On some relations between a Continuous Time Luenberger Productivity Indicator and the Solow Model

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Abstract

We introduce well-known microeconomics productivity measures in Solow models in discrete time or continuous time by adopting a Luenberger-type approach. In each framework, we derive the productivity indicators and the dynamical paths. Firstly, we show that the expression of the productivity indicators are similar to the well-known Solow residuals, allowing us to make an analogy between a firm's behaviour in a microeconomic setting and a country's behaviour in a macroeconomic setting. Secondly, we demonstrate that the properties of the paths are similar in both frameworks.

JEL: D24
Keywords: technical efficiency, technological progress, directional distance function, proportional distance.

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1 Introduction

Year 1957 is a landmark for the understanding of productivity. On the one hand, Solow (1957) proposed the first macroeconomic framework to evaluate the total factor productivity indices from a neoclassical aggregate production function. Solow's research initiated modern growth accounting research that will lead to distinguish 'separate contributions of technological change, capital accumulation, education and so forth in raising per capita income' (Allen, 1991, p 203). It is now well-established that productivity growth has two determinants: variation of technology efficiency (the so-called 'productivity gains') and technological changes linked to innovation.[1] On the other hand, Farrell (1957) proposed various measurements of productive efficiency at the firm level by using the concept of distance function. A firm is said to be technically inefficient if it can produce the same output with an equiproportional reduction in the use of all inputs. Procedures to measure inefficiency have since flourished in the literature. The Farrell Measure is essentially the inverse of the Shephard's (1970) distance function.

A generalization of the Shephard's distance function has been proposed known as the directional distance function to analyze both consumption and production theory. First, Luenberger (1992) defined the benefit function as a directional representation of preferences, thereby generalizing Shephard's input distance function defined in terms of a scalar output representing utility. He also introduces the shortage function which accounts for both input contractions and output improvements and which is dual to the profit function. Chambers et. al (1996-1998) relabel this same function as a directional distance function and since then it is commonly known by that name. They proved that the directional distance function, in the graph oriented case, is primal to the profit function that is dual to the directional distance function. Briec (1997), independently, proposed a proportional distance function. The directional distance function can be viewed as a local extension of the proportional distance function selecting a specific direction for any production vector.

The aim of this paper is to unify both approaches in simple dynamical frameworks. Our paper shows the analogy between technical efficiencies relative to production sets and the 'standard' theory of productivity measurement derived from Solow. Indeed, technical efficiency in Farrell's sense is the total factor productivity by a different name. Solow's productivity growth is the total factor productivity with a shift in technology. This correspondence is not new. Indeed, as noted by ten Raa and Mohnen (2002, p. 111), 'productivity is essentially the output-input ratop and therefore productivity growth the residual between output growth and input growth' in both approaches. Introducing the distance function concept in a dynamical framework has two main advantages. On the one hand, estimating total factor productivity growth does not require the specification of the production function. On

[1] In this paper, we adopt the most appropriate approach for the task in hand and, as a result, our TFP growth decomposition contains two components (growth in technical efficiency and growth in technical change). However, ast it was mentioned by an anonymous referee, in other settings other TFP growth decompositions may be suitable that contain at least one further component.

the other hand, evaluating production inefficiencies and therefore identifying possible aggregate gains become possible

To illustrate the above, let us assume that a firm is efficient at a given time period. If the set of production possibilities is increasing in time, then it may not remain efficient relative to the technical efficiency frontier at a later time period. As it is in the firm's interest to maintain efficiency, the firm has to change size between these two time periods. It does so by proceeding to a minimal transformation of factors and products. This 'ideal' change between two successive time periods can be evaluated by using the procedures derived from the directional an proportional framework. The firm's dynamical behavior can then be extrapolated by recurrence, although it remains dependent on its initial condition.

Though this paper proposes some analogy between the efficiency and Solow productivity concepts, it does not explain the origin of the technological progress and does not have microeconomic foundations. Specifically, we consider simple Solow-type models in which the technological progress is exogenous and the technical efficiency frontier is not specified. The dynamical analysis evaluates the impact of a firm's size change on productivity over a period of time, by simply actualizing observations over time. This enables us to treat these observations as cross-sectional data to estimate the 'actualized' frontier by envelopment. Although our concept of optimal paths is close to that derived in the traditional macroeconomic approach, it rests on the distance concept developed by Luenberger (1995) and Chambers, Chung and Färe (1996 and 1998). In other words, we integrate some essential and well-known tools of production theory in a dynamical context.

The paper is organized as follows. Section 2 collates the basic assumptions and definitions. We determine the dynamical path according to the rate of growth of the technological progress in discrete time in Section 3 and in continuous time in Section 4. Section 5 concludes.

2 Technology, technological progress and distance function

Firstly, this section introduces the assumptions on the production possibility set. Secondly, it presents the methods to evaluate efficiency changes relative to production frontiers, including the indicators used throughout the paper.

2.1 Assumptions on production technology

For $x, u \in \mathbb{R}^n_+$, $x \leq u \Rightarrow x_i \leq u_i, \forall\, i \in \{1, ..., n\}$. A production technology transforms input vectors $x = (x_1, ..., x_n) \in \mathbb{R}^n_+$ into output vectors $y = (y_1, ..., y_p) \in \mathbb{R}^p_+$. This production technology can be characterized by the input correspondence $L : \mathbb{R}^p_+ \longrightarrow$

$2^{\mathbb{R}^n_+}$ or the output correspondence $P : \mathbb{R}^n_+ \longrightarrow 2^{\mathbb{R}^p_+}$. For all $y \in \mathbb{R}^p_+$, $L(y)$ is the set of all input vectors that yield at least y. For all $x \in \mathbb{R}^n_+$, $P(x)$ is the set of all output vectors obtainable from a given input vector x. The graph of the technology T can be defined either from L or P by:

$$T = \{(x, y) \in \mathbb{R}^{n+p}_+ : x \in L(y)\} = \{(x, y) \in \mathbb{R}^{n+p}_+ : y \in P(x)\}.$$

As Färe, Grosskopf and Lovell (1985) and Shephard (1974), we assume that the technology representation satisfies the standard axioms of production:
T1: $(0, 0) \in T$; $(0, y) \in T \Rightarrow y = 0$.
T2: $T(x) = \{(u, y) \in T : u \leq x\}$ is a bounded set $\forall x \in \mathbb{R}^n_+$.
T3: $\forall\ (u, v) \in \mathbb{R}^{n+p}_+$; $(x, y) \in T$ and $(-u, v) \leq (-x, y) \Rightarrow (u, v) \in T$.
T4: T is a closed set.

Axiom T1 states that outputs cannot be produced without inputs. Although this axiom is realistic, it is not compulsory to maintain our results. Axiom T2 postulates that a finite input vector cannot lead to an infinite output vector. Strong disposability of both inputs and outputs is imposed by Axiom T3. Axiom T4 is a required condition to identify the set of efficient vectors of a subset of the frontier. To Axioms T1-T4 can be added Axiom T5 postulating the convexity of the production set, often used in empirical works (see for instance Charnes, Cooper and Rhodes (1978) and Banker, Charnes and Cooper (1984)).

T5: T is a convex set.

In what follows, we will assume that the technology T satisfies the axioms T1-T5, unless stated otherwise.

2.2 Technical efficiencies and distance functions

Any difference between actual production set and efficient set reveals efficient gains possibilities. In practice, these are evaluated by using distance functions that measure technical efficiencies relative to technology subsets.

