THE POINCARÉ EXPONENT AND THE DIMENSIONS OF KLEINIAN LIMIT SETS

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Abstract. We provide a proof of the (well-known) result that the Poincaré exponent of a non-elementary Kleinian group is a lower bound for the upper box dimension of the limit set. Our proof only uses elementary hyperbolic and fractal geometry.

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1. Kleinian groups, limit sets, and the Poincaré exponent

For integers $n \geqslant 2$, n-dimensional hyperbolic space can be modelled by the Poincaré ball

$$\mathbb{D}^n = \{z \in \mathbb{R}^n : |z| < 1\}$$

equipped with the hyperbolic metric d_H given by

$$|ds| = \frac{2|dz|}{1-|z|^2}.$$

The group of orientation preserving isometries of $(\mathbb{D}^n, d_H)$ is the group of conformal automorphisms of $\mathbb{D}^n$, which we denote by $\mathrm{con}^+(\mathbb{D}^n)$. A good way to get a handle on this group is to view it as the (orientation preserving) stabliser of $\mathbb{D}^n$ as a subgroup of the Möbius group acting on $\mathbb{R}^n \cup \{\infty\}$. This group consists of maps given by the composition of reflections in spheres.

A group $\Gamma \leqslant \mathrm{con}^+(\mathbb{D}^n)$ is called *Kleinian* if it is discrete. Kleinian groups generate beautiful fractal limit sets defined by

$$L(\Gamma) = \overline{\Gamma(0)} \setminus \Gamma(0)$$

where $\Gamma(0) = \{g(0) : g \in \Gamma\}$ is the orbit of 0 under Γ and the closure is the Euclidean closure. Discreteness of Γ implies that all Γ-orbits are locally

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finite in $\mathbb{D}^n$ and this ensures that $L(\Gamma) \subseteq S^{n-1}$. Here S^{n-1} is the 'boundary at infinity' of hyperbolic space. A Kleinian group is called *non-elementary* if its limit set contains at least 3 points, in which case it is necessarily an uncountable perfect set.

The *Poincaré exponent* captures the coarse rate of accumulation to the limit set and is defined as the exponent of convergence of the *Poincaré series*

$$P_\Gamma(s) = \sum_{g \in \Gamma} \exp(-s d_H(0, g(0))) = \sum_{g \in \Gamma} \left(\frac{1 - |g(0)|}{1 + |g(0)|} \right)^s$$

for $s \geqslant 0$. The *Poincaré exponent* is therefore

$$\delta(\Gamma) = \inf\{s \geqslant 0 : P_\Gamma(s) < \infty\}.$$

It is a simple exercise to show that the *Poincaré series* may be defined using the orbit of an arbitrary $z \in \mathbb{D}^n$ at the expense of multiplicative constants depending only on z. In particular, the exponent of convergence does not depend on the choice of z. (The definition above uses $z = 0$.) For more background on hyperbolic geometry and Kleinian groups see [B83, M88].

There has been a great deal of interest in computing or estimating the fractal dimension of the limit set $L(\Gamma)$ (as a subset of Euclidean space $\mathbb{R}^n$) and the Poincaré exponent plays a central role. We write $\dim_{\mathrm{H}}$, $\overline{\dim}_{\mathrm{B}}$, $\dim_{\mathrm{A}}$ to denote the Hausdorff, upper box, and Assouad dimensions respectively. These constitute three distinct and well-studied notions of fractal dimension. See [F14] for more background on dimension theory and fractal geometry, especially the box and Hausdorff dimensions, and [F20] for the Assouad dimension. For all non-empty bounded sets $F \subseteq \mathbb{R}^n$,

$$0 \leqslant \dim_{\mathrm{H}} F \leqslant \overline{\dim}_{\mathrm{B}} F \leqslant \dim_{\mathrm{A}} F \leqslant n.$$

For all non-elementary Kleinian groups,

$$\delta(\Gamma) \leqslant \dim_{\mathrm{H}} L(\Gamma)$$

and for non-elementary *geometrically finite* Kleinian groups,

$$\delta(\Gamma) = \dim_{\mathrm{H}} L(\Gamma) = \overline{\dim}_{\mathrm{B}} L(\Gamma).$$

See [B93] for more details on geometric finiteness. Roughly speaking it means that the Kleinian group admits a reasonable fundamental domain. The equality of Hausdorff dimension and Poincaré exponent in the geometrically finite setting goes back to Sullivan [S84], see also Patterson [P76]. The coincidence with box dimension in this case was proved (rather later) independently by Bishop and Jones [BJ97] and Stratmann and Urbański [SU96]. The fact that the Poincaré exponent is always a lower bound for the Hausdorff dimension (without the assumption of geometric finiteness) is due to Bishop and Jones [BJ97]. See the survey [S04]. In the presence of parabolic elements the Assouad dimension can be strictly greater than $\delta(\Gamma)$, even in the geometrically finite situation, see [F19].

