

DIOPHANTINE APPROXIMATION IN METRIC SPACE

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ABSTRACT. Diophantine approximation is traditionally the study of how well real numbers are approximated by rationals. We propose a model for studying Diophantine approximation in an arbitrary totally bounded metric space where the rationals are replaced with a countable hierarchy of “well-spread” points, which we refer to as *abstract rationals*. We prove various Jarník–Besicovitch type dimension bounds and investigate their sharpness.

1. Introduction

1.1. Diophantine approximation in a metric space. Let (F, d) be a non-empty totally bounded metric space, i.e. for every $\varepsilon > 0$, one can cover F with finitely many open ε -balls. For $q \in \mathbb{N}$, define *abstract rationals of level q* by fixing a maximal $(1/q)$ -separated set $P(q) \subseteq F$. By $(1/q)$ -separated we mean that for all distinct $p_1, p_2 \in P(q)$, $d(p_1, p_2) \geq 1/q$ and by *maximal* we mean it is impossible to add another point to the set and maintain the separation. Since F is totally bounded, $P(q)$ is a finite set for all q . One should think of $P(q)$ as playing a role in F similar to the role played by the set of rationals with denominator q inside $[0, 1]$. We refer to the set $\cup_q P(q)$ as a set of *abstract rationals*, noting that it is immediately a countable dense subset of F . For $t > 0$ let

$$\begin{aligned} F_t &= \{x \in F : \text{i.o. for some } p \in P(q) \ d(p, x) \leq 1/q^t\} \\ &= \bigcap_N \bigcup_{q > N} \bigcup_{p \in P(q)} B(p, 1/q^t) \end{aligned}$$

where $B(x, r)$ generally denotes the closed ball centred at $x \in X$ with radius $r > 0$. By the maximality of $P(q)$, if $t \leq 1$, then $F_t = F$ and so we are interested in $t > 1$. Computing (or estimating) the size (e.g. Hausdorff dimension) of F_t describes “how much of F can be far away from a chosen set of well spread out points”.

Classical Diophantine approximation may be cast in this language. For example, if $F = [0, 1]$ with the Euclidean distance and $P(q) = \mathbb{Z}/q \cap [0, 1]$ are the rationals with denominator q , then F_t is the set of t -well-approximable real numbers in $[0, 1]$. In this setting, Dirichlet’s theorem tells us that $F_2 = [0, 1]$, the Borel–Cantelli lemma tells us that F_t is a null set for all $t > 2$, and the Jarník–Besicovitch theorem gives

$$\dim_{\text{H}} F_t = 2/t$$

for all $t \geq 2$, where \dim_{H} denotes the Hausdorff dimension. That is, the exponent $t_0 = 2$ is a natural threshold beyond which it is sensible to study the dimensions of F_t . Our aim here is to establish analogous results in the general setting of abstract rationals.

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The inspiration for studying abstract rationals comes from problems of *intrinsic* Diophantine approximation. In such problems, one has a subset X of \mathbb{R}^n , and the goal is to understand how well its points can be approximated by rational points which also lie on X . The set X can for example be taken to be a manifold [FKMS18] or a fractal set [BFR11]. Questions on intrinsic Diophantine approximation on fractals have attracted much attention and there have been many interpretations and answers to the now-classical Mahler’s problem: ‘How well can irrationals in the Cantor set be approximated by rationals in the Cantor set?’ Interpretations include random rational points [BD16], intrinsic rationals [FMS18, FS14, LSV07, S21], and dynamical rationals [B19, AB].

In Section 2 we provide analogues to the Jarník–Besicovitch theorem in metric spaces using the language of abstract rationals. It turns out that the problem is subtle and the answer depends on how the abstract rationals are chosen. Despite the generality of the notion of abstract rationals, the bounds we prove in this general set-up are sharp in many cases of particular interest. For example, Theorem 2.1 below demonstrates that the upper bound for the dimension value in the Jarník–Besicovitch theorem holds in much more generality, replacing “2” with “ $\dim_{\mathbb{P}} F + 1$ ” where $\dim_{\mathbb{P}}$ denotes the *packing* dimension. We also show in Proposition 2.2 that in general one cannot replace “2” with “ $\dim_{\mathbb{H}} F + 1$ ” or “ $2 \dim_{\mathbb{H}} F$ ”, for example. In order to estimate $\dim_{\mathbb{H}} F_t$ in general we naturally encounter several different notions of fractal dimension, which we recall in the following subsection.

We obtain stronger dimension bounds in the special case when F is Ahlfors–David regular, see Theorem 2.7. In Section 3, during the investigation of the sharpness of the bounds in the Ahlfors–David regular case, we encounter the notions of *Dirichlet exponent* and *Dirichlet spectrum*. This is not to be confused with the Dirichlet spectrum defined using the optimal multiplicative constant (not exponent) in Dirichlet’s Theorem which has been considered in, for example, [AS]. By Dirichlet exponent we mean the threshold t_0 after which F_t does not (essentially) contain a ball. The definition is a natural extension of the classical Dirichlet exponent ($t_0 = 2$) to the setting of abstract rationals. We find that in general the Dirichlet exponent of F depends on the choice of abstract rationals, so we are occasioned to consider the collection of possible Dirichlet exponents for a given F —the Dirichlet spectrum of F . The main result of Section 2 and the mass transference principle of Beresnevich and Velani [BV06] allow us to bound the Dirichlet spectrum and reveal an interplay between Dirichlet exponents and the dimensions of F_t . In studying that interplay, we find that the dimension bounds from Theorem 2.7 are sharp.

1.2. Dimension theory. An important problem in the study of fractal objects is to characterise their dimension. There are many ways to define dimension and the relationship between the notions is especially interesting, and relevant to us. See [F14, F20] for more background on dimension theory and fractal geometry.

Given $s \geq 0$, the *s-dimensional Hausdorff measure* of a set X in a metric space is defined by

$$\mathcal{H}^s(X) = \liminf_{r \rightarrow 0} \left\{ \sum_i |U_i|^s : \{U_i\}_i \text{ is a countable } r\text{-cover of } X \right\}.$$

The *Hausdorff dimension* of X is then

$$\dim_{\mathbb{H}} X = \inf \left\{ s \geq 0 : \mathcal{H}^s(X) = 0 \right\} = \sup \left\{ s \geq 0 : \mathcal{H}^s(X) = \infty \right\}.$$

A subtle feature of the Hausdorff dimension is that sets of different sizes may be used in the covers and their contribution is weighted accordingly. The (upper) box dimension is a

simpler notion of dimension which avoids this subtlety. The *upper box dimension* of X is

$$\overline{\dim}_B X = \limsup_{r \rightarrow 0} \frac{\log N_r(X)}{-\log r},$$

where $N_r(X)$ is maximum cardinality of an r -separated subset of X . If the limsup is really a limit then we refer to it as the box dimension, written $\dim_B X$. Note that if X is not totally bounded then $N_r(X)$ may be infinite.

