DISTANCE BETWEEN NATURAL NUMBERS BASED ON THEIR PRIME SIGNATURE

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Abstract. We define a new metric between natural numbers induced by the ℓ_∞ norm of their unique prime signatures. In this space, we look at the natural analog of the number line and study the arithmetic function $L_\infty(N)$, which tabulates the cumulative sum of distances between consecutive natural numbers up to N in this new metric.

Our main result is to identify the positive and finite limit of the sequence $L_\infty(N)/N$ as the expectation of a certain random variable. The main technical contribution is to show with elementary probability that for $K = 1, 2$ or 3 and $\omega_0, \dots, \omega_K \geq 2$ the following asymptotic density holds

$$\lim_{n\to\infty} \frac{\left|\left\{M \leq n : \|M - j\|_\infty < \omega_j \text{ for } j = 0, \dots, K\right\}\right|}{n} = \prod_{p:\,\text{prime}} \left(1 - \sum_{j=0}^{K} \frac{1}{p^{\omega_j}}\right).$$

This is a generalization of the formula for k-free numbers, i.e. when $\omega_0 = \ldots = \omega_K = k$. The random variable is derived from the joint distribution when $K = 1$.

As an application, we obtain a modified version of the prime number theorem. Our computations up to $N = 10^{12}$ have also revealed that prime gaps show a considerably richer structure than on the traditional number line. Moreover, we raise additional open problems, which could be of independent interest.

1. Introducing distances on the prime grid

The natural numbers form a totally ordered set, traditionally visualized by plotting them as evenly spaced points on the positive half of the number line. This representation gives a clear idea of their magnitude but not much else. Another possible method of visualizing the natural numbers is called the Ulam spiral [23], where the numbers are placed in increasing order on the grid points of $\mathbb{Z}^2$ starting with 1 at $(0,0)$ and spiraling outwards, so $2 \to (1,0)$, $3 \to (1,1)$, $4 \to (0,1)$, $5 \to (-1,1)$, $6 \to (-1,0)$ and so on. Interestingly, prime numbers tend to line up along certain diagonal lines corresponding to specific quadratic polynomials, which could be explained by a conjecture of Hardy and Littlewood [11] (if proven to be true).

Another natural, off-the-number-line spatial representation is suggested by the fundamental theorem of arithmetic. Each natural number is a unique grid point in an infinite dimensional coordinate system "spanned" by the prime numbers, which we coin the *prime grid*. The position of two natural numbers in this grid leads us to define new distances between them. The goal is to study how this effects the natural analogs of the number line and the distribution of primes.

1.1. **The prime grid.** The factorization theorem states that every natural number N can be uniquely identified with an infinite sequence $\mathbf{i}^N = (i_1, i_2, \dots)$ of non-negative integers,

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called the prime signature of N, so that

$$N = p_1^{i_1} p_2^{i_2} \dots p_k^{i_k} \dots ,$$

where throughout the paper $\{p_k\}_{k=1}^{\infty}$ denote the prime numbers in ascending order. The number 1 is represented by the sequence $\mathbf{0} = (0, 0, \dots)$. This gives a bijection between the positive integers and the space of all infinite sequences that consist only of zeros except for a finite number of positive integers. We think of $\mathbf{0}$ as the origin of a coordinate system whose axis are indexed by the prime numbers and each natural number is a grid point on this *prime grid*. The addition of two signatures $\mathbf{i}^N + \mathbf{j}^M$ and also the multiplication by a scalar $n \in \mathbb{N}$ is done coordinate-wise. The former represents multiplication NM, while the latter raises N to the nth power.

Remark 1.1. *ne can canonically extend the grid to get a module over the integers by allowing negative integers in a signature as well to represent all positive rational numbers. In this case subtraction $\mathbf{i}^N - \mathbf{j}^M$, done coordinate-wise, represents division N/M.*

Another possible generalization is to consider supernatural numbers or Steinitz numbers, i.e. signatures with possibly infinitely many non-zero coordinates that may take the value ∞ as well. However, we do not pursue these directions and consider only signatures of natural numbers henceforth.

We consider the ℓ_∞ norm of the signatures of natural numbers, i.e.

$$\|N\|_\infty = \|\mathbf{i}^N\|_\infty = \max\{i_1, i_2, \dots\}.$$

The norm naturally induces the ℓ_∞ metric, also referred to as the Chebyshev distance, which for two natural numbers N and M with signatures $\mathbf{i}^N$ and $\mathbf{j}^M$ is defined as

$$d_\infty(N, M) = d_\infty(\mathbf{i}^N, \mathbf{j}^M) = \max\{|i_1 - j_1|, |i_2 - j_2|, \dots\}.$$

Recall, a natural number N is called k-free if $\|N\|_\infty < k$. Hence, the metric naturally collects k-free numbers: they are enclosed in the ball of radius $k-1$ (centered at $\mathbf{0}$). In particular, the unit ball consists of the square-free numbers. We refer to the numbers M for which $\|M\|_\infty = k$ as the Chebyshev contour at distance k. It is well-known that the number of k-free numbers up to M follow an asymptotic of $M/\zeta(k) + O(\sqrt[k]{M})$, where $\zeta(k)$ is the Riemann zeta function and O is the usual big-O notation. Hence, the Chebyshev contours follow the asymptotic

$$\lim_{n\to\infty} \frac{|\{M \leq n \text{ such that } \|M\|_\infty = k\}|}{n} = \frac{1}{\zeta(k+1)} - \frac{1}{\zeta(k)}. \tag{1.1}$$

The main technical contribution of the paper is to generalize this asymptotic density to sets concerning consecutive numbers like $\{M \leq n \text{ such that } \|M\|_\infty = k,\ \|M-1\|_\infty = \ell\}$. With this new distance between the natural numbers, we wish to study an analog of the number line and also the distribution of prime numbers.

1.2. **The number trail: an analog of the number line.** We focus on a particular object on the prime grid, a zigzag path that starts at $\mathbf{0}$ and crisscrosses through every single grid point on the prime grid in the order of the increasing sequence of the natural numbers. We term this path the *number trail* and define an arithmetic function $L_\infty(N)$

tabulating the total length of the number trail up to N using the chosen metric. That is

$$L_\infty(N) := \sum_{M=2}^{N} d_\infty(M, M-1) = \sum_{M=2}^{N} \max\{\|M\|_\infty, \|M-1\|_\infty\}, \tag{1.2}$$

where the second equality holds, since M and $M-1$ are always coprime. $L(1) = 0$ by definition. We think of it as an analog of the traditional number line, where $L(N) = \sum_{M=1}^{N} |M-(M-1)| = N$. The sequence $\{L_\infty(N)\}_{N=2}^{\infty}$ is indexed in the On-line Encyclopedia of Integer Sequences (OEIS) database as sequence A334573.

It follows from (1.2) that the growth of $L_\infty(N)$ depends on the deterministic sequence

$$\mathbf{\Omega} = \|2\|_\infty, \|3\|_\infty, \|4\|_\infty, \dots .$$

More precisely, the asymptotic in (1.1) is not enough, but rather the distribution of pairs $(\|M\|_\infty, \|M-1\|_\infty)$ determine L_∞. This is the content of our main technical result.

Theorem 1.2. *For any $\omega_0, \omega_1 \geq 2$, the following asymptotic density is satisfied:*

$$\lim_{n\to\infty} \frac{\left|\{M \leq n : \ \|M\|_\infty < \omega_0 \text{ and } \|M-1\|_\infty < \omega_1\}\right|}{n} = \prod_{p\text{: prime}} \left(1 - \frac{1}{p^{\omega_0}} - \frac{1}{p^{\omega_1}}\right). \tag{1.3}$$

Let $\pi(\omega_0, \omega_1)$ denote the limiting constant in (1.3), moreover, let N_n denote a random integer chosen uniformly at random from the set $\{1, 2, \dots, n\}$.

Then the sequence of random variables $\{(\|N_n\|_\infty, \|N_n - 1\|_\infty)\}_n$ has a distributional limit

$$(\|N_n\|_\infty, \|N_n-1\|_\infty) \xrightarrow{d} (X_0, X_1),$$

with probability mass function $\mathbb{P}\big((X_0, X_1) = (k, \ell)\big)$ equal to

$$\begin{cases} \pi(2,2), & \text{if } k = \ell = 1, \\ \pi(2, \ell+1) - \pi(2, \ell), & \text{if } k = 1, \ell \geq 2, \\ \pi(k+1, 2) - \pi(k, 2), & \text{if } k \geq 2, \ell = 1, \\ \pi(k+1, \ell+1) - \pi(k+1, \ell) - \pi(k, \ell+1) + \pi(k, \ell), & \text{if } k, \ell \geq 2. \end{cases}$$

In particular, the limit of the expectation of $Y_n := \max\{\|N_n\|_\infty, \|N_n-1\|_\infty\}$ is equal to

$$C_0 := \lim_{n\to\infty} \mathbb{E} Y_n = \mathbb{E}\max\{X_0, X_1\} = 2.288369512646\dots .$$

We point out that the formula in (1.3) has previously been shown by Brandes in [4] using a power sieve method. Prior to [4], related results to (1.3) have only been obtained for $\boldsymbol{\omega}$ with equal coordinates, i.e. tuples of k-free numbers. Carlitz [5] was the first to show that (1.3) holds for $\omega_0 = \omega_1 = 2$. Through a number of intermediate steps, such as [6, 12, 16], the most general result for r-tuples of k-free numbers is due to Reuss [21]. A comprehensive list of results with a similar flavor can be found in [24]. The powerful sieve and determinant methods used in these proofs also give bounds on the error term.

In contrast, the new proof we present for (1.3) to obtain the leading term is a short and completely elementary probabilistic approach. A further advantage of it is that the density in (1.3) naturally generalises to triplets $\boldsymbol{\omega} = (\omega_0, \omega_1, \omega_2)$. Namely, we prove in Section 2 using the probabilistic approach that for every $\omega_0, \omega_1, \omega_2 \geq 2$:

$$\lim_{n\to\infty} \frac{\left|\{M \leq n : \ \|M-j\|_\infty < \omega_j \text{ for } j = 0,1,2\}\right|}{n} = \prod_{p\text{: prime}} \left(1 - \sum_{j=0}^{2} \frac{1}{p^{\omega_j}}\right). \tag{1.4}$$

Section 2 contains the proof of Theorem 1.2 as well. Moreover, we comment on potentially extending to longer $\boldsymbol{\omega}$ sequences in Remark 2.3. From (1.3) and (1.4) the limiting joint distributions of $(\|N_n\|_\infty, \|N_n-1\|_\infty)$ and $(\|N_n\|_\infty, \|N_n-1\|_\infty, \|N_n-2\|_\infty)$ can be readily obtained by a standard differencing argument.

Returning now to our original motivation of determining the asymptotic growth rate of $L_\infty(N)$, from Theorem 1.2 it is natural to guess that this should be $cN+o(N)$ for some $0<c<\infty$. Moreover, $L_\infty(N)/N$ is an average of the consecutive values of $\max\{\|M\|_\infty, \|M-1\|_\infty\}$. Thus, with the probabilistic interpretation of C_0, it is reasonable to expect that in fact $c=C_0$. This is precisely our main result.

Theorem 1.3. *The limit*

$$\lim_{N\to\infty} \frac{1}{N} L_\infty(N) \quad \textit{exists and is equal to} \quad C_0. \tag{1.5}$$

Open Problem 1. *What is the order of magnitude of* $|L_\infty(N) - C_0 N|$*?*

From the raw data, we estimated the order of magnitude of the error $|L_\infty(N) - C_0 N|$ to be $O(N^{0.28})$, considerably smaller than the leading term. Notice that the function $\max\{\|N\|_\infty, \|N-1\|_\infty\}$ of N (and thus $L_\infty(N)$) is neither additive nor multiplicative. Hence, well-developed methods from probabilistic number theory are not available to obtain bounds on the error term. Current bounds from the sieve methods also can not account for this drop, indicating that this could be a challenging problem. Figure 1 below shows a list plot of the ratios $L_\infty(p_k)/p_k$ for every 10^5-th prime up to 10^{12} plotted against estimates on its fluctuation around C_0, see Subsection 3.1 for details.

