CALIBRATION OF THE CHAOTIC INTEREST RATE MODEL

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Abstract

In this thesis we establish a relationship between the Potential Approach to interest rates and the Market Models. This relationship allows us to derive the dynamics of forward LIBOR rates and forward swap rates by modelling the state price density. It means that we are able to secure the arbitrage-free condition and positive interest rate feature when we model the volatility drifts of those dynamics. On the other hand, we develop the Potential Approach, particularly the Hughston-Rafailidis Chaotic Interest Rate Model. The early argument enables us to infer that the Chaos Models belong to the Stochastic Volatility Market Models. In particular, we propose One-variable Chaos Models with the application of exponential polynomials. This maintains the generality of the Chaos Models and performs well for yield curves comparing with the Nelson-Siegel Form and the Svensson Form. Moreover, we calibrate the One-variable Chaos Model to European Caplets and European Swaptions. We show that the One-variable Chaos Models can reproduce the humped shape of the term structure of caplet volatility and also the volatility smile/skew curve. The calibration errors are small compared with the Lognormal Forward LIBOR Model, the SABR Model, traditional Short Rate Models, and other models under the Potential Approach. After the calibration, we introduce some new interest rate models under the Potential Approach. In particular, we suggest a new framework where the volatility drifts can be indirectly modelled from the short rate via the state price density.

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Contents

Abbreviation and Notation

- ATM: At the money,
- DM Statistics: Diebold-Mariano Statistics,
- FH: Flesaker and Hughston,
- FRA: Forward rate agreement,
- HJM: Heath, Jarrow and Morton,
- HW: Hull and White,
- IRS: Interest-Rate Swap,
- ITM: In the money,
- LFM: Lognormal Forward LIBOR Model,
- LMR: Log moneyness ratio,
- MLE: Maximum Likelihood Estimation Method,
- OTC: Over-the-counter,
- OTM: Out of the money,
- OU: Ornstein-Uhlenbeck,

- PFS: Payer Interest-Rate Swap,
- PS: European payer (call) Swaption,
- RFS: Receiver Interest-Rate Swap,
- RMSE: Root-Mean-Squared Error,
- RMSPE: Root-Mean-Squared Percentage Error,
- RS: European receiver (put) Swaption,
- SVM: Stochastic Volatility Market Model,
- ZBC: European call bond option,
- ZBP: European put bond option,
- f_{tT}: Instantaneous forward rate at t for the maturity T,
- m_{tT}: Stochastic discount factor at time t for the maturity T,
- r_t: Short rate at time t,
- B_t: Bank account at time t,
- $C_{a,b}(t)$: Associated numeraire of the forward swap rate defined by $C_{a,b}(t) := \sum_{i=a+1}^{b} \tau_i P_{t,T-i}$,
- F_{tTS}: Simply compounded forward (LIBOR) rate at time t for the expiry T and maturity S,
- $F_j(t) : F_{tT_{j-1}T_j}$,
- K: Strike rate,
- K_{ATM}: At the money strike rate,

- L_{tT}: Simply compounded (LIBOR) spot rate at time t for the maturity T,
- N: Nominal Value,
- P_{tT}: Discount bond at time t for the maturity T,
- $S_{a,b}(t)$: Forward swap rate at time t for the swaps between T_a and maturity T_b,
- V_t: State price density at time t,
- $\hat{V}_t$: Risk adjusted volatility at time t,
- W_t: Brownian Motion under the market measure $\mathbb{P}$ at time t,
- $\widetilde{W}_t$: Brownian Motion under the risk neutral measure $\mathbb{Q}$ at time t,
- Z_{tT}: Conditional expectation of the state price density, i.e., $Z_{tT} = \mathbb{E}_t[V_T]$,
- γ_{tTS}: Forward LIBOR rate volatility at time t for the expiry T and maturity S,
- $\hat{\gamma}_{a,b}(t)$: Forward swap rate volatility at time t for the swaps between T_a and maturity T_b,
- λ_t: Market price of risk at time t,
- ξ_t: Natural nemeraire at time t,
- π_{tT}: Time difference in years, i.e., $\pi_{tT} = T - t$,
- ρ_t: Change of measure density martingale,
- $\mathbb{P}$: Market measure/Real-World measure,
- $\mathbb{E}$: Expectation on the measure $\mathbb{P}$,
- $\mathbb{E}_t$: Conditional Expectation on the σ-field $\mathcal{F}_t$,

- $\mathbb{Q}$: Risk neutral measure,
- $\mathbb{E}^{\mathbb{Q}}$: Expectation on the measure $\mathbb{Q}$,
- $\mathbb{Q}^j$: T_j-forward adjusted measure,
- $\mathbb{E}^{\mathbb{Q}^j}$: Expectation on the measure $\mathbb{Q}^j$,
- $\mathbb{Q}^{a,b}$: Forward swap measure, having associated numeraire $C_{a,b}(t)$,
- D_t: Malliavin derivative with respect to t,
- $\sim$: distributed as,
- $\mathcal{N}(\mu, \sigma^2)$: Normal distribution with mean μ and variance σ^2,
- Φ: Standard normal cumulative distribution function,
- ρ: Standard normal density function.

Chapter 1

Introduction

In this thesis we present a quantitative analysis of the Interest Rate Markets. The motivation is to develop the Potential Approach to interest rates for valuing interest rate derivatives and to examine the Chaotic Approach. Since the Black-Scholes formula ([10]) was introduced in 1973, over-the-counter (OTC) derivatives have been actively traded in the financial markets. As can be observed from [2], the interest rate derivative market is the largest among all derivative markets, which accounts for 887.0 Trillions of US dollars and 73% of the total global derivatives for the whole of 2009; see Figure 1.1. There are three types of contracts in the OTC derivatives, namely; OTC swaps, OTC forwards and OTC options. In the interest rate derivative market, Interest rate swap belongs to OTC swap, Forward Rate Agreement belongs to OTC forward, and Interest Rate Cap and Floor, Swaption, Basis Swap and Bond Option belong to OTC option. The interest rate swap is most actively traded, which account for 691.1 Trillions of US dollars and 78% of the total interest rate derivatives for the whole of 2009. The value of interest rate options are 97.3 Trillions of US dollars and 14% of the total interest rate derivatives for the whole of 2009. As can be observed from Figure 1.2, the trading value of interest rate options has rapidly expanded during the last decade. The size of the interest rate option market is 6.1 times bigger in 2009 than the market in 1998. European Caps and European Swaptions are particularly popular interest rate options.

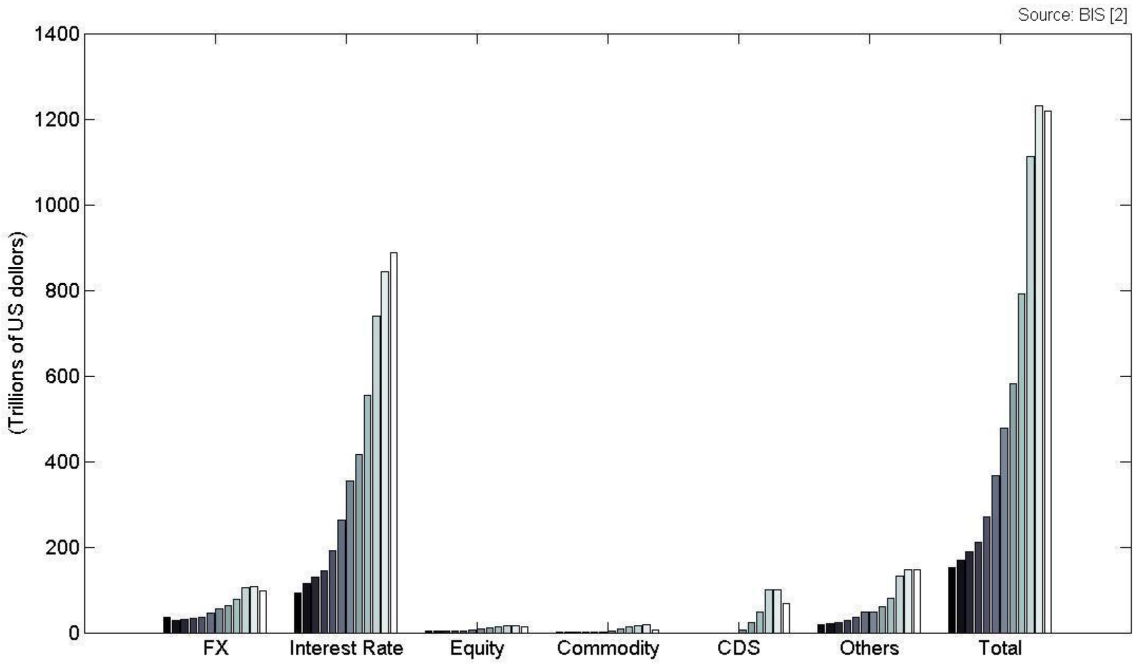

Figure 1.1: Notional amount outstanding of global over-the-counter derivatives

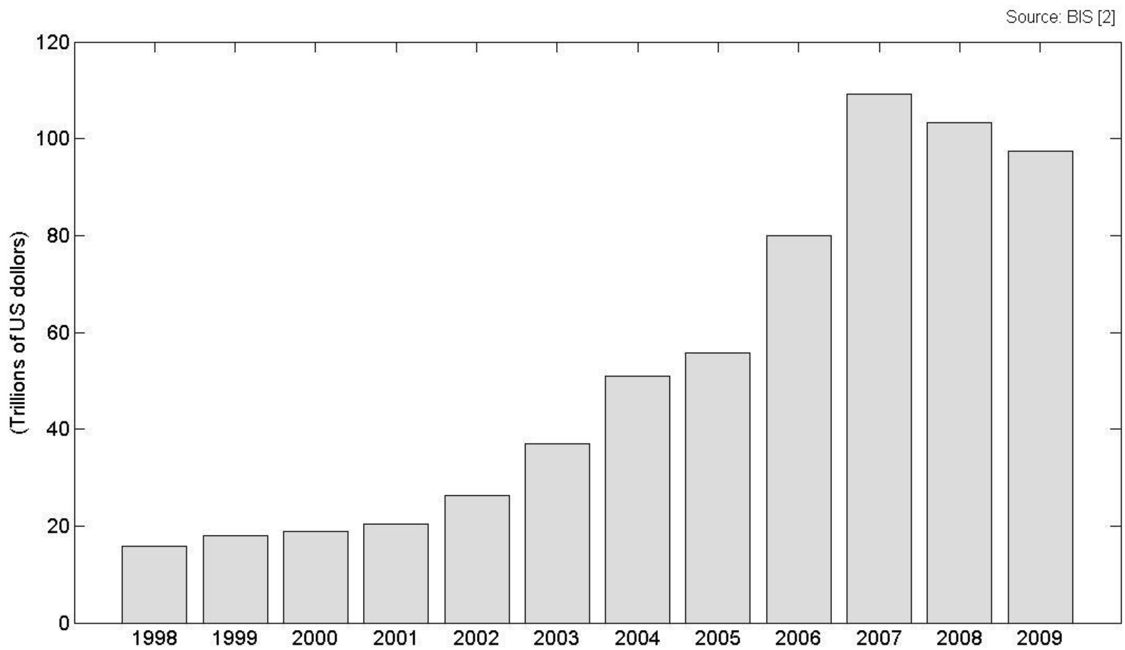

Figure 1.2: Notional amount outstanding of over-the-counter interest rate options

Research in this area has continuously evolved since the early 1970s. Rogers ([78]) claims the following requirements for a desirable interest rate model:

- Flexibility to reproduce most situations in the financial market.
- Simplicity to calibrate the model on European Options within reasonable time.
- Input parameters can be observed or estimated from the market.
- The model does not generate unrealistic values, such as negative interest rates or arbitrage opportunities.
- Good fitting ability into the market data.

However, there is still no model which can be thoroughly justified. The Short Rate Models were the first to be investigated within the modelling of interest rates. The first consistent Short Rate Model was the Vasicek Model ([92]) in 1977 and then later the Cox-Ingersoll-Ross Model ([27], hereafter referred as to CIR Model) in 1985. In these models we specify the dynamics of the short rate. However, the models were not correctly described, giving rather poor empirical results. As an extension of the Vasicek Model, the Hull-White Model ([49], hereafter referred as to HW Model) was introduced in 1990. Although the simplicity of the model still fascinates many practitioners in the market today, it generates negative interest rates, which is undesirable for valuing interest rate derivatives. Moreover, it does not fit well to the market data. Soon after the HW Model, a new general framework called "HJM Framework" ([45], also known as HJM Model) was published in 1992. In this framework, we specify the dynamics of the instantaneous forward rate where a drift condition ensures no arbitrage. Here, a one-factor HJM Framework with deterministic volatility is equivalent to the HW Model. The key problems in the HJM Framework were to ensure positive interest rates and compatibility to the Black formula (see, for example [14]). In other words, we wanted to find a drift condition for keeping the interest rate positivity and also generating log-normal behavior to the instantaneous forward rate.

As a result, the Lognormal Forward LIBOR Model (see [13], hereafter referred to as LFM, but also called BGM-Jamshidian Model) was introduced in 1997 to generate log-normal behavior to the forward LIBOR rate. It was the first compatible model with the Black formula and the first Market Model in which dynamics of a tradable asset is specified. This model was widely used and once accepted as a market standard model by the market practitioners. However, in the late 1990s a new dimension has been added to the interest rate options, that is, the smile/skew curve in the implied volatility of Caplets and Swaptions across the strike. The main problem of the LFM is that it does not generate an implied volatility smile/skew curve. It gives only a flat line, which contradicts the real market data. Since the LFM has been rejected for modelling the volatility smile, Local Volatility, Stochastic Volatility (hereafter referred to as SVM) and Jump Diffusion Market Models have been investigated in the area of quantitative finance for pricing interest rate derivatives. However, it was observed in [41] that the local volatility models have a crucial error for hedging derivatives, predicting volatility smiles and skews in the other direction. Moreover, although we would like to incorporate jump diffusion in the Market Models, it often causes a loss of computational speed. Therefore the current research trend focuses on the SVM. Among the SVM, the SABR Model ([41]) proposed in 2002 is the most appreciated model in the current financial market; hence, it is regarded as the market standard model (see for example, [76]). The advantages of the SABR Model are the following:

- Intuitive dynamics of the underlying assets, that is, forward LIBOR rates and forward swap rates.
- Stochasticity in the volatility drift, which property attains volatility smile curve.
- Simple implied volatility approximation forms, which gives very fast computa-

tional speed.

- Predicting volatility smiles and skews in the correct direction.

On the other hand, it also has some shortcomings:

- Inconsistency between Caplet and Swaption pricing formulas.
- Lack of mean reverting feature in the volatility term.[1]
- Failure to satisfy the arbitrage-free and positive interest conditions.
- It does not achieve a great fit to the ATM Options across maturity and tenor, as we will observe later.[2]
- Difficulty in pricing exotic options without applying the Monte Carlo Simulation.

To overcome those disadvantages some extensions of the SABR Model have been proposed, see for example [61], [76] and [81]. On the other hand, the Potential Approach has been investigated as a completely different method since the time when the Market Model was first considered in the 1990s. The methodology is to construct a potential process, that is, a supermartingale process which satisfies an asymptotic condition. We impose the potential process to work as the state price density, which is the inverse of the natural numeraire. Starting our argument from modelling the potential process, we secure the arbitrage-free and positive interest rate conditions. It was first introduced by Constantinides ([26]) in 1992 but the term "Potential Approach" was not coined until some five years later, by Rogers in [79]. Jin and Glasserman ([56]) derived the dynamics of the instantaneous forward rate under the Potential Approach, and showed that it belongs to the HJM Framework while the interest rate positivity is secured by the potential property of the state price density. Developing the Potential Approach,

[1] See, Mercurio and Morini ([61])

[2] Jump Diffusion Models have the same problem as observed in [29] where generated ATM implied volatilities are increasing with respect to time to maturity, which contradicts the real market data.

Flesaker and Hughston ([33]) announced the Rational Lognormal Model in 1996. In the Rational Lognormal Model we apply the Doob-Meyer decomposition (see for example [72]) and decompose the potential process into a martingale process and an increasing process, that is,

$$V_t = \mathbb{E}_t[A_\infty] - A_t, \quad \text{for} \quad t \geq 0.$$

The increasing process $(A_t)_{t\geq 0}$ is then specified by an integral of deterministic functions and a martingale process. However, their choice of A_t results in the LIBOR forward rate being bounded above and below, which renders it unable to price deep in the money and out of the money options. Hughston and Rafailidis continued to research along these same lines and announced the Chaotic Approach in the paper, "A Chaotic Approach to interest rate modelling" ([48]). They showed in this paper that the state price density is obtained through the conditional variance representation, that is,

$$V_t = \mathbb{E}_t\left[(X_\infty - \mathbb{E}_t\left[X_\infty\right])^2\right], \quad \text{for} \quad t \geq 0,$$

where X_∞ is unconstrained square integrable random variable. Application of the Wiener-Chaos expansion (see, for example [66] and [67]) gives a natural choice of the variable X_∞. In this model we are able to price deep in the money and out of the money options. Moreover, the model holds tractable pricing forms for the European Caplets and the European Swaptions. In [16], Brody and Hughston introduced the so-called "Coherent Interest rate Model" in which we also model the variable X_∞.

This thesis's main contributions to the literature are two fold; firstly, we develop the Potential Approach further. In particular, we extend an argument of Jin and Glasserman and derive the stochastic differential equations of the forward LIBOR rate and forward swap rate in the Potential Approach. In other words, we express the volatility drifts of the underlying assets in terms of the state price density. From this, we are able to incorporate the Potential Approach into the Market Models. Though for a long time these dynamics have been assumed to be arbitrary so that we obtain desirable distribution for the underlying assets, we now obtain a reasonable framework in the Market Model. Furthermore, we derive the LIBOR rate volatility and swap rate volatility in terms of the short rate. While expressing the volatility drifts by the short rate, we also specify the state price density by the short rate and the market price of risk. In particular, a positive short rate process secures the potential property of the state price density. In this framework, we are able to construct a Market Model by a Short Rate Model via the Potential Approach. Since we have the expression of the state price density, we are able to compute a discounted value of an interest rate derivative by the state price density. We particularly suggest to use the Affine Term Structure Models ([32]) in this framework where the conditional expectation of the stochastic discount factor can be explicitly solved. We show the Vasicek Model corresponds to the Shifted-lognormal Market Model ([35], also called Shifted BGM Model). Moreover, we also show that the Squared Gaussian Model ([69]) gives the forward LIBOR rate volatility Gaussian distributed.

Secondly, we particularly focus on the Chaotic Approach among the Potential Approach Models and calibrate the Chaos Models for yield curves, ATM Options and smile/skew curves of the implied volatility. Although in the original paper the tail of Wiener-Chaos expansion is truncated at the second term, we first make the argument without the truncation to keep the generality. In particular, we show the Chaotic Approach generates stochastic volatility. In other words, we show that the Chaos Models belong to the SVM. Here, we also notice that the Chaos Models give freedom to choose an initial yield curve. It means that we can calibrate the models on the yields and the options separately. In particular, we suggest a One-variable Winer-Chaos expansion where each chaos coefficient is a function of only one variable and derive the corresponding Chaos Models, which we call "One-variable Chaos Models". This specification allows simple analytical forms for all main processes. Moreover, we obtain the state price density in this framework by the polynomial of a Gaussian Process. Therefore, this method enables us to model a desirable probability density of the state price density. Furthermore, the One-variable Chaos Models also attain Caplets and Swaptions expressed by a polynomial function of the Brownian Motion. We first calibrate the One-variable Chaos Models on only yields. Among the literature the Nelson-Siegel Form ([64]) and the Svensson Form [86] are popular in the markets. When the initial curve can be freely chosen, traders often apply these forms to reproduce yield curves from the market. However, applying the exponential polynomial family to the One-variable Chaos coefficients also attains reasonable ability to fit to initial curve. Indeed it generates the instantaneous forward rate expressed by the following quotient form:

$$f_{0T} = \frac{\sum_{i=1}^{\infty}\left(\sum_{j=1}^{m_i}\widetilde{b}_{ij}e^{-c_{ij}T}\right)^2 T^{i-1}}{\int_T^{\infty}\sum_{i=1}^{\infty}\left(\sum_{j=1}^{m_i}\widetilde{b}_{ij}e^{-c_{ij}s}\right)^2 s^{i-1}ds}, \quad \text{for} \quad T \geq 0,$$

which is comparable with the the Björk and Christensen descriptive form ([12]). Since we obtain a fitting ability as good as the Svensson Form in the One-variable Chaos Model, we implement calibration on both yields and options by the Chaos Model so that we save a number of parameters. The option price calibration is firstly performed with ATM European Options. Then, we test the models on volatility smile/skew curves of the implied volatility, that is, away from the money options. We compare the calibration results with the LFM, the SABR Model, the traditional Short Rate Models, and the other models under the Potential Approach. From there, we show that One-variable Chaos Models have an outstanding ability to replicate the financial market data.

The thesis consists of ten chapters. The second chapter starts by reviewing the literature on the Potential Approach and the Marker Models, adding a few original results. Then we remind the reader about interest rate derivatives, using the book, Brigo and Mercurio ([14]) as a main reference. Among the Potential Approach Models, we review the Chaotic Approach, the Coherent Interest rate Model, the Rational Lognormal Model, and Constantinides Model. Among the Market Models, we review the LFM and the SABR Model. In the third chapter, we investigate the Potential Approach with particular motivation to establish a link with the Market Models. We develop the argument of Jin and Glasserman and derive the stochastic differential equations of the forward LIBOR rate and forward swap rate under the Potential Approach. As an example of the Potential Approach, we develop the Chaotic Approach and specify the chaos coefficients particularly for calibration work in the following chapter. Calibration works are split into three chapters, one each for yield calibration, ATM option price calibration, and smile calibration. We calibrate all models reviewed in the second chapter except for the Coherent Model and compare the performances with the One-variable Chaos Models. Some possible alternative models under the Potential Approach are proposed in the eighth chapter after the calibration. Here, we propose a new framework where we construct the volatility drifts from the short rate via the state price density. We then give our conclusions and address further works. Finally, in Chapter ten, some appendices containing relevant background information can be found.

Chapter 2

Literature Review

2.1 The Potential Approach

2.1.1 Introduction of the Potential Approach

In this section we mainly refer to [14], [48] and [51]. Let us consider a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_t)_{t\geq 0}$, where $\mathbb{P}$ is the market probability measure. Then we assume that there exists a continuous adapted process $(\xi_t)_{t\geq 0}$, called a "natural numeraire" such that for $t \geq 0$,

$$\frac{H_t}{\xi_t}, \quad \text{is a martingale under the market measure } \mathbb{P}$$

with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ for some continuous adapted non-dividend paying asset price process $(H_t)_{t\geq 0}$. To define the risk neutral measure, let us define a bank account process $(B_t)_{t\geq 0}$ using the following differential equation:

$$dB_t = r_t B_t dt, \tag{2.1.1}$$

where $(r_t)_{t\geq 0}$ is a progressively measurable stochastic process, such that $r_t > 0$ for all $t \geq 0$. This process is called the "short rate". The differential equation (2.1.1) has the solution

$$B_t = B_0 e^{\int_0^t r_s ds}, \quad \text{for} \quad t \geq 0.$$

The stochastic discount factor is defined by

$$m_{tT} := \frac{B_t}{B_T} = e^{-\int_t^T r_s ds}, \quad 0 \le t \le T < \infty. \tag{2.1.2}$$

Then, we may choose the bank account B_t to be the associated numeraire under the risk neutral measure $\mathbb{Q}$, that is, for $t \ge 0$,

$$\frac{H_t}{B_t}, \quad \text{is a martingale under the measure } \mathbb{Q}$$

with respect to the filtration $(\mathcal{F}_t)_{t\ge 0}$. This implies by (2.1.2) that, for $t \ge 0$,

$$m_{0t}H_t, \quad \text{is a martingale under the measure } \mathbb{Q}$$

with respect to the filtration $(\mathcal{F}_t)_{t\ge 0}$. It follows by the martingale property that

$$\mathbb{E}_t^{\mathbb{Q}}[m_{0T}H_T] = m_{0t}H_t \quad \text{for} \quad 0 \le t \le T < \infty.$$

In other words, we obtain that

$$H_t = \mathbb{E}_t^{\mathbb{Q}}[e^{-\int_t^T r_s ds} H_T] \quad \text{for} \quad 0 \le t \le T < \infty.$$

Since the measure $\mathbb{Q}$ is absolutely continuous with respect to $\mathbb{P}$ on $(\Omega, \mathcal{F})$, we obtain the Radon-Nikodym derivative

$$\rho := \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}}$$

where

$$\mathbb{Q}(A) = \int_A \rho(\omega) d\mathbb{P}(\omega), \quad \text{for all} \quad A \in \mathcal{F}.$$

Recall that the density ρ belongs to L^2 and is unique up to sets of measure zero. A martingale process $(\rho_t)_{t\ge 0}$ under $\mathbb{P}$ defined by

$$\rho_t := \mathbb{E}_t[\rho], \quad \text{for} \quad t \ge 0,$$

is called a "change of measure density martingale" or a "Radon-Nikodym derivative of $\mathbb{Q}$ relative to $\mathbb{P}$ restricted to $\mathcal{F}_t$". The latter terminology arises from the fact that

$$\rho_t = \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t}.$$

As is shown in Proposition 5.7 and Corollary 5.9 in [51], we have that for a continuous adapted process $(H_t)_{t\geq 0}$,

$$\mathbb{E}^{\mathbb{Q}}[H_t] = \mathbb{E}\left[\rho H_t\right] \quad \text{and} \quad \mathbb{E}_t^{\mathbb{Q}}\left[H_T\right] = \frac{\mathbb{E}_t\left[\rho H_T\right]}{\mathbb{E}_t\left[\rho\right]}, \quad 0 \leq t \leq T < \infty.$$

The reader might like to notice here that

$$\rho^{-1} = \left.\frac{d\mathbb{P}}{d\mathbb{Q}}\right|_{\mathcal{F}} \quad \text{and} \quad \mathbb{E}\left[H_t\right] = \mathbb{E}^{\mathbb{Q}}[\rho^{-1}H_t].$$

The tower property, (see [18]), gives us for each $0 \leq t \leq T < \infty$ that

$$\mathbb{E}^{\mathbb{Q}}[H_t] = \mathbb{E}\left[\rho_t H_t\right] \quad \text{and} \quad \mathbb{E}_t^{\mathbb{Q}}[H_T] = \mathbb{E}_t\left[\frac{\rho_T}{\rho_t} H_T\right]. \tag{2.1.3}$$

Therefore, the following equation is obtained:

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{H_t}{B_t}\right] = \mathbb{E}\left[\rho_t \frac{H_t}{B_t}\right], \quad \text{for} \quad t \geq 0. \tag{2.1.4}$$

On the other hand, from elementary martingale properties we have that

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{H_t}{B_t}\right] = \frac{H_0}{B_0} \quad \text{and} \quad \mathbb{E}\left[\frac{\xi_0}{B_0}\frac{H_t}{\xi_t}\right] = \frac{H_0}{B_0}, \quad \text{for} \quad t \geq 0.$$