The packing dimension of X may be defined via packing measure in an analogous way to the Hausdorff dimension, but we use a common alternative formulation in terms of the upper box dimension. We define the *packing dimension* of X by

$$\dim_P X = \inf \left\{ \sup_i \overline{\dim}_B X_i : X = \bigcup_{i=1}^{\infty} X_i \right\}.$$

It should be clear from the definitions that

$$\dim_H X \leq \dim_P X \leq \overline{\dim}_B X.$$

Finally, the *lower dimension* of X is defined by

$$\dim_L X = \sup \left\{ s \geq 0 : \text{there exists a constant } C > 0 \text{ such that,} \right. \\ \left. \begin{aligned} &\text{for all } 0 < r < R \leq |X| \text{ and } x \in X, \\ &N_r(B(x, R) \cap X) \geq C \left(\frac{R}{r} \right)^s \end{aligned} \right\}$$

where $|X|$ denotes the diameter of X . The lower dimension is the natural dual to the Assouad dimension and identifies the “thinnest part” of the space. If X is closed, then

$$\dim_L X \leq \dim_H X.$$

See [F20, Chapter 3] for more background on the lower dimension, including its relevance in a different area of Diophantine approximation, e.g. [F20, Section 14.2].

2. Main results

2.1. General bounds. Our first result provides general bounds for the dimension of F_t which hold independently of how the abstract rationals are chosen. The general lower bound assumes compactness of F . If F is not compact, then this lower bound does not hold. For example, if $F = [0, 1] \cap \mathbb{Q}$, then the packing dimension of $F = F_1$ is 0 and the lower dimension is 1.

Theorem 2.1. *For all $t \geq 1$*

$$\dim_H F_t \leq \frac{\dim_P F + 1}{t}$$

and, if F is compact,

$$\dim_P F_t \geq \frac{\dim_L F}{t}.$$

Proof. We begin with the general upper bound. Let $b > \dim_P F$ and let

$$F = \bigcup_i F_i$$

be a countable decomposition of F with the property that $\overline{\dim}_B F_i < b$ for all i . Temporarily fix an index i . We write

$$d(p, F_i) = \inf\{d(p, x) : x \in F_i\}.$$

For all $N \geq 1$

$$\bigcup_{q > N} \bigcup_{\substack{p \in P(q): \\ d(p, F_i) \leq 1/q}} \{B(p, 1/q^t)\}$$

provides a $2/N^t$ -cover of $F_t \cap F_i$. Therefore, for some constant $C_1 \geq 1$ depending on F , i and b ,

$$\begin{aligned} \mathcal{H}_{2/N^t}^s(F_t \cap F_i) &\leq \sum_{q > N} \sum_{\substack{p \in P(q): \\ d(p, F_i) \leq 1/q}} |B(p, 1/q^t)|^s \\ &= \sum_{q > N} (\#\{p \in P(q) : d(p, F_i) \leq 1/q\}) (2/q^t)^s \\ &\leq 2^s C_1 \sum_{q \geq 1} q^{b-ts} < \infty \end{aligned}$$

provided $s > \frac{b+1}{t}$. For such s

$$\mathcal{H}^s(F_t) \leq \sum_i \mathcal{H}^s(F_t \cap F_i) = 0$$

and the desired result follows.

Turning our attention to the general lower bound, let $0 < l < \dim_L F$ (we may assume the lower dimension is strictly positive). Fix a rapidly increasing sequence of integers q_k satisfying

$$(1) \quad q_1 q_2 \cdots q_k \leq \log(q_{k+1})$$

and construct a nested decreasing sequence of compact sets $E_k \subseteq F_t$ by setting $E_1 = B(z, q_1^{-t})$ with $z \in F$ chosen arbitrarily and then letting E_k be the union of the closed balls $B(p, q_k^{-t})$ with $p \in P(q_k)$ which lie completely inside E_{k-1} . The sets E_k are compact since F is compact and so by the Cantor intersection theorem

$$E = \bigcap_k E_k \subseteq F_t$$

with E non-empty and compact. We show $\dim_p E \geq l/t$, which completes the proof. Let μ be a probability measure supported on E constructed by assigning unit mass to E_1 and then, given a k -level cylinder $B(p, q_k^{-t})$ used in the construction of E_k , redistribute the mass of $B(p, q_k^{-t})$ equally among the $(k+1)$ -level cylinders lying inside. Since the q_k are rapidly increasing and $P(q_k)$ is $(1/q_k)$ -separated, using the definition of lower dimension, we may assume that there are at least

$$\left(\frac{q_k^{-t}}{q_{k+1}^{-1}} \right)^l$$

$(k+1)$ -level cylinders inside every k -level cylinder. For each k , consider $r = q_k^{-t}$. It follows that any ball $B(x, r)$ with $x \in E$ intersects at most one k -level cylinder. Recall that the centres

of the k -level cylinders are $(1/q_k)$ -separated. Therefore

$$\begin{aligned}
 \mu(B(x, r)) &\leq \left(\frac{q_1^{-t}}{q_2^{-1}}\right)^{-l} \left(\frac{q_2^{-t}}{q_3^{-1}}\right)^{-l} \cdots \left(\frac{q_{k-1}^{-t}}{q_k^{-1}}\right)^{-l} \\
 (2) \qquad &= q_1^{tl} (q_2 \cdots q_{k-1})^{tl-l} (q_k^{-1})^l \\
 &\leq (\log(q_k))^{tl} (q_k^{-1})^l \quad \text{by (1)} \\
 &= \left(\frac{|\log r|}{t}\right)^{tl} r^{l/t}.
 \end{aligned}$$

This proves that for all $x \in E$, the upper local dimension of μ at x is at least l/t . Applying standard estimates for packing dimension coming from local dimension of measures, e.g. [F97, Proposition 2.3], this proves that $\dim_{\text{P}} E \geq l/t$ as required. \square

2.2. Sharpness of general bounds.

2.2.1. Sharpness of upper bound. Comparing the upper bound from Theorem 2.1 with the classical Jarník–Besicovitch theorem, one might ask if the packing dimension can be replaced by the Hausdorff dimension, or indeed if an upper bound of the form $c \dim_{\text{H}} F/t$ is possible for some uniform constant c . The following proposition shows that this is not possible and demonstrates that the packing dimension should appear in bounding the Hausdorff dimension of F_t from above.

Proposition 2.2. *For all $t > 2$, there exists a compact set $F \subseteq [0, 1]$ and a choice of abstract rationals such that*

$$\dim_{\text{P}} F = 1$$

and

$$\dim_{\text{H}} F_t = \dim_{\text{H}} F = 2/t = \frac{\dim_{\text{P}} F + 1}{t} > \frac{\dim_{\text{H}} F + 1}{t}.$$

In particular we cannot replace the packing dimension by the Hausdorff dimension in the upper bound given in Theorem 2.1.