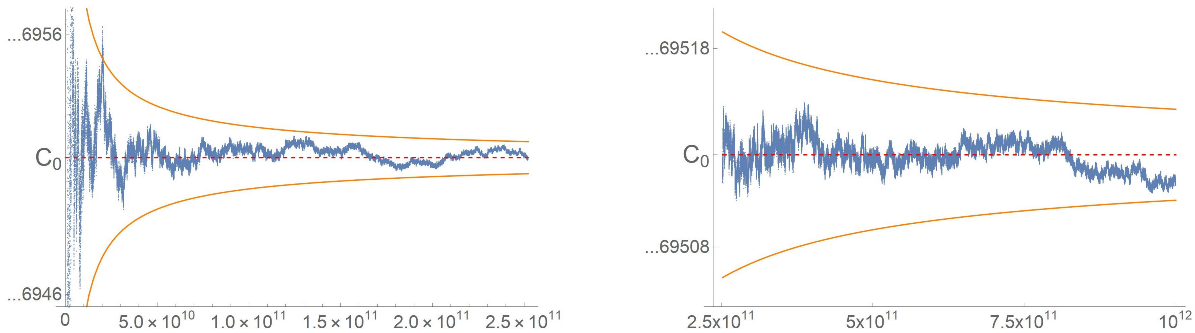


FIGURE 1. Ratio $L_\infty(p_k)/p_k$ for every 10^5-th prime up to 10^{12}. For reference, C_0 and $C_0 \pm N^{-0.72}$ are plotted as well. Each value on y-axis begins with $2.2883\ldots$.

1.3. **Distribution of primes on the number trail.** After establishing the asymptotic growth rate of $L_\infty(N)$, we turn to the effect that the new metric has on the distribution of primes. Let us modify the prime counting function to count the number of primes up to N on the number trail,

$$\pi_\infty(N) := \max\{k: \; L_\infty(p_k) \le N\}.$$

Of course, $\pi_\infty(L_\infty(p_k)) = k$. Combining Theorem 1.3 with the prime number theorem, we obtain a modified version for $\pi_\infty(N)$.

Theorem 1.4 (Modified Prime Number Theorem). *Let* $\mathrm{Li}(x)$ *denote the offset logarithmic integral function* $\int_2^x 1/\ln(y)\mathrm{d}y$. *Then*

$$\lim_{N\to\infty} \frac{\pi_\infty(N)}{N/\log N} = \lim_{N\to\infty} \frac{\pi_\infty(N)}{\mathrm{Li}(N)} = \frac{1}{C_0} = 0.43699236\ldots.$$

Proof. Using Theorem 1.3 we can write

$$\pi_\infty(N) = \max\{k:\ C_0 \cdot p_k + o(p_k) \le N\} = \max\{k:\ p_k \le N/C_0 + o(p_k)\}.$$

The prime number theorem further implies that

$$\pi_\infty(N) = \frac{(1+o(1))N/C_0}{\left(1 - \frac{\log C_0}{\log N}\right)\log N + o(1)}.$$

Dividing by $N/\log N$ and taking limit $N \to \infty$ concludes the proof. □

As in the case on the traditional number line the rate of convergence is very slow and $\mathrm{Li}(N)$ gives a better approximation to $C_0\pi_\infty(N)$ than $N/\log N$. In Table 1 we present the ratios for a few increasing values of p_k.

TABLE 1. Comparing $\pi_\infty(N)$ to $N/\log N$ and $\mathrm{Li}(N)$, for $N = L_\infty(p_k)$.

k	p_k	$\pi_\infty(N)\cdot\log(N)/N$	$\pi_\infty(N)/\mathrm{Li}(N)$
10^6	15 485 863	0.49053507 ...	0.46030511 ...
10^7	179 424 673	0.48303924 ...	0.45721224 ...
10^8	2 038 074 743	0.47735295 ...	0.45478434 ...
10^9	22 801 763 489	0.47294906 ...	0.45289153 ...
10^{10}	252 097 800 623	0.46942719 ...	0.45136754 ...
$2\cdot10^{10}$	518 649 879 439	0.46850132 ...	0.45096581 ...
$3\cdot10^{10}$	790 645 490 053	0.46798418 ...	0.45074112 ...

Our other interest lies in comparing the prime gap functions

$$\mathcal{D}_k^1 := L_\infty(p_{k+1}) - L_\infty(p_k) \quad\text{and}\quad \mathcal{D}_k^2 := \mathcal{D}_{k+1}^1 - \mathcal{D}_k^1$$

on the number trail to their counterparts $D_k^1 = p_{k+1} - p_k$ and $D_k^2 = D_{k+1}^1 - D_k^1$ on the traditional number line. In Section 4, we highlight some stark differences between them, such as the fact that $\mathcal{D}_k^1$ and $\mathcal{D}_k^2$ can take odd values as well.

1.4. **Other metrics.** In our opinion the ℓ_∞ metric is the most natural to use, however other metrics could be used as well. Another natural choice could be the ℓ_1 norm defined by $\|N\|_1 = \|\mathbf{i}^N\|_1 = \sum_k i_k$, which counts the total number of prime factors (with multiplicities) of N. This is the well-known additive arithmetic function $\Omega(N)$ in number theory. In particular, the numbers at unit distance from the origin in the ℓ_1 metric are the prime numbers. The balls with radius 2 or more don't have such a nice interpretation or asymptotic density as in the ℓ_∞ case, though.

The length of the number trail in this case is

$$L_1(N) = \sum_{M=2}^{N} d_1(M, M-1) = \sum_{M=2}^{N} \|M\|_1 + \|M-1\|_1 = \|N\|_1 + 2\sum_{M=2}^{N-1}\|M\|_1, \qquad (1.6)$$

where the second equality again holds, since M and $M-1$ are always coprime. Thus, the asymptotic growth of $L_1(N)$ is determined by the growth of $\sum_{M\leq N}\Omega(M)$.

Claim 1.5. *For every $N\geq 2$*

$$L_1(N)=2N\log\log N+2(A+B)N+o(N),$$

where A denotes the Meissel–Mertens constant and $B=\sum_k\big(p_k(p_k-1)\big)^{-1}$.

The ingredients for the proof can be found in for example [25]. We give the short argument here for completeness.

Proof. Since $\|N\|_1=O(\log\log N)$, it follows from (1.6) that $L_1(N)=2\sum_{M\leq N}\Omega(M)+o(N)$. Let p denote a prime and α a positive integer. Then

$$\sum_{p^\alpha\leq N}\frac{1}{p^\alpha}=\sum_{p\leq N}\frac{1}{p}+\sum_{\substack{p^\alpha\leq N\\ \alpha\geq 2}}\frac{1}{p^\alpha}=\sum_{p\leq N}\frac{1}{p}+\sum_{p\leq N}\frac{1}{p^2}\frac{1-o(1)}{1-1/p}=\log\log N+A+B+o(1). \quad (1.7)$$

Furthermore,

$$\sum_{M\leq N}\Omega(M)=\sum_{p^\alpha\leq N}\left[\frac{N}{p^\alpha}\right]=\sum_{p^\alpha\leq N}\frac{N}{p^\alpha}+o(N).$$

Combining this with (1.7) completes the proof. □

Structure of the paper. In the remainder of the paper, we only work with the ℓ_∞ norm. Section 2 contains all our results on limiting densities in $\mathbf{\Omega}$, including the proof of Theorem 1.2. We prove Theorem 1.3 in Section 3. Our results about the prime gap functions $\mathcal{D}_k^1$ and $\mathcal{D}_k^2$ are presented in Section 4.

An Appendix is included at the end with supplementary material. It includes raw numerical data of the histograms of $\mathcal{D}^1$ and $\mathcal{D}^2$ together with a detailed explanation of the Sagemath Python code used to generate the entire data set.

2. Limiting densities in $\mathbf{\Omega}$, proof of Theorem 1.2

In this section p is always a prime and let $N=N_n$ denote a random integer chosen uniformly at random from the set $\{1,2,\ldots,n\}$. The signature $\mathbf{I}=\mathbf{I}^N=(I_2,I_3,I_5,\ldots)$ of N is a random infinite dimensional vector, for which

$$N=\prod_p p^{I_p}.$$

When it is important to indicate that I_p is in the signature of N, we write I_p^N. It is well-known [2, Chapter 1.2] that as $n\to\infty$, the signature $\mathbf{I}$ tends in distribution

$$\mathbf{I}=(I_2,I_3,I_5,\ldots)\xrightarrow{d}\mathbf{Z}=(Z_2,Z_3,Z_5,\ldots),$$

where the Z_p are independent geometric random variables with distribution

$$\mathbb{P}[Z_p=k]=\left(1-\frac{1}{p}\right)\left(\frac{1}{p}\right)^k \quad \text{for } k=0,1,2,\ldots.$$

Moreover, Kubilius [14] was the first to show that the total variation distance

$$d_{\mathrm{TV}}\big(\mathcal{L}\big((I_p:p\leq b\big),\mathcal{L}\big((Z_p:p\leq b)\big)\big)\to 0 \quad \text{if } \frac{\log b}{\log n}\to 0.$$

Hence, one can immediately deduce the asymptotic in (1.1):

$$\begin{aligned}
&\lim_{n\to\infty} \frac{|\{M \le n \text{ such that } \|M\|_\infty = k\}|}{n} = \lim_{n\to\infty} \mathbb{P}[\|N_n\|_\infty = k] \\
&\quad = \lim_{n\to\infty} \mathbb{P}[\|N_n\|_\infty < k+1] - \mathbb{P}[\|N_n\|_\infty < k] \\
&\quad = \lim_{n\to\infty} \mathbb{P}[\forall\, p < n :\ I_p < k+1] - \mathbb{P}[\forall\, p < n :\ I_p < k] \\
&\quad = \mathbb{P}[\forall\, p :\ Z_p < k+1] - \mathbb{P}[\forall\, p :\ Z_p < k] = \prod_p \mathbb{P}[Z_p < k+1] - \prod_p \mathbb{P}[Z_p < k] \\
&\quad = \prod_p \left(1 - \frac{1}{p^{k+1}}\right) - \prod_p \left(1 - \frac{1}{p^k}\right) = \frac{1}{\zeta(k+1)} - \frac{1}{\zeta(k)}.
\end{aligned}$$

The main contribution of this section is to determine the asymptotics of the joint distributions

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{P}_n[\|N\|_\infty = \omega_0, \|N-1\|_\infty = \omega_1] \ \text{ and} \\
&\lim_{n\to\infty} \mathbb{P}_n[\|N\|_\infty = \omega_0, \|N-1\|_\infty = \omega_1, \|N-2\|_\infty = \omega_2] \text{ for } \omega_0, \omega_1, \omega_2 \ge 1,
\end{aligned}$$

where $\mathbb{P}_n$ indicates that N is chosen uniformly at random from $\{1, 2, \dots, n\}$.

We begin with the joint distribution of $(\|N_n\|_\infty, \|N_n - 1\|_\infty)$. Let us denote

$$\begin{aligned}
\Pi_n(\omega_0, \omega_1) &:= \mathbb{P}_n[\|N\|_\infty = \omega_0, \|N-1\|_\infty = \omega_1], \\
\Pi_n^<(\omega_0, \omega_1) &:= \mathbb{P}_n[\|N\|_\infty < \omega_0, \|N-1\|_\infty < \omega_1],
\end{aligned}$$

and

$$\pi(\omega_0, \omega_1) := \prod_p \left(1 - \frac{1}{p^{\omega_0}} - \frac{1}{p^{\omega_1}}\right).$$

Proposition 2.1. *The following asymptotics hold for* $\omega_0, \omega_1 \ge 2$*:*

$$\begin{aligned}
&\lim_{n\to\infty} \Pi_n(1,1) = \pi(2,2), \quad \lim_{n\to\infty} \Pi_n(1, \omega_1) = \pi(2, \omega_1 + 1) - \pi(2, \omega_1), \\
&\lim_{n\to\infty} \Pi_n(\omega_0, 1) = \pi(\omega_0 + 1, 2) - \pi(\omega_0, 2), \\
&\lim_{n\to\infty} \Pi_n(\omega_0, \omega_1) = \pi(\omega_0 + 1, \omega_1 + 1) - \pi(\omega_0 + 1, \omega_1) - \pi(\omega_0, \omega_1 + 1) + \pi(\omega_0, \omega_1).
\end{aligned}$$

