

Between the enhanced power graph and the commuting graph

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Abstract

The purpose of this note is to define a graph whose vertex set is a finite group G , whose edge set is contained in that of the commuting graph of G and contains the enhanced power graph of G . We call this graph the deep commuting graph of G . Two elements of G are joined in the deep commuting graph if and only if their inverse images in every central extension of G commute. We give conditions for the graph to be equal to either of the enhanced power graph and the commuting graph, and show that automorphisms of G act as automorphisms of the deep commuting graph.

KEYWORDS

Bogomolov multiplier, central extension, commuting graph, enhanced power graph, Schur multiplier

1 | INTRODUCTION

Let G be a finite group. Among a number of graphs defined on the vertex set G reflecting some algebraic properties of the group, two which have been studied are the following:

- the *commuting graph*, $\text{Com}(G)$, first considered by Brauer and Fowler [4] (but see below), in which two elements x and y are joined if they commute;
- the *enhanced power graph*, $\text{EPow}(G)$, first introduced by Aalipour et al. [1], in which elements x and y are joined if they generate a cyclic subgroup of G .

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Note that Brauer and Fowler were interested in the diameter of the graph, and so deleted the identity (which is joined to all other vertices) and also removed all the loops. For the same reason, many authors delete all vertices in the centre of G . In contrast, we wish to compare several graphs defined on a common vertex set, which we take to be the whole group G and we allow the loops.

It is clear that the edge set of the enhanced power graph is contained in that of the commuting graph. (that is, $\text{EPow}(G)$ is a *spanning subgraph* of $\text{Com}(G)$.) The purpose of this note is to define a graph whose edge set is between these two (contained in the commuting graph and containing the enhanced power graph). We will call it the *deep commuting graph*, for reasons which will hopefully become clear, and denote it by $\text{DCom}(G)$.

Remark. These are not the usual symbols used for these graphs in the literature, but since we have several different graphs in play, we hope that this choice of notation will help reduce confusion. This graph will refine the hierarchy introduced in [5], which also includes the *power graph* of G [15], a spanning subgraph of the enhanced power graph, and the *nongenerating graph* [9], which contains the commuting graph if G is nonabelian; these graphs will not be considered here.

The commuting graph has demonstrated its usefulness in group theory in the work of Brauer and Fowler. The deep commuting graph has no ready-made application as far as we are aware, but it does reach a little more deeply into the group structure, and we hope it will justify its introduction; as well, it links with other interesting topics such as the Bogomolov multiplier.

The definition requires some preliminary discussion of Schur covers and isoclinism.

2 | COVERS AND ISOCLINISM

In a group G , the *centre* $Z(G)$ is the subgroup consisting of elements which commute with every element of G . The *derived group* G' is the subgroup generated by all *commutators* $[x, y] = x^{-1}y^{-1}xy$ for $x, y \in G$.

A *central extension* of a finite group G is a finite group H with a subgroup $Z \leq Z(H)$ such that $H/Z \cong G$. We will think of G as a quotient of H , in other words, the image of a homomorphism whose kernel is Z . If $Z \leq Z(H) \cap H'$, it is called a *stem extension*. The subgroup Z is the *kernel* of the extension.

The deep commuting graph can be defined to have vertex set G , with an edge joining x and y if and only if the preimages of x and y commute in every central extension of G . (It is sometimes more natural to remove the loops, that is, to demand $x \neq y$, but we will not do that here.) However, we will give a more explicit definition shortly.

Schur [19] showed that every finite group has a unique *Schur multiplier* $M(G)$, the kernel of the largest possible stem extension of G . There are several other characterisations of $M(G)$: for example, it is the first homology group $H_1(G, \mathbb{Z})$, or the first cohomology group $H^1(G, \mathbb{C}^\times)$, or the quotient $(R \cap F')/[R, F]$ where G has a presentation F/R as the quotient of a free group F . The corresponding stem extension H is called a *Schur cover* of G ; it is not always uniquely defined by G . For example, the dihedral and quaternion groups of order 8 are both Schur covers of the Klein group of order 4.

