On Hölder maps and prime gaps

Haipeng Chen[1] and Jonathan M. Fraser[2]

[1]College of Big Data and Internet,
Shenzhen Technology University, Shenzhen, China.
hpchen0703@foxmail.com
[2]Mathematical Institute, University of St Andrews, UK
jmf32@st-andrews.ac.uk

April 16, 2021

Abstract

Let p_n denote the nth prime, and consider the function $1/n \mapsto 1/p_n$ which maps the reciprocals of the positive integers bijectively to the reciprocals of the primes. We show that Hölder continuity of this function is equivalent to a parametrised family of Cramér type estimates on the gaps between successive primes. Here the parametrisation comes from the Hölder exponent. In particular, we show that Cramér's conjecture is equivalent to the map $1/n \mapsto 1/p_n$ being Lipschitz. On the other hand, we show that the inverse map $1/p_n \mapsto 1/n$ is Hölder of all orders but not Lipschitz and this is independent of Cramér's conjecture.

Key words and phrases: primes, prime gaps, Cramér's conjecture, Hölder maps, Lipschitz maps.
Mathematics Subject Classification 2010: 11N05, 26A16.

1 Cramér's conjecture, Hölder maps, and our main result

Understanding the asymptotic properties of the primes is a fundamental and multifaceted problem in number theory. Let $\{p_n\}_{n=1}^{\infty}$ denote the set of primes where p_n is the nth prime number and $p_{n+1} > p_n$ for all n. Recall the *Prime Number Theorem* (PNT), which describes the asymptotic growth rate of p_n, and *Rosser's Theorem*, which bounds the nth prime by

$$n\left(\log n + \log\log n - \frac{3}{2}\right) \leq p_n \leq n\left(\log n + \log\log n - \frac{1}{2}\right)$$

for all $n \geq 20$. For further discussion of these results see [3, 6] and references therein.

A related problem is to consider the gaps between successive primes, see [2, 4, 5, 7]. *Cramér's conjecture* asserts that there should exist a constant $C > 0$ such that

$$p_{n+1} - p_n \leq C(\log p_n)^2$$

for all $n \geq 1$. In particular, using Rosser's theorem, Cramér's conjecture gives

$$p_{n+1} - p_n \leq C'(\log n)^2$$

for all $n \geq 1$ and a different constant C'. The main objective of this paper is to connect Cramér's conjecture to a problem concerning Hölder exponents of the natural map between the reciprocals of the positive integers and the reciprocals of the primes. This approach is motivated by various problems in metric geometry where one tries to understand a given metric space by identifying those spaces which are in the same bi-Lipschitz equivalence class. For example, bi-Lipschitz equivalence implies coincidence of familiar notions of fractal dimension such as Hausdorff, box and Assouad dimension. To this end we consider the bi-Hölder continuity of the map $1/n \mapsto 1/p_n$ mapping the reciprocals of the positive integers bijectively to the reciprocals of the primes. Our first result proves this map has a Hölder inverse of all orders. Recall that a map $f : X \to Y$ is *Hölder* (of order $\alpha \in (0,1)$) if there exists a constant $c \geq 1$ such that

$$|f(x) - f(y)| \leq c|x-y|^\alpha$$

for all $x, y \in X$. We assume here that X and Y are bounded subsets of Euclidean space, but this is not necessary in general. The map f is *bi-Hölder* if it is Hölder and has a Hölder inverse and *Lipschitz* if it satisfies the Hölder condition but with the optimal order $\alpha = 1$.