Proof. Observe that $\Pi_n(1,1) = \Pi_n^<(2,2)$. Moreover, the standard differencing technique implies that for $\omega_0, \omega_1 \ge 2$:

$$\begin{aligned}
\Pi_n(1, \omega_1) = \Pi_n^<(2, \omega_1 + 1) - \Pi_n^<(2, \omega_1), \quad \Pi_n(\omega_0, 1) = \Pi_n^<(\omega_0 + 1, 2) - \Pi_n^<(\omega_0, 2), \\
\Pi_n(\omega_0, \omega_1) = \Pi_n^<(\omega_0 + 1, \omega_1 + 1) - \Pi_n^<(\omega_0 + 1, \omega_1) - \Pi_n^<(\omega_0, \omega_1 + 1) + \Pi_n^<(\omega_0, \omega_1).
\end{aligned}$$

Thus, it is enough to look at the probability $\Pi_n^<(\omega_0, \omega_1)$ with $\omega_0, \omega_1 \ge 2$:

$$\Pi_n^<(\omega_0, \omega_1) = \mathbb{P}_n\big[\forall\, p < n :\ I_p^N < \omega_0,\ I_p^{N-1} < \omega_1\big] = \prod_{p<n} \mathbb{P}_n\big[I_p^N < \omega_0,\ I_p^{N-1} < \omega_1\big].$$

The last equality holds because 'divisibility by primes are independent events'. More formally, the Chinese remainder theorem implies that for distinct primes p and q the map

$$\begin{cases} \mathbb{Z}/pq\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z} \\ x \mapsto (x \mod p, x \mod q) \end{cases}$$

is a bijection, see discussion around [13, Proposition 1.3.7.]. Hence, it remains only to show that $\mathbb{P}_n\big[I_p^N < \omega_0,\, I_p^{N-1} < \omega_1\big] \to 1 - 1/p^{\omega_0} - 1/p^{\omega_1}$ as $n \to \infty$. Indeed,

$$\begin{aligned}
\mathbb{P}_n\big[I_p^N < \omega_0,\, I_p^{N-1} < \omega_1\big] &= \mathbb{P}_n\big[I_p^N < \omega_0\big] - \mathbb{P}_n\big[I_p^N < \omega_0,\, I_p^{N-1} \geq \omega_1\big] \\
&= \mathbb{P}_n\big[I_p^N < \omega_0\big] - \mathbb{P}_n\big[I_p^N = 0,\, I_p^{N-1} \geq \omega_1\big] \\
&= \mathbb{P}_n\big[I_p^N < \omega_0\big] - \sum_{\ell=0}^{p-1} \underbrace{\mathbb{P}_n\big[I_p^N = 0,\, I_p^{N-1} \geq \omega_1,\, N \equiv \ell \mod p\big]}_{\neq 0 \iff \ell=1} \\
&= \mathbb{P}_n\big[I_p^N < \omega_0\big] - \mathbb{P}_n\big[I_p^{N-1} \geq \omega_1 \big|\, I_p^N = 0,\, N \equiv 1 \mod p\big] \cdot \mathbb{P}_n\big[N \equiv 1 \mod p\big] \\
&= 1 - \frac{1}{p^{\omega_0}} + o(1) - \frac{1/p^{\omega_1} + o(1)}{1-(1-1/p)+o(1)} \cdot \frac{\lfloor n/p \rfloor}{n} \to 1 - \frac{1}{p^{\omega_0}} - \frac{1}{p^{\omega_1}}.
\end{aligned}$$

In the last equality we used that $\mathbb{P}_n\big[I_p \geq \omega\big] = 1/p^{\omega} + o(1)$ and that conditioned on $N \equiv 1 \mod p$, we have $I_p^{N-1} \geq 1$, hence, we divide $1/p^{\omega_1}$ by $1 - \mathbb{P}_n\big[I_p^{N-1} = 0\big]$. □

Proof of Theorem 1.2. In the proof of Proposition 2.1, we already established the relative density in (1.3) and the probability mass function of the limiting distribution (X_0, X_1). The formula for $C_0 = \mathbb{E}\max\{X_0, X_1\}$ is given by:

$$\begin{aligned}
C_0 = {}&\pi(2,2) + 2\sum_{k=2}^{\infty} k\,\big(\pi(2,k+1) - \pi(2,k)\big) \qquad (2.1)\\
&+ \sum_{k=2}^{\infty}\sum_{\ell=2}^{\infty} \max\{k,\ell\}\big(\pi(k+1,\ell+1) - \pi(k+1,\ell) - \pi(k,\ell+1) + \pi(k,\ell)\big).
\end{aligned}$$

Table 2 below contains approximations of the value of C_0 by using the first m prime numbers and truncating the sums at $\|\cdot\|_\infty \leq n$. □

TABLE 2. Approximating C_0 by using the first m prime numbers and truncating the sums at $\|\cdot\|_\infty \leq n$.

m	$n = 30$	$n = 40$	$n = 50$	$n = 60$
10^3	2.288361286306563	2.288361316070792	2.288361316108944	2.288361316108984
10^4	2.288368990545230	2.288369020309459	2.288369020347611	2.288369020347651
10^5	2.288369450036167	2.288369479800395	2.288369479838548	2.288369479838588
10^6	2.288369480701602	2.288369510465830	2.288369510503983	2.288369510504023
10^7	2.288369482843734	2.288369512607963	2.288369512646115	2.288369512646155
10^8	2.288369482843734	2.288369512607963	2.288369512646115	2.288369512646155

2.1. **Joint distribution of** $(\|N_n\|_\infty, \|N_n - 1\|_\infty, \|N_n - 2\|_\infty)$**.** The technique is analogous to the previous case, only the notation and formulas get more involved. Let

$$\begin{aligned}
\Pi_n(\omega_0,\omega_1,\omega_2) &:= \mathbb{P}_n[\|N\|_\infty = \omega_0, \|N-1\|_\infty = \omega_1, \|N-2\|_\infty = \omega_2], \\
\Pi_n^{<}(\omega_0,\omega_1,\omega_2) &:= \mathbb{P}_n[\|N\|_\infty < \omega_0, \|N-1\|_\infty < \omega_1, \|N-2\|_\infty < \omega_2],
\end{aligned}$$

and also define

$$\pi(\omega_0,\omega_1,\omega_2) := \prod_p \left(1-\frac{1}{p^{\omega_0}}-\frac{1}{p^{\omega_1}}-\frac{1}{p^{\omega_2}}\right) \text{ and } \Pi(\omega_0,\omega_1,\omega_2) := \lim_{n\to\infty} \Pi_n(\omega_0,\omega_1,\omega_2).$$

Proposition 2.2. *The following asymptotics hold for* $\omega_0,\omega_1,\omega_2 \geq 2$*:*

$$\begin{aligned}
\Pi(1,1,1) &= \pi(2,2,2),\\
\Pi(1,1,\omega_2) &= \pi(2,2,\omega_2+1) - \pi(2,2,\omega_2),\\
\Pi(1,\omega_1,1) &= \pi(2,\omega_1+1,2) - \pi(2,\omega_1,2),\\
\Pi(\omega_0,1,1) &= \pi(\omega_0+1,2,2) - \pi(\omega_0,2,2),
\end{aligned}$$

moreover,

$$\begin{aligned}
\Pi(1,\omega_1,\omega_2) &= \pi(2,\omega_1+1,\omega_2+1) - \pi(2,\omega_1+1,\omega_2) - \pi(2,\omega_1,\omega_2+1) + \pi(2,\omega_1,\omega_2),\\
\Pi(\omega_0,1,\omega_2) &= \pi(\omega_0+1,2,\omega_2+1) - \pi(\omega_0+1,2,\omega_2) - \pi(\omega_0,2,\omega_2+1) + \pi(\omega_0,2,\omega_2),\\
\Pi(\omega_0,\omega_1,1) &= \pi(\omega_0+1,\omega_1+1,2) - \pi(\omega_0+1,\omega_1,2) - \pi(\omega_0,\omega_1+1,2) + \pi(\omega_0,\omega_1,2),
\end{aligned}$$

and finally,

$$\begin{aligned}
\Pi(\omega_0,\omega_1,\omega_2) = \pi(\omega_0+1,\omega_1+1,\omega_2+1) &- \pi(\omega_0+1,\omega_1+1,\omega_2)\\
- \pi(\omega_0+1,\omega_1,\omega_2+1) &+ \pi(\omega_0+1,\omega_1,\omega_2) - \pi(\omega_0,\omega_1+1,\omega_2+1)\\
+ \pi(\omega_0,\omega_1+1,\omega_2) &+ \pi(\omega_0,\omega_1,\omega_2+1) - \pi(\omega_0,\omega_1,\omega_2).
\end{aligned}$$

Proof. With the same differencing argument as in the proof of Proposition 2.1, we can express the probabilities $\Pi_n(\omega_0,\omega_1,\omega_2)$ with the probabilities $\Pi_n^{<}(\omega_0,\omega_1,\omega_2)$. Moreover, the same argument implies that

$$\Pi_n^{<}(\omega_0,\omega_1,\omega_2) = \prod_{p<n} \mathbb{P}_n\big[I_p^N < \omega_0,\ I_p^{N-1} < \omega_1,\ I_p^{N-2} < \omega_2\big].$$

Hence, it is enough to show that for every $\omega_0,\omega_1,\omega_2 \geq 2$:

$$\mathbb{P}_n\big[I_p^N < \omega_0,\ I_p^{N-1} < \omega_1,\ I_p^{N-2} < \omega_2\big] \to 1 - \frac{1}{p^{\omega_0}} - \frac{1}{p^{\omega_1}} - \frac{1}{p^{\omega_2}} \quad \text{as } n\to\infty.$$

The argument is a proper adaptation of the one in the proof of Proposition 2.1:

$$\begin{aligned}
&\mathbb{P}_n\big[I_p^N < \omega_0,\ I_p^{N-1} < \omega_1,\ I_p^{N-2} < \omega_2\big]\\
&\quad = \mathbb{P}_n\big[I_p^N < \omega_0,\ I_p^{N-1} < \omega_1\big] - \mathbb{P}_n\big[I_p^N < \omega_0,\ I_p^{N-1} < \omega_1,\ I_p^{N-2} \geq \omega_2\big].
\end{aligned}$$

We know the first term tends to $1 - 1/p^{\omega_0} - 1/p^{\omega_1}$, thus, it is enough to show that the second term tends to $1/p^{\omega_2}$. The case $p = 2$ requires separate treatment.

First assume that $p \geq 3$:

$$\begin{aligned}
\mathbb{P}_n\big[I_p^N < \omega_0,\ I_p^{N-1} < \omega_1,\ I_p^{N-2} \geq \omega_2\big] &= \mathbb{P}_n\big[I_p^N = 0,\ I_p^{N-1} = 0,\ I_p^{N-2} \geq \omega_2\big]\\
&= \sum_{\ell=0}^{p-1} \underbrace{\mathbb{P}_n\big[I_p^N = 0,\ I_p^{N-1} = 0,\ I_p^{N-2} \geq \omega_2,\ N \equiv \ell \mod p\big]}_{\neq 0 \iff \ell=2}\\
&= \mathbb{P}_n\big[I_p^{N-2} \geq \omega_2 \big|\, I_p^N = 0,\ I_p^{N-1} = 0,\ N \equiv 2 \mod p\big] \cdot \mathbb{P}_n\big[N \equiv 2 \mod p\big]\\
&= \frac{1/p^{\omega_2} + o(1)}{1-(1-1/p)+o(1)} \cdot \frac{\lfloor n/p \rfloor}{n} \to \frac{1}{p^{\omega_2}}.
\end{aligned}$$

Now assume that $p=2:$

$$\mathbb{P}_n\big[I_p^N<\omega_0,\ I_p^{N-1}<\omega_1,\ I_p^{N-2}\geq\omega_2\big]=\mathbb{P}_n\big[1\leq I_p^N<\omega_0,\ I_p^{N-1}=0,\ I_p^{N-2}\geq\omega_2\big].$$