Thus, we obtain that

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{H_t}{B_t}\right] = \mathbb{E}\left[\frac{\xi_0}{B_0}\frac{H_t}{\xi_t}\right]. \tag{2.1.5}$$

On combining the equations (2.1.4) and (2.1.5), we find that

$$\mathbb{E}\left[\rho_t \frac{H_t}{B_t}\right] = \mathbb{E}\left[\frac{\xi_0}{B_0}\frac{H_t}{\xi_t}\right], \quad \text{for} \quad t \geq 0,$$

which allows us to express the change of measure density martingale in the following way:

$$\rho_t = \frac{B_t}{B_0}\frac{\xi_0}{\xi_t}, \quad \text{for} \quad t \geq 0.$$

This is indeed a martingale under the measure $\mathbb{P}$ and we have that $\rho_0 = 1$. Defining a process $(V_t)_{t\geq 0}$, called a "state price density", by setting, for each $t \geq 0$,

$$V_t := \frac{1}{\xi_t},$$

it follows that for $t \geq 0$

$$\rho_t = \frac{B_t V_t}{B_0 V_0}. \tag{2.1.6}$$

Considering a conditional expectation, because the relationship (2.1.3) gives us that

$$\mathbb{E}_t^{\mathbb{Q}}\left[\frac{B_t}{B_T}H_T\right] = \mathbb{E}_t\left[\frac{B_t}{B_T}\frac{\rho_T}{\rho_t}H_T\right], \quad 0 \leq t \leq T < \infty,$$

we find that

$$\mathbb{E}_t^{\mathbb{Q}}\left[\frac{B_t}{B_T}H_T\right] = \mathbb{E}_t\left[\frac{\xi_t}{\xi_T}H_T\right], \quad 0 \leq t \leq T < \infty. \tag{2.1.7}$$

Furthermore, recalling the definitions, we may express the risk-neutrally discounted value by

$$\mathbb{E}_t^{\mathbb{Q}}\left[m_{tT}H_T\right] = \frac{\mathbb{E}_t\left[V_T H_T\right]}{V_t}, \quad 0 \leq t \leq T < \infty.$$

In other words, we have that

$$H_t = \frac{\mathbb{E}_t\left[V_T H_T\right]}{V_t}, \quad 0 \leq t \leq T < \infty.$$

We now define the zero-coupon bond P_{tT}, also referred to as a "discount bond", which gives us a risk-free investment at time $t \geq 0$ for its holder securing the payment of one unit of currency at time $T \geq t$ without any intermediate payments. In other words, the discount bond process is a positive continuous adapted process $(P_{tT})_{0 \leq t \leq T < \infty}$ with the property $P_{TT} = 1$. Therefore, it may be defined by the state price density in the following way:

$$P_{tT} := \frac{\mathbb{E}_t\left[V_T\right]}{V_t}, \quad 0 \leq t \leq T < \infty. \tag{2.1.8}$$

For convenience, we define the conditional expectation of the state price density to be

$$Z_{tT} := \mathbb{E}_t\left[V_T\right], \quad 0 \leq t \leq T < \infty, \tag{2.1.9}$$

and express the discount bond by

$$P_{tT} = \frac{Z_{tT}}{V_t}, \quad 0 \leq t \leq T < \infty. \tag{2.1.10}$$

The simply compounded (LIBOR) spot rate L_{tT} (also referred as to LIBOR spot rate) may be expressed using the discount bond as follows:

$$1 = (1 + \tau_{tT} L_{tT}) P_{tT} \quad \text{for} \quad 0 \leq t \leq T < \infty,$$

where τ_{tT} is the time difference in years, i.e., $\tau_{tT} := T - t$. In other words, the LIBOR spot rate is defined by the discount bond as follows:

$$L_{tT} := \frac{1 - P_{tT}}{\tau_{tT} P_{tT}} \quad \text{for} \quad 0 \leq t \leq T < \infty.$$

Using expression (2.1.10), this can be expressed in the following way:

$$L_{tT} = \frac{1}{T - t} \left(\frac{V_t}{Z_{tT}} - 1 \right) \quad \text{for} \quad 0 \leq t \leq T < \infty. \tag{2.1.11}$$

As is observed in [57], the martingale process $(\rho_t)_{t \geq 0}$ can be modelled by the form:

$$\rho_t = e^{-\frac{1}{2} \int_0^t \lambda_s^2 ds - \int_0^t \lambda_s dW_s}, \quad \text{for} \quad t \geq 0$$

for a square-integrable process $(\lambda_t)_{t \geq 0}$ which satisfies the Novikov condition:

$$\mathbb{E} \left[e^{\frac{1}{2} \int_0^t \lambda_s^2 ds} \right] < \infty.$$

This process is called the "market price of risk". This satisfies the condition that $\mathbb{E}[\rho_t] = 1$. Thus we obtain

$$d\rho_t = -\lambda_t \rho_t dW_t. \tag{2.1.12}$$

Note here that Girsanov's theorem, (see [51]), allows us to introduce a Brownian Motion under the measure $\mathbb{Q}$ as follows:

$$\widetilde{W}_t := W_t + \int_0^t \lambda_s ds, \quad \text{for} \quad t \geq 0.$$

Therefore, using the relationship (2.1.6), we obtain that

$$\begin{aligned} V_t &= B_0 V_0 \frac{\rho_t}{B_t} \\ &= V_0 e^{-\int_0^t (r_s + \frac{1}{2} \lambda_s^2) ds - \int_0^t \lambda_s dW_s}. \end{aligned} \tag{2.1.13}$$

A positive supermartingale with the following asymptotic condition is called a "potential":

$$\lim_{T\to\infty} \mathbb{E}[V_T] = 0. \tag{2.1.14}$$

The Potential Approach is an interest rate modelling method that models the state price density process $(V_t)_{t\geq 0}$, such that the process is potential. Note here that because the process $(\rho_t)_{t\geq 0}$ is a martingale and the integral $\int_0^t r_s ds$ is an increasing function of t if the variable r_s is positive for all $0 \leq s \leq t$, we observe that the process $(V_t)_{t\geq 0}$ is a positive supermartingale. Moreover, because the state price density formed in (2.1.13) is positive process, the potential property of the state price density is secured under the positive short rate models. Furthermore, using this expression with (2.1.9) and (2.1.10), we may express for each $0 \leq t \leq T < \infty$ that

(2.1.15)

$$Z_{tT} = \mathbb{E}_t\Big[V_0 e^{-\int_0^T (r_s+\frac{1}{2}\lambda_s^2)ds - \int_0^T \lambda_s dW_s}\Big] \quad \text{and} \quad P_{tT} = \mathbb{E}_t\Big[e^{-\int_t^T (r_s+\frac{1}{2}\lambda_s^2)ds - \int_t^T \lambda_s dW_s}\Big].$$

Because we have by (2.1.3) that

$$Z_{tT} = \mathbb{E}_t\Big[B_0 V_0 \frac{\rho_T}{B_T}\Big] = \mathbb{E}_t^{\mathbb{Q}}\Big[B_0 V_0 \frac{\rho_t}{B_T}\Big] = B_0 V_0 \rho_t \mathbb{E}_t^{\mathbb{Q}}\Big[\frac{1}{B_T}\Big], \tag{2.1.16}$$

under the risk neutral measure we may express that

$$Z_{tT} = V_0 e^{-\int_0^t \frac{1}{2}\lambda_s^2 ds - \int_0^t \lambda_s dW_s} \mathbb{E}_t^{\mathbb{Q}}\Big[e^{-\int_0^T r_s ds}\Big].$$

It follows that

$$Z_{tT} = V_0 e^{-\int_0^t (\frac{1}{2}\lambda_s^2 + r_s)ds - \int_0^t \lambda_s dW_s} \mathbb{E}_t^{\mathbb{Q}}\Big[e^{-\int_t^T r_s ds}\Big] \quad \text{and} \quad P_{tT} = \mathbb{E}_t^{\mathbb{Q}}\Big[e^{-\int_t^T r_s ds}\Big]. \tag{2.1.17}$$

In order to obtain the dynamics of the state price density, first notice that on differentiating both sides of (2.1.6) we obtain that

$$d\rho_t = \frac{1}{B_0 V_0}(B_t dV_t + V_t dB_t).$$

From this, we infer that

$$dV_t = -\frac{V_t}{B_t}dB_t + \frac{B_0V_0}{B_t}d\rho_t.$$

Applying the expressions (2.1.1) and (2.1.12), we obtain that

$$dV_t = -r_tV_tdt - \frac{B_0V_0}{B_t}\lambda_t\rho_tdW_t.$$

Finally, from equation (2.1.6), we conclude that

$$dV_t = -r_tV_tdt - \lambda_tV_tdW_t. \tag{2.1.18}$$

2.1.2 Chaotic Interest Rate Model

For this section, we refer mostly to [48]. Also, unless stated otherwise, for the variable t we always assume that $t \geq 0$. Integrating the stochastic differential equation of the state price density (2.1.18), we obtain that

$$V_t = V_0 - \int_0^t r_sV_sds - \int_0^t \lambda_sV_sdW_s.$$

This gives us the following relationship:

$$V_t + \int_0^t r_sV_sds = V_0 - \int_0^t \lambda_sV_sdW_s. \tag{2.1.19}$$

By the property of the Itô integral, we notice that the right hand side of equation (2.1.19) is a martingale. This implies that the left hand side of equation (2.1.19) is also a martingale. Using the martingale property and the Monotone Convergence Theorem, (see [53]), it follows that

$$\mathbb{E}_t\Big[\int_0^\infty r_sV_sds\Big] = V_t + \int_0^t r_sV_sds. \tag{2.1.20}$$

Therefore, defining a positive process σ_t in the following way,

$$\sigma_t^2 := r_tV_t, \tag{2.1.21}$$

the state price density can be expressed in the following conditional expectation form:

$$V_t = \mathbb{E}_t\Big[\int_t^\infty \sigma_s^2 ds\Big]. \tag{2.1.22}$$

Defining X_∞ to be a square-integrable random variable by setting

$$X_\infty := \int_0^\infty \sigma_s dW_s,$$

it follows that

$$V_t = \mathbb{E}_t\left[(X_\infty - \mathbb{E}_t[X_\infty])^2\right].$$

The Wiener-Chaos expansion is then applied to let the random variable X_∞ be expressed by a sequence of deterministic functions $\{\phi_n\}_{n=0}^\infty$, called "chaos coefficients", as follows:

$$X_\infty = \int_0^\infty \left[\phi_1(s) + \int_0^s \phi_2(s,s_1)dW_{s_1} + \int_0^s\int_0^{s_1} \phi_3(s,s_1,s_2)dW_{s_2}dW_{s_1} + \cdots\right]dW_s.$$

Here, the Wiener-Chaos expansion can be proved using iteration of Ito Representation Theorem, as can be seen in [67] and [89]. Therefore, the variable σ_s can be expressed by the sum

$$\sigma_s = \phi_1(s) + \int_0^s \phi_2(s,s_1)dW_{s_1} + \int_0^s\int_0^{s_1} \phi_3(s,s_1,s_2)dW_{s_2}dW_{s_1} + \cdots. \tag{2.1.23}$$

We call the interest rate model a "k^{th}-order Chaos Model" when the chaos coefficients exist only up to order k.

2.1.3 Coherent Interest Rate Model

In [16], Brody and Hughston introduced the so-called "Coherent Interest Rate Models", in which the chaos expansion is formulated using only one function, i.e.,

$$\phi_1(s) = \psi(s), \quad \phi_2(s,s_1) = \psi(s)\psi(s_1), \quad \phi_3(s,s_1,s_2) = \psi(s)\psi(s_1)\psi(s_2), \cdots.$$

In this case we have that

$$X_\infty = \exp\Big(\int_0^\infty \psi(s)dWs - \frac{1}{2}\int_0^\infty \psi^2(s)ds\Big).$$

This forces the corresponding discount bond system to be deterministic. However, they use the fact that any element of the Hilbert Space $L^2(\Omega)$ may be represented as a superposition of coherent vectors to construct stochastic term structures. For example, for some constants $a, b \in \mathbb{R}$ and functions $\psi_1, \psi_2 \in L^2$ they suggest that

$$X_\infty = a \exp\Big(\int_0^\infty \psi_1(s) dWs - \frac{1}{2} \int_0^\infty \psi_1^2(s) ds \Big) + b \exp\Big(\int_0^\infty \psi_2(s) dWs - \frac{1}{2} \int_0^\infty \psi_2^2(s) ds \Big),$$

which produces a stochastic interest rate.

2.1.4 Rational Lognormal Model

We use the paper [33] and related literature ([38], [63] and [73]) as references for recalling the Rational Lognormal Model. Because the state price density is a potential, the Doob-Meyer decomposition states that there exists a unique increasing process $(A_t)_{t\geq 0}$ with $A_0 = 0$ such that for each $t \geq 0$ we have

$$V_t = \mathbb{E}_t[A_\infty] - A_t. \tag{2.1.24}$$

Indeed, in light of (2.1.22), we are able to express the increasing process $(A_t)_{t\geq 0}$ in the following way:

$$A_t = \int_0^t \sigma_s^2 ds, \quad \text{for} \quad t \geq 0. \tag{2.1.25}$$

In the Rational Lognormal Model we assume that the integrand is represented in the following form:

$$\sigma_s^2 = g_1(s) M_s + g_2(s), \quad 0 \leq s \leq t,$$

where g_1, g_2 are nonnegative deterministic functions of time, and M_t is a strictly positive continuous martingale such that $M_0 = 1$. Hence, we observe that the Rational Lognormal Model is comparable with the Chaotic Approach, because the difference is only the expression of σ_s, where in the Chaotic Approach we apply the Wiener-Chaos expansion as seen in (2.1.23). The reader might like to see the remark at the end of this section for a further discussion of this comparability. The form in the Rational Lognormal Model gives that the state price density is represented as

$$
\begin{aligned}
V_t =& \mathbb{E}_t \left[\int_t^\infty \left(g_1(s) M_s + g_2(s) \right) ds \right] \\
=& \int_t^\infty g_1(s) M_t ds + \int_t^\infty g_2(s) ds \\
=& G_1(t) M_t + G_2(t),
\end{aligned}
$$

where we denote for $t \geq 0$

$$
G_1(t) := \int_t^\infty g_1(s) ds, \quad \text{and} \quad G_2(t) := \int_t^\infty g_2(s) ds.
$$

Similarly, we obtain the conditional expectation of the state price density as follows:

$$
\begin{aligned}
Z_{tT} =& \mathbb{E}_t \left[\int_T^\infty \left(g_1(s) M_s + g_2(s) \right) ds \right] \\
=& \int_T^\infty g_1(s) M_t ds + \int_T^\infty g_2(s) ds \\
=& G_1(T) M_t + G_2(T),
\end{aligned}
$$

for $0 \leq t \leq T < \infty$. This implies that discount bonds are represented in the following way:

$$
P_{tT} = \frac{G_1(T) M_t + G_2(T)}{G_1(t) M_t + G_2(t)}, \quad \text{for} \quad 0 \leq t \leq T < \infty.
$$

Note that, as discussed in [19] and [33], discount bonds are bounded by

$$
\frac{G_1(T)}{G_1(t)} \leq P_{tT} \leq \frac{G_2(T)}{G_2(t)} \quad \text{or} \quad \frac{G_2(T)}{G_2(t)} \leq P_{tT} \leq \frac{G_1(T)}{G_1(t)}, \quad \text{for} \quad 0 \leq t \leq T < \infty,
$$

because the discount bond may be either decreasing or increasing or constant with respect to the stochasticity M_t for fixed t and T. As shown in [19], it follows that the short rate is also bounded by

$$
-\frac{\frac{\partial}{\partial t} G_1(t)}{G_1(t)} \leq r_t \leq -\frac{\frac{\partial}{\partial t} G_2(t)}{G_2(t)} \quad \text{or} \quad -\frac{\frac{\partial}{\partial t} G_2(t)}{G_2(t)} \leq r_t \leq -\frac{\frac{\partial}{\partial t} G_1(t)}{G_1(t)}, \quad \text{for} \quad t \geq 0.
$$

Therefore we are unable to price deep in the money and out of the money options in the Rational Lognormal Model.

Remark (Relationship with the Chaotic Approach)

When we apply the exponential martingale in the Rational Lognormal Model we may express the function σ_s^2 in the following way:

$$\sigma_s^2 = g_1(s)\exp\Big[-\frac{1}{2}\int_0^t \widetilde{\sigma}_u^2 du + \int_0^t \widetilde{\sigma}_u dW_u\Big] + g_2(s).$$

The Clark-Ocone formula, (see, for example [66] and [67]), allows us to express the functions in the integrand of the Wiener-Chaos expansion (2.1.23), where we denote the Malliavin derivative of t by D_t, in the following way:

$$\begin{aligned}&\phi_1(s) = \mathbb{E}[\sigma_s], \quad \phi_2(s,s_1) = \mathbb{E}[D_{s_1}[\sigma_s]], \quad \phi_3(s,s_1,s_2) = \mathbb{E}[D_{s_2}[D_{s_1}[\sigma_s]]],\\ &\phi_4(s,s_1,s_2,s_3) = \mathbb{E}[D_{s_3}[D_{s_2}[D_{s_1}[\sigma_s]]]], \quad \cdots .\end{aligned}$$

Hence, we find that

$$\begin{aligned}\phi_1(s) &= \mathbb{E}[\sigma_s],\\ \phi_2(s,s_1) &= g_1(s)\mathbb{E}\Big[\frac{1}{2\sigma_s}\exp\Big[\frac{1}{2}\int_0^t \widetilde{\sigma}_u^2 du + \int_0^t \widetilde{\sigma}_u dW_u\Big]\Big]\widetilde{\sigma}_{s_1},\\ \phi_3(s,s_1,s_2) &= g_1(s)\mathbb{E}\Big[-\frac{g_1(s)}{4\sigma_s^3}\Big(\exp\Big[-\frac{1}{2}\int_0^t \widetilde{\sigma}_u^2 du + \int_0^t \widetilde{\sigma}_u dW_u\Big]\Big)^2\\ &\quad + \frac{1}{2\sigma_s}\exp\Big[-\frac{1}{2}\int_0^t \widetilde{\sigma}_u^2 du + \int_0^t \widetilde{\sigma}_u dW_u\Big]\Big]\widetilde{\sigma}_{s_1}\widetilde{\sigma}_{s_2},\end{aligned}$$

and so on. In other words, we obtain that

$$\begin{aligned}\sigma_s =& \mathbb{E}[\sigma_s] + \int_0^s g_1(s)\mathbb{E}\Big[\frac{1}{2\sigma_s}\exp\Big[\frac{1}{2}\int_0^t \widetilde{\sigma}_u^2 du + \int_0^t \widetilde{\sigma}_u dW_u\Big]\Big]\widetilde{\sigma}_{s_1} dW_{s_1}\\ &+ \int_0^s\int_0^{s_1}\Big[g_1(s)\mathbb{E}\Big[-\frac{g_1(s)}{4\sigma_s^3}\Big(\exp\Big[-\frac{1}{2}\int_0^t \widetilde{\sigma}_u^2 du + \int_0^t \widetilde{\sigma}_u dW_u\Big]\Big)^2\\ &+ \frac{1}{2\sigma_s}\exp\Big[-\frac{1}{2}\int_0^t \widetilde{\sigma}_u^2 du + \int_0^t \widetilde{\sigma}_u dW_u\Big]\Big]\widetilde{\sigma}_{s_1}\widetilde{\sigma}_{s_2}\Big] dW_{s_2} dW_{s_1} + \cdots .\end{aligned}$$

We observe that the functions in the integrands are factorizable.

2.1.5 Constantinides Model

We consider the Constantinides Model ([26]), which represents the state price density in the following exponential form:

$$V_t = \exp\Big[-\Big(g+\frac{\sigma_0^2}{2}\Big)t + \sigma_0 W_0(t) + \sum_{i=1}^{N}(x_i(t)-\alpha_i)^2\Big],$$

where $x_i(t)$ for $1 \le i \le N$ are Ornstein-Uhlenbeck (hereafter, OU) processes defined by

$$dx_i(t) = -\lambda_i x_i(t)dt + \sigma_i dW_i(t),$$

where $W_0(t), W_1(t), \ldots, W_N(t)$ are mutually independent Wiener processes, and $g, \alpha_i, \sigma_0 \ge 0, \sigma_i > 0$ and $\lambda_i > 0$ are constants. To ensure that the interest rates are positive, the parameters are restricted by

$$\lambda_i > \sigma_i^2 \quad \text{for any} \quad i = 1, \ldots, N,$$

and

$$g - \sum_{i=1}^{N}\Big(\sigma_i^2 + \frac{\lambda_i \alpha_i^2}{2(1-\sigma_i^2/\lambda_i)}\Big) > 0.$$

It follows that the conditional expectation of the state price density can be represented as

$$\begin{aligned} Z_{tT} =& \mathbb{E}_t\Big[\exp\Big[-\Big(g+\frac{\sigma_0^2}{2}\Big)T + \sigma_0 W_0(T) + \sum_{i=1}^{N}(x_i(T)-\alpha_i)^2\Big]\Big] \\ =& \exp\Big[-\Big(g+\frac{\sigma_0^2}{2}\Big)T\Big]\mathbb{E}_t[\exp[\sigma_0 W_0(T)]]\prod_{i=1}^{N}\mathbb{E}_t[\exp[(x_i(T)-\alpha_i)^2]]. \end{aligned}$$

As we will see the Appendix, in the original paper [26], due to the facts that

$$\mathbb{E}_t[\exp[\sigma_0 W_0(T)]] = \exp\Big[\sigma_0 W_0(t) + \frac{\sigma_0^2}{2}(T-t)\Big],$$

and

$$\mathbb{E}_t[\exp[(x_i(T)-\alpha_i)^2]] = H_i^{-\frac{1}{2}}(T-t)\exp[\lambda_i(T-t) + H_i^{-1}(T-t)(x_t(t)-\alpha_i e^{\lambda_i(T-t)})^2]$$

where

$$H_i(T-t) := \frac{\sigma_i^2}{\lambda_i} + \Big(1 - \frac{\sigma_i^2}{\lambda_i}\Big)e^{2\lambda_i(T-t)} \tag{2.1.26}$$

for $i = 1, \dots, N$, we obtain that

$$\begin{aligned}
Z_{tT} =& \exp\Big[-\Big(g + \frac{\sigma_0^2}{2}\Big)T + \sigma_0 W_0(t) + \frac{\sigma_0^2}{2}(T-t)\Big] \\
&\times \prod_{i=1}^{N} H_i^{-\frac{1}{2}}(T-t)\exp[\lambda_i(T-t) + H_i^{-1}(T-t)(x_t(t) - \alpha_i e^{\lambda_i(T-t)})^2] \\
=& \Big(\Pi_{i=1}^N H_i(T-t)\Big)^{-\frac{1}{2}} \\
&\times \exp\Big[-gT - \Big(\frac{\sigma_0^2}{2}\Big)t + \sigma_0 W_0(t) + \sum_{i=1}^{N}\lambda_i(T-t) + \sum_{i=1}^{N} H_i^{-1}(T-t)(x_t(t) - \alpha_i e^{\lambda_i(T-t)})^2\Big].
\end{aligned}$$

Therefore, the bond price can be expressed for $0 \le t \le T < \infty$ by

$$\begin{aligned}
P_{tT} =& \Big(\Pi_{i=1}^N H_i(T-t)\Big)^{-\frac{1}{2}} \\
&\times \frac{\exp\Big[-gT - \Big(\frac{\sigma_0^2}{2}\Big)t + \sigma_0 W_0(t) + \sum_{i=1}^{N}\lambda_i(T-t) + \sum_{i=1}^{N} H_i^{-1}(T-t)(x_t(t) - \alpha_i e^{\lambda_i(T-t)})^2\Big]}{\exp\Big[-\Big(g + \frac{\sigma_0^2}{2}\Big)t + \sigma_0 W_0(t) + \sum_{i=1}^{N}(x_i(t) - \alpha_i)^2\Big]} \\
=& \Big(\Pi_{i=1}^N H_i(T-t)\Big)^{-\frac{1}{2}} \\
&\times \exp\Big[\Big(-g + \sum_{i=1}^{N}\lambda_i\Big)(T-t) + \sum_{i=1}^{N} H_i^{-1}(T-t)(x_i(t) - \alpha_i e^{\lambda_i(T-t)})^2 - \sum_{i=1}^{N}(x_i(t) - \alpha_i)^2\Big].
\end{aligned}$$

In particular, for the initial bond price, that is, when $t = 0$, the model gives us that

$$\begin{aligned}
P_{0T} =& \Big(\Pi_{i=1}^N H_i(T)\Big)^{-\frac{1}{2}} \\
&\times \exp\Big[\Big(-g + \sum_{i=1}^{N}\lambda_i\Big)T + \sum_{i=1}^{N} H_i^{-1}(T)\Big(x_i(0) - \alpha_i e^{\lambda_i T}\Big)^2 - \sum_{i=1}^{N}(x_i(0) - \alpha_i)^2\Big].
\end{aligned}$$

Jin and Glasserman ([56]) claim that this form cannot always give a good initial curve fitting. Note here that, because we restricted the parameters so that $\lambda_i > \sigma_i^2$, we infer for $0 \le t \le T < \infty$ that

$$\Big(1 - \frac{\sigma_i^2}{\lambda_i}\Big)e^{2\lambda_i(T-t)} \ge 1 - \frac{\sigma_i^2}{\lambda_i}.$$

Recalling the definition of the function H_i from (2.1.26), it follows that

$$H_i(T-t) \geq 1 \quad \text{for any} \quad i = 1, \ldots, N.$$

In particular, because we have for $T - t > 0$ that

$$\Big(1 - \frac{\sigma_i^2}{\lambda_i}\Big) e^{2\lambda_i(T-t)} > 1 - \frac{\sigma_i^2}{\lambda_i},$$

we find for $T - t > 0$ that

$$H_i(T-t) > 1 \quad \text{for any} \quad i = 1, \ldots, N.$$

2.2 Market Models

In this section we recall the Market Models, in particular the LFM using the literature [13], [14], [35] and [75], and the SABR Model using [14], [41], [49] and [76]. Though these Market Models do not belong to the Potential Approach, they may be comparable. This is because we are able to construct the forward LIBOR rate dynamics and forward swap rate dynamics from the Potential Approach, as we will show in Section 3.3. We will compare fitting ability of Chaos Models with the LFM and the SABR Model in the calibration chapters. Let us first recall the forward LIBOR rate and forward swap rate in this section.