Proof. Let W_t denote the set of t -well-approximable numbers in $[0, 1]$ in the classical setting. Then

$$\dim_{\text{H}} W_t = 2/t$$

and $\mathcal{H}^{2/t}(W_t) = \infty$. This is the Jarník–Besicovitch theorem, but we emphasise the infinitude of the Hausdorff measure which we need for what follows, see [BBDF09, Theorem 11]. By [F14, Theorem 4.10] there exists a compact subset $W_t^* \subseteq W_t$ such that

$$\dim_{\text{H}} W_t^* = 2/t.$$

Let

$$F = W_t^* \cup \{p/q : p \in \mathbb{Z}, q \in \mathbb{N}, |p/q - x| \leq 1/q \text{ for some } x \in W_t^*\}.$$

Then F is compact and

$$\dim_{\text{H}} F = \dim_{\text{H}} W_t = 2/t.$$

The abstract rationals $P(q)$ are then chosen to be $P(q) = \mathbb{Z}/q \cap F$. By construction $W_t^* \subseteq W_t \cap F = F_t$ and so

$$\dim_{\mathbb{H}} F_t = \dim_{\mathbb{H}} F = 2/t.$$

Moreover, by Theorem 2.1,

$$2/t = \dim_{\mathbb{H}} F_t \leq \frac{\dim_{\mathbb{P}} F + 1}{t}$$

which forces $\dim_{\mathbb{P}} F = 1$, completing the proof. \square

The next result is along similar lines but this time the construction works for arbitrary choices of abstract rationals.

Proposition 2.3. *For all integers $t \geq 2$, there exists a compact set $F = F(t) \subseteq [0, 1]$ such that*

$$F_s = F$$

for all $1 \leq s < t$ and all choices of abstract rationals $\mathcal{P} = \bigcup_q P(q)$. In particular, for all $t \geq 1$,

$$\supinf_F \frac{\dim_{\mathbb{H}} F_t}{\dim_{\mathbb{H}} F} = 1$$

where the supremum is taken over compact sets $F \subseteq [0, 1]$ with $\dim_{\mathbb{H}} F > 0$ and the infimum is taken over all choices of abstract rationals for F .

Proof. Consider an iterated function system (IFS) type construction based on the IFSs $\{x \mapsto x/2, x \mapsto (x+1)/2\}$ and $\{x \mapsto x/2\}$. See [F14] for background on IFS theory. Instead of using the same fixed IFS at each stage of the construction, here we choose one of these two IFSs at each stage. The attractor (or limit set) F of this construction now depends on the sequence determining which IFS is used at each stage. Let $N(k)$ be the number of times we choose the first IFS in the first k levels. Then it is standard that

$$\dim_{\mathbb{H}} F = \liminf_{k \rightarrow \infty} N(k)/k$$

and

$$\dim_{\mathbb{P}} F = \limsup_{k \rightarrow \infty} N(k)/k.$$

Let k_n be a very rapidly increasing sequence of integers, e.g. $k_n = n^n$. Fix an integer $t \geq 2$ and construct F inductively as follows. In the first tk_1 steps, begin by using the first IFS k_1 times, followed by the second IFS $(t-1)k_1$ times. In the next block of tk_2 steps use the first IFS k_2 times, followed by the second IFS $(t-1)k_2$ times. Continue in this way in blocks of length tk_n (for $n \geq 3$) by using the first IFS k_n times, followed by the second IFS $(t-1)k_n$ times. Then

$$\dim_{\mathbb{H}} F = 1/t < 1 = \dim_{\mathbb{P}} F.$$

Consider $x \in F$ and especially the cylinder (interval) E of level

$$M = t \sum_{n=1}^N k_n$$

containing x . (Just after the N th long run of the second IFS). This cylinder has size $2^{-M} \leq 2^{-tk_N}$. There are very large holes to the left and right of this cylinder, of size

$$> 2 \cdot 2^{-k_N \Delta}$$

where $\Delta > 1$ can be chosen arbitrarily provided N is large enough. Due to the presence of these holes, there must exist $p \in P(q) \cap E$ for some q with $1/q \geq 2^{-k_N \Delta}$. Since $x \in F$ was arbitrary, this ensures that the whole set F is contained in the

$$2^{-tk_N} = \left(2^{-k_N \Delta}\right)^{t/\Delta} \leq (1/q)^{t/\Delta}$$

neighbourhood of $P(q)$ for infinitely many q . Therefore,

$$F_{t/\Delta} = F$$

for all $\Delta > 1$. □

2.2.2. Sharpness of lower bound. Comparing the lower bound from Theorem 2.1 with the classical Jarník–Besicovitch theorem, one might ask if the lower dimension can be replaced by the Hausdorff dimension, or indeed if a lower bound of the form $(\dim_{\mathbb{L}} F + 1)/t$ or $c \dim_{\mathbb{H}} F/t$ is possible for some uniform constant c . The following proposition shows that these bounds are not possible in general and demonstrates that F_t ($t > 1$) can be very small, even countable, if the abstract rationals are “far away” from the large parts of F .

Proposition 2.4. *There exists a compact set $F \subseteq [0, 1]$ with $\dim_{\mathbb{H}} F = \dim_{\mathbb{P}} F > 0$ and such that, for some choice of abstract rationals, F_t is countable for all $t > 1$ and, for another choice of abstract rationals,*

$$\dim_{\mathbb{H}} F_t > 0$$

for all $t > 1$.

Proof. Let F_0 be the middle third Cantor set (see [F14, Example 4.5]) and F be $F_0 \cup C$ where C is the countable set consisting of a single point in the centre of each complementary interval of F_0 . Clearly $\dim_{\mathbb{H}} F = \dim_{\mathbb{P}} F = s = \log_3 2 > 0$. We choose the first set of abstract rationals such that $\cup_q P(q) = C$. More precisely, for $q \geq 1$, let $C_k \subseteq C$ be the set of centres of complementary intervals of length 3^{-k} where k is chosen such that $3^{-(k-1)}/2 \leq 1/q < 3^{-(k-2)}/2$. Then choose $P(q)$ to be a maximal $(1/q)$ -separated subset of C which contains C_k . Note that $P(q)$ is a maximal $(1/q)$ -separated subset of F by construction. Fix $t > 1$, $q \in \mathbb{N}$ and $p \in P(q)$ and let k be as above. Then

$$B(p, q^{-t}) \subseteq B(p, 3^{-t(k-2)}/2^t) = \{p\}$$

for sufficiently large q . More precisely, once q is large enough so that k is large enough to guarantee

$$3^{-t(k-2)}/2^t < 3^{-k}/2.$$

It follows that $F_t = C$ which is countable and, in particular,

$$\dim_{\mathbb{H}} F_t = \dim_{\mathbb{P}} F_t = 0.$$

For the second choice of abstract rationals, choose $P(q)$ such that it contains a maximal $(1/q)$ -separated subset of F_0 , which we denote by $P_0(q) \subseteq F_0$. Let $F_{0,t} = (F_0)_t \subseteq F_0$ be the set of points in F_0 which are i.o. t -approximable by abstract rationals $P_0(q)$. Then, since F_0 is s -Ahlfors–David regular, it follows from Theorem 2.7, which we prove later, that

$$\dim_{\mathbb{H}} F_t \geq \dim_{\mathbb{H}} F_{0,t} \geq s/t > 0$$

as required. □

The next result is an upgrade of the previous example to show that the lower bound $\dim_{\mathbb{L}} F/t$ from Theorem 2.1 is sharp in a stronger sense.

