In later chapters, I will argue that they differ not quantitatively but qualitatively.

In *The Metaphysicians of Meaning*, Makin argued that 'Frege moved into, and Russell out of, sensism, while their commitment to propositionalism remained constant' (2000: 142). Propositionalism is the view that propositions are the abstract and mind-independent complex entities that serve at once as the meanings of sentences, the bearers of truth, and the objects of propositional attitudes. Sensism is the species of propositionalism according to which at least some of the constituents of propositions are 'aboutness-shifters'. An aboutness-shifter is such that, whenever it occurs in a proposition, the proposition is not about it, but about what it denotes (cf. Russell 1903a §56).

According to Makin, Frege's senses and Russell's denoting concepts alike are aboutness-shifters. Hence Fregean thoughts and Russellian propositions differ only with respect to how many of their constituents are aboutness-shifters. According to the pre-1905 Russell only some are, to the post-1905 Russell none is. According to the pre-1890s Frege none is, to the post-1892 Frege, all are. Makin goes on to argue on this basis that Russell's 'Gray's Elegy argument' against his own denoting concepts applies *mutatis mutandis* to Frege's senses.

In this chapter I do not argue directly against Makin, nor claim anything that he might necessarily disagree with (with perhaps one exception). Rather, for the most part I merely aim to clarify the relationship between Russell's theory of denoting concepts, his theory of descriptions, and Frege's theory of sense.

In 5.1 I introduce Russell's denoting concepts and his 'ontological' view of the *true* variable as the ultimate ground for his theory of denoting. I contrast Russell's conception of variables with Frege's, whose account of quantification essentially contained a symbolic element.

In 5.2 I side with Geach against Dau on the interpretation of Russell's early theory of denoting. Geach argued that Russell's theory is formally inadequate. I will argue, first, that it is also materially inadequate, and, second, that Dau's interpretation cannot make sense of Russell's view of the variable.

In 5.3 I side with Makin's assessment of Russell's Gray's Elegy argument as central to his intentions in 'On denoting'. I argue that Frege would not be moved in the same way as Russell either by the Gray's Elegy argument or by his theory of descriptions. In particular, both Russell's theory of denoting and his theory of descriptions are simply orthogonal to Frege's theory of sense.

5.1 Denoting concepts

As we saw in chapter 3, Russell distinguished the following four propositions about Socrates: 'Socrates is human', 'Socrates has humanity', 'Socrates is-a man' and 'Socrates is a-man' (cf. 1903a §57). The last two disambiguate 'Socrates is a man'. All four involve the concept of humanity in some mode of occurrence, or some concept derived from it.

The first three propositions are (logically) equivalent. The first one, 'Socrates is human', is a subject-predicate proposition: *humanity* occurs there as a concept, and so the proposition is only about Socrates. The second one, 'Socrates has humanity', is a relational proposition: *humanity* occurs there as a term, and so the proposition is *both* about Socrates *and* about humanity, even if it is *not* about both Socrates and humanity. It could be expressed in the form 'Humanity characterizes Socrates'.

The third proposition, 'Socrates is-a man', also expresses a relation: that between Socrates and the class-concept *man*. The class-concept *man* is closely associated with the concept *humanity*, as it determines the class of men, i.e., the class of things that *are human* or that *have humanity*. In general, a class-concept is a concept that determines a class constituted by the terms that have the corresponding predicate. 'Socrates is-a man' is like 'Socrates has humanity' in so far as it is a relational proposition, but it is like 'Socrates is human' in so far as it is not about *humanity*.

According to Russell, the fourth proposition is *not* equivalent to the other three. Of course, extensionally, they are logically equivalent. But while the first three propositions are atomic, the fourth involves generality. 'Socrates is a-man' is relational, but it does *not* express a relation between Socrates and *humanity*, and it is *not* only about Socrates. Rather, it expresses 'the identity of Socrates with an ambiguous individual' (1903a §57 fn.). Ockham's theory of predication as identity immediately comes to mind (cf. 1.6 above). It is in connection with this use of class-concepts that Russell introduces *denoting concepts*.

‘A concept *denotes*’, Russell writes, ‘when, if it occurs in a proposition, the proposition is not *about* the concept, but about a term connected in a certain peculiar way with the concept’ (1903a §56). Hence ‘[i]f I say “I met a man”, the proposition is not about *a man*: this is a concept which does not walk the streets, but lives in the shadowy limbo of the logic-books. What I met was a thing, not a concept, an actual man with a tailor and a bank-account or a public-house and a drunken wife’ (1903a §56).

Denoting concepts are concepts expressed by denoting phrases. Denoting phrases are expressions formed by attaching ‘all’, ‘every’, ‘any’, ‘a(n)’, ‘some’ or ‘the’ to a word for a class-concept (1903a §58). Hence the concept (a class-concept) that forms part of a denoting concept occurs there as a concept and not as a term, if we may extend the usage of a mode of occurrence in this way.

While denoting phrases may be, and typically are, the *grammatical* subjects of the sentences in which they occur, denoting concepts are never the *logical* subjects of the propositions that those sentences express, which is to say they are *not* what those propositions are *about*. So for instance, although grammatically the phrase ‘All men’ is the subject of the sentence ‘All men are mortal’, the proposition that it expresses is *not* about the concept *all men*, which is then *not* its logical subject.

Denoting concepts are therefore peculiar in Russell’s conception of propositions. In typical cases there is no difference between what *may* come to constitute a proposition and what it *may* be about. In general, a proposition is about the terms that constitute it as long as they occur there as terms; but *any* term may occur as a term in a proposition. However, propositions that involve denoting concepts are not about them, but what they denote. Makin aptly calls them ‘aboutness-shifters’ (1995, 2000).

It remains to observe that the relation of denotation that obtains between denoting concepts and their denotata is, as Russell might put it, not psychological but logical. The sense in which denoting concepts denote is not the sense in which people denote things via their intentional use of words. Rather, it is a peculiar sense in which ‘concepts inherently and logically *denote*’ (1903a §56). Indeed ‘meaning, in the sense in which words have meaning, is irrelevant to logic’ (1903a §51).

For Russell the significance of denoting and denoting concepts lies in their prominence in mathematics. Denoting concepts formed with ‘the’ already play an indispensable role in definition generally. But Russell contrasts the definition of concepts with the definition of terms in this regard.

> The combination of concepts as such to form new concepts, of greater complexity than their constituents, is a subject upon which writers on logic have said many things. But the combination of terms as such, to form what by analogy may be called complex terms, is a subject upon which logicians, old and new, give us only the scantiest discussion. Nevertheless, the subject is of vital importance to the philosophy of mathematics, since the nature both of number and of the variable turns upon just this point. (1903a §58)

What Russell has in mind here is the definition of classes. In particular, it is to the definition of *infinite* classes (i.e., classes with infinitely many members) that denoting concepts are indispensable:

> With regard to infinite classes, say the class of numbers, it is to be observed that the concept *all numbers*, though not itself infinitely complex, yet denotes an infinitely complex object. This is the inmost secret of our power to deal with infinity. An infinitely complex concept, though there may be such, can certainly not be manipulated by the human intelligence; but infinite collections, owing to the notion of denoting, can be manipulated without introducing any concepts of infinite complexity. (1903a §72)

Yet Russell comes to acknowledge that 'the explicit mention of *any*, *some*, etc., need not occur in Mathematics: formal implication will express all that is required' (1903a §87). By this Russell *presumably* means that for any set-theoretic statement (and so *presumably* for any mathematical statement) there is a statement from quantification theory that is equivalent to it.

Now how can Russell hold *both* that denoting concepts are especially prominent in mathematics *and* that mathematics only requires formal implications (or quantification theory more generally)? The only way for him to be consistent on this point is to show that there is at least *implicit* mention of denoting concepts in statements of formal implication.

That is indeed his intention when he asks, 'How far do these equivalences [between set-theoretic and quantified statements] constitute definitions of *any*, *a*, *some*, and how far are these notions involved in the symbolism itself?' (1903a §87). Russell's answer to the second question is: very far indeed. In fact, he claims that quantificational counterparts of set-theoretic statements do *not* provide definitions of *any*, *a*, *some*, *because* these notions are *already* involved in the language of quantifiers and bound variables.

To be sure, Russell distinguishes two kinds of variable: the *true* or *formal variable*, and the *restricted variable*. This distinction does not coincide with the distinction between *real* and *apparent* variables. The latter concerns bound and free variables respectively, the former unrestricted and restricted quantification.

On Russell's view, typical denoting concepts or phrases such as 'any *F*' concern the *restricted* variable. By contrast, formal implications concern the *true* or *formal* variable. But the restricted variable *can* be defined in terms of formal implication. Hence, Russell allows that, in the case of the restricted variable, *any*, *some* and *a* are indeed 'definable in terms of formal implication'. Hence he seems to accept the parsing of what we might call restricted quantifiers in terms of unrestricted quantifiers, that is, of 'Any *F*' as '$\forall x$ (Fx …)'.

Yet, Russell could not bring himself to give up the idea that '$\forall x$ ($Fx \rightarrow Gx$)' and 'Any *F* is *G*' are 'equivalent, though not synonymous', presumably because he took them to be *about* different kinds of variables. That seems indeed to have been the reason why he resisted Peano's now standard paraphrase of 'the *F*s are included in the *G*s' (cf. 1903a §77).

Russell almost invariably expresses formal implications in the form '*Fx* implies *Gx*'. He thus appears to avoid intentionally reading the universal quantifier as '(for) any *x*'. Yet, if so, this is less a matter of logical hygiene than of rhetorical necessity. Russell aims to *establish* that '$\forall x$' involves the notion of 'any', and so he does not presuppose the issue at least verbally.

That said, Russell offers little more than a statement of a compulsion to think that the quantifier involves 'any'. Roughly, his idea is that, given that quantification essentially involves the variable, and the variable essentially involves the notion of 'any', quantification essentially involves the notion of 'any'.

The crucial premise is the second one. There Russell seems to run into a sort of equivocation. To recap, Russell allows that 'any *F*' is definable as '$\forall x$ (Fx …)', but *then* asks if '$\forall x$' is definable as 'any *x*'. This suggests that he is still treating the true variable as in some sense restricted. Now, either 'any *x*' is just a way to read '$\forall x$', or it is not. If it is not, then by parity of reasoning 'any *x*' should be equivalent to '$\forall x$ (x is x…)', in which case it is a kind of restricted variable. But if 'any *x*' is indeed just a way to read (aloud, as it were) '$\forall x$', then there remains no question to be asked. Russell thus seems to treat 'anything' invariably like 'any thing'. But his own account of formal implication as asserting a class of implications (cf. chapter 4) could have in fact provided him with a way to regard reading '$\forall x$' as 'any *x*' as a mere *façon de parler*.

What seems to be driving Russell's thinking here is simply his view of the variable itself as a denoting concept. For Russell, the variable is itself a kind of object, or at any rate denotes a kind of *variable* object.

Now this is precisely the sort of view that Frege repeatedly stigmatized as one of the 'logical defects in mathematics' (cf. his 1898/99) and as the reason why he 'should like to ban the expression "variable"' (Frege 1914: 81). In fact, not only did Frege not share Russell's view: he vehemently and explicitly opposed it. He criticized at length Russell's account of variables in *Principia Mathematica* (including variable propositions, variable functions and propositional functions) in a letter to Jourdain of 1914. Frege's basic complaint was that, although Russell then spoke of variables as symbols, he spoke of them as symbols with an *indeterminate* meaning. But as Frege wrote,

> To me this is quite mysterious. There are no undetermined men. [...] Instead of saying that 'the meaning of this sign is not determined', one should say that 'it is not determined what meaning this sign is to have'. Before it is established what meaning a sign is to have, one must not use the expression 'the meaning of this sign', and one must say neither that 'the meaning of this sign is determined' nor 'the meaning of this sign is not determined.' (Frege 1914)

As he put it in 'What is a function?', a variable is only part of the notation for the expression of generality, and so '[s]uch an expression must be considered in context':

> Let us take an example. 'If the number n is even, the[n] cos $n\pi = 1$.' Here only the whole has a sense, not the antecedent by itself nor the consequent by itself. The question whether the number n is even cannot be answered; no more can the question whether cos $n\pi = 1$. For an answer to be given, 'n' would have to be the proper name of a number, and in that case this would necessarily be a definite one. We write the letter 'n' in order to achieve generality. This presupposes that, if we replace it by the name of a number, both antecedent and consequent receive a sense. (Frege 1904b: 287)

One might think that the point Frege is making here only concerns the restricted variable, in which case Russell might agree with it. But in fact his point is entirely general. As he adds rhetorically, 'Must not every object be definite?' (1904b: 287) It is however worth pointing out that what Frege says about the *true* variable indeed coincides with what Russell says about the *restricted* variable. For Frege, then, what Russell says about the restricted variable is *all* there is to say.

In the end, Frege dismisses the sort of conception that Russell shared as a linguistic muddle arising from confusing uses of the adverb 'indefinitely' with uses of the adjective 'indefinite':

> Of course we may speak of indefiniteness here; but here the word 'indefinite' is not an adjective of 'number', but 'indefinitely' is an adverb, e.g., of the verb 'to indicate'. We cannot say that '*n*' designates an indefinite number, but we *can* say that it indicates numbers indefinitely. And so it is always when letters are used in arithmetic, except for the few cases (π, *e*, *i*) where they occur as proper names; but then they designate definite, invariable numbers. (Frege 1904b: 288)

There is no need to belabour Frege's point, as we take it to be obviously correct. But it does have a couple of significant consequences for our story. For Russell, the notation of quantifiers and bound variables itself presupposed denoting concepts, since for him the *true* variable was also a denoting concept of sorts. But just as there can be no doubt that Russell's theory of denoting was ultimately motivated by his view of variables, there can also be no doubt that this view was completely alien to Frege.

5.2 The 1903 theory of denoting

Now what *was* Russell's theory of denoting anyway? Russell presented it as a theory of the restricted variable, but we now know that it would be applicable to the true variable as well. On Russell's theory, distinctions of scope between quantifiers are replaced by differences between denoting concepts.

His exposition has three moments. First there is what Russell calls the extensional account of denoting, according to which the difference between denoting concepts concerns the *kind* of objects denoted (1903a §59). Then there is the intensional account, according to which it concerns the *way* in which objects are denoted (§60). Finally, in a long list of examples from different branches of mathematics, Russell illustrates how the theory is supposed to apply (§61).

Geach argued that Russell's application of his theory effectively undermines its purpose. Roughly, while Russell aims to explain multiple generality without invoking the notion of scope, his illustrations reintroduce it. Dau has challenged Geach's interpretation by suggesting that Russell meant only to provide a convention by which the relative scopes of quantifiers are marked in language. But in this section I reject Dau's claim by noting that his interpretation cannot account for Russell's conception of the *true* variable.

5.2.1 The extensional account

According to Russell's extensional account, different kinds of denoting concepts denote different kinds of objects of a peculiar sort. As his intensional account shows (cf. below), these new objects can in fact be dispensed with. However, a look at the extensional account will allow us to conclude that Russell's theory is applicable only where there is room for distinctions of scope.

There are six basic types of denoting concepts: 'all *F*s', 'every *F*', 'any *F*', 'an *F*', 'some *F*' and 'the *F*'. Russell's theory concerns the first five. 'The *F*' denotes a single object, but each of the remaining ones denote an 'absolutely peculiar' combination of simple objects, which is 'neither one nor many', with the exception of 'all *F*s', which simply denotes 'many terms' (cf. 1903a §59). Respectively, they denote a *numerical conjunction*, a *propositional conjunction*, a *variable conjunction*, a *variable disjunction*, and a *constant disjunction*. These correspond in turn to the different ways in which Russell claims Brown to combine with Jones in examples (1) to (5) below. In fact, supposing Brown and Jones to exhaust the class of men, we can take each pair of sentences to express equivalent statements:

Numerical conjunction/All *F*s:

(1a) *Brown and Jones* are two of Miss Smith's suitors.

(1b) *All men* are two of Miss Smith's suitors.

In (1a), the predicate does not apply distributively to Brown and Jones, but only collectively to both: neither is singly *two* of Miss Smith's suitors. Thus (1b) denotes 'the kind of combination […] which is characteristic of classes', to which, as we might say, second-level properties such as number apply. Hence the name, *numerical* conjunction.

Propositional conjunction/Every *F*:

(2a) *Brown and Jones* are paying court to Miss Smith.

(2b) *Every man* is paying court to Miss Smith.

In (2a), the predicate distributes over Brown and Jones: each is paying court to Miss Smith. The sentence is in fact equivalent to the conjunction of 'Brown is paying court to Miss Jones' and 'Jones is paying court to Miss Jones'. Hence (2b) denotes a *propositional* conjunction.

> *Variable conjunction*/Any *F*:
>
> (3a) If it was *Brown or Jones* you met, it was a very ardent lover.
>
> (3b) If it was *any man* you met, it was a very ardent lover.

(3) involves a mode of combination that 'seems half-way between a conjunction and a disjunction'. Although it is here given by a disjunction of names, (3a) is again equivalent to a conjunction of propositions, but Russell claims that the equivalence will not hold in more complex cases. Russell draws a comparison between this case and the following tautology: $(p \vee q) \rightarrow r \equiv (p \rightarrow r) \wedge (q \rightarrow r)$. However, neither '$(p \vee q) \rightarrow r$' nor '$(p \rightarrow r) \wedge (q \rightarrow r)$' are 'half-way' between anything. Indeed, as Geach observed, there's no more reason to call (3a) a disjunction than either of those (1962 §52). Nevertheless, the predicate 'If it was … you met, it was a very ardent lover' again distributes over Brown and Jones. Hence, Russell calls it a conjunction, but a *variable* one, because, to use his mannerism, it is irrelevant which term is chosen.

Russell's discussion of (2) and (3) is in fact vitiated by a poor choice of examples. Contrast (3b) with (3c):

> (3b) If you met *every* man, they were very ardent lovers.
>
> (3c) If you met *any* man, it was a very ardent lover.

The perceived difference between (3b) and (3c) can be easily explained as a difference in scope. The quantifier is intuitively understood to have narrow scope in (3b) and wide scope in (3c). Now contrast (2b) with (2c):

> (2b) Every man is paying court to Miss Smith.
>
> (2c) Any man is paying court to Miss Smith.

Here there is no clear distinction. This can be easily explained by the fact that there is no logical vocabulary in addition to the quantifier. In that respect, the predicate '…is paying court to Miss Smith' is simple. Hence Russell's example (2) provides no support for associating 'every' with the propositional conjunction. Moreover, while the more natural replacement for 'Brown and Jones' is 'every man' as in (3a), it is (3c) rather than (3b), hence 'any' rather than 'every', that can be naturally associated with a propositional conjunction. Russell should have therefore associated 'any' with the propositional conjunction and 'every' with the variable conjunction instead. But his not having done so may have been merely due to his choice of predicates of different complexity.

Variable disjunction/An *F*:

(4a) If it was one of Miss Smith's suitors, it must have been *Brown or Jones*.

(4b) If it was one of Miss Smith's suitors, it must have been *a man*.

Russell's discussion of the *variable disjunction* is by far the most cumbersome. Russell claims that (4a) is reducible neither to a conjunction nor to a disjunction of propositions, 'except in the very roundabout form: "if it was not Brown, it was Jones, and if it was not Jones, it was Brown", a form which rapidly becomes intolerable when the number of terms is increased beyond two'. He concludes, frustratingly, that 'a man' 'denotes a variable term, that is, whichever of the two terms we fix upon, it does not denote this term, and yet it does denote one or other of them' (1903a §59).

Bostock correctly points out that (4a) is (syntactically) ambiguous between '*it must have been that* ((if you met an A, it was a_1) or (if you met an A, it was a_2))' and '*it must have been that* (if you met an A, it was a_1) or *it must have been that* (if you met an A, it was a_2)' (2009: 57, fn. 12). He concludes that the example depends upon the force of 'must', and provides an alternative which does not (2009: 51, fn. 3).

However, Bostock is a little too uncharitable in taking Russell's wording too literally. In fact, it is clear from the context that 'must' simply indicates that Brown and Jones exhaust the possibilities. Russell clearly intends (4a) to be equivalent to 'If you met one of Miss Smith's suitors, then either you met Brown or you met Jones', which is certainly *not* equivalent to 'If you met one of Miss Smith's suitors, then you met Brown, or if you met one of Miss Smith's suitors, then you met Jones', as Bostock appears to suggest.

