COLLECTION FRAMES FOR DISTRIBUTIVE SUBSTRUCTURAL LOGICS

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Abstract. We present a new frame semantics for positive relevant and substructural propositional logics. This frame semantics is both a *generalisation* of Routley–Meyer ternary frames and a *simplification* of them. The key innovation of this semantics is the use of a single accessibility relation to relate *collections* of points to points. Different logics are modeled by varying the kinds of collections used: they can be sets, multisets, lists or trees. We show that collection frames on trees are sound and complete for the basic positive distributive substructural logic B^+, that collection frames on multisets are sound and complete for RW^+ (the relevant logic R^+, without contraction, or equivalently, positive multiplicative and additive linear logic with distribution for the additive connectives), and that collection frames on sets are sound for the positive relevant logic R^+. The completeness of set frames for R^+ is, currently, an open question.

§1. Ternary Relational Frames. The ternary relational semantics for relevant logics is a triumph. The groundbreaking results of Routley and Meyer [45–47] have significantly clarified our understanding of relevant logics.[1] After 20 years of viewing relevant logics with Hilbert-style axiomatisations, natural deduction systems and algebraic semantics, we finally had a truth-conditional semantics which modelled relevant logics in the same way that Kripke semantics provide models for normal modal logics and intuitionistic and intermediate logics.[2]

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[1] The ternary relational frame condition for conditionals was discovered independently both by Larisa Maksimova in the late 1960s and by Dana Scott in the early 1970s. Maksimova's strikingly early contributions [30] are discussed by Katalin Bimbó and Mike Dunn [8, see p. 43]. Scott's contributions are discussed by Brian Chellas, in a 1975 article [10, see p. 143 and notes 17 and 18]. Thanks to Lloyd Humberstone for bringing this reference to our attention.

For recent discussions of Routley and Meyer's early work on the ternary relational semantics, see papers by Bimbó and Dunn [9] and Ferenz [19].

[2] This is not to say that there weren't other point-based semantics for relevant logics, before the advent of the ternary relational semantics. The operational semantics, introduced by Urquhart [53], should not be omitted from any survey of semantics for relevant logics. It differs from the ternary relational semantics for R^+ (and related logics) by using a binary operation on points, rather than a ternary relation, to model the conditional. As a result,

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Propositions are modelled as sets of points, and connectives are interpreted as operations on such sets, some (namely the modal operators, intuitionistic conditional and negation, and in the case of relevant logics, relevant implication and the intensional conjunction, *fusion*) using accessibility relations on the class of points. In the case of the distinctively relevant conditional connective '$\to$', the two-place connective is naturally interpreted by a *three*-place accessibility relation, the eponymous ternary relation of the ternary relational semantics.

That a ternary relation should feature in a frame semantics for relevant logics should not have surprised anyone. The pieces had been in place for quite some time. Jónsson and Tarski's papers, from the 1950s, on Boolean algebras with operators [26, 27], showed how Boolean algebras with n-ary operators satisfying appropriate distributive laws can be concretely modelled as power set algebras where each n-place operator is interpreted using an $(n+1)$-place relation. Generalising these results from Boolean algebras to distributive lattices makes some of the details a little more complicated, but the picture is mostly unchanged. The details for how to make that generalisation of Jónsson and Tarski's work to arbitrary distributive lattices with operators—including relevant logics—were worked out by Dunn in his papers on gaggle theory in the early 1990s [14–17].[3] The picture is extremely natural and well motivated. The ternary relational semantics for relevant and substructural logics is powerful, and it has resulted in significant advances in our understandings of these logics.

Nonetheless, it cannot be said that the ternary relational semantics has met with anything like the reception of the Kripke semantics for modal and constructive logics. Some of the difference is no doubt due to the size of the respective audiences. Substructural and relevant logic is a boutique interest when compared to the modal industrial complex of the late twentieth and early twenty-first centuries. However, it

once disjunction is present in the language, this means that points cannot in general, be prime (supporting a disjunction $A \vee B$ only when supporting one of its disjuncts, one of A or B). To see why, take a point supporting $p \to (q \vee r)$ and *apply* it (using the binary application relation) to a point supporting p. By the interaction between the application operation and the conditional, the resulting point will satisfy $q \vee r$. For classical logic, this would be no problem, since $p \to (q \vee r)$ entails $(p \to q) \vee (p \to r)$. But this entailment fails in R^+ (and in intuitionistic logic). In the operational semantics for R^+, points need not be prime, and it turns out that points have much more of the flavour of arbitrary theories or propositions, rather than *special* theories like worlds or situations. In the Kripke models for intuitionist logic, and the ternary relational semantics for R^+, points are prime, and to evaluate a conditional like $p \to (q \vee r)$, given a point where p holds we may need to consider a *range* of points to evaluate $q \vee r$. At some of these points, q may be true, and at others r may be. In the Kripke semantics for intuitionist logic or the ternary relational semantics for relevant logics, points generate prime theories, the completeness theorem is hard work, and there is more we can learn from the distinctive structure of models.

Humberstone [23] shows that the operational semantics can be expanded to better model disjunction, with the addition of a second operation on points. In Humberstone's semantics, a disjunction $A \vee B$ is taken to be true at a point x just when $x = y + z$ where A is true at y and B is true at z. We gain the simplicity of a binary operation for the conditional (rather than a ternary relation) at the cost of a second binary operation for disjunction. See Humberstone's 2018 paper [25] for an extended discussion of this semantics. In the frame semantics that is our focus, the distributive lattice operators are modelled as intersection and union on sets of points, so using an operation for the conditional is out of the question.

3 Katalin Bimbó and J. Michael Dunn have written a comprehensive overview of gaggle theory, the theory of *Generalised Galois Logics* [7]. [18, chap. 12] provides a short introduction to gaggle theory.

seems to us that this does not explain all of the differences in the scale and quality of the reception of the respective semantic frameworks. Some of the relative dissatisfaction with the ternary relational semantics centres on philosophy and the question of the intelligibility of the semantics [2, 11]. We think those questions have been well dealt with in the literature, and that to a large degree the proof of this pudding is in the eating, rather than adding to the already long discussion of pudding interpretation. The ternary relational semantics is not problematic because it lacks interpretive power or philosophical intelligibility. The problem with the ternary relational semantics is that it is *fiddly*.

Consider Kripke semantics for modal logics. All you need to make a Kripke frame is a non-empty set of points, and a binary relation on those points. Nothing more. Propositions are modelled by sets of points. The Boolean operators correspond to the set functions of union, intersection and complementation, and the modal operators are simple universal or existential projections along the binary relation. This is simple, it is robust, and once you see it, you find this pattern everywhere. Structures for modal logics are ubiquitous.

Kripke semantics for intuitionistic logic is a little more complicated, but not by much. We must have a partial order on our set of points (or possibly a preorder) and propositions are sets of points closed upward along that order. Conjunction and disjunction are unchanged from the modal case, as intersection and union preserve the property of being upward closed. However, complementation, and the corresponding operation to model the material conditional, do not preserve the property of being closed, so they are replaced by operations that utilise the partial order and respect the upward closure condition. Again, this is all very straightforward. When you have an ordered collection of states, carrying information preserved along that order, constructive logic is a natural tool, and Kripke models for intuitionistic logic are correspondingly natural.

Now compare the general framework for substructural logics.[4] One natural presentation of the semantics takes this form: a frame is a 4-tuple $\langle P, R, \sqsubseteq, N\rangle$, where P is a non-empty set of points, R is a ternary relation on P, $\sqsubseteq$ is a binary relation on P, and N is a subset of P, where the following conditions are satisfied.

- $\sqsubseteq$ is a partial order.
- R is $\sqsubseteq$-downward preserved in the first two positions, and $\sqsubseteq$-upward preserved in the third. That is, if $Rxyz$ and $x^- \sqsubseteq x$, $y^- \sqsubseteq y$ and $z \sqsubseteq z^+$ then $Rx^-y^-z^+$.
- $y \sqsubseteq z$ if and only if there is some x where Nx and $Rxyz$.

Notice that these models have three distinct moving parts: the ternary relation R, the partial order $\sqsubseteq$, and the distinguished set N of points. Propositions are sets of points closed upward under the partial order $\sqsubseteq$. R is used to interpret the conditional connective '$\rightarrow$' (and the intensional conjunction '$\circ$', if present), while the set N of so-called *normal*, or regular, points is the set of points at which logical truths are taken hold.[5] The need for N is a distinctive feature of relevant logics, as logical truths

4 This presentation is taken from Restall's *Introduction to Substructural Logics* [41, chap. 6], but the choice of framework is irrelevant to the general point. No presentation of primitives is particularly less fiddly than any other.

5 There are many names for the points in N. We are following [37] in using the term 'normal', with its connections to modal logics. Along with 'regular', 'base points' and 'logical points' are used in the literature on relevant and substructural logics.

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(like, say, $p \to p$) need not hold at all points. Since, for example, $q \to (p \to p)$ is not a theorem of R^+, so some models feature have counterexamples to the conditional. Those models have at least one point where q is supported but $p \to p$ is not. But $p \to p$ is still a logical truth according to R^+. Logical truths are guaranteed to hold at *some* points (namely, those in N), but not necessarily at all points. So, our models have three distinct moving parts: $\sqsubseteq$ for providing our closure conditions for propositions,[6] R for modelling '$\to$' and '$\circ$', and N for modelling the logical truths.

We challenge anyone to find this kind of formal semantics to be as straightforward to *apply* as the Kripke semantics for modal and constructive logics. While it is relatively easy to find preorders or binary relations on sets under every bush, it is rather harder to see where ternary relations, partial orders and special sets of normal points are to be found. Perhaps they are there somewhere, but they do not seem particularly easy to spot. It is not for nothing that modal and constructive logics have been applied in many domains where relevant and substructural logics have not.[7]

It is true that the choice of primitives in the ternary frame semantics is somewhat arbitrary. We could take $\sqsubseteq$ to be defined in terms of N and R, but then the condition that it is a partial order (or a preorder) and that R is preserved along that order become even more complex and unnatural to state. In models for *some* of our logics (not all) we could impose the condition that $\sqsubseteq$ is the identity relation (and hence, all algebras of propositions arising out of such frames would be at least implicitly Boolean algebras, so this works only for logics conservatively extended with Boolean negation) [38]. It is possible, for some substructural logics, to trade in our set N for a single point g (and restrict our attention to so-called *reduced* models), cutting down further on the number of models generated, but the conceptual complexity remains [21, 48–51].

When you consider ternary relational models alongside point semantics for normal modal logics and constructive logics, the contrast is plain for all to see. Ternary relational models are significantly less elegant, and they have many different moving parts than Kripke models for modal and constructive logics. It is not for nothing that those of us working in the area have sought to simplify the semantics, but try as we might, significant complexity remains after such all such efforts [38, 39].[8]

In this paper we introduce a new class of models for positive relevant and substructural logics, which at the same time *generalises* and *simplifies* the ternary relational semantics. Collection frames generalise ternary relational frames in the sense that every ternary relational frame can be seen as a collection frame, but that there

6 Not all algebras of relevant logics are Boolean algebras (or more precisely, distributive lattices in which each element has a unique Boolean complement, that is for each x there is a y such that the meet of x and y is the bottom element of the lattice and the join of x and y is the top. This can be so even if the algebra has no operator that sends an element to its complement), so we wouldn't expect all of our frames too allow every subset of points to count as a proposition. However, these algebras are distributive lattices, so a partial order of this form is very natural.

7 This is not to say that the only way a logic finds its application is that a class of models for that logic is independently discovered in some domain. It is to say that this is one way that the tools of the logic may be applied. This is also not to say that there are no well motivated independent applications of the ternary frame semantics. Frames for the Lambek calculus, where the ternary relation arises out of string concatenation, for example, are one obvious case, though notice that in this case, the ternary relation collapses into a binary operation [12, 36].

8 Frames for some of the stronger logics seem to present particular challenges to simplification [43].

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are also collection frames that do not arise as ternary relational frames. Collection frames simplify ternary relational frames in the sense that there are significantly fewer independent parts and conditions connecting different components of the semantics. While the resulting models are not *quite* as simple as Kripke semantics for modal logics—some complexity is inevitable, given that we are aiming to model an intensional two-place connective—the gain in simplicity over the traditional presentation of the ternary relational frame semantics for relevant logics is significant.

Simplifying the semantics is one motivation for our work. The second motivation for an alternate approach to frames for these logics arises out of noticing the following fact: When we work with particular substructural logics—such as R^+, RW^+, and TW^+—it is very natural to consider not only the ternary relation R but its generalisations to more places: $R^2a(bc)d$ is defined as $(\exists x)(Rbcx \wedge Raxd)$, and $R^2(ab)cd$ is defined as $(\exists x)(Rabx \wedge Rxcd)$. In R^+ and RW^+, $R^2a(bc)d$ holds if and only if $R^2(ab)cd$ holds, so we can simplify our notation, and generalise further: for $n > 0$, we define R^n to be the $(n+2)$-ary relation on P, setting $R^1 = R$, and setting $R^{n+1}a_1a_2a_3 \cdots a_{n+3}$ to hold if and only if $(\exists x)(Ra_1a_2x \wedge R^n xa_3 \cdots a_{n+3})$. This generalisation into an arbitrary n-ary relation, where $n \geq 3$ is extremely natural, and conditions on R^2 and still higher orders of R play a role in the specification of various substructural logics.[9]

Our attempt to understand the phenomenon of higher order accessibility relations—and how they relate to each other—is the starting point for a new, *simpler* characterisation of frame semantics for substructural logics. In the next section we will start with one case, frames for the logics RW^+ and R^+. In later sections we will then branch out to a wider class of substructural logics.

§2. Multiset Frames. A guiding idea in ternary relational semantics for relevant logics is the notion of information application or combination. The ternary relation R relates the triple of points x, y, z (that is, $Rxyz$) if and only if applying the information in x to the information in y results in information that is in z. In the logics R^+ and RW^+, information application is commutative (applying x to y results in the same information as applying y to x), and associative (applying x to y and then applying the results to z results in the same things as applying x to a result of applying y to z). In models for R^+, combination is also idempotent, to the effect that the result of applying x to itself doesn't take you outside x (so we have $Rxxx$). Associativity and commutativity of application (or combination) means that we could simplify our ternary relation R by thinking of it not so much as a ternary relation where all three slots act independently, but rather, at least in the case of these logics, as a relation between unordered pairs of points on the one hand, and points on the other. The fact rendered as $Rxyz$ in the ternary semantics could instead be represented as

$$[x, y]Rz$$

where we have the (unordered) pair of x and y on the one hand, and the z on the other. The fact that this is an unordered pair, and not a *set* is important, because when we consider $Rxxz$ what we have is

$$[x, x]Rz,$$

[9] Mares' monograph *Relevant Logic* [31, p. 210] gives a definition of frames for R^+ using this generalisation of the ternary relation. This generalisation is also used by Meyer and Routley [34, p. 184], which introduces a notation similar to the multiset and list frames below.

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where x is applied *to itself*. But as far as order of application goes, $[x, y]Rz$ is the very same fact as $[y, x]Rz$. When it comes to associativity, what we have in models for RW^+, traditionally presented, is the following complex fact:

$$(\exists u)(Rxyu \wedge Ruzw) \text{ iff } (\exists v)(Ryzv \wedge Rxvw).$$

If we are willing to abuse notation a little more, what we have in this biconditional is two different ways of representing the one single fact

$$[x, y, z]Rw$$

to the effect that x, y and z together, combined in any order, are related to w. Collection frames arise from taking what was an abuse of notation literally. In collection frames, an accessibility relation relates *collections* of points to points.