Proposition 2.5. *For all $0 \leq L \leq H \leq 1$, there exists a compact set $F \subseteq [0, 1]$ with*

$$\dim_{\mathbb{L}} F = L$$

and

$$\dim_{\mathbb{H}} F = \dim_{\mathbb{P}} F = H$$

such that for some choice of abstract rationals,

$$\dim_{\mathbb{H}} F_t = L/t = \frac{\dim_{\mathbb{L}} F}{t}$$

for all $t > 1$.

Proof. This proposition is proved in the same way as Proposition 2.4 but with the middle third cantor set replaced by another central (self-similar) cantor set of Hausdorff and packing dimension H and with the single points placed in the complementary intervals replaced by scaled copies of a fixed set E with $\dim_{\mathbb{L}} E = \dim_{\mathbb{H}} E = L$. The set E could be another central cantor set, for example. Here a ‘central cantor set’ is a self-similar set contained in $[0, 1]$ generated by the IFS $\{x \mapsto \lambda x, x \mapsto \lambda x + (1 - \lambda)\}$ for $\lambda \in (0, 1/2)$. The scaling is done such that the diameter of a scaled copy of E is much smaller than the length of the complementary interval it is placed in. The abstract rationals are chosen to be contained in the countably many copies of E and within each copy E' of E must be chosen such that

$$(3) \quad \dim_{\mathbb{H}} E'_t = \frac{\dim_{\mathbb{H}} E'}{t}.$$

The details work the same way, but the challenge is in finding a way to choose the set E and abstract rationals to satisfy (3). We choose E to be an L -Ahlfors–David regular set with the abstract rationals chosen as in Proposition 3.4, which we prove later. \square

There are many other ways in which one could investigate sharpness of Theorem 2.1. For example, our results above are pointwise in the sense that for a given t a set F is constructed. It may be interesting to consider if one can construct sets witnessing sharpness for all $t \geq 1$ simultaneously. Another question of sharpness which we have not answered is if the Hausdorff dimension of F_t can be estimated from below in general.

Question 2.6. *Is it true that for all $t \geq 1$*

$$\dim_{\mathbb{H}} F_t \geq \frac{\dim_{\mathbb{L}} F}{t}?$$

2.3. The Ahlfors–David regular case. It turns out we can say more when F is an Ahlfors–David regular metric space. Recall that a compact set F is s -Ahlfors–David regular for some $s \geq 0$ if there exists a constant $C > 0$ such that

$$r^s/C \leq \mathcal{H}^s(B(x, r)) \leq Cr^s$$

for all $0 < r < 1$ and all $x \in F$. In this case it is not hard to see that

$$\dim_{\mathbb{L}} F = \dim_{\mathbb{H}} F = \dim_{\mathbb{P}} F = \overline{\dim_{\mathbb{B}}} F = s$$

and that F is doubling, see [F20, Theorem 6.4.1].

Theorem 2.7. *Suppose F is a compact s -Ahlfors–David regular. For all $t \geq 1$*

$$\frac{s}{t} \leq \dim_{\mathbb{H}} F_t \leq \frac{s+1}{t}.$$

Proof. Given the general bounds from Theorem 2.1, all that remains to establish is

$$\dim_{\mathbb{H}} F_t \geq \frac{s}{t}$$

which we prove now. The strategy is similar to the general lower bound, except that we need to control the measure at all scales, not just a sequence of scales. Let E and μ be as in the proof of Theorem 2.1. Given $r \in (0, q_1^{-t})$, let k be chosen uniquely such that

$$q_{k+1}^{-1} \leq r < q_k^{-1}.$$

Since $t \geq 1$ is fixed and the q_k are rapidly increasing, we may assume that either

$$(4) \quad q_{k+1}^{-1} \leq r < q_k^{-t} \leq q_k^{-1}$$

or

$$(5) \quad q_{k+1}^{-1} < q_k^{-t} \leq r < q_k^{-1}.$$

Consider the first case (4). There is a uniform constant C_3 such that an arbitrary ball $B(x, r)$ may intersect $C_3(r/q_{k+1}^{-1})^s$ many $(k+1)$ -level cylinders. This uses the fact that $P(q_{k+1})$ is $(1/q_{k+1})$ -separated and the fact that F is s -Ahfords-David regular. Moreover, each $(k+1)$ -level cylinder carries mass at most

$$(6) \quad q_1^{ts}(q_2 \cdots q_k)^{s(t-1)}(q_{k+1}^{-1})^s$$

by following the argument above (2). Therefore

$$\begin{aligned} \mu(B(x, r)) &\leq C_3(r/q_{k+1}^{-1})^s q_1^{ts}(q_2 \cdots q_k)^{s(t-1)}(q_{k+1}^{-1})^s && \text{by (6)} \\ &= C_3 q_1^{ts}(q_2 \cdots q_k)^{s(t-1)} r^s \\ &\leq C_3 (\log(q_k))^{ts} q_k^{s(t-1)} r^s && \text{by (1)} \\ &\leq C_3 |\log r|^{ts} (r^{-1/t})^{s(t-1)} r^s && \text{by (4)} \\ &= C_3 |\log r|^{ts} r^{s/t}. \end{aligned}$$

Now consider the second case (5). Since F is Ahlfors–David regular it is doubling and, therefore, there exists a uniform constant C_4 such that an arbitrary ball $B(x, r)$ intersects at most C_4 many k -level cylinders. This uses the fact that the centres of the k -level cylinders are $(1/q_k)$ -separated. Therefore, using (6) to estimate the mass of a k -level cylinder,

$$\begin{aligned} \mu(B(x, r)) &\leq C_4 q_1^{ts}(q_2 \cdots q_{k-1})^{s(t-1)}(q_k^{-1})^s \\ &\leq C_4 (\log(q_k))^{ts} (q_k^{-1})^s && \text{by (1)} \\ &\leq C_4 |\log r|^{ts} r^{s/t} && \text{by (5)}. \end{aligned}$$

Combining the two cases, we have proved that for an arbitrary $x \in E$ and $r \in (0, q_1^{-t})$

$$\mu(B(x, r)) \leq \max\{C_3, C_4\} |\log r|^{ts} r^{s/t}$$

and the result follows by the mass distribution principle, see [F14, Chapter 4]. \square

3. Dirichlet spectrum of Ahlfors–David regular spaces

Notice that in the classical setting of Diophantine approximation on $[0, 1]$ with P defined by

$$P(q) = \{0, 1/q, 2/q, \dots, 1\}, \quad (q \geq 1),$$

we have $s = 1$ and the *upper* bound in Theorem 2.7 gives the correct dimension of F_t , namely, $\dim_{\mathbb{H}} F_t = 2/t$, the dimension given in the Jarník–Besicovitch theorem. Therefore probing sharpness of the lower bound seems like the most pressing concern.

