



Polynomial time multiplication and normal forms in free bands [☆]

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ABSTRACT

We present efficient computational solutions to the problems of checking equality, performing multiplication, and computing minimal representatives of elements of free bands. A band is any semigroup satisfying the identity $x^2 \approx x$ and the **free band** $\text{FB}(k)$ is the free object in the variety of k -generated bands. Radoszewski and Rytter developed a linear time algorithm for checking whether two words represent the same element of a free band. In this paper we describe an alternate linear time algorithm for the same problem. The algorithm we present utilises a representation of words as synchronous deterministic transducers that lend themselves to efficient (quadratic in the size of the alphabet) multiplication in the free band. This representation also provides a means of finding the short-lex least word representing a given free band element with quadratic complexity.

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1. Introduction

This paper is about efficient computational solutions to the problems of checking equality, performing multiplication, and computing minimal representatives of elements of free bands. This paper arose from that of and Rytter [23], and is related to the papers of Neto and Sezinando [20,21]. We will discuss in more detail below the interactions of [20,21,23] and the current paper.

A **band** is any semigroup satisfying the identity $x^2 \approx x$ and a **free band** $\text{FB}(k)$ of rank k is the free object in the variety of k -generated bands. Bands have been widely studied in the literature, some highlights include: the lattice of varieties of bands is completely classified [1,7,9,10] and every such variety is defined by a single identity. Siekmann and Szabó [26] provide an infinite complete rewriting system for every free band; rewriting in further varieties of bands has also been considered in [14]. More recently, combinatorial structures such as hyperplane arrangements, interval greedoids, and matroids have been endowed with the structure of left regular bands (i.e. those bands belonging to the variety defined by the identity $xy \approx yx$), and this connection has been used in several disparate areas of mathematics; see, for example [16] and the references therein. Other recent references related to bands include [5,22,25].

In the context of the famous Burnside problem for semigroups, Green and Rees [11] showed that if every k -generated group, where $k \in \mathbb{N}$, satisfying the identity $x^{m-1} \approx 1$ is finite, then every k -generated semigroup satisfying the identity $x^m \approx x$ is finite. Brown [2] gave a simple proof of Green and Rees' result. In the case that $m = 2$, every k -generated group satisfying the identity $x^1 \approx 1$ is trivial, by virtue of which, every k -generated semigroup satisfying the identity $x^2 \approx x$ is

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finite. In particular, every finitely generated band is finite, including every free band $FB(k)$. The size of $FB(k)$ grows super-exponentially with k the number of generators; a closed form for $|FB(k)|$ is given in [12, Theorem 4.5.3], see also [13]. For example, the free band $FB(5)$ has size 2, 751, 884, 514, 765 $\approx 2^{41}$.

The ability to compute efficiently with elements in a free band is a fundamental prerequisite for any further computational tools for the study of bands; such as determining the least variety of bands containing a given band, or computing with finite band presentations. However, the vast number of elements renders it impossible to practically apply to free bands, with 5 or more generators, any of the known general purpose algorithms for finite semigroups such as, for example, the Froidure-Pin Algorithm [8] or the algorithms described in [6]. When it is possible to readily multiply and check equality of elements of a finitely generated semigroup S , the Froidure-Pin Algorithm [8] can be used to exhaustively enumerate S . Even assuming that the question of how to multiply and check the equality of elements in the free band is resolved, the Froidure-Pin Algorithm [8] requires too much space to compute $FB(k)$ with $k \geq 5$. If it were possible (and it is not) to store every one of the approximately 2^{41} elements of $FB(5)$ in a single bit, this would require more than 274 GB of RAM, a figure that is unlikely to be commonplace any time in the near future. The algorithms in [6] are most effective for semigroups containing large subgroups, but alas every subgroup of a band is trivial, so the algorithms from [6] are of little practical use for bands.

Semigroup presentations provide a possible alternative approach to computing with free bands, for example, using the Todd-Coxeter [27], Knuth-Bendix [15] or other rewriting methods. If A is any non-empty alphabet, then the collection A^+ of all finite non-empty words over A forms a semigroup where the operation is juxtaposition; A^+ is called the **free semigroup** on A . If $w \in A^+$, then the length of w is denoted by $|w|$. We denote the free band generated by A by $FB(A)$ and henceforth will not use the notation $FB(k)$ where $k \in \mathbb{N}$. One, not particularly illuminating, description of the free band $FB(A)$ on A is the quotient of A^+ by the least congruence \sim containing (w^2, w) for all $w \in A^+$. In other words, $FB(A)$ has the infinite presentation

$$\langle A \mid x^2 = x, x \in A^+ \rangle.$$

Of course, since $FB(A)$ is finite, it is finitely presented, although it is not immediately clear how to find a finite presentation when $|A| \geq 5$. Siekmann and Szabó [26] show that the following length reducing infinite conditional rewriting system for $FB(A)$ is complete:

$$\begin{aligned} x^2 &\longrightarrow x & (1) \\ xyz &\longrightarrow xz \text{ if } \text{cont}(y) \subseteq \text{cont}(x) = \text{cont}(z) & (2) \end{aligned}$$

where x, y, z range over all appropriate words in A^+ and where $\text{cont}(w) = \{b_1, b_2, \dots, b_n\}$ is the **content** of the word $w = b_1 b_2 \dots b_n \in A^+$. A simpler proof that the rewriting system of Siekmann and Szabó [26] is complete was given by Neto and Sezinando in [20, Theorem 6.1]. Although the papers [20,21,26] make a valuable contribution to the study of bands, their focus is mainly on the mathematical aspects. In particular, it is claimed in [20] that there is a polynomial-time solution for the word problem in all relatively free bands other than $FB(A)$, and that Siekmann and Szabó provide a quadratic time algorithm for solving the word problem in the free band in [26]. However, the algorithm is not explicitly given in [20], or elsewhere in the literature, nor is its time complexity formally analysed. Since Siekmann and Szabó's rewriting system is complete and length reducing, it can be used to solve the word problem by rewriting a given word w in $\mathcal{O}(|w|)$ steps. However, it is unclear what the time complexity of each rewriting step is, since the rewriting system is infinite, and so it is non-trivial to detect which rewrite rules apply to any given word w . This question is not addressed in [26]. It is possible to detect all subwords x^2 where x is primitive (not of the form $x = y^e$ for some $y \in A^+, e \geq 2$) of a given w in $\mathcal{O}(|w| \log |w|)$ time; see [4]. It is unclear how to efficiently find occurrences of xyz with $\text{cont}(y) \subseteq \text{cont}(x) = \text{cont}(z)$ in w . One approach might be to iterate through the subwords of w , to find a single subword of the form xyz where $\text{cont}(y) \subseteq \text{cont}(x) = \text{cont}(z)$. The time complexity of this approach is, at best, $\mathcal{O}(|w|^2)$ yielding an algorithm with time complexity at best $\mathcal{O}(|w|^3)$ for finding a normal form for w . Hence for words $u, v \in A^*$ solving the word problem using this approach has complexity at best $\mathcal{O}(|u|^3 + |v|^3)$.

Rather than using the infinite rewriting system given in (1), it is possible to compute the unique finite complete rewriting system for $FB(A)$ when $|A| < 5$ with respect to the short-lex reduction ordering on A^+ obtained from any linear order of A (using, for example, the Froidure-Pin Algorithm [8], as implemented in [17]); see Table 1.1. The Froidure-Pin Algorithm [8] stores every element of the semigroup it is enumerating, and so, as discussed above, it was not possible to compute a finite complete rewriting system for $FB(A)$ when $|A| \geq 5$. For reference, the time complexity of the Froidure-Pin Algorithm is $\mathcal{O}(|A| \cdot |FB(A)|)$.

With the preceding comments in mind, it seems unlikely that presentations or rewriting systems would allow for practical computation of the word problem, or normal forms, when $|A| \geq 5$. On the other hand, Radoszewski and Rytter's algorithm from [23] (as implemented in [19], and available in [17] and [18]) can test equality of words in $FB(A)$ when $|A| \geq 100$ with relative ease.

In this paper, we will show how to use a representation of elements of a free band as transducers in linear time algorithms for equality checking and the computation of normal forms, and an algorithm for multiplication that is quadratic

Table 1.1
Number of relations and total length of relations in the unique finite complete rewriting system of $FB(A)$ for some small values of $|A|$.

$ A $	$ FB(A) $	Number of relations	Total length of relations
1	1	1	1
2	6	4	20
3	159	45	468
4	332,380	11,080	217,072
5	2,751,884,514,765	?	?

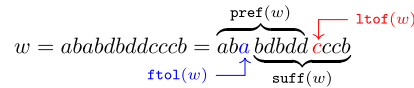


Fig. 1. An illustration of the functions pref , suff , ftol , and ltof applied to the word $w = ababdbddcccb$.

in the size of the alphabet and linear in the size of the representations of the elements; see below for a formal description of the complexity. This representation is similar to the “admissible maps” of [20,21] and the “factor automata” of [23].

The starting point for the development of the representation of free band elements by transducers (and the other representations in [20,21,23]), is the solution to the word problem given by Green and Rees in [11]; we will restate this description in Section 2. Recall that we denoted by \sim the least congruence on A^+ containing $\{(x^2, x) : x \in A^+\}$. We refer to the problem of determining whether or not $u \sim v$ for any $u, v \in A^+$ as **checking equality in the free band**. If $w \in A^+$, then we define $\text{pref}(w)$ to be the longest prefix of w containing one fewer unique letters than w itself (i.e. $\text{cont}(\text{pref}(w)) = \text{cont}(w) - 1$) and $\text{suff}(w)$ is defined analogously for suffixes. We denote the letter immediately after $\text{pref}(w)$ in w by $\text{ltof}(w)$ (from “last letter to occur first”), which is the unique letter in the content of w not in the content of $\text{pref}(w)$. The first letter $\text{ftol}(w) \in A$ to occur last in w is defined analogously with respect to $\text{suff}(w)$. Green and Rees [11] showed that $u \sim v$ if and only if $\text{pref}(u) \sim \text{pref}(v)$, $\text{ltof}(u) = \text{ltof}(v)$, $\text{ftol}(u) = \text{ftol}(v)$, and $\text{suff}(u) \sim \text{suff}(v)$. This is illustrated in Fig. 1.

It is trivial to check whether or not two letters in A are equal. If $\text{ltof}(u) = \text{ltof}(v)$ and $\text{ftol}(u) = \text{ftol}(v)$, then the Green-Rees Theorem reduces the problem of checking $u \sim v$ for $u, v \in A^+$ to the problem of checking $\text{pref}(u) \sim \text{pref}(v)$ and $\text{suff}(u) \sim \text{suff}(v)$ where $\text{pref}(u), \text{pref}(v) \in (A \setminus \{\text{ltof}(u) = \text{ltof}(v)\})^+$ and $\text{suff}(u), \text{suff}(v) \in (A \setminus \{\text{ftol}(u) = \text{ftol}(v)\})^+$. In particular, the size of the underlying alphabet is reduced by one, and therefore, by repeatedly applying the Green-Rees Theorem, we eventually reduce the size of the alphabet to one. Performing this naively requires $\mathcal{O}((|u| + |v|) \cdot 2^{|A|})$ time and $|u| + |v|$ space. A similar recursive approach was found by Gerhard and Petrich [9] to solve the word problem in relatively free bands (the free objects in sub-varieties of bands). Suppose that A is any fixed alphabet and that $u, v \in A^+$ are such that $|u| + |v| = k$ for some $k \in \mathbb{N}$. Neto and Sezinando [20] estimate the time complexity of Gerhard and Petrich’s approach for the free band to be $\mathcal{O}(2^k)$, which is worse than the naïve approach using the Green-Rees Theorem. However, for other relatively free bands, Neto and Sezinando [20] estimate the time complexity of Gerhard and Petrich’s approach to be $\mathcal{O}((k + 1)^{n-2})$ where $n \in \mathbb{N}$ is the height of the relatively free band in the lattice of all varieties of bands.

In contrast to the exponential time algorithms described in the previous paragraph, Radoszewski and Rytter [23] gave a $\mathcal{O}(|A| \cdot (|u| + |v|))$ time and $\mathcal{O}(|u| + |v|)$ space algorithm for checking equality in the free band on A . Their method constructs an acyclic automaton, called the **interval automaton**, for each element to be compared. In [23] it is shown that $u \sim v$ if and only if the corresponding interval automata recognize the same language. It is possible to check whether two acyclic automata recognize the same language in $\mathcal{O}(n)$ time where n is the total number of states in both automata; see, for example, [24]. At least from the perspective of the asymptotic time complexity, it seems extremely unlikely that a better solution than that of Radoszewski and Rytter exists for checking whether or not $u \sim v$ where $u, v \in A^+$. At best we could hope to replace the $|A|$ factor by a lesser function of $|A|$. But $|u| + |v|$ can be significantly larger than $|A|$ and so the $|A|$ factor does not contribute significantly to the overall time complexity in such cases.

Radoszewski and Rytter’s approach does not immediately give rise to an efficient representation of elements in the free band. In particular, the only representation of elements in the free band considered in [23] is by words in a free semigroup. Representing elements by words has the advantage of a very simple and (time) efficient multiplication algorithm: simply concatenate the words $u, v \in A^+$ in constant time and $\mathcal{O}(|u| + |v|)$ space. The main drawback of this representation is that the length of these words is unbounded. Of course, given any word $w \in A^+$ and any well-ordering \prec on A^+ , the equivalence class w/\sim contains a \prec -minimum word, and since $FB(A)$ is finite, this means that length of such minimum words is also bounded. Therefore, any method for finding such minimum words would provide a bound for the space required to store an element represented as a word. The rewriting system of [26] gives a way of finding the \prec -minimum word with respect to the short-lex ordering. As discussed above, in practice, this only works for alphabets A with $|A| \leq 4$. Assuming this was not the case, Neto and Sezinando’s claim in [21] implies that minimizing a word $w \in A^+$ has time complexity at best $\mathcal{O}(|w|^2)$.

In this paper, we derive an alternative space efficient representation for an element in the free band by using synchronous deterministic acyclic transducers based on the interval automata of Radoszewski and Rytter [23] and related to

the admissible maps of [20,21]. This representation lends itself to efficient solutions to the problems of checking equality and multiplication of elements of the free band, and to an algorithm for determining $\min(w)$ – the short-lex minimum representative of the \sim -class of a given word w . In Section 2, we establish some notation, and state the Green-Rees Theorem in terms of this notation. In Section 3, we describe the novel transducer based representation of elements of the free band that is the central notion of this paper. We also describe how Radoszewski and Rytter’s construction from [23] relates to these transducers. In Section 5, we describe how to check equality of the elements in the free band represented by transducers in $\mathcal{O}(|A| \cdot (|u| + |v|))$ space and time. The procedure for checking equality is called `EqualInFreeBand` and given in Algorithm 5.1. Although the $\mathcal{O}(|A| \cdot (|u| + |v|))$ space representation by transducers we present is worse than the $\mathcal{O}(|u| + |v|)$ space complexity of Radoszewski and Rytter in [23], we will show that the same representation can be used efficiently for multiplication and finding short-lex least words. In Section 6, we describe how to obtain a transducer representing the product of two free band elements x and y given transducers representing each of the elements. This algorithm is called `Multiply` and is given in Algorithm 6.2. `Multiply` has time and space complexity $\mathcal{O}(m + n + |A|^2)$ where m and n are the number of states in any transducers representing x and y . We will show that the number of states of the minimal transducer representing a free band element w/\sim is $|\min(w)||A|$ where $\min(w)$ is the length of the short-lex least word belonging to w/\sim . Hence the best case time and space complexity of the multiplication algorithm `Multiply` in Section 6 is $\mathcal{O}(|\min(u)||A| + |\min(v)||A| + |A|^2)$ and the worst case is $\mathcal{O}(|u||A| + |v||A| + |A|^2)$. Finally, in Section 7, we give the `MinWord` algorithm, Algorithm 7.2, for computing the short-lex least word $\min(w)$ in w/\sim given any transducer \mathcal{T} representing w/\sim . This algorithm has time and space complexity $\mathcal{O}(n \cdot |A|)$ where n is the number of states of \mathcal{T} .

We conclude the introduction by stating some open problems. Clearly, any algorithm deciding whether or not words $u, v \in A^*$ represent the same element of a free band has time complexity bounded below by $\Omega(|u| + |v|)$. On the other hand, `EqualInFreeBand` given in Algorithm 5.1 has time complexity $\mathcal{O}(|A| \cdot (|u| + |v|))$ time, so it is natural to ask if there is a way of reducing or removing the $|A|$ factor?

Question 1.1. *What is the lower bound for an algorithm checking the equality of two elements in the free band?*

In a similar vein, any algorithm computing $\min(w)$ for $w \in A^*$ must take at least $\Omega(|w|)$ time to read the input, if nothing else. The algorithm `MinWord` Algorithm 7.2 (together with the construction of the interval transducer that is the input to `MinWord`) solves this problem in $\mathcal{O}(|A| \cdot |w| + |A|^2 \cdot |\min(w)|)$ time.

Question 1.2. *What is the lower bound for an algorithm for finding $\min(w)$ given $w \in A^*$?*

A variety of bands is a collection of bands that is closed under taking subsemigroups, homomorphic images and arbitrary Cartesian products. Given that the lattice of varieties of bands is completely determined (as mentioned above), we ask the following question.

Question 1.3. *Are there analogues for any of the algorithms `EqualInFreeBand`, `Multiply`, and `MinWord` for the free object in other varieties of bands?*

If such algorithms do exist, then they could serve as an entry point for the further development of computational tools for finitely presented bands.

2. Preliminaries

In this section, we introduce some notation and preliminary results from the literature that will be used throughout the remainder of this paper.

As mentioned in the previous section, the Green-Rees Theorem [11] characterizes elements of the free band in terms of the functions `pref`, `suff`, `ltoF`, and `ftoL`. This characterisation is one of the key components in this paper, but we require slightly different (and more general) notation for these functions, which we now introduce.

If A is any non-empty alphabet, then the collection A^* of all finite words over A forms a monoid with juxtaposition being the operation. This monoid is called the **free monoid** over A ; the identity element of A^* is the empty word ε ; and $A^* = A^+ \cup \{\varepsilon\}$.

If X and Y are sets, then a **partial function** $f : X \rightarrow Y$ is just a function from some subset of X to Y . If $f : X \rightarrow Y$ is any partial function, and f is not defined at $x \in X$, then we will write $f(x) = \perp$ and we also assume that $f(\perp) = \perp$.

Definition 2.1. We define the partial function $\circ : A^* \times \{0, 1\}^* \rightarrow A^*$ such that for all $w \in A^*$ the following hold:

- (i) $\varepsilon \circ \alpha = \perp$ for all $\alpha \in \{0, 1\}^+$ and $w \circ \varepsilon = w$;
- (ii) $w \circ 0$ is the longest prefix of w such that $|\text{cont}(w \circ 0)| = |\text{cont}(w)| - 1$;
- (iii) $w \circ 1$ is the longest suffix of w such that $|\text{cont}(w \circ 1)| = |\text{cont}(w)| - 1$;
- (iv) if $\alpha = \beta\gamma \in \{0, 1\}^+$, for some $\beta \in \{0, 1\}$ and $\gamma \in \{0, 1\}^*$, then $w \circ \alpha = (w \circ \beta) \circ \gamma$.

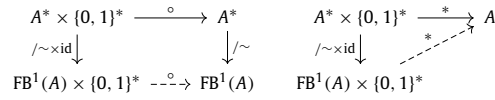


Fig. 2. Commutative diagrams giving the definition of the partial functions \circ and $*$ for the free band $\text{FB}^1(A)$.

Note that $w \circ 0 = \text{pref}(w)$ and $w \circ 1 = \text{suff}(w)$ in the notation of the Green-Rees Theorem.

Definition 2.2. We define the partial function $*$: $A^* \times \{0, 1\}^* \rightarrow A$ so that for all $w \in A^*$ the following hold:

- (i) $\varepsilon * \alpha = \perp$ and $w * \varepsilon = \perp$ for all $\alpha \in \{0, 1\}^+$;
- (ii) $w * 0 \in A$ is such that $w = (w \circ 0)(w * 0)w'$ for some $w' \in A^*$;
- (iii) $w * 1 \in A$ is such that $w = w''(w * 1)(w \circ 1)$ for some $w'' \in A^*$;
- (iv) if $\alpha = \beta\gamma \in \{0, 1\}^+$, for some $\beta \in \{0, 1\}$ and $\gamma \in \{0, 1\}^*$, then $w * \alpha = (w \circ \beta) * \gamma$.

As above, $w * 0 = \text{ltof}(w)$ and $w * 1 = \text{ftol}(w)$ in the notation of the Green-Rees Theorem.

We now state the Green-Rees Theorem using \circ and $*$ instead of pref , ltof , ftol , and suff ; for a proof see Lemma 4.5.1 in [12].

Theorem 2.3 (Green-Rees). Let $v, w \in A^+$. Then $v \sim w$ if and only if the following hold:

- (i) $\text{cont}(v) = \text{cont}(w)$;
- (ii) $v \circ \alpha \sim w \circ \alpha$ for all $\alpha \in \{0, 1\}$; and
- (iii) $v * \alpha = w * \alpha$ for all $\alpha \in \{0, 1\}$. \square

For technical reasons, it is preferable to consider the free band with identity adjoined; which we denote by $\text{FB}^1(A)$. Clearly $\text{FB}^1(A)$ is the quotient of the free monoid A^* by the least congruence containing (w, w^2) for all $w \in A^*$. Henceforth we will only consider $\text{FB}^1(A)$ and as such we write \sim without ambiguity to denote the least congruence on A^* containing (w, w^2) for all $w \in A^*$; and we will refer to $\text{FB}^1(A)$ as the free band.