Note that the sentence 'If you met one of Miss Smith's suitors, then either you met Brown or you met Jones' is equivalent instead both to 'If you met one of Miss Smith's suitors, then, if you didn't meet Brown, then you met Jones', and, obviously, to 'If you met one of Miss Smith's suitors, then, if you didn't meet Jones, then you met Brown'. This helps us understand Russell's 'very roundabout form' to express the variable disjunction above. (Strictly speaking, perhaps, it ought to have been: ' '…if it was not Brown, it was Jones', and '…if it was not Jones, it was Brown'.)

The variable disjunction contrasts with the *constant* disjunction exemplified by (5a), which is supposed to be 'equivalent to a disjunction of propositions, namely "Miss Smith will marry Brown, or she will marry Jones"' (1903a §59). Less helpfully, Russell adds that here 'either Brown is denoted, or Jones is denoted, but the alternative is undecided', and 'the disjunction denotes a particular one of them, though it may denote either particular one'.

Constant disjunction/Some *F*:

(5a) Miss Smith will marry *Brown or Jones*.

(5b) Miss Smith will marry *some man*.

By analogy with (2), Russell might have called the constant disjunction a *propositional disjunction* instead. His not having done so obscures the symmetry between (2) and (3) on the one hand and (4) and (5) on the other. That symmetry suggests associating the variable disjunction with an interpretation of the existential quantifier as having narrow scope in (4b) and the constant disjunction with an interpretation of the existential quantifier as having wide scope in (5b). In this case, that interpretation seems to be borne by the evidence. At least if we apply a similar test as before, the result is supported by (5) and consistent with (4).

Russell's use of the future tense in (5) allows us to read 'Miss Smith will marry…' either as a prediction, in which case the predicate is complex, or as the report of an intention, in which case it is an intentional context. Either way, it is thus possible to recognize a difference between (5b) and (5c):

(5b) Miss Smith will marry *some man*.

(5c) Miss Smith will marry *a man*.

It is indeed (5b) that reads like the propositional disjunction that (5a) is meant to exemplify.

There is no doubt that Russell should have used the same complex predicate throughout his illustrations. Yet in spite of his tortuous explanations and ill-chosen examples, Russell's principal claims are at least intelligible and minimally plausible (if corrected as suggested).

Supposing '*G*' to be complex, *G* applies distributively to *F*s in '*G*(any *F*)' and '*G*(some *F*)', but non-distributively in '*G*(all *F*s)', '*G*(every *F*)' and '*G*(an *F*)'. Hence '*G*(any *F*)' and '*G*(some *F*)' are rightly treated as equivalent to propositional conjunctions and disjunctions respectively. However, although we can consistently associate '*G*(every *F*)' and '*G*(an *F*)' with narrow-scope readings of the quantifiers, Russell never succeeds in explaining what the variable conjunctions and disjunctions that they are supposed to denote really are.

Now, '*G*' cannot be a simple predicate. We have already seen that Russell's example (2) does not in fact support associating the propositional conjunction either with 'every' or with 'any'. We can now add that this is because '*G*' will apply distributively to *F*s in *both* '*G*(any *F*)' *and* '*G*(every *F*)' if it is simple. That is, both are equivalent to a conjunction of propositions. Or at least there is no reason to suppose otherwise. The same holds of '*G*(some *F*)' and '*G*(an *F*)' *mutatis mutandis*. Hence Russell's theory is best understood as concerning complex predicates.

5.2.2 The intensional account

An odd feature of the extensional account is that the special kinds of objects that Russell introduces play no role in his explanations. It is therefore unsurprising that they disappear completely from the picture in the intensional account. In fact, this account confirms that Russell's 'absolutely peculiar' combinations of terms are entirely dispensable.

Russell now explains the differences between denoting concepts in terms of the *number* rather than the *kind* of objects they denote, and the *manner* in which they do so (cf. 1903a §60). Hence on the intensional account denoting concepts always denote simple terms, but a different number in each case and in a different way. Thus 'All *F*s' and 'every *F*' are said to denote *every F*, though the former denotes them 'taken all together', the latter 'severally instead of collectively'. 'Any *F*' denotes a single term, but in a way such that 'it is wholly irrelevant which', so that the term is said to be a *variable* one. 'Some *F*' also denotes a single object, but is then explained in terms of 'any': 'the term it denotes may be any term of the class'. As to 'an *F*', Russell just repeats that it denotes a variable disjunction, adding only that what holds of an *F* may be false of each *F*, so that it doesn't reduce to a disjunction of propositions. However, in his schematic summary of the theory, Russell explains the variable conjunction in terms of a disjunction as suggested above, and implies that 'an *F*' may after all denote a single *F*, by analogy with 'any *F*' and 'some *F*':

> In the case of a class *a* which has a finite number of terms—say a_1, a_2, a_3, . . a_n, we can illustrate these various notions as follows:
> (1) *All a*'s denotes a_1 and a_2 and . . . and a_n.
> (2) *Every a* denotes a_1 and denotes a_2 and . . . and denotes a_n.
> (3) *Any a* denotes a_1 or a_2 or . . . or a_n, where *or* has the meaning that it is irrelevant which we take.
> (4) *An a* denotes a_1 or a_2 or . . . or a_n, where *or* has the meaning that no one in particular must be taken, just as in *all a*'s we must not take any one in particular.
> (5) *Some a* denotes a_1 or denotes a_2 or . . . or denotes a_n, where it is not irrelevant which is taken, but on the contrary some one particular *a* must be taken. (1903a §61)

Although the intensional account improves little upon our understanding of variable conjunctions and disjunctions, it greatly simplifies Russell's theory. First, there are no special kinds of complex objects. Second, in most cases, those of 'any', 'some' and 'an', a single term is denoted. Hence, third, differences between denoting concepts are almost entirely explained in terms of the *way* they denote terms.

It is therefore surprising that Russell's considered view is to opt for the extensional account. He had asked, 'Is there one way of denoting six different kinds of objects, or are the ways of denoting different? And in the latter case, is the object denoted the same in all six cases, or does the object differ as well as the way of denoting it?' (1903a §59). Eventually he replied that 'whether there are different ways of denoting or not, the objects denoted by *all men, every man*, etc. are certainly distinct. It seems therefore legitimate to say that the whole difference lies in the objects, and that denoting itself is the same in all cases' (1903a §62).

Still, he conceded that 'There are, however, many difficult problems connected with the subject, especially as regards the nature of the objects denoted. […] we may doubt whether an ambiguous object is unambiguously denoted, or a definite object ambiguously denoted.' Perhaps Russell's preference for the extensional account lies in the fact that there is one respect in which the intensional account renders denoting concepts exceptional in his scheme of things. In general, for Russell words represent simple things, sentences represent complex things (propositions). If the intensional account were true, complex phrases would represent simple things.

Be that as it may, Peter Geach has taken up Russell's intensional account in order to draw an illuminating comparison between his theory of denoting concepts and Ockham's version of the theory of *suppositio*.

Geach begins by getting rid of the 'wild Realist metaphysics' (1962 §38) of Russell's extensional account altogether, and by noting that denoting concepts can be distinguished *solely* in virtue of the *way* objects are denoted (as opposed to in virtue *also* of their number). Then, just as Ockham had realized that signification was not strictly necessary to explain personal supposition, Geach eliminates denoting concepts themselves. For Russell, it is only via denoting concepts that denoting phrases denote, but Geach replaces that indirect relation of denotation with a direct relation of reference. He thus represents Russell's theory of denoting concepts as a theory of *referring phrases*. Geach acknowledges that 'Russell admittedly does not speak of different modes of reference', but his theory entails that different denoting phrases relate to some one given term differently anyhow, while each denoting phrase relates similarly to any other term (cf. 1962 §44).

Thus modified, Russell's theory can be regarded as of a piece with Medieval theories of supposition, or modes of reference. If '*A*' is a predicate and '*' is 'any', 'every', 'some' or 'a(n)', '**A*' is a referring phrase in which '*A*' may be said to refer impartially to each *A*, and '*' may be said to indicate the specific way in which it is referred to (cf. 1962 §37).

Geach then replaces Russell's propositional conjunctions and constant disjunctions with proper conjunctions and disjunctions of propositions (read: sentences), and his variable conjunctions and disjunctions with conjunctions and disjunctions of *names*.

Admittedly, the latter will look queer to the modern eye. There is indeed an asymmetry between these two ways of forming names.

On the one hand, although conjunctions and disjunctions of names are intuitively regarded as shorthand for conjunctions of propositions or predicates, as Geach observes there is a degree of arbitrariness from a grammatical point of view. Connectives such as 'or' and 'and' may be used to form expressions of some category from expressions of the same category, so that they can be construed as functions from names to names, as well as from predicates to predicates, and from sentences to sentences. Besides, complex names thus obtained may not be straightforwardly reducible to their predicative or sentential counterparts.

On the other hand, while it may be plausible to regard a conjunction of names as the subject of a collective predication, a disjunction of names should be taken to be intelligible merely for the sake of argument (cf. 1962: 92–3). At any rate Geach's disjunctions of names are certainly not worse off than Russell's variable disjunctions.

Finally, Geach replaces Russell's schematic summary with a set of truth-conditions for sentences involving four of the referring phrases as follows:

> If 'a_1, a_2, a_3, … ' is a complete list of proper names […], then:
> '*f*(an *A*)' is true iff '*f*(a_1 or a_2 or a_3 or …)' is true;
> '*f*(some *A*)' is true iff '*f*(a_1) or *f*(a_2) or *f*(a_3) or …' is true;
> '*f*(any *A*)' is true iff '*f*(a_1) and *f*(a_2) and *f*(a_3) and …' is true;
> '*f*(every *A*)' is true iff '*f*(a_1 and a_2 and a_3 and …)' is true. (1962 §49)

The rule for 'any *A*' corresponds both to *confused and distributive* (or simply *distributive*) supposition, and to Russell's propositional conjunction. The rule for 'some *A*' corresponds to *determinate* supposition, and to Russell's propositional disjunction. The rule for 'an *A*' corresponds to Ockham's rule for *merely confused* supposition, and to Russell's variable disjunction. The rule for 'every A' corresponds to Russell's variable conjunction. According to Geach, nothing like a 'conjunctive' supposition was widely recognized in the Middle Ages (§50), and so Russell's theory can in effect be thought to complement Ockham's.

Now there are no doubt a few discrepancies between Geach's truth-conditions and Russell's summary. For instance, 'all *A*s' drops out of Geach's account, and it is in fact Geach's rule for 'every *A*' that resembles Russell's line for 'all *A*s'. However it was Russell himself who noted that the numerical conjunction would be treated separately. Geach also associates 'any' rather than 'every' with Russell's own propositional conjunction, as Russell himself seems to do implicitly in his illustrations (cf. below). As Geach characteristically remarks, 'Russell's defective explanations do not count against the validity of his distinctions' (1962 §52–3).

Yet on the whole Geach's truth-conditions are faithful to the spirit if not the letter of Russell's explanations, and do seem to preserve the logical import of his theory. The rules for 'some *A*' and 'any *A*' capture the distributive behaviour of predicates, those for 'every A' and 'an A' their non-distributive behaviour. Together, they are vindicated inductively, as it were, as they generate the right truth-conditions for all of Russell's illustrations from §61, as Geach sets out to show.

5.2.3 The list of illustrations

Russell proceeds to provide a long list of 32 illustrations from mathematics that are meant to involve the 'various ways of combining terms'. For each statement about classes or series of real numbers involving at least two denoting phrases, he gives an equivalent statement involving *different* denoting phrases.

Russell sorts his examples into three groups. Groups (α) and (β) concern 'any', 'a', and 'some'; group (γ) concerns 'any', 'every', 'an' and 'some'. Group (α) shows how six possible relations between two classes arise from pairing any two of the three denoting concepts, and group (β) presents six analogous cases between series of real numbers. Group (γ) gives twenty possible relations between two classes arising from triples of any of the four denoting concepts. Here I pick an instance from each group, only for the sake of illustration. '*a*' and '*b*' stand in (α) for classes, in (β) for series of real numbers and in (γ) for classes of classes. Note that (γ4) supports interpreting 'any A' rather than 'every A' as the propositional conjunction as suggested above:

> (α) (2) Any *a* belongs to a *b*, *i.e.* the class *a* is contained in any class which contains all the *b*'s, or, is contained in the logical sum of all the *b*'s. […]
> (β) (3) Any *a* is less than some *b*, or, there is a term of *b* which is greater than all the *a*'s […]
> (γ) (4) Any term of some (or an) *a* belongs to every *b*, *i.e.* there is an *a* which is contained in the product of *b*. (1903a: 60–3)

At first sight, it might seem as if Russell is only concerned with showing how pervasive denoting concepts are in mathematics. But in fact he is at least as equally concerned with the *false* equivalences that his examples merely suggest:

> The above examples show that, although it may often happen that there is a mutual implication […] of corresponding propositions concerning *some* and *a*, or concerning *any* and *every*, yet in other cases there is no such mutual implication. Thus the five notions discussed in the present chapter are genuinely distinct, and to confound them may lead to perfectly definite fallacies. (1903a §61)

The 'perfectly definite fallacies' that Russell has in mind are scope fallacies involving multiple generality. In this respect, Russell is less specific than his Medieval predecessors. He does not formulate rules to avoid such fallacies in the terms of the theory of denoting. He no doubt could have done. But the problem would remain, that his theory, at least as presented, would have to be supplemented in order to achieve that aim. Neither of Russell's accounts concerned contexts with more than a single denoting concept. In particular, nothing so far said anything about contexts that involve propositional conjunctions and disjunctions, or variable conjunctions and disjunctions simultaneously. But Russell's illustrations do include simultaneous occurrences of both 'any' and 'some' on the one hand, and both 'every' and 'a(n)' on the other.

It was Geach who first noted (in print) that if Russell's theory is meant to apply to all cases involving more than one referring phrase, it will have to recognize their different scopes in many of those contexts. But once it does this, modes of reference become redundant. In short, the theory is bound to undermine its own motivation.

Let us first consider a case in which the theory seems to hold good. Intuitively, (6) demands that a single girl be the object of the love of each boy, while (7) allows that different girls may be so. That is precisely what the theory predicts.

(6) Every boy loves some girl.
(7) Any boy loves a girl.

Consider a domain (or, as Geach says, a small community) of two boys—Tom and John—and two girls—Mary and Kate. Applying the rule for 'every boy' in (6) gives (6a), the predicate being '…loves some girl'. Applying the rule for 'some girl' in (6a) gives (6b), the predicate being 'Tom and John love…':

(6a) Tom and John love some girl.
(6b) Tom and John love Mary, or Tom and John love Kate.

Likewise, we get (7a) from (7) by the rule for 'any boy', and (7b) from (7a) by the rule for 'a girl':

(7a) Tom loves a girl, and John loves a girl.
(7b) Tom loves Mary or Kate, and John loves Mary or Kate.

Note that if the rules had been applied in the reverse order, the result would have been the same. From (6) we would have gotten first 'Every boy loves Mary, or every boy loves Kate' and then 'Tom and John love Mary, or Tom and John love Kate'. From (7), first 'Any boy loves Mary or Kate', then 'Tom loves Mary or Kate, and John loves Mary or Kate'.

Either way, the results are congruent with the (restricted) particular quantifier being given wide scope in (6) and narrow scope in (7). Indeed, it might look as though Russell's theory was readily translatable into quantification theory. The rules for 'some' and 'any A' might seem to correspond to the particular and universal quantifiers being given wide scope, and the rules for 'an A' and 'every A' to their being given narrow scope. But so far we have only encountered cases that involve phrases the rules for which are equivalent to the quantifiers being given different scopes. But consider a case in which that is not so:

(8) Any boy loves some girl.

Applying first the rule for 'any boy' to (8) yields (8a), and then the rule for 'some girl' to (8a) yields (8b):

(8a) Tom loves some girl, and John loves some girl.
(8b) Tom loves Mary or Tom loves Kate, and John loves Mary or John loves Kate.

But applying first the rule for 'some girl' to (8) gives (8c), and then the rule for 'any boy' to (8c) gives (8d):

(8c) Any boy loves Mary, or any boy loves Kate.
(8d) Tom loves Mary and John loves Mary, or Tom loves Kate and John loves Kate.

(8b) and (8d) are obviously not equivalent. One has the form '$(p \vee q) \wedge (r \vee s)$', the other has the form '$(p \wedge r) \vee (q \wedge s)$'. Now, as Geach notes, Russell implicitly adopts a convention to the effect that, whenever 'some' occurs along with 'any', the rule for 'some' is to be applied first, with the effect that it has the widest scope. (8) can thus be unambiguously interpreted as (8d), demanding of a single girl that any boy loves her. For instance, in the following passage Russell clearly intends 'some moment' to be given wide, and 'a moment' narrow scope, relative to 'any moment':

> [...] a point lies between any point and any other point; but it would not be true of any one particular point that it lay between any point and any other point, since there would be many pairs of points between which it did not lie. […] Thus "some moment does not follow any moment" would mean that there was a first moment in time, while "a moment precedes any moment" means the exact opposite, namely, that every moment has predecessors. (1903a §60)

Incidentally this passage elucidates Russell's later claim that what is truly said of an *A* may be false of each *A*: a moment precedes any moment, but no moment precedes every moment. The same convention underlies Russell's use of any pairings of denoting phrases in his list of illustrations.

But the problem is not *just* that the convention adopted by Russell is not grounded on either of his accounts of denoting concepts. The problem is also that as soon as syntactical scope distinctions are allowed to have semantic import, they are by themselves sufficient to explain multiple generality. In particular, there is no need to recognize (independently objectionable) conjunctions and disjunctions of names. Russell's theory simply makes unnecessary distinctions. Geach's argument is in effect an argument from the indispensability of scope to the dispensability, as it were, of different modes of reference.

There is, however, a further problem. Geach showed Russell's original theory to be formally defective in that it can only be applied where scope distinctions are irrelevant, and when it is amended to cope with more complex cases, it becomes redundant. But in fact the theory is also materially inadequate.

To see why, let us go back to (6b) above. Consider its first disjunct, 'Tom and John love Mary'. In Russell's theory, 'Tom and John' is a variable conjunction, or at any rate it stands for what 'every boy' stands for in Geach's example. On Geach's reconstruction, it is at least meant to be a collective subject, or the subject of a collective predication. But as a matter of fact, 'Tom and John love Mary' is *not* a collective predication. Loving Mary is not something that Tom and John only collectively do. The sentence, that is, is equivalent to 'Tom loves Mary and John loves Mary'. But Russell's theory does *not* allow for that expansion. It would do so only if 'Tom and John' formed a propositional conjunction. Hence Russell's theory yields the wrong results even in those cases where it can be applied.

The problem that we mentioned at the end of our discussion of Russell's extensional account thus generalizes even to complex contexts.

Paolo Dau rejected Geach's interpretation and proposed an alternative that in fact avoids both objections. According to Dau (1986), Russell's theory concerns the quantificational force of different quantifier phrases in English. In particular, its sole purpose is to give a rule for distinguishing the relative scopes of such phrases (cf. Bostock 2009: 58). Namely, from widest to narrowest scope, they are ordered thus: 'some', 'any', 'a', and 'every' (1986: 143).

Like Geach, Dau ignores Russell's extensional account (1986: 142). He follows Geach in interpreting denoting phrases contextually (1986: 162), and in his correction of Russell's interpretation of 'every' and 'any' (1986: 144–5). He readily excuses Russell's few mistaken translations, and associates 'any' and 'some' with wide, and 'every' and 'a(n)' with narrow scope readings of the quantifiers (1986: 142). Dau also acknowledges that the convention that determines the relative scopes of 'some' and 'any' is nowhere explicitly stated by Russell, and can only be inferred inductively (1986: 143).

Unlike Geach, however, Dau *also* ignores the intensional account, and fastens almost exclusively upon Russell's illustrations. He thus reaches very different conclusions as to the point of Russell's analysis (1986: 161).