Now, recall the definition of the directional distance function introduced by Chambers, Chung and Färe (1996-1998) and $D_T : \mathbb{R}^n_+ \times \mathbb{R}^p_+ \times \mathbb{R}^n_+ \times \mathbb{R}^p_+ \longrightarrow \mathbb{R} \cup \{-\infty, \infty\}$ as follows:

$$D_T(x, y; h, k) = \begin{cases} \sup\limits_{\delta \in \mathbb{R}} \{\delta : (x - \delta h, y + \delta k) \in T\} & \text{if there is some } \delta \in \mathbb{R} \\ & \text{such that } (x - \delta h, y + \delta k) \in T \\ -\infty & \text{otherwise.} \end{cases} \tag{2.1}$$

This directional distance function indicates the maximal expansion in the direction of $g = (h, k)$ simultaneously in all inputs and output which still allow the production the weak efficient subset of the production set. By convention, the supremum of any real valued-function over an empty set is taken as $-\infty$. Therefore, if there is no real number δ such that $(x - \delta h, y + \delta k) \in T$ then $D_T(x, y; h, k) = -\infty$. In the trivial case where (x, y) is feasible and $g = 0$, we have $D_T(x, y; h, k) = +\infty$. This paper considers the proportional distance function based on simultaneous proportionate changes in inputs and outputs. Initiated by Briec (1997), this distance function is closely related to the directional distance function due to Chambers, Chung and Färe (1996,1998). A general taxonomy is proposed by Russell and Schworm (2011). This specification has the advantage to take into account a weighting scheme independent of the units of measurement. It can be justified in a macroeconomic framework as it could measure changes in per capita output relative to changes in per capita labor for instance.

The map $D_T^{\propto} : \mathbb{R}_+^{n+p} \times [0, 1]^{n+p} \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ defined by

$$D_T^{\propto}(x, y; \alpha, \beta) = \sup \quad \{\delta : (x - \delta\alpha \odot x, y + \delta\beta \odot y) \in T\} \tag{2.2}$$

is called *Proportional Distance Function*, where $\odot$ denotes the Hadamard coordinate-wise product defined by $\gamma \odot z = (\gamma_1 z_1, ..., \gamma_d z_d)$ for all $\gamma, z \in \mathbb{R}^d$. The proportional distance function measures the maximum proportion that inputs and outputs can be respectively decreased and increased to reach the boundary of the production set. The vector of parameters (α, β) represents a pre-assigned weighting scheme these proportional changes depend on. In particular,taking the direction

$$g = (h, k) = (\alpha \odot x, \beta \odot y) = (\alpha_1 x_1, ..., \alpha_n x_n, \beta_1 y_1, ..., \beta_p y_p) \tag{2.3}$$

we obtain:

$$D_T(x, y; \alpha \odot x, \beta \odot y) = D_T^{\propto}(x, y; \alpha, \beta). \tag{2.4}$$

Notice that, for all $(\alpha, \beta) \in [0, 1]^{n+p}$, this distance function is independent of the units of measurement chosen for the production technology. Equivalently, this means that it satisfies the commensurability condition (see Russell (1987)).

Figure 1 illustrates the paths followed by the proportional and directional distance functions with one input and one output. We have restricted the parameters of the proportional distance function and we require the weights α_i and β_j to be equal to one. In such a case the proportional distance function is

$$D_T^{\propto}(x, y; \mathbb{1}_n, \mathbb{1}_p) = \sup\{\delta : \big((1 - \delta)x, (1 + \delta)y\big) \in T\}, \tag{2.5}$$

where $\mathbb{1}_d$ denotes the d-dimensional unit vector for any positive natural number d. Three initial production vectors are (x, y), (x', y') and (x'', y'') with the arrows indicating the reference point. Note that the paths for the proportional distance function (indicated by a thick solid line) are not parallel while the paths of directional distance function are parallel.

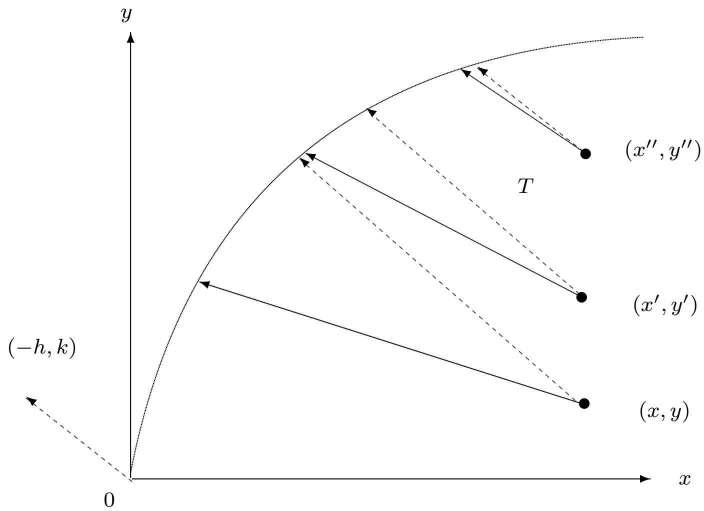

Figure 1: Directional and Proportional Distance Function.

An inefficiency index measures the 'distance' from the production vector to the frontier of T. Typically, the production point is compared to a particular point that is the reference vector on the boundary of T. The main issues addressed in formulating a specific inefficiency index are the selection of the reference vector corresponding to any production vector and the specification of the 'distance' between the production vector and the reference vector relative to the production set T. Suppose that $\delta^\star$ is solution of the program computing the proportional distance function in equation (2.2). The the reference vector is $(x^\star, y^\star) = (x - \delta^\star \alpha \odot x, y + \delta^\star \beta \odot y) = (x, y) + \delta^\star(-\alpha\odot, \beta \odot y)$. hence, the reference vector is

$$(x^\star, y^\star) = (x, y) + \left[D_T^\propto(x, y; \alpha, \beta)\right](-\alpha \odot x, \beta \odot y). \tag{2.6}$$

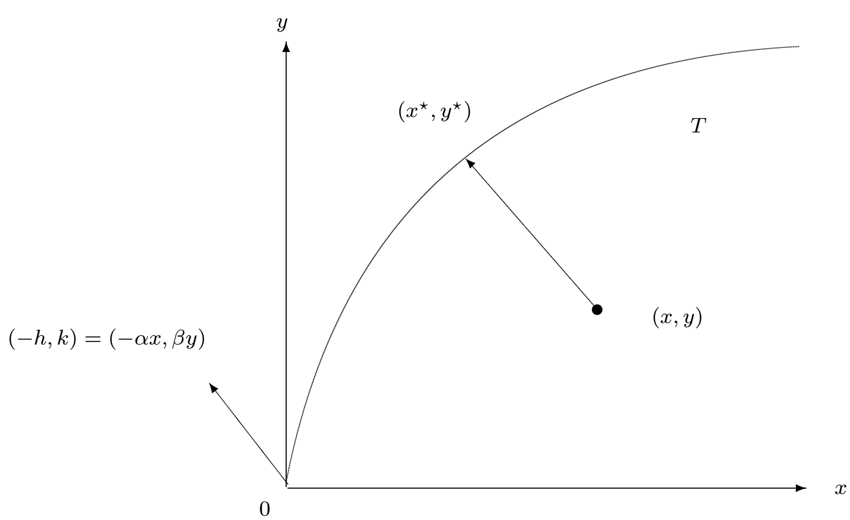

Figure 2: Proportional Distance Function and Reference Point.

Figure 2 shows how to determine the reference vector of the proportional distance function in a two-dimension diagram. Since $n = p = 1$, we have $\alpha \odot x = \alpha x$ and $\beta \odot y = \beta y$ with $\alpha > 0$ and $\beta > 0$. More precisely, the proportional distance $D_T^{\propto} : \mathbb{R}_+^2 \times [0,1]^2 \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ is defined by $D_T^{\propto}(x, y; \alpha, \beta) = \max\{\delta : (x - \delta\alpha\ x, y + \delta\beta\ y) \in T\}$. The coordinates of the optimal projection point $(x^\star, y^\star)$ are $(x - \alpha\delta^\star x, y + \beta\delta^\star y)$ where $\delta^\star = D_T^{\propto}(x, y; \alpha, \beta)$, from which the following proportional equality is deduced

$$\frac{x - x^\star}{\alpha x} = \frac{y^\star - y}{\beta y}. \tag{2.7}$$

Equality 2.7 indicates the necessary proportional changes in input and output in order to make the input-output vector (x, y) efficient.