Theorem 1.1. *For all $\varepsilon > 0$ there exists an integer $N(\varepsilon)$ such that, for all $m > n > N(\varepsilon)$, we have*

$$\left|\frac{1}{p_n} - \frac{1}{p_m}\right| \geq \frac{1}{6} \cdot \left|\frac{1}{n} - \frac{1}{m}\right|^{1+\varepsilon}.$$

We note that Theorem 1.1 is sharp in the sense that it cannot be 'upgraded' to a Lipschitz bound. For example, results on bounded gaps between primes, e.g. [7], show that

$$\liminf_{n\to\infty}(p_{n+1} - p_n) < \infty.$$

In particular, applying this result together with the PNT yields

$$\liminf_{n\to\infty} \frac{\frac{1}{p_n} - \frac{1}{p_{n+1}}}{\frac{1}{n} - \frac{1}{n+1}} = \liminf_{n\to\infty} \frac{n(n+1)}{p_n p_{n+1}} \cdot (p_{n+1} - p_n) = 0.$$

This proves that the map $1/p_n \mapsto 1/n$ is not Lipschitz.

Hölder continuity of the forward map is more subtle. Our next result shows that if m and n are 'sufficiently separated', then a bi-Lipschitz estimate can be derived, up to a logarithmic error.

Theorem 1.2. *For all $0 < \varepsilon < 1$ there exist an integer $N(\varepsilon)$ such that for all $n \geq N(\varepsilon)$ and $m > \frac{1+\varepsilon}{1-\varepsilon}n$, we have*

$$\frac{1}{(1+\varepsilon)^2} \cdot \left(\frac{1}{n} - \frac{1}{m}\right)\frac{1}{\log m} \leq \frac{1}{p_n} - \frac{1}{p_m} \leq (1+\varepsilon) \cdot \left(\frac{1}{n} - \frac{1}{m}\right)\frac{1}{\log n}.$$

It follows immediately from Theorem 1.2 that the forward map is actually Lipschitz continuous in the range $m \geq 2n$ for sufficiently large n.

Corollary 1.1. *For all $0 < \varepsilon < 1$ there exists an integer $N(\varepsilon)$ such that for all $n \geq N(\varepsilon)$ and $m > \frac{1+\varepsilon}{1-\varepsilon}n$, we have*

$$\frac{1}{p_n} - \frac{1}{p_m} \leq \frac{1+\varepsilon}{\log N(\varepsilon)}\left(\frac{1}{n} - \frac{1}{m}\right).$$

Hölder continuity of the forward map over the full range is related to a parametrised family of Cramér type bounds on prime gaps. This is the content of Theorems 1.3-1.4.

Theorem 1.3. *Suppose that for $\varepsilon \geq 0$, there exists a constant $c(\varepsilon)$ and an integer $N(\varepsilon) \geq 20$ such that, for all $n > N(\varepsilon)$,*

$$\left|\frac{1}{p_n} - \frac{1}{p_{n+1}}\right| \leq c(\varepsilon)\left|\frac{1}{n} - \frac{1}{n+1}\right|^{1-\varepsilon}.$$

Then, for all sufficiently large n, we have

$$p_{n+1} - p_n \leq 2c(\varepsilon)(n^\varepsilon \log n)^2.$$

Conversely, the weaker forms of Cramér's conjecture imply the forward map is Hölder of all orders.

Theorem 1.4. *Suppose that for $\varepsilon \geq 0$, there exists a constant $c(\varepsilon)$ and an integer $N(\varepsilon) \geq 20$ such that, for all $n > N(\varepsilon)$,*

$$p_{n+1} - p_n \leq c(\varepsilon)(n^\varepsilon \log n)^2.$$

Then, for all sufficiently large n, we have

$$\left| \frac{1}{p_n} - \frac{1}{p_{n+1}} \right| \leq c(\varepsilon) \left| \frac{1}{n} - \frac{1}{n+1} \right|^{1-\varepsilon}.$$

Moreover, for all $n < m < 2n$ with n sufficiently large, we have

$$\left| \frac{1}{p_n} - \frac{1}{p_m} \right| \leq c(\varepsilon) \left| \frac{1}{n} - \frac{1}{m} \right|^{1-2\varepsilon}.$$

Combining Theorems 1.3, 1.4 and Corollary 1.1, we note that the $\varepsilon = 0$ case shows that Cramér's conjecture is equivalent to the map $1/n \mapsto 1/p_n$ being Lipschitz.