Dau duly recognizes that his evidence is indirect. He acknowledges that Russell never speaks of 'scope' or 'quantificational force' (1986: 144) and may not be 'completely clear on all the issues involved' (1986: 152). But he is nevertheless sufficiently impressed by the overwhelming accuracy of Russell's 32 translations to conclude that 'in virtually all cases Russell follows the rules [he has] given' and that Russell is 'remarkably sensitive' to scope distinctions (1986: 150). This leads him to provide translations of all of Russell's examples into first-order logic (1986: 146–50). He then constructs formal semantics for each of his versions of Russell's ontology, and shows that they are equivalent to standard first-order logic (1986: 152–8). Dau concludes that Russell's early theory of denoting can be reconstructed in a logically cogent manner (1986: 133).

As an actual interpretation of Russell, however, Dau's proposal is extremely implausible, and this for two reasons.

First of all, Dau's very strategy to eschew the *content* of Russell's theory as he sees it (i.e., the extensional and intensional accounts of denoting) is simply question-begging. As Bostock observes, even Dau's own translations prove nothing about Russell's theory (2009: 60). Assuming as Dau does that Russell understands perfectly all three languages in question—English, predicate logic, and class theory—it will not be at all surprising if he is able to translate back and forth between them. Russell does adopt implicitly a convention as to the scope of quantifier phrases in English, so of course one can be extracted from his practice. But his doing so is no more than a reflection of that mastery.

Bostock argues further that 'Russell *cannot* have had any notion of scope that is at all similar to ours' (2009: 61). He takes the shortcomings of Russell's formalization of quantified logic in the *Principles* as evidence that he had not yet acquired in 1903 a clear understanding of the fundamental notions even of predicate logic. If so, Dau's approach is simply a non-starter. However, the possibility remains that Russell's lack of ingenuity in devising an adequate notation may *not* have been a measure of his understanding of the notions involved.

Secondly, Dau's central claim is hard to square with the fact, which he recognizes, that for Russell 'an account of the quantifiers presupposed his theory of denoting' (1986: 152). Dau acknowledges that Russell 'concedes that the quantificational paraphrase is equivalent to the original English sentence, but he also insists that the two are not synonymous' (Dau 1986: 152), and so that his own translations cannot be intended to reveal the meanings of denoting phrases (Bostock 2009: 61).

Russell's convention can be adapted to the *true* variable, that is, 'any x', 'every x', and such like. So, for instance, we could read 'An x kills every y' as '$\exists x\ \forall y\ Kxy$', 'An x kills any y' as '$\forall y\ \exists x\ Kxy$', etc. But the problem is that it is in terms of the theory of denoting that the notation of quantifiers and bound variables is supposed to be understood rather than conversely. For Russell, it is again 'any x' (or perhaps 'every x') that denotes a *variable individual*, in at least one of the senses of his alternative accounts of denoting. But it is precisely that part of Russell's theory that Dau completely excludes from his interpretation. It would therefore seem that an explanation of Russell's account of the variable is simply unavailable to him. It is simply *not* the case that Russell's theory of denoting reduces to a convention regarding the scopes of quantifier phrases in English.

There is no evidence that Russell ever became aware of the shortcomings of his theory of denoting. In the next section we shall enquire why he abandoned it nevertheless.

5.3 The 1905 theory of descriptions

In spite of its fame as a 'paradigm of philosophy' (cf. Ramsey 1931: 263), Russell's 'On denoting' is an extremely misleading paper. For instance, a reader of the *Principles of*

Mathematics will expect it to contain Russell's rejection of his early theory of denoting. A reader of Frege will expect it to contain Russell's rejection of his theory of sense. In what follows I will try to show that these are misconceptions fostered by Russell's grossly mistaken belief that his earlier theory was 'very nearly the same as Frege's' (1905a: 480, fn. 1).

Given Russell's greater focus on definite descriptions, it is easy to miss the fact that the theory that he puts forward in 'On denoting' does not concern definite descriptions specifically, but denoting phrases more generally. Here is Russell's early informal statement of the theory:

> *Everything*, *nothing*, and *something*, are not assumed to have any meaning in isolation, but a meaning is assigned to *every* proposition in which they occur. This is the principle of the theory of denoting I wish to advocate: that denoting phrases never have any meaning in themselves, but that every proposition in whose verbal expression they occur has a meaning (1905a: 481)

Russell's statement echoes, perhaps deliberately, Frege's claim in *Begriffsschrift* that 'The expression "every positive integer" just by itself, unlike "the number 20," gives no complete idea; it gets a sense only through the context of the sentence' (1879 §9). In effect, it is Russell's *new* theory of 'denoting phrases' that is essentially Frege's.

Russell sets out to provide contextual definitions for both 'primitive' and other denoting phrases, in terms of the 'primitive notions' 'is always true' and 'it is false that'. If we change Russell's primitive notions to '$\forall x$' and negation respectively, and add conjunction and the material conditional, we get the usual (i.e., Fregean) paraphrases for what we might call unrestricted and restricted quantifiers respectively:

G(everything) = $\forall x\, Gx$

G(nothing) = $\forall x\, \neg Gx$

G(something) = $\neg\forall x\, \neg Gx$

G(all/every F) = $\forall x\, (Fx \rightarrow Gx)$

G(no F) = $\forall x\, (Fx \rightarrow \neg Gx)$

G(a/some F) = $\neg\forall \mathrm{x}\, \neg(Fx \wedge Gx)$

The exception is of course definite descriptions. Rather than treating 'the *F*' as a complex name roughly as Frege had, Russell now regards it as another restricted quantifier. According to Russell, any sentence of the form '*G*(the *F*)' expresses *existence* and *uniqueness* claims:

$$G(\text{the } F) \quad = \quad \neg\forall x \neg ((Fx \wedge \forall y(Fy \rightarrow y{=}x)) \wedge Gx)$$

As Russell says, 'the above gives a reduction of all propositions in which denoting phrases occur to forms in which no such phrases occur' (1905a: 482). The reduction 'leaves "a man", by itself, wholly destitute of meaning, but gives a meaning to every proposition in whose verbal expression "a man" occurs' (1905a: 481). More generally, it implies that 'denoting phrases never have any meaning in themselves, but every proposition in whose verbal expression they occur has a meaning' (1905a: 480).

5.3.1 The Gray's Elegy argument

The mere statement of Russell's new theory already raises a question as to the significance that that theory could have possibly had *for Russell*. We, readers of the *Principles*, know that by 1903 Russell had already found quantificational paraphrases for all of his denoting phrases save definite descriptions (though see the discussion of definite descriptions in 5.3.2).

Now, in his *Appendix* to the *Principles* on Frege, Russell had made the following remark apropos of Frege's theory of quantification:

> [Frege] recognizes also, though he does not discuss, the oddities resulting from *any* and *every* and such words: thus he remarks that every positive integer is the sum of four squares, but 'every positive integer' is not a possible value of *x* in '*x* is the sum of four squares'. The meaning of 'every positive integer', he says, depends upon the context (Bs. p. 17)—a remark which is doubtless correct, but does not exhaust the subject. (1903a: §481)

The extent to which Russell had claimed Frege to be 'doubtless correct' concerned their shared understanding of quantification theory. But the extent to which that theory did not 'exhaust the subject' concerned what Russell took to be the problem of the variable.

A reasonable conjecture as to what 'On denoting' meant for Russell might therefore be that by 1905 he had changed his mind regarding the *true* variable. Once he hit upon his theory of descriptions, Russell found a way to paraphrase away *all* denoting phrases, and not just 'all', 'any', 'every', 'some' and 'a'. One might therefore expect that he could then finally regard quantification theory as effectively 'exhausting the subject' of generality, since he no longer had any need for the notion of a denoting concept at all.

Attractive as this suggestion may seem, it is strictly speaking false. Shortly after the publication of 'On denoting', Russell confessed to Moore that all that he had achieved in 1905 was a reduction of the problem of denoting to the problem of the variable. In a letter of 23 October, Moore had asked him what sort of entity the variable was:

> I was very interested in your article in 'Mind', and ended by accepting your main conclusions (if I understand them) though at first I was strongly opposed to one of them. What I should chiefly like explained is this. You say '*all* the constituents of propositions we apprehend are entities with which we have immediate acquaintance'. Have we, then, immediate acquaintance with the variable? and what sort of entity is it?

Russell replied only a couple of days later, on 25 October.

> I am glad you agreed to my main contentions in the article on Denoting. I admit that the question you raise about the variable is puzzling, as are all questions about it. The view I usually incline to is that we have immediate acquaintance with the variable, but it is not an entity. Then at other times I think it is an entity, but an indeterminate one. In the former view, there is still a problem of meaning and denotation as regards the variable itself. *I only profess to reduce the problem of denoting to the problem of the variable*. This latter is horribly difficult, and there seem equally strong objections to all the views I have been able to think of. (Russell 1994: xxxv, emphasis added)

Russell's claim that his new theory reduces propositions in which denoting phrases occur to propositions in which they do not occur (cf. above) is in fact ambiguous. It can mean either that the new propositions do not contain *any* denoting phrases, in which case the reduction could translate into the elimination of denoting concepts, or that the new propositions do not contain the *original* denoting phrases. Russell's letter suggests that he must have had in mind the *second* of these interpretations.

By 1905, then, Russell had not yet abandoned his 1903 views that the notation of quantifiers and bound variables itself involved the notion of 'any *x*', and that the latter denoted a variable individual (cf. 5.1 above). To Russell's mind, the theory of 'On denoting' was a *reductive* rather than a genuinely *eliminative* one. Russell succeeded in eliminating denoting phrases concerning the *restricted* variable, but only by reducing them to denoting phrases involving the *true* variable. Note that even Russell's primitive denoting phrases ('everything', etc.) do not yet concern the *true* variable, only the quantifiers themselves do.

Hence from the point of view of 'On denoting', there are *two* senses in which Frege's theory of quantification did not 'exhaust the subject' of generality. One is that Frege had *not* defined in terms of the true variable everything that Russell would count as a denoting phrase involving the restricted variable, namely definite descriptions. The other is that Frege simply had no account of the *true* variable as a denoting concept.

Russell must have therefore conceived his task to eliminate denoting concepts in two stages. The first stage consisted in defining *all* denoting phrases involving the restricted variable in terms of denoting phrases involving only the true variable. This was the task that the theory of descriptions brought to completion. The second stage would consist in eliminating the true variable. To this task, 'On denoting' could contribute only indirectly, and only in so far as the first task was a necessary step towards the second.

We now have the shape of the project of 'On denoting' in view. The theory of descriptions provides a definitive answer to the first stage of Russell's aim to eliminate denoting concepts. But this leaves us with another question. If Russell still clung to his old view of variables, why did he wish to eliminate denoting concepts in the first place?

Makin has forcefully and convincingly argued that the central passage from 'On denoting', and the one that reveals Russell's true intentions in the paper, is the so called 'Gray's Elegy argument' (hereafter 'GEA'). The GEA is in effect Russell's argument against denoting concepts of any kind. It therefore contains the answer to our question.

Russell's argument has many intricate details, but it is enough for current purposes just to lay down its critical moves and fundamental claims.

We begin with the broadest characterization of the nature of denoting concepts as aboutness-shifters. To recap, for Russell, for a proposition to be about something is for that thing to be one of its constituents, namely a term occurring there as a term. That thing will be its logical subject. The qualification 'logical' is not otiose, as sometimes what grammar suggests to be the subject of a proposition is not in reality what it is about. This is the case with denoting concepts. A denoting concept is such that, whenever it occurs in a proposition, even if it is expressed by the denoting phrase that constitutes the grammatical subject of the corresponding sentence, the proposition is not about it but what it denotes.

Now, a little reflection shows that there cannot be propositions *about* denoting concepts. Again, for a proposition to be about something *is* for that thing to be one of its constituents. Hence, a proposition *about* a denoting concept would have to contain it as a constituent. But, by definition, if a denoting concept occurs in a proposition as a constituent, that proposition is *not* about it. Hence no proposition *can* be about any denoting concept that occurs in it. For instance, if our denoting concept is *every man* and our proposition '*Every man* denotes every man', and if the denoting concept occurs twice in the proposition, then that proposition is simply equivalent to 'Every man denotes every man', which is at best silly, if not nonsensical.

There is, of course, an exception to Russell's general model of aboutness, and that is precisely the case of denoting concepts. The things that denoting concepts denote are precisely *not* among the constituents of the propositions where those concepts occur, and yet they are what those propositions are about. Hence, a seemingly obvious way out of the problem would be to construct a proposition out of a *second* denoting concept, say d_2, that could be about the intended denoting concept, say d_1. In this way, the target denoting concept d_1 would *not* be a constituent of the proposition, in which case the proposition would no longer be prevented from being about it.

However, the GEA is precisely devised to show that any attempt to construct some such proposition must ultimately fail. Russell considers several alternatives, which he presumably takes to be exhaustive, and claims in each case that they do not work.

To give but one instance, a natural choice for d_2 would be a denoting concept formed with the help of the corresponding denoting phrase. Rather than saying 'Every man denotes every man', we could say 'The denoting concept (expressed by) "Every man" denotes every man', where indeed the denoting concept *every man* occurs only once. However, this proposition will not do for Russell, because 'the relation of meaning and denotation is not merely linguistic through the phrase: there must be a logical relation involved, which we express by saying that the meaning denotes the denotation' (Russell 1905a: 486). (The term 'meaning' is Russell's preferred term for denoting concepts in 'On denoting', cf. 5.3.2 below.) The problem is that, although the new proposition is indeed about the intended denoting concept, it does not express the *logical* relation that obtains between denoting concepts and what they denote.

Other cases are more complicated, but we need not enter into those. Suffice it to say that we now have Russell's *reason* to eliminate denoting concepts altogether. Yet, if his own testimony to Moore is to be trusted, he only had the means to eliminate those denoting concepts that were expressed by means of the *restricted variable*.

Or did he? The first step of Russell's project to eliminate denoting concepts was necessary, and *for Russell* it was not sufficient. But as we saw in 5.1, Russell's conception of the true variable was greatly confused. Hence that first step would have been sufficient for nearly everyone else. 'On denoting' *in fact* completes Russell's elimination of denoting concepts in spite of his own assessment. But then again it does so only because there ought to have been no such task in the first place.

Frege, who never held Russell's view of variables, would therefore not have been moved by the GEA to adopt the theory of descriptions as a means to eliminate denoting concepts. This prompts two further questions.

One is whether Frege would be moved to adopt the theory of descriptions by any other reason. This is the question that we shall consider next in 5.3.2.

The other is whether the GEA would move him in any other way. Makin thought that it should. In particular, Makin argued that the GEA revealed the incoherence of Frege's sense-meaning distinction. Two points are worth making in this regard.

There is a striking similarity between the GEA and Frege's paradox of the concept *horse*. But Frege and Russell drew very different conclusions from the latter. Whenever we try to say something about a concept, Frege thought, we end up saying something about an object (in *his* sense of a concept, i.e., a function, cf. chapter 2). For Frege, this showed only that we sometimes fail to say what we mean, or indeed *seem* to mean something when in fact we do not mean anything at all. For Russell, however, the same 'paradox' in fact became an argument *against* the essentially predicative nature of concepts, indeed *for* his own doctrine of the alternative modes of occurrence of concepts (cf. 1903a §§49, 481, 483).

By analogy, supposing that the GEA could show the impossibility of referring to a sense, Frege might just accept that impossibility, while holding on to the sense-meaning distinction. And in a way he did just that: for Frege, senses are not objects of reference *per se*, only objects of *oblique* or *indirect* reference. So as long as reference to sense is guaranteed to be parasitic upon intentional contexts, then, Frege could have been unimpressed by the

GEA. The natural way to implement this strategy would be to systematically represent two expressions (or their meanings) as having the same sense (or mode of presentation) in so far as they have can be replaced *salva veritate* in intentional contexts (cf. chapter 7).

Now Makin would object to this, his ground being that for Frege, no less than for Russell, the relationship that obtains between sense and reference has nothing *essentially* to do with language or cognition. In particular, such a strategy would be tantamount to the strategy that Russell considered and rejected of referring to a denoting concept via the denoting phrase that expresses it, thus making the relation of denoting 'linguistic through the phrase'.

However, while Frege no doubt explained the relationship between words and their meanings in terms of the relationship between their *senses* and their meanings rather than the other way around, Makin's assumption is *at best* (for see chapter 7) only plausible in the atomic case. Note that Frege's account of generality *is*, as it were, 'linguistic through the phrase': again, for him, variables are just bits of language. Not that that account involves the sense-meaning distinction. On the contrary, it is only *Russell's* account of the variable that involves the meaning-denotation distinction. *Regardless* of one's views concerning sense, what Frege's account of generality shows is that he could never have had a *general* ambition to explain thought independently of language.

5.3.2 Extensional contexts

Famously, Russell drew inductive support for the theory of descriptions from its ability to solve three logical puzzles. As he put it, 'the evidence for the above theory is derived from the difficulties which seem unavoidable if we regard denoting phrases as standing for genuine constituents of the propositions in whose verbal expressions they occur' (1905a: 482). Although Russell must have conceived his puzzles to concern the restricted variable, his argument would have been enough for those who, like Frege, did not share his view of the variable.

Two of Russell's puzzles concern extensional contexts, one concerns intentional contexts. They are the problem of empty terms, the problem of negative existential claims, and the problem of non-trivial identities. In each case, Russell's strategy was to distinguish the *scope* (Russell's term) of the description relative to the context C where it occurs. Since uniqueness will never be at stake, and given the equivalence '$\exists x = \neg\forall x \neg$', let us make the following abbreviations:

$$\exists! x\, Fx \quad = \quad \exists x\, (Fx \wedge \forall y (Fy \rightarrow y{=}x))$$

$$G(\text{the } F) \quad = \quad \exists! x\, (Fx \wedge Gx)$$

On this analysis of 'G(the F)', there are multiple ways for a context C to embed or be embedded in it. Russell singles out two for attention. We may further define 'CG(the F)' and '$C(G$(the F))' as follows:

$$CG(\text{the } F) \quad = \quad \exists! x\, (Fx \wedge CGx)$$

$$C(G(\text{the } F)) \quad = \quad C(\exists! x\, (Fx \wedge Gx))$$

In 'CG(the F)' the description has what Russell calls a *primary occurrence* or wide scope, and in '$C(G$(the F))', it has a *secondary occurrence* or narrow scope.

Here is a statement of each of the puzzles:

Empty terms. Either the King of France is bald, or the King of France is not bald. (*Tertium non datur*.) Suppose that he is. An assertion about the King of France presupposes his existence. Since France is a Republic, there is no King of France, and so it is *not* true that the King of France is bald. Now suppose that he is not bald. Again, an assertion about the King of France presupposes his existence. But there is none, and so it is also *not* true that the King of France is *not* bald. This contradicts the principle that either a sentence or its negation must be true.

Negative existential claims. France is a Republic, we said, so there is no King of France. The King of France does not exist.[22] Again, this is an assertion about the King of France. Hence it should follow by existential generalization that the King of France exists. Contradiction. In general, to deny of something that it exists must always be self-contradictory.

Non-trivial identity. George IV wondered whether Walter Scott was the author of *Waverley*, which, indeed, he was. But presumably the same George IV *knew* that Scott was himself, and so did *not* wonder whether he was, in fact, himself. Now how can these two wonderings be distinct? According to 'Leibniz's law', co-referential terms may be

[22] Russell's actual example is unnecessarily complicated: the difference between A and B, when A=B.

interchanged *salva veritate*. Hence 'Walter Scott' may substitute 'the author of *Waverley*' in 'George IV wondered whether Walter Scott was the author of *Waverley*' so as to give 'George IV wondered whether Walter Scott was Walter Scott'. It therefore seems that there is no sense in which 'George IV wondered whether Walter Scott was the author of *Waverley*' can be true.

In the first two puzzles, the definite description occurs with negation. If we replace C with a sign for negation in the schema above, we have the following alternative:

$$\sim(G(\text{the } F)) \quad = \quad \sim\exists!x\,(Fx \wedge Gx)$$
$$\sim G(\text{the } F) \quad = \quad \exists!x\,(Fx \wedge \sim Gx)$$

We can call the first form an *external* negation and the second an *internal* one. If we now interpret 'The King of France is bald' as having the form '$\exists!x$ (x rules France $\wedge$ x is bald)' we face a similar choice regarding its negation. If we interpret 'The King of France is *not* bald' as its *external* negation, i.e. '$\sim\exists!x$ (x rules France $\wedge$ x is bald)', then '$\exists!x$ (x rules France $\wedge$ x is bald) $\vee$ $\sim\exists!x$ (x rules France $\wedge$ x is bald)' is an instance of '$p \vee \sim p$', but there is no puzzle because '$\sim\exists!x$ (x rules France $\wedge$ x is bald)' does not entail the existence of anything. But if we now interpret 'The King of France is *not* bald' as the *internal* negation of 'The King of France is bald', then *both* '$\exists!x$ (x rules France $\wedge$ x is bald)' *and* '$\exists!x$ (x rules France $\wedge$ $\sim$(x is bald))' entail the existence of the King of France. But now their disjunction is *not* an instance of '$p \vee \sim p$' and so, again, there is no puzzle.