Notice that in the remainder of the paper, it will be helpful to use a matrix formulation of the Hadamard product introducing two diagonal matrices $A = \text{diag}(\alpha)$ and $B = \text{diag}(\beta)$ respectively of dimension $n \times n$ and $p \times p$. Let I_n and I_p respectively denote the identity matrices of dimension n and p. It follows that the proportional distance function can then be defined as

$$D_T^{\propto}(x, y; \alpha, \beta) = \max\Big\{\delta : \big((I_n - \delta A)x, (I_p + \delta B)y\big) \in T\Big\} \tag{2.8}$$

From the definition proposed in (2.2), we can retrieve the two well-known Farrell measures of technical efficiency. The Debreu-Farrell input measure of technical efficiency $E^{\text{in}} : \mathbb{R}_+^{n+p} \longrightarrow \mathbb{R}_+ \cup \{+\infty\}$ is defined by

$$E^{\text{in}}(x, y) = \inf\{\lambda > 0 : \lambda x \in L(y)\} \tag{2.9}$$

The Debreu (1951)–Farrell (1957) index, measures the maximal radial contraction of the input vector consistent with production feasibility.

The Farrell output measure of technical efficiency $E^{\text{out}} : \mathbb{R}_+^{n+p} \longrightarrow \mathbb{R}_+ \cup \{-\infty, +\infty\}$ is defined by

$$E^{\text{out}}(x, y) = \sup\{\theta > 0 : \theta y \in P(x)\}. \tag{2.10}$$

It measures the maximal radial expansion of the output vector consistent with production feasibility.

In addition, if we set $\alpha = \mathbb{1}_n$ and $\beta = 0$, we have $D_T^{\propto}(x, y; \mathbb{1}_n, 0) = 1 - E^{\text{in}}(x, y)$. If we set $\alpha = 0$ and $\beta = \mathbb{1}_p$, we have $D_T^{\propto}(x, y; 0, \mathbb{1}_p) = E^{\text{out}}(x, y) - 1$.

To measure technical efficiency in simple macrodynamical frameworks with exogenous growth, let us give an intertemporal dimension to our framework (*cf.* Definition 2.2.2) by from the subset of the frontier T (*cf.*Definition 2.2.1). The following definition extends to the full input-output space the concept of input and output isoquant that are characterized from the input and output Farrell measures respectively.

Definition 2.2.1 *The subset*

$$\partial_{\alpha,\beta}^{\propto} T = \Big\{(x, y) \in T : \delta > 0 \Longrightarrow (x - \delta\alpha \odot x, y + \delta\beta \odot y) \notin T\Big\} \tag{2.11}$$

is called the oriented Graph-Isoquant of the production set T.

From Briec (1997), Definition 2.2.1, implies that

$$D_T^{\propto}(x, y; \alpha, \beta) = 0 \Leftrightarrow (x, y) \in \partial_{\alpha,\beta}^{\propto} T. \tag{2.12}$$

In other words, any input-output vector on the efficiency frontier, *i.e.* the oriented-graph isoquant, has a null distance. This means that any production vector of the output oriented isoquant cannot be improved both in input and in output. Equivalently, any reference vector belongs to the oriented isoquant.

Definition 2.2.2 *The set of production possibilities $T(t)$ for each time period t is defined by*

$$T(t) = \Big\{ \big(x(t), y(t)\big) \in \mathbb{R}_+^{n+p} : (x(t), y(t)) \text{ is possible at time period } t \Big\}.$$

The input-output vector $(x(t), y(t))$ can also be called the production unit. We will assume that $T(t)$ satisfies Axioms T1-T5 at each time period t, unless stated otherwise.

2.3 Technological progress and proportional Luenberger indicator

In this sub-section, we define the proportional Luenberger indicator,our productivity indicator based on the proportional distance function. [2] This indicator provides a flexible tool to account for both input contractions and output improvements when measuring efficiency (see Boussemart, Briec, Kerstens and Poutineau (2003, p. 393) and Briec, Kerstens and Peypoch (2012) for more information on Luenberger indexes and indicators).

In our framework, the set of production possibilities, defining the technological constraints supported by the producer, is assumed to increase in time for a given amount of input. In other words, if the producer adopts a given size in the market at a given time period, she will be able to do so again at a later time period. This is gathered in the following assumption.

Assumption (Growth of the set of production possibilities)

$$\forall\, t, s \geq 0 \quad \text{if} \quad s \geq t \quad \text{then} \quad T(t) \subset T(s).$$

This assumption means that any input-output vector that is feasible at the time-period t is feasible at the time period $s \geq t$. Equivalently, this means that there exists a technological progress at each time period.

The proportional Luenberger indicator is defined as:

[2] Our productivity indicator is a proportional version of the directional Luenberger indicator introduced by Chambers, Färe and Grosskopf (1996).

$$\begin{aligned} L_{t,t+1}\big(x(t), y(t), x(t+1), y(t+1); \alpha, \beta\big) \\ = \frac{1}{2}\Big[\Big(D^{\propto}_{T(t)}\big(x(t), y(t); \alpha, \beta\big) - D^{\propto}_{T(t)}\big(x(t+1), y(t+1); \alpha, \beta\big)\Big) \\ + \Big(D^{\propto}_{T(t+1)}\big(x(t), y(t); \alpha, \beta\big) - D^{\propto}_{T(t+1)}\big(x(t+1), y(t+1); \alpha, \beta\big)\Big)\Big]. \end{aligned} \tag{2.13}$$

This is a slight extension of the proportional indicator considered by Boussemart et.al. (2003), who implicitly assumed that $\alpha = \mathbb{1}_n$ and $\beta = \mathbb{1}_p$. An earlier definition was proposed by Chambers, Färe and Grosskopf (1996) in a directional context. Such a definition parallels that of the Malmquist index.

Indicator 2.13 is the arithmetic mean between the proportional changes in input and output observed at time period t and those observed at time period $t+1$. This allows us to avoid an arbitrary choice between base years. For instance, $D^{\propto}_{T(t+1)}\left(x(t), y(t); \alpha, \beta\right)$ indicates the necessary proportional changes in inputs and outputs for $(x(t), y(t))$ to be efficient at time period $t+1$. If $L = 0$, there are no productivity gains. If $L > 0$, there are productivity gains. If $L < 0$, there are productivity losses. As this Luenberger indicator evaluates the productivity change, we denote it by PCH. The latter can be decomposed into two components: the efficiency change ($EFCH$) of the proportional distance function between the two successive time periods and the technological change ($TECH$) measured by the arithmetic mean of the last two differences.[3] The motivation for finding sources or decompositions is to identify the sources of productivity growth in terms of 'catching up' and innovation, which can be given policy content. Nishimizu and Page (1982) used a parametric approach to decompose the Malmquist index into technical change and efficiency change. Färe et. al. (1994), followed up on this idea but implemented it using nonparametric linear programming techniques to estimate the distance functions. Hence, PCH can be expressed as:

$$PCH_{t,t+1} = EFCH_{t,t+1} + TECH_{t,t+1} \tag{2.14}$$

where

$$EFCH_{t,t+1} = D^{\propto}_{T(t)}\big(x(t), y(t); \alpha, \beta\big) - D^{\propto}_{T(t+1)}\big(x(t+1), y(t+1); \alpha, \beta\big) \tag{2.15}$$

and

$$\begin{aligned} TECH_{t,t+1} = \frac{1}{2}\Big[\Big(D^{\propto}_{T(t+1)}\big(x(t), y(t); \alpha, \beta\big) - D^{\propto}_{T(t)}\big(x(t), y(t); \alpha, \beta\big)\Big) \\ + \Big(D^{\propto}_{T(t+1)}\big(x(t+1), y(t+1); \alpha, \beta\big) - D^{\propto}_{T(t)}\big(x(t+1), y(t+1); \alpha, \beta\big)\Big)\Big] \end{aligned} \tag{2.16}$$

[3] As it was mentioned in our introduction, there are other set-ups though where the TFP growth decomposition has three components (returns to scale change, technical efficiency change and growth in technical change, allocative change). A few examples can be found in Epure et. al. (2011), Emrouznejad and Guo-Liang (2016) and Boussemart et. al. (2020).