2.2.1 Forward LIBOR rate

The forward rate agreement (FRA) is the name given to a contract in which the holder receives a fixed interest rate payment in a future time period. In other words, the contract sets a fixed interest rate at time t for the period between T and S, for $0 \leq t \leq T \leq S < \infty$, where the holder of the contract receives the fixed interest rate K and then pays a floating interest rate L_{TS} upon maturity at time S, i.e., the value of the contract at time S is given by

$$N\tau_{TS}(K - L_{TS}),$$

where N denotes a nominal value. Therefore the discounted value at time t is obtained in the following way:

$$\begin{aligned} FRA(t,T,S,\tau_{TS},N,K) =& N\mathbb{E}_t^{\mathbb{Q}}[e^{-\int_t^S r_s ds}\tau_{TS}(K-L_{TS})] \\ =& N\mathbb{E}_t^{\mathbb{Q}}\Big[e^{-\int_t^S r_s ds}\Big(\tau_{TS}K-\frac{1}{P_{TS}}+1\Big)\Big] \\ =& N(P_{tS}\tau_{TS}K+P_{tS})-N\mathbb{E}_t^{\mathbb{Q}}\Big[e^{-\int_t^S r_s ds}\Big(\frac{1}{P_{TS}}\Big)\Big], \end{aligned}$$

where $\mathbb{E}_t^{\mathbb{Q}}$ denotes the conditional expectation on the σ-field $\mathcal{F}_t$ under the risk neutral measure $\mathbb{Q}$. However, because

$$\begin{aligned} \mathbb{E}_t^{\mathbb{Q}}\Big[e^{-\int_t^S r_s ds}\Big(\frac{1}{P_{TS}}\Big)\Big] =& \mathbb{E}_t^{\mathbb{Q}}\Big[\mathbb{E}_T^{\mathbb{Q}}\Big[e^{-\int_t^S r_s ds}\Big(\frac{1}{P_{TS}}\Big)\Big]\Big] \\ =& \mathbb{E}_t^{\mathbb{Q}}\Big[\mathbb{E}_T^{\mathbb{Q}}\big[e^{-\int_t^S r_s ds}\big]\Big(\frac{1}{P_{TS}}\Big)\Big] \\ =& \mathbb{E}_t^{\mathbb{Q}}\Big[e^{-\int_t^T r_s ds}\mathbb{E}_T^{\mathbb{Q}}\big[e^{-\int_T^S r_s ds}P_{SS}\big]\Big(\frac{1}{P_{TS}}\Big)\Big] \\ =& \mathbb{E}_t^{\mathbb{Q}}\Big[e^{-\int_t^T r_s ds}P_{TS}\Big(\frac{1}{P_{TS}}\Big)\Big] \\ =& P_{tT}, \end{aligned}$$

we may instead express the discounted value as follows:

$$\begin{aligned} FRA(t,T,S,\tau_{TS},N,K) =& N(P_{tS}\tau_{TS}K-P_{tT}+P_{tS}) \\ =& NP_{tS}\tau_{TS}\Big[K-\frac{1}{\tau_{TS}}\Big(\frac{P_{tT}}{P_{tS}}-1\Big)\Big]. \end{aligned}$$

The simply compounded forward (LIBOR) rate F_{tTS} is defined so that

$$FRA(t,T,S,\tau_{TS},N,F_{tTS})=0, \quad \text{i.e.,} \quad F_{tTS}:=\frac{1}{\tau_{TS}}\left(\frac{P_{tT}}{P_{tS}}-1\right) \quad \text{for} \quad 0\le t\le T\le S<\infty.$$

We therefore conclude that the expression of the FRA is given by

$$FRA(t,T,S,\tau_{TS},N,K)=NP_{tS}\tau_{TS}(K-F_{tTS}).$$

2.2.2 Forward Swap Rate

There are two main types of Interest-Rate Swaps (IRS). The Receiver IRS (RFS) is the term for when a fixed interest rate is received and a floating interest rate is paid.

The opposite case is said to be the Payer IRS (PFS). The value of these contracts at a time $t \geq 0$ are expressed, respectively, by

$$RFS(t, \mathcal{T}, \tau, N, K) := N \sum_{i=a+1}^{b} \tau_i P_{tT_i}(K - F_{tT_{i-1}T_i}),$$

$$PFS(t, \mathcal{T}, \tau, N, K) := N \sum_{i=a+1}^{b} \tau_i P_{tT_i}(F_{tT_{i-1}T_i} - K),$$

where $\mathcal{T} =: \{T_a, T_{a+1}, \dots, T_b\}$ denotes a sequence of times, so that $T_a \leq T_{a+1} \leq \cdots \leq T_b$, and $\tau := \{\tau_{a+1}, \dots, \tau_b\}$ denotes the corresponding year fractions, that is, $\tau_i := T_i - T_{i-1}$ for $i = a+1, \cdots, b$. Here the floating leg reset dates are $\{T_a, T_{a+1}, \dots, T_{b-1}\}$ and the swap payment dates are $\{T_{a+1}, T_{\alpha+2} \dots, T_b\}$. We may express the RFS as a sum of FRA contracts in the following way:

$$\begin{aligned} RFS(t, \mathcal{T}, \tau, N, K) &= \sum_{i=a+1}^{b} FRA(t, T_{t-1}, T_i, \tau_i, N, K) \\ &= N \sum_{i=a+1}^{b} \big(\tau_i P_{tT_i} K - P_{tT_{i-1}} + P_{tT_i}\big) \\ &= N \sum_{i=a+1}^{b} \tau_i P_{tT_i} \left(K - \frac{P_{tT_a} - P_{tT_b}}{\sum_{i=a+1}^{b} \tau_i P_{tT_i}}\right). \end{aligned}$$

The Forward Swap Rate $S_{a,b}(t)$ is defined so that

$$RFS(t, \mathcal{T}, \tau, N, S_{a,b}(t)) = 0, \quad \text{i.e.,} \quad S_{a,b}(t) := \frac{P_{tT_a} - P_{tT_b}}{\sum_{i=a+1}^{b} \tau_i P_{tT_i}}.$$

Therefore, we conclude in the expression of the RFS that

$$RFS(t, \mathcal{T}, \tau, N, K) = N \sum_{i=a+1}^{b} \tau_i P_{tT_i} \big(K - S_{a,b}(t)\big).$$

By a similar argument, we obtain the expression

$$PFS(t, \mathcal{T}, \tau, N, K) = N \sum_{i=a+1}^{b} \tau_i P_{tT_i} \big(S_{a,b}(t) - K\big).$$

Note here that because the Swap Rate $SR(t, T_b)$ is defined by the forward swap rate with $t = T_a$, that is $SR(T_a, T_b) := S_{a,b}(T_a)$, it follows that

$$SR(T_a, T_b) = \frac{1 - P_{T_a T_b}}{\sum_{i=a+1}^{b} \tau_i P_{T_a T_i}}. \tag{2.2.1}$$

2.2.3 Lognormal Forward LIBOR Model

We first make the following notations for the forward LIBOR rate:

$$F_j(t) = F_{tT_{j-1}T_j}, \quad j = 1, 2, \ldots,$$

where $\{T_a, T_{a+1}, \ldots, T_{i-1}, T_i, \ldots, T_b\}$ is an increasing set of dates. Let $\mathbb{Q}^j$ be the forward measure for the maturity T_j, having the associated numeraire $P(\cdot, T_j)$. Considering a process $F_j(t)P(t, T_j)$, $t \geq 0$, for $j = 1, 2, \ldots$, that is,

$$F_j(t)P(t, T_j) = \frac{1}{T_j - T_{j-1}}[P(t, T_{j-1}) - P(t, T_j)],$$

we observe using the martingale property that

$$\mathbb{E}_t^{\mathbb{Q}^j}\Big[\frac{F_j(T_j)P(T_j, T_j)}{P(T_j, T_j)}\Big] = \frac{F_j(t)P(t, T_j)}{P(t, T_j)}.$$

Equivalently, we have that

$$\mathbb{E}_t^{\mathbb{Q}^j}[F_j(T_j)] = F_j(t), \quad \text{for} \quad j = 1, 2, \ldots.$$

Therefore we observe that $F_j(t)$ is a martingale under the measure $\mathbb{Q}^j$. The LFM defines the forward LIBOR rate dynamics by the following lognormal dynamics:

$$dF_j(t) = \sigma_j(t)F_j(t)dZ^j(t),$$

where $\sigma_j(t), t \geq 0$ is a deterministic process, and Z^j is a Brownian Motion under the measure $\mathbb{Q}^j$ with a correlation given by

$$dZ^i(t)dZ^j(t) = \rho_{ij}dt.$$

2.2.4 SABR Model

The SABR (stochastic $\alpha\beta\rho$) Model defines the forward LIBOR rate dynamics by the following stochastic differential equation:

$$dF_j(t) = v(t)F_j(t)^{\beta}dZ^j(t),$$

$$dv(t) = \epsilon v(t)dW^j(t), \quad v(0) = \alpha,$$

where $\beta \in (0,1]$, ϵ and α are some positive constants and Z^j and W^j are Brownian Motions under the measure $\mathbb{Q}^j$ with a correlation given by

$$dZ^j(t)dW^j(t) = \rho dt, \quad \text{for} \quad \rho \in [-1,1].$$

In the SABR Model we need to compute Swaption prices separately from the Caplet prices. Let us consider the forward swap rate $S_{a,b}(t)$, which is a martingale under the forward swap measure $\mathbb{Q}^{a,b}$, having associated numeraire $C_{a,b}(t) := \sum_{i=a+1}^{b} \tau_i P_{t,T-i}$. In the SABR Model we assume the following dynamics for the forward swap rate under the forward swap measure $\mathbb{Q}^{a,b}$:

$$dS_{a,b}(t) = \tilde{v}_{a,b}(t)S_{a,b}(t)^{\beta}dZ^{a,b}(t),$$

$$d\tilde{v}_{a,b}(t) = \epsilon\tilde{v}_{a,b}(t)dW^{a,b}(t), \quad \tilde{v}_{a,b}(0) = \alpha,$$

where $\beta \in (0,1]$, ϵ and α are some positive constants,and $Z^{a,b}$ and $W^{a,b}$ are Brownian Motions under the measure $\mathbb{Q}^{a,b}$ with a correlation

$$dZ^{a,b}(t)dW^{a,b}(t) = \rho dt, \quad \text{for} \quad \rho \in [-1,1].$$

Though we are using the same notations, $\alpha, \beta, \rho, \epsilon$, these do not correspond to the ones in the forward LIBOR rate dynamics.

2.3 Interest Rate Options

In this section we recall interest rate options using [14]. In particular, we consider European Bond Options, Caps, Caplets, Floors, Floorlets, and European Swaptions.

2.3.1 European Bond Options

The European Call/Put Bond Options are defined with a T-maturity bond $(P_{tT})_{0\leq t\leq T<\infty}$ and strike price $K \in \mathbb{R}^+$, respectively giving us the following payoff at reset date $t \geq 0$:

$$(P_{tT} - K)^+ \quad \text{and} \quad (K - P_{tT})^+.$$

From this, the discounted values of the European Bond Options at time s for $0 \leq s \leq t$ may be expressed using the risk neutral measure as follows:

$$ZBC(s,t,T,K) = \mathbb{E}_s^{\mathbb{Q}}[e^{-\int_s^t r_u du}(P_{tT} - K)^+] \quad \text{(Call Option)}$$

and

$$ZBP(s,t,T,K) = \mathbb{E}_s^{\mathbb{Q}}[e^{-\int_s^t r_u du}(K - P_{tT})^+] \quad \text{(Put Option)}.$$

These may be expressed using the market measure as follows:

$$ZBC(s,t,T,K) = \frac{1}{V_s}\mathbb{E}_s[V_t(P_{tT} - K)^+] = \frac{1}{V_s}\mathbb{E}_s[(Z_{tT} - KZ_{tt})^+]$$

and

$$ZBP(s,t,T,K) = \frac{1}{V_s}\mathbb{E}_s[V_t(K - P_{tT})^+] = \frac{1}{V_s}\mathbb{E}_s[(KZ_{tt} - Z_{tT})^+].$$

In particular, we have that the values of the European Bond Options at the settlement $s = 0$ (the time of buying the derivative) may be expressed using the market measure as follows:

$$ZBC(0,t,T,K) = \frac{1}{V_0}\mathbb{E}[(Z_{tT} - KZ_{tt})^+] \tag{2.3.1}$$

and

$$ZBP(0,t,T,K) = \frac{1}{V_0}\mathbb{E}[(KZ_{tt} - Z_{tT})^+]. \tag{2.3.2}$$

2.3.2 Caplet/Floorlet

The Caplet/Floorlet are defined with a T_j-maturity LIBOR rate $(L_{T_{j-1}T_j})_{0\leq T_{j-1}\leq T_j<\infty}$, strike price $K \in \mathbb{R}$ and a nominal value N (usually $N = 10^4$), respectively, giving us the following payoff at payment date $T_j \geq 0$:

$$N\tau_j(L_{T_{j-1}T_j} - K)^+ \quad \text{and} \quad N\tau_j(K - L_{T_{j-1}T_j})^+,$$

where T_{j-1} denotes the reset date (the time of exercising the option) and recall that $\tau_j = T_j - T_{j-1}$. The reader might like to notice here that the payment date is not the reset date. The discounted values at time s for $0 \leq s \leq T_{j-1}$ may be expressed using the risk neutral measure in the following way:

$$Cpl(s, T_{j-1}, T_j, \tau, N, K) = N\mathbb{E}_s^{\mathbb{Q}}\Big[e^{-\int_s^{T_j} r_u du}\tau_j(L_{T_{j-1}T_j} - K)^+\Big]$$

and

$$Fll(s, T_{j-1}, T_j, \tau, N, K) = N\mathbb{E}_s^{\mathbb{Q}}\Big[e^{-\int_s^{T_j} r_u du}\tau_j(K - L_{T_{j-1}T_j})^+\Big].$$

Then, using (2.1.7), we may also express using the T_j-forward measure that

$$\begin{aligned} Cpl(s, T_{j-1}, T_j, \tau, N, K) =& N\mathbb{E}_s^{\mathbb{Q}}\Big[\frac{B_s}{B_{T_j}}\tau_j(L_{T_{j-1}T_j} - K)^+\Big] \\ =& N\mathbb{E}_s^{\mathbb{Q}^j}\Big[\frac{P_{sT_j}}{P_{T_jT_j}}\tau_j(L_{T_{j-1}T_j} - K)^+\Big] \\ =& NP_{sT_j}\tau_j\mathbb{E}_s^{\mathbb{Q}^j}\left[(L_{T_{j-1}T_j} - K)^+\right] \end{aligned} \tag{2.3.3}$$

and similarly

$$Fll(s, T_{j-1}, T_j, \tau, N, K) = NP_{sT_j}\tau_j\mathbb{E}_s^{\mathbb{Q}^j}\left[(K - L_{T_{j-1}T_j})^+\right].$$

Recalling that $L_{T_{j-1}T_j} = \frac{1}{\tau_j}\big(\frac{V_{T_{j-1}}}{Z_{T_{j-1}T_j}} - 1\big)$ from (2.1.11), the discounted values at time s for $0 \le s \le T_{j-1}$ can be expressed using the market measure in the following way:

$$\begin{aligned}
Cpl(s, T_{j-1}, T_j, \tau, N, K) =& \frac{N}{V_s}\mathbb{E}_s\left[V_{T_j}\tau_j(L_{T_{j-1}T_j} - K)^+\right] \\
=& \frac{N}{V_s}\mathbb{E}_s\left[\frac{V_{T_j}}{Z_{T_{j-1}T_j}}\big(V_{T_{j-1}} - (1+K\tau_j)Z_{T_{j-1}T_j}\big)^+\right] \\
=& \frac{N}{V_s}\mathbb{E}_s\left[\mathbb{E}_{T_{j-1}}\left[\frac{V_{T_j}}{Z_{T_{j-1}T_j}}\big(V_t - (1+K\tau_j)Z_{T_{j-1}T_j}\big)^+\right]\right] \\
=& \frac{N}{V_s}\mathbb{E}_s\left[\big(Z_{T_{j-1}T_{j-1}} - (1+K\tau_j)Z_{T_{j-1}T_j}\big)^+\right]
\end{aligned}$$

and similarly

$$Fll(s, T_{j-1}, T_j, \tau, N, K) = \frac{N}{V_s}\mathbb{E}_s\left[\big((1+K\tau_j)Z_{T_{j-1}T_j} - Z_{T_{j-1}T_{j-1}}\big)^+\right].$$

In particular at the settlement $s = 0$ we find that

$$Cpl(0, T_{j-1}, T_j, \tau, N, K) = \frac{N}{V_0}\mathbb{E}\big[\big(Z_{T_{j-1}T_{j-1}} - (1+K\tau_j)Z_{T_{j-1}T_j}\big)^+\big] \tag{2.3.4}$$

and

$$Fll(0, T_{j-1}, T_j, \tau, N, K) = \frac{N}{V_0}\mathbb{E}\big[\big((1+K\tau_j)Z_{T_{j-1}T_j} - Z_{T_{j-1}T_{j-1}}\big)^+\big].$$

Therefore, it is evident that some care is needed in modelling the variable $Z_{T_{j-1}T_j}$ for pricing the Caplet and the Floorlet. Recalling that $L_{T_{j-1}T_j} = F_j(T_{j-1})$, when $K = K_{ATM}$ for $K_{ATM} = F_j(0)$ at the settlement $s = 0$, the Caplet and Floorlet are called "at-the-money" (ATM). If $K < K_{ATM}$, these are called "in-the-money" (ITM). If $K > K_{ATM}$, these are called "out-of-the-money" (OTM). Note here that we have the following relationship for the ATM Swaptions:

$$Cpl(0, T_{j-1}, T_j, \tau, N, K_{ATM}) = Fll(0, T_{j-1}, T_j, \tau_j, N, K_{ATM}).$$

The book [14] (page 41) and paper [88] show that the relationship between the European bond options and Caplet/Floorlet are for $0 \le s \le t \le T < \infty$, given by

$$Cpl(s, T_{j-1}, T_j, \tau, N, K) = N(1+K\tau_j)ZBP\big(s, T_{j-1}, T_j, \frac{1}{1+K\tau_j}\big) \tag{2.3.5}$$

and

$$(2.3.6) \qquad Fll(s, T_{j-1}, T_j, \tau, N, K) = N(1 + K\tau_j) ZBC\big(s, T_{j-1}, T_j, \frac{1}{1 + K\tau_j}\big).$$

The markets apply the implied volatility to estimate the volatility of the option. Let us recall the Black formula:

(2.3.7)

$$\begin{aligned} Bl(K, F_j(0), \upsilon_{T_{j-1}}) =& \mathbb{E}^{\mathbb{Q}^j}[(F_j(T_{j-1}) - K)^+] \\ =& F_j(0)\Phi\Big(\frac{ln(\frac{F_j(0)}{K}) + \frac{T_{j-1}\upsilon^2_{T_{j-1}}}{2}}{\sqrt{T_{j-1}}\upsilon_{T_{j-1}}}\Big) - K\Phi\Big(\frac{ln(\frac{F_j(0)}{K}) - \frac{T_{j-1}\upsilon^2_{T_{j-1}}}{2}}{\sqrt{T_{j-1}}\upsilon_{T_{j-1}}}\Big), \end{aligned}$$

where Φ denotes the standard normal cumulative distribution function, i.e.,

$$\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{X^2}{2}} dX.$$

Then the T_{j-1}-caplet can be expressed using expression (2.3.3) in the following way:

$$Cpl(0, T_{j-1}, T_j, \tau_j, N, K) = P(0, T_j)\tau_j Bl\left(K, F_j(0), \upsilon_{T_{j-1}}\right),$$

where $\upsilon_{T_{j-1}}$ is referred to as the "T_{j-1}-caplet implied volatility" or "forward forward volatility". An implied volatility curve of ATM Caplets, that is the function $T \to \upsilon_T$, is called the "term structure of (caplet) volatility". We see that the Black formula gives us a one-to-one correspondence between the option premium and the implied volatility. The T_{j-1}-caplet implied volatility is expressed in the LFM by

$$\upsilon_{T_{j-1}} = \sqrt{\frac{1}{T_{j-1}} \int_0^{T_{j-1}} \sigma_i^2(t) dt}.$$

In the SABR Model the closed form of the T_{j-1}-caplet implied volatilities $\upsilon_{T_{j-1}}$ are approximated by singular perturbation techniques as a function of the strike K and forward LIBOR rate $F_j(0)$ as follows:

$$\upsilon_{T_{j-1}} = \frac{\alpha\Big(1 + \Big[\frac{(1-\beta)^2\alpha^2}{24(F_j(0)K)^{1-\beta}} + \frac{\rho\beta\epsilon\alpha}{4(F_j(0)K)^{\frac{1-\beta}{2}}} + \epsilon^2\frac{2-3\rho^2}{24}\Big]T_{j-1}\Big)}{(F_j(0)K)^{\frac{1-\beta}{2}}\Big[1 + \frac{(1-\beta)^2}{24}\ln^2(\frac{F_j(0)}{K}) + \frac{(1-\beta)^4}{1920}\ln^4(\frac{F_j(0)}{K})\Big]} \frac{z}{x(z)},$$

where

$$z = \frac{\epsilon}{\alpha}(F_j(0)K)^{\frac{1-\beta}{2}} \ln\Big(\frac{F_j(0)}{K}\Big)$$

and

$$x(z) = \ln\Big(\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\Big).$$

Because the implied volatility form in the Potential Approach is not available, we need to estimate it from the premium using the Black formula. The books by Brigo and Mercurio ([14]) and James and Webber ([54]) claim that the term structure of caplet volatility has a humped shape in a moderate market condition. We observe this humped shape in the calibration chapter.

2.3.3 Cap/Floor

The Cap/Floor is a sum of Caplet/Floorlet contracts. Therefore the discounted values at time s for $0 \leq s \leq t$ are expressed in the following manner:

$$Cap(s, \mathcal{T}, \tau, N, K) := N \sum_{i=a+1}^{b} \mathbb{E}_s^{\mathbb{Q}}\Big[e^{-\int_s^{T_i} r_u du}\tau_i(L_{T_{i-1}T_i} - K)^+\Big],$$

$$Flr(s, \mathcal{T}, \tau, N, K) := N \sum_{i=a+1}^{b} \mathbb{E}_s^{\mathbb{Q}}\Big[e^{-\int_s^{T_i} r_u du}\tau_i(K - L_{T_{i-1}T_i})^+\Big],$$

where the reset dates are the times $\{T_a, T_{a+1}, \ldots, T_{b-1}\}$ and the payment dates are the times $\{T_{a+1}, T_{\alpha+2} \ldots, T_b\}$. These discounted values may be expressed using the market measure in the following way:

$$Cap(s, \mathcal{T}, \tau, N, K) = \frac{N}{V_s} \sum_{i=a+1}^{b} \mathbb{E}_s\big[\big(Z_{T_{i-1}T_{i-1}} - (1+K\tau_i)Z_{T_{i-1}T_i}\big)^+\big]$$

and

$$Flr(s, \mathcal{T}, \tau, N, K) = \frac{N}{V_s} \sum_{i=a+1}^{b} \mathbb{E}_s\big[\big((1+K\tau_i)Z_{T_{i-1}T_i} - Z_{T_{i-1}T_{i-1}}\big)^+\big].$$

In particular at the settlement $s = 0$ we have that

$$Cap(0, \mathcal{T}, \tau, N, K) = \frac{N}{V_0} \sum_{i=a+1}^{b} \mathbb{E}\big[\big(Z_{T_{i-1}T_{i-1}} - (1+K\tau_i)Z_{T_{i-1}T_i}\big)^+\big]$$

and

$$Flr(0,\mathcal{T},\tau,N,K) = \frac{N}{V_0}\sum_{i=a+1}^{b}\mathbb{E}\big[\big((1+K\tau_i)Z_{T_{i-1}T_i} - Z_{T_{i-1}T_{i-1}}\big)^+\big].$$

For example, a one year maturity Cap in the UK market contains three caplets, whose reset dates are at $3, 6$ and 9 months after the settlement date. Payment dates are at $6, 9$ and 12 months after the settlement date, that is, three months in arrears in the UK. Since we have that

$$Cap(s,\mathcal{T},\tau,N,K) - Flr(s,\mathcal{T},\tau,N,K) = N\sum_{i=a+1}^{b}\mathbb{E}_s^{\mathbb{Q}}\Big[e^{-\int_s^{T_i} r_u du}\tau_i(L_{T_{i-1}T_i} - K)\Big],$$

we obtain the following Put-Call Parity:

$$Cap(s,\mathcal{T},\tau,N,K) - Flr(s,\mathcal{T},\tau,N,K) = PFS(s,\mathcal{T},\tau,N,K).$$

Because we have that

$$PFS(s,\mathcal{T},\tau,N,S_{a,b}(s)) = 0,$$

we may infer that

$$Cap(s,\mathcal{T},\tau,N,S_{a,b}(s)) = Flr(s,\mathcal{T},\tau,N,S_{a,b}(s)). \tag{2.3.8}$$

Equation (2.3.8) yields that the ATM strike of Cap/Floor at the settlement $s = 0$ by the forward swap rate as is given as follows:

$$K_{ATM} = S_{a,b}(0).$$

In addition, from the relationship between European Call/Put of the zero-coupon bond and Caplet/Floorlet, the following is inferred:

$$Cap(s,\mathcal{T},\tau,N,K) = N\sum_{i=a+1}^{b}(1+K\tau_i)ZBP\big(s,T_{i-1},T_i,\frac{1}{1+K\tau_i}\big)$$

and

$$Flr(s,\mathcal{T},\tau,N,K) = N\sum_{i=a+1}^{b}(1+K\tau_i)ZBC\big(s,T_{i-1},T_i,\frac{1}{1+K\tau_i}\big).$$

2.3.4 European Swaptions

A European payer Swaption (also called a "European Call Swaption") is an option to have a PFS contract at a future time, which corresponds to the swaption maturity. Therefore the value of the payer Swaption at the maturity date $t \geq 0$ is defined by

$$\begin{aligned} PS(t, \mathcal{T}, \tau, N, K) :=& \Big(PFS(t, \mathcal{T}, \tau, N, K) \Big)^+ \\ =& N \sum_{i=a+1}^{b} \tau_i P_{tT_i} \big(S_{a,b}(t) - K \big)^+. \end{aligned}$$

Here we note that the length of the underlying IRS, $T_b - T_a$, is called the "tenor" of the swaption. Usually the markets apply the first reset date for the swaption maturity date, that is $t = T_a$, which implies that

$$PS(t, \mathcal{T}, \tau, N, K) = N \sum_{i=a+1}^{b} \tau_i P_{T_a T_i} \big(S_{a,b}(T_a) - K \big)^+.$$

Hence, the discounted price of the payer swaption at time s, for $0 \leq s \leq t$, is formulated by

$$PS(s, \mathcal{T}, \tau, N, K) = N \mathbb{E}_s^{\mathbb{Q}} \Big[e^{-\int_s^{T_a} r_u du} \sum_{i=a+1}^{b} \tau_i P_{T_a T_i} \big(SR(T_a, T_b) - K \big)^+ \Big].$$

Let us recall that $C_{a,b}(t) := \sum_{i=a+1}^{b} \tau_i P_{t,T_i}$ is an associated numeraire of the forward swap measure $\mathbb{Q}^{a,b}$. Then, using (2.1.7), we may also express that

$$\begin{aligned} PS(s, \mathcal{T}, \tau, N, K) =& N \mathbb{E}_s^{\mathbb{Q}} \Big[\frac{B_s}{B_{T_a}} C_{a,b}(T_a) \big(SR(T_a, T_b) - K \big)^+ \Big] \\ =& N \mathbb{E}_s^{\mathbb{Q}} \Big[\frac{C_{a,b}(s)}{C_{a,b}(T_a)} C_{a,b}(T_a) \big(SR(T_a, T_b) - K \big)^+ \Big] \\ =& N C_{a,b}(s) \mathbb{E}_s^{\mathbb{Q}} \Big[\big(SR(T_a, T_b) - K \big)^+ \Big]. \end{aligned} \tag{2.3.9}$$