In a similar way, if we interpret 'The King of France exists' as '$\exists!x$ (x rules France $\wedge$ x is bald)', we can distinguish between its *external* negation '$\sim\exists!x$ (x rules France $\wedge$ x exists)' and its *internal* negation '$\exists!x$ (x rules France $\wedge$ $\sim$(x exists))'. But it is only the latter that is self-contradictory. The puzzle disappears as negative existential claims may now be interpreted so as to be true. It is one thing to say *of* the King of France that he does not exist, quite another to say that nothing rules over France (or more than one thing does).

In short, Russell's puzzles are solved by noting that '$\sim G$(the F)' is false but does not contradict 'G(the F)', and that although '$\sim$(G(the F))' does contradict 'G(the F)', it is not false.

These two puzzles are certainly related, but the puzzle about negative existential claims is not so much directed against Frege as it is against Meinong. Russell claims that

Meinong's view on empty names entails a breach of the law of contradiction. He characterizes it as regarding 'any grammatically correct denoting phrase as standing for an object' (1905a: 482). According to Russell's Meinong, objects divide into existing and subsisting ones, so that some objects do not exist, but still subsist. 'The King of France' denotes the King of France—but since there is at present no King of France, he does not exist and so subsists. It is clear how Russell's solution escapes this result.

Russell then turns to the Fregean alternative based on the distinction between sense (Russell: meaning) and meaning (Russell: denotation). As if anticipating a later discussion, Russell shifts from talk of 'standing for' to 'aboutness' in connection with Fregean meaning. Russell writes that Frege distinguishes 'in a denoting phrase, two elements, which we may call the *meaning* and the *denotation*' (1905a: 483). Here he is again imprecise: on Frege's view, this is true only of definite descriptions. Still, Russell reports Frege's view correctly when he writes that, according to Frege, an expression such as 'the King of England' has both a sense and a reference, and so, by parity of form, so should 'the King of France'. But since there is no King of France, 'the King of France' at least has no reference. Russell concludes that 'the King of France is bald' must be nonsense for Frege.

This does not quite follow. In 'On sense and meaning', Frege allowed expressions to have sense but no meaning, in which case 'The King of France is bald' would be meaningful, but, perhaps, lack a truth-value. In *Grundgesetze*, however, Frege followed a different course. There he adopted a convention according to which sentences with definite descriptions in subject position come out true when uniqueness fails, and false when existence fails. Russell complained that 'this procedure, though it may not lead to actual logical error, is plainly artificial' (1905a: 484). But with respect to the puzzle of empty terms, Frege's convention had exactly the same effect as systematically interpreting the negation of '*G*(the *F*)' as *external.*

A better argument would have been for Russell to stress the parity of form between definite descriptions and similar denoting phrases. There is no question whether 'The *F* is *G*' is *somehow* committed to the existence of *F*s. But there *may* be a question whether 'The *F* is *not G*' *is* so committed. If it is not, then it is an external negation of '*G*(the *F*)'. If it is, then it is an internal one. But '*G*(the *F*)' is not unlike '*G*(some *F*)' in this respect. There is no question whether 'Some *F* is *G*' is committed to the existence of *F*s, but there *might* be a question whether 'Some *F* is *not G*' is so committed. Now for Frege, as indeed for Aristotle, 'Some *F* is not *G*' is not necessarily committed to the existence of *F*s, but it *may* be. Hence, 'Some *F* is *G*' has both an external and an internal negation, or contradictory and contrary statements. 'Some *F* is *G*' does not merely presuppose the existence of *F*s, but actually entails it. And so might 'The *F* is *G*'. Indeed, from Frege's point of view, Russell's analysis of definite descriptions looks more like the extension of a familiar strategy to a new, similar case, than the adoption of a radically different theory.

It is worth emphasizing in this connection that although 'the *F*' becomes a quantifier on Russell's analysis, it is a *defined* quantifier. Definite descriptions are indeed analysed in terms wholly available to Frege. A Fregean language (with his *Grundgesetze* description operator) is in fact able to express everything that a Russellian language is able to express, though not conversely. A Russellian language is essentially expressively weaker than a Fregean one. Frege's description and abstraction operators are both undefined in *Grundgesetze*. But it is only the abstraction operator that introduces reference to new objects in the language. Were it not for their different philosophies of arithmetic, then, the disagreement between Frege and Russell over empty names would have been relatively superficial.

5.3.3 Intentional contexts

Things are very different when we come to the third puzzle, which is a version of Frege's puzzle about identity. Formally, Russell's solution is identical to the puzzles about negation. The question is to what extent it consists in an alternative to Frege's theory of sense.

In the puzzle about George IV's curiosity, the definite description is embedded in the expression of the report of a wondering. If we interpret 'The author of *Waverley* is Scott' as '$\exists!x\ (Ax \wedge x{=}s)$', letting '$W$' abbreviate 'George IV wondered whether' leaves us with the following alternative:

$$W(S(\text{the } A)) \quad = \quad W\ (\exists!x\ (Ax \wedge x{=}s))$$

$$WS(\text{the } A) \quad = \quad \exists!x\ (Ax \wedge W\ (x{=}s))$$

Hence 'George IV wondered whether the author of *Waverley* is Scott' is ambiguous between these two interpretations. Russell now tries to solve the puzzle as follows. There is indeed a sense in which 'George IV wonders whether the author of *Waverley* is Scott' cannot be true, and a sense in which it can.

On the interpretation of 'George IV wondered whether the author of *Waverley* is Scott' in which the description has a primary occurrence (or wide scope), the sentence is equivalent to the result of applying Leibniz's law to it, and so must be false. That is, '$\exists! x$ (x wrote *Waverley* ∧ *W* (x = Scott))' is meant to be equivalent to '*W* (Scott = Scott)'.

It is therefore the interpretation in which the description has a secondary occurrence (or narrow scope) that adequately expresses George IV's curiosity. That is, it is '*W* ($\exists! x$ (x wrote *Waverley* ∧ x = Scott)' that expresses the sense in which 'George IV wondered whether the author of *Waverley* is Scott' may be true.

Now, Russell's interpretation of the primary occurrence of the description, i.e., '$\exists! x$ (x wrote *Waverley* ∧ *W* (x = Scott))', raises two problems.

Russell rightly claims that Leibniz's law cannot be applied there, since that law concerns singular terms, and 'the author of *Waverley*' is now a quantifier. However, he allows the substitution of the *variable* for 'Scott' in '*W* (x = Scott)'. It therefore becomes unclear why similar substitutions within the same context are not allowed, in particular in '*W* ($\exists! x$ (x wrote *Waverley* ∧ x = Scott)'.

At the same time, Russell claims that '$\exists! x$ (x wrote *Waverley* ∧ *W* (x = Scott))' would be true in a context where George IV saw Scott at a distance and asked, 'Is that Scott?' However, *that* sentence does *not* express George IV's curiosity in *that* situation, for it cannot express his curiosity in *any* situation. As Russell meant it to be interpreted, the sentence is equivalent to '*W* (Scott = Scott)'. In that situation, George IV's curiosity could be expressed by, say, '*W* ($\exists! x$ (x is at a distance ∧ x = Scott)', but here again the description has a secondary occurrence.

It is therefore far from clear whether Russell succeeded in providing *any* interpretation of 'George IV wondered whether the author of *Waverley* is Scott' in which it *can* be true.

Note in addition that Russell's account would in any case be inapplicable to 'George IV wondered whether Scott wrote *Waverly*', where there are simply no definite descriptions. As a matter of fact, Russell offered *no* explanation as to why 'Scott' itself, or any other *simple* expression, might *not* be replaced by *any* singular term in *any* context. By contrast, Frege's alternative proposal based on the sense-meaning distinction was not so constrained (cf. chapter 7).

This was no accident. For Russell, there are *no* such failures of substitution, no such exceptions to Leibniz's law, genuine or *prima facie*. That is to say that for him there are no *intensional* contexts, even apparently. In particular, for Russell, intentional contexts, with a 't', are not intensional, with an 's'. This is why Russell's solution to Frege's puzzle could be *indifferent* to the fact that '*W*' was an intentional context at all, and in effect treated it exactly like negation.

Hence, unlike Frege, Russell could not have taken, as it were, intensionality as the characteristic or defining mark of the intentional. In that sense, Russell's theory of descriptions and Frege's theory of sense were *not* alternative theories about the *same* sort of phenomenon. Rather, they had radically different conceptions of what that phenomenon consists in. In the next, final couple of chapters, we will try to provide a sense of how far their differences went in this regard.

6 Propositions

During the fifteen years between 1903 and 1918, Russell proposed and defended two alternative theories of judgement. Between 1903 and 1906/7, he held a dual-relation theory of judgement according to which judgement is a binary relation between a mind and a proposition. Russell then abandoned it for the multiple-relation theory that Wittgenstein would criticize in the *Tractatus*.

The two theories are often contrasted as radically distinct. In this chapter, I emphasize their continuity, and argue that it is what is common to both that is ultimately responsible for their downfall.

I introduce Russell's dual-relation theory of judgement in 6.1. In 6.2 and 6.3 I argue that the problem of falsehood is best understood in terms of Russell's realization that his early theory would not be able to jointly satisfy different requirements that a theory of judgement ought to satisfy. I then introduce Russell's multiple-relation theory of judgement in 6.4 and clarify its relationship to the former theory in 6.5. Sections 6.6 and 6.7 concern Russell's attempts to define truth. In 6.8 and 6.9 I side with Johnston's (2012) and Sullivan and Johnston's (2018) accounts of Wittgenstein's and Ramsey's objections to the multiple-relation theory. I conclude that, on *both* of Russell's theories, it is because judgement is conceived as a relation that it is formally unsuited to express *that* something is the case.

6.1 Judgement as a dual relation

Someone's judging that something is so is an event, a mental act: *that* she so judges, a fact. Russell's theories of judgement concern the form of such facts.

Russell's first theory was the dual-relation theory introduced in the *Principles of Mathematics*. He developed it further in his 1904 review of 'Meinong's theory of complexes and assumptions' and in a 1905 manuscript that remained unpublished titled 'The nature of truth'. But as Sullivan and Johnston note (2018: 4), the elements of the second theory appear as early as the 1904 review. In fact, Russell's discomfort with the early theory goes further back and shows already in the *Principles*.

Russell's second, multiple-relation theory was first properly sketched in a paper of 1906/7 for the *Aristotelian Society*, and by 1910, in a reworking of the third section of the same paper into a chapter for *Philosophical Essays*, Russell had fully endorsed it. He reformulated it first in 1912 for the *Problems of Philosophy*, following objections by G. F.

Stout, and then again in 1913 after he had changed his views about relations. However, as he famously abandoned his manuscript for *Theory of Knowledge* as a result of criticisms by Wittgenstein, the 1913 version of the multiple-relation theory remained unpublished.

According to Russell's first theory of judgement, the fact that one judges something is a *relational complex*. As we saw in chapter 3, terms combine into simple propositions, and simple propositions combine into complex ones. A judgement complex is in turn the combination of a term and a proposition. Judgement is therefore a relation between a judging subject (or mind) and the proposition that is the object of her judgement. A judgement complex thus has the form $J(s, p)$, where J is a simple two-place relation, s is a simple term, and p a complex term.

For Russell, the truth-value of a judgement is derived from the truth-value of the proposition judged. Propositions, that is, are the primary bearers of truth. A subject's judgement that *p* is true if, and only if, the proposition *p* is true. Russell's judgement that Caesar died, say, is true if the proposition *Caesar died* is true, and false otherwise.

Russell uses (apparently indifferently) both locutions 'the proposition *p*' and 'the proposition *that p*' in formulating his views. This is no accident. The letter '*p*' is schematic for a sentence in 'the proposition that *p*', and for a complex term in 'the proposition *p*'. Which role is sanctioned by Russell's understanding of propositions is precisely one of the points at issue. We should therefore remain neutral on this point when reporting Russell's views.

A proposition, we must remember, is not just a complex to which a mind is related in a judgement complex: it is a complex made up of the constituents of reality. There is, on Russell's view, no gap between thought and reality: no gap, as it were, between the content of one's mind and what makes up the world. A fact, according to him, is just what the proposition judged is if what is judged is true: it is, that is, a true proposition.

Russell thus combined a dual-relation theory of judgement with an identity theory of truth. But to say that he did this may be misleading, since it suggests that he could have chosen otherwise. In fact, both his theory of judgement and his theory of truth derive from his conception of propositions. Again, propositions are combinations of terms, which make up the world—and therefore also what a particular kind of term, a mind, is related to in judgement.

On this view, the relation between mind and world is, from a logical point of view, just like any other relation between two terms. For instance, the spatial relation that obtains between myself and my desk when it is in front of me would be of the same kind as the relation that obtains between my desk and my thinking of it, as I think that it is front of me.

This comes out in Russell's characterization of his *identity* theory as a *correspondence* theory of truth: 'The view that truth is the quality of belief in facts, and falsehood the quality of other beliefs, is a form of the correspondence theory, i.e., of the theory that truth means the correspondence of our ideas with reality' (1906/7: 45). Indeed, although in a highly peculiar sense, identity is a particular kind of correspondence: the correspondence between something and itself.

Russell's world is thus completely constituted independently of judgement. The world does contain judgement complexes, since there happen to be judging minds: but only accidentally so. The contents of their judgements are at any rate prior to them. Minds attach to facts when they 'judge them' (and if they didn't, they wouldn't). The relationship between minds and the objects of their judgements is therefore external to both. Such is Russell's picture of cognitive relations generally.

Now, if a fact is a true proposition and a proposition is as it were a bit of the world, it might seem as if a fact just *is* that bit of the world, and hence that a true proposition is just a proposition. Russell did not quite say this, though. For him, truth was a simple and indefinable property of propositions, which are facts when they have it.

The problem nevertheless arose that what *seemed* to make up the content of a judgement also *seemed* to make it true, in which case no judgement could be false. Russell did not address the problem of falsehood explicitly in the *Principles*, but his discussion of logical assertion clearly foreshadowed it.

Russell sometimes took logical assertion to be that quality which distinguished true propositions from false ones:

> The question is: How does a proposition differ by being actually true from what it would be as an entity if it were not true? It is plain that true and false propositions are entities of a kind, but that true propositions have a quality not belonging to false ones, a quality which, in a non-psychological sense, may be called being *asserted*. (Russell 1903a §38)

At the same time, he implied that logical assertion was that in virtue of which something is constituted as a proposition. The 'verb, when used as a verb, embodies the unity of the proposition' (1903a §54), but that 'ultimate notion of assertion, given by the verb […] is lost as soon as we substitute a verbal noun' (1903a §52). It would seem to follow, not only that only true propositions are logically asserted, but also that only true propositions are logically unified, i.e., propositions at all.

Logical assertion thus seemed to perform a double role in the *Principles*, as it was meant to account *both* for the unity *and* for the truth of propositions. Such a view would immediately imply that there could be no ('unified') false propositions. As Russell acknowledged, there are indeed 'grave difficulties in forming a consistent theory on this point' (1903a §38). These 'grave difficulties' would soon give way to outright incoherence.

In 1906/7 and 1910 Russell considered an objection to his early theory that would eventually lead him to abandon it. He expressed it in the form of a dilemma: 'The difficulty of the view we have been hitherto considering was that it compelled us either to admit objective falsehoods, or to admit that when we judge falsely there is nothing that we are judging' (1910: 177). On the dual-relation theory of judgement, then, we are bound to recognize either that there are 'objective falsehoods' (which Russell also called 'fictions' and 'non-facts'), or else that false judgements have no objects.

This became known in the literature as the problem of falsehood. Each horn of the dilemma raises a specific issue, concerning which Russell was never entirely explicit. We would therefore do well to discuss them separately. Somewhat arbitrarily, we can call the first horn 'the problem of false propositions', the second 'the problem of false judgements'. Let us begin with the latter.

6.2 False judgements

As the *Principles* half-anticipated, the existence of false propositions is not easy to accommodate within Russell's conception of propositions as fact-like entities, or complex terms. There, we saw, Russell came dangerously close to a view on which what constituted something as a proposition also made that proposition true—or at any rate on which a fact seemed to be both what a true proposition is, and the complex term that is the proposition.

The problem of false judgements begins with the supposition, natural within this framework, that fact, complex and proposition are indeed the same. If a judgement that *p* is false, *p* is not true, and so there is no fact *p*. But where there is no fact, there can be no complex, and so no proposition to be judged in the first place. 'When we believe truly,' Russell wrote, 'our belief is to have an object which is a fact, but when we believe falsely, it can have no object' (1906/7: 46).

The problem begins, but does not end, with that supposition. Russell does *not* (immediately at any rate) infer from this assumption that there would be no false judgements, or that since there are only true propositions, there could only be true judgements. Rather, he says that false judgements would have no objects. And according to him the problem with this is that it would introduce an 'intrinsic difference' between true and false judgements, with the effect that the truth of any judgement would be discoverable *a priori* just by analysing its nature:

> We cannot maintain [the view that judgments consist in a relation to a single object] with regard to true judgments while rejecting it with regard to false ones, for that would make an intrinsic difference between true and false judgments, and enable us (what is obviously impossible) to discover the truth or falsehood of a judgment merely by examining the intrinsic nature of the judgment. (Russell 1910: 177)

Now, why should this follow? What Russell seems to envisage here is a sort of disjunctivism about judgement. On that view, a true judgement would be a (dual) relation between a mind and a fact. In the absence of a fact, no such relation would hold, and so a false judgement would have to be, for instance, a property of a mind. But why should knowledge of the form one's judgement takes on particular occasions be subjectively available? After all, for all that Russell had told us so far, one might perfectly well *merely think* that one is judging truly when one is in fact judging falsely.

The question, in a way, answers itself. Russell demands that intrinsic features of a judgement be subjectively available, and that among them be its form. But rather than an arbitrary demand, this is simply a reflection, within Russell's theory, of the requirement that the *content* of a judgement be available to the one who judges, and that it be possible for her *not* to know whether her judgement is true. (Indeed, typically, whenever one's judgement is not trivial, for all one knows, one's judgement may turn out to be false.) But this imposes an obvious constraint on what an object of judgement may be if it is to be properly regarded as a content. It is not enough that judgements have objects, or that they may be true or false: the content of a judgement may remain the same, even if its truth-value does not. In other words, the content of a judgement is internal to it: its truth-value is not.

Russell was thus right *both* to identify the content of a judgement with the condition the satisfaction of which would make it true, *and* to recognize that the truth-value of a judgement may be contingent to it. We should indeed like to say that judgement is internally related to the *possibility* of its being true, but not to its *actual* truth.

But what he came to realize was that the sort of thing which according to the dual-relation theory are the objects of judgement could not satisfy these two conditions. On Russell's theory, the condition of the truth of a judgement is *both* its object *and* what makes it true. Any difference between a truth-maker and a truth-bearer therefore collapses. But his conception of propositions as facts or complex terms meant that their truth would simply amount to their existence. Since propositions were the primary bearers of truth, that meant that no judgement could have its truth-value contingently. And since they were also the objects of judgement, that also meant that one could not know what one judged without also knowing that one judged truly.

6.3 False propositions

Let us now turn to the first horn of the dilemma. False judgements have no objects, Russell adds, 'unless there are objective non-facts': 'The people who believe that the sun goes round the earth seem to be believing *something*, and this something cannot be a fact. Thus, if beliefs always have objects, it follows that there are objective non-facts' (Russell 1906/7: 46). While true judgements relate minds and facts, then, false judgements relate minds and complexes of another sort. Disjunctivism about judgement thus gives way to disjunctivism about the *objects* of judgement. Judgement is always a relation between a mind and a proposition, but propositions divide into two kinds: true and false.