Notice also that in the classical setting, we have

$$\begin{aligned} 2 &= \sup\{d : F_d = [0, 1]\} = \inf\{d : \text{Leb}(F_d) = 0\} \\ &= \sup\{d : F_d \text{ contains a ball}\} = \sup\{d : F_d \text{ essentially contains a ball}\}, \end{aligned}$$

where ‘ F_d essentially contains a ball’ means that F_d contains a ball up to a Lebesgue null set. As a generalisation of this, define the following: for a given s -Ahlfors–David regular set F and choice P of abstract rationals in F , define the *Dirichlet exponent*

$$\begin{aligned} d(P) &= \sup\{d : F_d \text{ contains a ball up to } \mathcal{H}^s\text{-measure } 0\} \\ &= \sup\{d : (\exists \text{ball } B \subset F) \quad \mathcal{H}^s(B \setminus F_d) = 0\} \end{aligned}$$

and the *Dirichlet spectrum*

$$\sigma(F) = \{d(P) : P \text{ is a choice of abstract rationals in } F\}.$$

We can then ask the following.

Question 3.1. *For a given s -Ahlfors–David-regular F , what is the nature of the set $\sigma(F)$, and to what extent does the formula*

$$\dim_{\mathbb{H}} F_t = \frac{d(P)s}{t} \quad (t \geq d(P))$$

hold for a given choice P of abstract rationals in F ?

In exploring this question, we will find, in particular, that the dimension bounds in Theorem 2.7 are sharp. First, let us report what Theorem 2.7 and the Mass Transference Principle say immediately about Question 3.1.

Proposition 3.1. *If F is a locally compact, s -Ahlfors–David-regular metric space, then $\sigma(F) \subset [1, 1 + 1/s]$, and for any choice P of abstract rationals in F we have*

$$\dim_{\mathbb{H}} F_t \geq \frac{d(P)s}{t} \quad (t \geq d(P)).$$

Proof. Clearly, since for any P one has $F = F_1$, we have $d(P) \geq 1$. On the other hand, suppose $\mathcal{H}^s(F_d) > 0$. Then $\dim_{\mathbb{H}} F_d \geq s$, which, by Theorem 2.7, implies that $s \leq (s + 1)/d$, hence $d \leq 1 + 1/s$. Taking supremum over all such d shows that $d(P) \leq 1 + 1/s$, so we have shown $d(P) \in [1, 1 + 1/s]$ for all P .

Now suppose $d < d(P) \leq t$. It follows that F_d essentially contains a closed ball $F' = B(x', r) \subset F$, and we may suppose that $\mathcal{H}^s(S(x', r)) = 0$, where $S(x', r)$ denotes the sphere of radius r centered at x' . (After all, at most countably many members of the uncountable disjoint union $F' = \bigcup_{0 < \rho \leq r} S(x', \rho)$ may have positive measure, so if we must, we may shrink

r to a value for which $S(x', r)$ has zero measure.) The ball F' is itself an s -Ahlfors–David-regular metric space. Define a hierarchy of points $P' = P|_{F'}$ in F' by

$$P'(q) = \begin{cases} P(q) \cap F' & \text{if } P(q) \cap F' \neq \emptyset \\ x' & \text{if } P(q) \cap F' = \emptyset. \end{cases}$$

Note that as soon as q is large enough we have $P'(q) = P(q) \cap F'$, hence $P'(q)$ is $(1/q)$ -separated but it may not be maximal. That is, P' may not constitute abstract rationals in F' as we have defined the term. Still, let us use F'_t to denote all the points in F' that are t -approximable by P' . Notice then that

$$(7) \quad (F_d \cap F') \setminus F'_d \subset S(x', r),$$

for, if $x \in F_d$ and $d(x', x) < r$, then for all q large enough, every $p \in P(q)$ with $d(x, p) \leq q^{-d}$ must lie in F' . All such p are in $P'(q)$, so we have $x \in F'_d$. Importantly for us, (7) implies that $\mathcal{H}^s(F' \setminus F'_d) = 0$.

For any $t \geq d(P)$, we have $F'_t \subset F_t$. Also, since $\mathcal{H}^s(F' \setminus F'_d) = 0$, we have that for any ball $B \subset F'$,

$$\mathcal{H}^s(F'_d \cap B) = \mathcal{H}^s(B).$$

Since $d < t$, we have $ds/t < s$, so the mass transference principle [BV06, Theorem 3] implies that for any ball $B \subset F'$ we have

$$\mathcal{H}^{ds/t}(F'_t \cap B) = \mathcal{H}^{ds/t}(B) = \infty.$$

Therefore $\dim_{\mathbb{H}} F'_t \geq ds/t$, hence $\dim_{\mathbb{H}} F_t \geq ds/t$. Since $d < d(P)$ was arbitrary, the result follows. \square

Remark. We could of course ask Question 3.1 when F is not Ahlfors–David regular. There, Theorem 2.1 would guarantee that

$$1 \leq d(P) \leq \frac{\dim_{\mathbb{P}} F + 1}{\dim_{\mathbb{H}} F}.$$

However, clearly no such dimension formula as the one in Question 3.1 holds in general, even with s replaced by $\dim_{\mathbb{H}} F$.

We can now refine Question 3.1 by asking what kind of subset $\sigma(F) \subset [1, 1 + 1/s]$ is. The next results exhibit cases where $\sigma(F) = [1, 1 + 1/s]$. We will also show that for certain spaces F and any $d \in \sigma(F)$, there exist abstract rationals witnessing $d(P) = d$ such that $\dim_{\mathbb{H}} F_t = ds/t$ for all $t \geq d$. That is, the full spectrum $\sigma(F)$ can be realized by abstract rationals satisfying a general Jarnik–Besicovitch theorem.

In particular, we show that the bounds from Theorem 2.7 are indeed sharp and, moreover, the dimensions of F_t can depend on the choice of abstract rationals, even in the Ahlfors–David regular setting.

Proposition 3.2. *For $F = [0, 1]$ we have $\sigma(F) = [1, 2]$ and for any $d \in \sigma(F)$ there is a choice P of abstract rationals with $d(P) = d$ such that*

$$\dim_{\mathbb{H}} F_t = \frac{d(P)}{t} \quad (t \geq d(P)).$$

Proof. The case $d(P) = 2$ is achieved by the classical setting of the Jarnik–Besicovitch theorem so it remains to consider $d \in [1, 2)$. We first prove the case $d = 1$, which provides a roadmap to the other cases.