In the geometrically infinite setting, $\delta(\Gamma) < \dim_{\mathrm{H}} L(\Gamma) < \overline{\dim}_{\mathrm{B}} L(\Gamma)$ is possible, and it is an intriguing open problem to determine if $\dim_{\mathrm{H}} L(\Gamma) = \overline{\dim}_{\mathrm{B}} L(\Gamma)$ for all finitely generated Γ for $n \geqslant 4$. For $n = 3$, Bishop and Jones proved that for finitely generated, geometrically infinite Γ, $\dim_{\mathrm{H}} L(\Gamma) = \overline{\dim}_{\mathrm{B}} L(\Gamma) = 2$, see [BJ97]. This result was extended by Bishop to *analytically finite* Γ [B96, B97]. Falk and Matsuzaki characterised the upper box dimension of an arbitrary non-elementary Kleinian group in terms of the *convex core entropy* [FM15]. This can also be expressed as the exponent of convergence of an 'extended Poincaré series', but is more complicated to introduce.

Proving the general inequality $\delta(\Gamma) \leqslant \dim_{\mathrm{H}} L(\Gamma)$ involves carefully constructing a measure supported on the limit set and applying the mass distribution principle. Our investigation begins with the following question: since (upper) box dimension is a simpler concept than Hausdorff dimension, can we prove the weaker inequality $\delta(\Gamma) \leqslant \overline{\dim}_{\mathrm{B}} L(\Gamma)$ using only elementary methods? We provide a short and self-contained proof of this estimate in the sections which follow. It is instructive to think about why our proof fails to prove the equality $\delta(\Gamma) = \overline{\dim}_{\mathrm{B}} L(\Gamma)$ in general and what sort of extra assumptions on Γ would be needed to 'upgrade' the proof to yield this stronger conclusion.

The (upper) box dimension of a non-empty bounded set $F \subseteq \mathbb{R}^n$ can be defined in terms of the asymptotic behaviour of the volume of the r-neighbourhood of F. Given $r > 0$ the r-neighbourhood of F is denoted by F_r and consists of all points in $\mathbb{R}^n$ which are at Euclidean distance less than or equal to r from a point in F. Write V_E to denote the Euclidean volume, that is, n-dimensional Lebesgue measure. If $V_E(F) = 0$, then $V_E(F_r) \to 0$ as $r \to 0$. The upper box dimension of F captures this rate of decay and is defined formally by

$$\overline{\dim}_{\mathrm{B}} F = n - \liminf_{r \to 0} \frac{\log V_E(F_r)}{\log r}.$$

Another elementary proof of the estimate $\delta(\Gamma) \leqslant \overline{\dim}_{\mathrm{B}} L(\Gamma)$, at least for $n = 2, 3$, can be found in [B96, Lemmas 2.1 and 3.1]. Here the connection is made via 'Whitney squares'.

2. Proof of dimension estimate

Let Γ be an arbitrary non-elementary Kleinian group acting on the Poincaré ball and $\delta(\Gamma)$ denote the associated Poincaré exponent. We prove the following (well-known) inequality:

$$\delta(\Gamma) \leqslant \overline{\dim}_{\mathrm{B}} L(\Gamma). \tag{2.1}$$

Throughout we write $A \lesssim B$ to mean there is a constant $c > 0$ such that $A \leqslant cB$. Similarly, we write $A \gtrsim B$ if $B \lesssim A$ and $A \approx B$ if $A \lesssim B$ and $A \gtrsim B$. The implicit constants may depend on Γ and other fixed parameters, but it will be crucial that they never depend on the scale $r > 0$ used to compute the box dimension or on a specific element $g \in \Gamma$.

2.1. **Elementary estimates from hyperbolic geometry.** Since Γ is non-elementary, it is easy to see that it must contain a loxodromic element, h. Loxodromic elements have precisely two fixed points on the boundary at infinity. Let $z \in \mathbb{D}^n$ be a point lying on the (doubly infinite) geodesic ray joining the fixed points of h. We may assume z is not fixed by any elliptic elements in Γ since it is an elementary fact that the set of elliptic fixed points is discrete. Choose $a > 0$ such that the set

$$\{B_H(g(z), a)\}_{g \in \Gamma}$$

is pairwise disjoint, where $B_H(g(z), a)$ denotes the closed hyperbolic ball centred at $g(z)$ with radius a. To see that such an a exists recall that the orbit $\Gamma(z)$ is locally finite. As such, a can be chosen such that $B_H(z, 2a)$ contains only one point from the orbit $\Gamma(z)$, namely z itself. Then the pairwise disjointness of the collection $\{B_H(g(z), a)\}_{g \in \Gamma}$ is guaranteed since if $y \in B_H(g_1(z), a) \cap B_H(g_2(z), a)$ for distinct $g_1, g_2 \in \Gamma$, then

$$d_H(z, g_1^{-1} g_2(z)) = d_H(g_1(z), g_2(z)) \leqslant d_H(g_1(z), y) + d_H(y, g_2(z)) \leqslant 2a$$

which gives $g_1^{-1} g_2(z) \in B_H(z, 2a)$, a contradiction.