Expression 2.16 does not necessitate the specification of the production function. As well-known in the literature, the latter can be estimated from non parametric techniques. Note that, currently, we do not assume that firms are efficient. However, in such a case, we have $EFCH_{t,t+1} = 0$ and the productivity indicator boils down to its technological change component. Notice that there are other set-ups though where the TFP growth decomposition has three components (returns to scale change, technical efficiency change and growth in technical change). There are lots of TFP growth decomposition studies that do this and the authors could provide some examples.

2.4 Example

Let us consider a Cobb-Douglas production function with constant returns to scale. The expression of the technology at time period t is given by

$$T(t) = \Big\{(x(t), y(t)) \in \mathbb{R}_+^{n+1} : y(t) \leq a(t) \prod_{i=1}^{n} x_i(t)^{\gamma_i} \Big\} \text{ with } \gamma_n > 0 \text{ and } \sum_{i=1}^{n} \gamma_i = 1. \tag{2.17}$$

We assume that all firms are efficient, *i.e.* $y(t) = a(t) \prod_{i=1}^{n} x_i(t)^{\gamma_i}$.

Let us first set $\alpha = \mathbb{1}_n$ and $\beta = 0$. As $D^{\propto}_{T(t)}\big(x(t), y(t); \alpha, \beta\big) = 1 - \frac{a(t) \prod_{i=1}^{n} x_i(t)^{\gamma_i}}{y(t)}$ and $y(t) = a(t) \prod_{i=1}^{n} x_i(t)^{\gamma_i}$, it is easy to show that the proportional efficiency change is null, *i.e.* $EFCH_{t,t+1} = 0$. The technological change is equal to

$$TECH_{t,t+1} = \frac{1}{2}\Big[\frac{a(t+1) - a(t)}{a(t+1)} + \frac{a(t+1) - a(t)}{A(t)}\Big]. \tag{2.18}$$

As a result, $PCH_{t,t+1} = EFCH_{t,t+1} + TECH_{t,t+1} = TECH_{t,t+1}$. If $t \simeq t+1$, (2.18) becomes

$$TECH_{t,t+1} = \Big[\frac{a(t+1) - a(t)}{a(t+1)}\Big] \simeq \frac{\dot{a}(t)}{a(t)} \simeq \frac{d \log a(t)}{dt} = d \log a \tag{2.19}$$

Let us simplify the notation by denoting $\frac{d \log a(t)}{dt}$ by $d \log a$.

Recall that in macrodynamical frameworks, when the production function is specified, the technological change can be deduced from the Log-transformation of (2.17). It yields

$$\log\left(a(t)\right) = \log y(t) - \sum_{i=1}^{n} \gamma_i \log\left(x_i\left(t\right)\right) \tag{2.20}$$

and differentiating on both side, we obtain the discrete time approximation of the technological progress as suggested by Solow:

$$d \log\left(a\right) = d \log\left(y\right) - \sum_{i=1}^{n} \gamma_i d \log\left(x_i\right) \tag{2.21}$$

This is the so-called Solow residual. As $PCH_{t,t+1} = TECH_{t,t+1} = d \log A$, we can deduce that the Solow apprach and the Farrell approach are equivalent.

Let us now set $\alpha = \mathbb{1}_n$ and $\beta = \mathbb{1}_p$. The expression of the proportional distance function becomes

$$D^{\propto}_{T(t)}\big(x(t), y(t); \alpha, \beta\big) = \max_{\delta}\Big\{\delta \geq 0; (1+\delta)\, y(t) \leq a(t) \prod_{i=1}^{n} \left((1-\delta)\, x_i(t)\right)^{\gamma_i} \Big\} \quad (2.22)$$

$$= \max_{\delta}\Big\{\delta \geq 0; \frac{(1+\delta)}{(1-\delta)} y(t) \leq a(t) \prod_{i=1}^{n} \left(x_i(t)\right)^{\gamma_i} \Big\} \quad (2.23)$$

$$= \frac{a(t) \prod_{i=1}^{n} \big(x_i(t)^{\gamma_i} - y(t)\big)}{a(t) \prod_{i=1}^{n} \big(x_i(t)^{\gamma_i} + y(t)\big)} \quad (2.24)$$

from which the expressions of $EFCH_{t,t+1}$, $TECH_{t,t+1}$ and $PCH_{t,t+1}$ can be calculated.

3 Dynamical path in discrete time

In this section, we assume that the size of an efficient firm at time period t evolves in such a way that it is still efficient at the successive time period. In what follows, the output vector at time period $t+1$ always 'realizes' its proportional distance of the previous time period t in our framework.[4] Without this assumption, the output vector relatively efficient to the technical efficiency frontier at time period t would not be automatically efficient at time period $t+1$. It is only a size change that allows the decision unit to be on the frontier at the following time period. If the output vector systematically follows this transformation rule between two successive time periods, then there exists an optimal dynamical behavior.

3.1 Notations and definitions

The technical efficiency indicator of the input-output $(x(t), y(t))$ at time period s is the distance $D^{\propto}_{T(s)}(x(t), y(t); \alpha, \beta)$. This indicator depends on $T(s)$, *i.e.* the production set at a given time period.

Let $\mathfrak{T} \subset \mathbb{R}_+$. The *family of production sets* $\{T(t)\}_{t\in\mathfrak{T}}$ is the set of all production sets $T(t)$ for $t \in \mathfrak{T}$. This definition is independent on the time characteristics. It can be used in discrete time ($t \in \mathbb{N}$) or in continuous time ($t \in \mathbb{R}_+$). The family $\{T(t)\}_{t\in\mathfrak{T}}$ is clearly embedded because of Axiom T1.

The *dynamical path* $\big\{(x(t), y(t))\big\}_{t\in\mathfrak{T}}$ of the production unit (x, y) is the set of the successive positions of the production unit (x, y) over time.

The optimal dynamical path of a production unit always depends on the choice of the technical efficiency indicator. In our context, the dynamical path of a vector

[4] This assumption is not original. A similar assumption can be found in Caves, Christensen and Diewert (1982), Färe, Grosskopf, Lindgren and Roos (1992), and Tulkens and Vanden Eeckaut (1995).

is said to be 'optimal proportional' if its outcome is an optimal movement between two successive time periods for a given proportional distance. The reference vector defined in equation (2.6) plays an important role in the definition of an optimal proportional dynamical path.