Russell argued inductively both for and against false propositions. On the one hand, logical complexity seemed to require their existence. Certain true propositions, he said, 'contain false propositions as constituent parts'. For instance, 'either the earth goes round the sun, or it does not' is true and is apparently compounded of two constituents, yet one of them must be false (1906/7: 48). On the other hand, treating propositions generally as 'single things' leads to paradoxes such as that of the liar (cf. 1906/7: 46). But Russell's eventually decisive reason for rejecting false propositions was simply that their existence 'is in itself almost incredible: we feel that there could be no falsehood if there were no minds to make mistakes' (1910: 176).

As Cartwright observed, Russell's change of heart in this regard is rather extraordinary (cf. his 1987). After all, he had countenanced these 'false Objectives', 'fictions' or 'non-facts' all along. And it is not as if Russell had found a new argument against their existence: rather, he now simply 'stared incredulously' at them. As Sullivan and Johnston remark, 'Plainly, someone's sense of reality can change': 'The early theory's propositions were, we should recall, *abstract* entities. In Russell's rejection of them they have become concrete entities mysteriously lacking in substance—"curious shadowy things" [(Russell 1918: 55)], the ghosts of departed facts' (Sullivan and Johnston 2018: 7).

Cartwright argued that Russell's rejection of false propositions should be understood in light of the problem of propositional unity (1987: 82–4). This is correct as far as it goes. As we saw in 6.2, Russell did come to think that the existence of false propositions was simply unintelligible within his framework. What this framework renders 'unintelligible' is not '*what* the difference [between true and false propositions] consists in, but *how there can be* any difference' (Sullivan and Johnston 2018 §2.2).

However, if that were all there is to be said, the first horn of the dilemma would simply collapse into the second: if false judgements had fictions as objects, then, since there are no fictions, they would have no objects. But in fact Russell seems to have recognized a further difficulty associated with the admission of false propositions.

Let us then, for the sake of argument, leave to one side the question of how there *could* be false propositions. On the assumption that there *are* false propositions, true and false propositions form a disjunctive class. 'If we accept the view that there are objective falsehoods,' Russell writes, 'we shall oppose them to facts, and make *truth* the quality of facts, *falsehood* the quality of their opposites, which we may call fictions. Then facts and fictions together may be called *propositions*' (1906/7: 48). For Russell, it would indeed be a mistake to say that the *same* proposition could be, as it were, true at some worlds, false at others:

> There is some difficulty in avoiding a fallacious form of question, namely: How does a proposition which is true differ from what *it* would be if it were false? We cannot rightly put the question in this form, because if it is true it is true, and it is not possible that *it* should be false. The only reason it seems possible is that we so often do not know whether a proposition is true or false, so that, in a subjective sense, either alternative is possible. (Russell 1905b: 503–4)

The 'fallacious form of question' was of course precisely the one he had posed in 1903. Although Russell's propositions may be true or false, they are not, as Wittgenstein might put it, bipolar: it would not have been possible for a true proposition to be false and vice-versa.

Now, truths and falsehoods would form a disjunctive class if, for instance, truth and falsehood were among their constituents. Again, in the *Principles* Russell seemed to have this sort of view in mind when he wrote that, while an unasserted proposition 'has an external relation to truth or falsehood (as the case may be)', an asserted one 'in some way or other contains its own truth or falsehood as an element' (1903a §52). But if that were so, we would again be in possession of a criterion by which to tell *a priori*, by mere analysis, whether a judgement was true, this time by inspecting the constituents of its object (cf. 1910: 173):

> If truth and falsehood were respectively constituents of true and false propositions, we could tell by inspection (at least as a rule) whether a proposition is true or false. […] Generally, if true and false propositions differ in any property, this property must consist in the presence or absence of some relation to something else, not in an intrinsic quality discoverable by analysis; for if there were a difference in any intrinsic quality, we could, as soon as we new this quality, discriminate true from false propositions by mere analysis, which in general is plainly impossible. (Russell 1905b: 504)

Consistently with his identity theory of truth, Russell assumes that there is no gap between what constitutes an object of judgement and what is judged. If truth and falsehood were constituents of propositions, they would be among the constituents of the object of one's judgement. One could then know whether one judged truly by knowing whether one judged a truth: and one would know whether one judged a truth by knowing what it was that one judged. But again, one *may* not know whether one judges truly just by knowing what one judges. Therefore, truth and falsehood cannot be constituents of propositions.

Russell inferred from this that truth and falsehood would have to be 'ultimate, and no account can be given of what makes a proposition true or false' (1906/7: 48). He therefore concluded that truth and falsehood were 'brute' properties of propositions. As he put it, 'some propositions are true and some false, just as some roses are red and some white' (1904a: 523).

The problem now, Russell thought, was that accepting the existence of false propositions, alongside and metaphysically on a par with true ones, would seem 'to leave our preference for truth a mere unaccountable prejudice' (1904a: 523). Or as he later put it, in addition to the incredibility of objective falsehoods, the dual-relation theory 'has the further drawback that it leaves the difference between truth and falsehood quite inexplicable' (1910: 176). Russell's quick remarks here are rather opaque. Sullivan and Johnston interpret these words as pointing towards the unintelligibility of false propositions (2018 §2.2), but, as we saw, doing so would mean falling back into the other horn of Russell's dilemma.

At one point in his review of Meinong, Russell is a bit more explicit. The problem seems to concern the normativity of truth for judgement (or belief):

> as for the preference which most people [...] feel in favour of true propositions, this must be based, apparently, upon an ultimate ethical proposition: 'It is good to believe true propositions, and bad to believe false ones'. This proposition, it is to be hoped, is true; but if not, there is no reason to think that it we do ill in believing it. (Russell 1904a: 524)

Russell, then, again rightly, recognizes that, as we might say, judgement aims at truth. But nothing on the picture we have been considering allows him to say that a *good* judgement is a *true* judgement. Truth, he implies, is in *some* sense normative for judgement: but he has nothing better to offer than a prescription to believe true propositions, rather than false ones. By itself, however, a true judgement is just as good as a false one.

On this understanding of propositions, then, judgement does not *internally* aim at truth. Indeed if it can be said to aim at anything at all, we would have to say that it aims *indifferently* at truth and falsehood. Judgements achieve what they are set up to achieve—namely, attaching to a complex—whether or not that complex is a true proposition. Supposing a true proposition to be the condition of the truth of a judgement, then it is *not* also *the* condition of its success.

To recap, we began by characterizing Russell's conception of cognitive relations as external relations between minds and propositions as an aspect of his realism in 6.1. Next we identified two features of judgement that Russell deemed to be essential to it. At the same time, we tried to portray the dilemma for Russell's dual-relation theory of judgement known in the literature as the problem of falsehood as exhibiting the tension between that conception and those features. We finally concluded that it is Russell's conception of cognitive relations that gives substance to the complaint that Russell's propositions are objects.

Russell correctly discerned, not only that judgements may be true or false, but that one may not know whether one's judgement is true, and that judgement aims at truth. What the problem of falsehood taught him was that these two requirements on judgement could not be jointly satisfied within his dual-relation theory.

As Russell formulated that problem, either false judgements have no objects, or there must be false propositions. Russell conceived of propositions as fact-like complex terms or objects. On one understanding, this conception makes it indeed hard to see how there could be false propositions, and hence how false judgements could have objects. But commentators have rightly been puzzled by Russell's sudden change of heart. Our interpretation attempts to make sense of the dilemmatic form of Russell's formulation of the problem.

If there were no false propositions, Russell could think that judgement aimed at truth, since the condition that would have to be satisfied in order that a judgement be true would also be its object. Hence, the condition of its truth would also be the condition of its success. By the same reason, though, a judgement would have an object if, and only if, it was true. It would therefore be impossible for one to know the object of her judgement without knowing whether it was true. On this assumption, then, the first of Russell's requirements on judgement fails to be satisfied.

If on the other hand there could be true propositions as well as false ones, the object of a judgement would be either a fact or a fiction respectively. Since truth and falsehood are not constituents of propositions, knowledge of the object of a judgement would not entail knowledge of its truth-value. The first of Russell's requirements would therefore be satisfied. But now the second one would not. The condition of the success of a judgement would now be only one of two possible objects. But then judgement would not specifically aim at one of them, namely the one that made it true.

6.4 Judgement as a multiple relation

By 1910, Russell had completely abandoned the dual-relation theory of judgement. The multiple-relation theory had its first complete sketch in 1906/7. From then on, Russell mostly talked of belief rather than judgement. The change was justified in so far as belief is to judgement what a dispositional mental state is to the act that manifests it.

Russell now sought to explain the possibility of false belief without invoking false propositions, allowing beliefs to 'depend on minds for their *existence*' but not 'for their *truth*' (1912: 75). His strategy was to turn truth into a property of beliefs rather than their objects, and to define the objects of belief independently of that property. 'Thus a belief, if this view is adopted', he wrote,

> will not consist of one idea with a complex object, but will consist of several related ideas. That is, if we believe (say) that A is B, we shall have the ideas of A and of B, and these ideas will be related in a certain manner; but we shall not have a single complex idea which

> can be described as the idea of 'A is B'. A *belief* will then differ from an idea or presentation by the fact that it will consist of several interrelated ideas. Certain ideas standing in certain relations will be called the belief that so-and-so. (Russell 1906/7: 46)

On the multiple-relation theory, then, belief is no longer a dyadic relation between a mind and a complex: rather, it is a polyadic or 'multiple' relation between the believing mind and the terms that would constitute that complex. Since different complexes may have a different number of terms, belief relations have different numbers of argument-places. And since Russell's logic does not contemplate multigrade relations, on his view belief will in fact be 'systematically ambiguous' between different multiple-relations for each number of terms in corresponding complexes.

To put it in arithmetical terms, Russell's attempt to define belief so as not to exclude the possibility of false belief was in effect to find the 'highest common factor' between true and false beliefs. Since the object of a belief is no longer something that would exist only if the belief were true, there is no longer an obstacle to false beliefs having the same objects as true ones. In the old theory, *s*'s judgement that *a*R*b* could be represented as *J*(*s*, *a*R*b*): in the new theory it would be represented as *J*(*s*, *a*, R, *b*).

Not only did Russell replace a dual-relation theory of judgement (or belief) with a multiple-relation theory, he also replaced his identity theory of truth with a correspondence theory more suitably so-called, since this time the corresponding items are distinct. When a belief is true, there is 'in reality' a complex made up of the elements of the belief, or the things that it mentions; if there is no such complex, the belief is false:

> In the event of the objects of the ideas standing in the corresponding relation, we shall say that the belief is true, or that it is belief in a *fact*. In the event of the objects not standing in the corresponding relation, there will be no objective complex corresponding to the belief, and the belief is belief in nothing, though it is not "thinking of nothing", because it is thinking of the objects of the ideas which constitute the belief. (Russell 1906/7: 46–7)

In other words, *J*(*s*, *a*, R, *b*) is true, if, and only if, there is a complex constituted by *a*, R and *b*. The primary truth-bearers are now beliefs, and complexes are invoked only as what make them true. These complexes can still be called facts, but, as it were, the other half of what was once a disjunctive class of propositions disappears. False beliefs are therefore allowed to have objects which are not fictions. While the multiple-relation theory of judgement ensures the identity of the objects of true and false beliefs, the correspondence theory of truth introduces an asymmetry between them.

6.5 Perception as a dual relation

At times Russell talked as if the multiple-relation theory of judgement eliminated propositions altogether. Propositions, he wrote, or rather 'that'-clauses, are incomplete symbols:

> It seems evident that the phrase 'that so and so' has no complete meaning by itself, which would enable it to denote a definite object as (e.g.) the word 'Socrates' does. We feel that the phrase 'that so and so' is essentially incomplete, and only acquires full significance when words are added so as to express a judgement, e.g. 'I believe that so and so,' 'I deny that so and so,' 'I hope that so and so'. (Russell 1910: 175)

Russell adds that 'if we can avoid regarding "that so and so" as an independent entity, we shall escape a paradox' (1910: 175–6). One assumes that he means the sort of liar-like paradox that he had considered under the guise of 'the man who believes that all his beliefs are mistaken, and whose other beliefs are certainly all mistaken' (1906/7: 46).

Strictly speaking, however, and especially as an account of how the new theory relates to the old one, this is misleading. The new theory eliminates propositions *as objects of judgement*, but they survive as their verifying or truth-making complexes, i.e., *as facts*. Recall that, on the new theory, the objects of judgement are the terms that constitute the complex that exists if the judgement is true: but that complex is just what the old theory called a proposition. (One indeed wonders if non-semantic but structurally similar paradoxes involving facts might not still arise.) As it occurs in such contexts as 'the fact that so and so', the symbol 'that so and so' is *not* incomplete.

Propositions also survive as the objects of perception. In perception, Russell wrote, 'the actual fact or objective complex is before the mind' (1906/7: 47). He therefore proposed a dual-relation theory of perception alongside his multiple-relation theory of judgement. Falsehood had been the *bête noire* of the dual-relation theory of judgement. But Russell thought that there is no such thing as a false perception, and hence that there could be no corresponding problem for perception. Unlike judgement or belief, perception is not 'liable to error': it is infallible. Whenever there seem to be mistakes in perception, e.g., in dreams or hallucinations, 'what is wrong is a judgement based upon the perception' (1910: 181).

Russell's distinction between judgement and perception can be generalized. Any 'infallible' mental state or act would be a dual relation between a mind and a fact: any 'fallible' one, a multiple relation between a mind and the constituents of the fact that would make it true (or otherwise satisfy or fulfil it, to cover such cases as hope). As Russell intends it, 'This distinction between perception and judgement is the same as the distinction between intuition and discursive knowledge' (1906/7: 47).

Now, according to Russell, what perception apprehends is what makes a judgement true: 'the existence of this complex object gives the condition for the truth of the judgement' (1910: 183). Russell thus hoped that combining the two theories might help explaining how perception grounds judgement: 'perception, unlike belief, apprehends the fact itself, and thus may, without being belief, be a valid ground of belief' (1906/7: 49).

Russell does not say what sort of 'ground' he has in mind. But on his theory, perception cannot ground belief in the same sense in which a belief can ground another belief, since perception and belief cannot have the same objects. That was in fact the entire point of his distinction.

As Ramsey would later show, for Russell to maintain a dual-relation theory of perception effectively undercut his own motivation for the multiple-relation theory of judgement. Ramsey imagines a scenario where *p* is the not in fact the case, but someone believes that it is, and is therefore in a position to believe in addition that someone else perceives it. Of course, both beliefs are false, since ~*p*. Ramsey agrees with Russell that perception is infallible in so far as it is *factive*: one cannot really perceive something if it is not in fact the case. In his example, '*p*' is 'The knife is to the right of the book' and '~*p*' is implied by 'The knife is to the left of the book':

> Let us for simplicity take the case of perception and, assuming for the sake of argument that it is infallible, consider whether 'He perceives that the knife is to the left of the book' can really assert a dual relation between a person and a fact. Suppose that I who make the assertion cannot myself see the knife and book, that the knife is really to the right of the book, but that through some mistake I suppose that it is on the left and that he perceives it to be on the left, so that I assert falsely 'He perceives that the knife is to the left of the book'. Then my statement, though false, is significant, and has the same meaning as it would have if it were true; this meaning cannot therefore be that there is a dual relation

> between the person and something (a fact) of which 'that the knife is to the left of the book' is a name, because there is no such thing. (Ramsey 1927: 140)

Ramsey's point seems to be the following. The purpose of the multiple-relation theory of judgement was to allow for the identity between the objects of true and false beliefs. It is therefore assumed that in general a true belief has the same object as it would have if it were false. Presumably, there may be false beliefs about perception: in fact, that possibility is required by Russell's own explanation of apparently 'fallible' instances of perception. In such circumstances, since there is no fact there to be perceived, the belief is a multiple relation between the believer, the would-be perceiver, a relation between the perceiver and the terms that would make up the fact that would have been perceived if it existed, and those terms themselves. But then, since the object of the belief is the same as it would be if the belief were true, perception cannot be a dual relation between a perceiver and a fact, even when that fact exists. It must therefore be a multiple relation no less than belief. Ramsey of course assumes that 'perceives' means the same whether or not it is embedded in 'believes that'.

What Ramsey's argument brings out especially vividly is that Russell's understanding of the 'infallibility' of perception (and knowledge) does not really amount to factivity. Turning to the formal mode, for Russell, it is as if it is not the truth but the *significance* of '*s* perceives (that) *p*' that entails '*p*' (cf. Sullivan and Johnston 2018 §4.2). That Russell failed to distinguish between these two cases further clarifies the sense in which, for him, propositions are still objects even at this stage. If *p* is false, then it does not exist, and so neither can the complex '*s* perceives *p*'.

Hence the fact that Russell kept a dual-relation theory of perception reveals that his multiple-relation theory did not represent a deep theoretical departure from his older view of judgement. To recap, the propositions that once served him as both truth-bearers and truth-makers now served him only as truth-makers. But more importantly, his general view of cognitive relations did not fundamentally change. Before, minds attached to facts (or fictions): now, they sometimes attached to facts (in perception), sometimes to several (simple) objects (in belief). But they remained in both kinds of cases external relations between minds and the objects of the relations in question. Russell said that according to the multiple-relation theory, if there were no minds to make mistakes, there would be no falsehoods. But in the sense of the dual-relation theory, there *would* still be truths even then. Just as before, the world would remain essentially the same had there been no judging minds or thinking subjects.

6.6 Truth as correspondence

In his early sketch of the multiple-relation theory, Russell said only that, in a belief that A is B, A and B are related 'in a certain manner', and the belief is true if A and B stand 'in the corresponding relation' (1906/7: 46–7; cf. above). A problem would arise, however, as soon as he attempted to define 'the "correspondence" which constitutes truth' (1910: 183).

As Russell observed in 1910, the judgement that A loves B differs from the judgement that B loves A, yet they involve exactly the same terms: A, B and the relation of loving. The problem, that is, is that, for any belief or judgement complex, there is more than one complex made up of its objects (as different complexes may have the same terms), but only one that makes it true. Russell had failed to establish the uniqueness of the correspondence between a judgement and its truth-condition. He would try (and fail) to overcome this difficulty in three alternative ways in 1910, 1912 and 1913.

6.6.1 Direction

In 1910 Russell proposed that, as it occurs in a judgement, 'the relation must not be abstractly before the mind, but must be before it as proceeding from A to B rather than from B to A' (1910: 183), so that it can have there the same 'direction' or 'sense' as it has in the judgement's verifying complex:

> We may distinguish two 'senses' of a relation according as it goes from A to B or from B to A. Then the relation as it enters into the judgement must have a 'sense', and in the corresponding complex it must have the same 'sense'. Thus the judgement that two terms have a certain relation R is a relation of the mind to the two terms and the relation R with the appropriate sense: the 'corresponding' complex consists of the two terms related by the relation R with the same sense. (Russell 1910: 184)

However, as G. F. Stout would soon point out to him, this first solution would not work. On Russell's theory there is little sense to be made of a relation having a direction at all when it occurs in judgement (cf. Geach 1957 §13). After all, it is the entire point of the multiple-relation theory that subordinate relations do *not* appear in judgement *as relating* their terms at all. Hence, they cannot appear there 'as proceeding' from one to the other.

Russell duly acknowledged the objection and soon came up with a simple way out of it. In 1912, he proposed that the order in which terms may appear in judgement, by which judgements themselves may differ, is determined by the judgement relation itself:

> It will be observed that the relation of judging has what is called a 'sense' or 'direction'. We may say, metaphorically, that it puts its objects in a certain *order*, which we may indicate by the order of the words in the sentence. [….] Othello's judgement that Cassio loves Desdemona differs from his judgement that Desdemona loves Cassio, in spite of the fact that it consists of the same constituents, because the relation of judging places the constituents in a different order in the two cases. Similarly, if Cassio judges that Desdemona loves Othello, the constituents of the judgement are still the same, but their order is different. This property of having a 'sense' or 'direction' is one which the relation of judging shares with all other relations. (Russell 1912: 74)

Unlike the relations subordinate to it, the judgement relation itself occurs in a judgment complex as a (relating) relation. It is therefore capable of ordering the terms that it relates. Hence the judgements that *a*R*b* and *b*R*a* may be distinguished simply in virtue of the order in which the judgement relation orders their terms. Indeed, *J*(*s*, *a*, R, *b*) and *J*(*s*, *b*, R, *a*) might be said to be, or be identified by, different ordered quadruples.