Recalling the definition of the swap rate from (2.2.1), this discounted price may be expressed using the market measure in the following way:

$$\begin{aligned}PS(s,\mathcal{T},\tau,N,K) =& \frac{N}{V_s}\mathbb{E}_s\Big[V_t\Big(1 - P_{tT_b} - K\sum_{i=a+1}^{b}\tau_i P_{tT_i}\Big)^+\Big]\\ =& \frac{N}{V_s}\mathbb{E}_s\Big[Z_{tt}\Big(1 - \frac{Z_{tT_b}}{Z_{tt}} - K\sum_{i=a+1}^{b}\tau_i \frac{Z_{tT_i}}{Z_{tt}}\Big)^+\Big]\\ =& \frac{N}{V_s}\mathbb{E}_s\Big[\Big(Z_{tt} - Z_{tT_b} - K\sum_{i=a+1}^{b}\tau_i Z_{tT_i}\Big)^+\Big].\end{aligned}$$

In particular at the settlement $s = 0$ we have that

$$PS(0,\mathcal{T},\tau,N,K) = \frac{N}{V_0}\mathbb{E}\Big[\Big(Z_{tt} - Z_{tT_b} - K\sum_{i=a+1}^{b}\tau_i Z_{tT_i}\Big)^+\Big]. \tag{2.3.10}$$

Similarly, the discounted value of a European receiver Swaption (sometimes called a "European Put Swaption") with the same underlying IRS is formulated by

$$RS(s,\mathcal{T},\tau,N,K) = N\mathbb{E}^{\mathbb{Q}}_s\Big[e^{-\int_s^{T_a} r_u du}\sum_{i=a+1}^{b}\tau_i P_{T_aT_i}\big(K - SR(T_a,T_b)\big)^+\Big].$$

Using the market measure this can be expressed in the following way:

$$RS(s,\mathcal{T},\tau,N,K) = \frac{N}{V_s}\mathbb{E}_s\Big[\Big(K\sum_{i=a+1}^{b}\tau_i Z_{tT_i} - Z_{tt} + Z_{tT_b}\Big)^+\Big].$$

In particular, at the settlement $s = 0$ we have that

$$RS(0,\mathcal{T},\tau,N,K) = \frac{N}{V_0}\mathbb{E}\Big[\Big(K\sum_{i=a+1}^{b}\tau_i Z_{tT_i} - Z_{tt} + Z_{tT_b}\Big)^+\Big].$$

When $K = K_{ATM}$ for $K_{ATM} = S_{a,b}(0)$ at the settlement $s = 0$, the payer Swaption is ATM. This is precisely the same as we had for the Cap and the Floor. Note here that we have the following relationship for the ATM Swaptions:

$$PS(0,\mathcal{T},\tau,N,K_{ATM}) = RS(0,\mathcal{T},\tau,N,K_{ATM}).$$

The volatility of the Swaption may be estimated by the Black formula. Recalling the Black formula from (2.3.7) we may express for the Swaption with the maturity T_a and the tenor $T_b - T_a$ that

$$\begin{aligned} PS(0, \mathcal{T}, \tau, N, K_{ATM}) =& NC_{a,b}(0)\mathbb{E}^{\mathbb{Q}^{a,b}}[(S_{a,b}(T_a) - K)^+] \\ =& NC_{a,b}(0) Bl\left(K, S_{a,b}(0), v_{a,b}(T_a)\right), \end{aligned}$$

where $v_{a,b}$ is referred to as the "Swaption implied volatility". The dynamics of the SABR Model gives us the following Swaption implied volatility form:

$$v_{a,b} = \frac{\alpha\Big(1 + \Big[\frac{(1-\beta)^2\alpha^2}{24(S_{a,b}(0)K)^{1-\beta}} + \frac{\rho\beta\epsilon\alpha}{4(S_{a,b}(0)K)^{\frac{1-\beta}{2}}} + \epsilon^2\frac{2-3\rho^2}{24}\Big]T_a\Big)}{(S_{a,b}(0)K)^{\frac{1-\beta}{2}}\Big[1 + \frac{(1-\beta)^2}{24}\ln^2(\frac{S_{a,b}(0)}{K}) + \frac{(1-\beta)^4}{1920}\ln^4(\frac{S_{a,b}(0)}{K})\Big]}\frac{z}{x(z)},$$

where

$$z = \frac{\epsilon}{\alpha}(S_{a,b}(0)K)^{\frac{1-\beta}{2}}\ln\Big(\frac{S_{a,b}(0)}{K}\Big)$$

and

$$x(z) = \ln\Big(\frac{\sqrt{1 - 2\rho z + z^2} + z - \rho}{1-\rho}\Big).$$

Though we are using the same notations, $\alpha, \beta, \rho, \epsilon$, these do not correspond to the ones in the Caplet implied volatility.

Chapter 3

Further investigations of the Potential Approach

We develop the argument of Jin and Glasserman ([56]) in which the stochastic differential equations of the variable Z_{tT} and the instantaneous forward rate f_{tT} in the Potential Approach, in particular Flesaker-Hughston Positive Interest Framework ([33], hereafter referred to as the FH Framework), are expressed. While Jin and Glasserman proposed the relationship between the Potential Approach and HJM Framework, we propose a relationship between the Potential Approach and the Market Model. In other words, we construct the dynamics of the forward LIBOR rate F_{tTS} and the forward swap rate $S_{a,b}(t)$ in the Potential Approach. In Chapter 8 we make further investigation and show that the Market Model can be constructed from the Short Rate Model via the Potential Approach.

3.1 Forms of the main processes

Let us first recall the definition of the instantaneous forward rate. This is defined to be

$$f_{tT} := -\frac{\partial}{\partial T} lnP_{tT}, \quad 0 \leq t \leq T < \infty.$$

In the Potential Approach, the form of the discount bond is given by the expression (2.1.10), i.e.,

$$P_{tT} = \frac{Z_{tT}}{Z_{tt}}.$$

It follows that

$$f_{tT} = \frac{M_{tT}}{Z_{tT}}, \quad 0 \leq t \leq T < \infty,$$

where we define a variable M_{tT} to be

$$M_{tT} := -\frac{\partial}{\partial T} Z_{tT}, \quad 0 \leq t \leq T < \infty. \tag{3.1.1}$$

The variable Z_{tT} is positive because it is a conditional expectation of the positive variable V_T with respect to $\mathcal{F}_t$, for $0 \leq t \leq T < \infty$. In addition, the supermartingale property of the state price density ensures that the variable M_{tT} is positive for any $0 \leq t \leq T < \infty$. Therefore, under the Potential Approach the positivity of interest rates is secured. The short rate, zero-coupon yield (continuously-compounded spot interest rate), forward LIBOR rate and forward swap rate may be expressed using their definitions for $0 \leq t \leq T \leq S < \infty$ in the following way:

$$r_t := f_{tt} = \frac{M_{tt}}{V_t},$$

$$y_{tT} := -\frac{1}{T-t} ln P_{tT} = -\frac{1}{T-t} ln \left(\frac{Z_{tT}}{V_t} \right),$$

$$F_{tTS} = \frac{1}{S-T} \left(\frac{P_{tT}}{P_{tS}} - 1 \right) = \frac{1}{S-T} \left(\frac{Z_{tT}}{Z_{tS}} - 1 \right) \tag{3.1.2}$$

and

$$S_{a,b}(t) = \frac{Z_{tT_a} - Z_{tT_b}}{\sum_{i=a+1}^{b} \tau_i Z_{tT_i}}. \tag{3.1.3}$$

In particular, at the initial time $t = 0$ we have that

$$V_0 = Z_{00}, \quad P_{0T} = \frac{Z_{0T}}{V_0}, \quad f_{0T} = \frac{M_{0T}}{Z_{0T}}, \tag{3.1.4}$$

$$y_{0T} = -\frac{1}{T} ln\left(\frac{Z_{0T}}{V_0}\right), \quad L_{0T} = \frac{1}{T}\left(\frac{V_0}{Z_{0T}} - 1\right),$$

$$F_{0TS} = \frac{1}{S-T}\left(\frac{Z_{0T}}{Z_{0S}} - 1\right) \quad \text{and} \quad S_{a,b}(0) = \frac{Z_{0T_a} - Z_{0T_b}}{\sum_{i=a+1}^{b} \tau_i Z_{0T_i}}.$$

The following can be directly deduced from (3.1.1) and (3.1.4):

$$\mathbb{E}[V_t] = P_{0t}V_0, \quad \mathbb{E}[Z_{tT}] = P_{0T}V_0, \quad \mathbb{E}[M_{tT}] = -\frac{\partial}{\partial T}P_{0T}V_0 = f_{0T}P_{0T}V_0. \tag{3.1.5}$$

Applying the form of the state price density expressed in (2.1.22), the Conditional Fubini Theorem (See [89]) gives us that

$$V_t = \int_t^\infty \mathbb{E}_t[\sigma_s^2]ds \quad \text{and} \quad Z_{tT} = \int_T^\infty \mathbb{E}_t[\sigma_s^2]ds. \tag{3.1.6}$$

From definition (3.1.1), we obtain that

$$M_{tT} = \mathbb{E}_t[\sigma_T^2], \quad 0 \le t \le T < \infty. \tag{3.1.7}$$

Then, all main processes can be formulated using the variable σ_t as follows:

$$P_{tT} = \frac{\int_T^\infty \mathbb{E}_t[\sigma_s^2]ds}{\int_t^\infty \mathbb{E}_t[\sigma_s^2]ds}, \quad f_{tT} = \frac{\mathbb{E}_t[\sigma_T^2]}{\int_T^\infty \mathbb{E}_t[\sigma_s^2]ds}, \quad r_t = \frac{\sigma_t^2}{\int_t^\infty \mathbb{E}_t[\sigma_s^2]ds}, \tag{3.1.8}$$

$$y_{tT} = -\frac{1}{T-t} ln\left(\frac{\int_T^\infty \mathbb{E}_t[\sigma_s^2]ds}{\int_t^\infty \mathbb{E}_t[\sigma_s^2]ds}\right), \quad L_{tT} = \frac{1}{T-t}\frac{\int_t^T \mathbb{E}_t[\sigma_s^2]ds}{\int_T^\infty \mathbb{E}_t[\sigma_s^2]ds},$$

$$F_{tTS} = \frac{1}{S-T}\frac{\int_T^S \mathbb{E}_t[\sigma_s^2]ds}{\int_S^\infty \mathbb{E}_t[\sigma_s^2]ds} \quad \text{and} \quad S_{a,b}(t) = \frac{\int_{T_a}^{T_b} \mathbb{E}_t[\sigma_s^2]ds}{\sum_{i=a+1}^{b} \tau_i \int_{T-i}^\infty \mathbb{E}_t[\sigma_s^2]ds}.$$

The initial prices, that is, the prices at $t = 0$, are then given by the following expressions:

$$P_{0T} = \frac{\int_T^\infty \mathbb{E}[\sigma_s^2]ds}{\int_0^\infty \mathbb{E}[\sigma_s^2]ds}, \quad f_{0T} = \frac{\mathbb{E}[\sigma_T^2]}{\int_T^\infty \mathbb{E}[\sigma_s^2]ds}, \quad r_0 = \frac{\sigma_0^2}{\int_0^\infty \mathbb{E}[\sigma_s^2]ds},$$

$$y_{0T} = -\frac{1}{T} ln\left(\frac{\int_T^\infty \mathbb{E}[\sigma_s^2]ds}{\int_0^\infty \mathbb{E}[\sigma_s^2]ds}\right), \quad L_{0T} = \frac{1}{T}\frac{\int_0^T \mathbb{E}[\sigma_s^2]ds}{\int_T^\infty \mathbb{E}[\sigma_s^2]ds},$$

$$F_{0TS} = \frac{1}{T}\frac{\int_T^S \mathbb{E}[\sigma_s^2]ds}{\int_S^\infty \mathbb{E}[\sigma_s^2]ds} \quad \text{and} \quad S_{a,b}(0) = \frac{\int_{T_a}^{T_b} \mathbb{E}[\sigma_s^2]ds}{\sum_{i=a+1}^{b} \tau_i \int_{T-i}^\infty \mathbb{E}[\sigma_s^2]ds}.$$

3.2 Dynamics of the main processes

3.2.1 Dynamics of the discount bond

For the conditional expectation of the state price density, because it is a martingale with respect to $\mathcal{F}_t$, we may express using Martingale Representation Theorem (see for example, [68]) that

$$dZ_{tT} = \hat{V}_{tT} Z_{tT} dW_t \quad \text{for} \quad 0 \leq t \leq T < \infty, \tag{3.2.1}$$

for a unique $\mathcal{F}_t$-measurable random variable $\hat{V}_{tT}$, which is called the "risk-adjusted volatility". It follows that

$$Z_{tT} = P_{0T} V_0 \exp\Big[-\frac{1}{2}\int_0^t \hat{V}_{uT}^2 du + \int_0^t \hat{V}_{uT} dW_u \Big] \tag{3.2.2}$$

and

$$V_t = P_{0t} V_0 \exp\Big[-\frac{1}{2}\int_0^t \hat{V}_{ut}^2 du + \int_0^t \hat{V}_{ut} dW_u \Big], \tag{3.2.3}$$

where we recall that $\mathbb{E}[Z_{tT}] = P_{0T}V_0$ and $\mathbb{E}[V_t] = P_{0t}V_0$. Hence, we observe that modelling the risk adjusted volatility is equivalent to modelling the state price density. On the other hand, applying the Itô product formula, (see, for example [60] and [68]), we obtain in this case that

$$dP_{tT} = d\Big(\frac{Z_{tT}}{V_t}\Big) = Z_{tT} d\Big(\frac{1}{V_t}\Big) + \frac{1}{V_t} dZ_{tT} + d\Big(\frac{1}{V_t}\Big) dZ_{tT}.$$

Applying Itô's Lemma, (we refer the reader to [57] or [60] for the statement of Itô's Lemma), we obtain that

$$\begin{aligned} d\Big(\frac{1}{V_t}\Big) =& \frac{1}{V_t^3}(dV_t)^2 - \frac{1}{V_t^2} dV_t \\ =& (r_t + \lambda_t^2)\frac{1}{V_t} dt + \lambda_t \frac{1}{V_t} dW_t. \end{aligned} \tag{3.2.4}$$

Therefore, we obtain that for $0 \leq t \leq T < \infty$

$$\begin{aligned} dP_{tT} =& (r_t + \lambda_t^2)\frac{Z_{tT}}{V_t} dt + \lambda_t \frac{Z_{tT}}{V_t} dW_t + \hat{V}_{tT}\frac{Z_{tT}}{V_t} dW_t + \lambda_t \hat{V}_{tT}\frac{Z_{tT}}{V_t} dt \\ =& \big(r_t + \lambda_t(\lambda_t + \hat{V}_{tT})\big) P_{tT} dt + (\lambda_t + \hat{V}_{tT}) P_{tT} dW_t. \end{aligned} \tag{3.2.5}$$

The dynamics allows us to express the discount bond in the following form:

$$P_{tT} = P_{0T} \exp\Big[\int_0^t \Big(r_u + \lambda_u(\lambda_u + \hat{V}_{uT}) - \frac{1}{2}(\lambda_u + \hat{V}_{uT})^2\Big)du + \int_0^t (\lambda_u + \hat{V}_{uT})dW_u\Big].$$

However, because we suppose that $P_{tt} = 1$, it follows that

$$\begin{aligned} P_{tT} =& \frac{P_{0T} \exp\Big[\int_0^t \Big(r_u + \lambda_u(\lambda_u + \hat{V}_{uT}) - \frac{1}{2}(\lambda_u + \hat{V}_{uT})^2\Big)du + \int_0^t(\lambda_u + \hat{V}_{uT})dW_u\Big]}{P_{0t} \exp\Big[\int_0^t \Big(r_u + \lambda_u(\lambda_u + \hat{V}_{ut}) - \frac{1}{2}(\lambda_u + \hat{V}_{ut})^2\Big)du + \int_0^t(\lambda_u + \hat{V}_{ut})dW_u\Big]} \\ =& \frac{P_{0T}}{P_{0t}} \exp\Big[-\frac{1}{2}\int_0^t (\hat{V}_{uT}^2 - \hat{V}_{ut}^2)du + \int_0^t (\hat{V}_{uT} - \hat{V}_{ut})dW_u\Big]. \end{aligned}$$

It can also be expressed in the following way:

$$P_{tT} = P_{0T} V_0 V_t^{-1} \exp\Big[-\frac{1}{2}\int_0^t \hat{V}_{uT}^2 du + \int_0^t \hat{V}_{uT} dW_u\Big], \quad 0 \le t \le T < \infty. \tag{3.2.6}$$

On the other hand, denoting the T-maturity discount bond volatility by

$$\Omega_{tT} := \lambda_t + \hat{V}_{tT}, \quad 0 \le t \le T < \infty, \tag{3.2.7}$$

it follows from (3.2.5) that

$$dP_{tT} = (r_t + \lambda_t \Omega_{tT}) P_{tT} dt + \Omega_{tT} P_{tT} dW_t,$$

which corresponds to the dynamics in [34] and [48].

3.2.2 Dynamics of the instantaneous forward rate

Let us recall the random variable M_{tT} defined in (3.1.1) is a martingale with respect to $\mathcal{F}_t$. Therefore, applying Martingale Representation Theorem, the stochastic differential equation of M_{tT} can be expressed as follows:

$$dM_{tT} = \eta_{tT} M_{tT} dW_t, \tag{3.2.8}$$

for a unique $\mathcal{F}_t$-measurable random variable η_{tT}. We here obtain that

$$M_{tT} = \mathbb{E}[M_{tT}] \exp\Big[-\frac{1}{2}\int_0^t \eta_{uT}^2 du + \int_0^t \eta_{uT} dW_u\Big].$$

Using (3.1.5) and (3.1.7) it follows that

$$M_{tT} = h_T \hat{M}_{tT}, \quad \sigma_t^2 = h_t \hat{M}_{tt}, \quad \text{for} \quad 0 \le t \le T < \infty, \tag{3.2.9}$$

where we have defined

$$h_T := f_{0T} P_{0T} V_0 \quad \text{and} \quad \hat{M}_{tT} := \exp\Big[-\frac{1}{2}\int_0^t \eta_{uT}^2 du + \int_0^t \eta_{uT} dW_u \Big].$$

Note here that, applying these variables to expression (2.1.22), we may construct the FH Framework, that is,

$$V_t = \int_t^\infty h_s \hat{M}_{ts} ds \quad \text{and} \quad Z_{tT} = \int_T^\infty h_s \hat{M}_{ts} ds, \quad 0 \le t \le T < \infty.$$

Now, applying the Itô product formula, we obtain that

$$df_{tT} = d\Big(\frac{M_{tT}}{Z_{tT}}\Big) = M_{tT} d\Big(\frac{1}{Z_{tT}}\Big) + \frac{1}{Z_{tT}} dM_{tT} + d\Big(\frac{1}{Z_{tT}}\Big) dM_{tT}.$$

However, because Itô's Lemma gives us that

$$\begin{aligned} d\Big(\frac{1}{Z_{tT}}\Big) &= \frac{1}{Z_{tT}^3}(dZ_{tT})^2 - \frac{1}{Z_{tT}^2} dZ_{tT} \\ &= \frac{\hat{V}_{tT}^2}{Z_{tT}} dt - \frac{\hat{V}_{tT}}{Z_{tT}} dW_t, \end{aligned}$$

we obtain that

$$\begin{aligned} df_{tT} &= M_{tT}\Big(\frac{\hat{V}_{tT}^2}{Z_{tT}} dt - \frac{\hat{V}_{tT}}{Z_{tT}} dW_t\Big) + \frac{1}{Z_{tT}} dM_{tT} + \Big(\frac{\hat{V}_{tT}^2}{Z_{tT}} dt - \frac{\hat{V}_{tT}}{Z_{tT}} dW_t\Big) dM_{tT} \\ &= \hat{V}_{tT}\big(\hat{V}_{tT} - \eta_{tT}\big) f_{tT} dt - \big(\hat{V}_{tT} - \eta_{tT}\big) f_{tT} dW_t. \end{aligned} \tag{3.2.10}$$

This implies that for $0 \le t \le T < \infty$,

$$f_{tT} = f_{0T} \exp\Big[\frac{1}{2}\int_0^t [\hat{V}_{uT}^2 - \eta_{uT}^2] du - \int_0^t [\hat{V}_{uT} - \eta_{uT}] dW_u\Big]$$

and

$$r_t = f_{0t} \exp\Big[\frac{1}{2}\int_0^t [\hat{V}_{ut}^2 - \eta_{ut}^2] du - \int_0^t [\hat{V}_{ut} - \eta_{ut}] dW_u\Big].$$

However, since for each $0 \le t \le T < \infty$, we have that

$$
\begin{aligned}
\frac{\partial}{\partial T}\hat{V}_{tT} =& \frac{\partial}{\partial T}\frac{D_t[Z_{tT}]}{Z_{tT}} \\
=& \frac{1}{Z_{tT}}\Big(\frac{Z_{tT}\frac{\partial}{\partial T}D_t[Z_{tT}] - D_t[Z_{tT}]\frac{\partial}{\partial T}Z_{tT}}{Z_{tT}}\Big) \\
=& \frac{M_{tT}}{Z_{tT}}\Big(\frac{D_t[-M_{tT}]}{M_{tT}} + \frac{D_t[Z_{tT}]}{Z_{tT}}\Big) \\
=& \big(\hat{V}_{tT} - \eta_{tT}\big) f_{tT},
\end{aligned}
\tag{3.2.11}
$$

it follows that equation (3.2.10) can be translated to

$$
df_{tT} = \hat{V}_{tT}\frac{\partial}{\partial T}\hat{V}_{tT}dt - \frac{\partial}{\partial T}\hat{V}_{tT}dW_t, \tag{3.2.12}
$$

which corresponds to the one given in [56]. Because we have that

$$
\int_t^T \frac{\partial}{\partial s}\hat{V}_{ts}ds - \lambda_t = \hat{V}_{tT}, \quad 0 \le t \le T < \infty,
$$

the dynamics of the instantaneous forward rate may also be expressed as follows:

$$
df_{tT} = \Big(\int_t^T \frac{\partial}{\partial s}\hat{V}_{ts}ds - \lambda_t\Big)\frac{\partial}{\partial T}\hat{V}_{tT}dt - \frac{\partial}{\partial T}\hat{V}_{tT}dW_t.
$$

Notice that these dynamics satisfy the arbitrage free condition. Now, integrating the dynamics in (3.2.12), we obtain that for $0 \le t \le T < \infty$,

$$
f_{tT} = f_{0T} + \int_0^t \hat{V}_{uT}\frac{\partial}{\partial T}\hat{V}_{uT}du - \int_0^t \frac{\partial}{\partial T}\hat{V}_{uT}dW_u \tag{3.2.13}
$$

and

$$
r_t = f_{0t} + \int_0^t \hat{V}_{ut}\frac{\partial}{\partial t}\hat{V}_{ut}du - \int_0^t \frac{\partial}{\partial t}\hat{V}_{ut}dW_u. \tag{3.2.14}
$$

Note here that we may deduce from the definition $\sigma^2 := r_t V_t$ that

$$
\sigma_t^2 = \underbrace{\Big(f_{0t} + \int_0^t \hat{V}_{ut}\frac{\partial}{\partial t}\hat{V}_{ut}du - \int_0^t \frac{\partial}{\partial t}\hat{V}_{ut}dW_u\Big)}_{=r_t}\underbrace{P_{0t}V_0\exp\Big[-\frac{1}{2}\int_0^t \hat{V}_{ut}^2du + \int_0^t \hat{V}_{ut}dW_u\Big]}_{=V_t}.
$$

which corresponds to equation (25) in [56]. The process $(\sigma_t)_{t\geq 0}$ can also be expressed in the following way:

$$\begin{aligned}\sigma_t^2 =& \underbrace{f_{0t}\exp\Big[\frac{1}{2}\int_0^t[\hat{V}_{ut}^2-\eta_{ut}^2]du-\int_0^t[\hat{V}_{ut}-\eta_{ut}]dW_u\Big]}_{=r_t}\underbrace{P_{0t}V_0\exp\Big[-\frac{1}{2}\int_0^t\hat{V}_{ut}^2du+\int_0^t\hat{V}_{ut}dW_u\Big]}_{=V_t}\\ =&f_{0t}P_{0t}V_0\exp\Big[-\frac{1}{2}\int_0^t\eta_{ut}^2du+\int_0^t\eta_{ut}dW_u\Big]\\ =&h_t\hat{M}_{tt},\end{aligned}$$

which corresponds to the expression given in (3.2.9).