If $v, w \in A^*$ are such that $v \sim w$, then $\text{cont}(v) = \text{cont}(w)$ by Theorem 2.3. Hence we may define the **content** of w/\sim in $\text{FB}^1(A)$ as being $\text{cont}(w)$; we denote this by $\text{cont}(w/\sim)$ also.

The partial function $\circ : A^* \times \{0, 1\}^* \rightarrow A^*$ induces a partial function from $\text{FB}^1(A) \times \{0, 1\}^*$ to $\text{FB}^1(A)$. Abusing our notation, we will also denote the induced partial function by \circ . Specifically, $\circ : \text{FB}^1(A) \times \{0, 1\}^* \rightarrow \text{FB}^1(A)$ is defined by

$$(w/\sim) \circ \alpha = (w \circ \alpha)/\sim$$

for all $w \in A^*$ such that $w \circ \alpha$ is defined; see Fig. 2.

The domain of the function $\circ : \text{FB}^1(A) \times \{0, 1\}^* \rightarrow \text{FB}^1(A)$ is $\{(x, \alpha) \in \text{FB}^1(A) \times \{0, 1\}^+ : |\alpha| \leq |\text{cont}(x)|\}$. It follows from Theorem 2.3 that $\circ : \text{FB}^1(A) \times \{0, 1\}^* \rightarrow \text{FB}^1(A)$ satisfies the analogue of Definition 2.1 where $w \in A^*$ is replaced by w/\sim in $\text{FB}^1(A)$. Similarly, we may define $*$: $\text{FB}^1(A) \times \{0, 1\}^* \rightarrow A$ by

$$(w/\sim) * \alpha = w * \alpha$$

for every $w \in A^*$ such that $w * \alpha$ is defined; see Fig. 2. Theorem 2.3 ensures that $*$ is well-defined, and satisfies the appropriate analogue of Definition 2.2.

3. Representation by synchronous transducers

If $w \in A^*$ is arbitrary, then naively computing all possible values of $w \circ \alpha$ and $w * \alpha$ for $\alpha \in \{0, 1\}^*$ has exponential time complexity because there are an exponential number $2^{|\text{cont}(w)|}$ of $\alpha \in \{0, 1\}^*$ such that $w \circ \alpha$ and $w * \alpha$ are defined.

In this section, we introduce a means of representing an element $x = w/\sim$ of $\text{FB}^1(A)$ which will permit us to efficiently compute all the possible values of $w \circ \alpha$ and $w * \alpha$ for $\alpha \in \{0, 1\}^*$ in Section 5.

We will associate to every word $w \in A^*$ a partial function $f_w : \{0, 1\}^* \rightarrow A^*$. This will yield a bijection from $\text{FB}^1(A)$ to the partial functions with domain $\{0, 1\}^k$ and codomain A^k for $1 \leq k \leq |A|$.

Definition 3.1. For a given $w \in A^*$, define the partial function $f_w : \{0, 1\}^* \rightarrow A^*$ recursively by letting $f_\varepsilon(\varepsilon) = \varepsilon$ and $f_\varepsilon(\alpha) = \perp$ for all $\alpha \in \{0, 1\}^+$. If $w \in A^+$, then we define $f_w : \{0, 1\}^* \rightarrow A^*$ by $f_w(\varepsilon) = \perp$ and $f_w(\alpha) = (w * \beta) f_{w \circ \beta}(\gamma)$ for all $\alpha \in \{0, 1\}^+$ where $\alpha = \beta\gamma$ for some $\beta \in \{0, 1\}$ and $\gamma \in \{0, 1\}^*$.

For $\alpha = \alpha_1 \dots \alpha_n \in \{0, 1\}^*$, we write $\alpha_{(i,j)} = \alpha_i \alpha_{i+1} \dots \alpha_j$ to be the subword starting at index $i \in \{1, \dots, n\}$ and ending at index $j \in \{i, \dots, n\}$. If $j < i$, then we define $\alpha_{(i,j)} = \varepsilon$. By Definition 3.1,

$$f_w(\alpha\beta) = \left(\prod_{i=1}^n (w * \alpha_{(1,i)}) \right) f_{w \circ \alpha}(\beta)$$

for all $\alpha, \beta \in \{0, 1\}^*$ where $\alpha = \alpha_1\alpha_2 \cdots \alpha_n$ and $\alpha_i \in \{0, 1\}$ for $1 \leq i \leq n$. As an example, we compute $f_{abac}(010)$ as follows:

$$\begin{aligned} f_{abac}(010) &= (abac * 0) f_{abac \circ 0}(10) = cf_{aba}(10) = c(aba * 1) f_{aba \circ 1}(0) \\ &= cbf_a(0) = cb(a * 0) f_{a \circ 0}(\varepsilon) = cbaf_\varepsilon(\varepsilon) = cba. \end{aligned}$$

In a similar way, it is possible to determine $f_{abac}(\alpha)$ for any $\alpha \in \{0, 1\}^3$:

$$\begin{aligned} f_{abac}(000) &= cba, & f_{abac}(001) &= cba, & f_{abac}(010) &= cba, & f_{abac}(011) &= cba, \\ f_{abac}(100) &= bca, & f_{abac}(101) &= bca, & f_{abac}(110) &= bac, & f_{abac}(111) &= bac. \end{aligned}$$

We will show that the partial function f_w can be realized by a transducer for every $w \in A^*$. For the sake of completeness, we next give a definition of a synchronous deterministic transducer.

Definition 3.2. A **deterministic synchronous transducer** is a septuple $\mathcal{T} = (Q, \Sigma, \Gamma, q_0, T, \odot, \otimes)$ where

- (i) Q is a non-empty set called the **set of states**;
- (ii) Σ is a set called the **input alphabet**;
- (iii) Γ is a set called the **output alphabet**;
- (iv) $q_0 \in Q$ is called the **initial state**;
- (v) $\emptyset \neq T \subseteq Q$ is called the **set of terminal states**;
- (vi) a partial function $\odot : Q \times \Sigma \rightarrow Q$ called the **state transition function**;
- (vii) a partial function $\otimes : Q \times \Sigma \rightarrow \Gamma$ called the **letter transition function**;

where for all $q \in Q, \alpha \in \Sigma, q \odot \alpha \neq \perp$ is if and only if $q \otimes \alpha \neq \perp$.

Since the only type of transducers we consider are deterministic synchronous transducers, we will write “transducer” to mean “deterministic synchronous transducer”.

The definition of \odot can be extended recursively for inputs in $Q \times \Sigma^*$ by letting $q \odot \varepsilon = q$ and $q \odot \alpha\beta = (q \odot \alpha) \odot \beta$ for all $\alpha \in \Sigma, \beta \in \Sigma^*$. Similarly, we may extend \otimes to inputs in $Q \times \Sigma^*$ by letting $q \otimes \varepsilon = \perp$ and $q \otimes \alpha\beta = (q \odot \alpha) \otimes \beta$ for all $\alpha \in \Sigma, \beta \in \Sigma^*$. Let $\mathcal{T} = (Q, \Sigma, \Gamma, q_0, T, \odot, \otimes)$ be a transducer. Then the **language accepted** by \mathcal{T} is $L(\mathcal{T}) = \{\alpha \in \Sigma^* : q_0 \odot \alpha \in T\}$. We say that a partial function $f : \Sigma^* \rightarrow \Gamma^*$ is **realized by** \mathcal{T} if and only if $f(\alpha) = q_0 \otimes \alpha$ for all $\alpha \in L(\mathcal{T})$ and $f(\alpha) = \perp$ for all $\alpha \notin L(\mathcal{T})$. If A is any set, and $w \in A^*$, then there are many transducers realising the function f_w given in Definition 3.1; one such transducer is defined as follows.

Example 3.3. Let $w \in A^*$ be arbitrary and let $k = |\text{cont}(w)|$. Then we define the transducer

$$\left(\{0, 1\}^{\leq k}, \{0, 1\}, A, \varepsilon, \{0, 1\}^k, \otimes, \odot \right)$$

with states $\{0, 1\}^{\leq k} = \bigcup_{i=0}^k \{0, 1\}^i$, input alphabet $\{0, 1\}$, output alphabet A , initial state ε , terminal states $\{0, 1\}^k$, and state and letter transition functions given respectively by:

$$q \odot x = qx \quad q \otimes x = w * (qx).$$

It is routine to verify that this transducer realises f_w . A picture of the corresponding transducer realizing f_{abac} can be found in Fig. 3.

The relevance of the functions f_w to free bands is established in the next result.

Lemma 3.4. Let $v, w \in A^*$. Then $v \sim w$ if and only if $f_v = f_w$.

Proof. If $|\text{cont}(v)| \neq |\text{cont}(w)|$, then $\text{cont}(v) \neq \text{cont}(w)$, so $v \not\sim w$ by Theorem 2.3. Also $f_v \neq f_w$, since the domains of f_v and f_w disagree. Therefore we only need to prove the lemma in the case $|\text{cont}(v)| = |\text{cont}(w)|$.

We proceed by induction on the size of the content. Clearly if $|\text{cont}(w)| = |\text{cont}(v)| = 0$, then $v = w = \varepsilon$ and $f_v = f_w = f_\varepsilon$, so the statement holds. Assume that the statement of the lemma holds for all $v, w \in A^*$ with $|\text{cont}(v)| = |\text{cont}(w)| < k$ for some $k \geq 1$.

Let $v, w \in A^*$ with $|\text{cont}(v)| = |\text{cont}(w)| = k$. If $v \sim w$, then, by Theorem 2.3, $v \circ \beta \sim w \circ \beta$ and $v * \beta = w * \beta$ for all $\beta \in \{0, 1\}$. By the inductive hypothesis, since $|\text{cont}(v \circ \beta)|, |\text{cont}(w \circ \beta)| < k$, it follows that $f_{v \circ \beta} = f_{w \circ \beta}$ for all $\beta \in \{0, 1\}$. But then $f_v(\alpha) = (v * \beta) f_{v \circ \beta}(\gamma) = (w * \beta) f_{w \circ \beta}(\gamma) = f_w(\alpha)$ for all $\alpha \in \{0, 1\}^k$, so $f_v = f_w$.

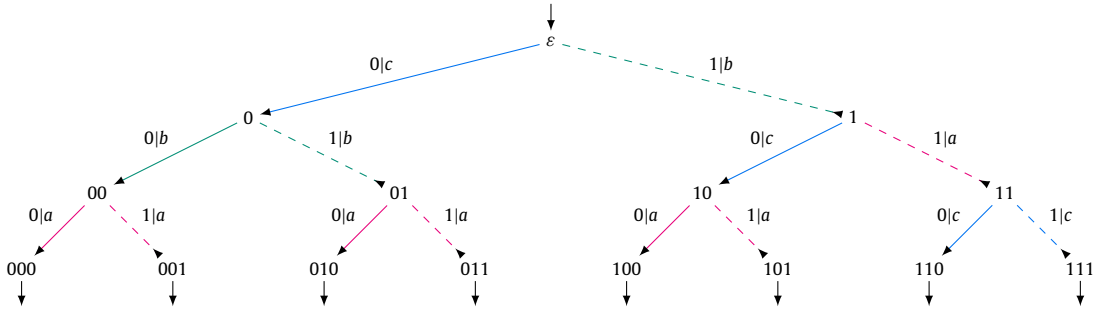


Fig. 3. A transducer realizing f_{abac} . Note that transitions on input 0 are drawn with a solid arrow and transitions on input 1 are drawn with a dashed and reversed arrow for visual differentiation. In subsequent diagrams we will omit the node labels and part of the edge labels for the sake of brevity.

On the other hand, if $f_v = f_w$, then $(v * \beta)f_{v \circ \beta}(\gamma) = (w * \beta)f_{w \circ \beta}(\gamma)$ for all $\beta \in \{0, 1\}$, $\gamma \in \{0, 1\}^{k-1}$. But then $v * \beta = w * \beta$ for all $\beta \in \{0, 1\}$, and $f_{v \circ \beta}(\gamma) = f_{w \circ \beta}(\gamma)$ for all $\gamma \in \{0, 1\}^{k-1}$, so $f_{v \circ \beta} = f_{w \circ \beta}$ for all $\beta \in \{0, 1\}$. Hence, by the inductive hypothesis, $v \circ \beta \sim w \circ \beta$ for all $\beta \in \{0, 1\}$. Finally, by Theorem 2.3, $v \circ 0 \sim w \circ 0$ implies $\text{cont}(v \circ 0) = \text{cont}(w \circ 0)$ so that $\text{cont}(v) = \{v * 0\} \cup \text{cont}(v \circ 0) = \{w * 0\} \cup \text{cont}(w \circ 0) = \text{cont}(w)$. It follows, again by Theorem 2.3, that $v \sim w$. \square

It follows from Lemma 3.4 that if $x \in \text{FB}^1(A)$, then we may denote by f_x any of the functions f_w where $w \in A^*$ and $w/\sim = x$ without ambiguity. In the same vein, we say that a transducer \mathcal{T} **represents** $x \in \text{FB}^1(A)$ if \mathcal{T} realizes f_x .

For a transducer $\mathcal{T} = (Q, \Sigma, \Gamma, q_0, T, \odot, \otimes)$ and a state $p_0 \in Q$, we define the **induced subtransducer rooted at** p_0 to be the transducer $\mathcal{T}' = (Q, \Sigma, \Gamma, p_0, T, \odot, \otimes)$, i.e. the transducer with the same states, transitions and terminal states, but whose initial state is p_0 instead of q_0 . Let g be the function realized by \mathcal{T} , and let g' be the function realized by the subtransducer \mathcal{T}' . It can be shown that if $q_0 \odot \alpha = p_0$ for some $\alpha \in \Sigma^*$, then $g'(\beta)$ is the suffix of $g(\alpha\beta)$ of length $|\beta|$ for all $\beta \in \Sigma^*$ such that $g(\alpha\beta) \neq \perp$; and where $g'(\beta) = \perp$ otherwise.

If a transducer \mathcal{T} represents $x \in \text{FB}^1(A)$, it would be convenient if the subtransducer rooted at $q_0 \odot \alpha$ represented $x \circ \alpha$ for any $\alpha \in \{0, 1\}^*$, wherever $x \circ \alpha$ is defined. This is indeed the case, which we establish in the next lemma. This lemma will allow us to determine properties of elements of the free band by considering the structure of a representative transducer directly, rather than considering the function realized by that transducer.

Lemma 3.5. *Let $\mathcal{T} = (Q, \{0, 1\}, A, q_0, T, \odot, \otimes)$ be a transducer representing $x \in \text{FB}^1(A)$. If $\alpha \in \{0, 1\}^*$ is such that $q_0 \odot \alpha = q$, then the induced subtransducer rooted at q realizes $f_{x \circ \alpha}$.*

Proof. Let $x \in \text{FB}^1(A)$, let $\mathcal{T} = (Q, \{0, 1\}, A, q_0, T, \odot, \otimes)$ be a transducer realizing the function f_x , let $q \in Q$ be reachable, and let $\alpha \in \{0, 1\}^*$ be such that $q_0 \odot \alpha = q$.

Suppose that g is the function realized by the induced subtransducer rooted at q . To compute g for $\beta \in \{0, 1\}^*$ we take the suffix of $f_x(\alpha\beta)$ obtained by removing the first $|\alpha|$ output letters. But, by the definition of f_x ,

$$f_x(\alpha\beta) = \left(\prod_{i=1}^{|\alpha|} x * \alpha_{(1,i)} \right) f_{x \circ \alpha}(\beta).$$

Since $\prod_{i=1}^{|\alpha|} x * \alpha_{(1,i)}$ is the length $|\alpha|$ prefix of $f_x(\alpha\beta)$, $g(\beta) = f_{x \circ \alpha}(\beta)$ for all $\beta \in \{0, 1\}^*$. In particular, g is the function $f_{x \circ \alpha}$ realized by the induced subtransducer rooted at q . Therefore the subtransducer rooted at q represents $x \circ \alpha$. \square

Let $\mathcal{T}_x = (Q, \{0, 1\}, A, q_0, T, \odot, \otimes)$ be a transducer representing $x \in \text{FB}^1(A)$. If we define the partial function $\Psi : Q \rightarrow \text{FB}^1(A)$ by

$$\Psi(q) = \begin{cases} x \circ \alpha & \text{if } q = q_0 \odot \alpha \text{ for some } \alpha \in \{0, 1\}^* \\ \perp & \text{otherwise,} \end{cases}$$

then it follows from Lemma 3.5 that the diagrams in Fig. 4 commute. Consequently, we may refer to a state q in a transducer as **representing** the element $\Psi(q)$ of $\text{FB}^1(A)$.

4. Connection with the admissible maps of Neto and Sezinando

In [21] the elements of a relatively free band are represented as maps from an infinite binary tree into the powerset of the generating set of the band. In this section we briefly explain the interaction, for free bands, between Neto and Sezinando's representation and the transducer representation described in the previous section.

A map $g : \{0, 1\}^* \rightarrow \mathcal{P}(A)$ is called **admissible** [21, Definition 3.7] if the following hold:

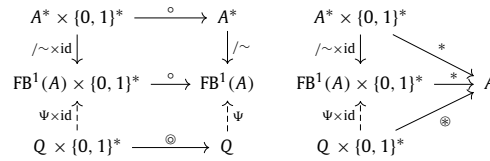


Fig. 4. Commutative diagrams giving correspondence between states in a transducer and elements of the free band.

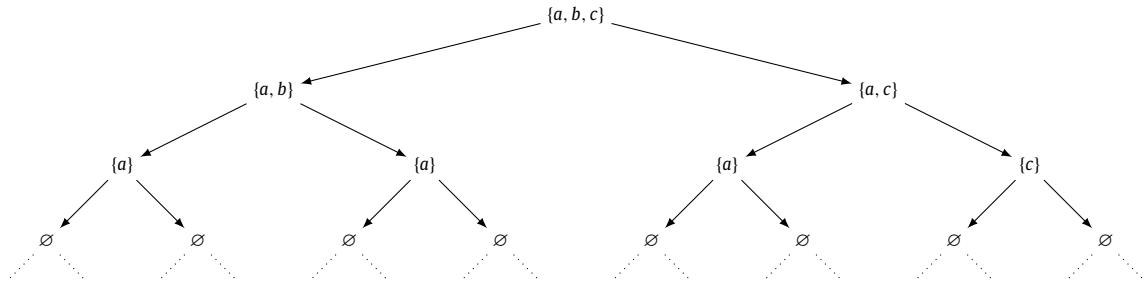


Fig. 5. A visualisation of the admissible map g_{abac} as an infinite binary tree where $g_{abac}(\varepsilon)$ is the label of the root node, the label of the left child of a vertex α is $g_{abac}(\alpha 0)$ and the label of the right child is $g_{abac}(\alpha 1)$.

- (i) for every $\alpha, \beta \in \{0, 1\}^*$, if α is a prefix of β , then $g(\alpha) \supseteq g(\beta)$; and
- (ii) for every $\alpha \in \{0, 1\}^*$, $\gamma \in \{0, 1\}$, if $g(\alpha) \neq \emptyset$, then $|g(\alpha\gamma)| = |g(\alpha)| - 1$.

In [21, Theorem 3.8] the authors show that there is a bijective correspondence between all admissible maps into $\mathcal{P}(A)$ and the elements of $FB^1(A)$. The correspondence associates to every $x \in FB^1(A)$ the admissible map $g_x : \{0, 1\}^* \rightarrow \mathcal{P}(A)$ given by

$$g_x(\alpha) = \begin{cases} \text{cont}(x \circ \alpha) & \text{if } x \circ \alpha \neq \perp \\ \emptyset & \text{otherwise.} \end{cases}$$

The free monoid $\{0, 1\}^*$ can be thought of as the set of vertices of an infinite binary tree, where the left child of a vertex α is $\alpha 0$ and the right child is $\alpha 1$. An admissible map assigns to each vertex of the infinite binary tree a content, such that moving down the tree decreases the content by a single letter; see Fig. 5 for an example, compared to Fig. 3. The correspondence between the transducers defined in Example 3.3 and admissible maps can be formalised as follows. The proof is omitted for the sake of brevity.

Theorem 4.1. *Let $x \in FB^1(A)$, and let $(\{0, 1\}^{\leq k}, \{0, 1\}, A, \varepsilon, \{0, 1\}^k, \otimes, \odot)$ be the transducer representing x from Example 3.3. Then $q * \beta = g_x(q) \setminus g_x(q\beta)$ for every non-terminal $q \in \{0, 1\}^{\leq k}$ and $\beta \in \{0, 1\}$. \square*

Several of the results in [20,21] are similar to those appearing in Section 7, we will highlight these similarities where possible. The approach we follow in this paper benefits from the minimization ideas of Radoszewski and Rytter [23], which were not present in Neto and Sezinando [20,21].

5. Equality checking

In this section we give a method using transducers for checking equality in the free band. The algorithm we describe in this section is heavily based on that of Radoszewski and Rytter from [23]. Roughly speaking, in [23], to compare two words $u, v \in A^*$ for equality in the free band, the authors construct an automaton, called the **interval automaton** that has $\mathcal{O}((|u| + |v|) \cdot |A|)$ states. Equality of u and v in the free band is then established by checking the equivalence of two states corresponding to u and v within the interval automaton. This is done without storing the entire automaton, but only the relevant parts at every step. In this section, we adapt this idea, constructing one transducer for each word and then comparing the transducers. The principal difference is that we retain the whole transducer, which leads to worse space complexity than Radoszewski and Rytter’s approach, but has the benefit of permitting us to define multiplication in Section 6 and obtain the shortlex least word in a \sim -class in Section 7.