In light of Theorem 1.4, it is natural to ask for which $\varepsilon > 0$ are these weak Cramér bounds known to hold. Note that Bertrand's postulate may be regarded as the first step in this line of research, verifying the case $\varepsilon = 1$. To the best of our knowledge the state of the art here is provided by Baker, Harman and Pintz [1] who proved that the interval $[n, n + n^{0.525}]$ always contains a prime for sufficiently large n. In particular, combining this with Theorem 1.4, Corollary 1.1 and Theorem 1.1 yields the following corollary.

Corollary 1.2. *For all $\varepsilon > 0$, there exists a constant $C = C(\varepsilon) > 0$ such that, for all $m > n \geq 1$,*

$$C^{-1} \left| \frac{1}{n} - \frac{1}{m} \right|^{1+\varepsilon} \leq \left| \frac{1}{p_n} - \frac{1}{p_m} \right| \leq C \left| \frac{1}{n} - \frac{1}{m} \right|^{0.475}.$$

Proof. The result of Baker, Harman and Pintz [1] together with the PNT implies that for some constant $C' > 0$ we have

$$p_{n+1} - p_n \leq p_n^{0.525} \leq C'(n^{0.525/2} \log n)^2$$

for sufficiently large n. Applying Theorem 1.4 (with $\varepsilon = 0.525/2$ and so $1 - 2\varepsilon = 0.475$) proves the desired upper bound for sufficiently large n and $n < m < 2n$. Corollary 1.1 takes care of the case when $m \geq 2n$. Theorem 1.1 provides the lower bound for sufficiently large n. Finally, the result follows by ensuring C is chosen large enough to also deal with the small n. □

Theorem 1.1 is proved in Section 2. Theorem 1.2 is proved in Section 3. Finally, Theorems 1.3 and 1.4 are proved in Section 4.

2 Proof of Theorem 1.1: Hölder continuity of inverse map

In this section, we prove Theorem 1.1, which uses Rosser's theorem and the convex version of Jensen's inequality.

Proof of Theorem 1.1. Fix $\varepsilon > 0$. For sufficiently large n, we have

$$2(\log n)\log(n+1) \le (n(n+1))^{\frac{\varepsilon}{2}}.$$

Thus, by Rosser's theorem, for sufficiently large n

$$\left|\frac{1}{p_n} - \frac{1}{p_{n+1}}\right| \ge \frac{1}{p_n p_{n+1}} \ge \frac{1}{n(n+1)} \cdot \frac{1}{2(\log n)\log(n+1)} \ge \left(\frac{1}{n(n+1)}\right)^{1+\frac{\varepsilon}{2}} = \left|\frac{1}{n} - \frac{1}{n+1}\right|^{1+\frac{\varepsilon}{2}}.$$

We now consider two cases, assuming that n is sufficiently large for the above to hold.
Case 1. $n < m \le 2n$**.** It follows from the convex version of Jensen's inequality that

$$\begin{aligned}
\left|\frac{1}{p_n} - \frac{1}{p_m}\right| &= \left|\frac{1}{p_n} - \frac{1}{p_{n+1}}\right| + \cdots + \left|\frac{1}{p_{m-1}} - \frac{1}{p_m}\right| \\
&\ge \left|\frac{1}{n} - \frac{1}{n+1}\right|^{1+\frac{\varepsilon}{2}} + \cdots + \left|\frac{1}{m-1} - \frac{1}{m}\right|^{1+\frac{\varepsilon}{2}} \\
&= (m-n) \cdot \frac{1}{m-n} \cdot \left(\left|\frac{1}{n} - \frac{1}{n+1}\right|^{1+\frac{\varepsilon}{2}} + \cdots + \left|\frac{1}{m-1} - \frac{1}{m}\right|^{1+\frac{\varepsilon}{2}}\right) \\
&\ge (m-n) \cdot \left(\frac{1}{m-n}\right)^{1+\frac{\varepsilon}{2}} \cdot \left(\frac{1}{n} - \frac{1}{m}\right)^{1+\frac{\varepsilon}{2}} \\
&= \left(\frac{1}{m-n}\right)^{\frac{\varepsilon}{2}} \cdot \left(\frac{1}{n} - \frac{1}{m}\right)^{1+\frac{\varepsilon}{2}}.
\end{aligned}$$