However, Jones and Wiegold [13] showed that a relation weaker than uniqueness holds, namely that of isoclinism. This means that the commutation maps γ from $H/Z(H) \times H/Z(H)$ to H' defined by $\gamma(Z(H)x, Z(H)y) = [x, y]$ are essentially the same in the two groups. (Note that this is independent of the choice of coset representatives.) More formally, two groups H_1 and H_2 are said to be *isoclinic* if there exist isomorphisms $\phi : H_1/Z(H_1) \rightarrow H_2/Z(H_2)$ and $\psi : H'_1 \rightarrow H'_2$ such that, for all $x, y \in H_1$,

$$\psi(\gamma_1(Z(H_1)x, Z(H_1)y)) = \gamma_2(\phi(Z(H_1)x), \phi(Z(H_1)y)), \quad (1)$$

where γ_1 and γ_2 are the commutation maps associated with H_1 and H_2 ; in other words, the isomorphisms respect the commutation map. The pair (ϕ, ψ) is an *isoclinism* from H_1 to H_2 .

For example, if H is any group and A an abelian group, then H and $H \times A$ are isoclinic.

Now an isoclinism maps the set of element-wise commuting pairs of cosets of $Z(H_1)$ in H_1 to the corresponding set in H_2 . So we have, as noted in [18], that if H_1 and H_2 are isoclinic groups of the same order, then their commuting graphs are isomorphic. In particular, Schur covers of a group G have isomorphic commuting graphs. More generally, if two groups G_1 and G_2 are isoclinic, then $\text{Com}(G_i)$ is the lexicographic product [11] of a graph Γ which is the same for both groups with a complete graph on $Z(G_i)$.

It is clear that in (1) we may replace $Z(H_i)$ with any subgroup Z_i in the centre of H_i provided that the groups H_1/Z_1 and H_2/Z_2 are isomorphic. Thus, if two central extensions of the same group G are isoclinic (cf. [2, Definition III.1.1]), then the induced commuting graphs on a transversal of Z_1 in H_1 and on a transversal of Z_2 in H_2 are isomorphic.

We note also that any central extension is isoclinic to a stem extension [2, Proposition III.2.6], so we lose nothing by considering stem extensions in what follows. As a temporary notation, if $G = H/Z$ with $Z \leq Z(H)$, we define the *relative commuting graph* of G by Z as the graph with vertex set G , in which x and y are joined if their inverse images in H commute.

Definition The *deep commuting graph* of G , denoted by $\text{DCom}(G)$, is the relative commuting graph of G with respect to its Schur multiplier $M(G)$.

We note that in this definition the drawing of edges might depend on choosing a Schur cover, but the resulting graphs are all isomorphic. We will show shortly that the edge set is in fact independent of the chosen Schur cover. The name is meant to suggest that x and y are joined if the commutator of their preimages in a stem extension of G lies as deep as possible in the extension.

Example 1. As noted, the Klein group of order 4 has two Schur covers up to isomorphism, the dihedral and quaternion groups of order 8. In fact there are three different covers isomorphic to the dihedral group, since any one of the three involutions in the Klein group may be the one that lifts to an element of order 4 in the cover, but a unique cover isomorphic to the quaternion group. However, in all cases, the commuting graph of the cover is the lexicographic product of a loopy star $K_{1,3}$ with a complete graph of order 2; so the deep commuting graph of the Klein group is isomorphic to $K_{1,3}$ (which is in fact equal to the enhanced power graph of this group).

Remark. The relative commuting graph defines a map from quotient groups of the Schur multiplier to spanning subgraphs of the commuting graph on G ; this map is order-reversing (but as we will see, not one-to-one).

3 | COMMUTING GRAPH AND COMMUTING PROBABILITY

As a preliminary to the next section, we note a connection between the commuting graph and the commuting probability of a group G .