Definition 3.1.1 *Let $\{T(t_k)\}_{k=0,\dots,m}$ be a family of production sets with $t_0 < t_1 < t_2 < \cdots < t_m$ satisfying Axioms T1-T4. Let $\{(x(t_k), y(t_k))\}_{k=0,\dots,m}$ be a dynamical path, with $(x(t_k), y(t_k)) \in T(t_k)$ for each k. If at each time period t_k, $(x(t_{k+1}), y(t_{k+1}))$ is the reference vector of $(x(t_k), y(t_k))$ at time period t_{k+1}, then the dynamical path $\{x(t_k), y(t_k)\}_{k=0,\dots,m}$ is optimal proportional.*

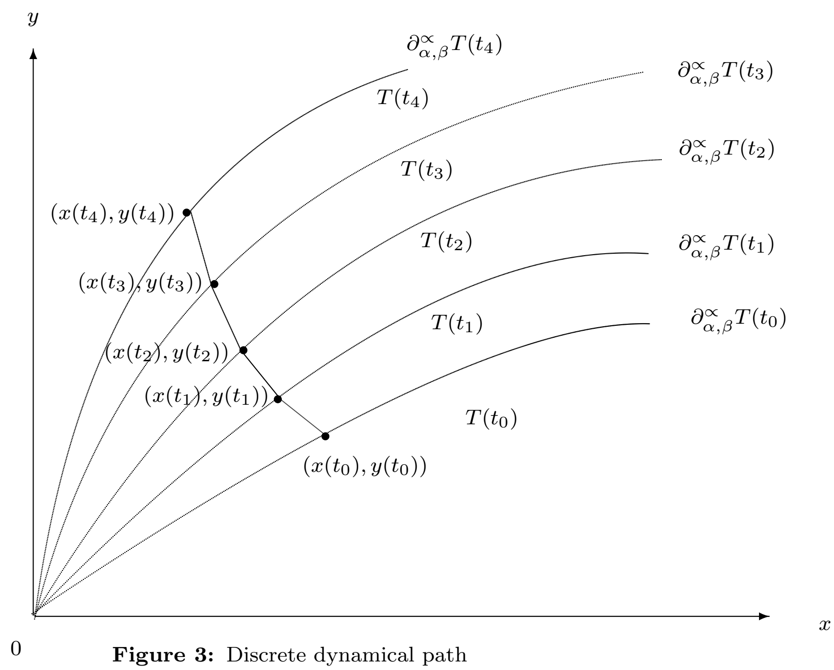

Figure 3: Discrete dynamical path

3.2 Time interval between two periods and regularity

Definition 3.1.1 allows us to introduce the concept of regularity, commonly used in the macroeconomics literature on technological progress.

Definition 3.2.1 *An optimal proportional path is θ-regular if there is $\theta > 0$ such that*

$$D^{\propto}_{T(t_{k+1})}(x(t_k), y(t_k); \alpha, \beta) = \theta$$

for all $k = 0, ..., m-1$.

The regularity specified in Definition 3.2.1 is similar to that considered in growth models, and this is an implicit assumption of the Solow model. In these latter, regularity is associated with the dynamical paths characterising technological progress through a linear time-dependent exponential function. The production process is of course modeled over time and the productivity measures evaluate by how much output results from a particular level of inputs. This specification can be studied in terms of variation rate of the factors or that of the products. Nishimizu and Page (1982) used a similar idea to allow for inefficiency in the Solow for which technical change and productivity change were identical. In the following lemma the growth rate can only be defined for inputs and outputs that are not zero. To that end, let us denote for all $(x, y) \in \mathbb{R}_+^{n+p}$, $I(x) = \{i \in [n] : x_i > 0\}$ and $J(y) = \{j \in [p] : y_j > 0\}$.

Lemma 3.2.2 *Let $T(t)$ and $T(s)$ be two production possibility sets at time periods t and s with $s \geq t$, both satisfying Axioms T1-T4. Let $(x(t), y(t))$ and $(x(s), y(s))$ be respectively efficient production units at time periods t and s. Suppose that $((x(s), y(s))$ is the reference vector of $(x(t), y(t))$ regarding to the production set $T(s)$. We have for all $(i, j) \in I(x(t)) \times J(y(t))$,*

$$D^{\propto}_{T(s)}(x(t), y(t); \alpha, \beta) = \frac{1}{\alpha_i} \frac{x_i(t) - x_i(s)}{x_i(t)} = \frac{1}{\beta_j} \frac{y_j(s) - y_j(t)}{y_j(t)}.$$

Moreover,

(1) if $\alpha = \mathbb{1}_n$ and $\beta = 0$, then $D^{\propto}_{T(s)}(x(t), y(t); \mathbb{1}_n, 0)$ is equal to the growth rate of the products between two successive time periods;

(2) if $\alpha = 0$ and $\beta = \mathbb{1}_p$, then $D^{\propto}_{T(s)}(x(t), y(t); 0, \mathbb{1}_p)$ is equal to the decline rate of the factors between two successive time periods;

(3) if $\alpha = \mathbb{1}_n$ and $\beta = \mathbb{1}_p$, then $D^{\propto}_{T(s)}(x(t), y(t); \mathbb{1}_n, \mathbb{1}_p)$ is equal to the decline rate of the factors and the growth rate of the products between two successive time periods.

The possible irregularity of the time interval does not prevent us from constructing a recurrent process determining the dynamics of the production set. Let A and B be respectively the diagonal matrices of α and β.

Lemma 3.2.3 *Let $\{T(t_k)\}_{k=0,\dots,m}$ be a family of production sets with $t_0 < t_1 < t_2 < \cdots < t_m$ satisfying Axioms T1-T4. Let $\{(x(t_k), y(t_k)\}_{k=0,1,\dots,m}$ be an optimal proportional dynamical path. Then we have at time period t_m*

$$x(t_m) = \Big[\prod_{k=0}^{m-1} (I_n - \theta_k A)\Big] x(t_0) \text{ and } y(t_m) = \Big[\prod_{k=0}^{m-1} (I_p + \theta_k B)\Big] y(t_0)$$

where $\theta_k = D^{\propto}_{T(t_{k+1})}(x(t_k), y(t_k); \alpha, \beta)$ for all $k = 0, ..., m - 1$.

As the regular time interval is a special case of irregular time intervals, the following two corollaries can be deduced from Lemma 3.2.3. Both definitions coincide in the case where the number of time periods is 2. Notice that in the following the dynamical path is indexed on the set of integers.

Corollary 3.2.4 *Let $\{T(t)\}_{t\in\mathbb{N}}$ be a family of production sets satisfying Axioms T1-T4. Suppose that $\{(x(t), y(t)\}_{t\in\mathbb{N}}$ is an optimal proportional dynamical path. If $\{(x(t), y(t)\}_{t\in\mathbb{N}}$ is θ-regular, then we have at each time period t*

$$x(t) = (1-\theta)^t x(0) \text{ and } y(t) = (1+\theta)^t y(0)$$

where θ is the decline rate of the factors and the growth rate of product between two successive time periods.

The above expression reminds us the usual specifications in simple macrodynamical frameworks. The simplest Solow framework has only one product, the output per capita. The latter grows according to the technological progress, *i.e.* $y_t = (1+\theta)^t F(x_0)$ where F is the aggregate production function and θ the growth rate of the product at each time period. If we consider a slightly more sophisticated model in which we assume a decline rate of the factors and a growth rate of the product, the per capita output grows according to $y_t = (1+\theta)^t F((1-\sigma)^t x_0)$.

The second corollary identifies the correspondence between regularity and proportionality.

Corollary 3.2.5 *Let $\{T(t)\}_{t\in\mathbb{N}}$ be a family of production sets satisfying Axioms T1-T4. Let $\{x(t), y(t)\}_{t\in\mathbb{N}}$ be an optimal proportional dynamical path. If $\{(x(t), y(t)\}_{t\in\mathbb{N}}$ is θ-regular, the two following properties apply.*

(1) *For each time period t, the factors are used in the same proportion.*

(2) *For each time period t, the products are produced in the same proportion.*

In the case of non-regular path, *i.e.* $s \neq t$, one can extend the definition of the Luenberger indicator as:

$$\begin{aligned} &L_{t,s}\big(x(t), y(t), x(s), y(s); \alpha, \beta\big) \\ &\quad = \frac{1}{2(s-t)}\Big[\Big(D^{\propto}_{T(t)}\big(x(t), y(t); \alpha, \beta\big) - D^{\propto}_{T(t)}\big(x(s), y(s); \alpha, \beta\big)\Big) \\ &\qquad + \Big(D^{\propto}_{T(s)}\big(x(t), y(t); \alpha, \beta\big) - D^{\propto}_{T(s)}\big(x(s), y(s); \alpha, \beta\big)\Big)\Big] \end{aligned} \tag{3.1}$$