This shifted perspective on R comes with advantages. Not only will this relation R do the job of the original ternary relation, in the case where the multiset has two elements, and not only can it represent R^2 and relations of higher arities with larger multisets. It *also* has the capacity to represent the binary relation $\sqsubseteq$ in the case where the collection being related is a *singleton*, and it also represents the predicate N, in the case where the collection being related is the *empty multiset*. The translation manual is straightforward:

$$\begin{array}{lrll} \text{(F1)} & Nx & \textit{becomes} & [\,]Rx, \\ \text{(F2)} & x \sqsubseteq y & \textit{becomes} & [x]Ry, \\ \text{(F3)} & Rxyz & \textit{becomes} & [x, y]Rz. \end{array}$$

What was represented by three different fundamental concepts in traditional Routley–Meyer frames becomes three different aspects of one underlying relation. The conditions linking N, $\sqsubseteq$ and (ternary) R become corollaries of the fundamental structure of the one multiset relation R.

To make things explicit, a collection frame for RW^+ has a non-empty set P of points and a single accessibility relation R on $M(P) \times P$, where $M(P)$ is the class of *finite multisets* of elements of P. Since multisets are not in very wide use,[10] we would do well to be explicit about them and their properties.

Definition 1 (Finite multisets, ground). *A multiset is a collection in which order is irrelevant, but multiplicity of membership is relevant. There are various ways to formally define the notion. One way is this: a finite multiset of objects taken from some class P can be represented as a function $m : P \to \omega$ where $m(x) = 0$ for all but finitely many values of* x*. If* x *is in P, then $m(x)$ is the number of times x is a member of the multiset* m*. The multisets m_1 and m_2 from P are identical if they have the same members to the same multiplicities: that is, $m_1 = m_2$ if and only if $m_1(x) = m_2(x)$ for each x in P.*

For any two multisets m_1 and m_2, their UNION *is the multiset with function $m_1 + m_2$. We also write '$m_1 \cup m_2$' using the traditional notation for union. Note, however, that $m_1 \cup m_1$ is now not (typically) the same multiset as m_1.*

We say that $m_1 \leq m_2$ (a generalisation of the subset relation to multisets) if $m_1(a) \leq m_2(a)$ for all a *in P.*

[10] The papers "Multisets and Relevant Implication I" and "II" by Meyer and McRobbie [32, 33] are accounts of multisets and their importance in the proof theory of relevant logics. Grattan-Guinness has a helpful discussion of the history of accounts of multisets in late Nineteenth and Twentieth Century mathematics [22].

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We use the familiar bracket notation for multisets: for example, $[a, a, b]$ *is the multiset where* $m(a) = 2$ *and* $m(b) = 1$ *and* $m(x) = 0$ *for every other value of* x. *So,* $[a, b] \cup [a, c, c] = [a, a, b, c, c]$.

As with sets, we will use the symbol '$\in$' for multiset membership. Here, '$x \in m$' will be taken to mean that $m(x) > 0$, *that is, the object* x *is in the multiset* m *a non-zero number of times.*

For any multiset m *on* P, *its* GROUND $g(m)$ *is the sub*set *of* P *consisting of all objects* x *with non-zero multiplicity in* m, *that is,* $g(m) = \{x \in P \mid m(x) > 0\}$.

Now we know enough about multisets for us to introduce the *multiset* semantics for RW^+ and for R^+. As we have already indicated, a collection frame consists of a set P of points (with at least one member), and a relation R on $M(P) \times P$, which relates multisets of points to points. Henceforth, we will call relations R on $M(P) \times P$ *multiset relations*.

The intended application of R in a multiset frame is straightforward: XRy holds when, and only when, the information in the points X *taken together* also holds in y. There are aspects, in this reading, of the partial order from constructive logics, and just like that case, there must be at least *some* condition on this relation for such an interpretation to make sense. The relation R cannot be entirely arbitrary. In the case of the semantics for constructive logic, there are two parts to the constraint on the order relation. First, that it be reflexive, and second, that it is transitive.[11] In the case of multiset relations for frames for RW^+, the condition has much the same form: a transitivity component and a reflexivity component. The strictest and most natural form of reflexivity would be we require that the information in the singleton multiset of points $[x]$ is indeed carried by the x itself. This says very little about combining points, of course. For transitivity, we require that combination *compose* in a straightforward manner: if XRy and $[y] \cup YRz$ then $(X \cup Y)Rz$.[12] However, we require something stronger than just composition in this direction: we also require its converse. That is, if $(X \cup Y)Rz$ then we can find some 'value' y where XRy and $([y] \cup Y)Rz$. We call these two conditions *compositionality* because we think of R as a generalised *combination* relation, selecting for each collection of points the single points which are suitable to represent it. The compositionality condition says that this relation can be composed or decomposed piecewise. So, we have the following definition:

DEFINITION 2 (Compositionality). *A multiset relation* R *on* $M(P) \times P$ *is said to be* compositional *if and only if for all multisets* X *and* Y *and for all points* z,

$$(\exists y)(XRy \text{ and } ([y] \cup Y)Rz) \text{ iff } (X \cup Y)Rz.$$

In addition, a compositional multiset relation is reflexive *iff for all points* x, *we have*

$$[x]Rx.$$

We break the compositionality condition into two parts, the left to right direction we will call *Transitivity*, for obvious reasons. The right to left direction we will call

[11] We could add the condition that it is anti-symmetric, though this is in no way essential for the models to give us intuitionistic logic.

[12] This is a generalised form of transitivity, much like those discussed for consequence relations by Ripley [44].

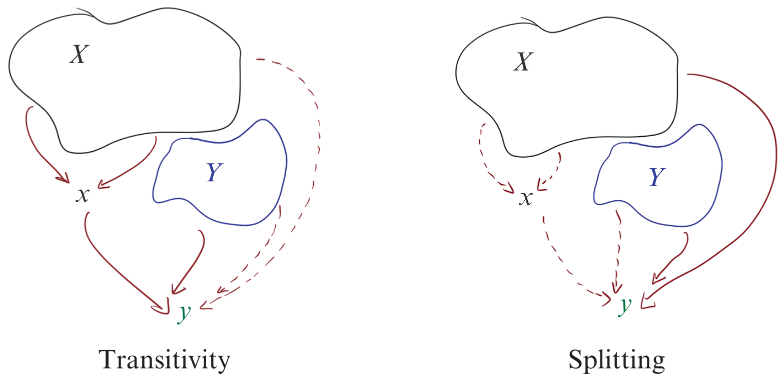

Fig. 1. The two directions of compositionality.

Splitting.[13] These two parts of the condition play different roles in exploring the properties of this semantics, so we will highlight these roles by mentioning at each point whether *Transitivity* or *Splitting* is being appealed to.

The intuitions behind the two directions are represented in Figure 1. The intuition behind *Transitivity* is that if one can combine the information in X to obtain x and combine the information in Y together with x to obtain y, represented by the solid lines, then one could have just as well have used the information in the combination of X and Y to obtain y, represented by the broken lines. If we restrict our attention to the case where $X = [x]$ and $Y = [\,]$ then we see that *Transitivity* gives us the transitivity of the binary relation $\lambda x.\lambda y.[x]Ry$ on points.

The intuition behind *Splitting* is that if one can obtain y from some information Z, which can be split into components X and Y, then one could evaluate the X portion to obtain something, x, which can be combined with the information in Y to obtain y.[14] If we restrict our attention to the case where $X = [x]$ and $Y = [\,]$, then *Splitting* gives us the *density* of the binary relation $\lambda x.\lambda y.[x]Ry$. That is, if $[x]Rz$ then there is some y where $[x]Ry$ and $[y]Rz$. Notice that the density of this relation holds automatically in the case where reflexivity holds, but this condition is strictly weaker than reflexivity.[15]

Since this binary relation $\lambda x.\lambda y.[x]Ry$ is so important in our frames, we will reserve special notation for it. In ternary frames the usual notation is '$\sqsubseteq$'. Since our frames will not require reflexivity (but we will *allow* it), let us write '$\sqsubset$' for this binary relation induced by the multiset relation R. We have seen proved the following lemma.

LEMMA 3. *If R is a compositional multiset relation then the induced binary relation $\sqsubset$ (given by setting $x \sqsubset y$ iff $[x]Ry$) is transitive and dense.*

Before we continue spelling out the semantics, we would do well to pause to consider some examples of simple multiset relations, and their properties.

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13 We thank an anonymous referee for suggesting this name.

14 The similarity with the rules of *Identity* and *Cut* in a single conclusion sequent calculus ($A \succ A$, and from $\Gamma \succ A$ and $A, \Gamma' \succ B$ to infer $\Gamma, \Gamma' \succ B$) should not be surprising. Like *Cut*, the second component of the compositionality condition is the appropriate kind of *transitivity* condition on the relation R.

15 The relation $<$ on $\mathbb{Q}$ or $\mathbb{R}$ is dense, but not reflexive, as one obvious example.

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EXAMPLE 4 (Compositional multiset relations on ω). *Here are some examples of compositional multiset relations on the set ω of natural numbers.*

[*The* PRODUCT] *XRy if and only if y is the product of all the members of X.*[16] (*This is genuinely and distinctively a* multiset *relation, which distinguishes repeated elements in the multiset. For this relation,* $[2, 2]R4$ *holds, but* $[2]R4$ *does not.*) *This is compositional, which fact is left to the reader.*

[*Some* PRODUCT] *XRy if and only if y is* some *product of the members of X, using each instance in X at most once.* (*Unlike* the product, *this relation is not functional.*

[*The* SUM, AND *Some* SUM] *In the same way, the relation R given by setting XRy iff $\Sigma X = y$ is compositional* (*given that we set* $\Sigma[\,] = 0$), *as is the relation given by setting XRy iff $\Sigma X' = y$ for some $X' \leq X$. As with the product relations, one is functional, and the other is not. Each of the relations discussed so far makes essential use of the multiset structure. The multiset* $[2, 2]$ *is related to different numbers in each case, than the singleton multiset* $[2]$. *In the next example, the multiplicity of members makes no difference at all.*

[MAXIMUM-OR-ZERO-IF-EMPTY] *In this case, XRy if and only if y is the largest member of X, and is* 0 *if X is empty. This satisfies the reflexivity condition, as well as* Transitivity *and* Splitting.

[THE EMPTY RELATION] *Another multiset relation, trivially compositional, is the* empty *multiset relation. It is straightforward to verify that this relation satisfies both the* Transitivity *and the* Splitting *conditions. Of course, this relation fails to be reflexive, unlike the other relations we have considered so far.*

That is a range of compositional multiset relations on ω. Not every multiset relation, however, is compositional.

EXAMPLE 5 (Non-compositional multiset relations on ω). *These relations fail to be compositional in different ways.*

[LARGER THAN THE PRODUCT OF] *XRy holds if and only if $y > \Pi X$. Clearly this is not reflexive. While* Splitting *holds, the* Transitivity *direction of compositionality fails.*

[LARGEST TWO] *XRy if and only if y is one of the largest two elements of X. This relation fails* transitivity.

[MEMBERSHIP] *XRy if and only if $y \in X$. This relation enjoys* Transitivity *but not* Splitting.

Although membership is not a compositional multiset relation on $M(P) \times P$, it is compositional if we restrict our attention to inhabited[17] *multisets.* (*We will discuss this restricted form of compositionality below.*)

[BETWEEN] *XRy iff y occurs between the smallest and the largest members of X, inclusive. So* $[2, 4]$ *is related to* 2 *and to* 4 *and to* 3 *but to no other number. This, like*

16 What is *the* product of all the members of [ ]? A moment's reflection shows that the natural answer is to declare $\Pi[\,]$ to be 1. Then, for *any* two multisets X and Y, $\Pi(X \cup Y) = \Pi X \times \Pi Y$.

17 A multiset is inhabited iff it has at least one member at multiplicity at least one. It is (at least if we ignore constructivist distinctions) the positive synonym for the negatively defined 'non-empty'.

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membership, is compositional on the inhabited multisets but not the full collection of multisets.

We will end this series of examples with two more compositional relations, this time, on the rational numbers $\mathbb{Q}$ and the reals, $\mathbb{R}$, rather than on ω, so we have scope for examples of non-reflexive but dense order relations.

EXAMPLE 6 (Non-reflexive multiset relations). *These examples of multiset relations make use of the density of the underlying order $<$ on $\mathbb{Q}$ and on $\mathbb{R}$.*

[LARGER THAN] XRy *if and only if* $y > x$ *for each* $x \in X$. *So,* $[\ ]Ry$ *for* every y (*in this case, the condition is vacuously satisfied*). *This relation satisfies* Transitivity *and* Splitting *but not reflexivity.*

In this case, the relation makes no distinction between multisets with the same ground. $[2, 2]$ *is related to all the numbers greater than* 2, *as is* $[2]$ *and* $[2, 2, 2]$.

[LARGER THAN THE SUM OF] *Here,* XRy *if and only if* y *is larger than the sum of all the members of* X (*counting their multiplicities, as in the case of the* sum *relation given previously*). *As before, we set* $\Sigma[\ ] = 0$. *While this fails to be reflexive, it is compositional.*

This flock of examples was longer than it strictly needed to be, if not for one thing. A complaint about the ternary relational semantics is that examples are hard to come by, hard to construct and above all, hard to *picture*. That there is such a list of naturally occurring examples of compositional multiset relations, both reflexive and irreflexive, and which exhibit significantly different behaviours, but are straightforward both to reason with and to *understand*, goes quite some way towards answering that complaint.

It is disappointing, however, that *membership* and *betweenness* failed to count as compositional relations. In fact, as we noted, those multiset relations *are* compositional if we restrict our attention to the class $M'(P)$ of inhabited multisets of points. We can make this notion precise in a definition.

DEFINITION 7 (Compositional inhabited-multiset relations). *A relation* R *on* $M'(P) \times P$ *is said to be compositional if and only if for all multisets* X *and* Y *where* $X \neq [\ ]$, *and for all points* z,

$$(\exists y)(XRy \textit{ and } ([y] \cup Y)Rz) \textit{ iff } (X \cup Y)Rz.$$

This is the appropriate definition of compositionality for a relation on inhabited multisets. You may wonder why, in this definition, X inhabited, but Y is allowed to be empty. Isn't that outside the spirit of restricting our attention to inhabited multisets? This is a natural restriction of compositionality to this setting, because it is the smallest modification to the condition that ensures that the left relatum of any R-fact is nonempty. (Since we require that X be inhabited, for XRy to make sense in this context, this is enough to guarantee that $X \cup Y$ is also inhabited, and $[y] \cup Y$ is inhabited by design.) A satisfying upshot of this result is the fact under this condition (allowing for Y to be empty), the proof of Lemma 15 works in the case of inhabited-multiset relations, too. That special case of *transitivity*, spelled out, is this: XRy and $[y]Rz$ implies XRz. We have also appealed to this condition in the proof Lemma 3. We will also see below, when we turn to more general structures, like lists and trees, that the general form of compositionality involves trading in a single *item* in a structure (here, a member of a multiset) for another structure. In the case of a multiset, *any* multiset with a member y can be written in the form $X \cup [y]$. For this representation

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to work, in general, we need to allow the case where X is empty, even if our attention is fixed on inhabited multisets, for we may wish to trade in the y in a *singleton* multiset $[y]$ for some other multiset.

EXAMPLE 8 (Compositional inhabited-multiset relations). *With this expanded definition, we can enlarge our class of models even further. We have already seen that* MEMBERSHIP *and* BETWEEN *give us compositional relations on inhabited multisets. So are these*:

> [MAXIMUM, AND MINIMUM] MAXIMUM-OR-ZERO-IF-EMPTY *is a compositional multiset relation on* ω. *Without the need to have a maximum for* $[\,]$, *we can remove the "or-zero-if-empty" dodge, and restrict our attention to the largest member of the multiset. Or the smallest, if we choose, and the result is a compositional inhabited-multiset relation.*
>
> [*the* SUM, AND *some* SUM ON SUBSETS OF ω] *If we no longer have the requirement that the empty multiset* $[\,]$ *have a sum, then given any subset* S *of* ω, *closed under addition* (*so if* $x, y \in S$, *then so is* $x + y$) *we can define a compositional inhabited-multiset relations* R *and* R' *on* S, *setting* XRy *iff* $y = \Sigma X$, *and* $XR'y$ *iff* $y = \Sigma X'$ *where* X' *is an inhabited multiset where* $X' \leq X$. *For example, we can let* $S = \{1, 2, 3, \ldots\} = \omega \backslash \{0\}$ *to provide a very different kind of model, once* 0 *is left out of the domain.*
>
> [*the* PRODUCT, AND *some* PRODUCT ON SUBSETS OF ω] *In exactly the same way, we can generate models defining* R *on subsets of* ω *closed under product, without having to include* 1 *as the product of the empty multiset.*

In what follows, we will consider both compositional *multiset* relations and, at times, compositional *inhabited-multiset* relations. For any compositional multiset relation, its restriction to inhabited multisets is, of course, also compositional. For the converse, we have the following lemma, which shows that there is a way to extend a compositional inhabited multiset relation R on $M'(P) \times P$ to a compositional multiset relation on $M(P \cup \{\infty\}) \times (P \cup \{\infty\})$, where we add a new 'point at infinity' to our point set.