First consider the case $\omega_0=2:$

$$\begin{aligned}\mathbb{P}_n\big[I_p^N=1,\ I_p^{N-1}=0,\ I_p^{N-2}\geq\omega_2\big]&=\mathbb{P}_n\big[I_p^N=1,\ I_p^{N-1}=0,\ I_p^{N-2}\geq\omega_2,\ N\equiv 0 \mod p\big]\\ &=\mathbb{P}_n\big[I_p^{N-2}\geq\omega_2\big|\ I_p^N=1,\ I_p^{N-1}=0,\ N\equiv 0 \mod p\big]\\ &\quad\cdot\mathbb{P}_n\big[I_p^N=1,\ I_p^{N-1}=0\big|\ N\equiv 0 \mod p\big]\cdot\mathbb{P}_n\big[N\equiv 0 \mod p\big]\\ &=\frac{1/p^{\omega_2}+o(1)}{1-(1-1/p)(1+1/p)+o(1)}\cdot\frac{(1-1/p)1/p+o(1)}{1-(1-1/p)+o(1)}\cdot\frac{\lfloor n/p\rfloor}{n}\to\frac{1}{p^{\omega_2}},\end{aligned}$$

where we also used that $1-1/2=1/2$. Finally, assume $\omega_0>2:$

$$\begin{aligned}&\mathbb{P}_n\big[I_p^N=1,\ I_p^{N-1}=0,\ I_p^{N-2}\geq\omega_2\big]\\ &\quad=\mathbb{P}_n\big[I_p^N=1,\ I_p^{N-1}=0,\ I_p^{N-2}\geq\omega_2\big]+\mathbb{P}_n\big[2\leq I_p^N<\omega_0,\ I_p^{N-1}=0,\ I_p^{N-2}\geq\omega_2\big].\end{aligned}$$

The second probability is equal to 0. Indeed, both N and $N-2$ can not be divisible by 4 (recall $\omega_2\geq 2$). The other term is exactly the same as in the previous point, where $\omega_0=2$. This concludes the proof. □

Remark 2.3. *Without checking all the details, we believe the asymptotic density*

$$\lim_{n\to\infty}\frac{\big|\big\{M\leq n:\ \|M-j\|_\infty<\omega_j \text{ for } j=0,1,2,3\big\}\big|}{n}=\prod_{p:\text{ prime}}\left(1-\sum_{j=0}^{3}\frac{1}{p^{\omega_j}}\right)$$

still holds for all quadruplets $\omega_0,\omega_1,\omega_2,\omega_3\geq 2$. The proof goes through without difficulty, except that $p=3$ needs to be handled separately as well (besides $p=2$). Even for $\omega_0=\omega_1=\omega_2=\omega_3=2$ we get the correct density of 0, agreeing with the fact that four consecutive numbers can never all be square-free. See Section 2.2 for details on these 'forbidden words' in $\mathbf{\Omega}$.

For sequences with length $k\geq 5$, the asymptotic density clearly can not be equal to $\prod_p\big(1-\sum_{j=0}^{k-1}p^{-\omega_j}\big)$ in general. For sequences of positive density, additional multiplicative factors depending on the sequence can be expected to appear coming from considerations about small primes.

2.2. **Forbidden words in $\mathbf{\Omega}$.** We consider $\mathbf{\Omega}$ as an infinite sequence of letters from the alphabet $\mathcal{A}=\mathbb{N}$. Let $\boldsymbol{\omega}=\omega_1,\dots,\omega_n$ denote a word of length $|\boldsymbol{\omega}|=n$ from $\mathcal{A}$. There is a set $\mathcal{F}$ of forbidden words which never appear in $\mathbf{\Omega}$. Observe that any subsequence of consecutive symbols of length 2^n must contain at least one element with $\|\cdot\|_\infty\geq n$ for any $n>1$. This defines

$$\mathcal{F}=\bigcup_{n=1}^{\infty}\mathcal{F}_n=\bigcup_{n=1}^{\infty}\big\{\boldsymbol{\omega}:\ |\boldsymbol{\omega}|=2^{n+1},\ \omega_i<n+1 \text{ for } 1\leq i\leq 2^{n+1}\big\}. \tag{2.2}$$

It describes additional structure present in the sequence $\mathbf{\Omega}$. Though, it does not influence the value of C_0 because it depends only on two consecutive symbols and the shortest forbidden word has length four.

Open Problem 2. *Does every finite length $\boldsymbol{\omega}\notin\mathcal{F}$ appear in $\mathbf{\Omega}$, which does not contain any $\boldsymbol{\tau}\in\mathcal{F}$ as a subsequence of consecutive symbols?*

We believe the answer is affirmative. Here is a simple argument to try to find a specific $\boldsymbol{\omega}$ in $\boldsymbol{\Omega}$. Consider an arbitrary $\boldsymbol{\omega} = \omega_1, \ldots, \omega_n \notin \mathcal{F}$, which does not contain any $\boldsymbol{\tau} \in \mathcal{F}$ as a subsequence of consecutive symbols. Choose any n distinct primes $\mathbf{p} = (p_1, \ldots, p_n)$ and look at the system of congruences

$$x + i - 1 \equiv 0 \mod p_i^{w_i}, \quad i = 1, \ldots, n. \tag{2.3}$$

The Chinese remainder theorem implies that there exists a unique x between 1 and $M = \prod_{i=1}^n p_i^{w_i}$ which satisfies (2.3). Then of course $\|x+i-1\|_\infty \geq \omega_i$. If all are equalities, then we found $\boldsymbol{\omega}$ in $\boldsymbol{\Omega}$. If not, then we can try with $x+kM$ for some positive integer k, since all such numbers satisfy (2.3). It is unclear whether such a k always exists. It need not exist for all choices of $\mathbf{p}$. For example, $\boldsymbol{\omega} = 1, 1, 1$ never appears in $\boldsymbol{\Omega}$ with $\mathbf{p} = (2, 3, 5)$, but when $\mathbf{p} = (5, 2, 7)$ it does with $x = 5$. For illustration, we give some non-trivial examples of $\boldsymbol{\omega}$ in $\boldsymbol{\Omega}$ in Table 3.

TABLE 3. Selected non-trivial $\boldsymbol{\omega}$ with their place of appearance in $\boldsymbol{\Omega}$

$\boldsymbol{\omega}$	$x + kM$	k	$\mathbf{p}$
$17, 30$	27 699 975 238 617 792 512	1	$(2, 3)$
$1, 15, 3, 14$	18 890 469 353 465 057 219 498	7	$(2, 3, 5, 7)$
$1, 2, 2, 1, 3, 5, 2, 1$	93 377 215 627 231 323	16	$(3, 2, 5, 7, 11, 13, 17, 19)$

We conjecture a stronger statement claiming that all such $\boldsymbol{\omega}$ have a strictly positive limiting density in $\boldsymbol{\Omega}$.

Conjecture 1. *Assume $\boldsymbol{\omega} = (\omega_0, \omega_1, \ldots, \omega_k) \notin \mathcal{F}$ is a finite length word which does not contain any $\boldsymbol{\tau} \in \mathcal{F}$ as a subsequence of consecutive symbols. Then the limit*

$$\lim_{n\to\infty} \frac{\left|\{M \leq n : \ \|M - j\|_\infty = \omega_j \text{ for } j = 0, 1 \ldots, k\}\right|}{n}$$

exists and is strictly positive.

3. ASYMPTOTIC GROWTH OF $L_\infty(N)$, PROOF OF THEOREM 1.3

Recall, $L_\infty(N) = \sum_{M=2}^N \max\{\|M\|_\infty, \|M-1\|_\infty\}$ and the notation

$$\Pi_N(\omega_0, \omega_1) = \frac{\left|\{M \leq N : \ \|M\|_\infty = \omega_0, \|M-1\|_\infty = \omega_1\}\right|}{N}.$$

Proof of Theorem 1.3. It is enough to approximate $L_\infty(N)$ from below with the sequence

$$L_k(N) := \sum_{M=2}^{N} \max\{\|M\|_k, \|M-1\|_k\},$$

where $\|M\|_k := \min\{\|M\|_\infty, k\}$. On one hand, the limit

$$C_k := \lim_{N\to\infty} \frac{1}{N} L_k(N) = \sum_{\omega_0=1}^{k} \sum_{\omega_1=1}^{k} \max\{\omega_0, \omega_1\} \lim_{N\to\infty} \Pi_N(\omega_0, \omega_1) \tag{3.1}$$

exists, because the value of $\lim_{N\to\infty} \Pi_N(\omega_0, \omega_1)$ is well-defined from Proposition 2.1. Moreover, the sequence C_k is non-decreasing and is bounded from above by

$$\lim_{N\to\infty} \frac{1}{N} L_\infty(N) \le \lim_{N\to\infty} \frac{2}{N} \sum_{M=2}^{N} \|M\|_\infty \overset{(1.1)}{=} 2\sum_{\ell=1}^{\infty} \ell\Big(\frac{1}{\zeta(\ell+1)} - \frac{1}{\zeta(\ell)}\Big) = 2\sum_{\ell=1}^{\infty}\Big(1 - \frac{1}{\zeta(\ell)}\Big),$$

which is an absolutely convergent series. Hence, C_k has a limit and comparing (3.1) with (2.1) we see that $\lim_{k\to\infty} C_k = C_0$.

On the other hand, there is an absolute constant $c > 0$ for which

$$\frac{L_k(N)}{N} \le \frac{L_\infty(N)}{N} \le \frac{L_k(N)}{N} + 2\sum_{\ell>k}\sum_{p:\,\text{prime}} \frac{\ell}{p^\ell} \le \frac{L_k(N)}{N} + \sum_{p:\,\text{prime}} \frac{c}{p^{k+1}}.$$

First letting $N \to \infty$ and then $k \to \infty$, we conclude that $\lim_{N\to\infty} L_\infty(N)/N = C_0$. □

3.1. **Direct computation of L_∞.** In order to analyze the distribution of prime gaps $\mathcal{D}_k^1 = L_\infty(p_{k+1}) - L_\infty(p_k)$, we kept track of the values of L_∞ for all primes. See Appendix A for the complete program code with explanations. To obtain the list plot in Figure 1, we calculated the ratios $L_\infty(p_k)/p_k$ for every 10^5-th prime between 1 and 10^{12}. This gave a set of 376 079 data points. The minimal value is 2.28836250 and the maximal is 2.288371417, giving a difference of 8.9×10^{-6}. The list plot revealed that the fluctuations of the ratios diminished quite rapidly around C_0 from Theorem 1.2.

To get an estimate on the order of magnitude of the error term, we calculated

$$\alpha(N) := \frac{\log |L_\infty(N) - C_0 \cdot N|}{\log N}$$

for our data points, and obtained the list plot shown in Figure 2. The maximum value is 0.298731, obtained at 848 321 917. The last place where it exceeds 0.28 is at 20 571 786 113. The plot shows that most frequently the value lies around 0.22, but this may be misleading, since $N = 10^{12}$ is not too large. We do conjecture that $L_\infty(N) = C_0 \cdot N + O(N^{0.28})$.

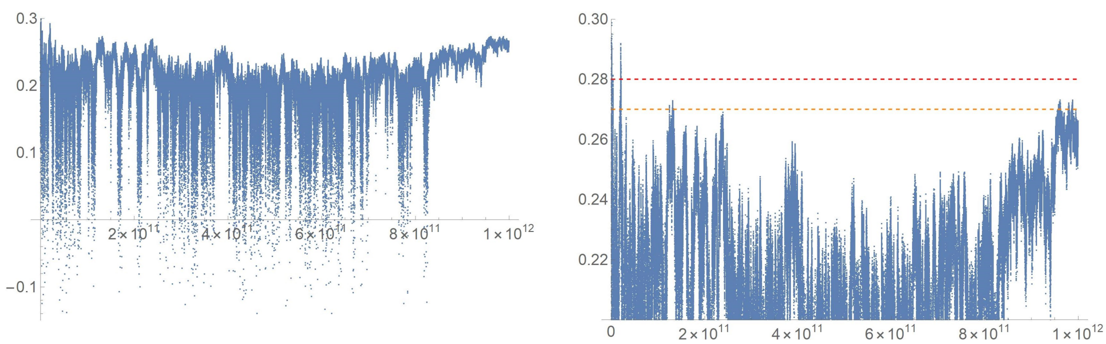


FIGURE 2. Value of $\alpha(p_k)$ for every 10^5-th prime between 1 and 10^{12} : full-range of values on left, restricting to $\alpha(p_k) \in [0.2, 0.3]$ on right.