As in Section 2, we denote this proportional Luenberger indicator, $PCH_{t,s}$. The latter can be decomposed into two components: the proportional efficiency change, $EFCH_{t,s}$, and the proportional technological change, $TECH_{t,s}$. It can be expressed as

$$PCH_{t,s} = EFCH_{t,s} + TECH_{t,s} \tag{3.2}$$

where

$$EFCH_{t,s} = \frac{1}{(s-t)}\Big[D^{\propto}_{T(t)}\big(x(t), y(t); \alpha, \beta\big) - D^{\propto}_{T(s)}\big(x(s), y(s); \alpha, \beta\big)\Big] \tag{3.3}$$

and

$$TECH_{t,s} = \frac{1}{2(s-t)}\Big[\Big(D^{\propto}_{T(s)}\big(x(t), y(t); \alpha, \beta\big) - D^{\propto}_{T(t)}\big(x(t), y(t); \alpha, \beta\big)\Big) + \Big(D^{\propto}_{T(s)}\big(x(s), y(s); \alpha, \beta\big) - D^{\propto}_{T(t)}\big(x(s), y(s); \alpha, \beta\big)\Big)\Big] \tag{3.4}$$

Corollary 3.2.6 *Let $\{T(t_k)\}_{k=0,\ldots,m}$ be a family of production sets with $t_0 < t_1 < t_2 < \cdots < t_m$ satisfying Axioms T1-T4. Let $\{(x(t_k), y(t_k)\}_{k=0,1,\ldots,m}$ be an optimal proportional dynamical path. Then we have at time period t_m*

$$L\big(x(t_k), y(t_k), x(t_{k+1}), y(t_{k+1}); \alpha, \beta\big) = \theta_k$$

where $\theta_k = D^{\propto}_{T(t_{k+1})}(x\,(t_k)\,, y\,(t_k)\,; \alpha, \beta)$ for $k = 0, ..., m-1$.

We can conclude this subsection by stressing that the non-regularity assumption has not prevented us from constructing a recurrent process. However, it is not one parameter but several parameters which reflect the technological progress between successive time periods.

4 Dynamical path in continuous time

Let us assume that the time interval between two time periods is infinitely small. As we shall see, this alternative time specification does not alter too much the expression of the proportional Luenberger indicator, its properties and the dynamical path.

4.1 Definition and notations

In the following, we extend the notions of optimal proportional and optimal regular proportional path to the continuous case.

Definition 4.1.1 *Let $\{T(t)\}_{t\geq t_0}$ be a family of production sets satisfying Axioms T1-T4. A dynamical path of $\{(x(t), y(t))\}_{t\geq t_0}$ is a H-optimal proportional if and only if for all $t \geq t_0$ $(x(t), y(t)) \in \partial^{\propto}_{\alpha,\beta}T(t)$ and there exists a continuous map $H : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $\forall\tau > 0$ the Proportional Distance Function satisfies the relation*

$$D^{\propto}_{T(t+\tau)}(x(t), y(t); \alpha, \beta) = H(t, \tau)$$

where H is differentiable in 0.

This definition extends that one of optimal proportional path to the continuous case. One can deduce the following result. In general notice that for any $d \times d$ matrix M and for any real number t, the exponential of the matrix tM is defined as:

$$e^{tM} = \sum_{k=0}^{\infty} \frac{1}{k!} t^k M^k. \tag{4.1}$$

The next result is a continuous time extension of Lemma 3.2.3.

Proposition 4.1.2 *Let $\{T(t)\}_{t \geq t_0}$ be a family of production sets satisfying Axioms T1-T4. Let $\{(x(t), y(t))\}_{t \geq t_0}$ be a H-optimal proportional dynamical path and let H be the map associated to this optimal proportional path. Suppose that for all $t \geq t_0$, $H(t, \cdot)$ is right-differentiable in 0 and let $\theta(t)$ be its derivative. Then*

$$x(t) = e^{-(\int_{t_0}^{t} \theta(\tau) d\tau) A} x(t_0) \quad \textit{and} \quad y(t) = e^{(\int_{t_0}^{t} \theta(\tau) d\tau) B} y(t_0).$$

$\theta(t)$ can be interpreted as the instantaneous variation rate of factors or products. In the following, we consider the regular case where $\theta(t)$ has a fixed value.

Definition 4.1.3 *Let $\{T(t)\}_{t \geq t_0}$ be a family of production sets satisfying Axioms T1-T4. Let $\{(x(t), y(t))\}_{t \geq t_0}$ be a dynamical path of $\{T(t)\}_{t \geq t_0}$. Let $\{(x(t), y(t))\}_{t \geq t_0}$ be a H-optimal proportional path. The path $\{(x(t), y(t))\}_{t \geq t_0}$ is regular if there exits a positive real number $\theta > 0$ such that:*

$$\lim_{\tau \to 0_+} \frac{H(t, \tau)}{\tau} = \theta.$$

Under the condition of Definition 4.1.3, we say that the optimal proportional path is θ-regular.

The optimal path described in the above definition is said to be θ-regular path. We assume an instantaneous adaption of the production unit integrating technological progress. It is optimal as it characterizes the evolution of the technical efficiency frontier over time. The relation expressed in this definition is similar to that expressed in the discrete case. The main difference between both definitions rests on the instantaneous feature which prevents the production unit from admitting technological decay in the continuous case. Notice that, paralleling the discrete time case, we assume that $t_0 = 0$.

Lemma 4.1.4 *Let $\{T(t)\}_{t \geq 0}$ be a family of production sets satisfying Axioms T1-T4. Let $\{(x(t), y(t))\}_{t \geq 0}$ be an optimal proportional path. If $\{(x(t), y(t))\}_{t \geq 0}$ is θ-regular, we have*

$$x(t) = e^{-\theta t A} x(0) \quad \textit{and} \quad y(t) = e^{\theta t B} y(0)$$

where θ is the instantaneous variation rate of factors or products.

The dynamical path described above is similar to that in the regular interval time in Section 3. However, it is 'smoother', *i.e.* the factors and the product are 'continuously' used in the same proportion.

As we have used the concept of proportional distance function, we are able to consider different weights on the factors and the products. This allows us to have various and non constant proportions between input and output. For instance, in the case of a single product, the per capita output grows according to $y_t = e^{\rho t} F(e^{\sigma t} x_t)$ where F is the production function, ρ the growth rate of a product and σ the decline rate of factor at each time period.

When the coefficient associated with a factor (product) is null, the growth (decline) rate is also null. This means that the decision unit does not have any room for manoeuvre for the inputs (outputs) considered. We can then refer back to a dynamics where only the product grows exponentially and where there is only one exponential decay with respect to the inputs.

Finally, the case in which the factor weightings vary according to time could also be considered. This is often what is reflected in the exponential specifications of technological progress. The growth rate of technological progress is then not regular.