Russell could now *distinguish* any two given complexes in terms of the order of their terms, including judgement complexes. But this would not yet allow him to settle whether the terms of any two given complexes had the *same* order. Or rather, he would still be unable to settle whether the terms of two complexes *with a different number of terms* have the *same* order.

In the problematic case, which concerns a judgement and its truth-maker, the two complexes do have a different number of terms, and so are *already* different ordered tuples anyway. Let the sequence *a*-*b* represent R(*a*, *b*) and the sequence *s*-*a*-R-*b* represent *J*(*s*, *a*, R, *b*). Some criterion is needed to say that *a* and *b* have *the same order* in *a*-*b* as they have in *s*-*a*-R-*b*. The problem is that, although *b* follows *a* in both cases, it does not follow *a immediately* in *s*-*a*-R-*b*.

Syntactically, as it were, there might be a way around the problem. It is easy enough to define a function leading from each judgement complex to its truth-making complex and back again. For instance, given an *expression* for a judgement complex, erase 'J', 's', any brackets and commas, and concatenate the remaining terms. Conversely, given an *expression* for a complex, write a comma after each term but the last, write 'J' at the beginning, followed by a comma, then 's', etc. In order to extend this treatment beyond the atomic case, a notation in which scope is indicated by the order of connectives, such as Polish notation, might be required. In this way, each judgement complex could be associated with a unique truth-condition via their expressions.

Now Russell would surely have none of this purported solution, if only because it is syntactic. As such, it would make the relationship between judgement complexes and their truth-conditions depend essentially on their linguistic expression. But in any case, he would soon reject the approach on quite independent grounds.

6.6.2 Positions

By 1913 Russell had abandoned the conception of relations that informed his 1912 solution, on grounds that have since become familiar (cf. 1913: 87; see also chapter 4). In 1912, Russell presupposed the view that is now called 'directionalism'. On that view, the way its terms are ordered is intrinsic to a relation. Relations are fully characterized by, and may indeed be identified with, sets of ordered pairs (or tuples). However, directionalism is guilty of double-counting facts or propositions that involve asymmetric relations. In order to see this, let R be an asymmetric relation, and R' its converse. R is asymmetric if, for every x and y, if xRy, then $\sim yRx$. Its converse R' is that relation which holds of y and x when R holds of x and y: $yR'x$ if, and only if, xRy. Trivially, converse relations are identical only if they are symmetrical: so, $R \neq R'$. Assuming that propositions with different constituents are different, $aRb \neq bR'a$. With a=Desdemona, b=Cassio and R=loves, it follows that 'Desdemona loves Cassio' and 'Cassio is loved by Desdemona' express different propositions. But, arguably, they do not.

Russell now thought that we owe the talk of the direction of a relation to 'misleading suggestions of the order of words in speech or writing', and that for each pair of apparently distinct converse relations there is but a single 'neutral' or 'pure' relation, which 'ceases to demand terms in order to be intelligible' (1913: 88).

Russell thus replaced the notion of direction with that of a '*position* in a complex with respect to the relating relation'. According to this new 'positionalist' view, each relation is equipped with a fixed number of positions, which are occupied by the terms it relates. In a two-place relation such as xRy, 'x' could be called the subject-position and 'y' the object-position. On this view, we can say that the facts that Desdemona loves Cassio and that Cassio loves Desdemona differ in that Desdemona occupies the subject-position in the loving relation in the former and its object-position in the latter, while Cassio occupies the object-position of the same relation in the former and its subject-position in the latter.

Again, judgement is not peculiar among relations. So just as in 1912 it was the judgement relation itself that ordered its terms, in 1913 it 'positioned' its terms. If we let it be represented by '$J(x, y, z, w)$', we can say that 'Othello judges that Desdemona loves Cassio' differs from 'Othello judges that Cassio loves Desdemona' in that, while Othello occupies the x-position in both, Desdemona occupies the y-position in the first and the w-position in the second, and Cassio occupies the w-position in the second and the y-position in the first.

Once again, Russell could now *distinguish* any given complexes with the same terms in terms of the positions that their terms occupied in each. In particular, he could distinguish any given judgement complexes. But just as before, this would not yet allow him to determine whether the terms of any given two complexes occupied the *relevant* (*same*?) corresponding positions in each. Again, in the problematic case, the complexes have a different number of terms, and different 'relating' relations with different positions. For instance, in order to associate '$R(a, b)$' with '$J(s, a, R, b)$' as its truth-condition, we would like to be able to say that a occupies the same position in both. But that is exactly what we cannot say, at least without reintroducing the notion of order, since R and J are different relations with different positions.

Russell therefore sought to distinguish any given complexes, not in terms of the positions occupied by their terms in each, but in virtue of their constituents alone. His first step was to define *permutative* and *non-permutative* belief complexes.

A judgement, or belief, is *non-permutative* if 'no different belief results from permuting the objects' (Russell 1913: 144). Here we may have two cases: either no belief results from the permutation, or the same does.

If no different belief results from the permutation, the belief is said to be a *heterogeneous complex* (1913: 123). Objects of heterogeneous complexes cannot be permuted because they belong to different types. For instance, no 'logically possible complex' results from shuffling the constituents of complexes that involve one-place relations, and so any belief made true by a subject-predicate complex is heterogenous.

The same belief results from permuting its objects if the complex that makes it true is *symmetric*, even if it is *homogeneous*. The simplest case of a homogeneous symmetric complex is one which involves a two-place symmetric relation.

For any non-permutative belief, then, there is a unique 'logically possible' complex made up of its objects, and so a unique candidate to make that belief true. And so Russell's original definition of truth in terms of correspondence holds non-problematically for non-permutative beliefs in general: a 'non-permutative belief is said to be *true* when there is a complex consisting of its objects; otherwise it is said to be false' (1913: 144–5).

A belief is *permutative*, however, if different beliefs may result from permuting its objects. These are beliefs which are made true by complexes that involve asymmetric relations. In this case, more than one complex may be constituted by the objects of the belief, and so more than one complex may be associated with it. And so there is more than a single candidate to make the belief true. The challenge, therefore, as Russell puts it, is, '[w]hen several complexes can be formed of the same constituents, to find associated complexes unambiguously determined by their constituents' (1913: 145).

The task that Russell set himself to carry out was a *reduction* of *unsymmetric homogeneous* complexes to *heterogeneous* ones, so as to eliminate permutative complexes altogether. One of our examples above should suffice to give an idea of what he had in mind. Suppose that 'Desdemona loves Cassio' is analysed as 'Desdemona is lover in a loving complex and Cassio is loved in the same complex'. The *analysans* is now a molecular proposition: or rather, it is the existentially quantified proposition 'there is a complex *C* such that Desdemona is lover in *C* and Cassio is loved in *C*'. But, according to Russell, Desdemona and *C* belong to different types. So even though 'Desdemona is lover in *C*' is a relational proposition, the relation it involves relates terms from different types. But this means that the complex is heterogenous. And the same holds for 'Cassio is loved in *C*'. Russell generalizes the strategy, which in effect entails the elimination of homogeneous asymmetric relations altogether.

Russell never extended the strategy beyond the atomic case. Pincock (2008) has forcefully argued that, had he attempted to do it, he would have encountered the problem of falsehood in a new guise. Less convincingly, Pincock also argued that it was Russell's failure to eliminate unsymmetric homogeneous complexes that dictated the collapse of the multiple-relation theory (as Ricketts had already suggested in 1996), and that it was Wittgenstein who made him aware of the resurgence of the problem of falsehood. It is however overwhelmingly more likely that Wittgenstein's critique of the multiple-relation theory concerned a rather different and more basic question.

6.7 Forms

Russell came to realise that his account of judgement would not be complete even if his reduction of unsymmetric homogeneous complexes was carried through successfully, i.e., even if he succeeded in assigning a unique truth-condition to each judgement.

In his manuscript for *Theory of Knowledge*, rather than judgement or belief, Russell preferred to speak more generally of understanding, since it 'is the most comprehensive and fundamental of propositional cognitive relations', as it is implied by all others: 'we cannot believe or disbelieve or doubt a proposition without understanding it' (1913: 110).

But the most significant change in his 1913 version of the multiple-relation theory came with his introduction of the *forms* of complexes as constituents of understanding (or judgement, or belief). In order to understand a proposition, Russell now thought, it is not enough to be acquainted with its terms. It is also necessary to be acquainted with its *form*, or the way those terms are supposed to combine:

> Let us take as an illustration some very simple proposition, say '*A* precedes *B*', where *A* and *B* are particulars. In order to understand this proposition, it is not necessary that we should believe it, or that it should be false. It is obviously necessary that we should know what is meant by the words which occur in it, that is to say, we must have acquaintance with *A* and *B* and the relation 'preceding'. It is also necessary to know how these three terms are meant to be combined; and this, as we say in the last chapter, requires acquaintance with the general form of the dual complex. (Russell 1913: 110–1)

Hence the general form of a judgement complex is now '$J(S, F, x_1, x_2, \ldots, x_n)$', where S is the judging subject, F is the form of its corresponding truth-making complex, and x_1, x_2, …, x_n are the latter's terms (cf. 1913: 144). Russell continues:

> But this is by no means enough to enable us to understand the proposition; in fact, it does not enable us to distinguish 'A precedes *B*' from '*B* precedes A'. (Russell 1913: 111)

Hence being acquainted with the form of a proposition, as well as its terms, is necessary but not sufficient for understanding it, if the belief is permutative. But it is also insufficient if the belief is *not* permutative, including if it is a homogeneous and symmetric. As Russell noted,

> These difficulties are not an essential part of the difficulty of discovering what is meant by 'understanding a proposition'. We shall do well, therefore, to take examples which do not introduce 'sense'. Among dual relations, there are two sorts of such examples, (1) those where the two terms are logically different, so that no proposition results from interchanging them, as in the above instance of '*A* precedes in a'; (2) those where the relation is symmetrical, and the complex is unchanged by interchanging the terms, as in '*A* resembles *B*'. The first class of examples introduce special difficulties of their own; we will therefore consider the second class. (Russell 1913: 112)

Hence the problem to which the introduction of forms purported to be a solution was *different* from the problem to which the reduction of unsymmetric homogeneous complexes purported to be a solution. But what exactly *was* that problem? As Russell explained it, it was the problem of *uniting in thought* the elements that may or may not be *actually* united in reality:

> I held formerly that the objects alone sufficed, and that the 'sense' of the relation of understanding would put them in the right order; this, however, no longer seems to me to be the case. Suppose we wish to understand '*A* and *B* are similar'. It is essential that our thought should, as is said, 'unite' or 'synthesize' the two terms and the relation; but we cannot *actually* unite them, since either *A* and *B* are similar, in which case they are already united, or they are dissimilar, in which case no amount of thinking can force them to become united. The process of 'uniting' which we *can* effect in thought is the process of bringing them into relation with the general form of dual complexes. (Russell 1913: 116)

Prior to the introduction of forms, then, even if we were able to associate a belief-complex with its truth-making complex, the elements of that complex would still *not* be united in belief in same way as they would be united in the complex (if it existed). Therefore two complexes could be uniquely associated without any of them reflecting *how* the elements of the other are united. But that means that a belief-complex did not have the appropriate relationship towards its truth-making complex. For a belief that *p* is precisely a belief *that p*. That is, the elements that make up *p* must be united in the belief that *p* as they are in *p*, if the belief is indeed a belief that *p*.

Now this raises two related questions.

The first question is that it is far from clear how the introduction of forms can make any difference in this regard. Again, relations that occur as terms in judgement complexes do not occur there as relating anything. But then neither do forms themselves.

And if forms do not relate other terms, it is hard to see how they can play any role in uniting them in thought. To be sure, they do not. What Russell claims is that judgement (or understanding) brings the elements of the complex into relation with its form. But this still does not entail that those elements are united in the judgement-complex *as they would be* in the truth-making complex.

The second question is that, if, on the other hand, forms *could* make such a difference, then there would seem to remain no question as to which complex made a belief true, since merely identifying the belief-complex would suffice to identifying its truth-making complex. But then there would be no need to reduce unsymmetric homogeneous complexes to heterogeneous ones at all. Something's gone amiss with Russell's strategy.

Russell seems to have been pressed at least on the first issue even before he abandoned his manuscript:

> More simply, in order to understand '*A* and *B* are similar', we must know what is to be done with *A* and *B* and similarity, i.e. what it is for two terms to have a relation; that is, we must understand the form of the complex which must exist if the proposition is true. I do not know how to make this point more evident, and I must therefore leave it to the reader's inspection, in hopes that he will arrive at the same conclusion. (Russell 1913: 116)

We can speculate with some confidence who Russell's exasperating reader may have been.

6.8 Wittgenstein's critique

In the spring of 1913, Wittgenstein presented Russell with an objection to his theory of judgement that, in his own words, 'paralysed' him. The magnitude of the event became legendary owing no doubt, at least in part, to Russell's own inflated account of the matter. Here is how he would recount the episode to Lady Ottoline Morrell in 1916:

> Do you remember that […] I wrote a lot of stuff about Theory of Knowledge, which Wittgenstein criticized with the greatest severity? His criticism, tho' I don't think you realised it at the time, was an event of first-rate importance in my life, and affected everything I have done since. I saw he was right, and I saw that I could not hope ever again to do fundamental work in philosophy. My impulse was shattered, like a wave dashed to pieces against a breakwater. I became filled with utter despair […] (Russell 1975: 267)

In his *Autobiography*, Russell added a footnote saying that he 'soon got over this mood' (1975: 706). And the tone of the letter certainly contrasts with the candour of his remark in *The Philosophy of Logical Atomism* that 'the theory of judgment which I set forth once in print some years ago was a little unduly simple' (1918: 59). The truth probably lies somewhere in the middle. But the fact remains that Wittgenstein's objection did cause Russell to abandon his 1913 draft for *Theory of Knowledge*.

The fact that there is no exact record of the objection, or at least of how Russell understood it, has led to a great deal of speculation. Griffin (1985) took it to be a form of the problem of direction; Pincock (2008), a version of the old problem of falsehood; and Hanks (2007), a version of the old problem of propositional unity. But the evidence favours Johnston's (2012) interpretation of the objection as concerning the fact that Russell's theory is unable to represent the objects of judgement as combined.

In a letter to Russell dated June 1913, Wittgenstein wrote:

> … I can now express my objection to your theory of judgement exactly: I believe it is obvious that, from the proposition 'A judges that (say) *a* is in a relation R to *b*', if correctly analysed, the proposition 'a R b.v.~a R b' must follow directly *without the use of any other premiss*. This condition is not fulfilled by your theory. (Wittgenstein 1961: 122)

And a note dated 22.7.13 reads:

> … I am very sorry to hear that my objection to your theory of judgement paralyses you. I think it can only be removed by a correct theory of propositions. (Wittgenstein 1961: 122)

The June letter presupposes an earlier formulation of the objection. In particular, Wittgenstein's emphasis on 'without the use of any other premiss' suggests that Russell had already tried to address it. It could indeed be that the 'additional premise' had been his introduction of forms. Either way, the objection reappears in the *Tractatus* in a slightly different form:

> The correct explanation of the form of the proposition 'A judges *p*' must show that it is impossible to judge a nonsense. (Russell's theory does not satisfy this condition.) (Wittgenstein TLP: 5.5422)

However, the Tractarian variation derives from the *Notes on Logic*, which also date from 1913 (though, according to Potter, from later in the year, cf. his 2009: Appendix A):

> Every right theory of judgement must make it impossible for me to judge that this table penholders the book. Russell's theory does not satisfy this requirement. (1961: 103)

> The proper theory of judgment must make it impossible to judge nonsense. (1961: 95)

It is therefore possible that the *Notes* version is the one that is presupposed by the June letter. The evidence suggests at any rate that the two *could* be alternative versions of the *same* objection. More importantly, the two formulations can *in fact* be interpreted as alternative expressions, in the material and formal modes respectively, of what is essentially the same objection. If so, the difference between them would indeed be, as Potter has suggested, merely 'cosmetic' (2009: 126).

In the formal mode, the objection may be understood as amounting to little more than that 'p' must be a sentence in 'A judges that p'. When Wittgenstein writes that '$p \vee \sim p$' should follow from 'A judges that p', he may well be imitating 'Russell's modes of expression' (Potter 2009: 127) for the sake of making himself understood. In the *Principles of Mathematics*, Russell had taken 'p is a proposition' to be equivalent to '$p \rightarrow p$' (cf. 1903a §16), which, as Potter notes, is equivalent to '$p \vee \sim p$'. Now, Wittgenstein would no doubt reject that equation: '$p \rightarrow p$' obviously *presupposes* the meaningfulness of 'p'. But that would in turn explain his emphasis on 'without the use of any other premiss'. It would indeed be spurious to add the premise that 'p' is a sentence. (As it were, formation rules are descriptive.)

In the material mode, Wittgenstein's objection translates into Johnston's claim that subordinate relations must appear in judgement *as relating*. Otherwise, as he argues, nonsense *could* be judged. Ever since the *Principles of Mathematics* Russell had been committed to the principle that any two terms occurring as terms in propositions may be substituted *salva congruitate*. Since relations appear in judgement complexes as terms, they may be replaced by, say, an individual occurring in some other proposition. Hence, one may easily obtain nonsense from any given meaningful judgement.

People sometimes balk at Wittgenstein's insistence that one cannot judge a nonsense. But the requirement that 'p' be a meaningful sentence in 'A judges that p' is nothing but an instance of the more general requirement that a meaningful whole must have meaningful parts. In order that a logically compound sentence make sense, that is, its compounds must make sense. The same indeed holds for negation and the propositional connectives more generally. As Wittgenstein put it in the *Notes*,

> When we say A judges that etc., then we have to mention a whole proposition which A judges. It will not do either to mention only its constituents, or its constituents and form, but not in the proper order. This shows that a proposition itself must occur in the statement that it is judged; however, for instance, "not-p" may be explained, the question what is negated must have a meaning. (Wittgenstein 1961: 94)

Now, it might seem as if Wittgenstein's point about compositionality simply begs the question against Russell. After all, Russell could accept the general principle: what he denied was that '*p*' is logically a part of 'A judges *p*'.

However, this reply on Russell's behalf suggests that '*p*' is an incomplete symbol, whereas Russell's *expressed* view was merely that 'that *p*' was an incomplete symbol. Indeed, as Wittgenstein put it in 1913, it is neither the sentence nor its meaning that is an incomplete symbol, but the *word* 'meaning' (cf. 1961: 124–5). On Wittgenstein's view, what Russell's theory entails is not that *sentences* have no meaning, but that their meanings are not objects. What must be 'paraphrased away' is not '*p*', but such phrases as 'the meaning of "*p*"' and 'the proposition *p*'. Hence rather than, say, 'the meaning of "*p*" is the proposition *p*', one should simply say "*p*' says that *p*' (cf., e.g., Johnston and Sullivan 2018 §3.4).

6.9 Ramsey's diagnostic

The first question that Russell's introduction of forms raised was that they would make no difference with regard to uniting the elements of thought in the required way. But although the question was prompted by forms, the problem is *independent* of them, and so applies to the earlier versions of the theory just as well. By showing why his strategy would not rule out the possibility of judging nonsense, Wittgenstein finally made it clear to Russell why that was so.

Whatever 'uniting' is done in judgement is done *by* judgement, *not* forms, which occur there as terms. To repeat Russell's own words, 'The process of 'uniting' which we *can* effect in thought is the process of bringing [terms] into relation with the general form of dual complexes' (1913: 116, cf. above). But that means that they are *not* related in judgement as they are in its corresponding complex, since in that complex they would be related by a relation that occurs in judgement as a term.

The second question was that, assuming that forms could have somehow accomplished their task, there would seem to remain no work to be done by Russell's definition of truth. The two questions come together in Ramsey's diagnostic of the whole situation. For Ramsey, Russell had completely misconceived his approach to judgement.

In order to avoid a commitment to false propositions, Russell had defined the content of a judgement by reference to what, from the point of view of his earlier theory, could be the common object of true and false judgements alike, i.e., not a proposition but its *terms*. But by doing this, as Wittgenstein showed, Russell had deprived himself of a specification of the *content* of the judgement, in so far as the content of a judgement is the condition of its truth. It may well be said that on Russell's theory judgements have *objects* (plural) but no *object* (singular); that is, no *content*. But having gone so far, Russell saw the need to specify, *in addition* to his analysis of a given judgement, the complex that made it true. Hence judgements would now have truth-conditions, but bizarrely their truth-conditions would not be their contents.