We start this section by recasting the interval automaton for $w \in A^*$ as a transducer using the notation established in the previous section. Let $w \in A^+$. A **content- k subword** of w is a subword v of w such that $|\text{cont}(v)| = k$. A content- k subword is called **prefix maximal** if it is not a proper prefix of any other content- k subword. Similarly, a subword is **suffix maximal** if it is not the proper suffix of any content- k subword. There are at most $|w|$ prefix maximal content- k subwords

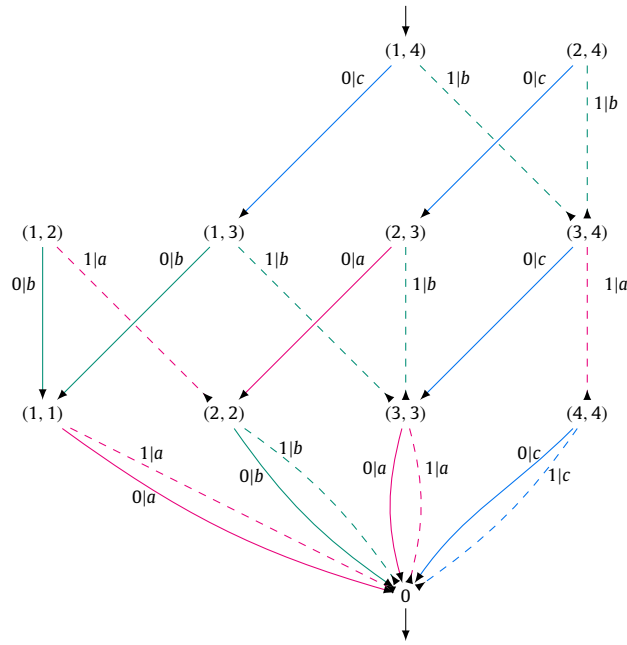


Fig. 6. The interval transducer realizing f_{abac} .

of w for any given k , since each letter of w can be the start of at most one such word. An analogous statement holds for suffix maximal content- k subwords also.

Let $w = a_1 \cdots a_n \in A^+$ where $a_1, \dots, a_n \in A$ and $k \in \mathbb{N}_0$. Then we define the partial function

$$\text{RIGHT}_k : \{1, \dots, n\} \longrightarrow \{1, \dots, n\}$$

such that $w_{(i, \text{RIGHT}_k(i))}$ is the unique prefix maximal content- k subword of w starting at index i if such a subword exists, and $w_{(i, \text{RIGHT}_k(i))} = \perp$ otherwise. The partial function $\text{LEFT}_k : \{1, \dots, n\} \longrightarrow \{1, \dots, n\}$ is defined dually.

The **interval transducer** of $w \in A^*$ is defined as follows:

$$\mathcal{T} = (Q, \{0, 1\}, A, q_0, T, \odot, \otimes)$$

where

- (i) the set of states Q consists of a symbol 0 and all of the pairs $(i, j) \in \{1, \dots, n\} \times \{1, \dots, n\}$ such that $w_{(i, j)}$ is either a prefix maximal or suffix maximal content- k subword of w for some $k \in \mathbb{N}$;
- (ii) the initial state is $q_0 = (1, n)$ (corresponding to the whole word w);
- (iii) the set of terminal states is $T = \{0\}$;
- (iv) if $(i, j) \in Q$ where $|\text{cont}(w_{(i, j)})| = k > 1$, then the state and letter transition functions are given by

$$(i, j) \odot 0 = (i, \text{RIGHT}_{k-1}(i)) \quad (i, j) \odot 1 = (\text{LEFT}_{k-1}(j), j)$$

$$(i, j) \otimes 0 = a_{\text{RIGHT}_{k-1}(i)+1} \quad (i, j) \otimes 1 = a_{\text{LEFT}_{k-1}(j)-1};$$

- (v) $(i, j) \in Q$ where $|\text{cont}(w_{(i, j)})| = 1$, then

$$(i, j) \odot \alpha = 0 \quad (i, j) \otimes \alpha = a_i;$$

- (vi) $0 \odot \alpha = 0 \otimes \alpha = \perp$ for all $\alpha \in \{0, 1\}$.

The transducer has a linear number of states in $|A| \cdot |w|$ since $|Q| \leq 2 \cdot |A| \cdot |w| + 1$, since there are at most $|w|$ prefix or suffix maximal content- k subwords for any given k , and no prefix or suffix maximal content- k subwords for $k \geq |A|$. The proof that \mathcal{T} realizes f_w is the same as that given in [23, Lemmas 1 and 2], and for the sake of brevity, we will not duplicate the proof here. An algorithm is given in [23, Theorem 2] that constructs the interval transducer \mathcal{T} for $w \in A^*$ in $\mathcal{O}(|A| \cdot |w|)$ time; we denote this algorithm by `IntervalTransducer` (see Fig. 6 for the interval transducer realizing f_{abac}).

A transducer is **trim** if every state belongs to some path starting at the initial state and ending at a terminal state. Any trim transducer \mathcal{T} realizing f_w for $w \in A^+$ is acyclic, since the domain of f_w is finite. A linear time algorithm for

minimizing acyclic deterministic synchronous transducers, such as those in this paper, can be derived from Revuz [24]; we refer to this algorithm as `Minimize`.

We say that transducers $\mathcal{T} = (Q, \Sigma, \Gamma, q_0, T, \odot, \otimes)$ and $\mathcal{T}' = (Q', \Sigma', \Gamma', q'_0, T', \odot', \otimes')$ are **isomorphic** if the input and output alphabets coincide, $\Sigma = \Sigma', \Gamma = \Gamma'$, and there exists a bijection $f : Q \rightarrow Q'$ such that $f(q_0) = q'_0, f(T) = T', f(q \odot \alpha) = f(q) \odot' \alpha$ and $q \otimes \alpha = f(q) \otimes' \alpha$ for all $q \in Q, \alpha \in \Sigma$. In other words, two transducers are isomorphic if they differ only by the labelling of states.

Just as with automata, the minimal deterministic synchronous transducer realizing a given function is unique up to isomorphism. It is also possible to check if two trim deterministic synchronous transducers are isomorphic in linear time in the number of transitions of the transducers. Essentially we treat the synchronous transducer as an automaton by making each transition be labelled by the pair containing both the input and output symbol, then apply the isomorphism algorithm for connected trim automata. We will call this algorithm `TrimTransducerIsomorphism`.

The algorithms `TrimTransducerIsomorphism`, `Minimize`, and `IntervalTransducer` can now be combined into an algorithm for checking equality in the free band, Algorithm 5.1.

Algorithm 5.1: `EqualInFreeBand`.

Arguments : words $v, w \in A^*$;
Returns : if $v \sim w$ or not.
1 Let $\mathcal{T} = \text{Minimize}(\text{IntervalTransducer}(v))$
2 Let $\mathcal{T}' = \text{Minimize}(\text{IntervalTransducer}(w))$
3 return `TrimTransducerIsomorphism`($\mathcal{T}_v, \mathcal{T}_w$)

The total time and space complexity of this algorithm is $O((|v| + |w|) \cdot |A|)$. A version of `EqualInFreeBand` is present in [23, Theorem 2]. However, the algorithm described in [23] achieves a better space complexity ($\mathcal{O}(|v| + |w|)$) than Algorithm 5.1 by processing the transducers \mathcal{T} and \mathcal{T}' layer by layer. Whereas in the method described here, we store the full transducers, and so our algorithm has complexity $\mathcal{O}((|v| + |w|) \cdot |A|)$.

The preceding discussion also allows us to specify the size of the minimal transducer. Recall that if A is any set, then the **shortlex ordering** on A^* is defined by $u \leq v$ if $|u| < |v|$ or if $|u| = |v|$ and u is lexicographically less than v (with respect to some underlying ordering on A) for all $u, v \in A^*$. For $w \in A^+$ let $\min(w) \in A^+$ denote the shortlex least word such that $w \sim \min(w)$. Applying `IntervalTransducer` to $\min(w)$ gives a transducer \mathcal{T} with at most $2 \cdot |\min(w)| \cdot |A| + 1$ states realizing $f_{\min(w)} = f_w$. The minimal transducer realizing f_w will have no more states than \mathcal{T} ; we record this in the next theorem.

Theorem 5.1. *If A is any non-empty set, and $w \in A^*$ is arbitrary, then the minimal transducer realizing f_w has at most $2 \cdot |\min(w)| \cdot |A| + 1$ states. \square*

Note that it is not necessary to compute (or even know) $\min(w)$ when constructing the minimal transducer realizing f_w .

6. Multiplication

In this section, we describe how to multiply elements of the free band using transducers representing the elements. Suppose that \cdot denotes the multiplication within $\text{FB}^1(A)$ for some alphabet A . One approach to finding a transducer representing $x \cdot y$ given $x, y \in \text{FB}^1(A)$ arises from the algorithms described in Section 5. If $v, w \in A^*$ represent $x, y \in \text{FB}^1(A)$, respectively, then a transducer for $x \cdot y$ is given by `IntervalTransducer`(vw). This construction has complexity $\mathcal{O}((|v| + |w|) \cdot |A|)$. Concatenating words, in all but the most trivial cases, produces a word that is not a minimal representative of the product. Therefore repeated multiplications using words can quickly lead to very long representatives. In this section, we give an algorithm that constructs a transducer $\mathcal{T}_{x \cdot y}$, representing the product $x \cdot y$, from any transducers \mathcal{T}_x and \mathcal{T}_y representing $x, y \in \text{FB}^1(A)$, respectively. The complexity of this approach is $\mathcal{O}(|\mathcal{T}_x| + |\mathcal{T}_y| + |A|^2)$ where $|\mathcal{T}_x|$ and $|\mathcal{T}_y|$ are the number of states of transducers \mathcal{T}_x and \mathcal{T}_y respectively. By Theorem 5.1, if \mathcal{T}_x and \mathcal{T}_y are minimal transducers, then `Minimize`($\mathcal{T}_{x \cdot y}$) can be found in $\mathcal{O}((|\min(x)| + |\min(y)| + |A|) \cdot |A|)$ time and $|\text{Minimize}(\mathcal{T}_{x \cdot y})| \leq 2 \cdot |A| \cdot |\min(x \cdot y)| + 1$. The number of states in the minimal transducer for $\mathcal{T}_{x \cdot y}$ is hence within a constant multiple of the minimum possible length of any word in A^* representing $x \cdot y$. This resolves the issue of the representative length blow-up in repeated multiplications.

In the following series of results, we establish some necessary properties of the product $x \cdot y$ and show how to compute these properties.

Lemma 6.1. *Let $x, y \in \text{FB}^1(A)$ with $x, y \neq \varepsilon$. Then*

$$(x \cdot y) \circ 0 = \begin{cases} x \cdot (y \circ 0) & \text{if } y * 0 \notin \text{cont}(x) \\ (x \cdot (y \circ 0)) \circ 0 & \text{otherwise,} \end{cases} \quad (x \cdot y) * 0 = \begin{cases} y * 0 & \text{if } y * 0 \notin \text{cont}(x) \\ (x \cdot (y \circ 0)) * 0 & \text{otherwise,} \end{cases}$$

$$(x \cdot y) \circ 1 = \begin{cases} (x \circ 1) \cdot y & \text{if } x * 1 \notin \text{cont}(y) \\ ((x \circ 1) \cdot y) \circ 1 & \text{otherwise,} \end{cases} \quad (x \cdot y) * 1 = \begin{cases} x * 1 & \text{if } x * 1 \notin \text{cont}(y) \\ ((x \circ 1) \cdot y) * 1 & \text{otherwise.} \end{cases}$$

Proof. We will prove the first two statements, the second two statements follow by a similar argument.

Let $x, y \in \text{FB}^1(A)$ and let $v, w \in A^*$ such that $v/\sim = x$ and $w/\sim = y$. Then $(vw)/\sim = x \cdot y$ is a representative of $x \cdot y$. Hence it suffices by the commutative diagram in Fig. 2 to prove the result for v and w instead of x and y . By assumption $x \neq \varepsilon$ and $y \neq \varepsilon$, and so $v * \alpha$ and $v \circ \alpha$ are defined for all $\alpha \in \{0, 1\}$ and similarly for w .

If $w * 0 \notin \text{cont}(v)$, then $|\text{cont}(v(w \circ 0))| = |\text{cont}(vw)| - 1$, since $w * 0 \notin \text{cont}(v)$ by assumption, and $w * 0 \notin \text{cont}(w \circ 0)$ by definition. Furthermore, $v(w \circ 0)$ is the largest prefix of vw with content strictly contained in $\text{cont}(vw)$, since the next letter in vw is $w * 0$. So $(vw) \circ 0 = v(w \circ 0)$ and $(vw) * 0 = w * 0$.

If $w * 0 \in \text{cont}(v)$, then $\text{cont}(v(w \circ 0)) = \text{cont}(vw)$. Since $v(w \circ 0)$ is a prefix with the same content as vw , the first letter to occur last in vw must occur somewhere within $v(w \circ 0)$. Therefore $(vw) \circ 0 = (v(w \circ 0)) \circ 0$ and $(vw) * 0 = (v(w \circ 0)) * 0$. \square

Recall that Theorem 2.3 essentially allows us to reconstruct $z \in \text{FB}^1(A)$ given $\text{cont}(z)$, $z \circ \alpha$, and $z * \alpha$ for all $\alpha \in \{0, 1\}$. Consider the case when $z = x \cdot y$ for some $x, y \in \text{FB}^1(A)$. We know that $\text{cont}(x \cdot y) = \text{cont}(x) \cup \text{cont}(y)$. Lemma 6.1 allows us to compute $(x \cdot y) \circ \alpha$ and $(x \cdot y) * \alpha$ for $\alpha \in \{0, 1\}$. However, using Lemma 6.1 naively to compute $x \cdot y$ requires exponential time, since in order to compute $(x \cdot y) \circ 0$ and $(x \cdot y) \circ 1$ we must compute $x \cdot (y \circ 0)$ and $(x \circ 1) \cdot y$. We will show that it is possible to reduce the complexity by iterating Lemma 6.1 and considering the possible values of $(x \cdot y) \circ \alpha$ for all $\alpha \in \{0, 1\}^*$. In Corollary 6.5, we will show that:

$$(x \cdot y) \circ \alpha \in \{(x \circ 1^i) \cdot (y \circ 0^j) : 0 \leq i \leq |\text{cont}(x)|, 0 \leq j \leq |\text{cont}(y)|\} \cup \{x \circ \beta : \beta \in \{0, 1\}^*\} \cup \{y \circ \gamma : \gamma \in \{0, 1\}^*\}$$

for all $\alpha \in \{0, 1\}^*$. So, to compute $(x \cdot y) \circ \alpha$ for $\alpha \in \{0, 1\}^*$, requires $\mathcal{O}(|\text{cont}(x)| \cdot |\text{cont}(y)|) = \mathcal{O}(|A|^2)$ products of the form $(x \circ 1^i) \cdot (y \circ 0^j)$, and the finitely many values in

$$\{x \circ \beta : \beta \in \{0, 1\}^*\} \cup \{y \circ \gamma : \gamma \in \{0, 1\}^*\}$$

which are represented in the given transducers for x and y . We will show in Lemma 6.4 that the values of $((x \circ 1^i) \cdot (y \circ 0^j)) \circ \alpha$ and $((x \circ 1^i) \cdot (y \circ 0^j)) * \alpha$ for $\alpha \in \{0, 1\}$ can be derived by considering how $\text{cont}(x \circ 1^i)$ and $\text{cont}(y \circ 0^j)$ interact. Taken together, this will allow us to construct a transducer representing $x \cdot y$, by taking the transducers representing x and y , adding $\mathcal{O}(|A|^2)$ states, and the necessary transitions corresponding to the products $\{(x \circ 1^i) \cdot (y \circ 0^j) : 0 \leq i \leq |\text{cont}(x)|, 0 \leq j \leq |\text{cont}(y)|\}$.

We will make use of the following partial functions.

Definition 6.2. Let $K_0, K_1 : \text{FB}^1(A) \times \text{FB}^1(A) \rightarrow \mathbb{N}$ be partial functions defined by:

$$K_0(x, y) = \begin{cases} \min\{k \in \mathbb{N} : y * 0^k \neq \perp \text{ and } y * 0^k \notin \text{cont}(x)\} & \text{if such } k \text{ exists} \\ \perp & \text{otherwise,} \end{cases}$$

$$K_1(x, y) = \begin{cases} \min\{l \in \mathbb{N} : x * 1^l \neq \perp \text{ and } x * 1^l \notin \text{cont}(y)\} & \text{if such } l \text{ exists} \\ \perp & \text{otherwise,} \end{cases}$$

for every $x, y \in \text{FB}^1(A)$.

The following lemma gives a recursive way of calculating the values of $K_0(x, y)$ and $K_1(x, y)$ for any $x, y \in \text{FB}^1(A)$.

Lemma 6.3. Let $x, y \in \text{FB}^1(A)$. Then:

$$K_0(x, y) = \begin{cases} 1 & \text{if } y * 0 \neq \perp \text{ and } y * 0 \notin \text{cont}(x) \\ 1 + K_0(x, y \circ 0) & \text{if } y * 0 \in \text{cont}(x) \text{ and } K_0(x, y \circ 0) \neq \perp \\ \perp & \text{otherwise,} \end{cases}$$

$$K_1(x, y) = \begin{cases} 1 & \text{if } x * 1 \neq \perp \text{ and } x * 1 \notin \text{cont}(y) \\ 1 + K_1(x \circ 1, y) & \text{if } x * 1 \in \text{cont}(y) \text{ and } K_1(x \circ 1, y) \neq \perp \\ \perp & \text{otherwise.} \end{cases}$$

Proof. We will only prove the lemma for K_0 ; the proof for K_1 is analogous.

If $y * 0 \neq \perp$ and $y * 0 \notin \text{cont}(x)$, then, by definition, $K_0(x, y) = 1$ since there is no positive integer less than 1.

If $y * 0 \neq \perp$ and $y * 0 \in \text{cont}(x)$, then $K_0(x, y) = \perp$ or $K_0(x, y) \geq 2$. If $K_0(x, y \circ 0) \neq \perp$, then $K_0(x, y \circ 0)$ is the least positive integer such that $(y \circ 0) * 0^{K_0(x, y \circ 0)} \neq \perp$ and $(y \circ 0) * 0^{K_0(x, y \circ 0)} \notin \text{cont}(x)$. But, by the definition of \circ and $*$, $(y \circ 0) * 0^{K_0(x, y \circ 0)} = y * 0^{1+K_0(x, y \circ 0)}$ and so $k = 1 + K_0(x, y \circ 0)$ is the least positive integer such that $y * 0^k \neq \perp$ and $y * 0^k \notin \text{cont}(x)$, as required. If $K_0(x, y \circ 0) = \perp$, then there is no $k \geq 1$ such that $y * 0^{1+k} \neq \perp$ and $y * 0^{1+k} \notin \text{cont}(x)$. This implies that there is no $k \geq 2$ such that $y * 0^k \neq \perp$ and $y * 0^k \notin \text{cont}(x)$. But $K_0(x, y) = \perp$ or $K_0(x, y) \geq 2$, and since the latter does not hold, we conclude that $K_0(x, y) = \perp$.

Finally, if $y * 0 = \perp$, then $y * 0^k = \perp$ for all $k \geq 1$, so by definition $K_0(x, y) = \perp$. \square

With the preceding lemma in mind, we now consider $((x \circ 1^i) \cdot (y \circ 0^j)) \circ \alpha$ and $((x \circ 1^i) \cdot (y \circ 0^j)) * \alpha$ for $\alpha \in \{0, 1\}$.

Lemma 6.4. *Let $x, y \in \text{FB}^1(A)$ and let $0 \leq i \leq |\text{cont}(x)|$ and $0 \leq j \leq |\text{cont}(y)|$. If $i \neq |\text{cont}(x)|$ or $j \neq |\text{cont}(y)|$, then*

$$\begin{aligned} ((x \circ 1^i) \cdot (y \circ 0^j)) \circ 0 &= \begin{cases} (x \circ 1^i) \cdot (y \circ 0^{j+k}) & \text{if } k = K_0(x \circ 1^i, y \circ 0^j) \neq \perp \\ x \circ 1^i 0 & \text{otherwise,} \end{cases} \\ ((x \circ 1^i) \cdot (y \circ 0^j)) * 0 &= \begin{cases} y * 0^{j+k} & \text{if } k = K_0(x \circ 1^i, y \circ 0^j) \neq \perp \\ x * 1^i 0 & \text{otherwise,} \end{cases} \\ ((x \circ 1^i) \cdot (y \circ 0^j)) \circ 1 &= \begin{cases} (x \circ 1^{i+k}) \cdot (y \circ 0^j) & \text{if } k = K_1(x \circ 1^i, y \circ 0^j) \neq \perp \\ y \circ 0^j 1 & \text{otherwise,} \end{cases} \\ ((x \circ 1^i) \cdot (y \circ 0^j)) * 1 &= \begin{cases} x * 1^{i+k} & \text{if } k = K_1(x \circ 1^i, y \circ 0^j) \neq \perp \\ y * 0^j 1 & \text{otherwise.} \end{cases} \end{aligned}$$

On the other hand, if $i = |\text{cont}(x)|$ and $j = |\text{cont}(y)|$, then $((x \circ 1^i) \cdot (y \circ 0^j)) \circ \alpha = \perp$ and $((x \circ 1^i) \cdot (y \circ 0^j)) * \alpha = \perp$ for $\alpha \in \{0, 1\}$.

Proof. We will show that if the result holds for $(x \circ 1^{i+1}) \cdot (y \circ 0^j)$ and $(x \circ 1^i) \cdot (y \circ 0^{j+1})$, then it holds for $(x \circ 1^i) \cdot (y \circ 0^j)$. For the base case, let $i = |\text{cont}(x)|$ or $j = |\text{cont}(y)|$.

If $i = |\text{cont}(x)|$ and $j = |\text{cont}(y)|$, then, since every application of \circ reduces the content of x by 1, $|\text{cont}(x \circ 1^i)| = |\text{cont}(x)| - i = 0$ and so $x \circ 1^i = \varepsilon$. Similarly $y \circ 0^j = \varepsilon$. It follows that $(x \circ 1^i) \cdot (y \circ 0^j) = \varepsilon$ and so $((x \circ 1^i) \cdot (y \circ 0^j)) \circ \alpha$ and $((x \circ 1^i) \cdot (y \circ 0^j)) * \alpha$ are undefined for $\alpha \in \{0, 1\}$.