4.2 The proportional Luenberger indicator

In continuous time, the proportional Luenberger indicator is equal to

$$\begin{aligned} & L\big(x(t), y(t), x(t+\tau), y(t+\tau); \alpha, \beta\big) \\ & = \frac{1}{2\tau}\Big[\Big(D^{\propto}_{T(t)}\big(x(t), y(t); \alpha, \beta\big) - D^{\propto}_{T(t)}\big(x(t+\tau), y(t+\tau); \alpha, \beta\big)\Big) \\ & \qquad + \Big(D^{\propto}_{T(t+\tau)}\big(x(t), y(t); \alpha, \beta\big) - D^{\propto}_{T(t+\tau)}\big(x(t+\tau), y(t+\tau); \alpha, \beta\big)\Big)\Big] \end{aligned} \tag{4.2}$$

In the case of a Cobb-Douglas production function with constant returns to scale with $\alpha = \mathbb{1}_n$ and $\beta = 0$, $L\big(x(t), y(t), x(t+\tau), y(t+\tau); \alpha, \beta\big) = TECH_{t,t+\tau}$ where

$$TECH_{t,t+\tau} = \frac{1}{2}\Big[\frac{a(t+\tau) - a(t)}{a(t)} + \frac{a(t+\tau) - a(t)}{a(t+\tau)}\Big]. \tag{4.3}$$

If the path $\{x_t\}_{t \geq t_0}$ is optimal and A is assumed differentiable at time period t, we obtain

$$\lim_{\tau \to 0} L\big(x(t), y(t), x(t+\tau), y(t+\tau); \alpha, \beta\big) = \frac{da(t)/dt}{a(t)}. \tag{4.4}$$

If the proportional distance function is continuously differentiable in time periods s and t, the proportional Luenberger indicator in continuous time is then defined by:

$$\mathcal{L}\big(x(t), y(t); \alpha, \beta\big) = - \left.\frac{\partial D^{\propto}_{T(t)}\big(x(s), y(s); \alpha, \beta\big)}{\partial s}\right|_{s=t} \tag{4.5}$$

Lemma 4.2.1 *Let $\{T(t)\}_{t \geq t_0}$ be a family of production sets satisfying Axioms T1-T4. Let $\{(x(t), y(t))\}_{t \geq t_0}$ be a dynamical path. If $D^{\propto}_{T(t)}\big(x(s), y(s); \alpha, \beta\big)$ is continuously differentiable in time periods s and t, then*

$$\mathcal{L}\big(x(t), y(t); \alpha, \beta\big) = - \left.\frac{dD^{\propto}_{T(s)}\big(x(s), y(s); \alpha, \beta\big)}{ds}\right|_{s=t} + \left.\frac{\partial D^{\propto}_{T(s)}\big(x(t), y(t); \alpha, \beta\big)}{\partial s}\right|_{s=t}.$$

From this Lemma, we derive the following corollary defining the proportional Luenberger indicator when the time interval is infinitively small.

Corollary 4.2.2 *Let $\{T(t)\}_{t \geq t_0}$ be a family of production sets satisfying Axioms T1-T4. Let $\{(x(t), y(t))\}_{t \geq t_0}$ be a dynamical path. If $D^{\propto}_{T(t)}\big(x(s), y(s); \alpha, \beta\big)$ is continuously differentiable in time periods s and t, then*

$$\lim_{\tau \to 0} L\big(x(t), y(t), x(t+\tau), y(t+\tau); \alpha, \beta\big) = \mathcal{L}\big(x(t), y(t); \alpha, \beta\big).$$

It follows that the proportional efficiency change in continuous time denoted $CEFCH$ is equal to

$$CEFCH = - \left. \frac{dD^{\propto}_{T(s)}\big(x(s), y(s); \alpha, \beta\big)}{ds} \right|_{s=t} \tag{4.6}$$

while the proportional technological change in continuous time denoted $CTECH$ is

$$CTECH = \left. \frac{\partial D^{\propto}_{T(s)}\big(x(t), y(t); \alpha, \beta\big)}{\partial s} \right|_{s=t} \tag{4.7}$$

Corollary 4.2.3 *Let $\{T(t)\}_{t \geq 0}$ be a family of production sets satisfying Axioms T1-T4. Let $\{(x(t), y(t))\}_{t \geq t_0}$ be an optimal and θ-regular dynamical path.*

If $D^{\propto}_{T(t)}\big(x(s), y(s); \alpha, \beta\big)$ is continuously differentiable in time periods s and t, then

$$\mathcal{L}\Big(x(0)e^{-\theta t A}, y(0)e^{\theta t B}; \alpha, \beta\Big) = \theta.$$

This corollary states that a unique parameter characterizes the technological progress.

5 Conclusion

We have introduced Farrell technical efficiencies in simple Solow models by adopting a Luenberger-type approach. This introduction has allowed us to make an analogy between a firm's behavior in a microeconomic setting and a country's behavior in a macroeconomic setting, both in a discrete time framework and a continuous time framework. In both cases, we were able to estimate the total factor productivity without having specified the production function. We were also able to evaluate production inefficiencies and therefore identify possible aggregate gains.

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Appendix

Proof of Lemma 3.2.2: To simplify the notation, let us denote $d(t, T(s))$ the expression of the proportional distance $D^{\propto}_{T(s)}(x(t), y(t); \alpha, \beta)$. We know

$$\big(x(s), y(s)\big) = (x(t), y(t)) + \left[D^{\propto}_{T(s)}(x(t), y(t); \alpha, \beta)\right](-\alpha \odot x(t), \beta \odot y(t)).$$

The above expression can be rewritten as

$$x(s) = x(t) - d(t, T(s))\alpha_i x(t) \quad \text{and} \quad y(s) = y(t) + d(t, T(s))\beta_j y(t) \text{ for all } i \text{ and } j.$$

The expression of the distance functions then follows.

Let $\alpha = \mathbb{1}_n$, $\beta = \mathbb{1}_p$ and $(x(s), y(s)) = \big([1 - d(t, T(s))]x(t), [1 + d(t, T(s))]y(t)\big)$ be the reference of $(x(t), y(t))$ at time period s. We can easily deduce that for all $i \in I(x(t))$ and all $j \in J(y(t))$

$$\frac{-x_i(s) + x_i(t)}{x_i(t)} = d(t, T(s)) \text{ and } \frac{y_j(s) - y_j(t)}{y_j(t)} = d(t, T(s)).\Box$$

Proof of Lemma 3.2.3: By definition of the proportional distance, we have

$$x(t_m) = (\mathbb{1}_n - \theta_{m-1}\alpha) \odot x(t_{m-1}) \textit{ and } y(t_m) = (\mathbb{1}_p + \theta_{m-1}\beta) \odot y(t_{m-1}).$$

Equivalently,

$$x(t_m) = (I_n - \theta_{m-1}A)x(t_{m-1}) \textit{ and } y(t_m) = (I_p + \theta_{m-1}B)y(t_{m-1}).$$

We also have

$$x(t_{m-1}) = (I_n - \theta_{m-2}A)x(t_{m-2}) \textit{ and } y(t_{m-1}) = (1_p + \theta_{m-2}B)y(t_{m-2}).$$

By recurrence, we deduce the result.□

Proof of f Corollary 3.2.4: From Lemma 3.2.3, if we consider the family of production sets $\{T(t)\}_{t \in \{t_0, t_1, \ldots\ldots, t_m\}}$ and an identical θ we have

$$x(t_m) = \left[\prod_{k=0}^{m-1}(I_n - \theta I)^k\right] x(t_0) \textit{ and } y(t_m) = \left[\prod_{k=0}^{m-1}(I_p + \theta I)^k\right] y(t_0).$$

Setting $t_0 = 0$, $t_1 = 1$, *etc.* yields

$$x(t) = (1 - \theta)^t x(0) \textit{ and } y(t) = (1 + \theta)^t y(0).\Box$$

Proof of Corollary 3.2.5:

(1) From Lemma 3.2.3, $x(t) = (1 - \theta)^t x(0)$ at time period t. It then results $\forall i, j \in I(x(0))^2$,

$$\frac{x_i(t)}{x_j(t)} = \frac{(1 - \theta)^t x_i(0)}{(1 - \theta)^t x_j(0)} = \frac{x_i(0)}{x_j(0)}.$$

In other words, the proportion by which each factor is used is constant over time.