We will use the simple volume estimate

$$V_E(B_H(g(z), a)) \approx (1 - |g(z)|)^n \tag{2.2}$$

for all $g \in \mathrm{con}^+(\mathbb{D}^n)$, where the implicit constants are independent of g and z, but depend on a and n. This follows since $B_H(g(z), a)$ is a *Euclidean* ball with diameter comparable to $1 - |g(z)|$ (most likely not centred at $g(z)$). To derive this explicitly it is useful to recall the (well-known and easily derived) formula for hyperbolic distance

$$d_H(0, w) = \log \frac{1 + |w|}{1 - |w|}, \qquad (w \in \mathbb{D}^n).$$

The next result says that if $1 - |g(z)|$ is small, then the image of a fixed set under g must be contained in a comparably small neighbourhood of the limit set. This is the only point in the proof where the fact that the group is non-elementary is used. It is instructive to find an example of an elementary group where the conclusion fails.

Lemma 2.1. *Let a, z be as above. There exists a constant $c > 0$ depending only on Γ, a and z such that if $g \in \Gamma$ is such that $1 - |g(z)| < 2^{-k+1}$ for a positive integer k, then*

$$B_H(g(z), a) \subseteq L(\Gamma)_{c 2^{-k}}.$$

Proof. The idea is that there must be a loxodromic fixed point close to $g(z)$ and loxodromic fixed points are necessarily in the limit set. Indeed, $g(z)$ lies on the geodesic ray joining the fixed points of the loxodromic map ghg^{-1}.

These fixed points are the images of the fixed points of h under g and at least one of them must lie in the smallest Euclidean sphere passing through $g(z)$ and intersecting the boundary S^{n-1} at right angles. This uses the fact that geodesic rays are orthogonal to the boundary and g is conformal. The diameter of this sphere is

$$\lesssim 1 - |g(z)| < 2^{-k+1}$$

and the result follows, recalling that the Euclidean diameter of $B_H(g(z), a)$ is $\approx 1 - |g(z)|$. $\square$

2.2. **Estimating the Poincaré series using the limit set.** Let $s > t > \overline{\dim}_{\mathrm{B}} L(\Gamma)$. Then by definition

$$V_E(L(\Gamma)_r) \lesssim r^{n-t} \tag{2.3}$$

for all $0 < r < c/2$ with implicit constant independent of r but depending on t and where c is the constant from Lemma 2.1. Then

$$\begin{aligned}
P_\Gamma(s) &\approx \sum_{g \in \Gamma} \left(\frac{1 - |g(z)|}{1 + |g(z)|} \right)^s \\
&\approx \sum_{k=1}^{\infty} \sum_{\substack{g \in \Gamma: \\ 2^{-k} \leqslant 1 - |g(z)| < 2^{-k+1}}} (1 - |g(z)|)^s \\
&\approx \sum_{k=1}^{\infty} 2^{-k(s-n)} \sum_{\substack{g \in \Gamma: \\ 2^{-k} \leqslant 1 - |g(z)| < 2^{-k+1}}} (1 - |g(z)|)^n \\
&\lesssim \sum_{k=1}^{\infty} 2^{-k(s-n)} \sum_{\substack{g \in \Gamma: \\ 2^{-k} \leqslant 1 - |g(z)| < 2^{-k+1}}} V_E(B_H(g(z), a)) \qquad \text{(by (2.2))} \\
&\lesssim \sum_{k=1}^{\infty} 2^{-k(s-n)} V_E(L(\Gamma)_{c2^{-k}}) \qquad \text{(by Lemma 2.1 and choice of } a) \\
&\lesssim \sum_{k=1}^{\infty} 2^{-k(s-n)} 2^{-k(n-t)} \qquad \text{(by (2.3))} \\
&= \sum_{k=1}^{\infty} 2^{-k(s-t)} \\
&< \infty.
\end{aligned}$$

Therefore $\delta(\Gamma) \leqslant s$ and letting $s \to \overline{\dim}_{\mathrm{B}} L(\Gamma)$ proves (2.1).

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