For each $q \in [2^k, 2^{k+1})$ let

$$P(q) = \left\{ \frac{a}{2^k} : a = 0, \dots, 2^k \right\}$$

be the rationals with denominator 2^k . Note that $P(q)$ is a maximal $(1/q)$ -separated set in F . In this setting we have $d(P) = 1$, which in particular shows $1 \in \sigma(F)$. This will follow from Proposition 3.1 together with the upper bound for the dimension of F_t which we prove now. The lower bound

$$\dim_{\mathbb{H}} F_t \geq \frac{1}{t} \quad (t \geq 1),$$

already follows from Theorem 2.7. Note that

$$F_t \subset \bigcup_{k \geq 1} \bigcup_{a=0}^{2^k} \left[\frac{a}{2^k} - \frac{1}{2^{tk}}, \frac{a}{2^k} + \frac{1}{2^{tk}} \right].$$

For all $\tau > 1/t$ we have that

$$\sum_{k \geq 1} \sum_{a=0}^{2^k} \left(\frac{1}{2^{tk-1}} \right)^\tau \ll \sum_{k \geq 1} 2^{(1-\tau t)k},$$

which converges. (The notation \ll means that \leq holds up to a positive absolute constant factor.) This implies $\mathcal{H}^\tau(F_t) < \infty$ and $\dim_{\mathbb{H}} F_t \leq \tau$. Therefore $\dim_{\mathbb{H}} F_t \leq 1/t$.

Now suppose $d \in (1, 2)$ and let $\alpha > 1$ be such that $d = 1 + 1/\alpha$. For $q \in \mathbb{N}$ with $k^\alpha \leq q < (k+1)^\alpha$, let

$$P(q) = \left\{ \frac{a}{\lfloor k^\alpha \rfloor} : a = 0, \dots, \lfloor k^\alpha \rfloor \right\}.$$

We claim that $d(P) = d$. In fact, this is a consequence of the recently proved Duffin–Schaeffer conjecture [KM], which states that if $\psi(n) : \mathbb{N} \rightarrow \mathbb{R}$ is such that

$$\sum_q \psi(q) \varphi(q) = \infty,$$

then for almost every $x \in [0, 1]$ there are infinitely many $p/q \in \mathbb{Q}$ such that

$$\left| x - \frac{p}{q} \right| < \psi(q).$$

Here and throughout φ is the Euler totient function. Consider the function

$$\psi(q) = \begin{cases} q^{-d} & \text{if } q = \lfloor k^\alpha \rfloor, k \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \sum_q \psi(q) \varphi(q) &= \sum_k \frac{1}{\lfloor k^\alpha \rfloor^d} \varphi(\lfloor k^\alpha \rfloor) \\ &\geq \sum_k \frac{\varphi(\lfloor k^\alpha \rfloor)}{k^{\alpha+1}} \\ &\geq \sum_k \frac{\varphi(\lfloor k^\alpha \rfloor)}{k^{\alpha+1}} \\ &\gg \sum_k \frac{1}{k \log \log(k^\alpha)} = \infty, \end{aligned}$$

this last estimate coming from the growth estimate $\varphi(n) \gg n/\log\log n$ [HW]. We may then conclude that for almost every $x \in [0, 1]$ there are infinitely many q such that x is within q^{-d} of some $p \in P(q)$. That is, F_d has full measure, hence $d(P) \geq d$. On the other hand, if we had defined ψ by $q^{-d'}$ for some $d' > d$, then the above sum would have converged, and the Borel–Cantelli lemma would imply that $F_{d'}$ had measure 0. This shows that $d(P) = d$, as claimed.

Proposition 3.1 now gives

$$\dim_{\mathbb{H}} F_t \geq \frac{d(P)}{t}$$

for every $t \geq d$. The corresponding upper bound comes from a cover of F_t by intervals as in the previous case. Here the relevant series is

$$\sum_{k \geq 1} \sum_{\alpha=0}^{\lfloor k^\alpha \rfloor} (k^{-\alpha t})^\tau \ll \sum_{k \geq 1} k^{\alpha(1-t\tau)},$$

which converges as long as

$$\tau > \frac{1+1/\alpha}{t} = \frac{d}{t},$$

which leads to the upper bound and establishes

$$(8) \quad \dim_{\mathbb{H}} F_t = \frac{1+1/\alpha}{t} = \frac{d(P)}{t}.$$

Since $d \in (1, 2)$ was arbitrary, we are done. \square

We investigate Dirichlet exponents for examples of Ahlfors–David regular F with dimension less than 1.

Proposition 3.3. *Let F be the middle third Cantor set, whose dimension is $s = \log_3 2$. Then for any $d \in [1, 1 + 1/s]$ there exists a choice of abstract rationals such that $d(F) = d$ and*

$$\dim_{\mathbb{H}} F_t = \frac{d}{t} s \quad (t \geq d).$$

Proof. Recall that the Cantor set is $F = \bigcap_{n \geq 0} E_n$, where

$$E_0 = [0, 1],$$

$$E_1 = [0, 1/3] \cup [2/3, 1],$$

$$E_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 4/9] \cup [8/9, 1],$$

and in general E_n is a union of 2^n “basic intervals” which are obtained by removing the open middle thirds of the basic intervals constituting E_{n-1} . Note that the basic intervals of E_n have length 3^{-n} .

(i) *The case $d = 1$.* First, for $3^n \leq q < 3^{n+1}$ let $P(q)$ be the endpoints of the basic intervals in the n th construction set E_n , of which there are 2^{n+1} . Theorem 2.7 tells us that for any $t \geq 1$, $\dim_{\mathbb{H}} F_t \geq s/t$. For the upper bound, note that F_t is covered by

$$F_t \subset \bigcup_{n \geq m} \bigcup_{p \in P(3^n)} [p - 3^{-nt}, p + 3^{-nt}]$$

for any $m > 0$. For each $n \geq m$, this cover has 2^{n+1} intervals, each of length $2 \cdot 3^{-nt}$. So, for $\tau \geq 0$, the cost of the corresponding τ -dimensional cover is

$$\sum_{n \geq m} 2^{n+1} \cdot 2^\tau \cdot 3^{-nt\tau},$$

which converges as long as $2 < 3^{t\tau}$, that is, if $\tau > s/t$. This shows the other inequality in dimension, hence $\dim_H F_t = s/t$.

Notice that in this example we have that for every $x \in F$ there are infinitely many q for which there is $p \in P(q)$ satisfying $|x - p| \leq 1/(3q)$. The previous paragraph shows that as soon as we increase the exponent beyond 1, the dimension drops. Therefore, the $d(F) = 1$ with this choice of P .