4. PRIME GAPS ON THE NUMBER TRAIL

On the traditional number line the most widely used representation for the distribution of prime numbers are the prime gap functions

$$D_k^1 = p_{k+1} - p_k \quad \text{and} \quad D_k^2 = D_{k+1}^1 - D_k^1.$$

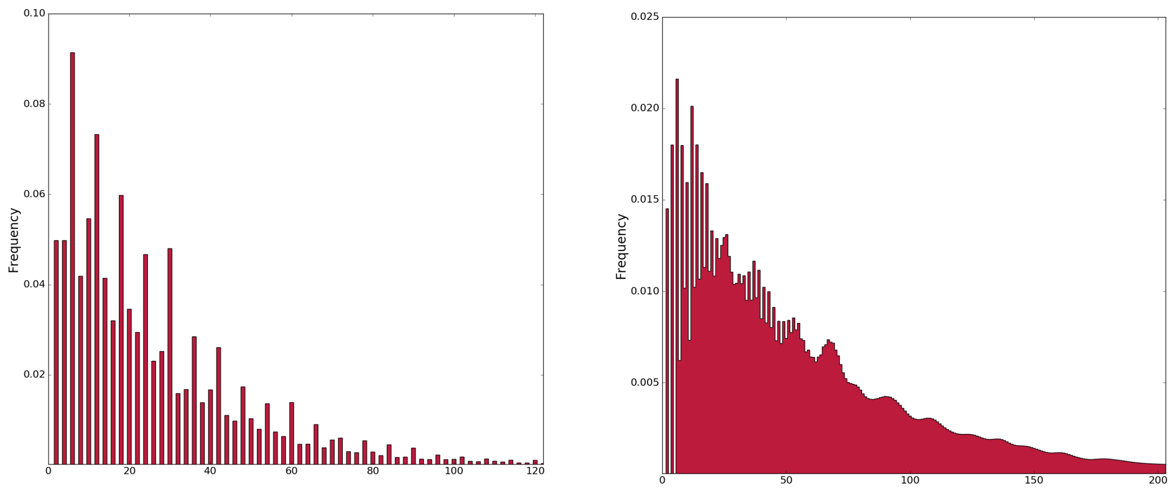
Especially D_k^1 has continuously received immense attention with results establishing large gaps between primes [7, 26], small gaps [9, 10], and limit points of $D_k^1/\log p_k$ [3, 18]. Zhang [27] made a big breakthrough by proving that D_k^1 was bounded from above by a constant for infinitely many k. The original constant of 7×10^7 has been greatly reduced by work of the Polymath Project [19, 20] and Maynard [15]. This list only gives a glimpse, it is far from being exhaustive.

On the other hand, there are long-standing conjectures which are still open today. The twin prime conjecture asserts that $D_k^1 = 2$ for infinitely many k. Polignac's conjecture is even stronger, stating that for any positive even number N there are infinitely many k such that $D_k^1 = N$.

Here we propose an alternative approach to study the distribution of prime numbers by looking at the prime gaps along the number trail. Analogous to the prime gap functions D^1 and D^2, we define the differences $\mathcal{D}^1$ and $\mathcal{D}^2$ along the number trail with L_∞ to be

$$\mathcal{D}_k^1 := L_\infty(p_{k+1}) - L_\infty(p_k) \quad \text{and} \quad \mathcal{D}_k^2 := \mathcal{D}_{k+1}^1 - \mathcal{D}_k^1.$$

Histograms already reveal stark differences between D^1, D^2 and $\mathcal{D}^1, \mathcal{D}^2$. Figure 3 shows the histograms of D^1 and $\mathcal{D}^1$ side-by-side taking prime numbers up to $N \leq 10^{12}$, while Figure 4 shows D^2 and $\mathcal{D}^2$. Appendix B contains tables of the numerical data used in the figures.

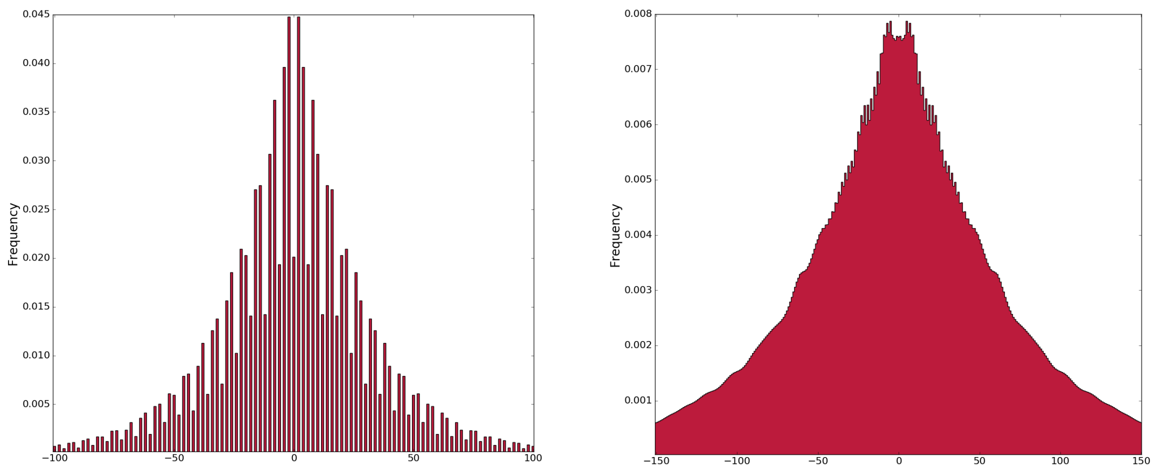


(A) $D_k^1 = p_{k+1} - p_k$ (B) $\mathcal{D}_k^1 = L_\infty(p_{k+1}) - L_\infty(p_k)$

FIGURE 3. Histogram of first order differences between consecutive prime numbers up to $N \leq 10^{12}$ on the number line (left) and on the number trail (right).

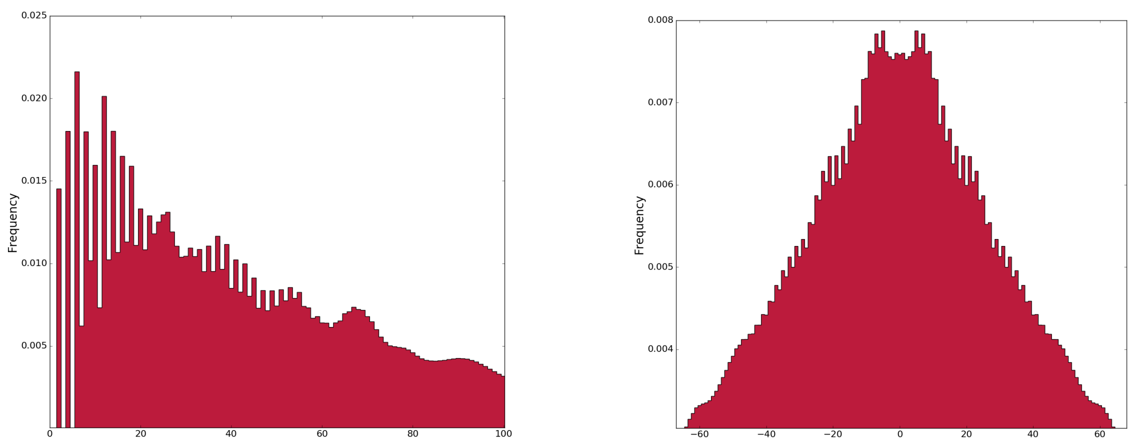
The semi-regular spiked structure of the histograms of D^1 and D^2 has been shown [1] to be attributed to the fact that for every prime number p, except 2, $p = \pm 1 \mod 6$. These spikes are replaced by an intricate, non-repeating fine structure on the $\mathcal{D}^1$ and $\mathcal{D}^2$ histograms with numerous, differently shaped local peaks. Most notably, as becomes better visible in Figure 5 (B) the cap of the $\mathcal{D}^2$ histogram is not the highest peak, it has two tiny local maxima at ± 1, but there are two more, symmetrical and significantly higher maxima at ± 5 and $\pm 7$the cap of the $\mathcal{D}^2$ histogram is not the highest peak, it has two tiny local maxima at ± 1, but there are two more, symmetrical and significantly higher maxima at ± 5 and ± 7.

Equally apparent difference between D^1, D^2 and $\mathcal{D}^1, \mathcal{D}^2$ is that the former only take even values (except for the gap between 2 and 3), whereas the latter can also take odd values.



(A) $D_k^2 = D_{k+1}^1 - D_k^1$ (B) $\mathcal{D}_k^2 = \mathcal{D}_{k+1}^1 - \mathcal{D}_k^1$

FIGURE 4. Histogram of second order differences between consecutive prime numbers up to $N \leq 10^{12}$ on the number line (left) and on the number trail (right).



(A) $\mathcal{D}_k^1$, close-up (B) $\mathcal{D}_k^2$, close-up

FIGURE 5. Close-up of $\mathcal{D}_k^1$ and $\mathcal{D}_k^2$, showing the local maxima.

Claim 4.1. *$\mathcal{D}^1$ takes arbitrarily large values. However, $\mathcal{D}^1$ never takes the values 3 or 5.*

Proof. The first assertion simply follows from the fact that $L_\infty(p_{k+1}) - L_\infty(p_k) \geq p_{k+1} - p_k = D_k^1$ and D_k^1 takes arbitrarily large values.

For the second assertion we can assume that $p_k \neq 2$ (since $\mathcal{D}_1^1 = 1$). Since $\mathcal{D}_k^1 \geq p_{k+1} - p_k$, $\mathcal{D}_k^1$ could equal 3 if and only if $p_{k+1} = p_k + 2$ and similarly $\mathcal{D}_k^1$ could equal 5 if and only if $p_{k+1} = p_k + 2$ or $p_{k+1} = p_k + 4$.

If $p_{k+1} = p_k + 2$, then by definition (1.2):

$$\mathcal{D}_k^1 = \max\{\|p_k\|_\infty, \|p_k + 1\|_\infty\} + \max\{\|p_k + 1\|_\infty, \|p_k + 2\|_\infty\} = 2\|p_k + 1\|_\infty, \qquad (4.1)$$

which is clearly an even number.

If $p_{k+1} = p_k + 4$, then either $p_k + 1$ or $p_k + 3$ is divisible by 4, hence has ℓ_∞ norm at least 2 and so $\mathcal{D}_k^1 \geq 2 + 2 + 1 + 1 > 5$. □

Polignac's Conjecture for $\mathcal{D}_k^1$ can be formulated as follows.

Conjecture 2 (Modified Polignac's Conjecture)**.** *For any $N \in \mathbb{N} \setminus \{1, 3, 5\}$ there are infinitely many k such that $\mathcal{D}_k^1 = N$.*

It is also not difficult to characterize $\mathcal{D}^1$ for the first few small values.

Claim 4.2. *$\mathcal{D}_k^1$ equals (a) 2, (b) 4, or (c) 6 for some k if and only if*

(a) p_{k+1} and p_k are twin prime and $p_k + 1$ is a square-free number,
(b) p_{k+1} and p_k are twin prime and $\|p_k + 1\|_\infty = 2$,
(c) p_{k+1} and p_k are twin prime and $\|p_k + 1\|_\infty = 3$ OR $p_{k+1} = p_k + 4$ and $\|p_k + 2\|_\infty = 1$.
In general, if p_{k+1} and p_k are twin prime and $\|p_k + 1\|_\infty = n$ then $\mathcal{D}_k^1 = 2n$.

Proof. We saw at the end of the proof of Claim 4.1 that if $p_{k+1} - p_k \geq 4$ then $\mathcal{D}_k^1 > 5$. Hence, if $\mathcal{D}_k^1 = 2$ or 4, then p_{k+1} and p_k must be twin prime. Moreover, (4.1) implies that $\|p_k + 1\|_\infty = 1$ or 2, respectively. If $\mathcal{D}_k^1 = 6$ then either p_{k+1} and p_k are twin prime and $\|p_k + 1\|_\infty = 3$ or $p_{k+1} = p_k + 4$ ($p_{k+1} \geq p_k + 6$ not possible). If $p_{k+1} = p_k + 4$ then it further follows that one of $p_k + 1$ or $p_k + 3$ has ℓ_∞ norm equal to 2 and so $p_k + 2$ must be square-free.

The other directions and the last claim are just trivial calculations. □

Another key difference between D^1 and $\mathcal{D}^1$ is that $\mathcal{D}^1$ makes a distinction between twin primes, according to the norm of the number between them. We compared the relative frequency amongst the first K twin primes (p_k, p_{k+1}) of the norm $\|p_k + 1\|_\infty = \ell$ to the limiting distribution of the norm of a number between two square-free numbers given by $\lim_{n\to\infty} \Pi_n(1, \ell, 1) / \sum_k \Pi_n(1, k, 1)$ from Proposition 2.2.

Table 4 shows that the relative frequencies of $\|p_k + 1\|_\infty = \ell$ settle around specific constants which are not equal to the ones given by Proposition 2.2. Most notably, the relative frequencies are not strictly decreasing, the relative frequency of $\|p_k + 1\|_\infty = 2$ is larger than the one for $\|p_k + 1\|_\infty = 1$. A first step toward determining these constants would be to study the limiting distribution of $\|p_k + 1\|_\infty = \ell$ along the subsequence of primes.