Let G be a finite group. Then G acts on itself by conjugation; the stabiliser of a point x is its centraliser (the set of neighbours of x in the commuting graph), while the orbits are conjugacy classes. So the Orbit-counting Lemma shows that the number of conjugacy classes is equal to the average valency of the commuting graph (including a loop at each vertex). Dividing through by $|G|$, we see that the proportion of all ordered pairs which commute is equal to the ratio of the number of conjugacy classes to $|G|$. This fraction is called the *commuting probability* of G , see [6]. We denote it by $\kappa(G)$, and note that it is the edge-density of the directed commuting graph, that is, a commuting graph with each edge replaced by a pair of directed edges going in opposite directions (and with a loop at each vertex).

Remark that if $1 \rightarrow Z \rightarrow H \rightarrow G \rightarrow 1$ is a central extension of G then the number of ordered commuting pairs in H equals $t|Z|^2$, where t is the number of ordered commuting pairs in transversal of Z . Therefore, $\kappa(H) = \frac{t|Z|^2}{|H/Z|^2 \cdot |Z|^2} = \frac{t}{|G|^2}$. This is used in the proofs below.

Proposition 3.1. *Suppose that $\pi_i : H_i \rightarrow G$ are epimorphisms corresponding to central extensions of G for $i = 1, 2$. Suppose that there exists an epimorphism $\phi : H_1 \rightarrow H_2$ such that $\pi_2 \circ \phi = \pi_1$. Then $\kappa(H_1) \leq \kappa(H_2)$.*

Proof. This follows from the fact that, for $x, y \in H_1$, if x and y commute then $\phi(x)$ and $\phi(y)$ commute in H_2 . \square

Corollary 3.2. *The commuting probability of a Schur cover of a group G is not greater than that of any central extension of G .*

Proof. This holds because any central extension is isoclinic to a stem extension, and any stem extension is a homomorphic image of some Schur cover [14, Theorem 2.1.22]. \square

Though it is not necessary for our argument, we note that Eberhard [6] proved that values of the commuting probability for finite groups are well-ordered by the reverse of the usual order on the unit interval. Thus, in the situation of the above proposition, either $\kappa(H_1) = \kappa(H_2)$, or $\kappa(H_1) \leq \kappa(H_2) - \epsilon$, where ϵ depends only on $\kappa(H_2)$.

4 | PROPERTIES OF THE DEEP COMMUTING GRAPH

The two theorems of this section show that the edge set of the deep commuting graph is independent of the chosen Schur cover, and that it lies between the edge sets of the enhanced power graph and the commuting graph.

Theorem 4.1. *Let G be a finite group.*

- (a) *If H_1 and H_2 are Schur covers of G , then H_1 and H_2 define the same edge sets of the deep commuting graph (in other words, for $x, y \in G$, the inverse images of x, y in H_1 commute if and only if the inverse images in H_2 commute).*
- (b) *Two elements of G are joined in the deep commuting graph if and only if their inverse images in every central extension of G commute.*
- (c) *The deep commuting graph of G is invariant under the automorphism group of G .*

Proof.

- (a) Let H_1 and H_2 be Schur covers of G , having projections $\pi_i : H_i \rightarrow G$ with kernels Z_i for $i = 1, 2$. Following Jones and Wiegold [13], we define

$$K = \{(h_1, h_2) \in H_1 \times H_2 : \pi_1(h_1) = \pi_2(h_2)\}, \quad (2)$$

and let ϕ_i be the coordinate projection of $H_1 \times H_2$ onto H_i restricted to K , for $i = 1, 2$. Clearly $\pi_1 \circ \phi_1 = \pi_2 \circ \phi_2$. Moreover, K is a central extension of G with projection $\pi_1 \circ \phi_1$ and kernel $Z_1 \times Z_2$.