Since $n < m \le 2n$, we have

$$\left(\frac{1}{m-n}\right)^{\frac{\varepsilon}{2}} \cdot \left(\frac{1}{n} - \frac{1}{m}\right)^{1+\frac{\varepsilon}{2}} = \left(\frac{1}{m-n}\right)^{\frac{\varepsilon}{2}} \cdot \left(\frac{1}{n} - \frac{1}{m}\right)^{-\frac{\varepsilon}{2}} \cdot \left(\frac{1}{n} - \frac{1}{m}\right)^{1+\varepsilon} \ge \left(\frac{1}{n} - \frac{1}{m}\right)^{1+\varepsilon}.$$

Case 2 $m > 2n$**.** It follows from Rosser's Theorem that for sufficiently large n

$$\left|\frac{1}{p_n} - \frac{1}{p_m}\right| \ge \left|\frac{1}{(3/2)n\log n} - \frac{1}{2n\log 2n}\right| \ge \frac{1}{6} \cdot \frac{1}{n\log n} \ge \frac{1}{6} \cdot \left(\frac{1}{n}\right)^{1+\varepsilon} \ge \frac{1}{6} \cdot \left|\frac{1}{n} - \frac{1}{m}\right|^{1+\varepsilon}.$$

Taking case 1 and 2 together proves the result. □

3 Proof of Theorem 1.2: Lipschitz continuity of forward map

We first provide an estimate for gaps between primes which will be used in the subsequent proof. We remark that the estimate is false if m and n are not assumed to be sufficiently separated. For example, results on bounded gaps between primes, e.g. [7], show that

$$\liminf_{n\to\infty}(p_{n+1} - p_n) < \infty.$$

In particular, $p_{n+1} - p_n$ cannot be bounded below by any function which grows without bound, such as $\log n$.

Proposition 3.1. *For all* $0 < \varepsilon < 1$*, there exists an integer* $N(\varepsilon)$ *such that for all* $n \ge N(\varepsilon)$ *and* $\frac{1+\varepsilon}{1-\varepsilon}n < m$*, we have*

$$(m-n)\log n \le p_m - p_n \le (1+\varepsilon)(m-n)\log m.$$

Proof. Fix $0 < \varepsilon < 1$. By Rosser's Theorem there exists an integer $N(\varepsilon)$ such that for any $n > N(\varepsilon)$, we have

$$n(\log n + (1-\varepsilon)\log\log n) \le p_n \le n(\log n + \log\log n)$$

and

$$1+\log n \le \left(1+\frac{\varepsilon}{4}\right)\log n, \qquad \log\log n + \frac{1}{\log n} \le \frac{\varepsilon}{2}\log n.$$

Then, for any $m \ge \frac{1+\varepsilon}{1-\varepsilon}n$, we obtain

$$(m\log m - n\log n) + ((1-\varepsilon)m\log\log m - n\log\log n) \le p_m - p_n$$

and

$$p_m - p_n \le (m\log m - n\log n) + (m\log\log m - (1-\varepsilon)n\log\log n)\,.$$

Consider the upper bound. Let $c_1(\varepsilon) = \frac{3}{2} - \frac{\varepsilon}{2} > 1$ noting that

$$m\log\log m - (1-\varepsilon)n\log\log n \le c_1(\varepsilon)\left(m\log\log m - n\log\log n\right),$$

and

$$\frac{c_1(\varepsilon)-1}{c_1(\varepsilon)-1+\varepsilon} = \frac{1-\varepsilon}{1+\varepsilon} \ge \frac{n\log\log n}{m\log\log m}.$$