If $i = |\text{cont}(x)|$ and $1 \leq j < |\text{cont}(y)|$. This implies that $x * 1^{i+k} = (x \circ 1^i) * 0^k = \varepsilon * 0^k$ is undefined for all $k \geq 1$. So $K_1(x \circ 1^i, y \circ 0^j)$ is undefined and

$$\begin{aligned} ((x \circ 1^i) \cdot (y \circ 0^j)) \circ 1 &= \varepsilon \cdot (y \circ 0^j) \circ 1 = (y \circ 0^j) \circ 1 = y \circ 0^j 1 \\ ((x \circ 1^i) \cdot (y \circ 0^j)) * 1 &= \varepsilon \cdot (y \circ 0^j) * 1 = (y \circ 0^j) * 1 = y * 0^j 1. \end{aligned}$$

Since $j < |\text{cont}(y)|$, $y \circ 0^j \neq \varepsilon$ so that $y \circ 0^{j+1}$ is defined and $y \circ 0^{j+1} \notin \text{cont}(x \circ 1^i) = \emptyset$. Hence by Lemma 6.3, $K_0(x \circ 1^i, y \circ 0^j) = 1$. In this case, since $x \circ 1^i = \varepsilon$,

$$((x \circ 1^i) \cdot (y \circ 0^j)) \circ 0 = (x \circ 1^i) \cdot (y \circ 0^{j+1}) \text{ and } ((x \circ 1^i) \cdot (y \circ 0^j)) * 0 = y * 0^{j+1},$$

as required. We have established the base case when $i = |\text{cont}(x)|$, the proof when $j = |\text{cont}(y)|$ is analogously.

Suppose that $0 \leq i < |\text{cont}(x)|$, $0 \leq j < |\text{cont}(y)|$ and assume that the conclusion of the lemma holds for $(x \circ 1^{i+1}) \cdot (y \circ 0^j)$ and $(x \circ 1^i) \cdot (y \circ 0^{j+1})$. Since $i < |\text{cont}(x)|$ and $j < |\text{cont}(y)|$, $x \circ 1^i, y \circ 0^j \neq \varepsilon$ and we can apply Lemma 6.1 to $(x \circ 1^i) \cdot (y \circ 0^j)$. The value of $y * 0^{j+1}$ is defined since $y \circ 0^j \neq \varepsilon$. If $y * 0^{j+1} \notin \text{cont}(x \circ 1^i)$, then $K_0(x \circ 1^i, y \circ 0^j) = 1$ by Lemma 6.3. But then Lemma 6.1 implies that

$$\begin{aligned} ((x \circ 1^i) \cdot (y \circ 0^j)) \circ 0 &= (x \circ 1^i) \cdot ((y \circ 0^j) \circ 0) = (x \circ 1^i) \cdot (y \circ 0^{j+1}) \\ ((x \circ 1^i) \cdot (y \circ 0^j)) * 0 &= (y \circ 0^j) * 0 = y * 0^{j+1}, \end{aligned}$$

as required. If $y * 0^{j+1} \in \text{cont}(x \circ 1^i)$, then, by Lemma 6.1,

$$\begin{aligned} ((x \circ 1^i) \cdot (y \circ 0^j)) \circ 0 &= ((x \circ 1^i) \cdot ((y \circ 0^j) \circ 0)) \circ 0 = ((x \circ 1^i) \cdot (y \circ 0^{j+1})) \circ 0 \\ ((x \circ 1^i) \cdot (y \circ 0^j)) * 0 &= ((x \circ 1^i) \cdot ((y \circ 0^j) \circ 0)) * 0 = ((x \circ 1^i) \cdot (y \circ 0^{j+1})) * 0 \end{aligned}$$

and so by the inductive hypothesis on $(x \circ 1^i) \cdot (y \circ 0^{j+1})$ the following holds

$$\begin{aligned} ((x \circ 1^i) \cdot (y \circ 0^j)) \circ 0 &= \begin{cases} (x \circ 1^i) \cdot (y \circ 0^{j+1+k}) & \text{if } k = K_0(x \circ 1^i, y \circ 0^{j+1}) \neq \perp \\ x \circ 1^i 0 & \text{otherwise,} \end{cases} \\ ((x \circ 1^i) \cdot (y \circ 0^j)) * 0 &= \begin{cases} y * 0^{j+1+k} & \text{if } k = K_0(x \circ 1^i, y \circ 0^{j+1}) \neq \perp \\ x * 1^i 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Lemma 6.3 implies that $K_0(x \circ 1^i, y \circ 0^j) = 1 + K_0(x \circ 1^i, y \circ 0^{j+1})$ if $K_0(x \circ 1^i, y \circ 0^{j+1})$ is defined and $K_0(x \circ 1^i, y \circ 0^j)$ is undefined otherwise. Therefore

$$\begin{aligned} ((x \circ 1^i) \cdot (y \circ 0^j)) \circ 0 &= \begin{cases} (x \circ 1^i) \cdot (y \circ 0^{j+k}) & \text{if } k = K_0(x \circ 1^i, y \circ 0^j) \neq \perp \\ x \circ 1^i 0 & \text{otherwise,} \end{cases} \\ ((x \circ 1^i) \cdot (y \circ 0^j)) * 0 &= \begin{cases} y * 0^{j+k} & \text{if } k = K_0(x \circ 1^i, y \circ 0^j) \neq \perp \\ x * 1^i 0 & \text{otherwise} \end{cases} \end{aligned}$$

which is exactly what we set out to show.

An analogous proof establishes the conclusion of the lemma for $((x \circ 1^i) \cdot (y \circ 0^j)) \circ 1$ and $((x \circ 1^i) \cdot (y \circ 0^j)) * 1$, completing the induction. \square

Lemma 6.4 implies that for every $\alpha \in \{0, 1\}$ one of the following holds: $((x \circ 1^i) \cdot (y \circ 0^j)) \circ \alpha = (x \circ 1^{i'}) \cdot (y \circ 0^{j'})$ for some i', j' ; $((x \circ 1^i) \cdot (y \circ 0^j)) \circ \alpha = x \circ \beta$ for some $\beta \in \{0, 1\}^*$; or $((x \circ 1^i) \cdot (y \circ 0^j)) \circ \alpha = y \circ \gamma$ for some $\gamma \in \{0, 1\}^*$. By iteratively applying Lemma 6.4, we extend this observation in the following corollary.

Corollary 6.5. *If $(x \cdot y) \circ \alpha \neq \perp$ for some $\alpha \in \{0, 1\}^*$, then*

$$\begin{aligned} (x \cdot y) \circ \alpha \in \{ & (x \circ 1^i) \cdot (y \circ 0^j) : 0 \leq i \leq |\text{cont}(x)|, 0 \leq j \leq |\text{cont}(y)| \} \\ & \cup \{x \circ \beta : \beta \in \{0, 1\}^*\} \cup \{y \circ \gamma : \gamma \in \{0, 1\}^*\}. \quad \square \end{aligned}$$

Lemma 6.4 and Corollary 6.5 allow us to describe a transducer representing the product $x \cdot y$ given transducers for x and y as follows. Let $x, y \in \text{FB}^1(A)$ and let $\mathcal{T}_x = (Q_x, \{0, 1\}, A, q_x, T_x, \odot_x, \otimes_x)$ and $\mathcal{T}_y = (Q_y, \{0, 1\}, A, q_y, T_y, \odot_y, \otimes_y)$ be any transducers representing x and y , respectively. We define:

$$\mathcal{T}_{x,y} = (Q, \{0, 1\}, A, q_0, T, \odot, \otimes) \tag{3}$$

where:

- (i) the set of states $Q = Q_x \cup Q_y \cup Q'$ consists of all the states of \mathcal{T}_x and \mathcal{T}_y together with the states $Q' = \{(i, j) : 0 \leq i \leq |\text{cont}(x)| \text{ and } 0 \leq j \leq |\text{cont}(y)|\}$, where $\{(i, j) : 0 \leq j \leq |\text{cont}(y)|\}$ and $\{(i, j) : 1 \leq i \leq |\text{cont}(x)|\}$ are identified with the original states $\{q_y \odot_y 0^j : 0 \leq j \leq |\text{cont}(y)|\}$ and $\{q_x \otimes_x 1^i : 0 \leq i \leq |\text{cont}(x)|\}$. Each state in $(i, j) \in Q'$ corresponds to the element $(x \circ 1^i) \cdot (y \circ 0^j)$ in Lemma 6.4;
- (ii) the initial state is $q_0 = (0, 0)$;
- (iii) the terminal states are exactly those of \mathcal{T}_x and \mathcal{T}_y , i.e. $T = T_x \cup T_y$;
- (iv) the state and letter transition functions \odot and \otimes for the states in Q_x and Q_y are defined exactly as their counterparts in \mathcal{T}_x and \mathcal{T}_y ;
- (v) if $(i, j) \in Q'$, then the state transition and letter transition functions are given by

$$\begin{aligned} (i, j) \odot 0 &= \begin{cases} (i, j+k) & \text{if } k = K_0(x \circ 1^i, y \circ 0^j) \neq \perp \\ q_x \otimes_x 1^i 0 & \text{otherwise,} \end{cases} \\ (i, j) \otimes 0 &= \begin{cases} q_y \otimes_y 0^{j+k} & \text{if } k = K_0(x \circ 1^i, y \circ 0^j) \neq \perp \\ q_x \otimes_x 1^i 0 & \text{otherwise,} \end{cases} \\ (i, j) \odot 1 &= \begin{cases} (i+k, j) & \text{if } k = K_1(x \circ 1^i, y \circ 0^j) \neq \perp \\ q_y \odot_y 0^j 1 & \text{otherwise,} \end{cases} \\ (i, j) \otimes 1 &= \begin{cases} q_x \otimes_x 1^{i+k} & \text{if } k = K_1(x \circ 1^i, y \circ 0^j) \neq \perp \\ q_y \otimes_y 0^j 1 & \text{otherwise.} \end{cases} \end{aligned}$$

The next lemma establishes that $\mathcal{T}_{x,y}$ is indeed a transducer.

Lemma 6.6. *If $x, y \in \text{FB}^1(A)$ are arbitrary, and \mathcal{T}_x and \mathcal{T}_y are transducers representing x and y respectively, then $\mathcal{T}_{x,y}$ defined in (3) is a transducer.*

Proof. It suffices to show that the state transition function \odot only takes values in Q and that \odot and \otimes are well-defined.

Clearly, if $(i, j) \odot \alpha \in Q_x \cup Q_y$, then $(i, j) \odot \alpha \in Q$. On the other hand, if $(i, j) \odot 0 = (i, j+k)$ where $k = K_0(x \circ 1^i, y \circ 0^j) \neq \perp$, then $y * 0^{j+k} \neq \perp$ so $j+k \leq |\text{cont}(y)|$ and $(i, j+k) \in Q'$. Similarly, if $(i, j) \odot 1 = (i+k, j)$, then it can be shown that $i+k \leq |\text{cont}(x)|$ and so $(i, j) \odot 1 \in Q'$ also.

To show that \odot and \otimes are well-defined, it suffices to show that for all $\alpha \in \{0, 1\}$, $(|\text{cont}(x)|, j) \odot \alpha = (q_y \odot_y 0^j) \odot \alpha$, $(|\text{cont}(x)|, j) \otimes \alpha = (q_y \odot_y 0^j) \otimes \alpha$ and the analogous statements hold for $(i, |\text{cont}(y)|)$ and $q_x \otimes_x 1^i$.

If $i = |\text{cont}(x)|$ and $1 \leq j \leq |\text{cont}(y)|$, then $x * 1^i = \varepsilon$ and so $x * 1^{i+k} = (x \circ 1^i) * 0^k = \perp$ for all $k \geq 1$. But then

$$(|\text{cont}(x)|, j) \odot 1 = q_y \odot_y 0^j 1 = (q_y \odot_y 0^j) \odot 1$$

$$(|\text{cont}(x)|, j) \otimes 1 = q_y \otimes_y 0^j 1 = (q_y \odot_y 0^j) \otimes 1.$$

If $y \circ 0^{j+1} \neq \perp$, then $y \circ 0^{j+1} \notin \text{cont}(x \circ 1^i) = \emptyset$ and so $K_0(x \circ 1^i, y \circ 0^j) = 1$. In this case,

$$(|\text{cont}(x)|, j) \odot 0 = (|\text{cont}(x)|, j+1) = q_y \odot_y 0^{j+1} = (q_y \odot_y 0^j) \odot 0$$

$$(|\text{cont}(x)|, j) \otimes 0 = q_y \otimes_y 0^{j+1} = (q_y \odot_y 0^j) \otimes 0.$$

If, on the other hand, $y \circ 0^{j+1} = \perp$, then $y \circ 0^{j+k} = \perp$ for all $k \geq 1$. Therefore

$$(|\text{cont}(x)|, j) \odot 0 = q_x \otimes_x 1^{|\text{cont}(x)|} 0 = (q_x \otimes_x 1^{|\text{cont}(x)|}) \odot 0$$

$$(|\text{cont}(x)|, j) \otimes 0 = q_x \otimes_x 1^{|\text{cont}(x)|} 0 = (q_x \otimes_x 1^{|\text{cont}(x)|}) \otimes 0.$$

Since $q_x \otimes_x 1^{|\text{cont}(x)|}$ represents $x \circ 1^{|\text{cont}(x)|} = \varepsilon$, and \mathcal{T}_x represents x , it follows that $q_x \otimes_x 1^{|\text{cont}(x)|}$ is terminal and so $(|\text{cont}(x)|, j) \odot 0 = \perp$ and $(|\text{cont}(x)|, j) \otimes 0 = \perp$. But this is exactly what we wanted, since $y \circ 0^{j+1}$ is undefined, $y \circ 0^j = \varepsilon$, so $q_y \odot_y 0^j$ is terminal and therefore $(q_y \odot_y 0^j) \odot 0 = \perp$ $(q_y \odot_y 0^j) \otimes 1 = \perp$ as well. \square

To show that $\mathcal{T}_{x \cdot y}$ does indeed represent $x \cdot y$ we require the following lemma.

Lemma 6.7. *Let $\mathcal{T} = (Q, \{0, 1\}, A, q_0, T, \odot, \otimes)$ be a transducer and let $\Psi : Q \rightarrow \text{FB}^1(A)$ be any partial function such that $\Psi(q_0) \neq \perp$; $\Psi(q) = \varepsilon$ if and only if $q \in T$; and Ψ satisfies the commutative diagram in Fig. 4. Then \mathcal{T} represents $\Psi(q_0)$.*

Proof. We show that for every $q \in Q$ with $\Psi(q) \neq \perp$, the induced subtransducer rooted at q realizes the function $f_{\Psi(q)}$. Since $\Psi(q_0) \neq \perp$ and the induced subtransducer rooted at q_0 realizes the same function as \mathcal{T} , this will imply that \mathcal{T} realizes $f_{\Psi(q_0)}$ and so represents $\Psi(q_0)$, as required. Note that for every state $q \in Q$, if $\Psi(q) \neq \perp$, then either: $\Psi(q) = \varepsilon$, and $q \odot 0 = q \odot 1 = \perp$; or $\Psi(q) \neq \varepsilon$, $q \odot 0 \neq \perp$, and $q \odot 1 \neq \perp$. This is because Ψ respects the commutative diagram in Fig. 4, and the only element of $\text{FB}^1(A)$ with $x \circ 0 = x \circ 1 = \perp$ is ε .

We proceed by induction on $|\text{cont}(\Psi(q))|$. For the base case, let $q \in Q$ be such that $\Psi(q) \neq \perp$ and $|\text{cont}(\Psi(q))| = 0$. Then $\Psi(q) = \varepsilon$ and since Ψ respects the diagram in Fig. 4, it follows that $q \odot \alpha = \varepsilon \odot \alpha = \perp$ for all $\alpha \in \{0, 1\}$. By assumption, $\Psi(q) = \varepsilon$ if and only if q is terminal. It follows that the induced subtransducer rooted at q realizes the function which maps ε to itself and every other input to \perp , which is exactly f_ε .

Now fix $1 \leq k \leq |\text{cont}(\Psi(q_0))|$ and assume that the statement holds for all $q \in Q$ with $\Psi(q) \neq \perp$ and $|\text{cont}(\Psi(q))| = k-1$. Let $q \in Q$ with $\Psi(q) \neq \perp$ be such that $|\text{cont}(\Psi(q))| = k$. Since $k \geq 1$, $\Psi(q) \neq \varepsilon$ and so $q \odot \alpha \neq \perp$ for all $\alpha \in \{0, 1\}$. But then $|\text{cont}(\Psi(q \odot \alpha))| = |\text{cont}(\Psi(q) \circ \alpha)| = |\text{cont}(\Psi(q))| - 1$ for all $\alpha \in \{0, 1\}$. It follows that the induced subtransducer rooted at $q \odot \alpha$ realizes $f_{\Psi(q \odot \alpha)} = f_{\Psi(q) \circ \alpha}$ for all $\alpha \in \{0, 1\}$. The subtransducer rooted at q realizes the partial function $f : \{0, 1\}^* \rightarrow A^*$ given by $f(\varepsilon) = \perp$ and $f(\alpha\beta) = (q \odot \alpha) f_{\Psi(q \odot \alpha)}(\beta)$ for all $\alpha \in \{0, 1\}$ and $\beta \in \{0, 1\}^*$. But this is exactly the partial function $f_{\Psi(q)}$. \square

We now prove that the transducer $\mathcal{T}_{x \cdot y}$ represents $x \cdot y$ as claimed.

Theorem 6.8. *Let $x, y \in \text{FB}^1(A)$ and let \mathcal{T}_x and \mathcal{T}_y be transducers representing x and y , respectively. Then the transducer $\mathcal{T}_{x \cdot y}$ defined in (3) represents $x \cdot y$.*

Proof. Suppose that $\mathcal{T}_x = (Q_x, \{0, 1\}, A, q_x, T_x, \odot_x, \otimes_x)$, that $\mathcal{T}_y = (Q_y, \{0, 1\}, A, q_y, T_y, \odot_y, \otimes_y)$, and that $\mathcal{T}_{x \cdot y} = (Q, \{0, 1\}, A, q_0, T, \odot, \otimes)$.

To show that $\mathcal{T}_{x \cdot y}$ represents $x \cdot y$ we will define a function $\Psi : Q \rightarrow \text{FB}^1(A)$ such that $\Psi(q_0) = x \cdot y$, Ψ satisfies the commutative diagrams given in Fig. 4, and $\Psi(q) = \varepsilon$ if and only if $q \in T$. It will follow from this that $\mathcal{T}_{x \cdot y}$ realizes $f_{x \cdot y}$ and therefore $\mathcal{T}_{x \cdot y}$ represents $x \cdot y$ by Lemma 6.7.

Let $\Psi_x : Q_x \rightarrow \text{FB}^1(A)$ and $\Psi_y : Q_y \rightarrow \text{FB}^1(A)$ be the functions realizing the commutative diagram in Fig. 4 for \mathcal{T}_x and \mathcal{T}_y respectively. We define $\Psi : Q \rightarrow \text{FB}^1(A)$ by

$$\Psi(q) = \begin{cases} (x \circ 1^i) \cdot (y \circ 0^j) & \text{if } q = (i, j) \in Q' \\ \Psi_x(q) & \text{if } q \in Q_x \\ \Psi_y(q) & \text{if } q \in Q_y. \end{cases}$$

Since we identified the states $\{(|\text{cont}(x)|, j) : 0 \leq j \leq |\text{cont}(y)|\}$ and $\{(i, |\text{cont}(y)|) : 1 \leq i \leq |\text{cont}(x)|\}$ with the states $\{q_y \circledast_y 0^j : 0 \leq j \leq |\text{cont}(y)|\}$ and $\{q_x \circledast_x 1^i : 0 \leq i \leq |\text{cont}(x)|\}$ within Q , we must check that Ψ is well-defined. In particular, we must show that $\Psi((i, |\text{cont}(y)|)) = \Psi(q_x \circledast_x 1^i)$ and $\Psi((|\text{cont}(x)|, j)) = \Psi(q_y \circledast_y 0^j)$ for all relevant values of i and j . Every application of \circ reduces the content of y by 1 letter, and so $|\text{cont}(y \circ 0^{|\text{cont}(y)|})| = |\text{cont}(y)| - |\text{cont}(y)| = 0$. In other words, $y \circ 0^{|\text{cont}(y)|} = \varepsilon$. But then $\Psi((i, |\text{cont}(y)|)) = (x \circ 1^i) \cdot (y \circ 0^{|\text{cont}(y)|}) = (x \circ 1^i) \cdot \varepsilon = x \circ 1^i$ for all i . On the other hand, since \mathcal{T}_x represents x , $\Psi(q_x \circledast_x 1^i) = \Psi_x(q_x \circledast_x 1^i) = \Psi_x(q_x) \circ 1^i = x \circ 1^i$ for all i . Therefore $\Psi((i, |\text{cont}(y)|)) = \Psi(q_x \circledast_x 1^i)$ for all i . Similarly, $\Psi((|\text{cont}(x)|, j)) = \Psi(q_y \circledast_y 0^j)$ for all j , and so Ψ is well-defined.

Next, we establish that the 3 conditions of Lemma 6.7 hold for Ψ . By the definition of Ψ , $\Psi(q_0) = \Psi((0, 0)) = (x \circ 1^0) \cdot (y \circ 0^0) = x \cdot y \neq \perp$.