LEMMA 9. *If* R *is a compositional inhabited-multiset relation on* $M'(P) \times P$, *and* $\infty \notin P$, *then the multiset relation* $R^\times$ *on* $M(P \cup \{\infty\}) \times (P \cup \{\infty\})$, *defined as follows, is compositional.*

$$XR^\times z \text{ iff } \begin{cases} z = \infty, & \text{if } X \backslash \infty = [\,], \\ (X \backslash \infty) R z, & \text{if } X \backslash \infty \neq [\,]. \end{cases}$$

Furthermore, if R *is reflexive, then so is* $R^\times$.

(In the definition of $R^\times$ we use the notation '$X \backslash y$' for the multiset formed by removing *all* instances of y from X. So, for example, $[a, b, b, c, c] \backslash c = [a, b, b]$. We reserve '$X \backslash Y$' for the multiset formed by removing the number of occurrences in Y from X, so $[a, b, b, c, c] \backslash [c] = [a, b, b, c]$.)

Proof. Let's suppose that $(X \cup Y) R^\times z$, in order to find some y where $YR^\times y$ and $(X \cup [y]) R^\times z$. By definition $(X \cup Y) R^\times z$ holds if and only if $z = \infty$ (if $(X \cup Y) \backslash \infty = [\,]$) or $((X \cup Y) \backslash \infty)) Rz$ (otherwise). Let's take these cases in turn. If $(X \cup Y) \backslash \infty = [\,]$ then clearly $X \backslash [\,]$ and $Y \backslash [\,]$, so in this case, both $YR^\times \infty$ and $(X \cup [\infty]) R^\times \infty$, as desired. So, now consider the second case: we have $((X \cup Y) \backslash \infty)) Rz$ and $(X \cup Y) \backslash \infty \neq [\,]$. We aim to find some y where $YR^\times y$ and

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$(X \cup [y])R^\times z$. If $Y \backslash \infty = [\,]$, then we choose ∞ for y. We have, then, $YR^\times \infty$ and since $((X \cup Y)\backslash\infty))Rz$, we have $(X\backslash\infty)Rz$, so we have $(X \cup \{\infty\})R^\times z$ as desired. On the other hand, if Y has some element other than ∞, since $((X \cup Y)\backslash\infty))Rz$, we have $((X\backslash\infty) \cup (Y\backslash\infty))Rz$, and since R is compositional, there is some y where $(Y\backslash\infty)Ry$ and $((X\backslash\infty) \cup [y])Rz$, which gives us $YR^\times y$ and $(X \cup [y])R^\times z$ as desired.

Now for the second half of the compositionality condition for $R^\times$, suppose that there is some y where $YR^\times y$ and $(X \cup [y])R^\times z$. We aim to show that $(X \cup Y)R^\times z$. If $YR^\times y$ then either $y = \infty$ and Y contains at most ∞, or otherwise $(Y\backslash\infty)Ry$. In the first case, $(X \cup [y])R^\times z$ tells us that $(X \cup [\infty])R^\times z$, which means either that $(X\backslash\infty)Rz$, or X also contains at most ∞ and then $z = \infty$. In the either of these cases, we have $(X \cup Y)R^\times z$, as desired. So, let's suppose $y \neq \infty$. In that case we have $(Y\backslash\infty)Ry$, and then, since $(X \cup [y]))R^\times z$, we have $((X \cup [y])\backslash\infty)Rz$, and by the compositionality of R, $((X \cup Y)\backslash\infty)Rz$, which gives $(X \cup Y)R^\times z$, as desired.

Finally, $R^\times$ is reflexive follows immediately from the reflexivity of R and the fact that $[\infty]R^\times\infty$. □

With this result, it is possible for us to use examples like membership and betweenness as compositional multiset relations, with the full complement of logical resources, including the set of normal points, identified as those related to the empty multiset [ ].

Now we are in a position to define multiset frames and models. We will begin with the more standard ternary relational frames for RW^+.

DEFINITION 10 (Ternary relational RW^+ frames, models). *A ternary relational frame for* RW^+ *is a quadruple* $\langle P, R, \sqsubseteq, N\rangle$ *obeying the following conditions.*

1. $\sqsubseteq$ *is a partial order.*
2. *If* $x \sqsubseteq w$, $y \sqsubseteq u$, $v \sqsubseteq z$, *and* $Rwuv$, *then* $Rxyz$.
3. $y \sqsubseteq z$ *iff* $\exists x \in N$, $Rxyz$.
4. *If* $x \in N$ *and* $x \sqsubseteq y$, *then* $y \in N$.
5. $Rxyz$ *only if* $Ryxz$.
6. $Rwxyz$ *only if* $Rw(xy)z$.

To get a ternary relational frame for R^+, *one adds the condition that if* $Rxyz$, *then* $Rxyyz$.

A ternary relational model is a quintuple $\langle P, R, \sqsubseteq, N, \Vdash\rangle$ *where the first four components make up a frame and the final component is a binary relation between* P *and the set of atoms such that if* $x \Vdash p$ *and* $x \sqsubseteq y$, *then* $y \Vdash p$. *This is extended to the whole language according to the following clauses.*

- $x \Vdash A \wedge B$ *iff* $x \Vdash A$ *and* $x \Vdash B$.
- $x \Vdash A \vee B$ *iff* $x \Vdash A$ *or* $x \Vdash B$.
- $x \Vdash A \to B$ *iff for each* y, z *where* $Rxyz$, *if* $y \Vdash A$ *then* $z \Vdash B$.
- $x \Vdash A \circ B$ *iff for some* y, z *where* $Rxyz$, *both* $y \Vdash A$ *and* $z \Vdash B$.
- $x \Vdash \mathsf{t}$ *iff* $x \in N$.
- $x \Vdash \bot$ *never.*

Next, we define multiset frames.

DEFINITION 11 (Multiset frame). *A* MULTISET FRAME $\langle P, R\rangle$ *is an inhabited set* P *of points together with a compositional multiset relation* R *on* P.

This definition is, in one sense, starkly simpler than the traditional frame semantics for RW^+, in that the three elements N, $\sqsubseteq$ and the ternary relation R are subsumed into

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one fundamental relation, the compositional multiset relation. They are also more general, because we consider not only models in which $\sqsubset$ is reflexive (as it is in ternary relational frames), but the more general class of frames allowing for the underlying order relation $\sqsubset$ to be non-reflexive, or even irreflexive. In fact, we allow as a frame the case where R is the *empty* relation. So, this is a wider class of frames. The multiset frames subsume the traditional ternary relational frames for RW^+, following the conditions (F1), (F2), and (F3) from Section 2. The one relation in a multiset frame encodes the three different moving parts of a ternary frame. We have the following fact:

LEMMA 12. *Each ternary frame $\langle P, R, \sqsubseteq, N\rangle$ for RW^+ determines a reflexive multiset frame $\langle P, R'\rangle$, defined by setting*:

- $[\,]R'x$ *iff* $x \in N$,
- $[x]R'y$ *iff* $x \sqsubseteq y$,
- $[x, y]R'z$ *iff* $Rxyz$,
- *If* Y *is a multiset of size two or more,* $([x] \cup Y)R'z$ *iff for some* y, $YR'y$ *and* $[x, y]R'z$.

Proof. We first need to show that the definition is R' coherent: that the third clause, to the effect that $[x, y]R'z$ iff $Rxyz$, that the last clause, according to which $([x] \cup Y)R'z$ iff for some y, $YR'y$ and $[x, y]R'z$, could both hold. For the third clause, we need to be sure that $Rxyz$ holds iff $Ryxz$ holds, since $[x, y] = [y, x]$, lest the clause give inconsistent guidance as about $[x, y]R'z$. But in any ternary frame $\langle P, R, \sqsubseteq, N\rangle$ for RW^+, we have $Rxyz$ iff $Ryxz$, so this clause is coherent.

For the last clause, if $[x] \cup Y$ is the same multiset as $[x'] \cup Y'$, we need to show that

$$(\exists y)(YR'y \wedge [x, y]R'z) \text{ if and only if } (\exists y')(Y'R'y' \wedge [x', y']R'z)$$

in order to ensure that this clause also gives consistent guidance concerning R'. We prove this by induction on the size of $[x] \cup Y$. When Y has size 2, this reduces to the case $(\exists y)([x_2, x_3]R'y \wedge [x_1, y]R'z$ iff $(\exists y')([x_1, x_3]R'y' \wedge [x_2, y']R'z)$, but given the definition of R' on two-element multisets in terms of the ternary R, this reduces to the biconditional $(\exists y)(Rx_2x_3y \wedge Rx_1yz)$ iff $(\exists y)(Rx_1x_3y' \wedge Rx_2y'z)$, but this is the biconditional between $R^2x_1(x_2x_3)z$ and $R^2x_2(x_1x_3)z$, which indeed holds in our RW^+ frame.

Suppose the equivalence has been proved for all multisets of size n (where $n > 2$) and we have a multiset $[x_1] \cup Y = [x_2] \cup Y'$ of size $n + 1$. Let Z be such that $Z \cup [x_2] = Y$ and $Z \cup [x_1] = Y'$. Note that we may assume $x_1 \neq x_2$, as otherwise the case is trivial. We wish to show that

$$(\exists y)(([x_2] \cup Z)R'y \wedge [x_1, y]R'z) \text{ iff } (\exists y')(([x_1] \cup Z)R'y' \wedge [x_2, y']R'z).$$

By the inductive hypothesis, $(\exists y)(([x_2] \cup Z)R'y \wedge [x_1, y]R'z)$ is equivalent to

$$(\exists y)(\exists w)(ZR'w \wedge [w, x_2]R'y \wedge [x_1, y]R'z).$$

From the definition of R', the latter two conjuncts suffice for $Rx_1(x_2w)z$, which is equivalent to $Rx_2(x_1w)z$, as in the base case. Therefore,

$$(\exists y')(\exists w)(ZR'w \wedge [w, x_1]R'y' \wedge [x_2, y']R'z),$$

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which in turn is equivalent, by the inductive hypothesis, to

$$(\exists y')(([x_1] \cup Z)R'y' \wedge [x_2, y']R'z).$$

So, we have shown by induction that the definition is coherent.

Now, it suffices to show that R', so defined, is reflexive and compositional. Reflexivity follows from the reflexivity of $\sqsubseteq$, and *Transitivity* follows straightforwardly from the definition of R' itself, albeit with many cases to check. It remains to show that *Splitting* holds.

We want to show that if $(X \cup Y)R'z$, then there is some y where $XR'y$ and $([y] \cup Y)R'z$. Given the definitions, we need to consider the cases where $X \cup Y$ has zero, one, two, or more elements. In the case where $X \cup Y$ is empty, then we have $[]R'z$. So, then we have $[]R'z$ and $[z]R'z$, as desired.

If $X \cup Y$ has size 1, then there are two subcases. <u>Subcase</u>: X is $[x]$. By assumption we have $[x]R'z$, so we then have $[x]R'x$, by Reflexivity, and $[x]R'z$, satisfying *Splitting*. <u>Subcase</u>: X is empty and Y is $[x]$. Since $[x]R'z$, $x \sqsubseteq z$, so there is some $y \in N$ such that $Ryxz$. We then have $[]R'y$ and $[y, x]R'z$, satisfying *Splitting*.

Suppose $X \cup Y$ has size 2. <u>Subcase</u>: X is empty. We need a y such that $[]R'y$ and $[y, y_1, y_2]R'z$. Since $y_1 \sqsubseteq y_1$, there is a $u \in N$ such that Ruy_1y_1. By assumption we have Ry_1y_2z, so it follows that Ruy_1y_2z, which is $[uy_1y_2]R'z$, as desired. <u>Subcase</u>: X is [x] and Y is $[y_1]$. In this subcase we have $[x, y_1]R'z$. Since $[x]R'x$, it follows that there is a y such that $[x]R'y$ and $[y, y_1]R'z$, namely x. <u>Subcase</u>: X is $[x_1, x_2]$ and Y is empty. By assumption we have $[x_1, x_2]R'z$ and we need a y such that $[x_1, x_2]R'y$ and $[y]R'z$. Since $[z]R'z$, we can simply take z as y.

Suppose $X \cup Y$ has size 3 or greater. <u>Subcase</u>: X is empty. The argument is similar to the subcase of the previous case where X is empty. <u>Subcase</u>: Y is empty. The argument is similar to the subcase of the previous case where Y is empty. <u>Subcase</u>: X and Y are inhabited, so $X = [x] \cup X'$ and $Y = [y_1, \dots, y_n]$ and $([x] \cup X' \cup [y_1, \dots, y_n])R'z$. From the definition of R', it follows that for some z_1, $([x] \cup X' \cup [y_2, \dots, y_n])R'z_1$ and $[y_1, z_1]R'z$. Repeated use of the definition results in $z_2, \dots, z_n$ such that $[y_1, z_1]R'z$, $[y_2, z_2]R'z_1$, ..., $[y_n, z_n]R'z_{n-1}$, and $([x] \cup X')R'z_n$. Repeated use of *Transitivity* then yields $[y_1, \dots, y_n, z_n]R'z$, so we can let z_n be the desired y.

All of the cases have been covered, so we conclude that R' obeys *Splitting*.

So, the lemma is proved. □

Now let us turn to consider what it is for a formula to hold at a point in a multiset frame. Given our understanding of the relation R, if $[x]Ry$ then the information in x also holds in y. So, if a formula holds at x, it is given by the multiset consisting of $[x]$ alone. But then, it should also hold at y, since the information given by $[x]$ is (perforce, according to R at least) also true at y, and there is nothing else in $[x]$ to take together with x. So, an appropriate *heredity condition* for truth-at-a-point in a multiset frame is given by the multiset relation R:

DEFINITION 13 (Heredity). *A relation $\Vdash$ between points and formulas is* HEREDITARY *along R for some class $\mathcal{F}$ of formulas if and only if whenever $[x]Ry$ (that is, when $x \sqsubseteq y$) and $x \Vdash A$ then $y \Vdash A$, for each formula A in $\mathcal{F}$.*

Given a hereditary relation $\Vdash$ for all *atomic* formulas on a multiset frame, we can extend it to a hereditary relation on all formulas in the language of RW^+ as follows:

DEFINITION 14 (Truth-at-a-point in a multiset model). *For any multiset frame $\langle P, R\rangle$ and a hereditary relation $\Vdash$ defined on atomic formulas in our language, we extend the relation $\Vdash$ to the whole vocabulary, defining $x \Vdash A$ recursively as follows*:

- $x \Vdash A \wedge B$ *iff* $x \Vdash A$ *and* $x \Vdash B$.
- $x \Vdash A \vee B$ *iff* $x \Vdash A$ *or* $x \Vdash B$.
- $x \Vdash A \to B$ *iff for each* y, z *where* $[x, y]Rz$, *if* $y \Vdash A$ *then* $z \Vdash B$.
- $x \Vdash A \circ B$ *iff for some* y, z *where* $[y, z]Rx$, *both* $y \Vdash A$ *and* $z \Vdash B$.
- $x \Vdash \mathsf{t}$ *iff* $[\,]Rx$.
- $x \Vdash \bot$ *never.*

LEMMA 15. *In any multiset frame $\langle P, R\rangle$, the evaluation relation $\Vdash$ defined above, between points and arbitrary formulas is hereditary along R.*

Proof. We aim to show that whenever $[x]Ry$ and $x \Vdash A$ then $y \Vdash A$. This is an easy induction on the structure of the formula A. The result holds by fiat for atomic formulas, and the induction step is trivial for conjunctions and disjunctions.