The lower bound can be handled similarly. Let $c_2(\varepsilon) = \frac{1-\varepsilon}{2} \in (0,1)$ noting that

$$(1-\varepsilon)m\log\log m - n\log\log n \ge c_2(\varepsilon)(m\log\log m - n\log\log n),$$

and

$$\frac{1-c_2(\varepsilon)-\varepsilon}{1-c_2(\varepsilon)} = \frac{1-\varepsilon}{1+\varepsilon} \ge \frac{n\log\log n}{m\log\log m}.$$

Then, by mean value theorem applied to $x\log x$ and $x\log\log x$, we obtain

$$(1+\log n)(m-n) \le m\log m - n\log n \le (1+\log m)(m-n)$$

and

$$\left(\log\log n + \frac{1}{\log n}\right)(m-n) \le m\log\log m - n\log\log n \le \left(\log\log m + \frac{1}{\log m}\right)(m-n).$$

Thus for the upper bound, we have

$$p_m - p_n \le (m-n)(1+\log m) + c_1(\varepsilon)(m-n)\left(\log\log m + \frac{1}{\log m}\right) \le (1+\varepsilon)\cdot(m-n)\log m,$$

Similarly, for the lower bound, we have

$$p_m - p_n \ge (m-n)(1+\log n) + c_2(\varepsilon)(m-n)\left(\log\log n + \frac{1}{\log n}\right) \ge (m-n)\log n,$$

completing the proof. □

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Fix $\varepsilon > 0$. By appealing to Rosser's Theorem we may choose an integer $N'(\varepsilon) \ge N(\varepsilon)$ where $N(\varepsilon)$ is the constant from Proposition 3.1 such that, for any $n \ge N'(\varepsilon)$, $n\log n \le p_n \le (1+\varepsilon)n\log n$.

Consider $n \ge N'(\varepsilon)$ and $m \ge \frac{1+\varepsilon}{1-\varepsilon}n$. For the upper bound, it follows from Proposition 3.1 that

$$\frac{1}{p_n} - \frac{1}{p_m} \;=\; \frac{p_m - p_n}{p_m p_n} \le (1+\varepsilon)\cdot\frac{(m-n)\log m}{mn\log m\log n} \le (1+\varepsilon)\left(\frac{1}{n} - \frac{1}{m}\right)\frac{1}{\log n}.$$

The lower bound is similar. It also follows from Rosser's Theorem and Proposition 3.1 that

$$\frac{1}{p_n} - \frac{1}{p_m} = \frac{p_m - p_n}{p_m p_n} \ge \frac{1}{(1+\varepsilon)^2}\cdot\frac{(m-n)\log n}{mn\log m\log n} = \frac{1}{(1+\varepsilon)^2}\cdot\left(\frac{1}{n} - \frac{1}{m}\right)\frac{1}{\log m}$$

completing the proof. □

4 Proofs of Theorems 1.3-1.4: Hölder continuity of forward map and Cramér type estimates

Theorem 1.3 shows that Hölder continuity of the forward map $1/n \mapsto 1/p_n$ implies a weak form of Cramér's conjecture. This follows easily from Rosser's theorem.

Proof of Theorem 1.3. It follows from our assumption that for all $n > N(\varepsilon)$,

$$\frac{p_{n+1} - p_n}{p_n p_{n+1}} \leq c(\varepsilon) \left(\frac{1}{n(n+1)} \right)^{1-\varepsilon}.$$

Therefore, applying Rosser's Theorem, for sufficiently large n

$$\begin{aligned} p_{n+1} - p_n &\leq c(\varepsilon) \cdot p_n p_{n+1} \cdot \left(\frac{1}{n(n+1)} \right)^{1-\varepsilon} \\ &\leq 2c(\varepsilon) \cdot n^2 (\log n)^2 \cdot \left(\frac{1}{n} \right)^{2-2\varepsilon} \\ &\leq 2c(\varepsilon) \cdot (n^\varepsilon \log n)^2 \end{aligned}$$

as required. □

Theorem 1.4 provides a converse to the above, and requires a little more to prove.