(ii) *The case $d = 1 + 1/s$.* For $n \geq 0$, let

$$k(n) = k = \left\lfloor 1 + n \left(1 + \frac{1}{s} \right) \right\rfloor,$$

that is, $k(n)$ is the greatest integer with the property that each basic interval of E_n contains at most $3^{n+1} - 3^n$ basic intervals of E_k . Denote by

$$E_n^L = \bigcup_{\substack{I \subset E_n \\ \text{basic interval}}} \{\min I\} = \{\ell_1, \dots, \ell_{2^n}\}$$

the set of left endpoints ℓ_i of the basic intervals of E_n . For $3^n \leq q < 3^{n+1}$ put

$$m_0 = \left(q - 3^n \pmod{2^{k-n}} \right) + 1$$

and define

$$P(q) = \left(E_n^L + \ell_{m_0} \right) \cup \{\text{enough endpoints of } E_n \text{ to get maximality}\}.$$

In particular, $P(3^n)$ consists exactly of the endpoints of the basic intervals of E_n , as in the previous example. (See Figure 1 for an illustration of $P(1)$ through $P(9)$.) Importantly, we have, for $3^n \leq q < 3^{n+1}$

$$2^{n-1} = \frac{\#P(3^n)}{2} \leq \#P(q) \leq \#P(3^n) = 2^n.$$

The ‘‘Dirichlet theorem’’ here is: For every $x \in F$ and $n \in \mathbb{N}$, there exists $q \leq 3^{n+1}$ and $p \in P(q)$ such that

$$|x - p| \leq 3^{-k} \leq \frac{1}{q^{k/(n+1)}}.$$

Since

$$\frac{k(n)}{n+1} \sim 1 + \frac{1}{s},$$

the Dirichlet exponent here is $d(F) = 1 + 1/s$.

By Proposition 3.1 we have $\dim_H F_t \geq d(P)s/t = \frac{s+1}{t}$ for all $t \geq d(P)$. For the upper bound, we cover

$$F_t \subset \bigcup_{q \geq 3^{m_0}} \bigcup_{p \in P(q)} B(p, q^{-t}) \subset \bigcup_{q \geq 3^{m_0}} \bigcup_{p \in P(q)} B(p, 3^{-nt})$$

The cost of the corresponding τ -dimensional cover is

$$\sum_{q \geq 3^{m_0}} \sum_{p \in P(q)} 3^{-nt\tau} = \sum_{n \geq m_0} 2^k 3^{-nt\tau} \leq \sum_{n \geq m_0} 2^{n(1+1/s)} 3^{-nt\tau},$$

which converges if $\tau > (s+1)/t$. This leads to the desired upper bound, so we have

$$\dim_H F_t = \frac{s+1}{t} = \frac{d(F)}{t} s.$$

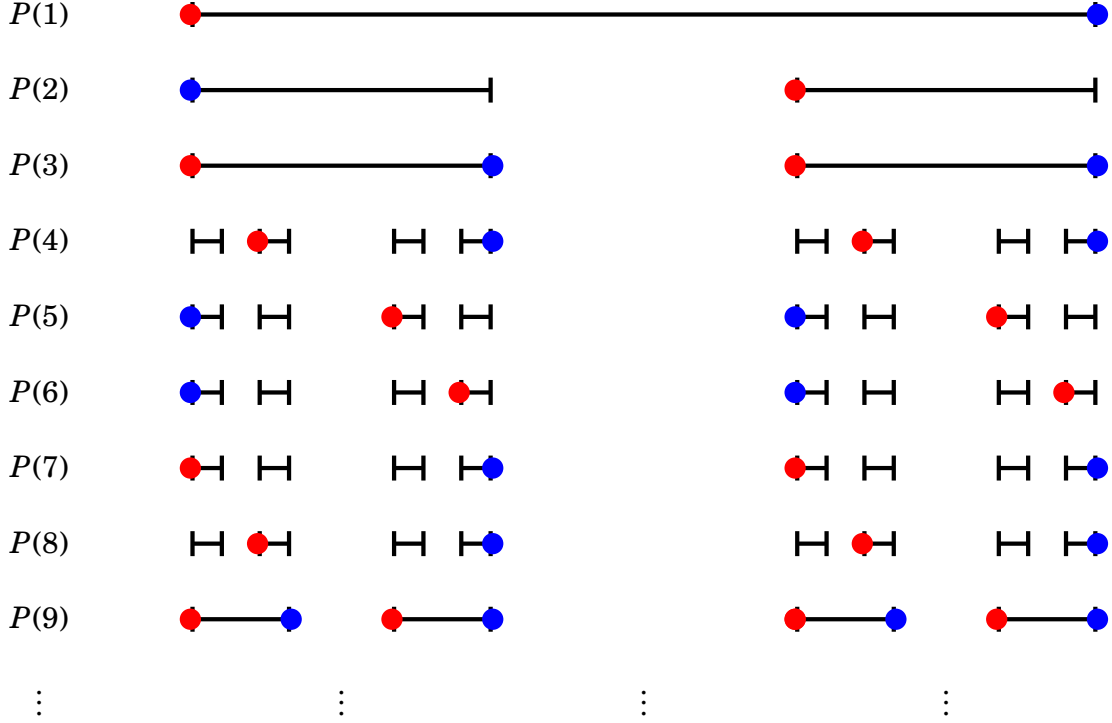


FIGURE 1. The red dots are the elements of $(E_n^L + \ell_{m_0})$ and the blue dots are the endpoints of E_n we have included to achieve maximality. United they are $P(q)$.

(iii) *The intermediate case.* Let $d \in (1, 1 + 1/s)$. We now modify the previous construction by using instead

$$k(n) = k = \lfloor nd \rfloor$$

and keeping the rest of the definitions intact. The same Dirichlet theorem from the previous construction continues to hold, but now we have

$$\frac{k(n)}{n+1} \sim d,$$

so that $d(F) \geq d$.

For the other bound, let $d' > d$, and note that $F_{d'}$ is covered by

$$F_{d'} \subset \bigcup_{q \geq q_0} \bigcup_{p \in P(q)} B(p, q^{-d'})$$

for any q_0 , and therefore the measure of $F_{d'}$ is bounded by

$$\begin{aligned} \sum_{q \geq q_0} \sum_{p \in P(q)} q^{-d's} &\leq \sum_{n \geq n_0} \sum_{3^n \leq q < 3^{n+1}} \sum_{p \in P(q)} q^{-d's} \\ &\leq \sum_{n \geq n_0} \sum_{3^n \leq q < 3^{n+1}} 2^n 3^{-nd's} \\ &\leq \sum_{n \geq n_0} 2^{k-n} 2^n 3^{-nd's} \\ &= \sum_{n \geq n_0} 3^{(k-nd')s}. \end{aligned}$$

This sum converges because $k/n \sim d$ and $d' > d$, so by sending $n_0 \rightarrow \infty$ we see that $F_{d'}$ has zero measure. Therefore $d(F) = d$ as wanted.