We now show that $\Psi(q) = \varepsilon$ if and only if $q \in T = T_x \cup T_y$. Since, for any $q \in Q_x$, $\Psi_x(q) = \varepsilon$ if and only if $q \in T_x$, and the analogously for Ψ_y , it follows that, for all $q \in Q_x \cup Q_y$, $\Psi(q) = \varepsilon$ if and only if $q \in T$. Let $q = (i, j) \in Q'$. Then $\text{cont}(\Psi(q)) = \text{cont}((x \circ 1^i) \cdot (y \circ 0^j)) = \text{cont}(x \circ 1^i) \cup \text{cont}(y \circ 0^j)$. If $\Psi(q) = \varepsilon$, then $|\text{cont}(x \circ 1^i)| = |\text{cont}(y \circ 0^j)| = 0$. But $|\text{cont}(x \circ 1^i)| = |\text{cont}(x)| - i$, and so $i = |\text{cont}(x)|$. Similarly $j = |\text{cont}(y)|$. Therefore q is the state that was identified with both $q_x \circledast_x 1^{|\text{cont}(x)|}$ and $q_y \circledast_y 0^{|\text{cont}(y)|}$. The former state is terminal in \mathcal{T}_x since $\Psi_x(q_x \circledast_x 1^{|\text{cont}(x)|}) = \Psi_x(q_x) \circ 1^{|\text{cont}(x)|} = x \circ 1^{|\text{cont}(x)|} = \varepsilon$.

It remains to show that Ψ satisfies the commutative diagram in Fig. 4. It suffices to show that $\Psi(q \circledast \alpha) = \Psi(q) \circ \alpha$ and $q \circledast \alpha = \Psi(q) * \alpha$ for all $q \in Q$ and $\alpha \in \{0, 1\}$, since the recursive definitions of \circledast , \circledast , \circ and $*$ imply the result for all $\alpha \in \{0, 1\}^*$. If $q \in Q_x$, then $q \circledast \alpha = q \circledast_x \alpha \in Q_x$ and $q \circledast \alpha = q \circledast_x \alpha \in A$ for all $\alpha \in \{0, 1\}$ since \circledast and \circledast coincide with \circledast_x and \circledast_x on Q_x . This implies that

$$\Psi(q \circledast \alpha) = \Psi(q \circledast_x \alpha) = \Psi_x(q \circledast_x \alpha) = \Psi_x(q) \circ \alpha = \Psi(q) \circ \alpha,$$

and

$$q \circledast \alpha = q \circledast_x \alpha = \Psi_x(q) * \alpha = \Psi(q) * \alpha$$

for all $\alpha \in \{0, 1\}$. Hence the equalities represented in the commutative diagram are satisfied for $q \in Q_x$. A similar argument shows that the commutative diagram holds for $q \in Q_y$ also.

Suppose that $q = (i, j) \in Q'$. If $k = K_0(x \circ 1^i, y \circ 0^j) \neq \perp$, then by applying Lemma 6.4

$$\begin{aligned} \Psi((i, j) \circledast 0) &= \Psi((i, j+k)) = (x \circ 1^i) \cdot (y \circ 0^{j+k}) = ((x \circ 1^i) \cdot (y \circ 0^j)) \circ 0 = \Psi(q) \circ 0 \\ (i, j) \circledast 0 &= q_y \circledast_y 0^{j+k} = \Psi_y(q_y) * 0^{j+k} = y * 0^{j+k} = ((x \circ 1^i) \cdot (y \circ 0^j)) * 0 = \Psi(q) * 0. \end{aligned}$$

On the other hand, if $K_0(x \circ 1^i, y \circ 0^j) = \perp$, then

$$\begin{aligned} \Psi((i, j) \circledast 0) &= \Psi(q_x \circledast_x 1^i) = \Psi_x(q_x \circledast_x 1^i) = \Psi_x(q_x) \circ 1^i = x \circ 1^i = ((x \circ 1^i) \cdot (y \circ 0^j)) \circ 0 = \Psi(q) \circ 0 \\ (i, j) \circledast 0 &= q_x \circledast_x 1^i = \Psi_x(q_x) * 1^i = x * 1^i = ((x \circ 1^i) \cdot (y \circ 0^j)) * 0 = \Psi(q) * 0. \end{aligned}$$

Hence $\Psi(q \circledast 0) = \Psi(q) \circ 0$ and $q \circledast 0 = \Psi(q) * 0$ for all $q \in Q'$. Similarly, it is routine to verify that $\Psi(q \circledast 1) = \Psi(q) \circ 1$ and $q \circledast 1 = \Psi(q) * 1$ for all $q \in Q'$. \square

We will in Algorithm 6.2 turn the definition in (3) into an algorithm for computing $\mathcal{T}_{x,y}$ from \mathcal{T}_x and \mathcal{T}_y . Algorithm 6.2 involves computing $(i, j) \circledast \alpha$ and $(i, j) \circledast \alpha$ for $\alpha \in \{0, 1\}$, which in turn require us to calculate $K_\alpha(x \circ 1^i, y \circ 0^j)$.

We cannot compute K_α directly, since it is defined on pairs of elements in $\text{FB}^1(A)$ and these have no representation here other than transducers. In particular, computing transducers (or another representation of) the input values $x \circ 1^i$ and $y \circ 0^j$ defeats the purpose of the algorithm presented in this section. However, if $x, y \in \text{FB}^1(A)$ are fixed, then Lemma 6.3 tells us that

$$K_0(x \circ 1^i, y \circ 0^j) = \begin{cases} 1 & \text{if } y * 0^{j+1} \neq \perp \text{ and } y * 0^{j+1} \notin \text{cont}(x \circ 1^i) \\ 1 + K_0(x \circ 1^i, y \circ 0^{j+1}) & \text{if } y * 0^{j+1} \in \text{cont}(x \circ 1^i) \text{ and } K_0(x \circ 1^i, y \circ 0^{j+1}) \neq \perp \\ \perp & \text{otherwise} \end{cases}$$

$$K_1(x \circ 1^i, y \circ 0^j) = \begin{cases} 1 & \text{if } x * 1^{i+1} \neq \perp \text{ and } x * 1^{i+1} \notin \text{cont}(y \circ 0^j) \\ 1 + K_1(x \circ 1^{i+1}, y \circ 0^j) & \text{if } x * 1^{i+1} \in \text{cont}(y \circ 0^j) \text{ and } K_1(x \circ 1^{i+1}, y \circ 0^j) \neq \perp \\ \perp & \text{otherwise.} \end{cases}$$

We can effectively compute, as follows, the values $K_\alpha(x \circ 1^i, y \circ 0^j)$ for any $i, j \in \mathbb{N}_0$ from any transducers \mathcal{T}_x and \mathcal{T}_y representing x and y , respectively. If q_x and q_y are the initial states and \circledast_x and \circledast_y are the letter transition functions of \mathcal{T}_x and \mathcal{T}_y , respectively, then $q_x \circledast_x 1^i = x * 1^i$ and $q_y \circledast_y 0^j = y * 0^j$ for all i, j , and so we can compute $K_\alpha(x \circ 1^i, y \circ 0^j)$ given $i, j \in \mathbb{N}_0$. For the remainder of this section we suppose that $x, y \in \text{FB}^1(A)$ are fixed. We define $\overline{K}_\alpha : \mathbb{N}_0 \times \mathbb{N}_0 \rightarrow \mathbb{N}$ by

Table 6.1

The functions $\overline{K_0}$ (left) and $\overline{K_1}$ (right) of $x = eaec, y = bcacbcd \in FB^1(a, b, c, d, e)$. Compute_{K_0} and Compute_{K_1} fill in their respective tables column-by-column, starting from the right, filling each column from bottom to top.

i	0	1	2	3
$\text{cont}(x \circ 1^i)$	$\{a, e, c\}$	$\{e, c\}$	$\{c\}$	\emptyset
0	d	1	1	1
1	a	3	1	1
2	c	2	2	1
3	b	1	1	1
4	\perp	\perp	\perp	\perp
j	$q_y \otimes 0^{j+1}$			

j	0	1	2	3	4
$\text{cont}(y \circ 0^j)$	$\{b, c, a, d\}$	$\{b, c, a\}$	$\{b, c\}$	$\{b\}$	\emptyset
0	a	1	1	1	2
1	e	1	1	1	1
2	c	1	1	\perp	\perp
3	\perp	\perp	\perp	\perp	\perp
i	$q_x \otimes 1^{i+1}$				

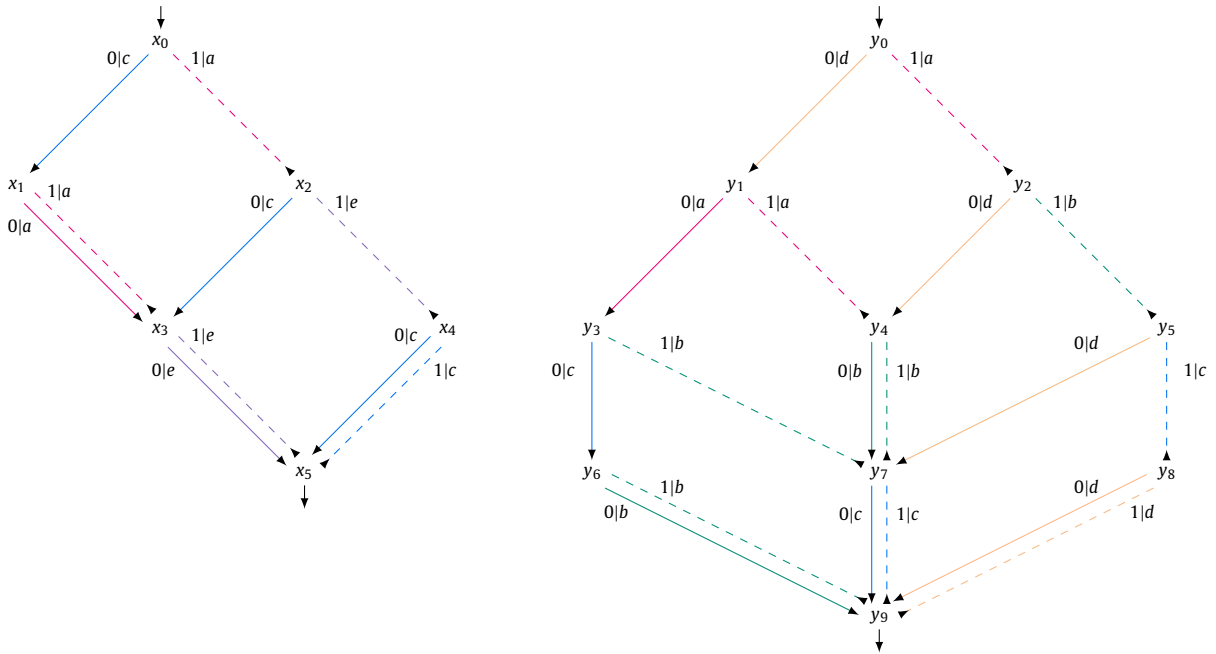


Fig. 7. The minimal transducers for $x = eaec$ and $y = bcacbcd$.

$$\overline{K_\alpha}(i, j) = K_\alpha(x \circ 1^i, y \circ 0^j).$$

Computing $\overline{K_\alpha}(i, j)$ naively by considering all possible values $k \geq 1$ and checking if $y * 0^{j+k}$ is defined, for example, requires at least $\mathcal{O}(|A|)$ steps per $(i, j) \in Q'$. This would require a total of $\mathcal{O}(|A|^3)$ to compute $\overline{K_\alpha}(i, j)$ for all $(i, j) \in Q'$. However, by utilizing Lemma 6.3 and traversing (i, j) in reverse lexicographic order, we can precompute the values of $\overline{K_\alpha}(i, j)$ for all $(i, j) \in Q'$ in $\mathcal{O}(|A|^2)$ time. Storing the values will take $\mathcal{O}(|A|^2)$ space and we can then retrieve each value in $\mathcal{O}(1)$ time. This is exactly what we do to compute $\overline{K_0}$ in Compute_{K_0} given in Algorithm 6.1. A function Compute_{K_1} to compute $\overline{K_1}$ can be defined analogously.

To illustrate, if $x = eaec$ and $y = bcacbcd$ are elements in $FB^1(a, b, c, d, e)$, then the minimal transducers representing each x and y are shown in Fig. 7 and the values of $\overline{K_0}$ and $\overline{K_1}$ are shown in Table 6.1.

Lemma 6.9. Let $x, y \in FB^1(A), \alpha \in \{0, 1\}$ and let \mathcal{T}_x and \mathcal{T}_y be transducers representing x and y , respectively. Then $\text{Compute}_{K_\alpha}(\mathcal{T}_x, \mathcal{T}_y) = \overline{K_\alpha}$ and $\text{Compute}_{K_\alpha}$ has time complexity $\mathcal{O}(|A|^2)$.

Proof. We prove the lemma for Compute_{K_0} , the proof for Compute_{K_1} is analogous.

We begin by showing that the value of c equals $\text{cont}(x \circ 1^i)$ at the start of each iteration of the loop in line 3. When $i = |\text{cont}(x)|$ at the start of the first loop, $|\text{cont}(x \circ 1^i)| = |\text{cont}(x)| - i = 0$ and so $\text{cont}(x \circ 1^i) = \emptyset = c$. Assume that $c = \text{cont}(x \circ 1^i)$ holds for some $0 < i \leq |\text{cont}(x)|$ at the start of the loop in line 3. Since $i \neq 0$, in line 12 we set c to be $c \cup \{q_x \otimes_x 1^i\}$. Since \mathcal{T}_x represents x , $q_x \otimes_x 1^i = x * 1^i$ and so $c \cup \{q_x \otimes_x 1^i\} = \text{cont}(x \circ 1^i) \cup \{x * 1^i\} = \text{cont}(x \circ 1^{i-1})$, as required.

Since $c = \text{cont}(x \circ 1^i)$ for every value of i , and \mathcal{T}_y represents y , it follows that checking $q_y \otimes_y 0^{j+1} \neq \perp$ and $q_y \otimes_y 0^{j+1} \notin c$ in line 5 is the same as checking $y * 0^{j+1} \neq \perp$ and $y * 0^{j+1} \notin \text{cont}(x \circ 1^i)$.

Algorithm 6.1: Compute K_0 .

Arguments : a pair of transducers $\mathcal{T}_x = (Q_x, \{0, 1\}, A, q_x, T_x, \otimes_x, \otimes_x)$ and $\mathcal{T}_y = (Q_y, \{0, 1\}, A, q_y, T_y, \otimes_y, \otimes_y)$ representing some $x, y \in \text{FB}^1(A)$ respectively.

Returns : the partial function \overline{K}_0 .

```

1 Let  $\overline{K}_0: \mathbb{N}_0 \times \mathbb{N}_0 \rightarrow \mathbb{N}$  be such that  $\overline{K}_0(i, j) = \perp$  for all  $i, j$ .
2 Let  $c \leftarrow \emptyset$  # The content of  $x$  seen so far
3 for  $i \in \{|\text{cont}(x)|, \dots, 0\}$  do
4   for  $j \in \{|\text{cont}(y)|, \dots, 0\}$  do
5     if  $q_y \otimes_y 0^{j+1} \neq \perp$  and  $q_y \otimes_y 0^{j+1} \notin c$  then
6       |  $K_0(i, j) := 1$ 
7     else if  $q_y \otimes_y 0^{j+1} \neq \perp$  and  $\overline{K}_0(i, j+1) \neq \perp$  then
8       |  $\overline{K}_0(i, j) := 1 + \overline{K}_0(i, j+1)$ 
9     end if
10  end for
11  if  $i \neq 0$  then
12    |  $c \leftarrow c \cup \{q_x \otimes_x 1^i\}$ 
13  end if
14 end for
15 return  $\overline{K}_0$ 

```

We will show that $\overline{K}_0(i, j) = K_0(x \circ 1^i, y \circ 0^j)$ at the end of the loop that starts in line 4 for all values of i and j . Suppose that $0 \leq i \leq |\text{cont}(x)|$ is arbitrary and $j = |\text{cont}(y)|$. Then $y * 0^{j+1} = \perp$ and so $K_0(x \circ 1^i, y \circ 0^j) = \perp$ and the checks in line 5 and line 7 both fail. Since the value of $\overline{K}_0(i, j)$ is not changed during the first iteration of the loop, $\overline{K}_0(i, j) = \perp = K_0(x \circ 1^i, y \circ 0^j)$ when $j = |\text{cont}(y)|$. Assume that at the start of the loop in line 4, $\overline{K}_0(i, j+1) = K_0(x \circ 1^i, y \circ 0^{j+1})$ for some i and j . Then we can translate the checks and assignments in lines 5-9 into

$$\overline{K}_0(i, j) = \begin{cases} 1 & \text{if } y * 0^{j+1} \neq \perp \text{ and } y * 0^{j+1} \notin \text{cont}(x \circ 1^i) \\ 1 + K_0(x \circ 1^i, y \circ 0^{j+1}) & \text{if } y * 0^{j+1} \in \text{cont}(x \circ 1^i) \text{ and } K_0(x \circ 1^i, y \circ 0^{j+1}) \neq \perp \\ \perp & \text{otherwise} \end{cases}$$

the right hand of which is exactly the definition of $K_0(x \circ 1^i, y \circ 0^j)$. So $\overline{K}_0(i, j)$ is correct at the end of the loop starting in line 4. This completes the proof that the values of \overline{K}_0 are correct by the end of the algorithm.

To show that the time complexity is $\mathcal{O}(|A|^2)$, note that the values of $\text{cont}(x), \text{cont}(y), q_y \otimes_x 0^j$ and $q_x \otimes_x 1^i$ for $1 \leq i \leq |\text{cont}(x)|$ and $1 \leq j \leq |\text{cont}(y)|$ can be precomputed in $\mathcal{O}(A)$ time. The assignments in lines 1 and 2, the checks in lines 5 and 7, and the assignment in line 12 are all constant time. The lines within the loops are executed at most $(|\text{cont}(x)| + 1)(|\text{cont}(y)| + 1) \in \mathcal{O}(|A|^2)$ times, as required. \square

The multiplication algorithm `Multiply` for transducers is given in Algorithm 6.2. Informally, we start with an initially empty transducer, then copy all the states and transitions from \mathcal{T}_x and \mathcal{T}_y into $\mathcal{T}_{x,y}$ so that $\mathcal{T}_{x,y}$ coincides with each of \mathcal{T}_x and \mathcal{T}_y on the states Q_x and Q_y respectively. We then add all the states $(i, j) \in Q'$ to $\mathcal{T}_{x,y}$, and assign their transitions according to Lemma 6.4. Although, in some sense, this informal description of the algorithm is not far away from the pseudo-code given in Algorithm 6.2, we require the precise formulation given in Algorithm 6.2 to describe its time complexity in Theorem 6.10. It is clear from the definition of $\mathcal{T}_{x,y}$ in (3), and the preceding discussion, that the value returned by `Multiply`($\mathcal{T}_x, \mathcal{T}_y$) is $\mathcal{T}_{x,y}$.

Theorem 6.10. *Let $x, y \in \text{FB}^1(A)$, $\alpha \in \{0, 1\}$, and let \mathcal{T}_x and \mathcal{T}_y be transducers representing x and y respectively. Then `Multiply` has time complexity $\mathcal{O}(|\mathcal{T}_x| + |\mathcal{T}_y| + |A|^2)$ where $|\mathcal{T}_x|$ and $|\mathcal{T}_y|$ are the numbers of states in \mathcal{T}_x and \mathcal{T}_y , respectively.*

Proof. Note that line 1 is constant time. Since there are at most twice as many transitions as states in any transducer, line 2 requires $\mathcal{O}(|\mathcal{T}_x| + |\mathcal{T}_y|)$ time. Line 3 takes $\mathcal{O}(|A|^2)$ time by Lemma 6.9.

As we already noted in the proof of Lemma 6.9 we can precompute the values $q_x \otimes_x 1^i 0, q_y \otimes_y 0^j 1, q_x \otimes_x 1^i$, and $q_y \otimes_y 0^j$ for $1 \leq i \leq |\text{cont}(x)|$ and $1 \leq j \leq |\text{cont}(y)|$ in $\mathcal{O}(|A|)$ time. Since \overline{K}_α is also precomputed, each of the checks and assignments in lines 6-16 can be done in constant time. This means that the for loop in lines 4-18 takes a total of $\mathcal{O}(|A|^2)$ time.

Line 19 takes at most $\mathcal{O}(|\mathcal{T}_x| + |\mathcal{T}_y|)$ time since each transducer can have no more terminal states than states in total. Lines 21 and 22 are at most $\mathcal{O}(A)$ time, since we can remove a state in constant time.

Finally, since we can check if a state is equal to $(|\text{cont}(x)|, j)$ or $(i, |\text{cont}(y)|)$ in constant time, lines 25-29 are constant time. So lines 23-31 take time proportional to the total number of states in \mathcal{T} which by construction is $\mathcal{O}(|\mathcal{T}_x| + |\mathcal{T}_y| + |A|^2)$. Therefore the overall time complexity is $\mathcal{O}(|\mathcal{T}_x| + |\mathcal{T}_y| + |A|^2)$, as required. \square

Example 6.11. Let $x = eaec$ and $y = bcacbcd \in \text{FB}^1(a, b, c, d, e)$. We illustrate the operation of `Multiply` in four steps. First, we construct Q' , initially with no transitions. In the second step, the values in the tables for \overline{K}_0 and \overline{K}_1 in Table 6.1 are

Algorithm 6.2: Multiply.

Arguments : a pair of transducers $\mathcal{T}_x = (Q_x, \{0, 1\}, A, q_x, T_x, \otimes_x, \oplus_x)$ and $\mathcal{T}_y = (Q_y, \{0, 1\}, A, q_y, T_y, \otimes_y, \oplus_y)$ representing $x, y \in \text{FB}^1(A)$ respectively,

Returns : the product transducer $\mathcal{T}_{x,y} = (Q, \{0, 1\}, A, q_0, T, \otimes, \oplus)$.