But as Ramsey observed already in his review of the *Tractatus*, Wittgenstein had reduced 'the question as to the analysis of judgement, to which Mr Russell has at various times given different answers, to the question "What is it for a proposition token to have a certain sense?"' (1923: 274–5). But 'if we can answer our question we incidentally solve the problem of truth' (1923: 275).

Ramsey could thus diagnose Russell's strategy as being as it were upside down. For to analyse a judgement is simply to determine what would be the case if the judgement were true, that is, what is the truth-condition of the sentence that expresses it. But as soon as that question is addressed, there is no work left to be done by Russell's definition of truth.

Hence, as Johnston and Sullivan conclude, the import of Wittgenstein's and Ramsey's so-called 'identity' theories of truth was not an identity between two kinds of entity as Russell's had been, but an identity between two kinds of *question*:

> To ask after the content of a proposition, and to ask what is required for its truth, are, according to Wittgenstein's identity theory, two ways of asking the same thing. If we have an answer to the first of these questions, then there is nothing that an answer to the second need, or even can, add to it. (Johnston and Sullivan 2018 §3.4)

A belief is always a belief *that p*, and a belief that *p* is true if *p*, false otherwise (cf. Ramsey 1923: 275, 1927: 142–3). The expression of a belief expresses *also* what makes it true: its content is given by the condition of its truth.

Russell himself later acknowledged that his then former theory was 'a little unduly simple' because it allowed for the possibility 'of putting the subordinate verb on a level with its terms as an object term in the belief':

> I did then treat the object verb as if one could put it as just an object like the terms, as if one could put 'loves' on a level with Desdemona and Cassio as a term for the relation 'believe'. That is why I have been laying such an emphasis […] on the fact that there are two verbs at least. (Russell 1918: 59)

But it is not as if Russell could easily fit this solution into his picture. Far from it. For a subordinate relation to appear in judgement as relating would mean for Russell to fall back into the dual-relation theory of judgement. The problem that judgement represented for Russell could indeed not be dealt with by making a simple tweak in his theory, but concerned the very substance of his worldview. He was thus spot on when he wrote that belief was a completely 'new beast in our zoo':

> You cannot get in space any occurrence which is logically of the same form as belief. […] I have got on here to a new sort of thing, a new beast for our zoo, not another member of our former species but a new species. The discovery of this fact is due to Mr. Wittgenstein. (Russell 1918: 58)

To recap, there were only at bottom terms and relations in Russell's world. Hence belief had to be a relation and its objects terms. On his first theory, it was a relation between a simple and a complex term. On his second theory, it was a relation between several simple terms. In both cases, belief would be, from a logical point of view, an external relation between those terms, not unlike a spatial one (cf. chapter 3). In the next, final chapter, we represent the opacity of Russell's propositions as a further aspect of the same picture.

7 Thoughts

Russell regarded Frege's views on judgement along the lines of his own dual-relation theory. In particular, he regarded Fregean thoughts as essentially the same kind of thing as propositions. Some commentators have taken Russell's lead in their assessment of Frege. Again, Makin, for one, has claimed that while Frege and Russell held different views at different times about the constituents of propositions, 'their commitment to propositionalism remained constant' (2000: 142).

In this chapter, I argue that the sense in which Frege's thoughts are objects is not fundamental, and so that Russell's and Frege's views on judgement are only superficially similar. I introduce Frege's grounds for distinguishing sense from meaning in 7.1 and 7.2, and discuss the sense in which sense is objective in 7.3 and 7.4. I draw an analogy between Frege's definition of numbers as objects and his account of the identity of thoughts in 7.5. I argue in 7.6 that the transparency of thoughts is peculiar even among logical objects, and in 7.7 that the opacity of propositions is a defining feature of their nature as objects.

7.1 Cognitive value

Frege introduces the notion of a thought with the sense-reference distinction in 'On sense and reference' (1892b). Famously, that paper belongs to the triad of articles in which he presents the three distinctive doctrines of the philosophical logic that informs the formal logic of *Grundgesetze*. Frege identifies concepts with one-place functions in 'Function and concept' (1891) and distinguishes concept and object in 'On concept and object' (1892a). These two may be said to deepen ideas that were already implicit in *Grundlagen* and in *Begriffsschrift*. As he explained it in *Grundgesetze*,

> the essence of the function as opposed to the object is shown by means of sharper criteria than in my *Begriffsschrift*. From this results further the distinction between functions of the first and second level. As I have shown in my essay *Function und Begriff* (Jena, 1891), concepts and relations are functions in my extended meaning of the word […] (1893: x)

By contrast, his earlier undifferentiated notion of conceptual content 'now split up into what I call "thought" and "truth-value"; a consequence of the distinction between the sense and reference of a sign' (1893: x). To some extent, the sense-reference distinction does amount to a bifurcation of the old notion. But as we shall see, while the notion of cognitive value somehow guided Frege's thinking in *Begriffsschrift*, it did not yet do so as an element within the theory itself.

The structure of Frege's argument in 'On sense and reference' is not immediately laid out. Frege famously begins the paper with a discussion of identity statements. There he assumes that names (or singular terms) have reference and argues that they must also have sense. Later he assumes that sentences have sense and argues that they must also have reference. The later argument about sentences presupposes that names have both sense and reference, while the argument about names only presupposes (implicitly) that sentences have sense. The reader is therefore expected to be familiar with a notion of reference as it applies to names and with a notion of sense as it applies to sentences, then be persuaded that names have sense, and later that sentences have (truth-values as their) reference.

Sense is therefore only a technical notion as it applies to names, and reference only a technical notion as it applies to sentences. It would thus be misleading to choose either sense or reference as the one intuitive or pre-theoretical notion of meaning.

Now, what are the features of the meaning of names and sentences that readers are supposed to have grasped in advance? In the case of names, their reference is what they stand for, or the objects we ordinarily talk about. In the case of sentences, the answer turns on the notion of cognitive value. Differences in cognitive value entail differences in meaning for sentences, though this is not initially assumed to hold also for names.

Two sentences have different cognitive value if it is *possible* for someone (rationally) to hold the one as true but not the other:

> the thought in the sentence 'The morning star is a body illuminated by the Sun' differs from that in the sentence 'The evening star is a body illuminated by the Sun.' Anybody who did not know that the evening star is the morning star might hold the one thought to be true, the other false. (Frege 1892b: 32)

The same criterion underlies Frege's assumption that some identity statements may be cognitively valuable, or informative:

> *a*=*a* and *a*=*b* are obviously statements of differing cognitive value; *a*=*a* holds *a priori* and, according to Kant, is to be labelled analytic, while statements of the form *a*=*b* often contain very valuable extensions of our knowledge and cannot always be established *a priori*. (1892b: 26)

This assumption gives rise to what has become known in the literature as 'Frege's puzzle'.

7.2 Modes of presentation

Suppose, say, that Aristotle believed that Socrates taught Plato. Suppose, in addition, that Aristotle never knew Plato's birth name 'Aristocles', and that Plato *was*, in fact, Aristocles. If two things are the same, then whatever is true of the one is also true of the other. So, it is true of Aristocles that Aristotle believed that he was taught by Socrates. Hence, we should seemingly conclude, Aristotle believed that Socrates taught Aristocles. Only, he did not.

In general, the argument is meant to show that, whenever an expression occurs in a sentence within the scope of a psychological verb, it cannot be presumed that it may be substituted by another expression with the same meaning without any difference being made to the truth-value of the sentence. In short, arguments of the form '$f(a)$, $a=b$.: $f(b)$' are invalid if 'f' is an intentional context.

Frege's famous solution to the puzzle consists in positing an ambiguity. The terms 'a' and 'b' stand for an object in '$a=b$', but in '$f(a)$' and '$f(b)$' they stand for different modes of presentation of the same object. Despite appearances, the only co-referential (token) terms in the argument are those that flank the identity sign. The appearance of validity disappears if the argument is represented as, say, '$f(c)$, $a=b$.: $f(d)$', where c is the mode of presentation of a and d is the mode of presentation of b. Leibniz's law (as Frege quotes it, '*eadem sunt*, *quae sibi mutuo substitui*, *salva veritate*') fails to apply, but does not itself fail.

The mode of presentation of an object is a way an object may appear to one, that provides a way of thinking about the object. The notion can be spelled out in a variety of ways. But it is important to note that it is not the *existence* of modes of presentation that is in question in Frege's argument. It is *given* that there may be different ways of thinking about the same objects. What Frege argues is that modes of presentation may be relevant for ascertaining the truth of certain sentences or thoughts.

Modes of presentation of things may therefore be involved in the meaning of their names. Frege is thus led to recognize a component in the meaning of names associated with their cognitive significance. He calls it their sense. As he writes,

> A difference [in cognitive value] can arise only if the difference in the signs corresponds to a difference in the mode of presentation of that which is designated […] It is natural, now,

> to think of there being connected with a sign […], besides that which the sign refers, which may be called the reference of the sign, also what I should like to call the *sense* of the sign, wherein the mode of presentation is contained. (1892b: 26–7)

The meaning of a name thus bifurcates into the object that it stands for, or its reference, and the mode of presentation of the object associated with the name, or its sense.

Although Frege's solution is presented in technical terms, it is in fact intuitively plausible. In order to see that, it is enough to consider what it would take for an argument such as the above to go through: '*a*' and '*b*' would have to have the same sense, that is, they would have to be associated with the same mode, or modes, of presentation of the object. In our example, this would amount to Aristotle's having known Plato as Aristocles.

Things, then, may appear to one in different ways, or have different modes of presentation. Sometimes, Frege implies, the different ways in which things appear is relevant to the meanings of the expressions that concern them. The point can be made with regard to thought. The different ways in which we think about things is often relevant to what thoughts we have about them.

In *Begriffsschrift*, Frege had given an alternative explanation of the same phenomenon in terms of an interpretation of identity as co-reference. According to that interpretation, names 'appear *in propria persona*' (1879 §8) in identity statements, which are held to assert of two names that they stand for the same object. This view entails that names usually stand for their 'content', but exceptionally stand for themselves whenever they flank an identity sign.

There is a sense in which the old theory resembles the new one. On both, names have both an ordinary and as it were an extraordinary meaning. Their ordinary meaning is the object they stand for. Occasionally, according to the old theory, they stand for themselves; according to the new theory, for their mode of presentation.

But more important than what their extraordinary meaning is, is the context in which it becomes relevant. On the new theory, it is not statements of identity, but psychological or intentional contexts.

Looking at the two theories in this light allows us to see immediately what was wrong with Frege's early theory. On the *Begriffsschrift* view of 'equality of content', *every* argument of the form '$f(a)$, $a=b$.: $f(b)$' must be invalid, since it is *identity* that is counted as an intensional context. The view also makes nonsense of the 'basic laws of identity of content' that Frege formulates in *Begriffsschrift*. For instance, his first such law, formula (52), can be expressed in modern notation as '$c=d \rightarrow (f(c) \rightarrow f(d))$'. By Frege's own account of equality of content, the signs '*c*' and '*d*' are here used ambiguously, since they stand for different things in the antecedent and in the consequent (cf. White 1977).

By contrast, in Frege's later account, the notion of cognitive significance not only gives rise to the problem, but also informs its solution. Modes of presentation become a feature of the name's meaning.

7.3 Truth-values

Concerning names, then, Frege assumed that they have reference, and argued for their sense. But he also assumed that the objects that they designate have different modes of presentation. Frege only needed to argue that the sense of a name is one of the modes of presentation of the object it designates.

Likewise, though symmetrically, Frege assumes that sentences have sense, and argues for their reference. But he also presupposes that they may be true or false, i.e., that they have some one of the truth-values. What he needs to argue is therefore *not* that sentences have truth-values, but that they have reference *and* that their reference is their truth-value.

Frege argues first that, *if* sentences have reference (besides sense), *then* their reference is their truth-value. Roughly, he assumes that sentences express thoughts (besides being either true or false), and argues that, since thoughts cannot be what sentences refer to, their reference must be their truth-value.

> We now enquire concerning the sense and reference for an entire declarative sentence. Such a sentence contains a thought. Is this thought now to be regarded as its sense or its reference? Let us assume for the time being that the sentence has reference. If we now replace one word of the sentence by another having the same reference, but a different sense, this can have no bearing upon the reference of the sentence. Yet we can see that in such a case the thought changes […] The thought, accordingly, cannot be the reference of the sentence, but must rather be considered as the sense. (1892b: 32)

Frege's argument raises a couple of questions. First, he argues that, since thoughts are functions of modes of presentation rather than objects, they must be the senses, not the references, of sentences. But given the way thoughts and modes of presentation were introduced, this really should go without saying. Second, his conclusion that *therefore* the reference of sentences must be their truth-value simply appears to beg the question, as he considers no other alternative. However, he could. In particular, something more fine-grained than truth-values but less than thoughts (for instance, a coarser grained notion of a proposition) might also survive the substitution of co-referential sub-sentential expressions.

We have to recall, however, that Frege is not here arguing that sentences express thoughts, or that the former have truth-values, or that the latter are determined by their cognitive significance. Rather, he is only deciding, between thought and truth-value, which is sense, which is reference. In fact, what Frege is advancing is not so much a new doctrine about an antecedently understood notion of the reference of a sentence, as an altogether new conception of what reference is.

'By the truth value of a sentence', Frege writes, 'I understand the circumstance that it is true or false' (1892b: 34). This is an understatement. For Frege, each of these 'circumstances' is an object: the True and the False respectively. He adds rather implausibly that 'These two objects are recognized, if only implicitly, by everybody who judges something to be true—and so even by a sceptic' (1892b: 34). However, this contrasts with Frege's expressed reason for applying the notion of reference to sentences at all. *That*, Frege implies, arises only from a concern with truth:

> one could be satisfied with the sense, if one wanted to go no further than the thought. If it were a question only of the sense of the sentence, the thought, it would be unnecessary to bother with the reference of a part of the sentence [...] But now why do we want every proper name to have not only a sense, but also a reference? Why is the thought not enough for us? Because, and to the extent that, we are concerned with its truth value. (1892b: 33)

Again, the question is not whether sentences have truth-values, but whether the latter should be recognized as a component in their meaning. And against this, one might think that we are precisely able to understand sentences or grasp thoughts regardless of whether they are true. Frege's argument for recognizing the truth-values of sentences as their reference is therefore a recommendation for adopting a certain kind of explanation of their meaning, which we might call 'semantic'. On that kind of explanation, the reference of a sentence is a function of the reference of its parts, but the reference of its parts is in turn simply what contributes to determining its truth-value. (Thus has Dummett equated reference with semantic value.)

7.4 The objectivity of sense

Up until this point, Frege did not need to invoke the objectivity of sense. In fact, the notion of cognitive value that he introduced is subjective. As Evans observed, Frege's 'Intuitive Criterion of Difference' for thoughts, as he called it, 'can be brought to bear only when the same subject is entertaining the same thoughts at the same time' (1982: 21).

Evans formulates the criterion as follows: 'the thought associated with one sentence *S* as its sense must be different from the thought associated with another sentence *S'* as *its* sense, if it is possible for someone to understand both sentences at a given time while coherently taking different attitudes towards them' (1982: 19). Hence, it 'cannot by itself determine the identity and distinctness of thoughts' as the notion is meant to apply to what is thought or judged by 'different subjects' or by 'a single subject at different times'—it only 'imposes a tight restriction on acceptable answers' (1982: 21).

'By a thought', Frege however wrote, 'I understand not the subjective performance of thinking but its objective content, which is capable of being the common property of several thinkers' (1892b: 32, fn. E). There ought therefore to be a criterion of identity for the objective content of subjective acts of thinking.

In 1906 Frege would state his earlier criterion of difference in a more elaborate fashion than before:

> two sentences *A* and *B* can stand in such a relation that anyone who recognizes the content of *A* as true must thereby also recognize the content of *B* as true and, conversely, that anyone who accepts the content of *B* must straightaway accept that of *A*. (*Equipollence*). It is here being assumed that there is no difficulty in grasping the content of *A* and *B*. […] I assume there is nothing in the content of either […] that would have to be immediately accepted as true by anyone who had grasped it properly. (Frege 1906a: 197)

Here Frege formulates a condition that is roughly the contrapositive of the one given by Evans. But he immediately transforms it into a criterion of identity:

> So one has to separate off from the content of a sentence the part that alone can be accepted as true or rejected as false. I call this part the thought expressed by the sentence. It is the same in equipollent sentences of the kind given above. (Frege 1906a: 197–8)

Roughly, then, two sentences express the same thought if, and only if, anyone who accepts the one (must) immediately accept the other, that is, if it is not possible for someone (rationally) to hold opposing attitudes towards them (having understood both).

It is unclear, however, if this criterion *can* in fact be applied as more than a criterion of difference. In order that two thoughts be the same, Frege says, one must be in a position to accept the one *immediately* or *straightaway*, if one accepts the other. But accepting a thought or a sentence *immediately* by accepting another rather seems to *presuppose* that the two thoughts are the same. For any hesitation would entail that the thoughts are different.

Now, although he did not quite have an objective notion of cognitive value, Frege did have an objective notion of reference. Obviously, to count thoughts according to their reference alone would be pointless. As he wrote,

> If now the truth value of a sentence is its reference, then on the one hand all true sentences have the same reference and so, on the other hand, do all false sentences. From this we see that in the reference of the sentence all that is specific is obliterated. (1892b: 35)

In *Begriffsschrift*, however, Frege had defined conceptual content in terms of entailment. Conceptual content, he then thought, was that aspect of the content of a judgement that is relevant to logic. Frege defined it as an equivalence class on judgements. Two judgements have the same conceptual content if, and only if, the same inferences can be drawn from both, with the help of the same additional premises:

> there are two ways in which the content of two judgements may differ; it may, or it may not, be the case that all inferences that can be drawn from the first judgement when combined with certain other ones can always also be drawn from the second when combined with the same other judgements. The two propositions 'the Greeks defeated the Persians at Plataea' and 'the Persians were defeated by the Greeks at Plataea' differ in the former way […] Now I call the part of the content that is the same in both the *conceptual content*. Only *this* has significance for our symbolic language; we need therefore make no distinction between propositions that have the same conceptual content (Frege 1879 §3)

This definition introduces an objective component in relation to the previous one. Inferences between judgements, rather than their immediate acceptance, can be checked. A similar strategy for defining an objective notion of a thought therefore naturally suggests itself. That was exactly the strategy that Frege pursued in a letter to Husserl of 1906. There he formulated a criterion of identity for thoughts close to the one for conceptual content, both in form and in content:

> It seems to me that an objective criterion is necessary for recognizing a thought again as the same […]. Now it seems to me that the only possible means of deciding whether proposition *A* expresses the same thought as proposition *B* is the following […]. If *both* the assumption that the content of *A* is false and that the content of *B* true *and* the assumption that the content of *A* is true and that of *B* false lead to logical contradiction, and if this can be established without knowing whether the content of *A* or *B* is true or false, and without requiring other than purely logical laws for this purpose, then nothing can belong to the content of *A* […] which does not also belong to the content of *B* […] (Frege 1906c: 105–6)

In short, two sentences express the same thought if, and only if, they are provably equivalent. In a slightly earlier letter to Husserl, Frege added:

> After the assertoric force with which they may have been uttered is subtracted, equipollent propositions have something in common in their content, and this is what I call the thought they express. This alone is of concern to logic. The rest I call the colouring and the shading of the thought. […] All that would be needed would be a single standard proposition for each system of equipollent propositions, and thought could be communicated by such a standard proposition. (Frege 1906b: 102)

We can think of Frege's first criterion of identity for thoughts as so fine that, according to it, as it were, no two thoughts are the same. Frege's second criterion, however, is too coarse. Contrary to his intentions, on that criterion *many* different sentences of the concept-script will have to be sanctioned as expressing the same thought. Frege himself would later recognize many instances of this (in 1923).

At any rate, for our purposes, it is enough to keep in mind the shape of Frege's definition. It is an instance of logical abstraction. Thoughts are identified according to the schema:

The thought that p = the thought that q iff $E(p, q)$,

where p, q are sentences, and E is an equivalence relation on sentences or (psychological) acts of judgement.

7.5 The Julius Caesar problem

Frege's technique of defining objects by logical abstraction is most familiar from the case of numbers. To define the concept of cardinal number in purely logical terms was the task that he set himself in *Grundlagen*.