3.2.3 Dynamics of the short rate

Let us first make two further definitions, namely,

$$\hat{\alpha}_{tT} := \hat{V}_{tT}\frac{\partial}{\partial T}\hat{V}_{tT} \quad \text{and} \quad \hat{\sigma}_{tT} := -\frac{\partial}{\partial T}\hat{V}_{tT} \quad \text{for} \quad 0\leq t\leq T<\infty, \tag{3.2.15}$$

so that the dynamics (3.2.12) may be represented in the following way:

$$df_{tT} = \hat{\alpha}_{tT}dt + \hat{\sigma}_{tT}dW_t. \tag{3.2.16}$$

Proposition 20.5 in [11] states that the dynamics of the instantaneous (3.2.16) gives the following dynamics of the short rate:

$$dr_t = \Big(\hat{\alpha}_{tt} + \frac{\partial}{\partial T}f_{tT}\Big|_{T=t^+}\Big)dt + \hat{\sigma}_{tt}dW_t. \tag{3.2.17}$$

Although the proof of Proposition 20.5 can be found in [11], we attach a detailed proof in the Appendix. Recalling the definitions given in (3.2.15), we express the dynamics of the short rate in the following way:

$$dr_t = \Big(\hat{V}_{tT}\frac{\partial}{\partial T}\hat{V}_{tT}\Big|_{T=t^+} + \frac{\partial}{\partial T}f_{tT}\Big|_{T=t^+}\Big)dt - \Big(\frac{\partial}{\partial T}\hat{V}_{tT}\Big|_{T=t^+}\Big)dW_t. \tag{3.2.18}$$

Furthermore, by the relationship (3.2.11), this equality may be written in the following way:

$$dr_t = \Big(\lambda_t(\lambda_t+\eta_{tt})r_t + \frac{\partial}{\partial T}f_{tT}\Big|_{T=t^+}\Big)dt + (\lambda_t+\eta_{tt})r_t dW_t. \tag{3.2.19}$$

3.3 Relationship with the Market Models

In this section, we investigate the dynamics of the LIBOR forward rate and swap rate. In the Market Models we start the argument from modelling these dynamics. It is a convenient method, because these are the underlying assets of Caps, Floors, and Swaptions. Moreover, the market applies the corresponding option volatility using the Black formula, which is derived from the dynamics. Thus, expressing these dynamics in the Potential Approach enables us to relate the Potential Approach to the Market Models. We notice that the risk adjusted volatility $\hat{V}_{tT}$ corresponds to the variable $-\Sigma_{tT}$ in the Market Model as expressed in the book [35]. Indeed, this book mentions that specification of the function Σ_{tT} is equivalent to modelling the term structure, while, as we observed, the risk adjusted volatility $\hat{V}_{tT}$ specifies the dynamics of instantaneous forward rate as follows: For $0 \le t \le T < \infty$,

$$df_{tT} = \hat{V}_{tT}\frac{\partial}{\partial T}\hat{V}_{tT}dt - \frac{\partial}{\partial T}\hat{V}_{tT}dW_t.$$

In this section, we will observe that it also specifies the forward LIBOR rate dynamics and forward swap rate dynamics. Furthermore, we will find that the market price of risk and the bond volatility may be expressed respectively in the following way: For $0 \le t \le T < \infty$,

$$\lambda_t = -\hat{V}_{tt}, \quad \Omega_{tT} = \hat{V}_{tT} - \hat{V}_{tt}.$$

3.3.1 Dynamics of the forward LIBOR rate

Let us first make the following definition:

$$G_{tTS} := -\frac{1}{2}\int_0^t (\hat{V}_{uT}^2 - \hat{V}_{uS}^2)du + \int_0^t (\hat{V}_{uT} - \hat{V}_{uS})dW_u, \quad 0 \le t \le T \le S < \infty.$$

Then using (3.2.2) we find that

$$\frac{Z_{tT}}{Z_{tS}} = \frac{P_{0T}}{P_{0S}}e^{G_{tTS}}, \quad 0 \le t \le T \le S < \infty$$

and also by the forms (2.1.11) and (3.1.2), i.e.,

$$L_{tT} = \frac{1}{T-t}\left(\frac{Z_{tt}}{Z_{tT}} - 1\right) \quad \text{and} \quad F_{tTS} = \frac{1}{S-T}\left(\frac{Z_{tT}}{Z_{tS}} - 1\right),$$

we obtain that for $0 \leq t \leq T \leq S < \infty$,

$$L_{tT} = \frac{1}{T-t}\Big(\frac{1+TL_{0T}}{1+tL_{0t}}e^{G_{ttT}} - 1\Big) \quad \text{and} \quad F_{tTS} = \Big(F_{0TS} + \frac{1}{S-T}\Big)e^{G_{tTS}} - \frac{1}{S-T}.$$

Here we see that the form of the variable $\hat{V}_{tT}$ determines the distribution of the LIBOR and the forward LIBOR rates. Because Itô's Lemma gives us that

$$\begin{aligned}
de^{G_{tTS}} =& e^{G_{tTS}} dG_{tTS} + \frac{1}{2}e^{G_{tTS}}(dG_{tTS})^2 \\
=& e^{G_{tTS}}\Big(-\frac{1}{2}(\hat{V}_{tT}^2 - \hat{V}_{tS}^2)dt + (\hat{V}_{tT} - \hat{V}_{tS})dW_t\Big) + \frac{1}{2}e^{G_{tTS}}(\hat{V}_{tT} - \hat{V}_{tS})^2 dt \\
=& e^{G_{tTS}}\Big(-\frac{1}{2}(\hat{V}_{tT}^2 - \hat{V}_{tS}^2) + \frac{1}{2}(\hat{V}_{tT} - \hat{V}_{tS})^2\Big)dt + e^{G_{tTS}}(\hat{V}_{tT} - \hat{V}_{tS})dW_t \\
=& -e^{G_{tTS}}\hat{V}_{tS}(\hat{V}_{tT} - \hat{V}_{tS})dt + e^{G_{tTS}}(\hat{V}_{tT} - \hat{V}_{tS})dW_t,
\end{aligned}$$

we find that

$$d\Big(\frac{Z_{tT}}{Z_{tS}}\Big) = -\hat{V}_{tS}(\hat{V}_{tT} - \hat{V}_{tS})\frac{Z_{tT}}{Z_{tS}}dt + (\hat{V}_{tT} - \hat{V}_{tS})\frac{Z_{tT}}{Z_{tS}}dW_t.$$

Therefore, because we may infer that

$$\begin{aligned}
dF_{tTS} =& \frac{1}{S-T}d\Big(\frac{Z_{tT}}{Z_{tS}}\Big) \\
=& -\hat{V}_{tS}(\hat{V}_{tT} - \hat{V}_{tS})\frac{1}{S-T}\frac{Z_{tT}}{Z_{tS}}dt + (\hat{V}_{tT} - \hat{V}_{tS})\frac{1}{S-T}\frac{Z_{tT}}{Z_{tS}}dW_t,
\end{aligned} \tag{3.3.1}$$

we find that

$$dF_{tTS} = -\hat{V}_{tS}(\hat{V}_{tT} - \hat{V}_{tS})\Big(F_{tTS} + \frac{1}{S-T}\Big)dt + (\hat{V}_{tT} - \hat{V}_{tS})\Big(F_{tTS} + \frac{1}{S-T}\Big)dW_t. \tag{3.3.2}$$

By equation (3.2.7) this can be expressed by the discount bond volatility Ω_{tT} in the following way:

$$dF_{tTS} = (\lambda_t - \Omega_{tS})(\Omega_{tT} - \Omega_{tS})\Big(F_{tTS} + \frac{1}{S-T}\Big)dt + (\Omega_{tT} - \Omega_{tS})\Big(F_{tTS} + \frac{1}{S-T}\Big)dW_t.$$

We notice here that we may obtain the forward LIBOR rate shifted log-normally distributed when the process $(\hat{V}_{tT})_{0\leq t\leq T<\infty}$ is deterministic. The LIBOR rate volatility γ_{tTS}, which is defined by the equation

$$dF_{tTS} = -\hat{V}_{tS}\gamma_{tTS}F_{tTS}dt + \gamma_{tTS}F_{tTS}dW_t, \tag{3.3.3}$$

is computed from (3.3.1) as follows: For $0 \leq t \leq T \leq S < \infty$

$$\begin{aligned}\gamma_{tTS} =& (\hat{V}_{tT} - \hat{V}_{tS})\frac{1}{S-T}\frac{Z_{tT}}{Z_{tS}}\left[\frac{1}{S-T}\left(\frac{Z_{tT}}{Z_{tS}} - 1\right)\right]^{-1} \\ =& (\hat{V}_{tT} - \hat{V}_{tS})\frac{Z_{tT}}{Z_{tT} - Z_{tS}}.\end{aligned} \tag{3.3.4}$$

Hence we obtain that the forward LIBOR rate dynamics may be expressed in the following way:

$$dF_{tTS} = -\hat{V}_{tS}(\hat{V}_{tT} - \hat{V}_{tS})\frac{Z_{tT}}{Z_{tT} - Z_{tS}}F_{tTS}dt + (\hat{V}_{tT} - \hat{V}_{tS})\frac{Z_{tT}}{Z_{tT} - Z_{tS}}F_{tTS}dW_t.$$

Recalling the expression (3.2.7), the forward LIBOR rate volatility can be expressed by the discount bond P_{tT} and the discount bond volatility Ω_{tT} in the following way:

$$\gamma_{tTS} = (\Omega_{tT} - \Omega_{tS})\frac{P_{tT}}{P_{tT} - P_{tS}}. \tag{3.3.5}$$

The Black formula assumes that the LIBOR rate volatility γ_{tTS} is deterministic. In the LFM, we estimate the LIBOR rate volatility from the market by applying either non-parametric methods or parametric methods, assuming it to be deterministic in both cases. However, when we arbitrarily choose the LIBOR rate volatility, the state price density is not guaranteed to be a potential. Therefore the arbitrage free and positive interest conditions are no longer guaranteed. In addition, the assumption of the log-normality causes a problem for modelling the volatility smiles, as it allows only flat volatility line. Some other local volatility models have been suggested for the volatility smiles, such as the shifted BGM Model ([35]). In this case the dynamics of the forward LIBOR rate is expressed as follows:

$$dF_{tTS} = -\hat{V}_{tS}\hat{\gamma}_t(F_{tTS} + k_{TS})dt + \hat{\gamma}_t(F_{tTS} + k_{TS})dW_t, \tag{3.3.6}$$

where

$$\hat{\gamma}_t(F_{tTS} + k_{TS}) := (\hat{V}_{tT} - \hat{V}_{tS})\Big(F_{tTS} + \frac{1}{S-T}\Big), \quad 0 \le t \le T \le S < \infty,$$

and where $\hat{\gamma}_t$ and k_{TS} are deterministic functions such that $-\infty < k_{TS} \le \frac{1}{S-T}$. Unfortunately these local volatilities models have a crucial problem in hedging performance. The book [35] claims that the shift k_{TS} models volatility skew and stochasticity of the function $\hat{\gamma}_t$ models kurtosis. Therefore we would need to incorporate a stochastic property into the risk-adjusted volatility. However, as we observed already, modelling the process $(\sigma_t)_{t\ge 0}$ with stochastic property yields desirable features. In addition, as shown in [48], the Chaotic Approach has analytical formulas for both Caps/Floors and Swaptions. The SABR dynamics may be constructed from equation (3.3.20) as follows: For $\beta \in (0, 1]$,

(3.3.7)
$$\begin{aligned} dF_{tTS} =& [\cdots]dt + \gamma_{tTS} F_{tTS}^{1-\beta} F_{tTS}^{\beta} dW_t \\ =& [\cdots]dt + \upsilon_{tTS} F_{tTS}^{\beta} dW_t, \end{aligned}$$

where the quantity υ_{tTS} can be expressed as follows:

$$\upsilon_{tTS} := (S-T)^{\beta-1}(\hat{V}_{tT} - \hat{V}_{tS})\frac{Z_{tT}}{Z_{tS}}\Big(\frac{Z_{tS}}{Z_{tT} - Z_{tS}}\Big)^{\beta}.$$

3.3.2 Dynamics of the forward swap rate

As we observed in Section 3.1, the forward swap rate may be expressed in the following way:

$$S_{a,b}(t) = \frac{Z_{tT_a} - Z_{tT_b}}{\sum_{i=a+1}^{b} \tau_i Z_{tT_i}}.$$

Therefore it is certain that we are able to construct the dynamics of the forward swap rate only from the process $(Z_{tT})_{0\le t\le T<\infty}$. Let us first notice that

$$d(Z_{tT_a} - Z_{tT_b}) = (\hat{V}_{tT_a} Z_{tT_a} - \hat{V}_{tT_b} Z_{tT_b})dW_t,$$

$$d\Big(\sum_{i=a+1}^{b} \tau_i Z_{tT_i}\Big) = \Big(\sum_{i=a+1}^{b} \tau_i \hat{V}_{tT_i} Z_{tT_i}\Big)dW_t,$$

and

$$\Big(d\sum_{i=a+1}^{b}\tau_i Z_{tT_i}\Big)^2=\Big(\sum_{i=a+1}^{b}\tau_i\hat{V}_{tT_i}Z_{tT_i}\Big)^2 dt.$$

In addition, we infer using Itô's Lemma that

$$\begin{aligned}
d\Big(\frac{1}{\sum_{i=a+1}^{b}\tau_i Z_{tT_i}}\Big)=&\frac{1}{\Big(\sum_{i=a+1}^{b}\tau_i Z_{tT_i}\Big)^3}\Big(d\sum_{i=a+1}^{b}\tau_i Z_{tT_i}\Big)^2-\frac{1}{\Big(\sum_{i=a+1}^{b}\tau_i Z_{tT_i}\Big)^2}d\Big(\sum_{i=a+1}^{b}\tau_i Z_{tT_i}\Big)\\
=&\frac{\Big(\sum_{i=a+1}^{b}\tau_i\hat{V}_{tT_i}Z_{tT_i}\Big)^2}{\Big(\sum_{i=a+1}^{b}\tau_i Z_{tT_i}\Big)^3}dt-\frac{\sum_{i=a+1}^{b}\tau_i\hat{V}_t Z_{tT_i}}{\Big(\sum_{i=a+1}^{b}\tau_i Z_{tT_i}\Big)^2}dW_t.
\end{aligned}$$

Putting all this together, we are now able to compute the dynamics of the forward swap rate

$$\begin{aligned}
dS_{a,b}(t)=&(Z_{tT_a}-Z_{tT_b})d\Bigg(\frac{1}{\sum_{i=a+1}^{b}\tau_i Z_{tT_i}}\Bigg)+\frac{1}{\sum_{i=a+1}^{b}\tau_i Z_{tT_i}}d(Z_{tT_a}-Z_{tT_b})\\
&+d\Bigg(\frac{1}{\sum_{i=a+1}^{b}\tau_i Z_{tT_i}}\Bigg)d(Z_{tT_a}-Z_{tT_b})\\
=&\Bigg[(Z_{tT_a}-Z_{tT_b})\frac{\Big(\sum_{i=a+1}^{b}\tau_i\hat{V}_{tT_i}Z_{tT_i}\Big)^2}{\Big(\sum_{i=a+1}^{b}\tau_i Z_{tT_i}\Big)^3}-\frac{\sum_{i=a+1}^{b}\tau_i\hat{V}_t Z_{tT_i}}{\Big(\sum_{i=a+1}^{b}\tau_i Z_{tT_i}\Big)^2}(\hat{V}_{tT_a}Z_{tT_a}-\hat{V}_{tT_b}Z_{tT_b})\Bigg]dt\\
&-\Bigg[(Z_{tT_a}-Z_{tT_b})\frac{\sum_{i=a+1}^{b}\tau_i\hat{V}_t Z_{tT_i}}{\Big(\sum_{i=a+1}^{b}\tau_i Z_{tT_i}\Big)^2}-\frac{\hat{V}_{tT_a}Z_{tT_a}-\hat{V}_{tT_b}Z_{tT_b}}{\sum_{i=a+1}^{b}\tau_i Z_{tT_i}}\Bigg]dW_t\\
=&-\frac{\sum_{i=a+1}^{b}\tau_i\hat{V}_t Z_{tT_i}}{\sum_{i=a+1}^{b}\tau_i Z_{tT_i}}\Bigg[-\frac{\sum_{i=a+1}^{b}\tau_i\hat{V}_{tT_i}Z_{tT_i}}{\sum_{i=a+1}^{b}\tau_i Z_{tT_i}}+\frac{\hat{V}_{tT_a}Z_{tT_a}-\hat{V}_{tT_b}Z_{tT_b}}{Z_{tT_a}-Z_{tT_b}}\Bigg]S_{a,b}(t)dt\\
&\Bigg[-\frac{\sum_{i=a+1}^{b}\tau_i\hat{V}_t Z_{tT_i}}{\sum_{i=a+1}^{b}\tau_i Z_{tT_i}}+\frac{\hat{V}_{tT_a}Z_{tT_a}-\hat{V}_{tT_b}Z_{tT_b}}{Z_{tT_a}-Z_{tT_b}}\Bigg]S_{a,b}(t)dW_t.
\end{aligned}$$

Therefore the forward swap rate denoted by $\widetilde{\gamma}_{a,b}(t)$ may be expressed in the following way:

$$\widetilde{\gamma}_{a,b}(t):=\frac{\hat{V}_{tT_a}Z_{tT_a}-\hat{V}_{tT_b}Z_{tT_b}}{Z_{tT_a}-Z_{tT_b}}-\frac{\sum_{i=a+1}^{b}\tau_i\hat{V}_t Z_{tT_i}}{\sum_{i=a+1}^{b}\tau_i Z_{tT_i}}. \tag{3.3.8}$$

In other words, we have that

$$\widetilde{\gamma}_{a,b}(t) = \frac{\Omega_{tT_a}P_{tT_a} - \Omega_{tT_b}P_{tT_b}}{P_{tT_a} - P_{tT_b}} - \frac{\sum_{i=a+1}^{b}\tau_i\Omega_{tT_i}P_{tT_i}}{\sum_{i=a+1}^{b}\tau_i P_{tT_i}}. \tag{3.3.9}$$

We know from the definition that modelling the dynamics of the forward swap rate provides a Swaptions pricing formula. The Black formula computes the swaption premium, assuming that the variable $\widetilde{\gamma}_{a,b}(t)$ deterministic. Here we notice that the forward swap rate dynamics is comparable with the forward LIBOR rate dynamics expressed in (3.3.3). Indeed, the forward rate dynamics can be derived from the forward swap rate by setting $T = T_a$ and $S = T_{a+1} = T_b$, as could be expected from the definitions. Finally, the SABR dynamics may be constructed from there as follows:

$$\begin{aligned} dS_{a,b}(t) =& [\cdots]dt + \widetilde{\gamma}_{a,b}(t)S_{a,b}^{1-\beta}(t)S_{a,b}^{\beta}(t)dW_t \\ =& [\cdots]dt + \widetilde{v}_{a,b}(t)S_{a,b}^{\beta}(t)dW_t, \end{aligned} \tag{3.3.10}$$

where we denote the volatility term by

$$\widetilde{v}_{a,b}(t) := \Big(- \frac{\sum_{i=a+1}^{b}\tau_i\hat{V}_t Z_{tT_i}}{\sum_{i=a+1}^{b}\tau_i Z_{tT_i}} + \frac{\hat{V}_{tT_a}Z_{tT_a} - \hat{V}_{tT_b}Z_{tT_b}}{Z_{tT_a} - Z_{tT_b}}\Big)\Big(\frac{\sum_{i=a+1}^{b}\tau_i Z_{tT_i}}{Z_{tT_a} - Z_{tT_b}}\Big)^{\beta-1}.$$

3.3.3 Further investigation of the volatility drifts

We apply the Malliavin Calculus to express the volatility drifts by only the variable Z_{tT}. The Clark-Ocone formula states that a square integrable $\mathcal{F}_t$-measurable random variable F_t may be represented in the following way:

$$F_t = \mathbb{E}[F_t] + \int_0^t \mathbb{E}_s\big[D_s[F_t]\big]dW_s \quad \text{for} \quad t \geq 0, \tag{3.3.11}$$

where D_t denotes the Malliavin derivative with respect to t. Therefore, the Martingale Representation Theorem for the variable Z_{tT} can be interpreted in the following way:

$$Z_{tT} = Z_{0T} + \int_0^t \mathbb{E}_s\big[D_s[Z_{tT}]\big]dW_s \quad \text{for} \quad 0 \leq t \leq T < \infty.$$

Differentiating both sides, it follows that

$$dZ_{tT} = D_t[Z_{tT}]dW_t, \tag{3.3.12}$$

where we define $D_t[Z_{tT}]$ to be

$$D_t[Z_{tT}] := \lim_{s\to t^-} D_s[Z_{tT}] \quad \text{for} \quad 0 \le s \le t,$$

while we have that

$$\lim_{s\to t^+} D_s[Z_{tT}] = 0 \quad \text{for} \quad 0 \le t \le s.$$

The reader can also find this asymptotic argument from the literature [52] and [59]. Note here that, applying Proposition 5.6 from [67], (see also Proposition 4.1 from [7]), the following interchange is satisfied:

$$\mathbb{E}_t\big[D_t[Z_{tT}]\big] = D_t\big[\mathbb{E}_t[Z_{tT}]\big] \quad \text{for} \quad 0 \le t \le T < \infty. \tag{3.3.13}$$

Comparing equation (3.3.12) with expression (3.2.1), we obtain that

$$\hat{V}_{tT} = \frac{D_t[Z_{tT}]}{Z_{tT}} \quad \text{for} \quad 0 \le t \le T < \infty. \tag{3.3.14}$$

Recalling the dynamics of the state price density, which is expressed in (2.1.18), we may similarly obtain that

$$\lambda_t = -\frac{D_t[V_t]}{V_t} \quad \text{for} \quad t \ge 0. \tag{3.3.15}$$

Furthermore, because the discount bond volatility may be expressed in the following way:

$$\Omega_{tT} = \frac{D_t[P_{tT}]}{P_{tT}}, \quad \text{for} \quad 0 \le t \le T < \infty, \tag{3.3.16}$$

the quotient rule of the Malliavin derivative gives that

$$\Omega_{tT} = \frac{D_t[Z_{tT}]V_t - D_t[V_t]Z_{tT}}{V_t Z_{tT}} = \frac{D_t[Z_{tT}]}{Z_{tT}} - \frac{D_t[V_t]}{V_t} = \hat{V}_{tT} + \lambda_t. \tag{3.3.17}$$

Because we have that $D_t[1] = 0$, we obtain from (3.3.16) and (3.3.17) that

$$\Omega_{TT} = 0 \quad \text{and} \quad \hat{V}_{TT} = -\lambda_T \quad \text{for} \quad T \geq 0. \tag{3.3.18}$$

Inserting the risk-adjusted volatility expressed in (3.3.14) into equation (3.3.4), the forward LIBOR rate volatility may be also expressed in the following way: For $0 \leq t \leq T \leq S < \infty$

$$\begin{aligned}\gamma_{tTS} =& \Big(\frac{D_t[Z_{tT}]}{Z_{tT}} - \frac{D_t[Z_{tS}]}{Z_{tS}}\Big)\frac{Z_{tT}}{Z_{tT} - Z_{tS}} \\ =& \Big(D_t[Z_{tT}] - D_t[Z_{tS}]\frac{Z_{tT}}{Z_{tS}}\Big)\frac{1}{Z_{tT} - Z_{tS}} \\ =& \Big(D_t[Z_{tT}] - D_t[Z_{tS}]\Big(1 + \frac{Z_{tT} - Z_{tS}}{Z_{tS}}\Big)\Big)\frac{1}{Z_{tT} - Z_{tS}} \\ =& \frac{D_t[Z_{tT} - Z_{tS}]}{Z_{tT} - Z_{tS}} - \frac{D_t[Z_{tS}]}{Z_{tS}}.\end{aligned} \tag{3.3.19}$$

Therefore, we are now able to express the stochastic differential equation of the forward LIBOR rate expressed via the Potential Approach as follows:

(3.3.20)

$$dF_{tTS} = -\frac{D_t[Z_{tS}]}{Z_{tS}}\Big(\frac{D_t[Z_{tT} - Z_{tS}]}{Z_{tT} - Z_{tS}} - \frac{D_t[Z_{tS}]}{Z_{tS}}\Big)F_{tTS}dt + \Big(\frac{D_t[Z_{tT} - Z_{tS}]}{Z_{tT} - Z_{tS}} - \frac{D_t[Z_{tS}]}{Z_{tS}}\Big)F_{tTS}dW_t.$$

This allows us to infer that the dynamics of the forward LIBOR rate may be determined by the conditional expectation of the state price density, i.e., Z_{tT}. Note here that because we may express that

$$F_{tTS} = [\cdots]dt + D_t[F_{tTS}]dW_t$$

and

$$D_t[F_{tTS}] = D_t\Big[\frac{1}{S-T}\Big(\frac{Z_{tT}}{Z_{tS}} - 1\Big)\Big] = \frac{1}{S-T}\frac{Z_{tS}D_t[Z_{tT} - Z_{tS}] - (Z_{tT} - Z_{tS})D_t[Z_{tS}]}{(Z_{tS})^2},$$

the forward LIBOR rate volatility may also be computed in the following way:

$$\begin{aligned}\gamma_{tTS} =& \frac{D_t[F_{tTS}]}{F_{tTS}} = \frac{Z_{tS}D_t[Z_{tT} - Z_{tS}] - (Z_{tT} - Z_{tS})D_t[Z_{tS}]}{(Z_{tT} - Z_{tS})Z_{tS}} \\ =& \frac{D_t[Z_{tT} - Z_{tS}]}{Z_{tT} - Z_{tS}} - \frac{D_t[Z_{tS}]}{Z_{tS}},\end{aligned}$$

as expected. Similarly, the volatility of volatility η_{tTS} can be computed in the following way:

$$\eta_{tTS} = \frac{D_t[\gamma_{tTS}]}{\gamma_{tTS}} = \frac{D_t\big[\frac{D_t[F_{tTS}]}{F_{tTS}}\big]}{\frac{D_t[F_{tTS}]}{F_{tTS}}} = \frac{D_t\big[D_t[F_{tTS}]\big]}{D_t[F_{tTS}]} - \frac{D_t[F_{tTS}]}{F_{tTS}}. \tag{3.3.21}$$

Because we have that

$$\begin{aligned}
D_t[D_t[F_{tTS}]] =& \frac{1}{S-T} D_t\Big[\frac{Z_{tS}D_t[Z_{tT}-Z_{tS}] - (Z_{tT}-Z_{tS})D_t[Z_{tS}]}{(Z_{tS})^2}\Big] \\
=& \frac{1}{S-T}\bigg(\frac{D_t\big[Z_{tS}D_t[Z_{tT}-Z_{tS}] - (Z_{tT}-Z_{tS})D_t[Z_{tS}]\big]}{(Z_{tS})^2} \\
&- \frac{2D_t[Z_{tS}]\big(Z_{tS}D_t[Z_{tT}-Z_{tS}] - (Z_{tT}-Z_{tS})D_t[Z_{tS}]\big)}{(Z_{tS})^3}\bigg)
\end{aligned}$$

we find that

$$\frac{D_t\big[D_t[F_{tTS}]\big]}{D_t[F_{tTS}]} = \frac{D_t\big[Z_{tS}D_t[Z_{tT}-Z_{tS}] - (Z_{tT}-Z_{tS})D_t[Z_{tS}]\big]}{Z_{tS}D_t[Z_{tT}-Z_{tS}] - (Z_{tT}-Z_{tS})D_t[Z_{tS}]} - 2\frac{D_t[Z_{tS}]}{Z_{tS}}.$$

Therefore we conclude from equation (3.3.21) that

$$\begin{aligned}
\eta_{tTS} =& \frac{D_t\big[Z_{tS}D_t[Z_{tT}-Z_{tS}] - (Z_{tT}-Z_{tS})D_t[Z_{tS}]\big]}{Z_{tS}D_t[Z_{tT}-Z_{tS}] - (Z_{tT}-Z_{tS})D_t[Z_{tS}]} - \frac{D_t[Z_{tT}-Z_{tS}]}{Z_{tT}-Z_{tS}} - \frac{D_t[Z_{tS}]}{Z_{tS}} \\
=& \frac{D_t[\widetilde{Z}_{tTS}]}{\widetilde{Z}_{tTS}} - \frac{\widetilde{Z}_{tTS}}{Z_{tS}(Z_{tT}-Z_{tS})},
\end{aligned}$$

where we have defined

$$\widetilde{Z}_{tTS} := Z_{tS}D_t[Z_{tT}] - Z_{tT}D_t[Z_{tS}] \quad \text{for} \quad 0 \le t \le T \le S < \infty.$$

At this point we would also like to formulate the dynamics of the SABR volatility v_{tTS}. The volatility of volatility under the SABR dynamics may be found by

$$\frac{D_t[v_{tTS}]}{v_{tTS}}.$$

However, it seems we are not able to obtain a simple form. Using expression (3.3.14), the forward swap rate volatility form in (3.3.8) may be interpreted in the following way:

$$\widetilde{\gamma}_{a,b}(t) := \frac{D_t[Z_{tT_a} - Z_{tT_b}]}{Z_{tT_a} - Z_{tT_b}} - \frac{D_t\big[\sum_{i=a+1}^{b} \tau_i Z_{tT_i}\big]}{\sum_{i=a+1}^{b} \tau_i Z_{tT_i}}. \tag{3.3.22}$$

The volatility of volatility is computed by $\frac{D_t[\widetilde{\gamma}_{a,b}(t)]}{\widetilde{\gamma}_{a,b}(t)}$. Note here that because we may express that

$$dS_{a,b}(t) = [\cdots]dt + D_t[S_{a,b}(t)]dW_t,$$

we can also compute the swap rate volatility in the following way:

$$\begin{aligned}
\widetilde{\gamma}_{a,b}(t) =& \frac{D_t[S_{a,b}(t)]}{S_{a,b}(t)} = \Big[\frac{Z_{tT_a} - Z_{tT_b}}{\sum_{i=a+1}^{b} \tau_i Z_{tT_i}}\Big]^{-1} D_t\Big[\frac{Z_{tT_a} - Z_{tT_b}}{\sum_{i=a+1}^{b} \tau_i Z_{tT_i}}\Big] \\
=& \Big[\frac{Z_{tT_a} - Z_{tT_b}}{\sum_{i=a+1}^{b} \tau_i Z_{tT_i}}\Big]^{-1} \frac{(\sum_{i=a+1}^{b} \tau_i Z_{tT_i}) D_t[Z_{tT_a} - Z_{tT_b}] - (Z_{tT_a} - Z_{tT_b}) D_t[\sum_{i=a+1}^{b} \tau_i Z_{tT_i}]}{(\sum_{i=a+1}^{b} \tau_i Z_{tT_i})^2} \\
=& \frac{(\sum_{i=a+1}^{b} \tau_i Z_{tT_i}) D_t[Z_{tT_a} - Z_{tT_b}] - (Z_{tT_a} - Z_{tT_b}) D_t[\sum_{i=a+1}^{b} \tau_i Z_{tT_i}]}{(Z_{tT_a} - Z_{tT_b})(\sum_{i=a+1}^{b} \tau_i Z_{tT_i})} \\
=& \frac{D_t[Z_{tT_a} - Z_{tT_b}]}{Z_{tT_a} - Z_{tT_b}} - \frac{D_t[\sum_{i=a+1}^{b} \tau_i Z_{tT_i}]}{\sum_{i=a+1}^{b} \tau_i Z_{tT_i}}.
\end{aligned}$$

Chapter 4

Further investigations of the Chaotic Approach

In this chapter we develop the analysis of the Chaotic Approach. In the original paper [48] the focus is on the First Chaos, Second Chaos and Factorizable Second Chaos Models, where an investigation into pricing options is also given. We follow their discussion of these models, adding some new ideas. However, we also express the main processes without truncating the tail of the chaos expansion, and consider higher order Chaos Models, introducing One-variable Chaos Models. Furthermore, we suggest the exponential polynomials for the chaos coefficients and calibrate the Chaos Models in the following chapters.