- 1 Let $\mathcal{T} = (Q = \emptyset, \{0, 1\}, A, q_0 = \varepsilon, T = \emptyset, \otimes, \oplus)$ be the empty transducer.
- 2 Copy all the states and transitions from \mathcal{T}_x and \mathcal{T}_y into \mathcal{T}
- 3 $\overline{K}_0 \leftarrow \text{Compute}K_0(\mathcal{T}_x, \mathcal{T}_y)$ and $\overline{K}_1 \leftarrow \text{Compute}K_1(\mathcal{T}_x, \mathcal{T}_y)$
- 4 **for** $i \in \{|\text{cont}(x)|, \dots, 0\}$ **do**
- 5 **for** $j \in \{|\text{cont}(y)|, \dots, 0\}$ **do**
- 6 Add state (i, j) to \mathcal{T}
- 7 **if** $\overline{K}_0(i, j) \neq \perp$ **then**
- 8 $(i, j) \otimes 0 := (i, j + \overline{K}_0(i, j))$ and $(i, j) \oplus 0 := q_y \otimes_y 0^{j + \overline{K}_0(i, j)}$
- 9 **else**
- 10 $(i, j) \otimes 0 := q_x \otimes_x 1^i 0$ and $(i, j) \oplus 0 := q_x \otimes_x 1^i 0$
- 11 **end if**
- 12 **if** $\overline{K}_1(i, j) \neq \perp$ **then**
- 13 $(i, j) \otimes 1 := (i + \overline{K}_1(i, j), j)$ and $(i, j) \oplus 1 := q_x \otimes_x 1^{i + \overline{K}_1(i, j)}$
- 14 **else**
- 15 $(i, j) \otimes 1 := q_y \otimes_y 0^j 1$ and $(i, j) \oplus 1 := q_y \otimes_y 0^j 1$
- 16 **end if**
- 17 **end for**
- 18 **end for**
- 19 $q_0 \leftarrow (0, 0)$, $T \leftarrow T_x \cup T_y$
- 20 # Perform an identification of the relevant states.
- 21 Remove the states $(i, |\text{cont}(y)|)$ for all $i \in \{|\text{cont}(x)|, \dots, 0\}$
- 22 Remove the states $(|\text{cont}(x)|, j)$ for all $j \in \{|\text{cont}(y)|, \dots, 0\}$
- 23 **for** $q \in Q$ **do**
- 24 **for** $\alpha \in \{0, 1\}$ **do**
- 25 **if** $q \otimes \alpha = (i, |\text{cont}(y)|)$ for some i **then**
- 26 $q \otimes \alpha = q_x \otimes_x 1^i$
- 27 **else if** $q \otimes \alpha = (|\text{cont}(x)|, j)$ for some j **then**
- 28 $q \otimes \alpha = q_y \otimes_y 0^j$
- 29 **end if**
- 30 **end for**
- 31 **end for**
- 32 **return** \mathcal{T}

used to find the transitions between the states in Q' . For example, since $\overline{K}_0(0, 0) = 1$, we can add the 0-transition from $(0, 0)$ to $(0, 1)$ labelled by $q_y \otimes_y 0^1 = d$. Similarly, $\overline{K}_1(0, 0) = 2$, so there is a 1-transition from $(0, 0)$ to $(0, 2)$ labelled by $q_y \otimes 0^2 = e$. In the same way, the remaining transitions between the states in Q' can be defined whenever $\overline{K}_0(i, j) \neq \perp$ or $\overline{K}_1(i, j) \neq \perp$; see Fig. 8a. In the third step, we consider those states $(i, j) \in Q'$ such that $\overline{K}_0(i, j) = \perp$ and $\overline{K}_1(i, j) = \perp$. For example, $\overline{K}_1(2, 0) = \perp$ and so $(2, 0) \otimes 1 = y_0 \otimes_y 0^0 1 = y_2$ and $(i, j) \otimes 1 = y_0 \otimes_y 0^0 1 = a$. Similarly, $\overline{K}_1(2, 1) = \perp$, so $(2, 1) \otimes 1 = y_0 \otimes_y 0 1 = y_4$ and $(2, 1) \otimes 1 = y_0 \otimes_y 0 1 = a$; see Fig. 8b. Finally, the remaining transitions are essentially those from \mathcal{T}_x and \mathcal{T}_y but with some states identified; see Fig. 9.

7. Minimal word representative

In this section, we present an algorithm for determining the shortlex least word $\min(w)$ belonging to the \sim -class of $w \in A^*$. We will abuse our notation, by writing $\min(x)$ to mean the shortlex least word such that $\min(x)/\sim = x$, where $x \in \text{FB}^1(A)$. Clearly, $\min(w/\sim) = \min(w)$. Given a transducer \mathcal{T}_x representing $x \in \text{FB}^1(A)$, the algorithm for computing $\min(x)$, that we introduce in this section, has $\mathcal{O}(|\mathcal{T}_x| \cdot |A|)$ time complexity. This leads to a $\mathcal{O}(|w| \cdot |A| + |\min(w)| \cdot |A|^2)$ time algorithm for computing $\min(w)$ given $w \in A^*$.

To establish our algorithm we first show some technical results about the structure of $\min(x)$.

Lemma 7.1 (cf. [20, Theorem 3.9(a) and (b)]). *Let $w \in A^*$ and let $s = \min(w \circ 0)(w * 0)$ and $t = (w * 1) \min(w \circ 1)$. If $i \in \{1, \dots, |s|\}$ is such that $s_{(i, |s|)}$ is the longest suffix of s that is also a prefix of t , then $\min(w) = s_{(1, i-1)} t$.*

Lemma 7.1 says, in other words, that to find a minimal word representative, it suffices to minimize the prefix and suffix, and then overlap them as much as possible. To prove Lemma 7.1 we require the following.

Lemma 7.2 (cf. [20, Theorem 3.9(c)]). *For all $w \in A^*$ and for all $\alpha \in \{0, 1\}^*$, $\min(w \circ \alpha) = \min(w) \circ \alpha$.*

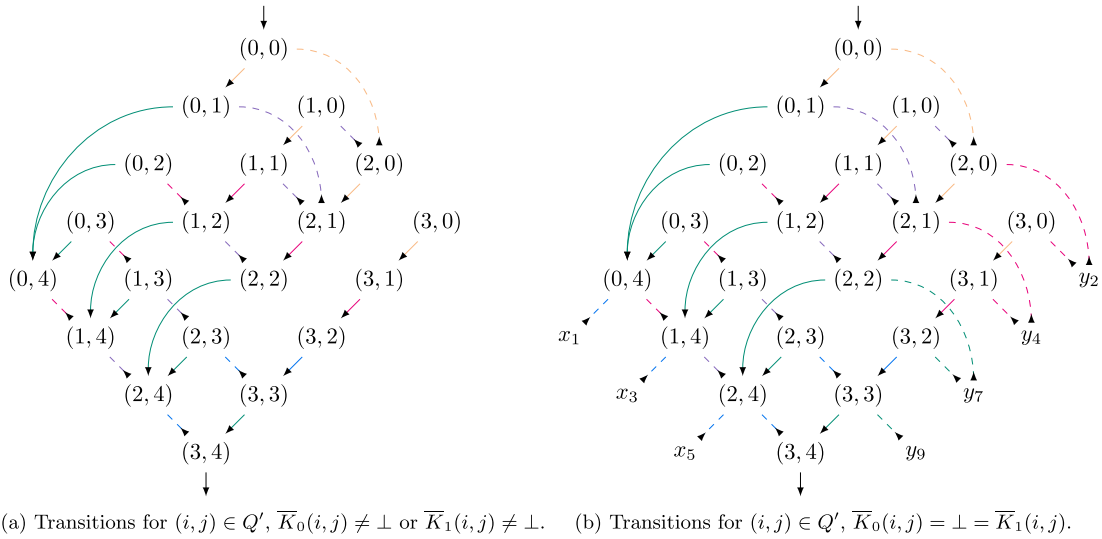


Fig. 8. The second and third steps in the construction of the product transducer \mathcal{T}_{xy} for $x = eaec$ and $y = bacbcd$ constructed from the minimal transducers \mathcal{T}_x and \mathcal{T}_y in Fig. 7. The pinkish-red transitions are labelled by a , the light-green ones by b , the blue ones by c , the yellow ones by d and purple ones by e . (For interpretation of the colours in the figure(s), the reader is referred to the web version of this article.)

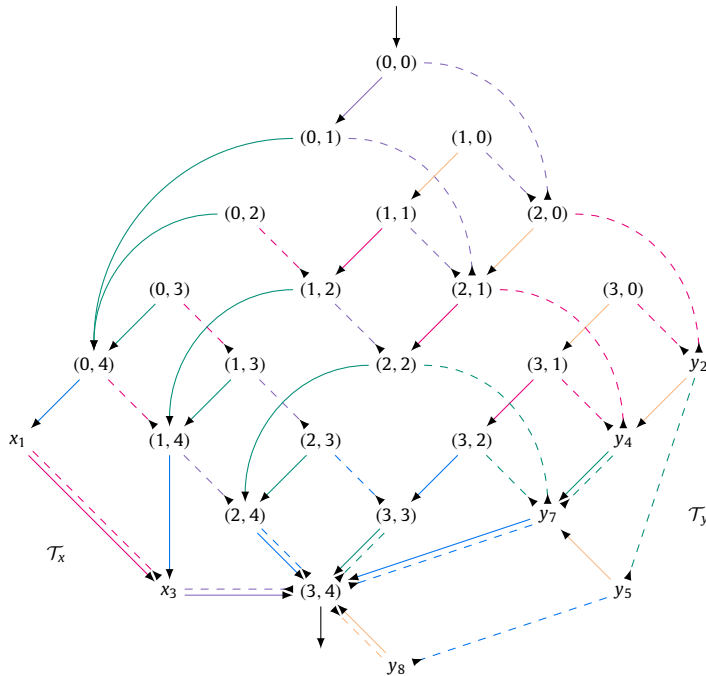


Fig. 9. The complete product transducer \mathcal{T}_{xy} for $x = eaec$ and $y = bacbcd$ constructed from the minimal transducers \mathcal{T}_x and \mathcal{T}_y in Fig. 7. The colours of the edges correspond to the generators labelling them in the same way as in Fig. 8.

Proof. Note that for all $w \in A^*$ and $\alpha \in \{0, 1\}^*$, there exist $u, v \in A^*$ such that $w = u(w \circ \alpha)v$. Since $w \circ \alpha \sim \min(w \circ \alpha)$ by definition, $w = u(w \circ \alpha)v \sim u \min(w \circ \alpha)v$ and $u \min(w \circ \alpha)v \leq u(w \circ \alpha)v = w$. Repeating this process for $\min(w)$ yields $u \min(\min(w) \circ \alpha)v \leq u(\min(w) \circ \alpha)v = \min(w)$. By the minimality of $\min(w)$, $u \min(\min(w) \circ \alpha)v = u(\min(w) \circ \alpha)v$ and so $\min(\min(w) \circ \alpha) = \min(w) \circ \alpha$. By the Green-Rees Theorem (Theorem 2.3), $\min(w) \circ \alpha \sim w \circ \alpha$ and so $\min(\min(w) \circ \alpha) = \min(w \circ \alpha)$. It follows that $\min(w) \circ \alpha = \min(w \circ \alpha)$ for all $w \in A^*$ and $\alpha \in \{0, 1\}^*$. \square

Proof of Lemma 7.1. By Lemma 7.2, it follows that $\min(w) \circ 0 = \min(w \circ 0)$ and $\min(w) \circ 1 = \min(w \circ 1)$. By the Green-Rees Theorem, $\min(w) * 0 = w * 0$ and $\min(w) * 1 = w * 1$. Hence from the definitions of \circ and $*$, $\min(w)$ has a prefix $s = (\min(w) \circ 0) (\min(w) * 0) = (\min(w \circ 0)) (w * 0)$ and a suffix $t = (w * 1) (\min(w \circ 1))$.

There are now two possible cases. Either s and t overlap within $\min(w)$ or they do not. If there is no overlap, then $\min(w) = su't$ for some $u' \in A^*$. Note, however, that $st = \min(w \circ 0)(w * 0)\min(w \circ 1) \sim w$ by the Green-Rees Theorem. So if s, t do not overlap within $\min(w)$, then $u' = \varepsilon$ and $\min(w) = st$.

Otherwise, if there is some overlap of s and t within $\min(w)$, then $\min(w) = s'v't'$ for some $s', t' \in A^*$ and $v' \in A^+$ such that $s'v' = s$ and $v't' = t$. If $s'', t'' \in A^*$ and $v'' \in A^+$ with $|v''| < |v'|$ are such that $s''v'' = s$ and $v''t'' = t$, then $\min(w) \sim s''v''t''$ and $|s''v''t''| < |s'v't'|$, contradicting the minimality of $\min(w)$. Therefore v' is the largest suffix of s that is also a prefix of t , as required. \square

In order to later use Lemma 7.1 in a computational manner, we will fully classify the possible types of overlap between s and t in Lemma 7.3. In general, there are many different suffixes of a given word that are also prefixes of other words, e.g. for $bbababa$, each of the proper suffixes $a, aba, ababa$ is a prefix of $ababaa$. Perhaps surprisingly, as it will turn out in Lemma 7.3, at most one suffix $s = \min(w \circ 0)(w * 0)$ (from Lemma 7.2) is a prefix of $t = (w * 1)\min(w \circ 1)$. This is key to the linear complexity of the algorithm presented in this section.

We require the following observation in the proof of Lemma 7.3. If $w \in A^*$, then $w * 0 \notin \text{cont}(w \circ 0)$, and so $\text{cont}(w) = \text{cont}(w \circ 0) \cup \{w * 0\}$. By repeatedly applying this, we get $\text{cont}(w) = \bigcup_{k=1}^{|\text{cont}(w)|} \{w * 0^k\}$. Hence if $a \in \text{cont}(w)$, then $a = w * 0^k$ for some k . An analogous statement holds when 0 is replaced by 1 .

Lemma 7.3 (cf. [20, Theorem 3.9(a), Lemma 4.1]). *Let $w \in A^*$, let $s = \min(w \circ 0)(w * 0)$, and let $t = (w * 1)\min(w \circ 1)$. Then there is at most one non-empty suffix v' of s that is also a prefix of t .*

(i) *If $w * 0 = w * 1$, then $v' = w * 0$ and so $\min(w) = \min(w \circ 0)(w * 0)\min(w \circ 1)$.*

*If $w * 0 \neq w * 1$, then there exist $k, l > 0$ such that $w * 01^k = w * 1$ and $w * 10^l = w * 0$ and either:*

- (ii) *$k = l$, $\min(w \circ 01^k) = \min(w \circ 10^k)$, and $v' = (w * 1)\min(w \circ 01^k)(w * 0)$ and so $\min(w)$ is the unique word with prefix $\min(w \circ 0)(w * 0)$ and suffix $(w * 1)\min(w \circ 1)$ whose overlap is $(w * 1)\min(w \circ 01^k)(w * 0)$; or*
- (iii) *no non-empty suffix of s is a prefix of t , and so $\min(w) = \min(w \circ 0)(w * 0)(w * 1)\min(w \circ 1)$.*

Proof. Assume that $s = s'v'$ and $t = v't'$ for some $s', v', t' \in A^*$.

(i) If $w * 0 = w * 1$, then we will show that $v' = w * 0$. Clearly, $v' = w * 0$ is a suffix of $s = (w \circ 0)(w * 0)$ and a prefix of $t = (w * 1)(w \circ 1)$.

Assume that there is another non-empty suffix v'' of s that is a prefix of t not equal to v' . Since v'' is distinct from $w * 0$, $|v''| > 1$. Then $v''t'' = t = (w * 1)\min(w \circ 1)$ for some $t'' \in A^*$, and so the first letter of v'' is $w * 1 = w * 0$. On the other hand, $s''v'' = s = \min(w \circ 0)(w * 0)$ for some $s'' \in A^*$, $|v''| > 1$, and so the first letter of v'' , $w * 0$, occurs somewhere in $\min(w \circ 0)$. But by definition, $\text{cont}(\min(w \circ 0)) = \text{cont}(w \circ 0) = \text{cont}(w) \setminus \{w * 0\}$, a contradiction.

So if $w * 0 = w * 1$, then $v' = w * 0$ is the only suffix of s that is also a prefix of t , which implies that $\min(w) = \min(w \circ 0)(w * 0)\min(w \circ 1)$.

For the remainder of the proof we suppose that $w * 0 \neq w * 1$. We establish that there is at most one non-empty suffix v' of s that is also a prefix of t .

Assume that such a suffix v' exists. Clearly $|v'| > 1$ since v' is non-empty and if $|v'| = 1$, then we would have $w * 0 = w * 1$, a contradiction.

Since $s'v' = s = \min(w \circ 0)(w * 0)$ and $v't' = t = (w * 1)\min(w \circ 1)$, it follows that $v' = (w * 1)v''(w * 0)$ for some $v'' \in A^*$. By considering the content of $\min(w \circ 0)$ and $\min(w \circ 1)$, we deduce that $\text{cont}(v'') = \text{cont}(w) \setminus \{w * 0, w * 1\}$. Since $w * 1$ is the first letter of v' and $w * 1$ does not occur anywhere else in v' , v' starts at the rightmost occurrence of $w * 1$ in s . Similarly, v' must end at the leftmost occurrence of $w * 0$ in t . In particular, v' is the unique suffix of s that is also a prefix of t .

Next, we establish the existence of the values k and l in the statement of the lemma. By the comment immediately before the lemma, and because $w * 1 \in \text{cont}(w) \setminus \{w * 0\} = \text{cont}(\min(w) \circ 0)$, there exists $k > 0$ such that $(w \circ 0) * 1^k = w * 1$. But $w * 01^k = (w \circ 0) * 1^k$ by definition, and so $w * 01^k = w * 1$. In a similar manner, there exists $l > 0$ such that $w * 10^l = w * 0$.

Before proving cases (ii) and (iii) of the lemma, we determine the rightmost occurrence of $w * 1$ in s and the leftmost occurrence of $w * 0$ in t . Since $w * 01^k = w * 1$, the definition of \circ implies that there is a prefix s'' of $\min(w) \circ 0$ with

$$\min(w) \circ 0 = s''(\min(w) * 01^k)(\min(w) \circ 01^k) = s''(w * 01^k)\min(w \circ 01^k) = s''(w * 1)\min(w \circ 01^k).$$

Since $(\min(w) \circ 0)(w * 0) = \min(w \circ 0)(w * 0) = s$, the word s'' is also a prefix of s , and so

$$s = s''(w * 1)\min(w \circ 01^k)(w * 0).$$

Since $w * 1 = w * 01^k = \min(w) * 01^k \notin \text{cont}(\min(w \circ 01^k))$ and $s = s''(w * 1) \min(w \circ 01^k)(w * 0)$, the rightmost occurrence of $w * 1$ in s is just after s'' . In a similar manner, $t = (w * 1) \min(w \circ 10^l)(w * 0)t''$ for some suffix t'' , and the leftmost occurrence of $w * 0$ in t is just before t'' .

(ii) If $k = l$ and $\min(w \circ 01^k) = \min(w \circ 10^k)$, then from the above $v' = (w * 1) \min(w \circ 01^k)(w * 0)$ is a suffix of s and a prefix of t , and hence the unique word with this property.

(iii) Assume that $k \neq l$ or $\min(w \circ 01^k) \neq \min(w \circ 10^k)$.

If there exists a non-empty suffix v' of s that is also a prefix of t , then since v' must begin at the rightmost occurrence of $w * 1$ in s , it follows that $v' = (w * 1) \min(w \circ 01^k)(w * 0)$. Similarly, by considering the content of $\min(w \circ 10^l)$, we can establish that $v' = (w * 1) \min(w \circ 10^l)(w * 0)$. Hence $(w * 1) \min(w \circ 01^k)(w * 0) = (w * 1) \min(w \circ 10^l)(w * 0)$ and so $\min(w \circ 01^k) = \min(w \circ 10^l)$. Since $|\text{cont}(w)| - (l + 1) = |\text{cont}(\min(w \circ 10^l))| = |\text{cont}(\min(w \circ 01^k))| = |\text{cont}(w)| - (k + 1)$, this implies that $k = l$, a contradiction. Therefore v' does not exist, and there is no non-empty suffix of s that is a prefix of t , as required. \square

Let $x \in \text{FB}^1(A)$ and let $\mathcal{T}_x = (Q, \{0, 1\}, A, q_0, T, \otimes, \oplus)$ be the minimal transducer representing x . Note that by the Green-Rees Theorem (Theorem 2.3), it follows that any two $v, w \in A^*$ with $v/\sim = w/\sim = x$ will be in the same case of Lemma 7.3, and if they are in case (ii), then they will have the same value for k . So we may refer to these cases applying to $x \in \text{FB}^1(A)$. If $q \in Q$ is any state and $y \in \text{FB}^1(A)$ is the element represented by q (i.e. y is not necessarily equal to x), then Lemma 7.3 can be applied to y . We say that Lemma 7.3 applies to a state q of \mathcal{T}_x to mean that it applies to the element of the free band represented by q .

In Algorithm 7.1, we present a $\mathcal{O}(|\text{cont}(x)|)$ time algorithm named `ClassifyCase` for determining which of the cases (i), (ii), or (iii) of Lemma 7.3 applies to a given $q \in Q$. If case (ii) applies, then the algorithm also finds $1 \leq k < |\text{cont}(x)|$ such that $\min(y \circ 01^k) = \min(y \circ 10^k)$ where $y \in \text{FB}^1(A)$ is represented by q . For convenience when presenting Algorithm 7.2 later, `ClassifyCase` returns both the case of Lemma 7.3 that applies to q and an integer. In case (ii) this integer is k , and in cases (i) and (iii) it is $|\text{cont}(y)|$ where $y \in \text{FB}^1(A)$ is represented by the argument $q \in Q$.

Algorithm 7.1: ClassifyCase.