It follows from Proposition 3.1, replacing (H_1, π_1) with $(K, \pi_1 \circ \phi_1)$ and setting $\phi = \phi_2$, that $\kappa(K) \leq \kappa(H_2)$. Since H_1 and H_2 are isoclinic, we have $\kappa(H_1) = \kappa(H_2)$. Corollary 3.2 shows that $\kappa(H_1) \leq \kappa(K)$. Thus two elements of H_i commute if and only if their inverse images in K commute.

Let Γ_1 and Γ_2 be the deep commuting graphs on G defined by H_1 and H_2 respectively. If $x, y \in G$ with inverse images $a, b \in K$, then we have

$$\begin{aligned} x \sim y \text{ in } \Gamma_1 &\Leftrightarrow [\phi_1(a), \phi_1(b)] = 1 \text{ in } H_1 \\ &\Leftrightarrow [a, b] = 1 \text{ in } K \\ &\Leftrightarrow [\phi_2(a), \phi_2(b)] = 1 \text{ in } H_2 \\ &\Leftrightarrow x \sim y \text{ in } \Gamma_2. \end{aligned}$$

- (b) Let H be a central extension of G and let Z be its kernel. Choose Schur cover G^* of G and let $\pi_1 : H \rightarrow G$ and $\pi_2 : G^* \rightarrow G$ be the corresponding projections. Define, similar to (2), $K = \{(h, g^*) \in H \times G^* : \pi_1(h) = \pi_2(g^*)\}$ and proceed as in item (a) to show that $\kappa(G^*) \leq \kappa(K) \leq \kappa(H)$. Hence, the transversal of $Z \times M(G)$ in K must have at least as many commuting pairs as the transversal of $M(G)$ in G^* . Hence, if the liftings of two elements of G to Schur cover G^* commute, their liftings to H must also commute.
- (c) The automorphism group of G permutes its Schur covers. □

Remark. Without using the above theorem, it is clear that if G has a unique Schur cover then the deep commuting graph is invariant under all group automorphisms. More generally, Fry [7] defined a class \mathcal{L} of groups G with the property that G has a Schur cover H such that all automorphisms of G lift to H (in other words, H is fixed by all automorphisms); these groups also clearly have the property that all group automorphisms preserve the deep commuting graph. An example of such a group is

the Klein group of order 4, which as we have seen has four Schur covers, three isomorphic to the dihedral group of order 8 and one to the quaternion group.

Theorem 4.2. *Let G be a finite group. Then*

$$E(\text{EPow}(G)) \subseteq E(\text{DCom}(G)) \subseteq E(\text{Com}(G)),$$

where $E(\Gamma)$ is the edge set of the graph Γ .

Proof. Let H be a Schur cover of G with kernel Z . Suppose that x and y are joined in the enhanced power graph of G . Then there is an element $z \in G$ such that $x, y \in \langle z \rangle$. Let a, b, c be preimages of x, y, z in H . Then $a, b \in \langle Z, c \rangle$, and this group is abelian since Z is central. So a and b commute, and x and y are joined in the deep commuting graph of G .

The other inclusion is trivial since if two elements commute then so do their images under a homomorphism. \square

Having these inclusions, it is natural to ask when equality holds. Aalipour et al. [1] characterised groups whose enhanced power graph and commuting graph are equal: these are precisely the groups having no subgroup $C_p \times C_p$ for any prime p (so the Sylow subgroups are cyclic or generalized quaternion). Our next job is to refine this.

Theorem 4.3. *The deep commuting graph of the finite group G is equal to its enhanced power graph if and only if G has the following property: Let H be a Schur cover of G , with $H/Z = G$. Then for any subgroup A of G , with B the corresponding subgroup of H (so $Z \leq B$ and $B/Z = A$), if B is abelian, then A is cyclic.*

Proof. Equality of these graphs requires that, if $x, y \in G$ and a, b are inverse images in H , if a and b commute, then $\langle x, y \rangle$ is cyclic. This is certainly implied by the condition of the theorem, with $A = \langle x, y \rangle$ and $B = \langle Z, a, b \rangle$. In the other direction, suppose that the graphs are equal, and let $A \leq G$ be such that its inverse image is abelian. Then for any $x, y \in A$, their inverse images in B commute, and so $\langle x, y \rangle$ is cyclic. But if a finite group A has the property that any two elements generate a cyclic subgroup, then A is cyclic. \square