Let $t \geq d$. Then, by Proposition 3.1 $\dim_{\mathbb{H}} F_t \geq \frac{ds}{t}$. For the upper bound we cover

$$F_t \subset \bigcup_{n \geq 1} \underbrace{\bigcup_{3^n \leq q \leq 3^{n+1}} \bigcup_{p \in P(q)} B(p, q^{-t})}_{\text{redundant}}$$

and observe that the indicated union has a redundancy in it owing to the fact that as q ranges through $(3^n, 3^{n+1}) \cap \mathbb{Z}$, $P(q)$ cycles with period 2^{k-n} . That is, we may more efficiently cover by

$$F_t \subset \bigcup_{n \geq 1} \bigcup_{q=3^n}^{3^n+2^{k-n}} \bigcup_{p \in P(q)} B(p, q^{-t})$$

The cost of a corresponding τ -dimensional cover is

$$\begin{aligned} \sum_{n \geq 1} \sum_{q=3^n}^{3^n+2^{k-n}} \sum_{p \in P(q)} \text{diam}(B(p, q^{-t}))^\tau &\ll \sum_{n \geq 1} \sum_{q=3^n}^{3^n+2^{k-n}} \sum_{p \in P(q)} q^{-t\tau} \\ &= \sum_{n \geq 1} \sum_{q=3^n}^{3^n+2^{k-n}} \#P(q) q^{-t\tau} \\ &\leq \sum_{n \geq 1} 2^{k-n} \#P(3^n) 3^{-nt\tau} \\ &\leq \sum_{n \geq 1} 2^k 3^{-nt\tau} \\ &\leq \sum_{n \geq 1} 3^{ks-nt\tau}. \end{aligned}$$

This sum converges if $\tau > ds/t$, therefore $\dim_{\mathbb{H}} F_t \leq ds/t$ and we are done. \square

The above result holds for more general Ahlfors–David regular sets than just the middle third Cantor set. Since it is especially useful to have examples with different dimensions having minimal Dirichlet exponent, see Proof of Proposition 2.5, we record the following partial extension.

Proposition 3.4. *Let F be the central Cantor set having dimension $s = -\log 2 / \log \lambda$, where $\lambda \in (0, 1/2)$. Then there is a choice P of abstract rationals such that $\dim_{\mathbb{H}} F_t = s/t$ for all $t \geq 1$.*

Proof. The proof of this statement is essentially the same as the proof of Proposition 3.3 (i). Using the analogous notation, for $\lambda^{-n} \leq q < \lambda^{-n-1}$ let

$$P(q) = E_n^L \cup \{\text{enough endpoints of } E_n \text{ to get maximality}\}.$$

Notice that $2^{n-1} \leq P(q) \leq 2^n$. Notice that $P(\lambda^{-n}) = E_n^L$ and every point of F is in the λ^n -neighbourhood of every $P(\lambda^{-n})$, so that $d(F) \geq 1$.

What is more, for any m ,

$$F_t \subset \bigcup_{n \geq m} \bigcup_{p \in P(\lambda^{-n})} [p - \lambda^{-nt}, p + \lambda^{-nt}].$$

This implies that

$$\mathcal{H}^\tau(F_t) \leq \sum_{m \geq n} 2^n \lambda^{-nt\tau},$$

which converges if $\tau > s/t$. This implies two things: for $t > 1$, we immediately obtain $\dim_{\mathbb{H}} F_t < \dim_{\mathbb{H}} F$ so that $d(F) = 1$. Further, we also see that $\dim_{\mathbb{H}} F_t \leq s/t$. Since Theorem 2.7 tells us that $\dim_{\mathbb{H}} F_t \geq s/t$, we have finished the proof of the theorem. \square

References

- [AS] R.K. Akhunzhanov and D.O. Shatskov. *On Dirichlet spectrum for two-dimensional simultaneous Diophantine approximation*, Mosc. J. Comb. Number Theory, **3**, No. 3–4, (2013), 5–23. [2](#)
- [AB] D. Allen and B. Bárány. *Hausdorff measures of shrinking targets on self-conformal sets.*, preprint, arXiv:1911.03410. [2](#)
- [B19] S. Baker. *An analogue of Khintchine’s theorem for self-conformal sets*. Math. Proc. Cambridge Philos. Soc. 167 (2019), no. 3, 567–597. [2](#)
- [BV06] V. Beresnevich and S. Velani. *A mass transference principle and the Duffin-Schaeffer conjecture for Hausdorff measures*. Ann. of Math. (2) 164 (2006), no. 3, 971–992. [2](#), [11](#)
- [BFR11] R. Broderick, L. Fishman and A. Reich. *Intrinsic approximation on Cantor-like sets, a problem of Mahler*. Mosc. J. Comb. Number Theory 1 (2011), no. 4, 3–12. [2](#)
- [BD16] Y. Bugeaud and A. Durand. *Metric Diophantine approximation on the middle-third Cantor set*. J. Eur. Math. Soc. (JEMS) 18 (2016), no. 6, 1233–1272. [2](#)
- [BBDF09] V. Beresnevich, V. Bernik, M. Dodson and S. Velani. *Classical metric Diophantine approximation revisited*, Analytic number theory, 38–61, Cambridge Univ. Press, Cambridge, (2009). [5](#)
- [F97] K. J. Falconer. *Techniques in Fractal Geometry*, John Wiley, (1997). [5](#)
- [F14] K. J. Falconer. *Fractal Geometry: Mathematical Foundations and Applications*, John Wiley & Sons, Hoboken, NJ, 3rd. ed., (2014). [2](#), [5](#), [6](#), [7](#), [9](#)
- [FMS18] L. Fishman, K. Merrill and D. Simmons. *Uniformly de Bruijn sequences and symbolic Diophantine approximation on fractals*. Ann. Comb. 22 (2018), no. 2, 271–293. [2](#)
- [FKMS18] L. Fishman, D. Kleinbock, K. Merrill and D. Simmons. *Intrinsic Diophantine approximation on manifolds: general theory*. Trans. Amer. Math. Soc. 370 (2018), no. 1, 577–599. [2](#)
- [FS14] L. Fishman and D. Simmons *Intrinsic approximation for fractals defined by rational iterated function systems: Mahler’s research suggestion*. Proc. Lond. Math. Soc. (3) 109 (2014), no. 1, 189–212. [2](#)
- [F20] J. M. Fraser. *Assouad dimension and fractal geometry*, CUP, Tracts in Mathematics Series, 222, (2020). [2](#), [3](#), [8](#)
- [HW] G. H. Hardy and E. M. Wright *An Introduction to the Theory of Numbers*, OUP, 6th Edition (2008). [13](#)
- [LSV07] J. Levesley, C. Salp and S. Velani. *On a problem of K. Mahler: Diophantine approximation and Cantor sets*. Math. Ann. 338 (2007), no. 1, 97–118. [2](#)
- [KM] D. Koukoulopoulos and J. Maynard *Duffin–Schaeffer Conjecture*, Ann. Math. 192 (1), (2020), 251–307. [12](#)
- [S21] J. Schleischitz. *On intrinsic and extrinsic rational approximation to Cantor sets*. Ergodic Theory Dynam. Systems 41 (2021), no. 5, 1560–1589. [2](#)

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