For conditionals, suppose $[x]Ry$ and $x \Vdash A \to B$. We wish to show that $y \Vdash A \to B$ too. Take u, v where $[y, u]Rv$. We wish to show that if $u \Vdash A$ then $v \Vdash B$. By compositionality, since $[x]Ry$ and $[y, u]Rv$, we have $[x, u]Rv$. Since $x \Vdash A \to B$, if $u \Vdash A$ then $v \Vdash B$ as desired.

Similarly, if $[x]Ry$ and $x \Vdash A \circ B$, we wish to show that $y \Vdash A \circ B$. So, we wish to find u, v where $[u, v]Ry$, $u \Vdash A$ and $v \Vdash B$. Since $x \Vdash A \circ B$, we have u, v where $[u, v]Rx$, $u \Vdash A$ and $v \Vdash B$. By compositionality, $[u, v]Rx$ and $([x] \cup [\,])Ry$ gives us $([u, v] \cup [\,])Ry$, i.e., $[u, v]Ry$ as desired.

Finally, if $[x]Ry$ and $x \Vdash \mathsf{t}$, then we have $[\,]Rx$. Notice that compositionality ensures that $[\,]Rx$ and $([x] \cup [\,])Ry$ give $([\,] \cup [\,])Ry$, i.e., from $[\,]Rx$ and $[x]Ry$, we have $[\,]Ry$, so if t holds at x and $[x]Ry$, then t holds at y too. □

So, evaluation relations on frames allow us to interpret formulas from the language of RW^+ or R^+ at points. Note that in the case for fusion, we needed to consider the multiset $[x] \cup [\,]$, which is the special case highlighted in the definition of compositionality for inhabited-multiset relations. We call the combination of a frame $\langle P, R\rangle$ and an evaluation relation $\Vdash$ on that frame a *model*, and we abuse notation slightly to think of the triple $\langle P, R, \Vdash\rangle$ as a model.

Another way to represent how formulas are evaluated at points in frames is, for each formula A, to collect together the points that support A. We use the notation $[\![A]\!]$ for the set $\{x \in P : x \Vdash A\}$, the *extension* of the formula A in the model. The results of this section show that the set $[\![A]\!]$ is upwardly closed along the relation $\sqsubset$, and the evaluation conditions for atomic formulas are simply that for each atomic formula p, its extension $[\![p]\!]$ is an upwardly closed set.

We pause to note that the evaluation conditions on *ternary* frames agree with those on multiset frames. In other words we have the following lemma:

LEMMA 16 (Model equivalence). *If $\langle P, R, \sqsubseteq, N, \Vdash\rangle$ is a ternary relational model for RW^+ (or R^+), then $\langle P, R', \Vdash\rangle$ is a multiset model defined on the multiset frame $\langle P, R'\rangle$.*

The proof is immediate, given that $[x, y]R'z$ iff $Rxyz$, and $[\,]R'x$ iff $x \in N$.

So, we have shown that reflexive multiset frames correspond tightly to ternary relational frames. We have also seen that compositional inhabited-multiset relations arise naturally as structures in the same general family as compositional multiset

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relation. A frame $\langle P, R\rangle$ which is furnished with an *inhabited*-multiset relation R can also be used to model our propositional vocabulary. Given an inhabited-multiset frame $\langle P, R\rangle$ and a hereditary evaluation relation $\Vdash$ on atomic formulas, we can extend $\Vdash$ to the propositional language *except for the Ackermann constant* t, in the manner given in Definition 14. The proof that $\Vdash$ so defined is heredity follows in exactly the same way. The only point at which the condition that R relate only inhabited multisets is violated in that proof is at the clause for t. The rest of the proof goes through as expected.

With multiset frames, we can model the relevant logic RW^+. To make this precise, we introduce the logic RW^+ by way of a sequent calculus. The calculus utilises sequents of the form $\Gamma \succ A$, where A is a *formula* and Γ is a *structure*, generated by the following grammar:

$$\Gamma := A \mid \varepsilon \mid (\Gamma, \Gamma) \mid (\Gamma; \Gamma).$$

In other words, a structure is a formula A, the *empty* structure ε, the *extensional* combination (Γ, Γ') of two structures, or the *intensional* combination $(\Gamma; \Gamma')$ of two structures. When presenting structures, we often omit the outer layer of parentheses (so A, B is a structure, as is $A; (B, C)$), but we do not omit interior parentheses: $A, (B, C)$ differs from $(A, B), C$ in the order of combination, even though they will end up having the same logical force, due to the structural rules of the proof calculus.[18] We will also treat binary structural connectives as binding less tightly than any formula connectives, so $A \to B; C$ will be $(A \to B); C$.

When specifying rules of inference, we use parentheses in another way: $\Gamma(A)$ is a structure with a particular subformula A singled out. Given $\Gamma(A)$, the structure $\Gamma(\Gamma')$ is found by substituting that instance of A by Γ'. The same goes for other structures. So, $\Gamma(\Gamma', \Gamma'')$ is a structure in which the structure Γ', Γ'' is found somewhere as a constituent, and the structure $\Gamma(\Gamma'', \Gamma')$ is found by reversing the order of Γ' and Γ'' inside that structure. For future reference, we will call the part of the structure $\Gamma(A)$ *around* the instance A the *context* of A in $\Gamma(A)$, and we will use the notation '$\Gamma(-)$' to refer to that context.

A derivation in this sequent calculus is a tree of sequents, of which every *leaf* is an axiom, where each transition is an inference rule. The fundamental rules in the sequent calculus are the axioms of *Identity* and the inference rule, *Cut*.[19]

[18] The technique, of allowing *two* forms of premise combination in sequents, is due to Dunn [13], details of which can be found in *Entailment* volume 1 [1, sec. 28.5] and one development of which is provided by Belnap, Dunn and Gupta [4]. For an extended introduction to sequent calculi of this form, consult Restall's *An Introduction to Substructural Logics*, Chapter 6 [41]. Bimbó [5] provides an introduction to these sorts of sequent systems in the specific context of relevant logics.

[19] *Id* and *Cut* are fundamental in the sense that they apply invariably to every formula, and to each structure without any discrimination. They appeal to no distinctive properties of any connectives or formulas (unlike the specific rules for each connective), or of any particular form of structural combination (unlike the structural rules). They appeal to formulas *as such*, and structures *as such*. Of course, a fundamental theorem of proof theory for sequent systems is that the rule of *Cut* can be eliminated, in the sense that any derivation using *Cut* can be transformed into a derivation in which *Cut* is not used. Appeals to *Id* for complex formulas can also be traded in for appeals only to atomic formulas. These matters, though important for the analysis of proof, are not central to our concerns here.

$$A \succ A \;\; Id \qquad \frac{\Gamma \succ A \quad \Gamma'(A) \succ B}{\Gamma'(\Gamma) \succ B}\, Cut.$$

The next series of rules are *structural rules*, governing *extensional* and *intensional* structure combination respectively. Extensional combination allows for commutativity and associativity (at arbitrary depth inside a structure), as well as contraction and weakening, while intensional combination allows for only commutativity and associativity. In addition, ε acts as an identity for intensional combination.

$$\frac{\Gamma(\Gamma', \Gamma'') \succ B}{\Gamma(\Gamma'', \Gamma') \succ B}\, EC \quad \frac{\Gamma(\Gamma', (\Gamma'', \Gamma''')) \succ B}{\Gamma((\Gamma', \Gamma''), \Gamma''') \succ B}\, EB \quad \frac{\Gamma(\Gamma', \Gamma') \succ B}{\Gamma(\Gamma') \succ B}\, EW \quad \frac{\Gamma(\Gamma') \succ B}{\Gamma(\Gamma', \Gamma'') \succ B}\, EK$$

$$\frac{\Gamma(\Gamma'; \Gamma'') \succ B}{\Gamma(\Gamma''; \Gamma') \succ B}\, IC \quad \frac{\Gamma(\Gamma'; (\Gamma''; \Gamma''')) \succ B}{\Gamma((\Gamma'; \Gamma''); \Gamma''') \succ B}\, IB \quad \frac{\Gamma(\Gamma') \succ B}{\Gamma(\varepsilon; \Gamma') \succ B}\, \varepsilon I \quad \frac{\Gamma(\varepsilon; \Gamma') \succ B}{\Gamma(\Gamma') \succ B}\, \varepsilon E.$$

The remaining rules are left and right rules for each connective. These are totally *modular*, in the sense that we can choose to include a connective or to leave it out. No rule for one connective requires the presence of any other connective in the vocabulary.

$$\frac{\Gamma(A, B) \succ C}{\Gamma(A \wedge B) \succ C}\, \wedge L \qquad \frac{\Gamma \succ A \quad \Gamma' \succ B}{\Gamma, \Gamma' \succ A \wedge B}\, \wedge R$$

$$\frac{\Gamma(A) \succ C \quad \Gamma(B) \succ C}{\Gamma(A \vee B) \succ C}\, \vee L \qquad \frac{\Gamma \succ A}{\Gamma \succ A \vee B}\, \vee R \quad \frac{\Gamma \succ B}{\Gamma \succ A \vee B}\, \vee R$$

$$\frac{\Gamma \succ A \quad \Gamma'(B) \succ C}{\Gamma'(A \to B; \Gamma) \succ C}\, \to L \qquad \frac{\Gamma; A \succ B}{\Gamma \succ A \to B}\, \to R \qquad \frac{}{\bot \succ C}\, \bot R$$

$$\frac{\Gamma(A; B) \succ C}{\Gamma(A \circ B) \succ C}\, \circ L \qquad \frac{\Gamma \succ A \quad \Gamma' \succ B}{\Gamma; \Gamma' \succ A \circ B}\, \circ R \qquad \frac{\Gamma(\varepsilon) \succ C}{\Gamma(\mathsf{t}) \succ C}\, \mathsf{t}L \qquad \frac{}{\varepsilon \succ \mathsf{t}}\, \mathsf{t}R.$$

For R^+, we add one more rule: contraction for intensional combination.

$$\frac{\Gamma(\Gamma'; \Gamma') \succ B}{\Gamma(\Gamma') \succ B}\, IW.$$

With *IW*, we can derive new sequents, which could not be derived without it. For example, we can derive $\varepsilon \succ (A \wedge (A \to B)) \to B$.

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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \succ A \quad B \succ B}{A \to B; A \succ B}{\to}L}{A \to B; (A, A \to B) \succ B}EK}{(A, A \to B); (A, A \to B) \succ B}EK}{A, A \to B \succ B}IW}{A \wedge (A \to B) \succ B}\wedge L}{\varepsilon; A \wedge (A \to B) \succ B}\varepsilon I}{\varepsilon \succ A \wedge (A \to B) \to B}{\to}R.$$

The proof theory for logics like RW^+ and R^+ is well known, and so is the ternary relational semantics. Given our perspective on collection frames, it is worth taking the time to reconsider the relationship between proofs and models. Consider the proof given above, of the sequent $\varepsilon \succ (A \wedge (A \to B)) \to B$. What does this say about R^+ models? It does not tell us that $(A \wedge (A \to B)) \to B$ holds at *every* point in those models, only that it holds at *normal* points, those points x where $[\,]Rx$. In other words, the sequent $\varepsilon \succ (A \wedge (A \to B)) \to B$ should tell us that

For every point x, if $[\,]Rx$ then $x \Vdash (A \wedge (A \to B)) \to B$.

Scanning back to our derivation to its second line, we have $A \to B; A \succ B$. This does not tell us that if $A \to B$ is true at a point and that A is true at that point, then B is true there too (if that were all the sequent said, the conditional would be *irrelevant*). The appropriate way to understand the 'cash value' of the derivation of this sequent according to our frames is that

For all x, y and z, if $x \Vdash A \to B$ and $y \Vdash A$, if $[x, y]Rz$ then $z \Vdash B$.

In the first of these cases, we have involved the R relation on *empty* multiset. In the second of these cases, we have used the R relation on a two-element multiset. The natural thing to consider when it comes to the sequent $(A \wedge (A \to B)) \succ B$, then, would be to understand the sequent as telling us this:

For all x and y, if $x \Vdash A \wedge (A \to B)$ and $[x]Ry$ then $y \Vdash B$.

This is how we will understand validity of sequents on our frames. A single-premise single-conclusion sequent $A \succ B$ is valid on a frame if and only if:

For all x and y, if $x \Vdash A$ and $[x]Ry$ then $y \Vdash B$.

This agrees with the traditional understanding of validity of a sequent $A \succ B$ on a frame (that for each point x, if $x \Vdash A$ then $x \Vdash B$ too) when that frame is *reflexive*. (Take any reflexive frame. If a sequent has a counterexample according to the old definition, that provides a counterexample in the new definition too, by the reflexivity of the frame. Conversely, if we had points x and y where $[x]Ry$ and A holds at x but B fails at y, then by heredity on our frame, B must also fail at x, since $[x]Ry$, and so we have a counterexample according to the traditional definition). This understanding of validity diverges only in cases where the frame is not reflexive.[20] Since non-reflexive frames are a proper generalisation of ternary relational frames, the question of how to interpret sequents on *them* is open. We have argued here that invoking R, and evaluating the LHS of our sequent at one point and the RHS at another is in keeping with how we have always interpreted *zero*-premise and *multiple*-premise sequents on ternary frames. It is also in keeping with the interpretation of conditionals in these frames. It would be surprising if the conditional-like notion of entailment in a relevant logic did not share in the features that the semantics ascribes to the conditional in that logic. So, we proceed with this new understanding of what it is for a sequent to be valid in a model.[21]

So, when is a sequent $\Gamma \succ A$ valid in some model $\langle P, R, \Vdash\rangle$? We have considered sequents of the form $\varepsilon \succ A$, those of the form $A \succ B$ and those of the form $A; B \succ C$. What about those involving the *extensional* combiner, the comma? When is the sequent $A, B \succ C$ valid in our frame? One candidate (generalising the case of the single formula on the left) is to say that whenever $x \Vdash A$ and $x \Vdash B$ then when $[x]Ry$, we have $y \Vdash C$. However, an equivalent way of formulating this claim will be more natural in our setting. Instead, we can say that $A, B \succ C$ is valid on a frame if and only if

> For all x, y and z, if $x \Vdash A$ and $y \Vdash B$, if $[x]Rz$ and $[y]Rz$, then $z \Vdash C$.

The parallel with the case for the semicolon is clear. We look for points where the LHS formulas are true, and we combine them, using R to locate where to check the RHS formula. Here we check C at all *common descendants* of x and of y, rather than those points found by *combining* x and y together. This choice allows us to give a particularly straightforward interpretation of the validity of sequents in our models. We start with the notion of the *shadow* cast by a structure in a model.

DEFINITION 17 (The shadow cast by a structure). *For a structure* Γ *its* shadow $\{\{\Gamma\}\}$ *in the model* $\langle P, R, \Vdash\rangle$ *is a set of points, defined recursively as follows*:

- $\{\{\varepsilon\}\} = \{x \in P : [\,]Rx\}$,
- $\{\{A\}\} = \{x \in P : (\exists y \in [\![A]\!])[y]Rx\}$,
- $\{\{\Gamma, \Gamma'\}\} = \{x \in P : (\exists y \in \{\{\Gamma\}\})(\exists z \in \{\{\Gamma'\}\})([y]Rx \wedge [z]Rx)\}$,
- $\{\{\Gamma; \Gamma'\}\} = \{x \in P : (\exists y \in \{\{\Gamma\}\})(\exists z \in \{\{\Gamma'\}\})[y, z]Rx\}$.