Arguments : a minimal transducer $\mathcal{T}_x = (Q, \{0, 1\}, A, q_0, T, \otimes, \oplus)$ representing some $x \in \text{FB}^1(A)$, and a state $q \in Q$,
Returns : the case c of Lemma 7.3 that applies to the element of $\text{FB}^1(A)$ represented by q , and the corresponding integer k if c is (ii).

```

1 if  $q \otimes 0 = q \otimes 1$  then
2   | return case (i) and  $|\text{cont}(\Psi(q))|$ 
3 end if
4  $u \leftarrow q \otimes 0, v \leftarrow q \otimes 1$ 
5 for  $k \in \{1, \dots, |\text{cont}(\Psi(q))|\}$  do
6   | if  $u \otimes 1 = q \otimes 1, v \otimes 0 = q \otimes 0$ , and  $u \otimes 1 = v \otimes 0$  then
7     | | return case (ii) and  $k$ 
8     | end if
9      $u \leftarrow u \otimes 1, v \leftarrow v \otimes 0$ 
10  | if  $u$  or  $v$  is terminal then
11    | | return case (iii) and  $|\text{cont}(\Psi(q))|$ 
12    | end if
13 end for
```

Theorem 7.4. Algorithm 7.1 is correct and has time complexity $\mathcal{O}(|A|)$.

Proof. Let $y = \Psi(q)$ be the element of $\text{FB}^1(A)$ represented by q . Since $q \otimes \alpha = y * \alpha$ for all $\alpha \in \{0, 1\}^*$, $q \otimes 0 = q \otimes 1$ in line 1 if and only if $y * 0 = y * 1$. If this holds, then case (i) of Lemma 7.3 applies to y and the algorithm returns in line 2.

Supposing that $q \otimes 0 \neq q \otimes 1$, we will show that $u = q \otimes 01^{k-1}$ and $v = q \otimes 10^{k-1}$ at the start of the loop in line 5 for every value of k . In line 4, the states u and v represent $y \circ 0$ and $y \circ 1$, respectively, so the statement is true when $k = 1$. In line 9 we set the values of u and v to be $u \otimes 1$ and $v \otimes 0$, respectively. So if $u = q \otimes 01^{k-1}$ and $v = q \otimes 10^{k-1}$ prior to line 9, then $u = q \otimes 01^k, v = q \otimes 10^k$ after line 9, as required.

Since u and v represent $y \circ 01^{k-1}$ and $y \circ 10^{k-1}$, respectively, by Lemma 3.5, $u \otimes 1 = v \otimes 0$ holds if and only if $y \circ 01^k = y \circ 10^k$. Additionally, $u \otimes 1 = (y \circ 01^{k-1}) * 1 = y * 01^k$ and $q \otimes 1 = y * 1$, and so $u \otimes 1 = q \otimes 1$ if and only if $y * 01^k = y * 1$. Similarly, $v \otimes 0 = q \otimes 0$ if and only if $y * 10^k = y * 0$. So the condition in line 6 holds if and only if we are in case (ii) of Lemma 7.3 and $y \circ 01^k = y \circ 10^k$.

Finally for the correctness of the algorithm, if we terminate at line 11, it follows that the condition of case (ii) does not hold for any k , and so we must be in case (iii).

For the complexity, $|\text{cont}(y)|$ can be computed once at the start of the algorithm in time $\mathcal{O}(|\text{cont}(y)|)$ by simply traversing \mathcal{T}_x starting from the state q until we reach a terminal state. The checks and operations done at each step of Algorithm 7.1 are constant time. The loop in line 5 will execute at most $|\text{cont}(y)| \leq |A|$ times. So the time complexity of the algorithm is $\mathcal{O}(|A|)$ as required. \square

We now present an algorithm `MinWord` for computing $\min(x)$ given a minimal transducer representing $x \in \text{FB}^1(A)$. The input to `MinWord` consists of:

- (i) a minimal transducer $\mathcal{T}_x = \{Q, \{0, 1\}, A, q_0, T, \odot, \otimes\}$ representing x ;
- (ii) a state $q \in Q$;
- (iii) a word $w \in A^*$ that is a prefix of $\min(x)$;
- (iv) a non-negative integer l ;
- (v) a partial function $B : Q \rightarrow \mathbb{N}_0 \times \mathbb{N}_0$.

In Lemma 7.5, we will show that $\text{MinWord}(\mathcal{T}_x, q, w, l, B)$ produces the minimum word corresponding to the element represented by the state q , assuming that the input satisfies certain constraints. It will follow that $\min(x) = \text{MinWord}(\mathcal{T}_x, q, \varepsilon, 0, B)$, where B is defined $B(q) = (0, 0)$ for all terminal states $q \in T$ and $B(q) = \perp$ otherwise.

The algorithm `MinWord` is stated formally in Algorithm 7.2.

Algorithm 7.2: `MinWord`.

```

Arguments   :  $\mathcal{T}_x, q, w, l, B$  as described above.
Returns    : a word  $w'$  and a partial function  $B'$ .
1  $s \leftarrow |w| - l + 1$ 
2 if  $q \in \text{dom}(B)$  then
3    $(i, j) \leftarrow B(q)$ 
4   return  $w w_{(i+j)}, B$ 
5 end if
6  $w, B \leftarrow \text{MinWord}(\mathcal{T}_x, q \odot 0, w, l, B)$ 
7  $c, k := \text{ClassifyCase}(\mathcal{T}_x, q)$ 
8 if  $c$  indicates that Lemma 7.3(i) applies then
9    $w \leftarrow w(q \otimes 0)$ 
10   $l' \leftarrow 0$ 
11 else if  $c$  indicates that Lemma 7.3(ii) applies then
12   $l' \leftarrow |w_{B(q \otimes 0 1^k)}|$ 
13 else
14   $w \leftarrow w(q \otimes 0)(q \otimes 1)$ 
15   $l' \leftarrow 0$ 
16 end if
17  $w, B \leftarrow \text{MinWord}(\mathcal{T}_x, q \odot 1, w, l', B)$ 
18  $B(q) := (s, |w|)$ 
19 return  $w, B$ 

```

Informally, the idea of `MinWord` is to build up $\min(x)$ from its constituent parts $\min(x) * 0 = x * 0$, $\min(x) * 1 = x * 1$, $\min(x) \circ 0$, and $\min(x) \circ 1$. Since $\min(x) \circ 0 = \min(x \circ 0)$ and similarly for 1, $\min(x) \circ 0$ and $\min(x) \circ 1$ can be computed recursively by considering the transducer states representing $x \circ 0$ and $x \circ 1$, respectively. In this way, the recursion proceeds by traversing the states of \mathcal{T}_x . The parameter q represents the current state in the recursion.

The parameter w represents the prefix of $\min(x)$ found so far. In `MinWord`, w is extended to be a longer and longer prefix of $\min(x)$ until the two words coincide.

The first recursive call in `MinWord` will compute $\min(x \circ 0)$. How $\min(x \circ 0)$ is combined with $x * 0$, $x * 1$ and $\min(x \circ 1)$ depends on the case of Lemma 7.3 that applies. If case (i) or (iii) applies, then there is no overlap in $\min(x \circ 0)$ and $\min(x \circ 1)$, and so having recursively computed them, $\min(x)$ is obtained by taking the appropriate concatenation of $\min(x \circ 0)$, $x * 0$, $x * 1$, and $\min(x \circ 1)$. On the other hand, if case (ii) of Lemma 7.3 occurs, then there is non-trivial overlap between $\min(x \circ 0)$ and $\min(x \circ 1)$. In this case, we find $\min(x \circ 0)$ initially and then compute $\min(x \circ 1)$ in a way which avoids recomputing the overlap.

This is where the fourth parameter l comes into play. We set l to be equal to the size of the overlap of Lemma 7.3 when computing `MinWord` of $x \circ 1$. The value of l indicates the size of the prefix $\min(x \circ 1^k)$ of $\min(x \circ 1)$ that has already been calculated, and allows us to pick up the calculation from there.

The function B is used to avoid repeatedly recomputing recursive calls; in other words, B implements the well-known memoization paradigm. In particular, the minimum word $\min(y)$ corresponding to the element y of the free band represented by each state $q \in Q$ is only computed once, and, in some sense, the value is stored in B . If the state q corresponding to y has been visited by `MinWord` already, then the word w contains $\min(y)$. Hence rather than storing $\min(y)$ itself, it is only necessary to store the start and end index of $\min(y)$ in w . More precisely, $B(q) = (i, j)$ if and only if $w_{(i,j)} = \min(y)$ where y is the element of the free band corresponding to the state q .

In order to prove that `MinWord` is correct, we require the following notation. A partial function $B : Q \rightarrow \mathbb{N}_0 \times \mathbb{N}_0$ is **closed under reachability** if $\text{dom}(B)$ contains all terminal states in Q , and $\{r \otimes \alpha : \alpha \in \{0, 1\}^*\} \subseteq \text{dom}(B)$ for all $r \in \text{dom}(B)$. A pair (w, B) where $w \in A^*$ and $B : Q \rightarrow \mathbb{N}_0 \times \mathbb{N}_0$ is closed under reachability is **min-compatible** $w_{B(r)} = \min(\Psi(r))$ for every $r \in \text{dom}(B)$.

Suppose that (w, B) is min-compatible and $w' \in A^*$ has w as a prefix. If $B(r) = (i, j)$ and $i \leq j$, since $w_{(i,j)} = \min(\Psi(r))$, then $j \leq |w|$ and $w_{(i,j)} \neq \varepsilon$. It follows that $w'_{(i,j)} = w_{(i,j)}$ since $w'_{(1,|w|)} = w$. If $B(r) = (i, j)$ with $i > j$, then $w_{(i,j)} = \varepsilon = \min(\Psi(r))$ and so $w'_{(i,j)} = \varepsilon = \min(\Psi(r))$ also. We have shown that (w', B) is min-compatible.

The next lemma provides a crucial component in the proof of correctness of `MinWord`.

Lemma 7.5. *Suppose that $x \in FB^1(A)$ and that $\mathcal{T}_x = (Q, \{0, 1\}, A, q_0, T, \odot, \otimes)$ is the minimal transducer representing x . If $w \in A^*$, $B : Q \rightarrow \mathbb{N}_0 \times \mathbb{N}_0$, $q \in Q$, and $l \in \mathbb{N}_0$ are such that there exists a $K \in \mathbb{N}_0$ where:*

- (i) (w, B) is min-compatible;
- (ii) $q \odot 0^K \in \text{dom}(B)$;
- (iii) $w_{(s,|w|)} = \min(\Psi(q \odot 0^K))$ where $s = |w| - l + 1$;

then the pair $(w', B') = \text{MinWord}(\mathcal{T}_x, q, w, l, B)$ is min-compatible, $q \in \text{dom}(B')$, w is a prefix of w' , and $w'_{(s,|w'|)} = \min(\Psi(q))$.

Proof. We will proceed recursively on the states reachable from $q \in Q$. It suffices to show that the lemma holds for those states $q \in Q$ that have no child states, and that if the statement holds for $q \odot 0$ and $q \odot 1$, then the statement holds for q also.

For the assumptions of the lemma to hold, $q \odot 0^K$ has to be defined. Therefore, if q has no child states, then $q \odot 0^K$ is defined if and only if $K = 0$. But in this case, $q \odot 0^0 = q \odot \varepsilon = q \in \text{dom}(B)$. Thus to establish this case it suffices to prove the lemma for all states $q \in \text{dom}(B)$.

Note that the s calculated in line 1 is exactly the s in (iii) of the lemma.

Suppose that $q \in \text{dom}(B)$. Then the check in line 2 will pass. In line 3 we set $(i, j) = B(q)$ so that $w_{(i,j)} = \min(\Psi(q))$ since (w, B) is min-compatible by assumption. In line 4 we return $w' = ww_{(i+l,j)}$ and $B' = B$. Since $q \in \text{dom}(B) = \text{dom}(B')$ and w is trivially a prefix of w' , in order to show the lemma holds, we need to ensure that (w', B') is min-compatible and that $w'_{(s,|w'|)} = \min(\Psi(q))$. For the former, the input pair (w, B) is min-compatible, w is a prefix of w' , and $B' = B$, and so, by the comments before the lemma, (w', B') is min-compatible also. For the latter, recall that $w_{(i,j)} = \min(\Psi(q))$ and $w_{(s,|w|)} = \min(\Psi(q \odot 0^K)) = \min(\Psi(q)) \circ 0^K$. Since $w_{(i,j)} \circ 0^K = \min(\Psi(q)) \circ 0^K = w_{(s,|w|)}$ is a prefix of $w_{(i,j)}$ with length $|w| - s + 1 = l$, it follows that $w_{(s,|w|)} = w_{(i,i+l-1)}$. This implies that $w_{(s,|w|)}w_{(i+l,j)} = w_{(i,i+l-1)}w_{(i+l,j)} = w_{(i,j)} = \min(\Psi(q))$. So the returned $w' = ww_{(i+l,j)}$ has the suffix $w'_{(s,|w'|)} = \min(\Psi(q))$ as required.

Next, we assume that the lemma holds for $q \odot 0$ and $q \odot 1$ and any $w \in A^*$, $l \in \mathbb{N}_0$, and $B : Q \rightarrow \mathbb{N}_0 \times \mathbb{N}_0$ that satisfy the hypotheses of the lemma.

We already covered the case when $q \in \text{dom}(B)$, so we may assume that $q \notin \text{dom}(B)$. The check in line 2 will now fail and in line 6 we calculate $\text{MinWord}(\mathcal{T}_x, q \odot 0, w, l, B)$. We will show that the assumptions of the lemma hold for this invocation of `MinWord`. The pair (w, B) is min-compatible, by assumption, and so condition (i) holds. Since $q \notin \text{dom}(B)$, $K \geq 1$ and so $(q \odot 0) \odot 0^{K-1} = q \odot 0^K \in \text{dom}(B)$, and condition (ii) holds. In the same vein, $w_{(s,|w|)} = \min(\Psi((q \odot 0) \odot 0^{K-1}))$ and condition (iii) holds. Therefore if $(w', B') = \text{MinWord}(\mathcal{T}_x, q \odot 0, w, l, B)$, then (w', B') satisfy the conclusion of the lemma, i.e. (w', B') is min-compatible, $q \odot 0 \in \text{dom}(B')$, the input w is a prefix of w' and $w'_{(s,|w'|)} = \min(\Psi(q \odot 0))$.

In line 7 we use `ClassifyCase` to calculate c and k . By Theorem 7.4, c represents the case of Lemma 7.3 that applies to $\Psi(q)$. Depending on the case, we will append some letters to w and set the value of the parameter l' in lines 8-16 and then calculate $\text{MinWord}(\mathcal{T}_x, q \odot 1, w, l', B)$ on line 17.

We will show that after calculating $\text{MinWord}(\mathcal{T}_x, q \odot 1, w, l', B)$ in line 17, that the following hold: (w, B) is min-compatible; $q \odot 1 \in \text{dom}(B)$; the input w is a prefix of the current w ; and $w_{(s,|w|)} = \min(\Psi(q))$. Having established that this holds in line 17, after executing line 18, $q \in \text{dom}(B)$ and B is still reachability closed, since $q \odot 0, q \odot 1 \in \text{dom}(B)$. Also (w, B) will be min-compatible since $B(q \odot 0^k) = (s, |w|)$ and $w_{(s,|w|)} = \min(\Psi(q))$. Hence w and B returned in line 19 will satisfy the conclusions of the lemma, and the proof will be complete.

Therefore it suffices to show that after computing $\text{MinWord}(\mathcal{T}_x, q \odot 1, w, l', B)$ in line 17, the following hold: (w, B) is min-compatible; $q \odot 1 \in \text{dom}(B)$; the input w is a prefix of the current w ; and $w_{(s,|w|)} = \min(\Psi(q))$. We will consider each of the cases as they apply to $\Psi(q)$ in Lemma 7.3 separately.

(i) In this case,

$$\min(\Psi(q)) = \min(\Psi(q) \circ 0)(\Psi(q) * 0) \min(\Psi(q) \circ 1) = \min(\Psi(q \odot 0))(q \otimes 0) \min(\Psi(q \odot 1))$$

and lines 9 and 10 are executed. After line 9, $w_{(s,|w|)} = \min(\Psi(q \odot 0))(q \otimes 0)$.

Since $l' = 0$, in line 17 we invoke $\text{MinWord}(\mathcal{T}_x, q \odot 1, w, 0, B)$. We now check that the assumptions of the lemma hold for this invocation. Since (w, B) is min-compatible after line 6 and we have only appended letters to w , (w, B) is min-compatible before line 17, and so condition (i) holds. Since \mathcal{T}_x is minimal, there exists K' such that $(q \odot 1) \odot 0^{K'}$ is terminal. Hence $\min(\Psi((q \odot 1) \odot 0^{K'})) = \varepsilon$ and $(q \odot 1) \odot 0^{K'} \in \text{dom}(B)$ since B is closed under reachability. This shows that condition (ii) holds. To see that condition (iii) holds, note that $l' = 0$ (line 10) and so $w_{(|w|-l'+1,|w|)} = w_{(|w|+1,|w|)} = \varepsilon = \min(\Psi((q \odot 1) \odot 0^{K'}))$.

We can recursively apply the lemma to conclude that if $(w', B') = \text{MinWord}(\mathcal{T}_x, q \odot 1, w, 0, B)$, then (w', B') is min-compatible, $q \odot 1 \in \text{dom}(B')$, w is a prefix of w' and $w'_{(|w|+1,|w'|)} = \min(\Psi(q \odot 1))$.

It remains to show that $w'_{(s,|w'|)} = \min(\Psi(q))$. Since w is a prefix of w' , it follows that $w'_{(1,|w|)} = w$ and so $w'_{(s,|w|)} = w_{(s,|w|)} = \min(\Psi(q \otimes 0))(q \otimes 0)$. Thus

$$w'_{(s,|w'|)} = w'_{(s,|w|)} w'_{(|w|+1,|w'|)} = \min(\Psi(q \otimes 0))(q \otimes 0) \min(\Psi(q \otimes 1)) = \min(\Psi(q)).$$

(ii) Since Lemma 7.3(ii) holds for $\Psi(q)$, $\min(\Psi(q))$ is the unique word that has prefix $\min(\Psi(q \otimes 0))(\Psi(q) * 0) = \min(\Psi(q \otimes 0))(q \otimes 0)$ and suffix $(\Psi(q) * 1) \min(\Psi(q \otimes 1)) = (q \otimes 1) \min(\Psi(q \otimes 1))$ such that this prefix and suffix overlap exactly on the word $(q \otimes 1) \min(\Psi(q \otimes 0) 1^k)(q \otimes 0)$, where k is the same as the value we computed in line 7 (by the correctness of `ClassifyCase`).

Since we are in case (ii), in line 12 we set $l' = |w_{B(q \otimes 0 1^k)}|$. Note that $q \otimes 0 1^k \in \text{dom}(B)$ since $q \otimes 0 \in \text{dom}(B)$ after line 6, and B is reachability-closed.

In line 17 we invoke `MinWord`($\mathcal{T}_x, q \otimes 1, w, l', B$). We now check that the assumptions of the lemma hold for this invocation. As in the proof when Lemma 7.3(i) held, (w, B) is min-compatible since it was after line 6 and we have only appended letters to w and left B unchanged. In other words, condition (i) of the current lemma holds.

It is part of the assumption of Lemma 7.3(ii) that $\min(\Psi(q \otimes 10^k)) = \min(\Psi(q \otimes 0 1^k))$. Hence since elements of the free band, such as $\Psi(q) \otimes 10^k$ and $\Psi(q) \otimes 0 1^k$, are equal if and only if their minimum word representatives are equal, it follows that $\Psi(q) \otimes 10^k = \Psi(q) \otimes 0 1^k$. Since \mathcal{T}_x is minimal, $q \otimes 10^k = q \otimes 0 1^k$ and so $(q \otimes 1) \otimes 0^k = q \otimes 10^k = q \otimes 0 1^k \in \text{dom}(B)$. This shows that condition (ii) of the hypothesis of the current lemma holds.

Finally, we will show that condition (iii) of the current lemma holds for `MinWord`($\mathcal{T}_x, q \otimes 1, w, l', B$). Since $\min(\Psi(q \otimes 0 1^k))$ is a suffix of $\min(\Psi(q \otimes 0))$ and, as we showed above, $w_{(s,|w|)} = \min(\Psi(q \otimes 0)) = \min(\Psi(q \otimes 0))$ after line 6, it follows that $w_{(|w| - |\min(\Psi(q \otimes 0 1^k))| + 1, |w|)} = \min(\Psi(q \otimes 0 1^k))$ as the only length $|\min(\Psi(q \otimes 0 1^k))|$ suffix of w . Note that $l' = |w_{B(q \otimes 0 1^k)}| = |\min(\Psi(q \otimes 0 1^k))|$ by min-compatibility. But then $w_{(|w| - l' + 1, |w|)} = \min(\Psi(q \otimes 0 1^k)) = \min(\Psi(q \otimes 10^k)) = \min(\Psi((q \otimes 1) \otimes 0^k))$ and condition (iii) holds.

So all the assumptions of the lemma hold and so we can recursively apply it to conclude that if $(w', B') = \text{MinWord}(\mathcal{T}_x, q \otimes 1, w, l', B)$ in line 17, then (w', B') is min-compatible, $q \otimes 1 \in \text{dom}(B')$, w is a prefix of w' and $w'_{(s',|w'|)} = \min(\Psi(q \otimes 1))$ where $s' = |w| - l' + 1$.