Proof of Theorem 1.4. Applying Rosser's Theorem, for sufficiently large n,

$$p_n \geq n \log n.$$

Fix $\varepsilon \geq 0$. It follows from this estimate and our assumption that for sufficiently large n

$$\begin{aligned} \left| \frac{1}{p_n} - \frac{1}{p_{n+1}} \right| &= \frac{p_{n+1} - p_n}{p_n p_{n+1}} \\ &\leq c(\varepsilon) \cdot \frac{(n^\varepsilon \log n)^2}{n \cdot (n+1) \cdot \log n \cdot \log(n+1)} \\ &\leq c(\varepsilon) \cdot \left(\frac{1}{n(n+1)} \right)^{1-\varepsilon} \\ &= c(\varepsilon) \left| \frac{1}{n} - \frac{1}{n+1} \right|^{1-\varepsilon}. \end{aligned}$$

Moreover, we can "upgrade" this estimate for $n < m < 2n$ using the concave version of Jensen's inequality. We obtain

$$\begin{aligned} \left| \frac{1}{p_n} - \frac{1}{p_m} \right| &= \left| \frac{1}{p_n} - \frac{1}{p_{n+1}} \right| + \cdots + \left| \frac{1}{p_{m-1}} - \frac{1}{p_m} \right| \\ &= (m-n) \cdot \frac{1}{m-n} \cdot \left(\left| \frac{1}{p_n} - \frac{1}{p_{n+1}} \right| + \cdots + \left| \frac{1}{p_{m-1}} - \frac{1}{p_m} \right| \right) \\ &\leq c(\varepsilon)\, (m-n) \cdot \frac{1}{m-n} \cdot \left(\left(\frac{1}{n} - \frac{1}{n+1} \right)^{1-\varepsilon} + \cdots + \left(\frac{1}{m-1} - \frac{1}{m} \right)^{1-\varepsilon} \right) \\ &\leq c(\varepsilon)\, (m-n) \cdot \left(\frac{1}{m-n} \right)^{1-\varepsilon} \left(\frac{1}{n} - \frac{1}{m} \right)^{1-\varepsilon} \\ &= c(\varepsilon)\, (m-n)^\varepsilon \cdot \left(\frac{1}{n} - \frac{1}{m} \right)^{\varepsilon} \cdot \left(\frac{1}{n} - \frac{1}{m} \right)^{1-2\varepsilon} \\ &\leq c(\varepsilon) \left(\frac{1}{n} - \frac{1}{m} \right)^{1-2\varepsilon} \end{aligned}$$

completing the proof. □

Acknowledgements.

H. Chen is thankful for the excellent atmosphere for research provided by the University of St Andrews. The research of H. Chen was funded by China Scholarship Council (File No. 201906150102). J. M. Fraser was financially supported by an EPSRC Standard Grant (EP/R015104/1) and a Leverhulme Trust Research Project Grant (RPG-2019-034). The authors thank an anonymous referee for making helpful comments.

References

[1] R. C. Baker, G. Harman, J. Pintz, *The difference between consecutive primes, II.* Proc. Lond. Math. Soc., 2001, 83(3), 532–562.

[2] K. Ford, B. Green, S. Konyagin, J. Maynard, T. Tao, *Long gaps between primes.* J. Amer. Math. Soc., 2018, 31(1), 65-105.

[3] G. H. Hardy, E. M. Wright, R. Heath-Brown, J. Silverman, *An Introduction to the Theory of Numbers.* Oxford University Press, 2008.

[4] J. Maynard, *Small gaps between primes.* Ann. of Math., 2015, 181(1), 383-413.

[5] J. Maynard, *Large gaps between primes.* Ann. of Math., 2016, 183(3), 915-933.

[6] J. B. Rosser, L. Schoenfeld, *Approximate formulas for some functions of prime numbers.* Illinois J. Math., 1962, 6(1), 64-94.

[7] Y. Zhang, *Bounded gaps between primes.* Ann. of Math., 2014, 179(3), 1121-1174.