When a structure is a single formula A, then $\{\{A\}\}$, the shadow it casts is not the formula's extension, $[\![A]\!]$, but rather, it is the set of points upward from some point in the extension. If R is reflexive, then $\{\{A\}\} = [\![A]\!]$, so where R is reflexive, the distinction between shadows and extensions makes no significant difference. In any model, whether reflexive or not, $\{\{A\}\} \subseteq [\![A]\!]$.

It is worth pausing to understand the behaviour of shadows in a specific non-reflexive frame. Consider the multiset frame $\langle \mathbb{R}, <\rangle$ with the multiset relation given by taking a multiset X of reals to relate to all and only those reals *larger* than each member of X.

[20] The idea of dropping reflexivity as a condition of Kripke frames has been studied in connection with relatives of intuitionistic logic [40, 54].

[21] You may wonder: What happens to the *traditional* understanding of validity on our frames? Isn't *that* notion of validity worth respecting, even on non-reflexive frames? Here we take succour in the fact that we can be *pluralists* about validity [3], even *relevant* validity. The fact that a frame provides more than one natural candidate for a notion of validity is, for us, a *feature*, not a *bug*.

Here, the underlying order $\sqsubset$ is the order $<$ on $\mathbb{R}$. So, the extension $[\![A]\!]$ of a formula must be upwardly closed on $\mathbb{R}$. So, an *extension* must have one of the forms $(-\infty, \infty)$, $[r, \infty)$, or (r, ∞) for some real r, or be empty. A *shadow*, on the other hand, cannot have the form $[r, \infty)$. If $[\![A]\!] = [r, \infty)$, then $\{\{A\}\} = (r, \infty)$, and if $[\![A]\!] = (r, \infty)$ then $\{\{A\}\} = (r, \infty)$ too. The possible values of shadows are $(-\infty, \infty)$, and (r, ∞) for each real r, and the empty set of reals.

It is also worth pausing to note that the notion of a shadow can be applied equally well in inhabited-multiset frames, provided that our structures do not contain the marker 'ε' for the empty structure. So, for the rest of this section, we will consider two kinds of models: those on multiset frames, and those on inhabited-multiset frames. The first kind will be models of the whole calculus, while inhabited-multiset frames can be used as models for the fragment of the proof calculus in which ε is absent: that is, the calculus without the rules εI, εE, $\mathsf{t}L$ and $\mathsf{t}R$. We will call the calculi for RW^+ and R^+ *without* ε, $\mathsf{RW}^+_{-\varepsilon}$ and $\mathsf{R}^+_{-\varepsilon}$ respectively, to make explicit the absence of sequents with ε.

We have seen that the shadow $\{\{A\}\}$ of a formula A is related to its extension $[\![A]\!]$ in a natural way. $x \in \{\{A\}\}$ iff there is some $y \in [\![A]\!]$ where $[y]Rx$ (that is, $y \sqsubset x$). This transition from extension to shadow is an operation on sets of points, and it is worth singling out with some notation.

DEFINITION 18 ($\sqsubset$ on sets of points). *$X_\sqsubset$ is defined as* $\{x \in P : (\exists y \in X) y \sqsubset x\}$.

So, this lemma is immediate:

LEMMA 19 (From extensions to shadows). $\{\{A\}\} = [\![A]\!]_\sqsubset$.

This operation satisfies two useful conditions.

LEMMA 20 ($\sqsubset$ is monotone and idempotent). *For any sets X and Y, if $X \subseteq Y$ then $X_\sqsubset \subseteq Y_\sqsubset$. Furthermore, $X_\sqsubset = X_{\sqsubset\sqsubset}$.*

Proof. For monotony, if $z \in X_\sqsubset$ then there is some $x \in X$ where $[x]Rz$. Since $x \in Y$, $z \in Y_\sqsubset$ too. For idempotence, we appeal to the density and transitivity of $\sqsubset$. If $z \in X_\sqsubset$ then since there is some $x \in X$ where $[x]Rz$ then by density there is some y where $[x]Ry$ (so $y \in X_\sqsubset$) and $[y]Rz$, ensuring that $x \in X_{\sqsubset\sqsubset}$. Conversely, if $z \in X_{\sqsubset\sqsubset}$ then there is some $y \in X_\sqsubset$ where $[y]Rz$ and some $x \in X$ where $[x]Ry$. By transitivity, $[x]Rz$, ensuring that $z \in X_\sqsubset$. □

The shadow of a *formula A* is the set of points above that formula's extension, $[\![A]\!]$. A shadow of a structure is not defined by taking the points above the extension of some formula, but nonetheless, it too is a fixed point for the operation $\sqsubset$.

LEMMA 21 (Shadows and order). *For each shadow $\{\{\Gamma\}\}$, we have $\{\{\Gamma\}\} = \{\{\Gamma\}\}_\sqsubset$.*

To prove this, it is simplest to characterise the sets fixed under $\sqsubset$ in general terms. We first prove a more general lemma, for which Lemma 21 is a corollary. For this, we need one more definition:

DEFINITION 22 (Closed upwards and open downwards). *A set X is* closed upwards *along $\sqsubset$ if whenever $x \in X$ and $x \sqsubset x'$ then $x' \in X$ too. A set X is* open downwards *along $\sqsubset$ if whenever $x \in X$, there is some $x' \sqsubset x$ where $x' \in X$ too.*

In the multiset frame $\langle \mathbb{R}, < \rangle$ discussed above, the intervals $[r, \infty)$ are closed upwards but not open downwards, while the intervals (r, ∞) are both closed upwards and

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open downwards along the order $<$. The properties of being closed upwards and open downwards are related to the operation $\sqsubset$ as follows:

LEMMA 23 (Open and closed sets). *If the relation $\sqsubset$ is transitive, then if X is closed upwards, then $X_{\sqsubset} \subseteq X$. If $\sqsubset$ is dense, then if X is open downwards, then $X \subseteq X_{\sqsubset}$.*

The proof is a simple matter of unpacking the definitions:

Proof. Suppose $\sqsubset$ is transitive and that X is closed upwards. Take $x \in X_{\sqsubset}$. So there is some $x' \in X$ where $x' \sqsubset x$. Since X is closed upwards, we have $x \in X$. Suppose $\sqsubset$ is dense and X is open downwards. Take $x \in X$. Since X is open downwards we have some $x' \in X$ where $x' \sqsubset x$. It follows that $x \in X_{\sqsubset}$. □

So, the sets X that are closed upwards and open downwards are fixed points for the operation $\sqsubset$. Since on any collection frame, $\sqsubset$ is transitive and dense, the shadow $\{\{\Gamma\}\}$ of any structure Γ is both closed upwards and open downwards, and is a fixed point for the operation $\sqsubset$.

Now we can return to the proof of Lemma 21.

Proof. Consider each kind of shadow, as given in Definition 17. A quick inspection of each clause shows that if R satisfies *Transitivity* and *Splitting*, then the shadow is closed upward and open downward. For one example, for $\{\{\varepsilon\}\}$, if $x \in \{\{\varepsilon\}\}$, for upward closure, assume that $x \sqsubset x'$. Since $[\,]Rx$ and $x \sqsubset x'$ we have $[\,]Rx'$ by transitivity, and $x' \in \{\{\varepsilon\}\}$. For downward openness, since $[\,]Rx$, by *Splitting* we have some x' where $[\,]Rx'$ (so $x' \in \{\{\varepsilon\}\}$) and $x' \sqsubset x$, as desired.

For the intensional composition case, if $x \in \{\{\Gamma; \Gamma'\}\}$, for upward closure, assume that $x \sqsubset x'$. Since we have $y \in \{\{\Gamma\}\}$ and $z \in \{\{\Gamma'\}\}$ where $[y, z]Rx$, and since $x \sqsubset x'$, by transitivity we have $[y, z]Rx'$, and $x' \in \{\{\Gamma; \Gamma'\}\}$ as desired. For downward openness, since $[y, z]Rx$, by *Splitting* we have some x' where $[y, z]Rx'$ and $[x']Rx$ (so $x' \in \{\{\Gamma; \Gamma'\}\}$) and $x' \sqsubset x$, as desired.

The other two cases follow in the same way, so we can declare this lemma proved. □

With this behaviour of shadows proved, we can see that the definition of the shadow of an extensional structure can be simplified. Since $\{\{\Gamma, \Gamma'\}\} = \{x \in P : (\exists y \in \{\{\Gamma\}\})[y]Rx\} \cap \{x \in P : (\exists y \in \{\{\Gamma'\}\})[z]Rx\} = \{\{\Gamma\}\}_{\sqsubset} \cap \{\{\Gamma'\}\}_{\sqsubset}$, we have the following consequence:

COROLLARY 24. $\{\{\Gamma, \Gamma'\}\} = \{\{\Gamma\}\} \cap \{\{\Gamma'\}\}$.

With the definition of a structure's shadow, the statement the condition for validity on a model is straightforward.

DEFINITION 25 (Model validity). *A sequent $\Gamma \succ A$ is valid in the model $\langle P, R, \Vdash \rangle$ if and only if $\{\{\Gamma\}\} \subseteq [\![A]\!]$. That is, the shadow cast by the structure Γ is restricted to the extension of the formula A.*

So, we are in a position to state our soundness theorem:

THEOREM 26 (RW^+ is sound for multiset frames). *Any RW^+ derivable sequent $\Gamma \succ A$ holds in each model $\langle P, R, \Vdash \rangle$ on a multiset frame. Furthermore, any $\mathsf{RW}^+_{-\varepsilon}$ derivable sequent holds in each model on an inhabited-multiset frame.*

To prove the soundness theorem, it helps to establish the following facts about shadows and contexts.

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LEMMA 27 (Contexts preserve order, and are prime). *If* $\{\{\Gamma\}\} \subseteq [\![A]\!]$, *then for any context* $\Gamma'(–)$, *we have* $\{\{\Gamma'(\Gamma)\}\} \subseteq \{\{\Gamma'(A)\}\}$. *In this sense, contexts are order preserving over valid sequents. Furthermore,* $\{\{\Gamma'(A \vee B)\}\} = \{\{\Gamma'(A)\}\} \cup \{\{\Gamma'(B)\}\}$, *so contexts are prime, and* $\{\{\Gamma'(\bot)\}\} = \{\{\bot\}\} = \emptyset$.

Proof. Both facts follow from an easy induction on the construction of the context $\Gamma'(–)$. An atomic context $\Gamma'(–)$ the hole '–' itself. In this case, primeness is trivial, and order preservation follows from the monotony and idempotence (Lemma 20). If $\{\{A\}\} \subseteq [\![B]\!]$, then by monotony, $\{\{A\}\}_\sqsubset \subseteq [\![B]\!]_\sqsubset$, but $[\![B]\!]_\sqsubset = \{\{B\}\}$, so $\{\{A\}\}_\sqsubset \subseteq \{\{B\}\}$, and since idempotence gives $\{\{A\}\} = \{\{A\}\}_\sqsubset$, we have $\{\{A\}\} \subseteq \{\{B\}\}$ as desired.

For the induction steps, $\Gamma'(–)$ either has the form $\Gamma''(–), \Gamma'''$, or $\Gamma''', \Gamma''(–)$, or $\Gamma''(–); \Gamma'''$, or $\Gamma'''; \Gamma''(–)$, in which case preservation and primeness follow immediately from the properties holding for the simpler context $\Gamma''(–)$.

For example, if $\{\{\Gamma''(\Gamma)\}\} \subseteq \{\{\Gamma''(A)\}\}$, then $\{\{\Gamma'; \Gamma''(\Gamma)\}\} = \{x \in P : (\exists y \in \{\{\Gamma'\}\})(\exists z \in \{\{\Gamma''(\Gamma)\}\})[y, z]Rx\}$, but since $\{\{\Gamma''(\Gamma)\}\} \subseteq \{\{\Gamma''(A)\}\}$, it follows that this set is a subset of $\{x \in P : (\exists y \in \{\{\Gamma'\}\})(\exists z \in \{\{\Gamma''(A)\}\})[y, z]Rx\}$, which is $\{\{\Gamma'; \Gamma''(A)\}\}$, as desired. Similarly, given that $\{\{\Gamma''(A \vee B)\}\} = \{\{\Gamma''(A) \cup \Gamma''(B)\}\}$, then $\{\{\Gamma'; \Gamma''(A \vee B)\}\} = \{x \in P : (\exists y \in \{\{\Gamma'\}\})(\exists z \in \{\{\Gamma''(A \vee B)\}\})[y, z]Rx\}$, which is equal to $\{x \in P : (\exists y \in \{\{\Gamma'\}\})(\exists z \in \{\{\Gamma''(A)\}\} \cup \{\{\Gamma''(B)\}\})[y, z]Rx\}$, which is $\{x \in P : (\exists y \in \{\{\Gamma'\}\})(\exists z \in \{\{\Gamma''(A)\}\})[y, z]Rx\} \cup \{x \in P : (\exists y \in \{\{\Gamma'\}\})(\exists z \in \{\{\Gamma''(B)\}\})[y, z]Rx\}$, which is in turn $\{\{\Gamma'; \Gamma''(A)\}\} \cup \{\{\Gamma'; \Gamma''(B)\}\}$. Finally, given that $\{\{\Gamma''(\bot)\}\} = \emptyset$, clearly $\{\{\Gamma'; \Gamma''(\bot)\}\} = \{x \in P : (\exists y \in \{\{\Gamma'\}\})(\exists z \in \{\{\Gamma''(\bot)\}\})[y, z]Rx\} = \{x \in P : (\exists y \in \{\{\Gamma'\}\})(\exists z \in \emptyset)[y, z]Rx\} = \emptyset$, as desired. □

Now we can return to our proof of the soundness theorem. As is usual, it is a straightforward induction on the length of a derivation. The technique is standard, and there are no surprises, despite the idiosyncratic interpretation of sequents to allow for the non-reflexive frames.[22]

Proof. We prove soundness by induction on the length of a derivation for the sequent $\Gamma \succ A$. The axiomatic sequent $A \succ A$ holds in every multiset frame and in every inhabited-multiset frame since $\{\{A\}\} \subseteq [\![A]\!]$. The sequent $\varepsilon \succ \mathrm{t}$ holds in every multiset frame, since in these frames we have $\{\{\varepsilon\}\} \subseteq [\![\mathrm{t}]\!]$.

For the *Cut* rule, suppose we have $\{\{\Gamma\}\} \subseteq [\![A]\!]$ and $\{\{\Gamma'(A)\}\} \subseteq [\![B]\!]$. We wish to show that $\{\{\Gamma'(\Gamma)\}\} \subseteq [\![B]\!]$. Here we appeal to the fact that the context Γ' preserves order. Since $\{\{\Gamma\}\} \subseteq [\![A]\!]$, we have $\{\{\Gamma'(\Gamma)\}\} \subseteq \{\{\Gamma'(A)\}\}$, and since $\{\{\Gamma'(A)\}\} \subseteq [\![B]\!]$, we have $\{\{\Gamma'(\Gamma)\}\} \subseteq [\![B]\!]$ as desired.

That the *extensional* structural rules preserve validity on frames is an immediate consequence of the fact that the outer context $\Gamma(–)$ preserves order, and the extensional structure is modelled by intersection of shadows. For example, for the weakening rule *EK*, since $\{\{\Gamma', \Gamma''\}\} = \{\{\Gamma'\}\} \cap \{\{\Gamma''\}\} \subseteq \{\{\Gamma'\}\}$, and since $\Gamma(–)$ preserves order, we know that if $\{\{\Gamma(\Gamma')\}\} \subseteq [\![B]\!]$ then we also have $\{\{\Gamma(\Gamma', \Gamma'')\}\} \subseteq [\![B]\!]$. In the same way, associativity, commutativity and contraction are assured.

Most of the *intensional* structural rules follow in the same way from the properties of multisets. For example, the associativity rule *IB* follows appeals to the compositionality of R. $\{\{(\Gamma'; \Gamma''); \Gamma'''\}\} = \{x \in P : (\exists y \in \{\{(\Gamma'; \Gamma'')\}\})(\exists z \in \{\{\Gamma'''\}\})[y, z]Rx\}$ unpacking the definition of $\{\{\Gamma'; \Gamma''\}\}$ this set is identical to $\{x \in P : (\exists y \in P)(\exists u \in$

[22] The soundness proof shows that although the frames may be non-reflexive, the resulting logic is, nonetheless, reflexive, in the sense discussed by French [20].