Famously, Frege considered and rejected first a contextual definition of cardinal numbers as second-level predicates (effectively as numerically definite quantifiers), and then a contextual definition of numbers as objects. The ground of his rejection in both cases was the so-called 'Julius Caesar problem'. He then proposed to define numbers explicitly in terms of extensions of concepts. The step would prove fatal. It was the axiom for extensions (or rather value-ranges, of which extensions are a special case) that he advanced in *Grundgesetze* that would lead to Russell's paradox.

Frege's contextual definition of numbers as objects is meant to provide a criterion of identity for cardinal numbers. The definition, also known as 'Hume's principle', runs as follows: the number of *F*s is the same as the number of *G*s if, and only if, there are just as many *F*s as there are *G*s, i.e., there is a one-one correspondence between the *F*s and the *G*s. In symbols,

$$\#F = \#G \text{ iff } F \sim G.$$

Now, why did Frege reject this definition? According to him, the criterion proposed does not settle the truth-value of every identity statement involving numbers. In particular, it only settles it when *both* terms flanking the identity sign have the form '$\#F$'; not, for instance, when the identity is 'mixed', or has the form '$\#F=q$', where 'q' is a simple name. In Frege's 'crude example' concerning the definition of numbers as quantifiers, q is Julius Caesar.

After mentioning Hume, he discusses a definition of directions by abstraction in terms of parallelism: the direction of *a* is identical with the direction of *b* if, and only if, *a* is parallel to *b* (cf. 1884 §65). The example he chooses to illustrate the 'Caesar problem' actually involves England and the direction of the Earth's axis. His definition of direction, he writes,

> will not, for instance, decide for us whether England is the same as the direction of the Earth's axis—if I may be forgiven an example which looks nonsensical. Naturally no one is going to confuse England with the direction of the Earth's axis; but that is no thanks to

> our definition of direction. That says nothing as to whether the proposition 'the direction of *a* is identical with *q*' should be affirmed or denied, except for the one case where *q* is given in the form of 'the direction of *b*'. (1884 §66)

Hale and Wright interpret the Julius Caesar problem as manifesting the fact that Frege's definition fails to characterize the concept of number as a *sortal* concept (see for instance the Introduction to their 2001). A sortal concept is one for which there is both a criterion of application and a criterion of identity. Frege's definition provides a criterion of identity for cardinal numbers, but not a criterion of application. It therefore serves to identify numbers only when it is settled in advance that it is numbers that are being identified.

There might be a temptation to circumvent the problem by restricting the applicability of the definition to numbers. But as Frege implies, that would be tantamount to giving the definition of direction in the form '*q* is a direction, if there is a line *b* whose direction is *q*'. 'But then', Frege writes,

> we have obviously come round in a circle. For in order to make use of this definition, we should have to know already in every case whether the proposition '*q* is identical with the direction of *b*' was to be affirmed or denied. (1884 §66)

Hale and Wright's *is* nevertheless an odd sort of diagnostic. It at least gives the (misleading) impression that a concept could have a criterion of *identity* but *not* a criterion of *application*. But sortal concepts are distinctive by having a criterion of identity *in addition to* a criterion of application, not the other way around. Presumably, *any* concept has a criterion of application. (Perhaps formal concepts are an exception, in so far as they apply to any object whatever.)

In short, having a criterion of application is *necessary* for having a criterion of identity at all. If Hale and Wright were right, then, the correct thing to say would be rather that Frege's definition fails to characterize the concept of number as a *sortal* concept because it fails to characterize it *as a concept*—which is to say that it simply fails to characterize it.

But then perhaps they *are* right. That is, maybe that is indeed how Frege saw the Caesar problem. Here is how the quotation before the last continues:

> What we lack is the concept of direction; for if we had that, then we could lay it down that, if *q* is not a direction, our proposition is to be denied, while if it is a direction, our original definition will decide whether it is to be denied or affirmed. (1884 §66)

Frege's proposal fails to define directions *simpliciter*, he seems to imply, *because* the condition it introduces is not defined for *everything* which is in fact a direction. The question remains, however, as to why that is so.

Let us go back to the beginning of §66. There Frege had written:

> In the proposition 'the direction of *a* is identical with the direction of *b*' the direction of *a* plays the part of an object, and our definition affords us a means of recognising this object as the same again, *in case it should happen to crop up in some other guise*, say as the direction of *b*. *But this means will not provide for all cases.* (1884 §66; emphasis added)

The implication here is that the direction of *a* can appear not only *as* the direction of *b*, but also under *other*, as yet unspecified, guises, not given in the form 'the direction of …'. For the direction of *a* not to be given in the form 'the direction of…' is therefore only a particular way of it not being given *as* the direction of *a*. The problem, then, is that we cannot be assured that 'England' is *not* one of the guises under which the direction of *a* may be given.

Likewise, for the number of *F*s to be given as the number of *G*s is just *one* way of its *not* being given as the number of *F*s. But, for Frege, the same number *need not* be given in the form 'the number of…' *at all*. In particular, Frege seems to think, we have no guarantee that 'Julius Caesar' is *not* one of these other guises in which the number of *F*s might appear.

Now, Frege still worked with his early undifferentiated notion of conceptual content in *Grundlagen*. But the point can be made, and indeed more aptly, by recourse to his sense-reference distinction. A guise in which an object appears is simply its mode of presentation on a particular occasion. And that objects have alternative modes of presentation is simply a *mark* of their being objects—even if they are logical objects, like numbers. Hume's principle therefore captures a *range* of modes of presentation of numbers, namely those given in the form 'the number of…', but not all, since that form, Frege would say, does not exhaust the ways in which numbers may be given. Hence, by Frege's standards at any rate, the definition does *not* provide a criterion of identity for numbers. A criterion of identity should precisely allow us to decide whether two objects are the same *in all cases* (cf. 1884 §62, quoted below).

7.6 A disanalogy with numbers

We finally reach the point of our short digression about numbers. Frege sketched a definition of thoughts by abstraction similar to his contextual definition of numbers as objects. But he did not seem to consider a version of the Julius Caesar problem for thoughts. It was not by chance that he did not. In what follows we shall enquire into why there could not be a version of that problem for thoughts, and how this shows that there is a sense in which thoughts, though objects, are not 'self-standing' in the same way as numbers (and other objects properly so-called).

To be self-standing is a form of independence. But there are at least two kinds of independence according to Frege.

In the first instance, to be self-standing is to be complete. This form of independence contrasts with unsaturatedness or incompleteness. In this sense, all objects are self-standing and contrast with concepts, or more generally with functions. We might indeed say that functions depend on objects for their existence, just as second-level quantification presupposes first-level quantification.

In another sense, to say that something is self-standing is to claim for it a criterion of identity. For Frege, a criterion of identity for an object must decide 'in all cases whether *a* is the same as *b*', thus providing for us to recognize the object 'again as the same' (cf. 1884 §62).

This is of course *not* to deny that there could be a relation between concepts that might do for concepts what identity does for objects, so long as it was of the appropriate level. For Frege, that relation would be co-extensiveness. Two concepts are the same, that is, if exactly the same objects fall under both. In symbols,

$$F = G \text{ iff } \forall x\, (Fx \leftrightarrow Gx).$$

Hence there being an analogue of a criterion of identity for concepts does not prevent them from being incomplete, and hence from failing to be self-standing in the first of the two senses distinguished above. On the contrary, an analogue of a criterion of identity for functions should reveal the way in which functions depend on objects.

Now, as Dummett observed, it is possible to draw traditional distinctions among objects within the *Fregean* category of objects, as Frege himself appeared to do with regard to concrete and abstract objects (cf. 1973: 258). One such distinction is Aristotle's distinction between things that are *in*, or as Dummett says, *of*, other things, i.e., non-substantial or accidental individuals, and things that are not, i.e., substantial individuals (cf. chapter 1).

Non-substantial individuals too might have criteria of identity. But one should expect that such criteria reveal the way in which they depend on substantial individuals, much as corresponding criteria for functions reveal the way in which *they* depend on objects.

Now, recall that in Frege's definition of numbers, numerical terms are given in the form 'the number of…'. Numbers figure there as numbers *of* something else, namely concepts. As Frege himself said, even if numbers are treated as objects rather than quantifiers, 'the content of a statement of number is an assertion about a concept' (cf. 1884 §46, 57).

This might suggest that numbers are non-substantial individuals. In that case, Hume's principle would reveal the manner of their dependence upon other objects. If the definition were meant to be interpreted predicatively, this might perhaps be indeed the right way to understand it.

However, Frege's definition is *not* meant to be so interpreted (otherwise his proof of the infinity of the natural numbers would not go through). And as the recognition by Frege of a problem such as the Julius Caesar problem reveals, numbers, for him, are *not* necessarily given *as* numbers *of* something. As we saw in 7.5, Hume's principle covers only one range of modes of presentation of numbers among (possibly) many. In some contexts, numbers may indeed appear, as it were, as substantial individuals. And in fact that *is* how they appear in arithmetic. The number two, for instance, is the number of authors of *Principia Mathematica*, but it is also the (unique) even prime.

Numbers contrast in this regard with modes of presentation. A mode of presentation is always and intrinsically a mode of presentation *of* some object. Likewise, a thought is always and intrinsically a thought *that* things are so; it may equally be said to be a thought *about* the things mentioned in its expression. (And for Frege thoughts are composed of modes of presentation, and are in fact *themselves* modes of presentation of truth-values.)

Unlike numbers, then, senses *could* be classified as non-substantial individuals. This does not mean that they are not objects, i.e., that they cannot be quantified over, or be the focus of our attention, or indeed referred to. But it does mean that their criteria of identity reveal how they depend on other, perhaps substantial, objects.

This already suggests that Frege's definitions of thoughts by abstraction should be read predicatively. (More on this below.) But more than that, it entails that modes of presentation must be given in the form 'the mode of presentation of…', and thoughts necessarily given in the form 'the thought that…'.

Frege's definition of thoughts by abstraction therefore succeeds in a way that his definition of numbers did not, even if it ultimately fails for other reasons. As we saw, numbers may have modes of presentation that fall outside the range of modes of presentation captured by Hume's principle. But thoughts have no modes of presentation that fall outside the range specified by 'the thought that'.

It is therefore not an accident that Frege never even seemed to consider a version of the 'Julius Caesar problem' for thoughts. Perhaps the same thought may appear under different guises, for instance, if it is expressed by different sentences. Hence the question may arise whether two thoughts are the same, and a criterion of identity for thoughts should allow us to answer that question. But all of those guises fall within the range specified by that criterion. Hence thoughts, unlike numbers, have no modes of presentation beyond the ways in which they can be expressed.

Thus senses are distinctive in two ways, even as possible objects of reference. First, they are dependent upon other (possibly independent) objects. And second, their manner of dependence is necessarily reflected in the manner in which they may be referred to. Any other way of referring to a sense must be parasitic upon a 'canonical' way of referring to it.

Frege's attempt to define thoughts by abstraction clearly indicates that he took them to be in *some* sense self-standing, and themselves possible objects of thought. That they depend upon other objects already shows that they are not self-standing in the same sense as Frege required numbers to be. That they have no 'hidden' kinds of modes of presentation further reveals that they are not, as it were, rightful inhabitants of the realm of reference.

7.7 The contrast with Russell

The minimal sense in which one may be in doubt as to whether two Fregean thoughts are the same derives from their definition as objects, not their original status as the cognitive significance of sentences. This contrasts sharply with Russell's propositions, the 'opacity' of which is constitutive, and, with it, the possibility of a version of the Caesar problem for propositions.

‘Truth’, Frege had told Russell in 1904, ‘is not a component part of a thought, just as Mont Blanc with its snowfields is not itself a component part of the thought that Mont Blanc is more than 4000 metres high’ (1904a: 163). About truth Russell agreed; about Mont Blanc, he famously replied,

> Concerning sense and meaning, I see nothing but difficulties which I cannot overcome. […] I believe that in spite of all its snowfields Mont Blanc itself is a component part of what is actually asserted in the proposition ‘Mont Blanc is more than 4000 metres high’. We do not assert the thought, for this is a private psychological matter: we assert the object of the thought, and this is, to my mind, a certain complex (an objective proposition, one might say) in which Mont Blanc is itself a component part. If we do not admit this, then we get the conclusion that we know nothing at all about Mont Blanc. This is why for me the *meaning* of a proposition is not the true, but a certain complex which (in the given case) is true. (Russell 1904b: 169)

In the passage above, Russell revealingly equates, not thoughts with propositions, but propositions with truth-values. In an earlier letter of 1903, he had written:

> I still do not quite share your opinion about sense and meaning. I should like to say the following about them. In all cases, both imagination and judgement have an object: what I call a ‘proposition’ can be the object of judgement, and it can be the object of imagination. There are therefore two ways in which we can think of an object, in case this object is a complex: we can imagine it, or we can judge it; yet the object is the same in both cases (e.g., when we say ‘the cold wind’ and when we say ‘The wind is cold’). To me, the judgement stroke therefore means a different way of being directed towards an object. Complexes are true or false: in judging, we aim at a true complex; but we may, of course, miss our aim. But truth is not a component part of the true, as *green* is a component part of a green tree. (Russell 1903b: 159)

Here we see Russell reiterating some of the claims that we attributed to him in 6.3, namely that propositions are complex objects, that judgement in some sense aims at truth, and that truth is not a constituent of propositions. But Russell makes explicit two further points. First, that to imagine or to judge a proposition is to imagine or to judge *something*, not to imagine or to judge *that* something is so. Second, that the same proposition is referred to by a sentence and its nominalization, which recalls his doctrine of assertion from the *Principles*.

In Ramsey's hands, the difference between a sentence and its nominalization corresponds to the difference between the expression of a fact and the description of an object. As he observed in 1927, a phrase such as 'the death of Caesar' can be used in two ways. Ordinarily, it is used as a description of a complex object, in this case the event of Caesar's death. But it may also be used as just an alternative expression of 'that Caesar died'. In such contexts as 'He was aware of the death of Caesar', Ramsey writes, even though the death of Caesar was the same event as his murder, the phrase is not interchangeable with 'the murder of Caesar', for one may know of Caesar's death without being aware that he was murdered. That is, 'He was aware of the death of Caesar' may be true and yet 'He was aware of the murder of Caesar' be false. Intentional contexts, then, 'the sort of case which occurs in the discussion of cognition' (1927: 141), are intensional: in such a context, 'He was aware of the death of Caesar' is equivalent to 'He was aware *that* Caesar died'.

Ramsey's discussion presupposes Russell's theory of definite descriptions. It is in terms of that theory that he establishes the connection between the two uses of the description. Russell had of course not yet developed that theory in 1904, but the two are independent. The fact that Russell did *not* draw Ramsey's distinction was not so much a failure on his part, as a sign that, in effect, for him, no context is genuinely intensional (cf. chapter 5). And this, in turn, is one aspect of Russell's view of cognitive relations as external relations between minds and propositions (cf. chapter 4).

Russell assimilates the nominalization of a sentence to the description of a complex, rather than to the expression of a fact in Ramsey's sense. For Russell, then, sentences too describe complexes. Hence, his propositions are 'opaque' not because they are 'coarse-grained thoughts', but because they are complex objects, which may appear to a mind in different ways.

Exactly the opposite holds in the case of Frege's thoughts. The sense in which two thoughts may not be known to be the same depends on the coarser notion of a thought implied the criterion of identity for thoughts. Fundamentally, though, thoughts are transparent, as dictated by the 'intuitive criterion of difference'. In turn, this manifests the fact that, despite being abstract objects, thoughts and thinking minds stand for Frege in an internal relation.

In chapter 6 we attributed the failure of Russell's theories of judgement to the fact that, on both theories, the relationship between a judgement and its putative content is external. What this meant was that, for Russell, judgments do not really have contents, but only objects. This was necessitated by the fact that any sort of complexity must for Russell follow the model of the combination of a relation and its terms. Hence to judge is fundamentally to judge *something*; to think, to think *of* something.

We have now completed the discussion by connecting this feature of Russellian judgement with the more familiar 'coarseness' of Russellian propositions. It is the fact that propositions are (complex) objects that grounds the possibility of there being a version of the Julius Caesar problem for propositions, that is, of their having aspects that are not immediately available when they are first grasped.

By contrast, Frege's thoughts are transparent. Their very nature as the cognitive significance of sentences, constituted by modes of presentation of objects, already implies that Frege's realism about thoughts cannot take the same form as Russell's realism about propositions. Thoughts are objects in so far as they are defined by abstraction. But their definition, which provides the only ground for questioning whether two thoughts are the same, reflects the way in which they are dependent upon other objects, namely by being thoughts *that* things are so. The sense, if any, in which judgement is a dual relation for Frege is therefore fundamentally different from the sense in which this holds for Russell, and within Frege's theory it is not fundamental at all.

Conclusion

Russell's world was a world of terms. Simple terms combined into complex ones, or atomic propositions. Atomic propositions combined into complex propositions. Complex propositions combined into yet more complex propositions.

This worldview provided Russell with a straightforward account of simple predication. In the simplest case, two simple terms, a thing and a predicate, or an individual and a universal, combine to form an atomic proposition. When his work into the logic of quantification forced him to recognize complex predication, he was therefore well-prepared not to mistake it for what it was not, i.e., *simple* predication.

We could thus establish our first main claim about Russell equally straightforwardly. For him, the attribution of a property to an individual is what is expressed by a simple predication in an atomic proposition.

However, Russell was *not* prepared to recognize complex predication for what *it* was, i.e., *predication*. Propositional functions were building blocks of quantified propositions, but not of their instances. But according to Russell, the only way in which something could be said to compose a proposition was by being one of its building blocks. Hence, propositional functions could in no way compose propositions.

In this respect, the way in which Russell regarded propositional functions contrasted with the way in which Frege viewed functions. For Frege, functions *could* compose propositions, or sentences, even if they would not compose them in the same way in which they would compose the quantified sentences of which they were instances. It was natural for Frege to conceive of functions alternatively as mere components and proper constituents (in Dummett's terms). Not only did he lack Russell's model of term composition, but his entire thought revolved around his account of quantification.

Because of this, one might expect Frege to lack an account of the composition of atomic sentences, and therefore of simple predication. Frege did recognize a category of simple predicates, but it is possible that those were complex in Dummett's sense. Besides, he seemed to appeal explicitly to his functional model of complexity in order to account for the composition of atomic propositions.

However, Frege's recognition of *a* category of simple predicates was nevertheless significant, in that it introduced an asymmetry among predicates relative to the priority of their formation with respect to atomic sentences. And his appeal to the functional model of complexity in the atomic case can at the very least be interpreted in light of the context principle.

The decisive evidence, however, came from his discussion of the so-called paradox of the concept *horse*. In that discussion, Frege was crystal clear that *his* notion of a function had been introduced anew for certain theoretical purposes, but had *not* eliminated the intuitive notion of a concept. And the same, we argued, could be said about properties. Hence Frege not only recognized complex predicates as predicates, he also recognized them as *complex*.

It was in this way that we arrived at our first main claim about Frege. That, for him, too, though less straightforwardly than for Russell, functions were *not* among the constituents of atomic propositions or sentences.

It is nevertheless likely that Frege did *not* have an account of simple predication. Yet, because of his adherence to the context principle, he would have been in a better position to do so than Russell, whose *building blocks* model of composition soon encountered the problem of propositional unity.

For Russell, propositions were again building blocks of logically compound propositions, and of propositions reporting psychological acts or states towards other propositions, i.e., propositional attitudes. In the case of judgement, this led into the problem of falsehood. But when Russell abandoned his dual-relation theory, however, he did not abandon his model of complexity. Hence judgement remained a relation between terms, although it was no longer a relation to a complex term. But as Wittgenstein and Ramsey showed, this relation could not yet be *the* relation of judgement. Russell's model of complexity was therefore suited to characterize external relations at best.

To some extent, Frege also conceived thoughts as objects. But that conception of thoughts was not backed by an ontology such as Russell's. At best, Frege's conception of thoughts as objects was again influenced by his functional model of complexity. But that model did not require thoughts to be counter-intuitively represented as opaque to the thinking subject. By contrast, Russell's account of propositions as complex terms did. The opacity of propositions was therefore little more than an epicycle of his model of term composition. We thus arrived at our second main contention about Frege and Russell. Only Frege's thoughts, not Russell's propositions, could be proper objects of thought.

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