5 | THE BOGOMOLOV MULTIPLIER

To explore equality of the commuting graph and deep commuting graph, we need to discuss the Bogomolov multiplier of a finite group G . This arose in connection with the work of Artin and Mumford on obstructions to Noether's conjecture on the pure transcendence of the field of invariants; but we do not need this background. We refer to [3, 12, 16] for further information.

Recall that $[x, y] = x^{-1}y^{-1}xy$ is the commutator of x and y , and let $x^y = y^{-1}xy$ be the conjugate of x by y . It is easy to check that in every group G the universal commuting identities $[xy, z] = [x^y, z^y][y, z]$, $[x, yz] = [x, z][x^z, y^z]$, and $[x, x] = 1$ hold. Based on this, Miller [17] defined the *nonabelian exterior square* of G , denoted by $G \wedge G$, as a group generated by symbols $x \wedge y$, $x, y \in G$ subject to relations

$$(xy) \wedge z = (x^y \wedge z^y)(y \wedge z), x \wedge (yz) = (x \wedge z)(x^z \wedge y^z), x \wedge x = 1.$$

It also was shown in [17] that $x \wedge y \mapsto [x, y]$ is a surjective homomorphism from $G \wedge G$ onto G' whose kernel is naturally isomorphic to $M(G)$, the Schur multiplier of G . Denote $M_0(G) := \langle x \wedge y \mid [x, y] = 1 \rangle$; then the group $B_0(G) := M(G)/M_0(G)$ is known as the Bogomolov multiplier of G . Observe that $B_0(G)$ is isomorphic to $M(G)$ if and only if they have the same cardinality.

Let H be a central extension of G , with $G = H/Z$. Clearly, if two elements of H commute, then their images under the projection from H to G also commute. We say that the extension is *commutation preserving* (for short, CP) if the converse holds.

It was shown by Jezernik and Moravec in [12, Theorem 4.2] that there exists a CP stem extension K of G with kernel isomorphic to $B_0(G)$. Indeed, $B_0(G)$ is the largest possible kernel of a CP stem extension of G . If $|B_0(G)| = |M(G)|$, then $B_0(G) \simeq M(G)$ and clearly K is also a Schur cover of G .

On the other hand, if $|B_0(G)| < |M(G)|$, then the Schur cover of G has larger cardinality than K . Since K is of maximal cardinality among commutation preserving stem extensions, the Schur cover cannot be commutation preserving.

So we have proved:

Theorem 5.1. *For the finite group G , we have $\text{DCom}(G) = \text{Com}(G)$ if and only if the Bogomolov multiplier of G is equal to the Schur multiplier.*

Corollary 5.2. *Let G be a finite group with $B_0(G) = 1$. Then $\text{DCom}(G) = \text{Com}(G)$ if and only if $M(G) = 1$.*

Remark. Kunyavskii [16] proved a conjecture of Bogomolov by showing that the Bogomolov multiplier of a finite simple group is trivial. Hence the above corollary applies to finite simple groups.

Remark. We observed that there is a map from quotients of $M(G)$ to spanning subgraphs of the commuting graph of G . We can now add to this observation: the quotient $M(G)/M_0(G) = B_0(G)$ maps to the full commuting graph, as well as every quotient $M(G)/Z$ with $M_0(G) \leq Z$. It may be that the map from quotients $M(G)/Z$ with $Z \leq M_0(G)$ is one-to-one; we have not investigated this.

6 | EXAMPLES

Example 2. We begin with an example of a group whose deep commuting graph lies strictly between the enhanced power graph and the commuting graph.