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$\{\{\Gamma'\}\})(\exists v \in \{\{\Gamma''\}\})([u, v]Ry \wedge [y, z]Rx)\}$. Applying compositionality, we see that $(\exists y \in P)([u, v]Ry \wedge [y, z]Rx)$ is equivalent to $[u, v, z]Rx$, using *Transitivity* in one direction and *Splitting* in the other. Thus, the set $\{\{(\Gamma'; \Gamma''); \Gamma'''\}\}$ simplifies (as expected) to $\{x \in P : (\exists u \in \{\{\Gamma'\}\})(\exists v \in \{\{\Gamma''\}\})(\exists z \in \{\{\Gamma'''\}\})[u, v, z]Rx\}$ where the left-associated structure $(\Gamma'; \Gamma''); \Gamma'''$ unwraps into the unassociated multiset $[u, v, z]$. A moment's reflection shows that the right-associated structure $\Gamma'; (\Gamma''; \Gamma''')$ unwraps to exactly the same set, so $\{\{(\Gamma'; \Gamma''); \Gamma'''\}\} = \{\{\Gamma'; (\Gamma''; \Gamma''')\}\}$, showing that the associativity structural rule *IB* is valid on frames. It is simpler to show that *IC* holds, since $\{\{\Gamma'; \Gamma''\}\} = \{\{\Gamma''; \Gamma'\}\}$ straightforwardly, given that $[y, z] = [z, y]$ for each y and z.

The εI and εE rules hold in models on multiset frames (but not in models on inhabited-multiset frames). Here, we have $\{\{\varepsilon; \Gamma'\}\} = \{\{\Gamma'\}\}$ since $\{\{\varepsilon\}\} = \{x \in P : [\,]Rx\}$ and so $\{\{\varepsilon; \Gamma'\}\} = \{x \in P : \exists y([\,]Ry \wedge \exists z \in \{\{\Gamma'\}\}[y, z]Rx)\}$. However, if $[\,]Ry$ and $[y, z]Rx$ then by transitivity, $[\,] \cup [z]Rx$, i.e., $[z]Rx$. And conversely, by *Splitting*, if $[z]Rx$ then $[\,] \cup [z]Rx$ and so, there is some y where $[\,]Ry$ and $[y, z]Rx$. So, our set $\{x \in P : \exists y([\,]Ry \wedge \exists z \in \{\{\Gamma'\}\}[y, z]Rx)\}$ *is* the set $\{x \in P : \exists z \in \{\{\Gamma'\}\}[z]Rx\}$, which is $\{\{\Gamma'\}\}$ itself, by Lemma 21.

It remains to verify the validity of each of the connective rules. The validity of the *left* rules for conjunction, disjunction, and fusion follow immediately from the truth conditions for these connectives and the fact that the context $\Gamma(\text{–})$ preserves order. For example, for $\circ L$, if we know that $\{\{\Gamma(A; B)\}\} \subseteq [\![C]\!]$ holds in the model, then since $\{\{A \circ B\}\} = \{x \in P : (\exists y \in [\![A \circ B]\!])[y]Rx\} = \{x \in P : (\exists w \in \{\{A\}\})(\exists v \in \{\{B\}\})[w, v]Rx\} = \{\{A; B\}\}$, and the context $\Gamma(\text{–})$ preserves order, it follows that $\{\{\Gamma(A \circ B)\}\} \subseteq [\![C]\!]$ too. The reasoning for the left rules for conjunction is similar, and so is the left rule for t when our attention is restricted to multiset frames.

The reasoning for the left rule for disjunction follows immediately from the primeness of the context $\Gamma(\text{–})$. If $\{\{\Gamma(A)\}\} \subseteq [\![C]\!]$ and $\{\{\Gamma(B)\}\} \subseteq [\![C]\!]$ then indeed $\{\{\Gamma(A \vee B)\}\} = \{\{\Gamma(A)\}\} \cup \{\{\Gamma(B)\}\} \subseteq [\![C]\!]$. The left rule for $\perp$ is trivial, given that $\{\{\perp\}\} = \emptyset$.

For the last left rule, for the conditional, to show that $\{\{\Gamma\}\} \subseteq [\![A]\!]$ and $\{\{\Gamma'(B)\}\} \subseteq [\![C]\!]$ ensures that $\{\{\Gamma'(A \to B; \Gamma)\}\} \subseteq [\![C]\!]$, we appeal to the fact that $\Gamma'(\text{–})$ preserves order. Using this fact, it suffices to show that $\{\{A \to B; \Gamma\}\} \subseteq [\![B]\!]$, for then we indeed have $\{\{\Gamma'(A \to B; \Gamma)\}\} \subseteq \{\{\Gamma'(B)\}\} \subseteq [\![C]\!]$ as desired. That $\{\{A \to B; \Gamma\}\} \subseteq [\![B]\!]$ follows from $\{\{\Gamma\}\} \subseteq [\![A]\!]$ by the definition of shadows for intensional combination. If $x \in \{\{A \to B; \Gamma\}\}$ then there are y and z where $y \in \{\{A \to B\}\}$ and $z \in \{\{\Gamma\}\}$ such that $[y, z]Rx$. Since $\{\{\Gamma\}\} \subseteq [\![A]\!]$ we have $z \Vdash A$. Since $y \in \{\{A \to B\}\}$ we have $y \Vdash A \to B$. It follows from $[y, z]Rx$ that $x \Vdash B$, i.e., $x \in [\![B]\!]$, as desired.

That completes the verification of the left connective rules. The right rules $\vee R$ and $\wedge R$ follow immediately from the truth conditions for the connectives, and we have already dealt with $\mathsf{t}R$ as an axiom. For $\to R$ and $\circ R$ the verification is also straightforward. For $\circ R$, if $\{\{\Gamma\}\} \subseteq [\![A]\!]$ and $\{\{\Gamma'\}\} \subseteq [\![B]\!]$, we wish to show that $\{\{\Gamma; \Gamma'\}\} \subseteq [\![A \circ B]\!]$. If $x \in \{\{\Gamma; \Gamma'\}\}$ then there are y, z where $[y, z]Rx$, $y \in \{\{\Gamma\}\}$ and $z \in \{\{\Gamma'\}\}$. So, we also have $y \in [\![A]\!]$ and $z \in [\![B]\!]$, so $x \in [\![A \circ B]\!]$ as desired. For $\to R$, suppose $\{\{\Gamma; A\}\} \subseteq [\![B]\!]$. To show that $\{\{\Gamma\}\} \subseteq [\![A \to B]\!]$, suppose we have $x \in \{\{\Gamma\}\}$. To show that $x \in [\![A \to B]\!]$, suppose we have a y where $y \Vdash A$ and $[x, y]Rz$. By *Splitting*, we have some y' where $[y]Ry'$ and $[x, y']Rz$. Since $y \in [\![A]\!]$ and $[y]Ry'$ we have $y \in \{\{A\}\}$, and since $x \in \{\{\Gamma\}\}$ and $[x, y']Rz$ we have $z \in \{\{\Gamma; A\}\}$, so $z \in [\![B]\!]$, as desired.

This completes the proof. Each rule of the sequent calculus is sound on multiset frames. So, if a sequent $\Gamma \succ A$ can be derived in RW^+, on any multiset frame (whether reflexive or not) we have $\{\{\Gamma\}\} \subseteq [\![A]\!]$. Furthermore, if that sequent can be derived in $\mathsf{RW}^+_{-\varepsilon}$, it also holds on any inhabited-multiset frame. □

It is worth remarking on the role of the *Transitivity* portion of compositionality in the proof of the soundness of the Cut rule. That case is handled by appeal to Lemma 27, the fact that contexts preserve order. Inspection of the proof of Lemma 27 reveals that it hinges on the monotony and idempotency of the $\sqsubset$ operator, Lemma 20. Showing that $\sqsubset$ is idempotent, in particular, appeals to the density and transitivity of $\sqsubset$. That appeal does not use the full *Transitivity* principle, but rather a restricted form involving only singletons on the left, much as density is a restricted form of the full *Splitting* principle. These observations suggest that collection frames that adopt only the restricted forms of *Transitivity* and *Splitting* may be of interest for the study of weaker logics.[23]

Before proceeding with further results, let's put this soundness proof to work, by showing how to use some of the frames we have constructed can provide counterexamples to sequents.

EXAMPLE 28 (Refuting $p \wedge (p \to q) \succ q$ and $s \succ r \to s$). *Start with $\langle \mathbb{R}, R\rangle$, where XRy iff $y > \Sigma X$. This is a non-reflexive frame on $\mathbb{R}$, in which the underlying order on points is $<$. So, extensions of formulas are the intervals $[r, \infty)$ or (r, ∞) closed or open at the left, together with $\mathbb{R}$ as a whole and the empty set. If we take $[\![p]\!] = [1, \infty)$ and $[\![q]\!] = [2, \infty)$, then we have $x \Vdash p \to q$ iff for each y, if $y \Vdash p$ (that is, if $y \geq 1$) and $x + y < z$, we have $z \Vdash q$ (that is, $z \geq 2$). It is easy to see that this obtains when $x \geq 1$, but if $x < 1$, we can find some value of* y, *(e.g., 1) and a value of* z *(e.g., $1 + x$) such that $x + y < z$ and $z \not\geq 2$. So, $[\![p \to q]\!] = [1, \infty)$. So, in particular, $1 \Vdash p \wedge (p \to q)$, and so, for example, $1\frac{1}{2} \in \{\{p \wedge (p \to q)\}\}$ and $1\frac{1}{2} \notin [\![q]\!]$. So this model provides a counterexample to the sequent $p \wedge (p \to q) \succ q$. As we would expect in at least some frames for* RW^+, *we have a violation of contraction.*

This frame also provides a counterexample to sequents involving failures of relevance, such as $s \succ r \to s$. If we set $[\![r]\!] = [-3, \infty)$ and $[\![s]\!] = [0, \infty)$ then it is easy to see that $1 \in \{\{s\}\}$, while $1 \notin [\![r \to s]\!]$, since $-3 \Vdash r$ and $[-3, 1]R-1$ (since $-3 + 1 = -2 < -1$) and $-1 \nVdash s$. These simple numerical frames provide the leeway to explore a number of the distinctive features of the substructural logic RW^+.

Another result that follows immediately is the fact that RW^+ is a non-conservative extension of $\mathsf{RW}^+_{-\varepsilon}$. The sequent $(A \to A) \to B \succ B$ is derivable in RW^+ as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \succ A}{\varepsilon; A \succ A}\,\varepsilon I}{\varepsilon \succ A \to A}\,{\to}R \qquad B \succ B}{(A \to A) \to B; \varepsilon \succ B}\,{\to}L}{\varepsilon; (A \to A) \to B \succ B}\,IC}{(A \to A) \to B \succ B}\,\varepsilon E.$$

[23] We would like to thank Dave Ripley for pushing us for clarify the issues discussed in this paragraph.

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This derivation makes use of ε. It might be asked whether *any* RW^+ derivation of this sequent must go through ε in this way. Inhabited-multiset frames give us an answer. This sequent is *not* derivable in $\mathsf{RW}^+_{-\varepsilon}$.[24]

EXAMPLE 29 (RW^+ is not conservative over $\mathsf{RW}^+_{-\varepsilon}$). *Consider $\langle P, R\rangle$ where P is the set $\{1, 2, 3, \ldots\}$ of positive natural numbers, and for inhabited multisets X, XRy iff $y = \Sigma X$. R, defined in this way, is both compositional and reflexive. This is an inhabited-multiset frame. The underlying order $\sqsubset$ is identity, so any set of points may be used as the extension of a formula. Define $\Vdash$ by setting $[\![p]\!] = P$ (so p is true everywhere) and $[\![q]\!] = P\backslash\{1\}$ (so q holds everywhere other than 1). In this model $[\![p \to p]\!] = P$, trivially. It follows that $(p \to p) \to q$ is true at every number $n \geq 1$, too, since for any such* n, *and for any $m \geq 1$ where $m \Vdash p \to p$ (i.e., for any $m \geq 1$) then $n + m \Vdash q$, since clearly, $n + m \geq 2$. So, we have a counterexample to our sequent $(p \to p) \to q \succ q$ on our model. In particular, we have $1 \in \{\{(p \to p) \to q\}\}$ while $1 \notin [\![q]\!]$.*

If we wish to model the stronger logic R^+, we must restrict our attention to a smaller class of multiset frames. In ternary relational semantics, the traditional frame condition to impose on RW^+ models to validate contraction is $Rxxx$. Its analogue in multiset frames is straightforward: $[x, x]Rx$. Once we have non-reflexive frames in view, however, we can see that this frame condition is not general enough. A more *general* form of contraction on ternary frames is this condition:

$$Rxyz \Rightarrow R^2(xy)xz$$

corresponding to the validity of the sequent $A \to (A \to B) \succ A \to B$. If we choose a normal point for y, then the condition becomes

$$x \sqsubseteq z \Rightarrow Rxxz$$

which, in the presence of reflexivity gives us $Rxxx$ for every x. In the *absence* of reflexivity, no such consequence need follow. In the multiset frame on $\mathbb{R}$ where we set XRy iff y is greater than every member of X, it is clear that whenever $x \sqsubset z$ (that is, $x < z$) we have $Rxxz$ (that is, $x < z$). However, we never have $Rxxx$ on this frame.

The appropriate understanding of contraction on arbitrary multiset frames, whether reflexive or not, is simple. A multiset rendering of the condition goes like this:

$$[x]Rz \Rightarrow [x, x]Rz.$$

The relation R is preserved when the multiset expands from one repetition of x to two. If R is compositional, this condition will continue to hold in a more general form, using the ground function g from Definition 1:

LEMMA 30 (Preservation for contracting relations). *Whenever R is compositional multiset relation where $[x]Rz \Rightarrow [x, x]Rz$ for every* x *and* z *then if XRy and X' is another multiset where $X \leq X'$ and $g(X) = g(X')$, then $X'Ry$ too.*

[24] In Humberstone's terms [24, sec. 1.2], logical frameworks apart from FMLA need to be considered for inhabited collection frames.
See Shay Logan's "Deep Fried Logic" [29] for more on $\mathsf{RW}^+_{-\varepsilon}$.

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Proof. Recall that $X \leq X'$ iff any object that is an element of X i times is a member of X' at *least* that many times. The constraint that $g(X') = g(X)$ means that the only elements with non-zero multiplicity in X' have non-zero multiplicity in X too. So, X' differs from X only by allowing elements that were *already* in X to be in X' more times. Since X and X' are finite multisets, if we prove that XRy implies $([x] \cup X)Ry$, when $x \in X$, we can repeat this process until we have built X' from X in a series of additions of single elements.

Now, if XRy and $x \in X$, then we have $([x] \cup (X \backslash x))Ry$. By *Splitting* there is some z where $[x]Rz$ and $([z] \cup (X \backslash x))Ry$. Since $[x]Rz$ we have $[x, x]Rz$, and so, by *transitivity*, $([x, x] \cup (X \backslash x))Ry$, i.e., $([x] \cup X)Ry$, as desired. Applying this process repeatedly, for each additional element in X', we see that $X'Ry$, and we have completed the proof. □

To show that R^+ is indeed sound for contracting multiset frames, we need to verify that on each model on such a frame $\{\{\Gamma\}\} \subseteq \{\{\Gamma; \Gamma\}\}$. But this is immediate: let's suppose that $x \in \{\{\Gamma\}\}$. Then by Lemma 21, there is some $[y]Rx$ where $y \in \{\{\Gamma\}\}$ too. Now, since $[y]Rx$, we have $[y, y]Rx$ and so, we have that $x \in \{\{\Gamma; \Gamma\}\}$. With this reasoning, the soundness result for R^+ on contracting multiset frames is proved.