4.1 Form of the main processes

Recalling the Wiener-Chaos expansion of the variable σ_s from (2.1.23), we may write this expression in the following way: For $0 \leq t \leq s_n \cdots \leq s_2 \leq s_1 \leq s < \infty$, we have that

$$(4.1.1)\quad \sigma_s = R_1(t,s) + \int_t^s R_2(t,s,s_1)dWs_1 + \int_t^s \int_t^{s_1} R_3(t,s,s_1,s_2)dWs_2dWs_1 + \cdots ,$$

where

$$R_1(t,s) = \phi_1(s) + \int_0^t \phi_2(s,s_1)dWs_1 + \int_0^t \int_0^{s_1} \phi_3(s,s_1,s_2)dWs_2dWs_1 + \cdots ,$$

$$R_n(t,s,s_1,\ldots,s_{n-1}) = \phi_n(s,s_1,\ldots,s_{n-1}) + \int_0^t \phi_{n+1}(s,s_1,\ldots,s_n)dWs_n + \int_0^t\int_0^{s_n} \phi_{n+2}(s,s_1,\ldots,s_{n+1})dWs_{n+1}dWs_n + \cdots, \quad \text{for} \quad n=2,3,\ldots.$$

Note here that we find that

$$\sigma_t = R_1(t,t), \quad D_t[\sigma_t] = R_2(t,t,t), \quad D_t[D_t[\sigma_t]] = R_3(t,t,t,t), \quad \cdots.$$

In addition, we find that the function R_n for each positive integer n is a martingale with respect to $\mathcal{F}_t$, since we have that

$$\mathbb{E}[R_1(t,s)] = \phi_1(s),$$
$$\mathbb{E}[R_n(t,s,s_1,\ldots,s_{n-1})] = \phi_n(s,s_1,\ldots,s_{n-1}), \quad \text{for} \quad n=2,3,\ldots,$$

and

$$dR_1(t,s) = R_2(t,s,t)dW_t,$$
$$dR_2(t,s,s_1) = R_3(t,s,s_1,t)dW_t,$$
$$\vdots$$
$$dR_{n-1}(t,s,s_1,\ldots,s_{n-1}) = R_n(t,s,s_1,\ldots,s_{n-2},t)dW_t.$$

Because the function R_n for each n is $\mathcal{F}_t$-measurable, it follows by the Itô isometry and by orthogonality that

$$(4.1.2) \quad \mathbb{E}_t[\sigma_s^2] = R_1^2(t,s) + \int_t^s R_2^2(t,s,s_1)ds_1 + \int_t^s\int_t^{s_1} R_3^2(t,s,s_1,s_2)ds_2ds_1 + \cdots,$$

which implies that

$$(4.1.3) \qquad \mathbb{E}[\sigma_s^2] = \phi_1^2(s) + \int_0^s \phi_2^2(s,s_1)ds_1 + \int_0^s\int_0^{s_1} \phi_3^2(s,s_1,s_2)ds_2ds_1 + \cdots.$$

(For another method to derive equation (4.1.2) we refer the reader to the Appendix.) Because we know from (2.1.22) that $V_t = \int_t^\infty \mathbb{E}_t[\sigma_s^2]ds$, we obtain that

$$(4.1.4) \quad V_t = \int_t^\infty \Big(R_1^2(t,s) + \int_t^s R_2^2(t,s,s_1)ds_1 + \int_t^s\int_t^{s_1} R_3^2(t,s,s_1,s_2)ds_2ds_1 + \cdots\Big)ds,$$

(4.1.5)

$$Z_{tT} = \int_T^\infty \Big(R_1^2(t,s) + \int_t^s R_2^2(t,s,s_1)ds_1 + \int_t^s \int_t^{s_1} R_3^2(t,s,s_1,s_2)ds_2ds_1 + \cdots\Big)ds,$$

(4.1.6)

$$P_{tT} = \frac{\int_T^\infty \Big(R_1^2(t,s) + \int_t^s R_2^2(t,s,s_1)ds_1 + \int_t^s \int_t^{s_1} R_3^2(t,s,s_1,s_2)ds_2ds_1 + \cdots\Big)ds}{\int_t^\infty \Big(R_1^2(t,s) + \int_t^s R_2^2(t,s,s_1)ds_1 + \int_t^s \int_t^{s_1} R_3^2(t,s,s_1,s_2)ds_2ds_1 + \cdots\Big)ds}$$

and

(4.1.7)

$$F_{tTS} = \frac{1}{S-T}\frac{\int_T^S \Big(R_1^2(t,s) + \int_t^s R_2^2(t,s,s_1)ds_1 + \int_t^s \int_t^{s_1} R_3^2(t,s,s_1,s_2)ds_2ds_1 + \cdots\Big)ds}{\int_S^\infty \Big(R_1^2(t,s) + \int_t^s R_2^2(t,s,s_1)ds_1 + \int_t^s \int_t^{s_1} R_3^2(t,s,s_1,s_2)ds_2ds_1 + \cdots\Big)ds}.$$

In particular, the initial curve may be drawn by setting

$$P_{0T} = \frac{\int_T^\infty \Big(\phi_1^2(s) + \int_0^s \phi_2^2(s,s_1)ds_1 + \int_0^s \int_0^{s_1} \phi_3^2(s,s_1,s_2)ds_2ds_1 + \cdots\Big)ds}{\int_0^\infty \Big(\phi_1^2(s) + \int_0^s \phi_2^2(s,s_1)ds_1 + \int_0^s \int_0^{s_1} \phi_3^2(s,s_1,s_2)ds_2ds_1 + \cdots\Big)ds}$$

and

(4.1.8) $$f_{0T} = \frac{\phi_1^2(T) + \int_0^s \phi_2^2(T,s_1)ds_1 + \int_0^T \int_0^{s_1} \phi_3^2(T,s_1,s_2)ds_2ds_1 + \cdots}{\int_T^\infty \Big(\phi_1^2(s) + \int_0^s \phi_2^2(s,s_1)ds_1 + \int_0^s \int_0^{s_1} \phi_3^2(s,s_1,s_2)ds_2ds_1 + \cdots\Big)ds}.$$

Let us now consider the Malliavin derivative of Z_{tT} and V_t in order to compute the market price of risk and the risk-adjusted volatility. We first notice that

$$\begin{aligned}
&D_t[R_1(t,s)]\\
=&D_t\Big[\phi_1(s) + \int_0^t \phi_2(s,s_1)dWs_1 + \int_0^t \int_0^{s_1} \phi_3(s,s_1,s_2)dWs_2dWs_1 + \cdots\Big]\\
=&D_t[\phi_1(s)] + D_t\Big[\int_0^t \phi_2(s,s_1)dWs_1\Big] + D_t\Big[\int_0^t \int_0^{s_1} \phi_3(s,s_1,s_2)dWs_2dWs_1\Big] + \cdots\\
=&0 + \phi_2(s,t) + \int_0^t \phi_3(s,t,s_2)dWs_2 + \cdots\\
=&R_2(t,s,t),
\end{aligned}$$

and for $n = 2, 3, \dots$

$$\begin{aligned}
&D_t[R_n(t,s,s_1,\dots,s_{n-1})]\\
=&D_t\Big[\phi_n(s,s_1,\dots,s_{n-1}) + \int_0^t \phi_{n+1}(s,s_1,\dots,s_n)dWs_n\\
&+\int_0^t\int_0^{s_2}\phi_{n+2}(s,s_1,\dots,s_{n+1})dWs_{n+1}dWs_n+\cdots\Big]\\
=&D_t[\phi_n(s,s_1,\dots,s_{n-1})] + D_t\Big[\int_0^t \phi_{n+1}(s,s_1,\dots,s_n)dWs_n\Big]\\
&+D_t\Big[\int_0^t\int_0^{s_2}\phi_{n+2}(s,s_1,\dots,s_{n+1})dWs_{n+1}dWs_n\Big]+\cdots\\
=&0+\phi_{n+1}(s,s_1,\dots,s_{n-1},t)+\int_0^t\phi_{n+2}(s,s_1,\dots,s_{n-1},t,s_{n+1})dWs_{n+1}+\cdots\\
=&R_{n+1}(t,s,s_1,\dots,s_{n-1},t).
\end{aligned}$$

Therefore, applying the chain rule of the Malliavin derivative, we obtain that

(4.1.9)

$$\begin{aligned}
D_t\big[\mathbb{E}_t[\sigma_s^2]\big] =&D_t[R_1^2(t,s)] + \int_t^s D_t[R_2^2(t,s,s_1)]ds_1 + \int_t^s\int_t^{s_1} D_t[R_3^2(t,s,s_1,s_2)]ds_2ds_1\\
&+\int_t^s\int_t^{s_1}\int_t^{s_2} D_t[R_4^2(t,s,s_1,s_2,s_3)]ds_3ds_2ds_1+\cdots\\
=&2R_1(t,s)D_t[R_1(t,s)] + \int_t^s 2R_2(t,s,s_1)D_t[R_2(t,s,s_1)]ds_1\\
&+\int_t^s\int_t^{s_1} 2R_3(t,s,s_1,s_2)D_t[R_3(t,s,s_1,s_2)]ds_2ds_1\\
&+\int_t^s\int_t^{s_1}\int_t^{s_2} 2R_4(t,s,s_1,s_2,s_3)D_t[R_4(t,s,s_1,s_2,s_3)]ds_3ds_2ds_1+\cdots\\
=&2R_1(t,s)R_2(t,s,t) + \int_t^s 2R_2(t,s,s_1)R_3(t,s,s_1,t)ds_1\\
&+\int_t^s\int_t^{s_1} 2R_3(t,s,s_1,s_2)R_4(t,s,s_1,s_2,t)ds_2ds_1+\cdots.
\end{aligned}$$

Because we here have that

$$D_t[V_t] = \int_t^\infty D_t\big[\mathbb{E}_t[\sigma_s^2]\big]ds \quad\text{and}\quad D_t[Z_{tT}] = \int_T^\infty D_t\big[\mathbb{E}_t[\sigma_s^2]\big]ds,$$

we obtain that

$$\begin{aligned}D_t[V_t] =& 2\int_t^{\infty}\Big(R_1(t,s)R_2(t,s,t)+\int_t^s R_2(t,s,s_1)R_3(t,s,s_1,t)ds_1\\&+\int_t^s\int_t^{s_1} R_3(t,s,s_1,s_2)R_4(t,s,s_1,s_2,t)ds_2ds_1+\cdots\Big)ds\end{aligned}$$

and

$$\begin{aligned}D_t[Z_{tT}] =& 2\int_T^{\infty}\Big(R_1(t,s)R_2(t,s,t)+\int_t^s R_2(t,s,s_1)R_3(t,s,s_1,t)ds_1\\&+\int_t^s\int_t^{s_1} R_3(t,s,s_1,s_2)R_4(t,s,s_1,s_2,t)ds_2ds_1+\cdots\Big)ds.\end{aligned}$$

Therefore the market price of risk and the risk-adjusted volatility formulated by (3.3.14) and (3.3.15) may be respectively expressed in the Chaotic Approach as follows:

(4.1.10)

$$\lambda_t = -\frac{2\int_t^{\infty}\Big[R_1(t,s)R_2(t,s,t)+\int_t^s R_2(t,s,s_1)R_3(t,s,s_1,t)ds_1+\cdots\Big]ds}{\int_t^{\infty}\Big[R_1^2(t,s)+\int_t^s R_2^2(t,s,s_1)ds_1+\int_t^s\int_t^{s_1}R_3^2(t,s,s_1,s_2)ds_2ds_1+\cdots\Big]ds}$$

and

(4.1.11)

$$\hat{V}_{tT} = \frac{2\int_T^{\infty}\Big[R_1(t,s)R_2(t,s,t)+\int_t^s R_2(t,s,s_1)R_3(t,s,s_1,t)ds_1+\cdots\Big]ds}{\int_T^{\infty}\Big[R_1^2(t,s)+\int_t^s R_2^2(t,s,s_1)ds_1+\int_t^s\int_t^{s_1}R_3^2(t,s,s_1,s_2)ds_2ds_1+\cdots\Big]ds}.$$

4.2 Modelling initial curves in the Chaotic Approach

We are able to secure freedom of modelling initial curves in the Chaotic Approach. Application of the Clark-Ocone formula to the variable σ_s^2, $s \geq 0$ gives us that

$$\sigma_s^2 = \mathbb{E}[\sigma_s^2] + \int_0^s \mathbb{E}_u[D_u[\sigma_s^2]]dW_u.$$

Then, taking conditional expectation with respect to $\mathcal{F}_t$, $0 \leq t \leq s < \infty$ for both sides of the equation, we obtain that

$$\mathbb{E}_t[\sigma_s^2] = \mathbb{E}_t[\mathbb{E}[\sigma_s^2]] + \mathbb{E}_t\Big[\int_0^s \mathbb{E}_u[D_u[\sigma_s^2]]dW_u\Big]. \qquad (4.2.1)$$

Because $\mathbb{E}[\sigma_s^2]$ is a constant and the Itô integral $\int_0^s \mathbb{E}_u[D_u[\sigma_s^2]]dW_u$ is a martingale, it follows that

$$\mathbb{E}_t[\sigma_s^2] = \mathbb{E}[\sigma_s^2] + \int_0^t \mathbb{E}_u[D_u[\sigma_s^2]]dW_u. \tag{4.2.2}$$

By (3.1.6), we obtain here that

$$Z_{tT} = \int_T^\infty \mathbb{E}[\sigma_s^2]ds + \int_T^\infty \int_0^t \mathbb{E}_u[D_u[\sigma_s^2]]dW_u ds.$$

We apply the Stochastic Fubini Theorem, (see [5] and [72]), to obtain that

$$Z_{tT} = P_{0T} \int_0^\infty \mathbb{E}[\sigma_s^2]ds + \int_0^t \int_T^\infty \mathbb{E}_u[D_u[\sigma_s^2]]dsdW_u, \tag{4.2.3}$$

where we recall from (3.1.5) that $\mathbb{E}[Z_{tT}] = P_{0T}V_0$ and that

$$V_0 = \int_0^\infty \mathbb{E}[\sigma_s^2]ds.$$

It follows that for $0 \le t \le T < \infty$

$$P_{tT} = \frac{P_{0T} \int_0^\infty \mathbb{E}[\sigma_s^2]ds + \int_0^t \int_T^\infty \mathbb{E}_u[D_u[\sigma_s^2]]dsdW_u}{P_{0t} \int_0^\infty \mathbb{E}[\sigma_s^2]ds + \int_0^t \int_t^\infty \mathbb{E}_u[D_u[\sigma_s^2]]dsdW_u}.$$

We now see the benefit of the chaos expansion, using (4.1.9), to compute the integrand as follows: For $0 \le u \le s < \infty$,

$$\begin{aligned}\mathbb{E}_u[D_u[\sigma_s^2]] = D_u[\mathbb{E}_u[\sigma_s^2]] =& 2\Big(R_1(u,s)R_2(u,s,u) + \int_u^s R_2(u,s,s_1)R_3(u,s,s_1,u)ds_1 \\ &+ \int_u^s \int_u^{s_1} R_3(u,s,s_1,s_2)R_4(u,s,s_1,s_2,u)ds_2ds_1 + \cdots\Big).\end{aligned}$$

Therefore, inserting this into equations (4.2.2) and (4.2.3), we obtain that

$$\begin{aligned}\mathbb{E}_t[\sigma_s^2] =& \mathbb{E}[\sigma_s^2] + \int_0^t 2\Big(R_1(u,s)R_2(u,s,u) + \int_u^s R_2(u,s,s_1)R_3(u,s,s_1,u)ds_1 \\ &+ \int_u^s \int_u^{s_1} R_3(u,s,s_1,s_2)R_4(u,s,s_1,s_2,u)ds_2ds_1 + \cdots\Big)dW_u\end{aligned}$$

and

(4.2.4)

$$Z_{tT} = P_{0T} \int_0^\infty \mathbb{E}[\sigma_s^2] ds + 2 \int_0^t \int_T^\infty \Big(R_1(u,s) R_2(u,s,u) + \int_u^s R_2(u,s,s_1) R_3(u,s,s_1,u) ds_1 + \int_u^s \int_u^{s_1} R_3(u,s,s_1,s_2) R_4(u,s,s_1,s_2,u) ds_2 ds_1 + \cdots \Big) ds dW_u.$$

The distribution form of the random variable Z_{tT} is crucial for pricing options as we will see later. The state price density is then formulated via $V_t = Z_{tt}$ in the following way:

(4.2.5)

$$V_t = P_{0t} \int_0^\infty \mathbb{E}[\sigma_s^2] ds + 2 \int_0^t \int_t^\infty \Big(R_1(u,s) R_2(u,s,u) + \int_u^s R_2(u,s,s_1) R_3(u,s,s_1,u) ds_1 + \int_u^s \int_u^{s_1} R_3(u,s,s_1,s_2) R_4(u,s,s_1,s_2,u) ds_2 ds_1 + \cdots \Big) ds dW_u.$$

Hence the discount bond may be formulated as follows: For $0 \le t \le T < \infty$,

$$P_{tT} = \frac{P_{0T} \int_0^\infty \mathbb{E}[\sigma_s^2] ds + 2 \int_0^t \int_T^\infty \big(R_1(u,s) R_2(u,s,u) + \int_u^s R_2(u,s,s_1) R_3(u,s,s_1,u) ds_1 + \cdots \big) ds dW_u}{P_{0t} \int_0^\infty \mathbb{E}[\sigma_s^2] ds + 2 \int_0^t \int_t^\infty \big(R_1(u,s) R_2(u,s,u) + \int_u^s R_2(u,s,s_1) R_3(u,s,s_1,u) ds_1 + \cdots \big) ds dW_u}$$

and the forward LIBOR rate may be formulated as follows: For $0 \le t \le T \le S < \infty$,

$$F_{tTS} = \frac{1}{S-T} \times \frac{(P_{0T} - P_{0S}) \int_0^\infty \mathbb{E}[\sigma_s^2] ds - 2 \int_0^t \int_T^S \big(R_1(u,s) R_2(u,s,u) + \int_u^s R_2(u,s,s_1) R_3(u,s,s_1,u) ds_1 + \cdots \big) ds dW_u}{P_{0S} \int_0^\infty \mathbb{E}[\sigma_s^2] ds + 2 \int_0^t \int_S^\infty \big(R_1(u,s) R_2(u,s,u) + \int_u^s R_2(u,s,s_1) R_3(u,s,s_1,u) ds_1 + \cdots \big) ds dW_u}.$$

These expressions allow us to calibrate the initial curve and options separately. However, as we observed in (4.1.3), we may also express the initial curve by the chaos coefficients in the following way: For $T \ge 0$,

$$P_{0T} = \frac{\int_T^\infty \Big[\phi_1^2(s) + \int_0^s \phi_2^2(s,s_1) ds_1 + \int_0^s \int_0^{s_1} \phi_3^2(s,s_1,s_2) ds_2 ds_1 + \cdots \Big] ds}{\int_0^\infty \Big[\phi_1^2(s) + \int_0^s \phi_2^2(s,s_1) ds_1 + \int_0^s \int_0^{s_1} \phi_3^2(s,s_1,s_2) ds_2 ds_1 + \cdots \Big] ds}. \tag{4.2.6}$$

Therefore, to save increasing the number of parameters we apply those chaos coefficients to model the initial curve, and at the same time calibrate options in later chapters.

4.3 First Chaos Model

In the First Chaos Model, that is, for $\sigma_t = \phi_1(t)$, we have that

$$R_1(t,s) = \phi_1(s) \quad \text{and} \quad R_n \equiv 0 \quad \text{for} \quad n = 2, 3, \ldots.$$

By examining the expressions (4.1.4) – (4.1.8), we obtain that for $0 \le t \le T \le S < \infty$

$$V_t = \int_t^\infty \phi_1^2(s)ds, \quad Z_{tT} = \int_T^\infty \phi_1^2(s)ds,$$

$$f_{tT} = \frac{\phi_1^2(T)}{\int_T^\infty \phi_1^2(s)ds} \quad \text{and} \quad F_{tTS} = \frac{1}{S-T}\frac{\int_T^S \phi_1^2(s)ds}{\int_S^\infty \phi_1^2(s)ds}.$$

Note here that because the process $(Z_{tT})_{0 \le t \le T < \infty}$ does not change over time $t \ge 0$, we infer that

$$Z_{tT} = Z_{0T} = V_T.$$

Moreover, we observe that the instantaneous forward rate and the forward LIBOR rate do not change over time $t \ge 0$. From the expressions (4.1.10) and (4.1.11), we obtain that

$$\lambda_t \equiv 0 \quad \text{and} \quad \hat{V}_{tT} \equiv 0 \quad \text{for any } 0 \le t \le T < \infty.$$

Thus, in light of (3.2.18), the short rate dynamics is given by

$$dr_t = \frac{\partial}{\partial T} f_{0T} dt,$$

which is deterministic, as was expected.

European call/put bond option

We now consider option pricing in the First Chaos Model to compare it with the higher order Chaos Models, although we know that the First Chaos Model gives only deterministic term structures. The deterministic term structure gives us that

$$ZBC(0,t,T,K) = \frac{1}{V_0}\mathbb{E}[V_t(P_{tT} - K)^+] = (P_{0T} - KP_{0t})^+$$

and

$$ZBP(0,t,T,K) = (KP_{0t} - P_{0T})^+.$$

Swaption

Because the First Chaos Model gives us a deterministic term structure, we find for the nominal $N = 1$ that

$$\begin{aligned} PS(0,\mathcal{T},\tau,1,K) =& \frac{1}{V_0}\mathbb{E}\Big[V_t\Big(1 - P_{tt_n} - K\sum_{i=1}^{n}\tau_i P_{tt_i}\Big)^+\Big] \\ =& \Big(P_{0t} - P_{0t_n} - K\sum_{i=1}^{n}\tau_i P_{0t_i}\Big)^+. \end{aligned}$$

Therefore when $K = K_{ATM}$ we obtain that

$$PS(0,\mathcal{T},\tau,N,K_{ATM}) = 0.$$

4.4 Second Chaos Model

Now we move to the Second Chaos Model, that is, we set $\sigma_t = \phi_1(t) + \int_0^t \phi_2(t,s_1)dW_{s_1}$. By the expressions in (4.1.4) – (4.1.8), we find in the Second Chaos Model, for $0 \le t \le T \le S < \infty$, that

$$\text{(4.4.1)} \qquad V_t = \int_t^\infty \Big(R_1^2(t,s) + \int_t^s R_2^2(t,s,s_1)ds_1\Big)ds,$$

$$Z_{tT} = \int_T^\infty \Big(R_1^2(t,s) + \int_t^s R_2^2(t,s,s_1)ds_1\Big)ds,$$

$$P_{tT} = \frac{\int_T^\infty \big(R_1^2(t,s) + \int_t^s R_2^2(t,s,s_1)ds_1\big)ds}{\int_t^\infty \big(R_1^2(t,s) + \int_t^s R_2^2(t,s,s_1)ds_1\big)ds}$$

and

$$F_{tTS} = \frac{1}{S-T}\frac{\int_T^S \big(R_1^2(t,s) + \int_t^s R_2^2(t,s,s_1)ds_1\big)ds}{\int_S^\infty \big(R_1^2(t,s) + \int_t^s R_2^2(t,s,s_1)ds_1\big)ds},$$

where

$$R_1(t,s) = \phi_1(s) + \int_0^t \phi_2(s,s_1)dWs_1, \quad R_2(t,s,s_1) = \phi_2(s,s_1).$$

Here we have that

$$M_{ts} = R_1^2(t,s) + \int_t^s R_2^2(t,s,s_1)ds_1, \quad 0 \le t \le s < \infty,$$

which corresponds to the expression in [48]. There it is stated that the random variable M_{ts} in the Second Chaos Model is a parametric family of squared Gaussians plus a constant. Now by the expressions (4.1.10) and (4.1.11) we infer in the Second Chaos Model that for $0 \le t \le T < \infty$,

$$\lambda_t = -\frac{2\int_t^\infty R_1(t,s)R_2(t,s,t)ds}{\int_t^\infty \big(R_1^2(t,s) + \int_t^s R_2^2(t,s,s_1)ds_1\big)ds} \quad \text{and} \quad \hat{V}_{tT} = \frac{2\int_T^\infty R_1(t,s)R_2(t,s,t)ds}{\int_T^\infty \big(R_1^2(t,s) + \int_t^s R_2^2(t,s,s_1)ds_1\big)ds}.$$

Recalling the definition of the variable η_{tT} from (3.2.8), we derive that

$$\eta_{tT} = \frac{2R_1(t,T)R_2(t,T,t)}{R_1^2(t,T) + \int_t^T R_2^2(t,T,s_1)ds_1}, \quad 0 \le t \le T < \infty.$$

Therefore, equation (3.2.19) allows us to form a stochastic volatility short rate dynamics in the following way:

$$dr_t = [\cdots]dt + 2\Big[\frac{R_2(t,t,t)}{R_1(t,t)} - \frac{\int_t^\infty R_1(t,s)R_2(t,s,t)ds}{\int_t^\infty \big(R_1^2(t,s) + \int_t^s R_2^2(t,s,s_1)ds_1\big)ds}\Big] r_t dW_t.$$

From equation (3.3.20), we are able to form a stochastic volatility forward LIBOR dynamics as follows:

$$dF_{tTS} = [\cdots]dt + 2\Bigg(\frac{\int_T^S R_1(t,s)R_2(t,s,t)ds}{\int_T^S \big(R_1^2(t,s) + \int_t^s R_2^2(t,s,s_1)ds_1\big)ds} - \frac{\int_S^\infty R_1(t,s)R_2(t,s,t)ds}{\int_S^\infty \big(R_1^2(t,s) + \int_t^s R_2^2(t,s,s_1)ds_1\big)ds}\Bigg) F_{tTS} dW_t.$$

Therefore we obtain a stochastic property in the volatility drift, which secure non-flat volatility curve.