Take $n \geq 8$, and let G be the symmetric or alternating group of degree n . Then it is known (see [10]) that the Schur multiplier of G has order 2. The alternating group has a unique Schur cover, while the symmetric group has two; but all three share the property that an involution in G which is a product of m transpositions lifts to an involution in the cover if $m \equiv 0 \pmod{4}$, while it lifts to an element of order 4 if $m \equiv 2 \pmod{4}$. Furthermore,

- G has a noncyclic abelian subgroup isomorphic to $C_3 \times C_3$, whose lift is isomorphic to $C_2 \times C_3 \times C_3$, that is, abelian; so $\text{DCom}(G) \neq \text{EPow}(G)$.

- The involutions

$$x = (1, 2)(3, 4)(5, 6)(7, 8) \text{ and } y = (1, 3)(2, 4)(5, 6)(7, 8)$$

are joined in $\text{Com}(G)$ but not in $\text{DCom}(G)$, so these graphs are also unequal.

Remark. We can formalise the argument in the first bullet point above. If G is a group and p a prime such that G has a subgroup isomorphic to $C_p \times C_p$ but p does not divide $|M(G)|$, then $\text{EPow}(G)$ and $\text{DCom}(G)$ are unequal.

Remark. Kunyavskii’s theorem [16] together with the preceding remark gives other examples of simple groups with all three graphs distinct, such as the Mathieu group M_{12} .

Example 3. There exists a group G of order $|G| = 2^6$ with nontrivial Schur multiplier such that its deep commuting graph coincides with its commuting graph. To see this, let $G = C_8 \rtimes Q_8$ be a split extension of a quaternion group by a cyclic group of order eight (this is `SmallGroup(64,182)` in the GAP [8] library). Using GAP we compute that its Schur multiplier equals $M(G) = C_2$.

Let H be a Schur cover. By the proof of the first inclusion in Theorem 4.2, if two elements x and y generate a cyclic group, then their preimages in H commute. On the other hand, with GAP computations one sees that, up to conjugacy, the group G has exactly 11 noncyclic abelian subgroups and each lifts to an abelian subgroup of H . So the Schur cover is commutation preserving.

In view of Theorem 5.1 this group G is an example of a group where Bogomolov and Schur multiplier are isomorphic and nontrivial. In particular, combined with Theorem 4.3 we have $\text{EPow}(G) \subsetneq \text{DCom}(G) = \text{Com}(G)$.

Example 4. The dihedral 2-group,

$$H = D_{2^{n+1}} = \langle a, b \mid a^{2^n} = b^2 = 1, a^b = a^{-1} \rangle$$

is a Schur cover of $G = D_{2^n}$ for $n \geq 3$.

The centre $Z = Z(H)$ of H is generated by $a^{2^{n-1}}$ so the commutator $[a^{2^{n-2}}, b] = a^{2^{n-1}}$ is central in H . That is, $a^{2^{n-2}}$ and b do not commute in H but they do commute in $G \simeq H/Z$. This shows that the cover is not CP. So $\text{DCom}(G)$ is strictly contained in $\text{Com}(G)$.

However, $\text{EPow}(G) = \text{DCom}(G)$ because if $x, y \in H = D_{2^{n+1}}$ commute and the group $\langle x, y \rangle$ is not cyclic, then xy^{-1} belongs to the centre of $D_{2^{n+1}}$ so in the quotient group $H/Z \simeq D_{2^n}$ we have $xZ = yZ$, that is, $\langle xZ, yZ \rangle$ is a cyclic group in G .

7 | FURTHER PROPERTIES

Unlike the enhanced power graph and commuting graph, the deep commuting graph of G does not have the property that the induced subgraph on a subgroup H is the deep commuting graph of H . The above groups provide examples. The deep commuting graph of $C_3 \times C_3$ consists of four loopy triangles with a common vertex; but the induced subgraph of the deep commuting graph of S_n on a $C_3 \times C_3$ subgroup is the complete graph on nine vertices.

We have not considered graph-theoretic properties of the deep commuting graph such as connectedness or chromatic number.

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