THEOREM 31 (R^+ is sound for contracting multiset frames). *Any sequent* $\Gamma \succ A$ *derivable in* R^+ *also holds in each model* $\langle P, R, \Vdash\rangle$ *on a contracting multiset frame.*

For completeness, we need to show that if a sequent holds in all multiset frames then it is derivable in RW^+, and that if a sequent holds in all contracting multiset frames, then it is derivable in R^+. As is usual, the most straightforward way to prove completeness is to prove the contrapositive, by showing that for any *underivable* sequent, one can find a counterexample in some frame. In the case of the ternary relational semantics, as with Kripke models for normal modal logics and intuitionistic logics, this is achieved by constructing the *canonical* frame [41, 45–47], whose points are prime theories,[25] and where the normal points are those theories containing all logical truths, where $\sqsubseteq$ is the subset relation, and where R is defined syntactically: $R\alpha\beta\gamma$ iff for each $A \to B \in \alpha$, if $A \in \beta$ then $B \in \gamma$. It is a standard result that membership is an evaluation relation on the canonical frame, defining $\alpha \Vdash A$ iff $A \in \alpha$, which gives us a relation satisfying the expected truth conditions, and that that any underivable sequent has a counterexample in the resulting canonical *model*. In addition, the RW^+ canonical frame satisfies the RW^+ conditions on the ternary relation, and the R^+ canonical frame satisfies the contraction condition. So, we can appeal to Lemma 16, to show that the canonical ternary relational model for RW^+ (or for R^+) will also provide a multiset model (or contracting multiset model), which gives exactly the same truth conditions on points, and so, counterexamples to the same sequents. So, we have completeness for free:

THEOREM 32 (Completeness for multiset frames). *Each sequent that holds on every reflexive multiset frame is derivable in* RW^+. *Furthermore, each sequent that holds on every contracting, reflexive multiset frame is derivable in* R^+.

25 A set of formulas is a theory iff it is closed under conjunction introduction and provable implication, and a theory is *prime* iff it contains a disjunction only if it contains at least one disjunct.

So, multiset frames provide an elegant, simple class of models for RW^+, unifying the parts of the ternary relational frames. The compositionality condition on the multiset relation R is a natural generalisation of the condition that inclusion ($\sqsubseteq$) be a preorder, to the general setting that we relate a *collection* of points to a point. The generalisation goes so far as to include models in which the underlying order is not even reflexive.

However, not all collections are *multisets*. In the rest of this paper, we will show that we can generalise these results to other kinds of collections in a natural way. We will start by considering *sets*.

§3. Set frames. Once you understand multiset frames, it is straightforward to define *set* frames. We start with the definition of compositionality for relations on $\mathcal{P}^*(P) \times P$, where $\mathcal{P}^*(P)$ is the set of finite subsets of P.

Definition 33 (Compositionality for set relations). *A relation R on $\mathcal{P}^*(P) \times P$ is said to be* compositional *if and only if for all sets X, Y and all points x and z,*

$$\textit{if } XRx \textit{ and } (\{x\} \cup Y)Rz \textit{ then } (X \cup Y)Rz.$$

Such a set relation R is reflexive *iff for all points $x \in P$, we have*

$$\{x\}Rx.$$

We have replaced talk of multisets of elements of P with finite subsets of P. The compositional multiset relations discussed in Example 4 can be all reframed as *set* relations. *Membership*, *Maximum*, *The Product*, *Some Product of* and *Between* can all be defined as *set* relations on ω, and each is set relation so defined is compositional.

The novelty with *set* relations, as opposed to multiset relations, is that they are, by construction, contracting. There is no difference at all between $\{x, x\}Ry$ and $\{x\}Ry$, and since by reflexivity, we have $\{x\}Rx$, it follows that $\{x, x\}Rx$ holds in every compositional set relation R. Once we define the notion of a set frame, and the corresponding notion of a set model, it will follow immediately that R^+ is sound for set models.

Definition 34 (Set frames and set models). *If P is an inhabited set and R is a compositional* set *relation on P, then $\langle P, R\rangle$ is said to be a* set frame. *If $\Vdash$ is a relation between the set P and the set of atomic formulas, which is hereditary along R (so if $x \Vdash p$ and $\{x\}Ry$ then $y \Vdash p$ too), then $\langle P, R, \Vdash\rangle$ is said to be a* set model, *where $\Vdash$ evaluates all formulas in the language of R^+ as follows*:

- $x \Vdash A \wedge B$ *iff* $x \Vdash A$ *and* $x \Vdash B$.
- $x \Vdash A \vee B$ *iff* $x \Vdash A$ *or* $x \Vdash B$.
- $x \Vdash A \to B$ *iff for each y, z where $\{x, y\}Rz$, if $y \Vdash A$ then $z \Vdash B$.*
- $x \Vdash A \circ B$ *iff for some y, z where $\{y, z\}Rx$, both $y \Vdash A$ and $z \Vdash B$.*
- $x \Vdash \mathsf{t}$ *iff* $\{\,\}Rx$.
- $x \Vdash \bot$ *never.*

As with multiset models, the evaluation relation $\Vdash$ on set models is hereditary across the relation R. And as with contracting multiset models, the logic R^+ is sound for *set*. The soundness proof for RW^+ can be rewritten, word-for-word, with set singletons and set union replacing multiset singletons and multiset union. Furthermore, any relation compositional *set* relation R satisfies the contraction condition *vacuously*,

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so the contraction rule preserves validity on all set models. We have the following soundness theorem for free:

THEOREM 35 (R^+ is sound for set frames). *Any* R^+ *derivable sequent* $\Gamma \succ A$ *holds in each model* $\langle P, R, \Vdash\rangle$ *on a set frame.*

A natural question arises. Is R^+ *complete* for set frames? Here, any completeness theorem will not be quite as straightforward as in the case for multiset frames and RW^+. We cannot simply take the canonical frame and show that it is a set frame. In general, contracting ternary frames (or contracting multiset frames) do not turn out to be equivalent to set frames. In any set frame we have $\{x, x\}Ry$ if and only if $\{x\}Ry$ trivially, but the corresponding biconditional—$Rxxy$ if and only if $x \sqsubseteq y$ (or $[x, x]Ry$ iff $[x]Ry$)—does *not* hold in all ternary relational frames for R^+, or on all contracting multiset frames. In general, only one direction of the biconditional holds.

EXAMPLE 36 (A ternary R^+ frame that isn't (equivalent to) a set frame). *The frame on the set* $P = \{0, a, b\}$ *of points with* $N = \{0\}$, *where* $\sqsubseteq$ *is identity and where* R *is defined with the following table*

R	0	a	b
0	0	a	b
a	a	ab	$0ab$
b	b	$0ab$	ab

is not equivalent to a set frame. To read the table, the values in the x *row and* y *column in the table are the different values of* z *such that* $Rxyz$. *So, the* ab *in the* a *row and* a *column indicates that* $Raaa$ *and* $Raab$. *It is not too difficult to check that this is a ternary* R^+ *frame (it is associative, commutative and contracting), but it does not satisfy the condition needed for equivalence to a set frame: that* $Rxxy$ *if and only if* $x \sqsubseteq y$. *Here,* $Raab$, *but* $a \not\sqsubseteq b$. *This is a ternary frame that is not equivalent to a set frame.*

What goes for this ternary frame can go for the canonical frame for R^+.[26] So, there is no guarantee that any canonical frame for R^+ will be (equivalent to) a set frame. This raises the question of whether set frames overgenerate, whether they determine a logic stronger than R^+. It might be thought that the stronger frame condition induced on a set frame means that the *mingle* axiom $p \to (p \to p)$ (which is equivalent to $(p \circ p) \to p$) holds on our frames. It fails, as the following example shows.

EXAMPLE 37 (A set frame counterexample to mingle). *Consider the inhabited-set frame on* ω, *where* XRy *iff* y *is in the interval bound by the set* X. *So, for example, the set* $\{0, 2\}$ *is related to* 0, 1 *and* 2 *but no other elements of* ω. *This is frame on inhabited sets. We can then extend this frame to construct a set frame using the technique of Lemma* 8, *by adjoining an element* ∞ *and choosing the* $R^\times$ *extension of the relation* R. *Here,* $\{\,\}R^\times\infty$ *and* $\{\infty\}R^\times\infty$ *and for every other set* X, $XR^\times z$ *iff* $(X\backslash\infty)Rz$, *to make this a model for the whole of* R^+, *including* $\mathtt{t}$. *In this model, the order relation* $\sqsubseteq$ *is the identity relation, since* $\{x\}R^\times y$ *iff* $y = x$, *for every* x *(including* ∞*). Let's take* $[\![p]\!] = \{0, 2\}$. *Then it is straightforward that* $0 \Vdash p$ *but* $0 \not\Vdash p \to p$, *since* $2 \Vdash p$ *and* $1 \not\Vdash p$

[26] In fact, this ternary R^+ frame *is* isomorphic to the canonical frame constructed from a small R^+ algebra on eight elements—the eight subsets of $\{0, a, b\}$, the propositions defined on that frame, and under the operations, conjunction, disjunction, conditional, fusion, $\mathtt{t}$ and $\bot$, defined by the truth conditions on that frame.

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and $\{0,2\}R^{\times}1$. *So, since* $\{0,\infty\}R^{\times}0$, *we have* $\infty \not\Vdash p \to (p \to p)$, *and since* $\{\ \}R^{\times}\infty$, *we see that* $p \to (p \to p)$ *fails at a normal point* (*at the* only *normal point*, ∞), *giving us a counterexample to the sequent* $\varepsilon \succ p \to (p \to p)$, *as desired.*

So, set frames are sound for R^+, but the standard techniques for completeness do not suffice to show completeness for R^+. It seems we must use another approach, or find some way that these frames overgenerate R^+. We will not, however, settle the question here, so we leave it as a topic for further research.[27]

[OPEN QUESTION] Is R^+ complete for the class of all set frames?

As with multiset frames, we can move from set frames to *inhabited*-set frames, if we loosen the requirement that the compositional relation R relates the *empty* set to points. All of the results concerning inhabited-multiset frames generalise to inhabited-set frames. Inhabited-set frames on the real plane are surprisingly straightforward to construct, and they have interesting properties of their own.[28] We have the following result.

THEOREM 38 ($\mathsf{R}^+_{-\varepsilon}$ is sound for set frames). *Any* $\mathsf{R}^+_{-\varepsilon}$ *derivable sequent* $\Gamma \succ A$ *holds in each model* $\langle P, R, \Vdash\rangle$ *on an inhabited-set frame.*

Again, the proof of this theorem comes essentially for free, once we recognise that the structural rule IW of intensional contraction satisfied on inhabited-set frames. We can use an inhabited-set frame to show that the sequent $(A \to A) \to B \succ B$ *also* fails in $\mathsf{R}^+_{-\varepsilon}$, so R^+ fails to be conservative over $\mathsf{R}^+_{-\varepsilon}$, just as RW^+ is not conservative over $\mathsf{RW}^+_{-\varepsilon}$.

EXAMPLE 39 (R^+ is not conservative over $\mathsf{R}^+_{-\varepsilon}$). *This time, consider the inhabited-set frame* $\langle P, R\rangle$ *where* $P = \{0,1,2\}$ *and* XRy *holds when* y *is* bounded *by the set* X. *In other words,* XRy *if and only if* $\min(X) \le y \le \max(Y)$. *In this frame, the underlying order is identity, so the relation is reflexive, and any set of points is a possible extension of a formula. Let* $[\![p]\!] = \{0,2\}$. *Then at* $p \to p$ *is satisfied nowhere, since for any point* x *you choose, there is some point* y (*choose* 2 *if* x *is* 0, *and choose* 0 *otherwise*) *where* $y \Vdash p$, *and where* $\{x,y\}R1$, *where* $1 \not\Vdash p$. *So, at every point we have a counterexample to the identity statement* $p \to p$.

This means that every point in our frame supports $(p \to p) \to q$, *since* $p \to p$ *fails everywhere. In particular,* $1 \Vdash (p \to p) \to p$, *while* $1 \not\Vdash p$, *so* $(p \to p) \to p \succ p$ *fails on this frame, and hence, it is not derivable in* $\mathsf{R}^+_{-\varepsilon}$. A fortiori, *neither is the sequent* $(p \to p) \to q \succ q$.

§4. List frames and tree frames. Different collections gather their elements in different ways. Sets collect elements with no regard to order or multiplicity. Multisets allow their members to occur repeatedly, but there is no record of the order of their arrival. It is natural to consider collections that keep track of both multiplicity and

27 Standefer [52] has shown that the logic of *functional* set frames is sound and complete with respect to Urquhart's semilattice logic, which is a proper extension of the $\{\to,\wedge,\vee\}$-fragment of R^+.

28 Elsewhere [42], Restall explores features of frames on geometric spaces, and options for extending geometric set frames with new points to bring in the empty set, in case one simply cannot do without t and without ε.

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order: *lists*. The list $\langle a, a, b, c\rangle$ is distinct from the list $\langle a, b, a, c\rangle$, both of which are distinct from the list $\langle a, b, c\rangle$.

The definition given for compositionality in set and multiset frames generalises nicely to the context of list frames, but we will need to be careful when doing so: the definitions were not attentive to matters of ordering, so we will need to pay attention to that here when defining what it is to replace an *element* y occurring in some list by another list. To this we turn, now.

DEFINITION 40 (List composition). *If the list X is $\langle x_1, \dots, x_n\rangle$ and the inhabited list $Y(y_j)$ is $\langle y_1, \dots, y_j, \dots, y_m\rangle$, then $Y(X)$ is $\langle y_1, \dots, y_{j-1}, x_1, \dots, x_n, y_{j+1}, \dots, y_m\rangle$.*

Given an inhabited set P, the set $L(P)$ is the set of all finite lists of elements from P.

DEFINITION 41 (Compositionality). *A list relation R on $L(P) \times P$ is said to be* compositional *if and only if for all lists X and Y and for all points z,*

$$(\exists y)(XRy \text{ and } (Y(y))Rz) \text{ iff } (Y(X))Rz.$$

A list relation R is reflexive *iff for all points x, we have*

$$\langle x\rangle Rx.$$

As with multisets, a compositional list relation between inhabited lists and points adds the requirement that X be inhabited to the preceding definition. ($Y(y)$ must of course be inhabited, though of course it may just be the singleton list $\langle y\rangle$.)

DEFINITION 42 (List frames and list models). *If P is an inhabited set and R is a compositional list relation on P, then $\langle P, R\rangle$ is said to be a* LIST FRAME. *(If R is an inhabited-list relation, then this is an* INHABITED-LIST *frame.) If $\Vdash$ is a relation between the set P and the set of atomic formulas, which is hereditary along R (so if $x \Vdash p$ and $\langle x\rangle Ry$ then $y \Vdash p$ too), then $\langle P, R, \Vdash\rangle$ is said to be a* LIST MODEL, *where $\Vdash$ evaluates all formulas in the language of R^+ as follows:*

- *$x \Vdash A \wedge B$ iff $x \Vdash A$ and $x \Vdash B$.*
- *$x \Vdash A \vee B$ iff $x \Vdash A$ or $x \Vdash B$.*
- *$x \Vdash A \to B$ iff for each y, z where $\langle x, y\rangle Rz$, if $y \Vdash A$ then $z \Vdash B$.*
- *$x \Vdash A \circ B$ iff for some y, z where $\langle y, z\rangle Rx$, both $y \Vdash A$ and $z \Vdash B$.*
- *$x \Vdash \bot$ never.*

If R *is a list relation, and not merely an inhabited-list relation, we can add the* $\mathtt{t}$ *clause.*

- *$x \Vdash \mathtt{t}$ iff $\langle\ \rangle Rx$.*

We can define validity for sequents on our models in the usual way. In fact, the definitions the extension $[\![A]\!]$ of a formula A carries over unchanged in the setting of list frames, and the definition of the shadow $\{\{\Gamma\}\}$ of a structure requires only one small tweak, given the move from multiset or set frames to list frames. References to multisets must be replaced by the corresponding references to *lists*, as follows:

- $\{\{\varepsilon\}\} = \{x \in P : \langle\ \rangle Rx\}$,
- $\{\{A\}\} = \{x \in P : (\exists y \in [\![A]\!])\langle y\rangle Rx\}$,
- $\{\{\Gamma, \Gamma'\}\} = \{x \in P : (\exists y \in \{\{\Gamma\}\})(\exists z \in \{\{\Gamma'\}\})\langle y\rangle Rx \wedge \langle z\rangle Rx\}$,
- $\{\{\Gamma; \Gamma'\}\} = \{x \in P : (\exists y \in \{\{\Gamma\}\})(\exists z \in \{\{\Gamma'\}\})\langle y, z\rangle Rx\}$.