4.4.1 Factorizable Second Chaos Model

In the Factorizable Second Chaos Model we simplify the Second Chaos Models as follows:

$$R_1(t,s) = \alpha_s + \int_0^t \beta_s \gamma_{s_1} dWs_1 \quad \text{and} \quad R_2(t,s,s_1) = \beta_s \gamma_{s_1},$$

for some square-integrable functions α, β, γ. Inserting this into the expression (4.4.1) we find that for $0 \le t \le T < \infty$ we have

$$Z_{tT} = \int_T^\infty \Big(\Big(\alpha_s + \beta_s \int_0^t \gamma_{s_1} dW s_1 \Big)^2 + \beta_s^2 \int_t^s \gamma_{s_1}^2 ds_1 \Big) ds.$$

To simplify the notation in what follows, define:

$$\hat{R}_t := \int_0^t \gamma_{s_1} dW s_1, \quad \hat{Q}_t := \int_0^t \gamma_{s_1}^2 ds_1,$$

$$A_t := \int_t^\infty (\alpha_s^2 + \beta_s^2 \hat{Q}_s) ds, \quad B_t := 2 \int_t^\infty \alpha_s \beta_s ds \quad \text{and} \quad C_t := \int_t^\infty \beta_s^2 ds \quad \text{for } t \ge 0.$$

We then find that for $0 \le t \le T < \infty$

$$\begin{aligned} Z_{tT} &= \int_T^\infty [(\alpha_s + \beta_s \hat{R}_t)^2 - \beta_s^2 \hat{Q}_t + \beta_s^2 \hat{Q}_s] ds \\ &= \int_T^\infty [\alpha_s^2 + \beta_s^2 \hat{Q}_s + 2\alpha_s \beta_s \hat{R}_t + \beta_s^2 (\hat{R}_t^2 - \hat{Q}_t)] ds \\ &= \int_T^\infty [\alpha_s^2 + \beta_s^2 \hat{Q}_s] ds + 2 \int_T^\infty \alpha_s \beta_s ds \hat{R}_t + \int_T^\infty \beta_s^2 ds (\hat{R}_t^2 - \hat{Q}_t) \\ &= A_T + B_T \hat{R}_t + C_T (\hat{R}_t^2 - \hat{Q}_t). \end{aligned} \tag{4.4.2}$$

Note here that because we may deduce that

$$\mathbb{E}[Z_{tT}] = A_T \quad \text{and} \quad V_0 = A_0 \quad \text{for} \quad 0 \le t \le T < \infty,$$

it follows that

$$A_T = P_{0T} A_0, \quad T \ge 0. \tag{4.4.3}$$

Hence, we find that

$$Z_{tT} = P_{0T} A_0 + B_T \hat{R}_t + C_T (\hat{R}_t^2 - \hat{Q}_t), \quad 0 \le t \le T < \infty. \tag{4.4.4}$$

Consequently, we further obtain that

$$P_{tT} = \frac{P_{0T} A_0 + B_T \hat{R}_t + C_T (\hat{R}_t^2 - \hat{Q}_t)}{P_{0t} A_0 + B_t \hat{R}_t + C_t (\hat{R}_t^2 - \hat{Q}_t)}, \quad 0 \le t \le T < \infty.$$

The initial value can be formed in the following way:

$$P_{0T} = \frac{A_T}{A_0}, \quad \text{for} \quad T \geq 0.$$

Note here that we have the following form of the risk-adjusted volatility:

$$\hat{V}_{tT} = \frac{B_T \gamma_t + 2C_T \hat{R}_t \gamma_t}{P_{0T} A_0 + B_T \hat{R}_t + C_T(\hat{R}_t^2 - \hat{Q}_t)}, \quad 0 \leq t \leq T < \infty.$$

European call/put bond option

We now recall that the option price is formulated by the expectation rule as follows:

$$ZBC(0, t, T, K) = \frac{1}{V_0} \mathbb{E}[(Z_{tT} - KZ_{tt})^+].$$

However, because we obtain from (4.4.4) that

$$Z_{tT} - KZ_{tt} = (P_{0T} - KP_{0t})A_0 + (B_T - KB_t)\hat{R}_t + (C_T - KC_t)(\hat{R}_t^2 - \hat{Q}_t),$$

we find that

$$ZBC(0, t, T, K) = \mathbb{E}\Big[\Big(P_{0T} - KP_{0t} + \frac{1}{A_0}(B_T - KB_t)\hat{R}_t + \frac{1}{A_0}(C_T - KC_t)(\hat{R}_t^2 - \hat{Q}_t)\Big)^+\Big].$$

Therefore, we may make the interpretation that the European bond option in the Factorizable Second Chaos Model is non-central chi-squared with mean $P_{0T} - KP_{0t}$. Furthermore, if we first define some notations,

$$\theta := \frac{\hat{R}_t}{\sqrt{\hat{Q}_t}} \sim \mathcal{N}(0, 1),$$

$$\hat{A} := (P_{0T} - KP_{0t}) - \frac{1}{A_0}(C_T - KC_t)\hat{Q}_t, \quad \hat{B} := \frac{1}{A_0}(B_T - KB_t)\sqrt{\hat{Q}_t}, \quad \hat{C} := \frac{1}{A_0}(C_T - KC_t)\hat{Q}_t,$$

we obtain that

$$ZBC(0, t, T, K) = \mathbb{E}\Big[\Big(\hat{A} + \hat{B}\theta + \hat{C}\theta^2\Big)^+\Big].$$

Applying the expectation rule, the option price is expressed as follows:

$$ZBC(0,t,T,K) = \frac{1}{\sqrt{2\pi}} \int_{\mathcal{P}_c(\theta)\geq 0} \mathcal{P}_c(\theta) e^{-\frac{\theta^2}{2}} d\theta,$$

where

$$\mathcal{P}_c(\theta) := \hat{A} + \hat{B}\theta + \hat{C}\theta^2 \quad \text{and} \quad \mathbb{E}[\mathcal{P}_c(\theta)] = P_{0T} - KP_{0t}.$$

For put options, letting $\mathcal{P}_p(\theta) = -\mathcal{P}_c(\theta)$, it follows that

$$ZBP(0,t,T,K) = \frac{1}{\sqrt{2\pi}} \int_{\mathcal{P}_p(\theta)\geq 0} \mathcal{P}_p(\theta) e^{-\frac{\theta^2}{2}} d\theta.$$

Let Φ be the standard normal cumulative distribution function and ρ be the standard normal density function, i.e.,

$$\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{X^2}{2}} dX \quad \text{and} \quad \rho(x) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

Then, we know that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} X e^{-\frac{X^2}{2}} dX = -\rho(x)$$

and

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} X^2 e^{-\frac{X^2}{2}} dX = \Phi(x) - x\rho(x).$$

If $\hat{C} = 0$ and $\hat{B} > 0$, it follows that

$$ZBC(0,t,T,K) = \frac{1}{\sqrt{2\pi}} \int_{y}^{\infty} (\hat{A} + \hat{B}\theta) e^{-\frac{\theta^2}{2}} d\theta = \hat{A}\Phi(-y) + \hat{B}\rho(y)$$

where $y = -\frac{\hat{A}}{\hat{B}}$. If $\hat{C} = 0$ and $\hat{B} < 0$, however, we instead obtain the expression

$$ZBC(0,t,T,K) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} (\hat{A} + \hat{B}\theta) e^{-\frac{\theta^2}{2}} d\theta = \hat{A}\Phi(y) - \hat{B}\rho(y).$$

Now, if $\hat{C} > 0$ and $\Delta \leq 0$ where $\Delta := \hat{B}^2 - 4\hat{A}\hat{C}$, then

$$\begin{aligned} ZBC(0,t,T,K) =& \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\hat{A} + \hat{B}\theta + \hat{C}\theta^2) e^{-\frac{\theta^2}{2}} d\theta \\ =& \mathbb{E}[\hat{A} + \hat{B}\theta + \hat{C}\theta^2] \\ =& P_{0T} - KP_{0t}. \end{aligned}$$

If $\hat{C} < 0$ and $\Delta \le 0$, then $ZBC(0,t,T,K) = 0$. It remains to consider the case where $\Delta > 0$. In this case, we obtain two roots:

$$z_1 := \frac{-\hat{B} - \sqrt{\Delta}}{2\hat{C}} \quad \text{and} \quad z_2 := \frac{-\hat{B} + \sqrt{\Delta}}{2\hat{C}}.$$

There are two further options to consider. Firstly, suppose that $\hat{C} > 0$ and $\Delta > 0$. Then $z_1 < z_2$ and

$$\begin{aligned} ZBC(0,t,T,K) =& \frac{1}{\sqrt{2\pi}} \int_{\{-\infty \le \theta \le z_1\} \cap \{z_2 \le \theta \le \infty\}} (\hat{A} + \hat{B}\theta + \hat{C}\theta^2) e^{-\frac{\theta^2}{2}} d\theta \\ =& (\hat{A} + \hat{C})(\Phi(z_1) + \Phi(-z_2)) - (\hat{B} + \hat{C} z_1)\rho(z_1) + (\hat{B} + \hat{C} z_2)\rho(z_2) \\ =& (P_{0T} - K P_{0t})(\Phi(z_1) + \Phi(-z_2)) - \frac{1}{2}(\hat{B} - \sqrt{\Delta})\rho(z_1) + \frac{1}{2}(\hat{B} + \sqrt{\Delta})\rho(z_2). \end{aligned}$$

Secondly, if $\hat{C} < 0$ and $\Delta > 0$, it follows that $z_1 > z_2$ and so

$$\begin{aligned} ZBC(0,t,T,K) =& \frac{1}{\sqrt{2\pi}} \int_{\{z_2 \le \theta \le z_1\}} (\hat{A} + \hat{B}\theta + \hat{C}\theta^2) e^{-\frac{\theta^2}{2}} d\theta \\ =& (P_{0T} - K P_{0t})(\Phi(z_1) - \Phi(z_2)) - \frac{1}{2}(\hat{B} - \sqrt{\Delta})\rho(z_1) + \frac{1}{2}(\hat{B} + \sqrt{\Delta})\rho(z_2) \end{aligned}$$

Swaption

Plugging the expression of the variable Z_{tT} into the pricing formula, we obtain that

$$\begin{aligned} Z_{tt} - Z_{tT_b} - K \sum_{i=a+1}^{b} \tau_i Z_{tT_i} =& \Big(A_t - A_{T_b} - K \sum_{i=a+1}^{b} \tau_i A_{T_i}\Big) - \Big(C_t - C_{T_b} - K \sum_{i=a+1}^{b} \tau_i C_{T_i}\Big)\hat{Q}_t \\ &+ \Big(B_t - B_{T_b} - K \sum_{i=a+1}^{b} \tau_i B_{T_i}\Big)\hat{R}_t - \Big(C_t - C_{T_b} - K \sum_{i=a+1}^{b} \tau_i C_{T_i}\Big)\hat{R}_t^2. \end{aligned}$$

Therefore, we find that the payer swaption price with the nominal $N = 1$ has the following form:

$$PS(0, \mathcal{T}, \tau, 1, K) = \mathbb{E}[(\widetilde{A} + \widetilde{B}\theta + \widetilde{C}\theta^2)^+],$$

where

$$\theta = \frac{\hat{R}_t}{\sqrt{\hat{Q}_t}} \sim \mathcal{N}(0,1), \quad \widetilde{A} = \Big(P_{0t} - P_{0T_b} - K \sum_{i=a+1}^{b} \tau_i P_{0T_i}\Big) - \frac{1}{A_0}\Big(C_t - C_{T_b} - K \sum_{i=a+1}^{b} \tau_i C_{T_i}\Big)\hat{Q}_t,$$

$$\widetilde{B} = \frac{1}{A_0}\Big(B_t - B_{T_b} - K\sum_{i=a+1}^{b}\tau_i B_{T_i}\Big)\sqrt{\hat{Q}_t} \quad \text{and} \quad \widetilde{C} = \frac{1}{A_0}\Big(C_t - C_{T_b} - K\sum_{i=a+1}^{b}\tau_i C_{T_i}\Big)\hat{Q}_t,$$

Therefore, the payer swaption under the Factorizable Second Chaos Model is also non-central chi-squared distributed. It has mean

$$\mathbb{E}[\widetilde{A} + \widetilde{B}\theta + \widetilde{C}\theta^2] = P_{0t} - P_{0T_b} - K\sum_{i=a+1}^{b}\tau_i P_{0T_i}.$$

Applying the expectation rule, we find that

$$PS(0, \mathcal{T}, \tau, 1, K) = \frac{1}{\sqrt{2\pi}}\int_{\mathcal{P}_{PS}(\theta)\geq 0}\mathcal{P}_{PS}(\theta)e^{-\frac{\theta^2}{2}}d\theta,$$

where we have defined that

$$\mathcal{P}_{PS}(\theta) := \widetilde{A} + \widetilde{B}\theta + \widetilde{C}\theta^2.$$

Further simplification can be achieved by considering the roots of the function $\mathcal{P}_{PS}$ and dividing into six scenarios as we have done for pricing a European call bond option:

$$PS(0, \mathcal{T}, \tau, 1, K) = \begin{cases} \widetilde{A}\Phi(-\widetilde{y}) + \widetilde{B}\rho(\widetilde{y}) \quad \text{if} \quad \{\widetilde{C} = 0\} \cap \{\widetilde{B} > 0\}, \\ \widetilde{A}\Phi(\widetilde{y}) - \widetilde{B}\rho(\widetilde{y}) \quad \text{if} \quad \{\widetilde{C} = 0\} \cap \{\widetilde{B} < 0\}, \\ \big(P_{0t} - P_{0T_N} - K\sum_{i=a+1}^{b}\tau_i P_{0T_i}\big) \quad \text{if} \quad \{\widetilde{C} > 0\} \cap \{\widetilde{\Delta} \leq 0\}, \\ 0 \quad \text{if} \quad \{\widetilde{C} < 0\} \cap \{\widetilde{\Delta} \leq 0\}, \\ \big(P_{0t} - P_{0T_b} - K\sum_{i=a+1}^{b}\tau_i P_{0T_i}\big)(\Phi(z_1) + \Phi(-z_2)) \\ -\frac{1}{2}(\widetilde{B} - \sqrt{\widetilde{\Delta}})\rho(z_1) + \frac{1}{2}(\widetilde{B} + \sqrt{\widetilde{\Delta}})\rho(z_2) \quad \text{if} \quad \{\widetilde{C} > 0\} \cap \{\widetilde{\Delta} > 0\}, \\ \big(P_{0t} - P_{0T_b} - K\sum_{i=a+1}^{b}\tau_i P_{0T_i}\big)(\Phi(z_1) - \Phi(z_2)) \\ -\frac{1}{2}(\widetilde{B} - \sqrt{\widetilde{\Delta}})\rho(z_1) + \frac{1}{2}(\widetilde{B} + \sqrt{\widetilde{\Delta}})\rho(z_2) \quad \text{if} \quad \{\widetilde{C} < 0\} \cap \{\widetilde{\Delta} > 0\}, \end{cases}$$

where

$$\widetilde{y} := -\frac{\widetilde{A}}{\widetilde{B}}, \quad \widetilde{\Delta} := \widetilde{B}^2 - 4\widetilde{A}\widetilde{C}$$

and z_1 and z_2 are the two roots of $\mathcal{P}_{PS}$, given by

$$z_1 = \frac{-\widetilde{B} - \sqrt{\widetilde{\Delta}}}{2\widetilde{C}} \quad \text{and} \quad z_2 = \frac{-\widetilde{B} + \sqrt{\widetilde{\Delta}}}{2\widetilde{C}}.$$

It follows that for the ATM Swaptions we have

$$PS(0,\mathcal{T},\tau,1,K_{ATM}) = \begin{cases} \widetilde{A}\Phi(-\widetilde{y}) + \widetilde{B}\rho(\widetilde{y}) \quad \text{if} \quad \{\widetilde{C}=0\}\cap\{\widetilde{B}>0\}, \\ \widetilde{A}\Phi(\widetilde{y}) - \widetilde{B}\rho(\widetilde{y}) \quad \text{if} \quad \{\widetilde{C}=0\}\cap\{\widetilde{B}<0\}, \\ 0 \quad \text{if} \quad \{\widetilde{C}\neq 0\}\cap\{\widetilde{\Delta}\leq 0\}, \\ -\frac{1}{2}(\widetilde{B}-\sqrt{\widetilde{\Delta}})\rho(z_1) + \frac{1}{2}(\widetilde{B}+\sqrt{\widetilde{\Delta}})\rho(z_2) \quad \text{if} \quad \{\widetilde{C}\neq 0\}\cap\{\widetilde{\Delta}>0\}. \end{cases}$$

4.5 Third Chaos Model

In the Third Chaos Model we have

$$M_{ts} = R_1^2(t,s) + \int_t^s R_2^2(t,s,s_1)ds_1 + \int_t^s \int_t^{s_1} R_3^2(t,s,s_1,s_2)ds_2ds_1, \quad 0 \leq t \leq s < \infty$$

where

$$R_1(t,s) = \phi_1(s) + \int_0^t \phi_2(s,s_1)dWs_1 + \int_0^t \int_0^{s_1} \phi_3(s,s_1,s_2)dWs_2dWs_1,$$

$$R_2(t,s,s_1) = \phi_2(s,s_1) + \int_0^t \phi_3(s,s_1,s_2)dWs_2 \quad \text{and} \quad R_3(t,s,s_1,s_2) = \phi_3(s,s_1,s_2).$$

Then, we obtain stochastic forms for $0 \leq t \leq T < \infty$:

$$V_t = \int_t^\infty \Big(R_1^2(t,s) + \int_t^s R_2^2(t,s,s_1)ds_1 + \int_t^s \int_t^{s_1} R_3^2(t,s,s_1,s_2)ds_2ds_1\Big)ds,$$

$$Z_{tT} = \int_T^\infty \Big(R_1^2(t,s) + \int_t^s R_2^2(t,s,s_1)ds_1 + \int_t^s \int_t^{s_1} R_3^2(t,s,s_1,s_2)ds_2ds_1\Big)ds,$$

$$\lambda_t = -\frac{2\int_t^\infty (R_1(t,s)R_2(t,s,t) + \int_t^s R_2(t,s,s_1)R_3(t,s,s_1,t)ds_1)ds}{\int_t^\infty (R_1^2(t,s) + \int_t^s R_2^2(t,s,s_1)ds_1 + \int_t^s \int_t^{s_1} R_3^2(t,s,s_1,s_2)ds_2ds_1)ds}$$

and

$$\hat{V}_{tT} = \frac{2\int_T^\infty [R_1(t,s)R_2(t,s,t) + \int_t^s R_2(t,s,s_1)R_3(t,s,s_1,t)ds_1)ds}{\int_T^\infty (R_1^2(t,s) + \int_t^s R_2^2(t,s,s_1)ds_1 + \int_t^s \int_t^{s_1} R_3^2(t,s,s_1,s_2)ds_2ds_1)ds}.$$

The dynamics (3.2.19) and (3.3.20) give us that

$$\begin{aligned} dr_t =& [\cdots]dt \\ &+ 2\Big[\frac{R_2(t,t,t)}{R_1(t,t)} - \frac{2\int_t^\infty [R_1(t,s)R_2(t,s,t) + \int_t^s R_2(t,s,s_1)R_3(t,s,s_1,t)ds_1]ds}{\int_t^\infty [R_1^2(t,s) + \int_t^s R_2^2(t,s,s_1)ds_1 + \int_t^s \int_t^{s_1} R_3^2(t,s,s_1,s_2)ds_2ds_1]ds}\Big] r_t dW_t \end{aligned}$$

and

$$
\begin{aligned}
dF_{tTS} =&[\cdots]dt\\
&+2\Big(\frac{\int_T^S[R_1(t,s)R_2(t,s,t)+\int_t^s R_2(t,s,s_1)R_3(t,s,s_1,t)ds_1]ds}{\int_T^S[R_1^2(t,s)+\int_t^s R_2^2(t,s,s_1)ds_1+\int_t^s\int_t^{s_1} R_3^2(t,s,s_1,s_2)ds_2ds_1]ds}\\
&-\frac{\int_S^\infty[R_1(t,s)R_2(t,s,t)+\int_t^s R_2(t,s,s_1)R_3(t,s,s_1,t)ds_1]ds}{\int_S^\infty[R_1^2(t,s)+\int_t^s R_2^2(t,s,s_1)ds_1+\int_t^s\int_t^{s_1} R_3^2(t,s,s_1,s_2)ds_2ds_1]ds}\Big)F_{tTS}dW_t.
\end{aligned}
$$

4.5.1 Factorizable Third Chaos Model

We consider the Factorizable Third Chaos Model, i.e.,

$$
M_{ts}=R_1^2(t,s)+\int_t^s R_2^2(t,s,s_1)ds_1+\int_t^s\int_t^{s_1} R_3^2(t,s,s_1,s_2)ds_2ds_1
$$

where

$$
R_1(t,s)=\alpha_s+\beta_sJ_1(t,\gamma)+\delta_sJ_2(t,\epsilon\zeta),\quad R_2(t,s,s_1)=\beta_s\gamma_{s_1}+\delta_s\epsilon_{s_1}J_1(t,\zeta),
$$

$$
R_3(t,s,s_1,s_2)=\delta_s\epsilon_{s_1}\zeta_{s_2},
$$

$$
J_1(t,\gamma):=\int_0^t\gamma_{s_1}dWs_1\quad\text{and}\quad J_2(t,\epsilon\zeta):=\int_0^t\int_0^{s_1}\epsilon_{s_1}\zeta_{s_2}dWs_2dWs_1,
$$

for some square-integrable functions $\alpha,\beta,\gamma,\delta,\epsilon$ and ζ. It follows that for $0\le t\le s<\infty$, we have that

$$
\begin{aligned}
M_{ts}=&\alpha_s^2+\beta_s^2J_1^2(t,\gamma)+\delta_s^2J_2^2(t,\epsilon\zeta)+2\alpha_s\beta_sJ_1(t,\gamma)+2\alpha_s\delta_sJ_2(t,\epsilon\zeta)+2\beta_s\delta_sJ_1(t,\gamma)J_2(t,\epsilon\zeta)\\
&+\int_t^s\Big(\beta_s^2\gamma_{s_1}^2+\delta_s^2\epsilon_{s_1}^2J_1^2(t,\zeta)+2\beta_s\delta_s\gamma_{s_1}\epsilon_{s_1}J_1(t,\zeta)\Big)ds_1+\int_t^s\int_t^{s_1}\delta_s^2\epsilon_{s_1}^2\zeta_{s_2}^2ds_2ds_1\\
=&\alpha_s^2+\int_t^s\beta_s^2\gamma_{s_1}^2ds_1+\int_t^s\int_t^{s_1}\delta_s^2\epsilon_{s_1}^2\zeta_{s_2}^2ds_2ds_1\\
&+2\alpha_s\beta_sJ_1(t,\gamma)+\int_t^s 2\beta_s\delta_s\gamma_{s_1}\epsilon_{s_1}ds_1J_1(t,\zeta)+2\alpha_s\delta_sJ_2(t,\epsilon\zeta)\\
&+\beta_s^2J_1^2(t,\gamma)+\int_t^s\delta_s^2\epsilon_{s_1}^2ds_1J_1^2(t,\zeta)+2\beta_s\delta_sJ_1(t,\gamma)J_2(t,\epsilon\zeta)+\delta_s^2J_2^2(t,\epsilon\zeta).
\end{aligned}
$$

Because we have that for $0\le t\le s<\infty$,

$$
\int_t^s\int_t^{s_1}\epsilon_{s_1}^2\zeta_{s_2}^2ds_2ds_1=\int_0^s\int_0^{s_1}\epsilon_{s_1}^2\zeta_{s_2}^2ds_2ds_1-\int_0^t\int_0^{s_1}\epsilon_{s_1}^2\zeta_{s_2}^2ds_2ds_1-\int_t^s\int_0^t\epsilon_{s_1}^2\zeta_{s_2}^2ds_2ds_1,
$$

we infer that

$$
\begin{aligned}
M_{ts} =& \alpha_s^2 + \int_0^s \beta_s^2 \gamma_{s_1}^2 ds_1 + \int_0^s \int_0^{s_1} \delta_s^2 \epsilon_{s_1}^2 \zeta_{s_2}^2 ds_2 ds_1 \\
&+ 2\alpha_s \beta_s J_1(t,\gamma) + 2\int_t^s \beta_s \delta_s \gamma_{s_1} \epsilon_{s_1} ds_1 J_1(t,\zeta) + 2\alpha_s \delta_s J_2(t,\epsilon\zeta) \\
&+ \beta_s^2 \Big(J_1^2(t,\gamma) - \int_0^t \gamma_{s_1}^2 ds_1 \Big) + \int_t^s \delta_s^2 \epsilon_{s_1}^2 ds_1 \Big(J_1^2(t,\zeta) - \int_0^t \zeta_{s_2}^2 ds_2 \Big) \\
&+ 2\beta_s \delta_s J_1(t,\gamma) J_2(t,\epsilon\zeta) + \delta_s^2 \Big(J_2^2(t,\epsilon\zeta) - \int_0^t \int_0^{s_1} \epsilon_{s_1}^2 \zeta_{s_2}^2 ds_2 ds_1 \Big).
\end{aligned}
$$