Open Problem 3. *Let P_n denote a random prime number chosen uniformly at random from the set $\{1, 2, \ldots, n\}$. Does $\|P_n - 1\|_\infty$ converge in distribution? If so, what is the distribution of the limiting random variable?*

TABLE 4. Comparing $\|N\|_\infty = \ell$ when $\|N \pm 1\|_\infty = 1$ (first column) to the relative frequencies $\|p_k + 1\|_\infty = \ell$ amongst the first K twin primes (p_k, p_{k+1}).

ℓ	if $\|N \pm 1\|_\infty = 1$	$K = 1 \times 10^8$	$K = 2 \times 10^8$	$K = 4 \times 10^8$	$K = 8 \times 10^8$
1	0.3889570915	0.29185522	0.29183928	0.291823840	0.29180058625
2	0.3156950811	0.36190949	0.36189782	0.361889505	0.361905095
3	0.1545003396	0.18599846	0.18601558	0.1860339925	0.186028890
4	0.0730237294	0.08559793	0.085607245	0.085600065	0.08560440375
5	0.0348180734	0.03929783	0.039304695	0.0393018075	0.039307170
6	0.0168170300	0.01835525	0.01835070	0.0183527025	0.01835503625

APPENDIX A. SAGEMATH PYTHON CODE USED TO CALCULATE $L_\infty(N)$

Computations were carried out with *Sagemath* 7.2 [22] running on 8 hyperthreaded 3.6 GHz i7 CPU cores, and using 32 GB of RAM and about 500 GB of harddisk storage on a 64-bit Linux desktop computer. It took about 80 days of total walltime to complete all computations up to the order of $N = 10^{12}$. Hardware resources were maximized by running the computations in 20 independent batches and utilizing a hand-tuned, load-balanced parallelization protocol. The code listing below is mostly self explanatory with comments and we provide detailed explanations where necessary, referring to line numbers.

(2) The only non-standard package we used is called *bignumpy* [8] and it is crucial for seamlessly handling very large *numpy* arrays. Our computations heavily utilize *numpy* array operations but these become prohibitive using standard *numpy* when the size of a *numpy* array exhausts available memory. *Bignumpy* provides file backed *numpy* array objects for *Python* utilizing the mmap/shared memory feature of Unix. *Bignumpy* allows for streamlined array operations on arrays multiple times larger than available memory, with a negligible slowdown compared to system swap.

(12) The *@parallel* decoration applied to the *calc_norm00* function invokes the parallel interface, which means that the single set of (N, M) function arguments will be replaced by a list of multiple $(N1, M1), (N2, M2), ..., (Nk, Mk)$ arguments, see (13) and (78). The parallel interface will then execute, simultaneously, multiple copies of *calc_norm00* running on multiple ($ncpus = 8$) CPU cores in parallel. The argument list is automatically divided up and distributed among the parallel processes, and in the end, the results are reassembled in a single list.

(22) This is the innermost loop—the rate limiting computation of integer factorization. The *factor* function (24) is a wrapper around the standard *PARI* factorization routine [17]. Note that we did not use specialized factorization algorithms suitable for certain classes of numbers, because the computation of $L_\infty(N)$ requires the factorization of every single number $1 - N$. The number of iterations in the innermost loop should be hand tuned to achive optimum balance between computation and data manipulation (see more about this below). Also note that determining whether or not a number N is prime can, of course, be calculated much faster than full factorization but *factor* has already been called on N.

(38-44) The input parameters *N_∗* allow flexibility to compute $L_\infty(N)$ depending on the hardware configuration. *N_min* and *N_max* define a particular segment, in our computations we used 20 segments each $5*10^{10}$ long to get to 10^{12}. The workflow is organized in three nested loops. We already mentioned the innermost loop (22), which includes multiple factorizations calculated in a single call to *calc_norm00*. The inner loop (77) iterates multiple calls to *calc_norm00* using the parallel interface as explained above (12). Note that the argument list (78) is a standard *Python* list, which has a significant memory footprint and, therefore, the balance between the chosen values of *N_inmost_loop* and *N_inner_loop* is crucial. i) Their product *N_maxmem* should be set such that the resulting *Python* list (78) fits comfortably in memory. The associated work arrays (47-59) are all *bignumpy* arrays and their size is not limiting. ii) *N_inmost_loop* should be set $>> 1$ to make sure that computation (factorization) dominates because the overhead of data manipulation associated with a single function call to *calc_norm00* in the parallel environment is significant. In fact, the maximum overall speedup using 8 CPU cores was limited to less than five fold.

(63) The core data set at the heart of our computations is comprised of the $L_\infty(N)$ values at prime "stops" along the number trail. The associated *bignumpy* array can be generated piece wise by concatenating consecuitve sub-arrays generated in a succession of batch calculations as noted above. Since the file associated with a *bignumpy* array is the exact binary copy of the array's memory image, these files can readily be concatenated using the UNIX *cat* command.

(70, 71, 113, 114) $L_\infty(N)$ is computed as the cumulative sum of the sequence of "hops" between consecutive numbers along the number trail and, therefore, every new batch computation needs two data points from the previous calculation to start with. One is the length of the last hop and the other is the current value of $L_\infty(N)$.

(73-101) The cumulative summation is carried out in the outer loop utilizing a number of *numpy* operations. i) First, the sequence of hops is computed for a continuous segment $N_{beg} - N_{end}$ by taking the maximum of every two adjacent infinity norm values (Chebyshev contour indices), and the values are stored in *Hop_Sequence_Arr* (88). ii) The cumulative sum is then computed in two steps (95, 96). (The current values of *last_hop_seq* and *cumsum* are saved for the next segment (86, 87, 97), see previous paragraph.) iii) Finally, *Prime_bIndex_Arr* is utilized as a binary mask (98) to keep only the prime stops and store them in *Prime_Stops_onL00_Arr* (100).

(78) Note that the output list generated from the return values of the parallel calculation (34) is not guaranteed to preserve the order of the input list, and must be sorted.

(117-137) Listing of the *last_hop_seq* and *cumsum* values printed at lines 113, 114 after completion of each of the 20 segments of the master calculation.

(142-155) This section of the code reads in the concatenated file *Prime_Stops_onL00__N=1-1000000000000.mmap* holding the $L_\infty(p)$ values at the prime "stops" and computes the $\mathcal{D}^1$ and $\mathcal{D}^2$ differences.