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With this, we can define validity on a model as before. The sequent $\Gamma \succ A$ is valid on $\langle P, R, \Vdash \rangle$ iff $\{\{\Gamma\}\} \subseteq [\![A]\!]$.

We have seen that logic RW^+ is sound and complete for multiset frames. The logic R^+ is sound and complete for multiset frames with contraction, and that R^+ is sound for set frames. A natural question is what logic is sound and complete for list frames. List frames will not validate the structural rules IC and IW, so the logic will be weaker than RW^+. One might think that list frames validate TW^+, a close relative of RW^+ that eschews the structural rules IC and IW, but that thought is not borne out, as we will show.

LEMMA 43. *The following structural rules are valid on list frames.*

$$\frac{\Gamma(\Gamma'; (\Gamma''; \Gamma''')) \succ B}{\Gamma((\Gamma'; \Gamma''); \Gamma''') \succ B}\, IB \qquad \frac{\Gamma((\Gamma'; \Gamma''); \Gamma''') \succ B}{\Gamma(\Gamma'; (\Gamma''; \Gamma''')) \succ B}\, IBc.$$

Proof. It is straightforward to show that $\{\{\Gamma'; (\Gamma''; \Gamma''')\}\} = \{\{(\Gamma'; \Gamma''); \Gamma'''\}\}$, given the associativity of list composition, and the compositionality of the relation R. The proof used for Theorem 26 carries over here with only notational changes, like so: $\{\{(\Gamma'; \Gamma''); \Gamma'''\}\} = \{x \in P : (\exists y \in \{\{(\Gamma'; \Gamma'')\}\})(\exists z \in \{\{\Gamma'''\}\})\langle y, z\rangle Rx\}$ unpacking the definition of $\{\{\Gamma'; \Gamma''\}\}$ this set is identical to $\{x \in P : (\exists y \in P)(\exists u \in \{\{\Gamma'\}\})(\exists v \in \{\{\Gamma''\}\})(\langle u, v\rangle Ry \wedge \langle y, z\rangle Rx)\}$. Applying compositionality, we see that $(\exists y \in P)(\langle u, v\rangle Ry \wedge \langle y, z\rangle Rx)$ is equivalent to $\langle u, v, z\rangle Rx$ so the set $\{\{(\Gamma'; \Gamma''); \Gamma'''\}\}$ simplifies (as expected) to $\{x \in P : (\exists u \in \{\{\Gamma'\}\})(\exists v \in \{\{\Gamma''\}\})(\exists z \in \{\{\Gamma'''\}\})\langle u, v, z\rangle Rx\}$ where the left-associated structure $(\Gamma'; \Gamma''); \Gamma'''$ unwraps into the unassociated $list\langle u, v, z\rangle$. Similarly, the right-associated structure $\Gamma'; (\Gamma''; \Gamma''')$ unwraps to exactly the same set, so $\{\{(\Gamma'; \Gamma''); \Gamma'''\}\} = \{\{\Gamma'; (\Gamma''; \Gamma''')\}\}$, showing that the associativity structural rule IB is valid on list frames. □

The structural rule IB is valid in ternary frames for TW^+, but the rule IBc is not. The latter rule can be used to derive the sequent $A \circ (B \circ C) \succ (A \circ B) \circ C$, which does not hold in TW^+.[29] Rather than the structural rule IBc, the usual structural rule paired with IB for TW^+ is the rule IB'.

$$\frac{\Gamma(\Gamma'; (\Gamma''; \Gamma''')) \succ B}{\Gamma((\Gamma''; \Gamma'); \Gamma''') \succ B}\, IB'.$$

This rule, despite its importance in the study of relevant logics, is not valid on inhabited list frames.

LEMMA 44. *The rule IB' is not valid on list frames.*

Proof. For the counterexample, let the frame be $\langle \omega, R\rangle$ on inhabited lists from ω, where $\langle x_1, \dots, x_n\rangle Ry$ iff $x_1 = y$. For this frame, $x \sqsubset y$ iff $x = y$, so any set of points is an extension, and this frame is reflexive. On this frame, set $[\![p]\!] = \{1\}$, $[\![q]\!] = \{2\}$ and $[\![r]\!] = \{3\}$. Let's check the validity of $(q; p); r \succ p \circ (q \circ r)$ on this model. Here, $\{\{q; p\}\} = \{x : \langle 2, 1\rangle Rx\} = \{2\}$, and so, $\{\{(q; p); r\}\} = \{x : \langle 2, 3\rangle Rx\} = \{2\}$, too. On the other hand, $[\![q \circ r]\!] = \{x : \langle 2, 3\rangle Rx\} = \{2\}$, and so, $[\![p \circ (q \circ r)]\!] = \{x : \langle 1, 2\rangle Rx\} = \{1\}$, and hence, $\{\{(q; p); r\}\} \not\subseteq [\![p \circ (q \circ r)]\!]$, and $(q; p); r \succ p \circ (q \circ r)$ is

[29] We will leave it to the reader to find a counterexample, for which we suggest using John Slaney's program `MaGIC` (http://users.cecs.anu.edu.au/~jks/magic.html).

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not valid on our model. Since it is derivable, using IB', this rule is not valid on list frames. □

This counterexample uses one natural way of forming inhabited list frames from a given set P of points and suggests another natural example.

> [First] Say that XRy iff $X = \langle x_1, \dots, x_n\rangle$ and $x = x_1$. $\langle x\rangle Rx$ clearly holds. It is only slightly more work to see that the compositionality condition, $\exists z(XRz$ and $Y(z)Ry)$ iff $Y(X)Ry$, holds.
> [Last] Say that XRy iff $X = \langle x_1, \dots, x_n\rangle$ and $x = x_n$.

Each compositional multiset relation R (on inhabited multisets, or on all multisets) can be lifted to a list relation (correspondingly, on inhabited lists, or all lists) too, where we set $XR'x$ if and only if $m(X)Rx$, where $m(X)$ is the multiset of members of the list X defined in the obvious way.[30] So, all of the other compositional multiset relations we have considered, like *sum*, *product*, *membership*, etc., transfer naturally to this setting, albeit without making any use of the distinctively *non-commutative* nature of the list structures being related. Another example of a functional compositional list relation is given by any semigroup.

Example 45 (Lifting a semigroup). *If $\langle P, *\rangle$ is a semigroup (if $*$ is an associative binary operation on P) then the inhabited-list relation R^*, given by setting $\langle x\rangle R^* y$ iff $x = y$ and $\langle x, x_1, \dots, x_n\rangle R^* y$ iff there is some z where $\langle x, x_1, \dots, x_n\rangle R^* z$ and $y = x * z$, is both compositional and functional. If, in addition, P is a monoid with identity* e, *then we can extend this to a list relation, setting $\langle\,\rangle R^* y$ iff $y = e$.*

The logic that is sound and complete for list is not TW^+, but rather the associative Lambek calculus.[31] Much work on associative Lambek calculus uses a different language than the one we have been considering, often with the addition of another conditional, $\leftarrow$ and without t.[32]

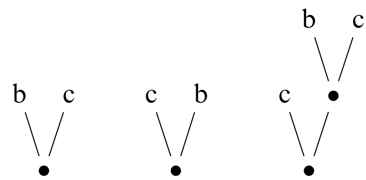
In the transition from multisets to lists, we noted that multisets take account of multiplicity but not order, whereas lists mind them both. There is still more structure to jettison. Lists are implicitly associative. For example, the list $\langle a, b, c\rangle$ is indifferent to whether it was formed by concatenating $\langle a\rangle$ with $\langle b, c\rangle$ or by concatenating $\langle a, b\rangle$ with $\langle c\rangle$. The final collections we will look at are ones that pay more attention to how the collections were formed, namely *trees*. We will focus on rooted binary-branching trees.

Leaf-labelled, rooted, binary-branching trees, or just *trees*, for the remainder of the section, are familiar objects. Given a set P of points, $T(P)$ is the set of all inhabited, finite trees where each node has exactly zero or two successors and each of whose leaves is labelled with an element of P. The trees will be oriented, so that they distinguish the left successor node from the right successor node. As an example, let $P = \{b, c\}$, then the following three (distinct) trees are elements of $T(P)$.

30 Here is the ‘obvious way’: $m(\langle\,\rangle) = [\,]$, $m(\langle x, X\rangle) = [x] \cup m(X)$.

31 Lambek [28] introduced two calculuses, one associative and one non-associative, the latter of which does not appear here.

32 For example, see [41, 307ff] or [35, 66ff]. See [12] for an early discussion of frames for Lambek calculus.



Rather than draw trees in a two-dimensional array, we will adopt a more compact notation, specifying the leaves of the tree by their labels. The example trees above would be represented as follows: (b, c), (c, b), and $(c, (b, c))$.[33] The following definition formalises this idea.

Definition 46 (Trees). *Given a set of points P, the binary trees over P are defined as follows.*

- *For all $x \in P$, x is a tree.*
- *If L and R are trees, (L, R) is a tree.*

To maintain the notational similarity with the other collections, we will use (x) for the singleton tree of x.

Definition 47 (Tree composition). *If X is an inhabited tree and $Y(x)$ is a tree with a distinguished leaf labelled x, then $Y(X)$ is the tree that results by replacing the leaf x with the tree X.*

As an example of tree composition, let X be (b, c) and let $Y(b)$ be (c, b). Then $Y(X)$ is $(c, (b, c))$, which is the rightmost tree in the diagram above, obtained by replacing the b node in the middle tree by the leftmost tree.

Definition 48 (Compositionality). *A tree relation* R *on $T(P) \times P$ is said to be* compositional *if and only if for all trees $X, Y \in T(P)$ and for all points z,*

$$(\exists y)(XRy \textit{ and } (Y(y))Rz) \textit{ iff } (Y(X))Rz.$$

A tree relation R *is* reflexive *iff for all points x, we have*

$$(x)Rx.$$

Definition 49 (Tree frame and tree model). *If P is an inhabited set and R is a compositional* tree *relation on P, then $\langle P, R\rangle$ is said to be a* tree frame. *If $\Vdash$ is a relation between the set P and the set of atomic formulas, which is hereditary along R (so if $x \Vdash p$ and $(x)Ry$ then $y \Vdash p$ too), then $\langle P, R, \Vdash\rangle$ is said to be a* list model, *where $\Vdash$ evaluates all formulas in the language of* R^+ *as follows:*[34]

- *$x \Vdash A \wedge B$ iff $x \Vdash A$ and $x \Vdash B$.*
- *$x \Vdash A \vee B$ iff $x \Vdash A$ or $x \Vdash B$.*
- *$x \Vdash A \to B$ iff for each y, z where $(x, y)Rz$, if $y \Vdash A$ then $z \Vdash B$.*
- *$x \Vdash A \circ B$ iff for some y, z where $(y, z)Rx$, both $y \Vdash A$ and $z \Vdash B$.*

33 The linear notation for trees has a natural connection to combinatory terms, and so to combinatory logic. For an introduction to combinatory logic, see [6]. We would like to thank an anonymous referee for pointing out this connection.

34 The clause for $\perp$ can be added. It is omitted here since the other propositional constant we have been considering, t, is not included.

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Tree frames are rather easy to come by. Here are two examples.

[GROUPOID] Let $(G, \cdot)$ be a groupoid. To define R, we will use a mapping τ from $T(G)$ to G as follows: $\tau((x)) = x$ and $\tau((X, Y)) = \tau(X) \cdot \tau(Y)$. Define R as follows: $(x)Rx$, for all x, and XRy iff $\tau(X) = y$. It is straightforward to see that R is a compositional.

[JOIN SEMI-LATTICE] Let $(S, +)$ be a join semi-lattice. Define $x \preceq y$ iff $x + y = y$. Set $(x)Rx$ and, adapting τ from the previous example, define XRy iff $\tau(X) \preceq y$. As in the previous example, it is straightforward to show that R is compositional.

We will now relate the tree frames to some more standard ternary frames. For this, we will introduce some notation using square brackets, which should not be confused for multisets as in earlier sections: Here, $Y[x]$ is to be understood as the tree Y with a distinguished leaf x, while $Y[x, y]$ is Y with two distinguished leaves, $Y[(x, y)]$ is Y with a distinguished pair of adjacent leaves (x, y), $Y[(x, y), (u, v)]$ with two distinguished pairs of adjacent nodes, and $Y[x, (y, z)]$ is a distinguished triple of leaves, where one is adjacent to a pair.

LEMMA 50. *Each ternary frame $\langle P, R, \sqsubseteq, N\rangle$ determines a reflexive tree frame $\langle P, R'\rangle$ by setting*:

- $(x)Ry$ *iff* $x \sqsubseteq y$.
- $(x, y)R'z$ *iff* $Rxyz$.
- *If Y is a tree with two or more leaves, then $Y[(x, y)]R'z$ iff for some* u, $Y(u)R'z$ *and* $(x, y)R'u$.

You will notice here that there is nothing in the tree frame that corresponds to the set N of normal points, since our trees are essentially inhabited.

Proof. The proof proceeds much as in the proof of Lemma 12. We need to verify that R' is coherent. There is nothing to check for clause 2.

To check the final clause, we need to prove that if $Y[(x, y)]$ is the same tree as $Y'[(x', y')]$, then

$$(\exists z)(Y(z)R'u \wedge (x, y)R'z) \text{ iff } (\exists z')(Y'(z')R'u \wedge (x', y')R'z')).$$

If Y has one leaf, then $x = y = x' = y'$, and the displayed biconditional is satisfied by the first and second clauses of the definition. If Y has either two or three leaves, then $x = x'$ and $y = y'$, and the displayed biconditional is satisfied.

Let $X[(x, y), (x', y')]$ be the tree Y with the two distinguished pairs of adjacent leaves (x, y) and (x', y'). Assume X has $n > 3$ leaves. Suppose $(\exists z)(X[z, (x', y')]R'u \wedge (x, y)R'z)$. The tree $X[z, (x', y')]$ has $n - 1$ leaves, so by the inductive hypothesis, this is equivalent to $(\exists z)(\exists z')(X[z, z']R'u \wedge (x, y)R'z \wedge (x', y')R'z')$. This, in turn, is equivalent to $(\exists z')(X[(x, y), z']R'u \wedge (x', y')R'z')$ by the inductive hypothesis, which establishes the desired biconditional.

The reflexivity condition on R' is satisfied by the reflexivity of $\sqsubseteq$ and an argument similar to that of Lemma 12 establishes the compositionality conditions. □

So, every ternary frame generates a tree frame. A straightforward inductive argument shows that the extensional structural rules are all sound for tree frames, as are the operational rules, excluding the rules for t and for ε. This suffices for the adequacy of the logic $\mathsf{B}^+_{-\varepsilon}$, given by the connective rules and the extensional structural rules, but

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without the intensional structural rules and IC, IB, and without εI or εE, with respect to tree frames.

THEOREM 51. *The logic* $\mathsf{B}^+_{-\varepsilon}$ *is sound and complete with respect to tree frames.*

To model the basic substructural logic B^+, we need to add εI and εE to our repertoire of rules, and to do this in a natural way corresponding to our treatments of lists, multisets and sets, we would need to allow an empty tree, $(\,)$, such that the tree $((\,), R)$ is *identical* to the tree R. We leave exploration of this, and further developments in collection frames, to future work.

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