It remains to show that $w'_{(s,|w'|)} = \min(\Psi(q))$ where $s = |w| - l' + 1$. Since w is a prefix of w' , it follows that $w'_{(s,|w|)} = w_{(s,|w|)} = \min(\Psi(q \otimes 0))$. This implies that

$$w'_{(s',|w|)} = w_{(s',|w|)} = w_{(|w| - l' + 1, |w|)} = \min(\Psi(q \otimes 0 1^k)) = \min(\Psi(q \otimes 0 1^{k-1})) \otimes 1$$

is a suffix of $w'_{(s,|w|)}$. So, by Lemma 7.3(ii),

$$w'_{(s'-1, s'-1)} = \min(\Psi(q \otimes 0 1^{k-1})) * 1 = \Psi(q \otimes 0 1^{k-1}) * 1 = \Psi(q) * 0 1^k = \Psi(q) * 1 = q \otimes 1.$$

Similarly, since $w'_{(s',|w|)} = \min(\Psi(q \otimes 10^{k-1})) \otimes 0$ is a prefix of $w'_{(s',|w'|)} = \min(\Psi(q \otimes 1))$, it follows that $w'_{(|w|+1, |w|+1)} = \Psi(q) * 10^k = \Psi(q) * 0 = q \otimes 0$.

Therefore $w'_{(s,|w'|)}$ is exactly the word with prefix $w'_{(s,|w|+1)} = \min(\Psi(q \otimes 0))(q \otimes 0)$ and suffix $w'_{(s'-1, |w'|)} = (q \otimes 1) \min(\Psi(q \otimes 1))$ whose common overlap is $w'_{(s'-1, |w|+1)} = (q \otimes 1) \min(\Psi(q \otimes 0 1^k))(q \otimes 0)$. Hence Lemma 7.3 implies that $w'_{(s,|w'|)} = \min(\Psi(q))$.

(iii) The proof in the case that Lemma 7.3(iii) holds is similar to the proof given above when Lemma 7.3(i) holds. In this case,

$$\min(\Psi(q)) = \min(\Psi(q \otimes 0))(\Psi(q) * 0)(\Psi(q) * 1) \min(\Psi(q \otimes 1)) = \min(\Psi(q \otimes 0))(q \otimes 0)(q \otimes 1) \min(\Psi(q \otimes 1)).$$

After line 6, $w_{(s,|w|)} = \min(\Psi(q \otimes 0))$ and because of the assumption of this case lines 14 and 15 are applied. After line 15, $w_{(s,|w|)} = \min(\Psi(q \otimes 0))(q \otimes 0)(q \otimes 1)$. So as in case (i) it remains to show that after line 17 we have appended $\min(\Psi(q \otimes 1))$ to w .

It is possible to verify, using the same argument as in the proof of case (i), that (w, B) is min-compatible and if K' is such that $(q \otimes 1) \otimes 0^{K'}$ is terminal, then $(q \otimes 1) \otimes 0^{K'} \in \text{dom}(B)$ and $w_{(|w| - l' + 1, |w|)} = w_{(|w|+1, |w|)} = \varepsilon = \min(\Psi(q \otimes 0^{K'}))$ (the latter equality holds since l' is defined to be 0 in line 15). In other words, the conditions of the lemma apply in line 17. Therefore we can recursively apply the lemma to conclude that if $(w', B') = \text{MinWord}(\mathcal{T}_x, q \otimes 1, w, 0, B)$, then (w', B') is min-compatible, $q \otimes 1 \in \text{dom}(B')$, w is a prefix of w' and $w'_{(|w|+1, |w'|)} = \min(\Psi(q \otimes 1))$.

It remains to show that $w'_{(s,|w'|)} = \min(\Psi(q))$ where $s = |w| - l' + 1$. Since w is a prefix of w' , it follows that $w'_{(s,|w|)} = w_{(s,|w|)} = \min(\Psi(q \otimes 0))(q \otimes 0)(q \otimes 1)$. Therefore

$$w'_{(s,|w'|)} = w'_{(s,|w|)} w'_{(|w|+1, |w'|)} = \min(\Psi(q \otimes 0))(q \otimes 0)(q \otimes 1) \min(\Psi(q \otimes 1)) = \min(\Psi(q)). \quad \square$$

For the purposes of the following theorem, we assume the RAM model of computation. This is quite a modest assumption in the sense that the time complexity is realized in the implementation of `MinWord` in [3]; see Appendix A for more details.

Corollary 7.6. Let $x \in FB^1(A)$, let $\mathcal{T}_x = (Q, \{0, 1\}, A, q, T, \odot, \otimes)$ be the minimal transducer representing x , and let $B : Q \rightarrow \mathbb{N}_0 \times \mathbb{N}_0$ be such that $B(q) = (0, 0)$ for all $q \in T$ and $B(q) = \perp$ on all other inputs. Then the first component of $\text{MinWord}(\mathcal{T}_x, q_0, \varepsilon, 0, B)$ is equal to $\min(x)$ and MinWord has time complexity $\mathcal{O}(|A| \cdot |Q| + |\min(x)|) = \mathcal{O}(|A|^2 \cdot |\min(x)|)$ and space complexity $\mathcal{O}(|Q| + |\min(x)|) = \mathcal{O}(|A| \cdot |\min(x)|)$.

Proof. We show that the conditions of Lemma 7.5 apply to $\text{MinWord}(\mathcal{T}_x, q_0, \varepsilon, 0, B)$ with the parameters as stated. The initial partial function B is defined only on the terminal states T of the minimal transducer \mathcal{T}_x . Hence B is reachability closed since terminal states have no child states. By assumption, $B(q) = (0, 0)$ for all $q \in T$ and $w = \varepsilon$, and so $w_{B(q)} = \varepsilon$. Since q is terminal, $q \odot \alpha$ and $q \otimes \alpha$ are undefined for all $\alpha \in \{0, 1\}^*$. In particular, $\min(\Psi(q)) = \varepsilon = w_{B(q)}$ for all $q \in T$ and so (w, B) is min-compatible. This establishes that Lemma 7.5(i) holds. Since $\Psi(q_0) = x$ and $|\text{cont}(x \circ 0^{|\text{cont}(x)|})| = |\text{cont}(x)| - |\text{cont}(x)| = 0$, $q_0 \odot 0^{|\text{cont}(x)|}$ is terminal and so $q_0 \odot 0^{|\text{cont}(x)|} \in \text{dom}(B)$. Therefore Lemma 7.5(ii) holds with $K = |\text{cont}(x)|$. Since $l = 0$ and $|w| = 0$, $s = |w| - l + 1 = 1$ and so $w_{(s, |w|)} = w_{(1, 0)} = \varepsilon = \min(\Psi(q \odot 0^{|\text{cont}(x)|}))$. This shows that Lemma 7.5(iii) holds. Hence the hypothesis of Lemma 7.5 is satisfied. Therefore if $(w', B') = \text{MinWord}(\mathcal{T}_x, q_0, \varepsilon, 0, B)$, then $w'_{(s, |w'|)} = w'_{(1, |w'|)} = w' = \min(x)$, as required.

For the time and space complexities, we start by elaborating some of the assumptions in the model of computation. In particular, in the RAM model, we may assume that the following operations are constant time for all $q \in Q$ and $\alpha \in \{0, 1\}$: access a letter in w , retrieve a value of $B(q)$, $q \odot \alpha$, or $q \otimes \alpha$, append a letter to w , and define a value $B(q)$. We may also assign w and/or B in constant time in lines 6, 9, 14, and 17 by modifying w and B in-place. Changing w or B in-place in a recursive call, will modify w and B in the original call, but this does not cause any issues since upon reentering the original function, in line 6 or 17, we immediately assign the returned w and B to the current w and B .

We denote by t_{total} the total number of steps taken by $\text{MinWord}(\mathcal{T}_x, q_0, \varepsilon, 0, B)$ with the inputs given in the statement of the corollary. We split $t_{\text{total}} = t_{\text{word}} + t_{\text{main}}$, where t_{word} is the total number of steps taken to write letters of w and t_{main} accounts for all of the other steps. Since w is only modified by appending letters, and appending a letter is constant time, it follows that $t_{\text{word}} \in \mathcal{O}(|\min(x)|)$.

In the remainder of the proof, we will exclude the time taken to append letters to w , since it is already accounted for in t_{word} . For t_{main} , note the role played by B . If a particular state q belongs to $\text{dom}(B)$, then the check in line 2 passes, and line 3 is $\mathcal{O}(1)$ and we return in line 4. If $q \notin \text{dom}(B)$, then, irrespective of which case of Lemma 7.3 applies, $B(q)$ is defined in line 18 and so $q \in \text{dom}(B)$ upon completion of the call to MinWord when q is first considered.

We denote by t_{body} the number of steps taken in MinWord excluding steps taken by the recursive calls in lines 6 and 17 (and excluding the time taken to append to w). For every state $q \in Q$ and every invocation of MinWord with input q , either $q \in \text{dom}(B)$ at a cost of $\mathcal{O}(1)$ or $q \notin \text{dom}(B)$ incurring t_{body} . The latter occurs precisely once as described in the previous paragraph. Hence $t_{\text{main}} \in \mathcal{O}(|Q| \cdot t_{\text{body}})$.

To determine t_{body} , note that apart from lines 7 and 12, the remaining steps are constant time. Line 7 takes $\mathcal{O}(|A|)$ steps as per Theorem 7.4. In line 12, we calculate $|w_{B(q \odot 01^k)}|$. By assumption $q \odot 01^k$ requires $\mathcal{O}(k + 1) = \mathcal{O}(|A|)$ steps since $k \leq |\text{cont}(x)| \leq |A|$. The subsequent lookup of $B(q \odot 01^k)$ is constant time. The calculation of $|w_{(i, j)}|$ for any i, j is constant time, since if $0 < i \leq j$, then $|w_{(i, j)}| = j - i + 1$ and otherwise $|w_{(i, j)}| = 0$. So the total for the calculation in line 12 is $\mathcal{O}(|A|)$ steps.

Therefore $t_{\text{main}} \in \mathcal{O}(|Q| \cdot t_{\text{body}}) = \mathcal{O}(|Q| \cdot |A|)$, and so the total time complexity is $\mathcal{O}(|\min(x)| + |Q| \cdot |A|)$. Since \mathcal{T}_x is minimal, by Theorem 5.1, $\mathcal{O}(|Q|) = \mathcal{O}(|\mathcal{T}_x|) = \mathcal{O}(|A| \cdot |\min(x)|)$. It follows that $\mathcal{O}(|\min(x)| + |Q| \cdot |A|) = \mathcal{O}(|\min(x)| + |\min(x)| \cdot |A|^2) = \mathcal{O}(|A|^2 \cdot |\min(x)|)$.

For the space complexity, storing \mathcal{T}_x already requires $\mathcal{O}(|\min(x)| \cdot |A|)$ space. Since w is a prefix of $\min(x)$, the space taken by w is $\mathcal{O}(|\min(x)|)$. The partial function B has at most $|Q|$ values where it is defined, and each $B(q)$ is a pair of integers. Hence B requires $\mathcal{O}(|Q|) = \mathcal{O}(|\min(x)| \cdot |A|)$ space. Thus in total $\mathcal{O}(|\min(x)| \cdot |A|)$ space is required. \square

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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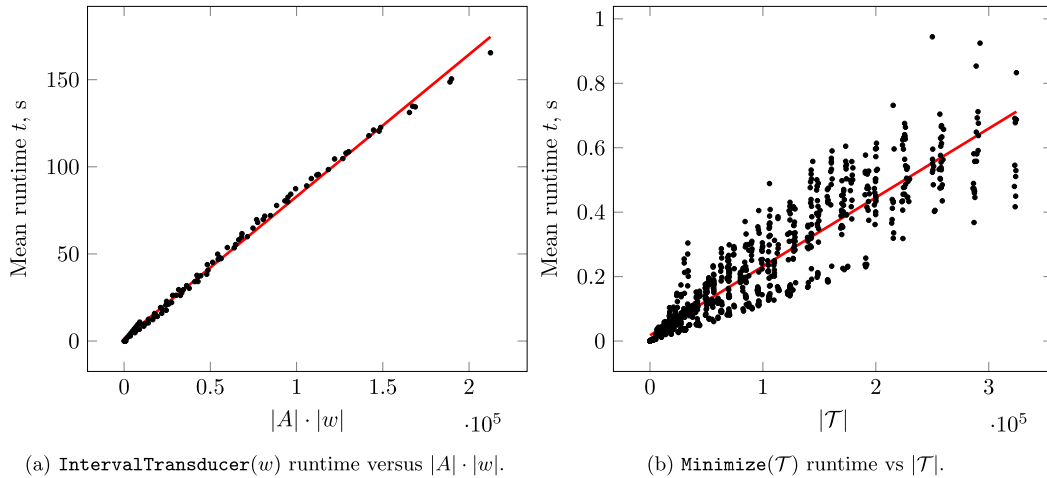


Fig. 10. Benchmarks for the IntervalTransducer and Minimize algorithms.

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Appendix A. Benchmarks

The Python package [3] implements the algorithms EqualInFreeBand, Multiply, and MinWord. In this section we describe some benchmarks for these algorithms. The purpose of [3] is to provide a reference implementation, and the benchmarks are included to demonstrate that even a relatively naïve implementation in a high-level language, such as Python, exhibits the asymptotic time complexity described in this paper. The implementation in [3] is not optimised, and the absolute times provided in this section can almost certainly be improved significantly.

All the benchmarks in this section were run on a 2021 MacBook Pro with an Apple M1 processor and 16 GB of RAM running Python 3.9.12. Instructions for how to reproduce the benchmarks in this section can be found in [3].

In order to benchmark our code, we generated the following set of sample data: for each alphabet of size $m \in \{2, 7, 12, \dots, 47\}$ and each length $l \in \{20, 520, 1020, \dots, 4520\}$ we generated a sample of 100 words of length l (selected uniformly at random) over an alphabet of size m using the Python `random` library for randomness. We will refer to these as the **word samples**. The interval transducer for each word in each sample was computed using `IntervalTransducer`; we refer to this as the **interval transducer sample**. Finally, each of the interval transducer samples was run through the `Minimize` algorithm, forming the **minimal transducer sample**. The benchmarks were produced using the `pytest-benchmark` Python library. Any quoted time is the mean of a number of runs of the benchmark using the same input data. This makes the results more reproducible and less susceptible to random variation in the device used to run them.

The algorithm `EqualInFreeBand` is the composition of the three algorithms defined in Section 5: `Minimize`, `IntervalTransducer`, and `TrimTransducerIsomorphism`.

Fig. 10a shows run times of `IntervalTransducer(w)` for the words w in the word samples described above. Each data point in Fig. 10a shows the cumulative run time for a sample consisting of 100 words. This is plotted against $|A| \cdot |w|$ which is the theoretical asymptotic complexity of the algorithm. The red line is the line of best fit obtained by running a linear regression in Python.

Fig. 10b shows the run times of `Minimize(T)` for transducers \mathcal{T} in the interval transducer samples. This is plotted against the number of states of $|\mathcal{T}|$ which is the theoretical asymptotic complexity of the `Minimize` algorithm. Each point represents a single transducer \mathcal{T} . The line of best fit is shown in red. The fit of the line in Fig. 10b might not seem very good, with a fairly wide range of times for the given transducers of each number of states. However the linear regression has an R^2 value of 0.8368, which is perhaps not as bad as it first appeared.

Fig. 11a shows run times of `TrimTransducerIsomorphism(T, T)` for a transducer \mathcal{T} in the interval transducer samples. This is plotted against $|\mathcal{T}|$ which is the theoretical asymptotic complexity of the algorithm. The line of best fit is shown in red. We chose to run the isomorphism of a transducer against itself since this represents the longest execution

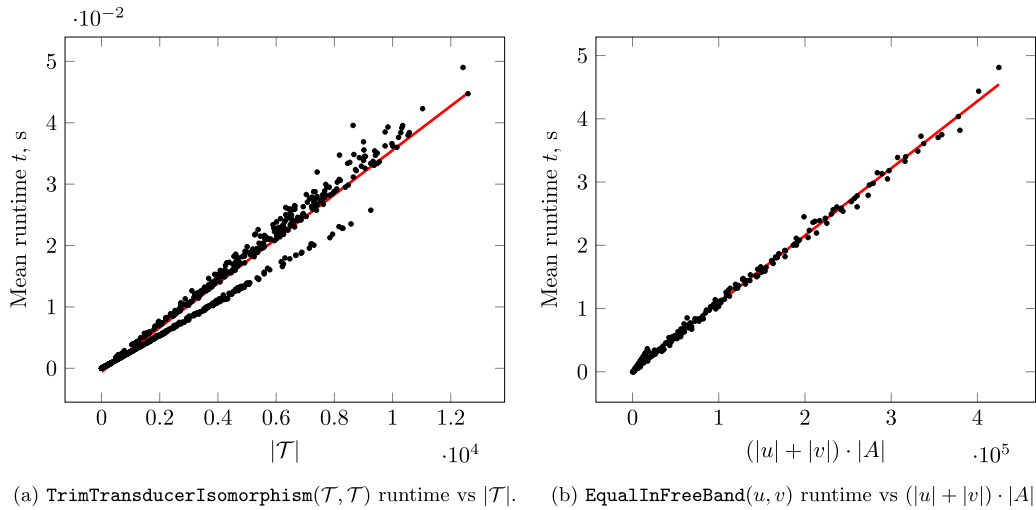


Fig. 11. Benchmarks for the `TrimTransducerIsomorphism` and `EqualInFreeBand` algorithms.

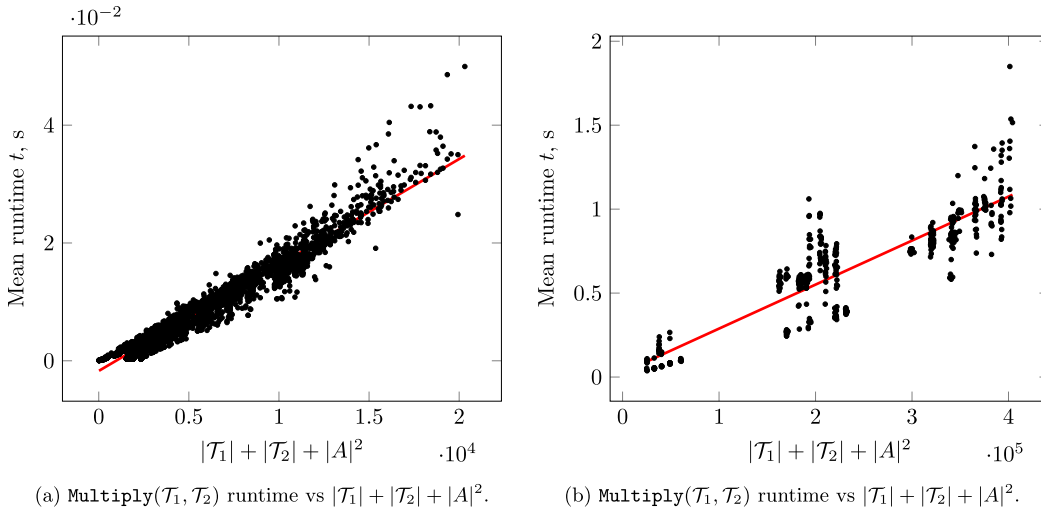


Fig. 12. Benchmarks for the `Multiply` algorithm when applied to minimal (left) and interval transducers (right).

path in the code. We believe the bifurcation present in the plot is due to details of the implementation and not an inherent property of the algorithm itself.

Fig. 11b shows the run times of `EqualInFreeBand`(u, v) for words u and v obtained by randomly sampling 10000 pairs from the word samples. This is plotted against $(|u| + |v|) \cdot |A|$ which is the theoretical asymptotic complexity of the algorithm. The line of best fit is shown in red. We chose to only sample 10000 pairs to reduce the total run time of the benchmark.

Fig. 12a shows the runtime of `Multiply`($\mathcal{T}_1, \mathcal{T}_2$) for transducers \mathcal{T}_1 and \mathcal{T}_2 obtained by randomly sampling 50 transducers from the minimal transducer sample and considering all possible pairs of these transducers (2500 pairs total). This is plotted against $|\mathcal{T}_1| + |\mathcal{T}_2| + |A|^2$ which is the theoretical asymptotic complexity of the algorithm. The line of best fit is shown in red. We chose to only sample 2500 pairs to reduce the total runtime of the benchmark. The data is not quite linear, we performed a linear regression on the log-log plot which estimated the exponent to be 1.13 (r -value 0.94), so the overall complexity of the implementation is $\mathcal{O}(n^{1.13})$, where $n = |\mathcal{T}_1| + |\mathcal{T}_2| + |A|^2$. Again, we believe that this discrepancy is related to details of the implementation.

Fig. 12b shows the runtime of `Multiply`($\mathcal{T}_1, \mathcal{T}_2$) for transducers \mathcal{T}_1 and \mathcal{T}_2 obtained by batching the interval transducer samples into 20 batches, randomly sampling 5 transducers from each batch and considering all possible pairs of these transducers in each batch (500 pairs total). This is plotted against $|\mathcal{T}_1| + |\mathcal{T}_2| + |A|^2$ which is the theoretical asymptotic complexity of the algorithm. The line of best fit is shown in red. We chose to only sample 500 pairs to reduce the total runtime of the benchmark.

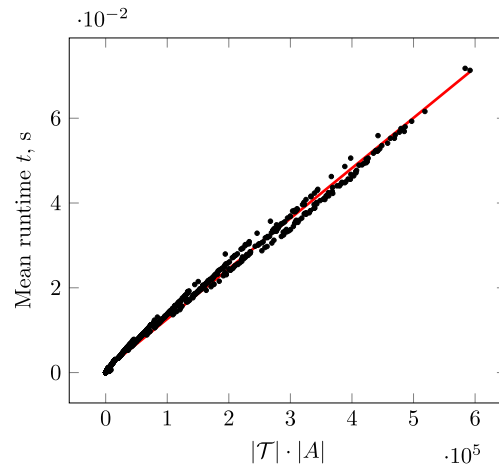


Fig. 13. $\text{MinWord}(\mathcal{T})$ runtime versus $|\mathcal{T}| \cdot |A|$.

Fig. 13 shows the runtime of $\text{MinWord}(\mathcal{T})$ for transducers \mathcal{T} in the minimal word samples. This is plotted against $|\mathcal{T}| \cdot |A|$ which is the theoretical asymptotic complexity of the algorithm. The line of best fit is shown in red.

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