(160) The final section computes and plots the $\mathcal{D}^1$ and $\mathcal{D}^2$ histograms.

```
,
1 import numpy as np
2 from bignumpy import bignumpy
3 ########################################################################
```

```

def find(name, path):
for root, dirs, files in os.walk(path):
if name in files:
return os.path.join(root, name)
####################################################################

# Calc for N: Chebyshev contour index (norm00)
@parallel(p_iter='multiprocessing', ncpus=8)
def calc_norm00(N,M):

N += 1                # N is passed as a zero based array index
norm00     = []
is_a_prime = []
if N==1:
norm00     = [0]
is_a_prime = [0]

for i in xrange( max( 2,N),N+M):               # innermost loop

f = factor( i)
tuples = [x for x   in f]
#primes = [p for p,e in f]
powers = [e for p,e in f]
norm00.append( max( powers))              # infinity norm
if len( tuples)==1 and tuples[0][1]==1:    # 'i' is a prime
is_a_prime.append( 1)
else:
is_a_prime.append( 0)

return norm00, is_a_prime
####################################################################

%timeit
N_min         = 450000000001
N_max         = 500000000000
N_primes      = prime_pi( N_max) - prime_pi( N_min-1)
N_outer_loop  = 5*10^3
N_inner_loop  = 10^4
N_inmost_loop = 10^3
N_maxmem      = N_inner_loop * N_inmost_loop          # max array size stored in memory
ext           = str(N_min)+"-"+str( N_max)+".mmap"    # file extension

# Chebyshev contour index array --create
```

```
Contour_Indx_mmap_fname = "Contour_Indx__N="+ext
Contour_Indx_Arr = bignumpy( Contour_Indx_mmap_fname, np.uint8, (N_maxmem,))

# Is it a prime? Binary index array --create
Prime_bIndex_mmap_fname = "Prime_bIndex__N="+ext
Prime_bIndex_Arr = bignumpy( Prime_bIndex_mmap_fname, np.uint8,  N_maxmem,))

# Hop sequence array --create
Hop_Sequence_mmap_fname = "Hop_Sequence__N="+ext
Hop_Sequence_Arr = bignumpy( Hop_Sequence_mmap_fname, np.uint8, (N_maxmem,))

# Temp array for data type casting --create
Temp1_Arr = bignumpy( 'Temp1.mmap', np.int64, (N_maxmem,))
Temp2_Arr = bignumpy( 'Temp2.mmap', np.int64, (N_maxmem,))

# L00 values at prime stops --create
Prime_Stops_onL00_mmap_fname = "Prime_Stops_onL00__N="+ext
Prime_Stops_onL00_Arr = bignumpy( Prime_Stops_onL00_mmap_fname, np.int64,
    (N_primes,))

incr = N_inner_loop * N_inmost_loop
stp  = N_inmost_loop
last_prime_stop = 0
last_hop_seq    = 11
cumsum          = 1029766280643
tim  = walltime()
for i in xrange( N_outer_loop):                            # OUTER LOOP

beg        = i *   incr + N_min-1     # array indx
end        = beg + incr
input_list = [(x,stp) for x in range(beg,end,stp)]    # inner loop
tuples = sorted( list( calc_norm00( input_list)))

output_list = [ x[1][0] for x in tuples ]
flat_list = [item for sublist in output_list for item in sublist]
Contour_Indx_Arr = np.fromiter( flat_list, np.uint8)
if beg==0:
Hop_Sequence_Arr[0] = 0
else:
Hop_Sequence_Arr[0] = max( Contour_Indx_Arr[0], last_hop_seq)
last_hop_seq = Contour_Indx_Arr[-1]
np.maximum( Contour_Indx_Arr[1:], Contour_Indx_Arr[:-1],
    out=Hop_Sequence_Arr[1:])

output_list = [ x[1][1] for x in tuples ]
```

```
flat_list   = [item for sublist in output_list for item in sublist]
Prime_bIndex_Arr = np.fromiter( flat_list, np.uint8)

Temp1_Arr[:] = Hop_Sequence_Arr
Temp1_Arr.cumsum( out=Temp2_Arr)
Temp2_Arr += cumsum
cumsum = Temp2_Arr[-1]
np.multiply( Prime_bIndex_Arr, Temp2_Arr, out=Temp1_Arr)  # binary mask
count_nonzero = np.count_nonzero( Temp1_Arr)
Prime_Stops_onL00_Arr[last_prime_stop:last_prime_stop+count_nonzero] = Temp1_Arr[np.nonzero( Temp1_Arr)]
last_prime_stop += count_nonzero

if( (i+1) % (N_outer_loop // 100)) == 0:
print str(int(100*float(i+1)/float(N_outer_loop))).rjust(3)+'%', 'Wall time = ', str(walltime(tim)).rjust(20)


#print Contour_Indx_Arr[...]
#print Prime_bIndex_Arr[...]
#print Hop_Sequence_Arr[...]

#print Prime_Stops_onL00_Arr[...]

print "last_hop_seq= ", last_hop_seq
print "cumsum=       ", cumsum
####################################################################

#                              last_hop_seq    cumsum
#            1-  50000000000  11              114418475903
#  50000000001- 100000000000  11              228836951528
# 100000000001- 150000000000  11              343255427228
# 150000000001- 200000000000  12              457673901868
# 200000000001- 250000000000  12              572092378372
# 250000000001- 300000000000  11              686510853733
# 300000000001- 350000000000  11              800929329310
# 350000000001- 400000000000  13              915347805373
# 400000000001- 450000000000  11             1029766280643
# 450000000001- 500000000000  12             1144184756086
# 500000000001- 550000000000  11             1258603231525
# 550000000001- 600000000000  12             1373021707238
# 600000000001- 650000000000  11             1487440183820
# 650000000001- 700000000000  11             1601858659108
# 700000000001- 750000000000  12             1716277134939
# 750000000001- 800000000000  14             1830695610669
```

```
# 800000000001- 850000000000  11            1945114085159
# 850000000001- 900000000000  11            2059532560893
# 900000000001- 950000000000  11            2173951036235
# 950000000001-1000000000000  12            2288369511216
####################################################################


N_max = 10^12
N_primes = prime_pi( N_max)
ext = str( N_max)+".mmap"
dir = "/where/to/find/my_files/"

Prime_Stops_onL00_mmap_fname = "Prime_Stops_onL00__N=1-"+ext
Prime_Stops_onL00_Arr = bignumpy( find( Prime_Stops_onL00_mmap_fname, dir), np.int64, (N_primes,))        # L00 at prime stops --read

Prime_diff1_onL00_mmap_fname = "Prime_diff1_onL00__N=1-"+ext
Prime_diff1_onL00_Arr = bignumpy( Prime_diff1_onL00_mmap_fname, np.int16, (N_primes-1,))                # --create
np.subtract( Prime_Stops_onL00_Arr[1:], Prime_Stops_onL00_Arr[:-1], out=Prime_diff1_onL00_Arr)

Prime_diff2_onL00_mmap_fname = "Prime_diff2_onL00__N=1-"+ext
Prime_diff2_onL00_Arr = bignumpy( Prime_diff2_onL00_mmap_fname, np.int16, (N_primes-2,))                # --create
np.subtract( Prime_diff1_onL00_Arr[1:], Prime_diff1_onL00_Arr[:-1], out=Prime_diff2_onL00_Arr)
####################################################################

# Plot data

import matplotlib.pyplot as plt

E_max    = 12
N_max    = 10^12
N_primes = prime_pi( N_max)

ext = str(N_max)+".mmap"
dir = "/where/to/find/my_files/"

Prime_diff1_onL00_mmap_fname = "Prime_diff1_onL00__N=1-"+ext
Prime_diff1_onL00_Arr = bignumpy( find( Prime_diff1_onL00_mmap_fname, dir), np.int16, ( N_primes-1,))                # --read

Prime_diff2_onL00_mmap_fname = "Prime_diff2_onL00__N=1-"+ext
```

```
Prime_diff2_onL00_Arr = bignumpy( find( Prime_diff2_onL00_mmap_fname, dir),
    np.int16, ( N_primes-2,))                # --read

slice_diff1 = [prime_pi( 10^x)-1 for x in range( E_max+1)]
slice_diff2 = [prime_pi( 10^x)-2 for x in range( E_max+1)]

bin_edges_Prime_diff1_onL00 = np.arange( np.amin(
    Prime_diff1_onL00_Arr),np.amax( Prime_diff1_onL00_Arr)+2)            #
    '+2' to make sure rightmost bin is empty
bin_edges_Prime_diff2_onL00 = np.arange( np.amin(
    Prime_diff2_onL00_Arr),np.amax( Prime_diff2_onL00_Arr)+2)

# Histogram raw data:

slice_indx = 12  # >= 1
sdiff1 = slice_diff1[slice_indx]
sdiff2 = slice_diff2[slice_indx]

hist, bin_edges = np.histogram( Prime_diff1_onL00_Arr[:sdiff1],
    bins=bin_edges_Prime_diff1_onL00)
start = np.amin( np.nonzero(hist))
end   = np.amax( np.nonzero(hist))
np.savetxt( 'Prime_diff1_onL00_histogram__N=1-'+str( 10^slice_indx)+'.txt',
    hist[start:end+1],      fmt='%12d')
np.savetxt( 'Prime_diff1_onL00_bin_edges__N=1-'+str( 10^slice_indx)+'.txt',
    bin_edges[start:end+1], fmt='%12d')

hist, bin_edges = np.histogram( Prime_diff2_onL00_Arr[:sdiff2],
    bins=bin_edges_Prime_diff2_onL00)
start = np.amin( np.nonzero( hist))
end   = np.amax( np.nonzero( hist))
np.savetxt( 'Prime_diff2_onL00_histogram__N=1-'+str(10^slice_indx)+'.txt',
    hist[start:end+1],      fmt='%12d')
np.savetxt( 'Prime_diff2_onL00_bin_edges__N=1-'+str(10^slice_indx)+'.txt',
    bin_edges[start:end+1], fmt='%12d')

# Individual plots:

slice_indx = 12  # >= 1
my_color = (188/255, 27/255, 60/255)

plt.figure(1)
plt.hist( Prime_diff1_onL00_Arr[:slice_diff1[slice_indx]],
    bins=bin_edges_Prime_diff1_onL00, color=my_color, normed=True,
    align='left', histtype='stepfilled')
```

```
plt.ylabel( 'Frequency', fontsize=20)
plt.yticks( fontsize=16)
plt.xticks( fontsize=16)

plt.figure(2)
plt.hist( Prime_diff2_onL00_Arr[:slice_diff2[slice_indx]],
    bins=bin_edges_Prime_diff2_onL00, color=my_color, normed=True,
    align='left', histtype='stepfilled')
plt.ylabel( 'Frequency', fontsize=20)
plt.yticks( fontsize=16)
plt.xticks( fontsize=16)

# Multiple plots:

my_colors = [(0/255, 153/255, 153/255), (252/255, 126/255, 0/255),
    (251/255, 2/255, 1/255), (102/255, 205/255, 204/255), (255/255, 175/255,
    103/255), (255/255, 102/255, 102/255), (0/255, 103/255, 102/255),
    (178/255, 86/255, 1/255), (175/255, 1/255, 2/255), (83/255, 165/255,
    161/255), (246/255, 113/255, 78/255)]

plt.figure(3)
plt.hist( (Prime_diff1_onL00_Arr[:slice_diff1[2]],
    Prime_diff1_onL00_Arr[:slice_diff1[3]],
    Prime_diff1_onL00_Arr[:slice_diff1[4]],
    Prime_diff1_onL00_Arr[:slice_diff1[5]],
    Prime_diff1_onL00_Arr[:slice_diff1[6]],
    Prime_diff1_onL00_Arr[:slice_diff1[7]],
    Prime_diff1_onL00_Arr[:slice_diff1[8]],
    Prime_diff1_onL00_Arr[:slice_diff1[9]],
    Prime_diff1_onL00_Arr[:slice_diff1[10]],
    Prime_diff1_onL00_Arr[:slice_diff1[11]], Prime_diff1_onL00_Arr),
    bins=bin_edges_Prime_diff1_onL00, color=my_colors, normed=False,
    align='left', log=True, stacked=True, histtype='stepfilled')
plt.ylabel( 'log Frequency', fontsize=20)
plt.yticks( fontsize=16)
plt.xticks( fontsize=16)

plt.figure(4)
```

```
plt.hist( (Prime_diff2_onL00_Arr[:slice_diff2[2]],
    Prime_diff2_onL00_Arr[:slice_diff2[3]],
    Prime_diff2_onL00_Arr[:slice_diff2[4]],
    Prime_diff2_onL00_Arr[:slice_diff2[5]],
    Prime_diff2_onL00_Arr[:slice_diff2[6]],
    Prime_diff2_onL00_Arr[:slice_diff2[7]],
    Prime_diff2_onL00_Arr[:slice_diff2[8]],
    Prime_diff2_onL00_Arr[:slice_diff2[9]],
    Prime_diff2_onL00_Arr[:slice_diff2[10]],
    Prime_diff2_onL00_Arr[:slice_diff2[11]], Prime_diff2_onL00_Arr),
    bins=bin_edges_Prime_diff2_onL00, color=my_colors, normed=False,
    align='left', log=True, stacked=True, histtype='stepfilled')
plt.ylabel( 'log Frequency', fontsize=20)
plt.yticks( fontsize=16)
plt.xticks( fontsize=16)

plt.show()
#####################################################################
```

Appendix B. Tabulated histograms of prime gaps

Table 5. Numerical data of histograms of $\mathcal{D}^1$ differences in column one for $N \leq 10^{2-12}$ tabulated in columns 2-12

$\mathcal{D}^1$	10^{12}	10^{11}	10^{10}	10^{9}	10^{8}	10^{7}	10^{6}	10^{5}	10^{4}	10^{3}	10^{2}
1	1	1	1	1	1	1	1	1	1	1	1
2	545836959	65482891	8000621	999952	128678	17238	2352	358	63	12	3
3	0	0	0	0	0	0	0	0	0	0	0
4	676962502	81197873	9918050	1238662	159572	21273	2962	460	83	14	4
5	0	0	0	0	0	0	0	0	0	0	0
6	812569885	97460711	11910458	1489205	191644	25567	3508	519	86	17	3
7	233485333	28008641	3421613	427267	54555	7218	1024	151	24	3	1
8	675924191	80872118	9851261	1225160	156491	20793	2882	439	70	13	1
9	382285674	45657562	5551012	689857	88083	11685	1644	241	34	10	2
10	599704882	71643947	8701213	1078829	137483	18181	2481	353	52	9	1
11	275043047	32785324	3977086	492050	62471	8189	1152	163	30	0	0
12	756641841	90135360	10920595	1351869	171761	22303	3026	420	63	4	1
13	384294766	45758060	5540132	683669	86297	11231	1547	239	30	6	1
14	677173536	80473232	9718049	1196206	150599	19575	2670	369	63	9	1
15	401018460	47543338	5723867	703177	88375	11521	1513	237	37	5	0
16	620181299	73474151	8841576	1083654	135551	17349	2271	311	49	13	3
17	424850184	50195576	6020340	735419	91699	11709	1583	224	27	4	1
18	597556733	70592210	8465897	1033639	128604	16478	2171	293	39	3	0
19	417182685	49185886	5882046	715767	88879	11340	1491	199	22	6	0
20	500276741	58952639	7052759	857727	106288	13622	1822	248	35	8	1
21	407144642	47774244	5683277	686540	84352	10560	1395	181	27	5	
22	484516599	56852120	6763219	816597	100701	12564	1623	198	28	3	
23	443439269	51900616	6153871	740041	90225	11240	1434	199	19	1	

Continued on next page

Table 5 – *Continued from previous page*

$\mathcal{D}^1$	10^{12}	10^{11}	10^{10}	10^9	10^8	10^7	10^6	10^5	10^4	10^3	10^2
24	470479351	55067706	6528288	784364	96024	12067	1551	198	26	1	
25	486683648	56773666	6700619	802366	97486	11964	1418	158	18	2	
26	492742414	57478966	6784519	812971	98737	12325	1526	196	30	3	
27	447830032	52126930	6134631	731493	88573	11033	1263	142	19	1	
28	415327048	48370523	5701776	680946	82323	10102	1263	165	19	2	
29	390303895	45329992	5320982	630812	76393	9259	1153	141	14	1	
30	392437767	45575573	5347835	635587	76130	9320	1167	133	21	3	
31	411121720	47566224	5555827	655573	78270	9417	1109	145	15	0	
32	391706876	45315055	5292173	625088	74690	8998	1063	119	8	2	
33	407916113	47009979	5467687	641287	75646	8971	1057	117	10	1	
34	357435898	41178422	4788218	561466	66411	7902	969	127	15	0	
35	415666860	47710551	5519737	643360	75712	9030	1089	120	9	2	
36	357761163	41026033	4741553	552125	64897	7588	901	95	6	1	
37	438008056	50053582	5759940	666505	77507	8914	1032	109	7	0	
38	362734703	41403690	4757877	548943	63884	7324	856	110	11	0	
39	419338332	47753944	5473818	631038	72661	8342	906	88	12	0	
40	319372932	36349710	4161936	478868	55499	6364	687	72	9	1	
41	384186364	43587533	4972936	570258	65417	7486	844	92	6	0	
42	310770326	35210712	4008763	458211	52511	5895	656	69	6	0	
43	375151082	42408631	4809769	548333	62368	7088	800	83	6	0	
44	301328314	34025495	3854173	438680	50067	5531	665	72	6	0	
45	342858056	38657690	4375235	495639	55826	6342	710	74	7	0	
46	274048457	30861067	3483910	394841	44666	4925	534	58	4	0	
47	314323046	35331031	3980879	448389	50260	5599	625	65	5	0	
48	268461025	30128434	3390308	381310	42656	4740	519	50	2	0	
49	313748331	35145253	3944099	442484	49301	5395	602	67	4	1	
50	279059340	31197050	3493274	390264	43288	4677	502	43	3		
51	316192097	35270467	3938108	438494	48632	5216	555	54	2		
52	290982321	32393863	3612103	401777	44226	4687	490	39	4		
53	321173394	35704308	3971568	440735	48442	5238	546	46	4		
54	296555755	32916809	3654866	404294	44375	4865	487	50	1		
55	310144069	34377195	3805054	419457	46030	4780	485	38	4		
56	278244969	30786869	3402596	375360	40765	4250	377	23	1		
57	274938973	30365686	3349189	367275	39855	4219	452	36	5		
58	251238297	27693610	3046692	332896	35716	3799	390	23	3		
59	255057585	28064094	3078949	334786	35881	3707	364	26	1		
60	240625481	26406855	2893001	314572	33529	3453	356	27	4		
61	240098350	26312891	2874685	311734	33303	3428	317	25	0		
62	230501322	25221244	2747421	296466	31590	3203	317	32	2		
63	240699750	26287308	2858713	308409	32549	3323	281	17	1		
64	245034155	26679075	2893017	309735	32703	3363	310	34	1		
65	261624047	28405280	3071254	328501	34295	3350	309	32	3		
66	266059296	28835866	3106800	331091	34325	3517	320	23	1		
67	276300134	29870623	3211217	341332	35612	3414	301	24	1		
68	271214452	29261959	3136401	331350	34448	3383	321	20	0		
69	269383864	29009679	3101086	327085	33620	3299	272	13	0		
70	254946493	27403933	2922126	306593	31439	3079	282	23	2		
71	243206891	26090253	2773428	290862	29670	3004	241	21	1		
72	225296489	24122260	2558040	267104	27016	2579	232	14	2		