(2) The proof is similar.□

Proof of Lemma 4.1.2: Let us assume that $(x(t+\tau), y(t+\tau))$ realizes the value $d(t, T(t+\tau))$ on $\partial^{\propto}_{\alpha,\beta} T(t+\tau)$. For all $(i,j) \in I(x(t)) \times J(y(t))$, we have

$$x_i(t+\tau) = x_i(t) - x_i(t)\alpha_i d(t, T(t+\tau)) \tag{5.1}$$

and

$$y_j(t+\tau) = y_j(t) + y_j(t)\beta_j d(t, T(t+\tau)). \tag{5.2}$$

As $\tau > 0$, we can divide expressions 5.1 and 5.2 by τ. We obtain

$$\frac{x_i(t+\tau) - x_i(t)}{\tau} = -\frac{H(t,\tau)}{\tau}\alpha_i x_i(t)$$

and

$$\frac{y_j(t+\tau) - y_j(t)}{\tau} = \frac{H(t,\tau)}{\tau}\beta_j y_j(t).$$

As $H(t,0) = d(t, T(t)) = 0$ and the function $H(t,\cdot)$ is right-differentiable at 0, we can evaluate the limit of the above both expressions when $\tau \to 0_+$

$$\lim_{\tau \to 0_+} \frac{x_i(t+\tau) - x_i(t)}{\tau} = -\theta(t)\alpha_i x_i(t)$$

and

$$\lim_{\tau \to 0_+} \frac{y_j(t+\tau) - y_j(t)}{\tau} = \theta(t)\beta_j y_j(t).$$

Similarly, we can assume that $(x(t), y(t)$ realizes the value $d(t-\tau, T(t))$ on $\partial^{\propto}_{\alpha,\beta} T(t)$. For all $(i,j) \in I(x(t)) \times J(y(t))$, we have:

$$x_i(t) = x_i(t-\tau) - x_i(t-\tau)\alpha_i d(t-\tau, T(t))$$

and

$$y_j(t) = y_j(t-\tau) + y_j(t-\tau)\beta_j d(t-\tau, T(t)).$$

Simple permutations yield:

$$\frac{x_i(t) - x_i(t-\tau)}{\tau} = -\frac{H(t,\tau)}{\tau}\alpha_i x_i(t-\tau)$$

and

$$\frac{y_j(t) - y_j(t-\tau)}{\tau} = \frac{H(t,\tau)}{\tau}\beta_j y_j(t-\tau).$$

As $d(t, T(t+\tau)) = H(\tau)$ and $\lim_{\tau \to 0_+} \frac{H(t,\tau)}{\tau} = \theta(t)$, we can evaluate the limits of both expressions. These are equal to

$$\lim_{\tau \to 0_+} \frac{x_i(t) - x_i(t-\tau)}{\tau} = -\theta(t)\alpha_i x_i(t)$$

and

$$\lim_{\tau \to 0_+} \frac{y_j(t) - y_j(t-\tau)}{\tau} = \theta(t)\beta_j y_j(t).$$

As the vectorial function $(x(t), y(t))$ is also left-differentiable at time period t, it is differentiable $\forall t \geq t_0$. As a result, for all $(i, j) \in I(x(t)) \times J(y(t))$,

$$\frac{dx_i(t)}{dt} = -\theta(t)\alpha_i x_i(t) \qquad \text{and} \qquad \frac{dy_j(t)}{dt} = \theta(t)\beta_j y_j(t).$$

We integrate both expressions from t_0 to t and we denote $x(t_0)$ by x_0 and $y(t_0)$ by y_0 to obtain

$$x(t) = e^{-(\int_{t_0}^{t} \theta(\tau)d\tau)A} x(t_0) \qquad \text{and} \qquad y(t) = e^{(\int_{t_0}^{t} \theta(\tau)d\tau)B} y(t_0).\square$$

Proof of Lemma 4.1.4: As $d(t, T(t+\tau)) = H(\tau)$ and $\lim_{\tau \to 0_+} H(\tau) = 0$, we can evaluate the limits of both expressions. These are equal to

$$\lim_{\tau \to 0_+} \frac{x_i(t) - x_i(t-\tau)}{\tau} = -\theta\alpha_i x_i(t)$$

and

$$\lim_{\tau \to 0_+} \frac{y_j(t) - y_j(t-\tau)}{\tau} = \theta\beta_j y_j(t).$$

As the vectorial function $(x(t), y(t))$ is also left-differentiable at time period t, it is differentiable $\forall t \geq t_0$. As a result, for all $(i, j) \in I(x(t)) \times J(y(t))$,

$$\frac{dx_i(t)}{dt} = -\theta\alpha_i x_i(t) \qquad \text{and} \qquad \frac{dy_j(t)}{dt} = \theta\beta_j y_j(t).$$

We integrate both expressions from t_0 to t and we denote $x(t_0)$ by x_0 and $y(t_0)$ by y_0 to obtain

$$x(t) = e^{-\theta t A} x(0) \qquad \text{and} \qquad y(t) = e^{\theta t B} y(0).\square$$

Proof of Lemma 4.2.1: Let $\Phi : [t_0, +\infty[\times [t_0, +\infty[$ be the map defined by

$$\Phi(s, t) = D^{\propto}_{T(s)}\big(x(t), y(t); \alpha, \beta\big).$$

It follows that $D^{\propto}_{T(s)}\big(x(s), y(s); \alpha, \beta\big) = \Phi(\xi(s))$ where $\xi(s) = (s, s)$. Differentiating in s yields

$$\left.\frac{d\Phi(s,s)}{ds}\right|_{s=t} = \left.\frac{\partial\Phi(s,t)}{\partial s}\right|_{s=t} + \left.\frac{\partial\Phi(s,t)}{\partial t}\right|_{s=t}.$$

From which we obtain

$$\left.\frac{dD^{\propto}_{T(s)}\big(x(s), y(s); \alpha, \beta\big)}{ds}\right|_{s=t} = \left.\frac{\partial D^{\propto}_{T(s)}\big(x(t), y(t); \alpha, \beta\big)}{\partial s}\right|_{s=t} + \left.\frac{\partial D^{\propto}_{T(s)}\big(x(t), y(t); \alpha, \beta\big)}{\partial t}\right|_{s=t}.$$

Since $\left.\frac{\partial D^{\propto}_{T(s)}\big(x(t),y(t);\alpha,\beta\big)}{\partial t}\right|_{s=t} = \left.\frac{\partial D^{\propto}_{T(t)}\big(x(s),y(s);\alpha,\beta\big)}{\partial s}\right|_{s=t}$, we deduce the result.□

Proof of Corollary 4.2.2: By definition, we have

$$\begin{aligned}
&L\big(x(t),y(t),x(t+\tau),y(t+\tau);\alpha,\beta\big) \\
&\quad = \frac{1}{2}\Big[\Big(\frac{D^{\propto}_{T(t)}\big(x(t),y(t);\alpha,\beta\big) - D^{\propto}_{T(t)}\big(x(t+\tau),y(t+\tau);\alpha,\beta\big)}{\tau}\Big) \\
&\qquad\quad + \Big(\frac{D^{\propto}_{T(t+\tau)}\big(x(t),y(t);\alpha,\beta\big) - D^{\propto}_{T(t+\tau)}\big(x(t+\tau),y(t+\tau);\alpha,\beta\big)}{\tau}\Big)\Big].
\end{aligned}$$

Since the proportional distance function is continuously differentiable, we deduce

$$\begin{aligned}
&\lim_{\tau\to 0}\frac{1}{2}\left[\frac{D^{\propto}_{T(t)}\big(x(t),y(t);\alpha,\beta\big) - D^{\propto}_{T(t)}\big(x(t+\tau),y(t+\tau);\alpha,\beta\big)}{\tau}\right] \\
&+ \lim_{\tau\to 0}\frac{1}{2}\left[\frac{D^{\propto}_{T(t+\tau)}\big(x(t),y(t);\alpha,\beta\big) - D^{\propto}_{T(t+\tau)}\big(x(t+\tau),y(t+\tau);\alpha,\beta\big)}{\tau}\right] \\
&= -\left.\frac{\partial D^{\propto}_{T(t)}\big(x(s),y(s);\alpha,\beta\big)}{\partial s}\right|_{s=t} \\
&= \mathcal{L}\big(x(t),y(t);\alpha,\beta\big).\Box
\end{aligned}$$