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Table 5 – *Continued from previous page*

$\mathcal{D}^1$	10^{12}	10^{11}	10^{10}	10^9	10^8	10^7	10^6	10^5	10^4	10^3	10^2
73	208369696	22276198	2361046	246199	24927	2467	223	20	0		
74	196338136	20936263	2209352	229402	23131	2200	198	12	0		
75	188391164	20046979	2110189	218518	21830	1943	156	10	0		
76	186442645	19783796	2074672	213678	21235	1923	166	10	2		
77	184682531	19558506	2046017	208819	20672	1945	179	12	1		
78	183172563	19350290	2022416	207385	20373	1881	148	7	0		
79	178996142	18878770	1964451	200764	19730	1845	135	9	0		
80	172704652	18184358	1890852	192546	18867	1728	138	7	0		

Table 6. Numerical data of histograms of $\mathcal{D}^2$ differences in column one for $N \leq 10^{2-12}$ tabulated in columns 2-12

$\mathcal{D}^2$	10^{12}	10^{11}	10^{10}	10^9	10^8	10^7	10^6	10^5	10^4	10^3	10^2
-60	124678820	13617925	1479775	159214	16731	1719	140	9	2		
-59	125417281	13698994	1493730	161701	17398	1755	162	20	3		
-58	125911011	13764630	1500292	162002	17011	1657	172	18	0		
-57	126923083	13894104	1516473	164037	17363	1785	182	18	0		
-56	128904390	14133007	1547182	168320	18180	1859	172	20	1		
-55	131234765	14416090	1578825	173076	18466	1934	185	14	0		
-54	134222294	14782109	1626127	177965	19312	2032	193	16	1		
-53	137524028	15185420	1676289	183678	19652	2153	231	15	1		
-52	140779422	15571660	1718999	188928	20752	2206	201	11	1		
-51	144402779	16017844	1776828	196358	21781	2337	214	21	4		
-50	147259221	16369755	1820487	201629	21894	2394	240	14	0		
-49	150647376	16780566	1870272	208691	23276	2460	258	21	1		
-48	152336894	16989890	1895485	210985	23352	2546	266	22	2		
-47	154896930	17306325	1936930	217237	24137	2622	272	30	1		
-46	154945318	17317334	1937210	216378	23987	2600	277	23	1		
-45	157288655	17615557	1976344	221746	24731	2679	281	23	4		
-44	157474057	17640425	1980331	222635	24737	2670	265	25	1		
-43	161440665	18131460	2041299	230497	25939	2801	319	34	3		
-42	161477126	18139987	2042247	230677	26217	2963	323	35	4	1	
-41	166326919	18758031	2124063	240927	27317	3107	338	41	5	0	
-40	166066810	18715681	2117644	240503	27342	2987	352	38	2	0	
-39	172416970	19515444	2219822	253386	28718	3232	321	34	2	0	
-38	172061911	19471804	2214742	252503	28497	3286	363	37	3	0	
-37	179636293	20424829	2335566	268516	30951	3562	378	45	2	0	
-36	177640738	20172741	2304548	264324	30438	3401	382	42	4	0	
-35	186402059	21262299	2443733	282221	32943	3845	429	56	7	0	
-34	183557880	20917244	2399966	277126	32006	3774	441	50	3	0	
-33	192595098	22050424	2542567	295787	34423	4026	458	50	6	0	
-32	187933156	21473919	2470177	286212	33446	3806	454	56	7	1	
-31	197513541	22680026	2625029	305443	35984	4232	477	56	7	0	
-30	192806659	22085709	2551227	296524	35018	4006	466	46	7	0	
-29	200679723	23085997	2679220	313792	37043	4416	538	61	6	0	
-28	196747898	22596987	2618029	306978	35944	4255	526	68	14	1	

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Table 6 – *Continued from previous page*

$\mathcal{D}^2$	10^{12}	10^{11}	10^{10}	10^9	10^8	10^7	10^6	10^5	10^4	10^3	10^2
-27	208373342	24066755	2807751	331861	39362	4796	620	66	6	0	
-26	207524517	23975103	2798797	330124	39711	4907	595	71	9	2	
-25	220660508	25634863	3011643	358793	43228	5235	652	72	12	1	
-24	218679154	25398221	2985061	355705	42957	5207	683	95	13	2	
-23	231838351	27072426	3201247	383263	46861	5759	698	93	10	0	
-22	227001776	26473195	3125624	374095	45595	5669	717	72	8	0	
-21	238581530	27942303	3315655	399252	49078	6132	772	93	9	0	
-20	225379188	26275304	3104488	371187	45314	5658	734	101	12	1	
-19	238927515	27983835	3319062	400185	49312	6217	745	98	12	4	
-18	228438575	26674122	3153715	379285	46182	5764	753	99	14	1	
-17	243203220	28525498	3391278	410922	50370	6350	821	124	21	2	
-16	235237016	27535204	3268627	393297	48604	6120	770	113	20	1	
-15	251141295	29529930	3521444	425681	52563	6507	855	108	13	0	
-14	245657729	28854709	3434713	415428	51396	6518	811	115	17	3	1
-13	261697331	30871436	3694997	449396	56052	7063	941	133	16	1	0
-12	253325086	29845788	3566511	433328	53567	6944	953	137	18	5	0
-11	273796317	32442924	3903869	479799	59764	7739	1018	146	16	3	1
-10	274354895	32570409	3933248	483817	61117	7859	1090	158	22	3	1
-9	286644016	34099599	4120679	508154	64219	8408	1124	162	25	5	0
-8	285492760	34048670	4132705	512306	65401	8539	1218	157	30	3	1
-7	294625344	35172331	4274938	530226	67242	8802	1238	178	36	10	0
-6	288344140	34440842	4187603	519919	66531	8756	1258	179	32	10	2
-5	296011555	35390375	4307092	534847	68057	9124	1258	161	29	4	1
-4	286503794	34243792	4167061	518758	66184	8826	1225	179	22	1	0
-3	284220703	33864884	4102245	508054	64912	8449	1142	167	18	3	0
-2	282984904	33788571	4106387	509356	65280	8448	1188	181	28	8	2
-1	285790536	34071252	4131348	511701	65077	8515	1203	163	20	4	1
0	285132051	34090579	4150019	517031	66320	8805	1190	170	23	8	1
1	285816973	34079474	4134226	512001	64923	8454	1166	180	22	4	0
2	282965478	33790755	4105123	510733	65120	8741	1243	164	24	4	1
3	284247018	33856898	4102916	508198	64138	8292	1171	182	29	5	2
4	286532237	34254862	4168340	517632	66368	8872	1297	203	44	7	1
5	296029198	35386325	4304568	534186	68239	8887	1188	172	19	3	1
6	288364738	34451448	4192449	522034	66343	8860	1169	166	29	6	1
7	294642143	35177434	4275015	530500	67604	8935	1228	168	24	6	1
8	285494550	34054118	4132141	513234	65554	8650	1166	152	27	5	1
9	286594727	34088658	4124488	508351	64462	8372	1148	169	30	7	1
10	274348420	32567912	3931836	484093	61559	7998	1120	152	26	3	1
11	273777740	32426966	3899784	477812	60258	7824	1110	164	20	1	0
12	253311421	29839307	3566489	433427	53589	6795	820	108	12	1	0
13	261679280	30866198	3694130	450241	55652	6897	898	124	24	5	1
14	245666156	28870503	3440583	416527	51179	6679	894	129	22	2	1
15	251104118	29523322	3520513	426857	52579	6798	862	110	15	2	
16	235222471	27532894	3263211	392940	48632	6059	786	93	9	0	
17	243217035	28529132	3394144	409862	50337	6288	822	101	18	2	
18	228433005	26670900	3154649	378424	46025	5816	738	97	14	2	
19	238932915	27978916	3321718	399985	49124	6348	795	101	9	1	
20	225357244	26282933	3105430	372688	45379	5720	683	92	20	0	
21	238581056	27934903	3314769	398874	49033	6267	801	106	10	3	

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Table 6 – *Continued from previous page*

$\mathcal{D}^2$	10^{12}	10^{11}	10^{10}	10^9	10^8	10^7	10^6	10^5	10^4	10^3	10^2
22	227016320	26480081	3126879	374803	45568	5598	702	115	15	1	
23	231864937	27070736	3199074	383370	46714	5774	756	103	9	1	
24	218674999	25405730	2986737	356172	43085	5357	638	78	10	2	
25	220654476	25636190	3013633	358464	43490	5245	656	63	13	0	
26	207550224	23975565	2797734	329791	39477	4686	556	66	6	0	
27	208353637	24063968	2807640	330900	39448	4718	613	74	10	0	
28	196716398	22593961	2621784	306740	36422	4337	502	49	6	2	
29	200688403	23081340	2678062	313512	37251	4541	539	79	9	1	
30	192822269	22086684	2552125	297179	34594	4073	467	52	4	0	
31	197517587	22669608	2621551	305250	36005	4255	493	61	8	0	
32	187932603	21464007	2468752	286987	33400	3951	494	60	4	0	
33	192596736	22046542	2542733	296132	34594	4077	458	48	1	0	
34	183600374	20928566	2398187	275541	31768	3651	440	60	5	0	
35	186373340	21266926	2441600	282075	32839	3936	451	45	1	0	
36	177654153	20160128	2302012	264274	30523	3503	392	45	6	1	
37	179653033	20422240	2334627	267943	30676	3620	403	36	4	0	
38	172064515	19474489	2214595	252591	28745	3279	338	38	5	0	
39	172423860	19521029	2221306	253670	28972	3311	373	37	5	0	
40	166104905	18724906	2117971	240611	27344	3091	371	43	3	0	
41	166323319	18749509	2123150	240448	27089	3053	338	47	4	0	
42	161484723	18142330	2044048	230015	25900	2796	310	29	1	0	
43	161411134	18136861	2044972	230317	26167	2940	338	30	3	1	
44	157504079	17645743	1979344	222051	24488	2675	286	27	3		
45	157289722	17615572	1977617	222128	24897	2697	278	27	1		
46	154924957	17311332	1937595	216315	23849	2569	266	18	1		
47	154898752	17306039	1937480	216551	23822	2619	299	25	3		
48	152317224	16983058	1896656	211348	23314	2505	236	34	3		
49	150644552	16780039	1867095	207637	22946	2471	238	21	3		
50	147271752	16360151	1817438	200919	22107	2344	225	20	0		
51	144377149	16013063	1776781	195943	21361	2212	224	11	1		
52	140821922	15572918	1723600	189431	20598	2164	207	23	0		
53	137514881	15179551	1672906	183860	20090	2121	213	23	1		
54	134230989	14783335	1627558	177736	19311	2036	199	21	1		
55	131236967	14423710	1582500	172696	18457	1861	175	17	1		
56	128906212	14137741	1546258	168321	17967	1829	186	13	1		
57	126912794	13892042	1514923	163999	17368	1781	178	14	0		
58	125903469	13766302	1499470	161793	17046	1688	158	10	0		
59	125423037	13700814	1490810	160879	17254	1804	152	11	1		
60	124678744	13607074	1477998	159281	16989	1674	156	22	2		

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