Therefore, we obtain that for $0 \le t \le T < \infty$,

(4.5.1)

$$
\begin{aligned}
Z_{tT} =& A_T + B_T J_1(t,\gamma) + C_{tT} J_1(t,\zeta) + D_T J_2(t,\epsilon\zeta) + E_T \big(J_1^2(t,\gamma) - Q_1(t,\gamma) \big) \\
&+ F_{tT} \big(J_1^2(t,\zeta) - Q_1(t,\zeta) \big) + G_T J_1(t,\gamma) J_2(t,\epsilon\zeta) + H_T \big(J_2^2(t,\epsilon\zeta) - Q_2(t,\epsilon\zeta) \big),
\end{aligned}
$$

where we have defined

$$
\begin{aligned}
A_T &:= \int_T^\infty \Big(\alpha_s^2 + \int_0^s \beta_s^2 \gamma_{s_1}^2 ds_1 + \int_0^s \int_0^{s_1} \delta_s^2 \epsilon_{s_1}^2 \zeta_{s_2}^2 ds_2 ds_1 \Big) ds, \\
B_T &:= 2\int_T^\infty \alpha_s \beta_s ds, \quad C_{tT} := 2\int_T^\infty \int_t^s \beta_s \delta_s \gamma_{s_1} \epsilon_{s_1} ds_1 ds, \\
D_T &:= 2\int_T^\infty \alpha_s \delta_s ds, \quad E_T := \int_T^\infty \beta_s^2 ds, \qquad (4.5.2) \\
F_{tT} &:= \int_T^\infty \int_t^s \delta_s^2 \epsilon_{s_1}^2 ds_1 ds, \quad G_T := 2\int_T^\infty \beta_s \delta_s ds, \quad H_T := \int_T^\infty \delta_s^2 ds, \\
Q_1(t,\gamma) &:= \int_0^t \gamma_{s_1}^2 ds_1 \quad \text{and} \quad Q_2(t,\epsilon\zeta) := \int_0^t \int_0^{s_1} \epsilon_{s_1}^2 \zeta_{s_2}^2 ds_2 ds_1.
\end{aligned}
$$

Note here that because we may deduce that

$$
\mathbb{E}[Z_{tT}] = A_T \quad \text{and} \quad V_0 = A_0 \quad \text{for} \quad 0 \le t \le T < \infty,
$$

it follows that

$$
A_T = P_{0T} A_0, \quad T \ge 0. \tag{4.5.3}
$$

Therefore we obtain that

(4.5.4)

$$
\begin{aligned}
Z_{tT} =& P_{0T} A_0 + B_T J_1(t,\gamma) + C_{tT} J_1(t,\zeta) + D_T J_2(t,\epsilon\zeta) + E_T \big(J_1^2(t,\gamma) - Q_1(t,\gamma) \big) \\
&+ F_{tT} \big(J_1^2(t,\zeta) - Q_1(t,\zeta) \big) + G_T J_1(t,\gamma) J_2(t,\epsilon\zeta) + H_T \big(J_2^2(t,\epsilon\zeta) - Q_2(t,\epsilon\zeta) \big)
\end{aligned}
$$

and

$$P_{0T} = \frac{A_T}{A_0}, \quad T \geq 0.$$

European call/put option

From (4.5.4), we obtain for the European call bond option that

$$\begin{aligned} Z_{tT} - KZ_{tt} =& (P_{0T} - KP_{0t})A_0 + [B_T - KB_t]J_1(t,\gamma) + [C_{tT} - KC_{tt}]J_1(t,\zeta) \\ &+ [D_T - KD_t]J_2(t,\epsilon\zeta) + [E_T - KE_t]\big(J_1^2(t,\gamma) - Q_1(t,\gamma)\big) \\ &+ [F_{tT} - KF_{tt}]\big(J_1^2(t,\zeta) - Q_1(t,\zeta)\big) \\ &+ [G_T - KG_t]J_1(t,\gamma)J_2(t,\epsilon\zeta) + [H_T - KH_t]\big(J_2^2(t,\epsilon\zeta) - Q_2(t,\epsilon\zeta)\big). \end{aligned}$$

Here we know from [60] (page 183), that

$$J_2(t,\epsilon\zeta) = \frac{1}{2}\Big(J_1(t,\epsilon)J_1(t,\zeta) - L(t,\epsilon\zeta)\Big), \quad \text{where} \quad L(t,\epsilon\zeta) := \int_0^t \epsilon_{s_1}\zeta_{s_1} ds_1.$$

Defining a standard normally distributed random variable θ by setting

$$\theta(\gamma) := \frac{J_1(t,\gamma)}{\sqrt{L(t,\gamma^2)}} \sim \mathcal{N}(0,1),$$

we have that

$$J_2(t,\epsilon\zeta) = \frac{1}{2}\Big(\sqrt{L(t,\epsilon^2)}\sqrt{L(t,\zeta^2)}\theta(\epsilon)\theta(\zeta) - L(t,\epsilon\zeta)\Big).$$

Therefore, a function for the call option defined by

$$\mathcal{P}_c(\theta(\gamma),\theta(\epsilon),\theta(\zeta)) := \frac{1}{A_0}(Z_{tT} - KZ_{tt}),$$

can be expressed as

$$
\begin{aligned}
&\mathcal{P}_c(\theta(\gamma),\theta(\epsilon),\theta(\zeta))\\
=&P_{0T}-KP_{0t}-\frac{1}{A_0}[E_T-KE_t]Q_1(t,\gamma)\\
&-\frac{1}{A_0}[F_{tT}-KF_{tt}]Q_1(t,\zeta)-\frac{1}{A_0}[H_T-KH_t]Q_2(t,\epsilon\zeta)\\
&+\frac{1}{A_0}[B_T-KB_t]\sqrt{L(t,\gamma^2)}\theta(\gamma)+\frac{1}{A_0}[C_{tT}-KC_{tt}]\sqrt{L(t,\zeta^2)}\theta(\zeta)\\
&+\frac{1}{2A_0}[D_T-KD_t]\Big(\sqrt{L(t,\epsilon^2)}\sqrt{L(t,\zeta^2)}\theta(\epsilon)\theta(\zeta)-L(t,\epsilon\zeta)\Big)\\
&+\frac{1}{A_0}[E_T-KE_t]L(t,\gamma^2)\theta^2(\gamma)+\frac{1}{A_0}[F_{tT}-KF_{tt}]L(t,\zeta^2)\theta^2(\zeta)\\
&+\frac{1}{2A_0}[G_T-KG_t]\sqrt{L(t,\gamma^2)}\theta(\gamma)\Big(\sqrt{L(t,\epsilon^2)}\sqrt{L(t,\zeta^2)}\theta(\epsilon)\theta(\zeta)-L(t,\epsilon\zeta)\Big)\\
&+\frac{1}{4A_0}[H_T-KH_t]\Big(\sqrt{L(t,\epsilon^2)}\sqrt{L(t,\zeta^2)}\theta(\epsilon)\theta(\zeta)-L(t,\epsilon\zeta)\Big)^2,
\end{aligned}
$$

where we observe that

$$
\mathbb{E}[\mathcal{P}_c(\theta(\gamma),\theta(\epsilon),\theta(\zeta))]=P_{0T}-KP_{0t}.
$$

It follows that

$$
\begin{aligned}
&\mathcal{P}_c(\theta(\gamma), \theta(\epsilon), \theta(\zeta)) \\
=&[P_{0T} - KP_{0t}] - \frac{1}{A_0}[E_T - KE_t]Q_1(t, \gamma) \\
&- \frac{1}{A_0}[F_{tT} - KF_{tt}]Q_1(t, \zeta) - \frac{1}{A_0}[H_T - KH_t]Q_2(t, \epsilon\zeta) \\
&- \frac{1}{2A_0}[D_T - KD_t]L(t, \epsilon\zeta) + \frac{1}{4A_0}[H_T - KH_t]L_1^2(t, \epsilon\zeta) \\
&+ \frac{1}{A_0}\Big([B_T - KB(t)]\sqrt{L(t, \gamma^2)} - \frac{1}{2}[G_T - KG_t]\sqrt{L(t, \gamma^2)}L(t, \epsilon\zeta)\Big)\theta(\gamma) \\
&+ \frac{1}{A_0}[C_{tT} - KC_{tt}]\sqrt{L(t, \zeta^2)}\theta(\zeta) \\
&+ \frac{1}{2A_0}\Big([D_T - KD_t] - [H_T - KH_t]L(t, \epsilon\zeta)\Big)\sqrt{L(t, \epsilon^2)}\sqrt{L(t, \zeta^2)}\theta(\epsilon)\theta(\zeta) \\
&+ \frac{1}{A_0}[E_T - KE(t)]L(t, \gamma^2)\theta^2(\gamma) + \frac{1}{A_0}[F_{tT} - KF_{tt}]L(t, \zeta^2)\theta^2(\zeta) \\
&+ \frac{1}{2A_0}[G_T - KG_t]\sqrt{L(t, \gamma^2)}\sqrt{L(t, \epsilon^2)}\sqrt{L(t, \zeta^2)}\theta(\gamma)\theta(\epsilon)\theta(\zeta) \\
&+ \frac{1}{4A_0}[H_T - KH_t]L(t, \epsilon^2)L(t, \zeta^2)\theta^2(\epsilon)\theta^2(\zeta).
\end{aligned}
$$

Denoting the coefficients of the function $\mathcal{P}_c$ respectively by $\{A, B, C, D, E, F, G, H\} \in \mathbb{R}^8$, we may write this in a more convenient fashion as

$$\mathcal{P}_c(x_1, x_2, x_3) = A + Bx_1 + Cx_3 + Dx_2x_3 + Ex_1^2 + Fx_3^2 + Gx_1x_2x_3 + Hx_2^2x_3^2.$$

From this, it follows that

$$ZBC(0, t, T, K) = \int\int\int_{\mathcal{P}_c(x_1,x_2,x_3)\geq 0} \mathcal{P}_c(x_1, x_2, x_3)\mathcal{D}_3(x_1, x_2, x_3)dx_3dx_2dx_1,$$

where $\mathcal{D}_3$ denotes the probability density of the trivariate standard normal distribution, that is,

$$\mathcal{D}_3(x_1, x_2, x_3) := \frac{\exp[w_1/2w_2]}{(2\pi)^{3/2}\sqrt{w_2}},$$

where

$$w_1 = x_1^2(\rho_{23}^2-1)+x_2^2(\rho_{13}^2-1)+x_3^2(\rho_{12}^2-1)+2[x_1x_2(\rho_{12}-\rho_{13}\rho_{23})+x_1x_3(\rho_{13}-\rho_{12}\rho_{23})+x_2x_3(\rho_{23}-\rho_{12}\rho_{13})]$$

and $w_2 = 1 - (\rho_{12}^2 + \rho_{13}^2 + \rho_{23}^2) + 2\rho_{12}\rho_{13}\rho_{23}$. For put options, letting $\mathcal{P}_p = -\mathcal{P}_c$, it follows that

$$ZBP(0,t,T,K) = \int\int\int_{\mathcal{P}_p(x_1,x_2,x_3)\geq 0} \mathcal{P}_p(x_1,x_2,x_3)\mathcal{D}_3(x_1,x_2,x_3)dx_3dx_2dx_1.$$

Swaption

We are able to express the Swaption pricing formula in the following way:

$$\begin{aligned}
&Z_{tt} - Z_{tT_b} - K\sum_{i=a+1}^{b}\tau_i Z_{tT_i}\\
=&\Big(A_t - A_{T_b} - K\sum_{i=a+1}^{b}\tau_i A_{T_i}\Big) + \Big(B_t - B_{t_n} - K\sum_{i=a+1}^{b}\tau_i B_{T_i}\Big)J_1(t,\gamma)\\
&+\Big(C_{tt} - C_{tT_b} - K\sum_{i=a+1}^{b}\tau_i C_{tT_i}\Big)J_1(t,\zeta) + \Big(D_t - D_{T_b} - K\sum_{i=a+1}^{b}\tau_i D_{T_i}\Big)J_2(t,\epsilon\zeta)\\
&+\Big(E_t - E_{T_b} - K\sum_{i=a+1}^{b}\tau_i E_{T_i}\Big)\Big(J_1^2(t,\gamma) - Q_1(t,\gamma)\Big)\\
&+\Big(F_{tt} - F_{tT_b} - K\sum_{i=a+1}^{b}\tau_i F_{tT_i}\Big)\Big(J_1^2(t,\zeta) - Q_1(t,\zeta)\Big)\\
&+\Big(G_t - G_{T_b} - K\sum_{i=a+1}^{b}\tau_i G_{T_i}\Big)J_1(t,\gamma)J_2(t,\epsilon\zeta)\\
&+\Big(H_t - H_{T_b} - K\sum_{i=a+1}^{b}\tau_i H_{T_i}\Big)\Big(J_2^2(t,\epsilon\zeta) - Q_2(t,\epsilon\zeta)\Big).
\end{aligned}$$

Therefore, a function for the European payer Swaption defined by

$$\mathcal{P}_{PS}(\theta(\gamma),\theta(\epsilon),\theta(\zeta)) := Z_{tt} - Z_{tT_b} - K\sum_{i=a+1}^{b}\tau_i Z_{tT_i},$$

can be expressed as

$$
\begin{aligned}
&\mathcal{P}_{PS}(\theta(\gamma),\theta(\epsilon),\theta(\zeta))\\
=&\Big(P_{0t}-P_{0T_b}-K\sum_{i=a+1}^{b}\tau_i P_{0T_i}\Big)-\frac{1}{A_0}\Big(E_t-E_{T_b}-K\sum_{i=a+1}^{b}\tau_i E_{T_i}\Big)Q_1(t,\gamma)\\
&-\frac{1}{A_0}\Big(F_{tt}-F_{tT_b}-K\sum_{i=a+1}^{b}\tau_i F_{tT_i}\Big)Q_1(t,\zeta)-\frac{1}{A_0}\Big(H_t-H_{T_b}-K\sum_{i=a+1}^{b}\tau_i H_{T_i}\Big)Q_2(t,\epsilon\zeta)\\
&-\frac{1}{2A_0}\Big(D_1(t)-D_{T_b}-K\sum_{i=a+1}^{b}\tau_i D_{T_i}\Big)L(t,\epsilon\zeta)+\frac{1}{4A_0}\Big(H_1(t)-H_{T_b}-K\sum_{i=a+1}^{b}\tau_i H_{T_i}\Big)L_1^2(t,\epsilon\zeta)\\
&+\frac{1}{A_0}\Big[\Big(B_t-B_{T_b}-K\sum_{i=a+1}^{b}\tau_i B_{T_i}\Big)-\frac{1}{2}\Big(G_t-G_{T_b}-K\sum_{i=a+1}^{b}\tau_i G_{T_i}\Big)L(t,\epsilon\zeta)\Big]\sqrt{L(t,\gamma^2)}\theta(\gamma)\\
&+\frac{1}{A_0}\Big(C_{tt}-C_{tT_b}-K\sum_{i=a+1}^{b}\tau_i C_{tT_i}\Big)\sqrt{L(t,\zeta^2)}\theta(\zeta)+\frac{1}{2A_0}\Big[\Big(D_t-D_{T_b}-K\sum_{i=1}^{n}\tau_i D_1(T_i)\Big)\\
&-\Big(H_t-H_{T_b}-K\sum_{i=a+1}^{b}\tau_i H_{T_i}\Big)L(t,\epsilon\zeta)\Big]\sqrt{L(t,\epsilon^2)}\sqrt{L(t,\zeta^2)}\theta(\epsilon)\theta(\zeta)\\
&+\frac{1}{A_0}\Big(E_t-E_{T_b}-K\sum_{i=a+1}^{b}\tau_i E_{T_i}\Big)L(t,\gamma^2)\theta^2(\gamma)+\frac{1}{A_0}\Big(F_{tt}-F_{tT_b}-K\sum_{i=a+1}^{b}\tau_i F_{tT_i}\Big)L(t,\zeta^2)\theta^2(\zeta)\\
&+\frac{1}{2A_0}\Big(G_t-G_{T_b}-K\sum_{i=a+1}^{b}\tau_i G_{T_i}\Big)\sqrt{L(t,\gamma^2)}\sqrt{L(t,\epsilon^2)}\sqrt{L(t,\zeta^2)}\theta(\gamma)\theta(\epsilon)\theta(\zeta)\\
&+\frac{1}{4A_0}\Big(H_t-H_{T_b}-K\sum_{i=a+1}^{b}\tau_i H_{T_i}\Big)L(t,\epsilon^2)L(t,\zeta^2)\theta^2(\epsilon)\theta^2(\zeta),
\end{aligned}
$$

where we observe that

$$\mathbb{E}[\mathcal{P}_{PS}(\theta(\gamma),\theta(\epsilon),\theta(\zeta))]=P_{0t}-P_{0T_b}-K\sum_{i=a+1}^{b}\tau_i P_{0T_i}.$$

Similarly, denoting the coefficients of the function $\mathcal{P}_c$ respectively by $\{A^*,B^*,C^*,D^*,E^*,F^*,G^*,H^*\}\in\mathbb{R}^8$, we write this as

$$\mathcal{P}_{PS}(x_1,x_2,x_3)=A^*+B^*x_1+C^*x_3+D^*x_2x_3+E^*x_1^2+F^*x_3^2+G^*x_1x_2x_3+H^*x_2^2x_3^2,$$

which is the same expression as the European bond call option, only with different coefficients. Applying the expectation rule, we express the initial price of the European payer Swaption with the nominal $N = 1$ as

$$PS(0, \mathcal{T}, \tau, 1, K) = \int\int\int_{\mathcal{P}_c(x_1,x_2,x_3)\geq 0} \mathcal{P}_{PS}(x_1, x_2, x_3)\mathcal{D}_3(x_1, x_2, x_3)dx_3dx_2dx_1.$$

4.5.2 Two-distribution functions Third Chaos Models

European call/put option

Case 1. $\gamma \equiv \zeta$

Assuming that $\gamma \equiv \zeta$ in the Factorizable Third Chaos Model, we are able to simplify the model slightly, so that it has two normal distribution functions:

$$\mathcal{P}_c(x_1, x_2) = A + (B + C)x_1 + Dx_1x_2 + (E + F)x_1^2 + Gx_1^2x_2 + Hx_1^2x_2^2.$$

Then, we find that the call option pricing formula is given by

$$ZBC(0, t, T, K) = \int\int_{\mathcal{P}_c(x_1,x_2)\geq 0} \mathcal{P}_c(x_1, x_2)\mathcal{D}_2(x_1, x_2)dx_2dx_1,$$

where $\mathcal{D}_2$ is the bivariate standard normal distribution density, i.e.

$$\mathcal{D}_2(x_1, x_2) := \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\Big[-\frac{x_1^2 - 2\rho x_1x_2 + x_2^2}{2(1-\rho^2)} \Big].$$

The integrand can be computed by checking that the condition $\mathcal{P}_c(x_1, x_2) \geq 0$ holds. If we assume that $\gamma \equiv \zeta \equiv 1$, we have that

$$L(t, \gamma^2) = L(t, \zeta^2) = t, \quad L(t, \epsilon\zeta) = \int_0^t \epsilon_{s_1} ds_1, \quad \theta(\gamma) = \theta(\zeta) = \frac{W_t}{\sqrt{t}}.$$

Case 2. $\gamma \equiv \epsilon$

Assuming that $\gamma \equiv \epsilon$ in the Factorizable Third Chaos Model, we are again able to simplify the model slightly, so that we have two normal distribution functions:

$$\mathcal{P}_c(x_1, x_3) = A + Bx_1 + Cx_3 + Dx_1x_3 + Ex_1^2 + Fx_3^2 + Gx_1^2x_3 + Hx_1^2x_3^2.$$

Case 3. $\epsilon \equiv \zeta$

Assuming that $\epsilon \equiv \zeta$ in the Factorizable Third Chaos Model, as expected, we are again able to simplify the model to have two normal distribution functions:

$$\mathcal{P}_c(x_1, x_2) = A + Bx_1 + Cx_2 + Ex_1^2 + (D+F)x_2^2 + Gx_1x_2^2 + Hx_2^4.$$

4.5.3 Two-variable Third Chaos Models

If we suppose that $\zeta \equiv 1$ in the Factorizable Third Chaos Model, we are able to slightly simplify the model, so that

$$L(t, \zeta^2) = t, \quad L(t, \epsilon\zeta) = \int_0^t \epsilon_{s_1} ds_1, \quad \theta(\zeta) = \frac{W_t}{\sqrt{t}}.$$

However, the call option function still has three variables:

$$\mathcal{P}_c(x_1, x_2, x_3) = A + Bx_1 + Cx_3 + Dx_2x_3 + Ex_1^2 + Fx_3^2 + Gx_1x_2x_3 + Hx_2^2x_3^2.$$

4.5.4 One-distribution function Third Chaos Models

European call/put option

Assuming further that $\gamma \equiv \epsilon \equiv \zeta$ in the Factorizable Third Chaos Model, we are able to simplify the model, so that it has a degree four polynomial form with respect to a unique normally distributed random variable:

$$\mathcal{P}_c(x) = A + (B+C)x + (D+E+F)x^2 + Gx^3 + Hx^4.$$

Therefore, in this case we are able to deduce that the option pricing form is given by

$$ZBC(0, t, T, K) = \frac{1}{\sqrt{2\pi}} \int_{\mathcal{P}_c(\theta) \geq 0} \mathcal{P}_c(\theta) e^{-\frac{\theta^2}{2}} d\theta.$$

Note here that, recalling the notations used, we can express the function $\mathcal{P}_c$ as follows:

$$\begin{aligned}
\mathcal{P}_c(x) =&[P_{0T} - KP_{0t}] - \frac{1}{A_0}[E_T - KE_t]Q_1(t,\gamma) - \frac{1}{A_0}[F_{tT} - KF_{tt}]Q_1(t,\zeta) \\
&- \frac{1}{A_0}[H_T - KH_t]Q_2(t,\epsilon\zeta) \\
&- \frac{1}{2A_0}[D_T - KD_t]L(t,\gamma^2) + \frac{1}{4A_0}[H_T - KH_t]L^2(t,\gamma^2) \\
&+ \frac{1}{A_0}\Big([B_T - KB_t] - \frac{1}{2}[G_T - KG_t]L(t,\gamma^2) + \frac{1}{A_0}[C_{tT} - KC_{tt}]\Big)\sqrt{L(t,\gamma^2)}x \\
&+ \frac{1}{A_0}\Big(\frac{1}{2}[D_T - KD_t] - \frac{1}{2}[H_T - KH_t]L(t,\gamma^2) + [E_T - KE_t] + [F_{tT} - KF_{tt}]\Big)L(t,\gamma^2)x^2 \\
&+ \frac{1}{2A_0}[G_T - KG_t]L^{\frac{3}{2}}(t,\gamma^2)x^3 \\
&+ \frac{1}{4A_0}[H_T - KH_t]L^2(t,\gamma^2)x^4.
\end{aligned}$$

Classifications

We are able to investigate the option price further by checking the roots of the function $\mathcal{P}_c$. Let us first simplify the notation by denoting the coefficients such that

$$\mathcal{P}_c(x) = \widetilde{A}_0 + \widetilde{A}_1 x + \widetilde{A}_2 x^2 + \widetilde{A}_3 x^3 + \widetilde{A}_4 x^4.$$

We define the roots of the function $\mathcal{P}_c$ to be $\{x_1, x_2, x_3, x_4\}$ where $-\infty < x_1 \leq x_2 \leq x_3 \leq x_4 < \infty$, and the number of distinct roots by $n \in \{0,1,2,3,4\}$. Recall that

$$ZBC(0,t,T,K) = \frac{1}{\sqrt{2\pi}} \int_{\mathcal{P}_c(\theta)\geq 0} \mathcal{P}_c(\theta) e^{-\frac{\theta^2}{2}} d\theta.$$

We now can describe all the different cases. To start, if $\{n = 4\} \cap \{\widetilde{A}_4 > 0\}$, we have that

$$ZBC(0,t,T,K) = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{x_1} \mathcal{P}_c(\theta) e^{-\frac{\theta^2}{2}} d\theta + \int_{x_2}^{x_3} \mathcal{P}_c(\theta) e^{-\frac{\theta^2}{2}} d\theta + \int_{x_4}^{\infty} \mathcal{P}_c(\theta) e^{-\frac{\theta^2}{2}} d\theta \right].$$

If $\{n = 4\} \cap \{\widetilde{A}_4 < 0\}$, we have that

$$ZBC(0,t,T,K) = \frac{1}{\sqrt{2\pi}} \left[\int_{x_1}^{x_2} \mathcal{P}_c(\theta) e^{-\frac{\theta^2}{2}} d\theta + \int_{x_3}^{x_4} \mathcal{P}_c(\theta) e^{-\frac{\theta^2}{2}} d\theta \right].$$

If $\{n=3\} \cap \{\widetilde{A}_4 > 0\} \cap \{\mathcal{P}_c\big(\frac{x_2+x_3}{2}\big) > 0\}$, and $\mathcal{P}_c$ has roots $\{x_1, x_2, x_3\}$, we have that

$$ZBC(0,t,T,K) = \frac{1}{\sqrt{2\pi}}\bigg[\int_{-\infty}^{x_1} \mathcal{P}_c(\theta)e^{-\frac{\theta^2}{2}}d\theta + \int_{x_2}^{\infty} \mathcal{P}_c(\theta)e^{-\frac{\theta^2}{2}}d\theta\bigg].$$

If $\{n=3\} \cap \{\widetilde{A}_4 > 0\} \cap \{\mathcal{P}_c\big(\frac{x_2+x_3}{2}\big) < 0\}$, and $\mathcal{P}_c$ has roots $\{x_1, x_2, x_3\}$, we have that

$$ZBC(0,t,T,K) = \frac{1}{\sqrt{2\pi}}\bigg[\int_{-\infty}^{x_2} \mathcal{P}_c(\theta)e^{-\frac{\theta^2}{2}}d\theta + \int_{x_3}^{\infty} \mathcal{P}_c(\theta)e^{-\frac{\theta^2}{2}}d\theta\bigg].$$

If $\{n=3\} \cap \{\widetilde{A}_4 < 0\} \cap \{\mathcal{P}_c\big(\frac{x_2+x_3}{2}\big) > 0\}$, and $\mathcal{P}_c$ has roots $\{x_1, x_2, x_3\}$, we have that

$$ZBC(0,t,T,K) = \frac{1}{\sqrt{2\pi}}\int_{x_2}^{x_3} \mathcal{P}_c(\theta)e^{-\frac{\theta^2}{2}}d\theta.$$

If $\{n=3\} \cap \{\widetilde{A}_4 < 0\} \cap \{\mathcal{P}_c\big(\frac{x_2+x_3}{2}\big) < 0\}$, and $\mathcal{P}_c$ has roots $\{x_1, x_2, x_3\}$, we have that

$$ZBC(0,t,T,K) = \frac{1}{\sqrt{2\pi}}\int_{x_1}^{x_2} \mathcal{P}_c(\theta)e^{-\frac{\theta^2}{2}}d\theta.$$

If $\{n=3\} \cap \{\widetilde{A}_4 = 0\} \cap \{\widetilde{A}_3 > 0\}$, and $\mathcal{P}_c$ has roots $\{x_1, x_2, x_3\}$, we have that

$$ZBC(0,t,T,K) = \frac{1}{\sqrt{2\pi}}\bigg[\int_{x_1}^{x_2} \mathcal{P}_c(\theta)e^{-\frac{\theta^2}{2}}d\theta + \int_{x_3}^{\infty} \mathcal{P}_c(\theta)e^{-\frac{\theta^2}{2}}d\theta\bigg].$$

If $\{n=3\} \cap \{\widetilde{A}_4 = 0\} \cap \{\widetilde{A}_3 < 0\}$, and $\mathcal{P}_c$ has roots $\{x_1, x_2, x_3\}$, we have that

$$ZBC(0,t,T,K) = \frac{1}{\sqrt{2\pi}}\bigg[\int_{-\infty}^{x_1} \mathcal{P}_c(\theta)e^{-\frac{\theta^2}{2}}d\theta + \int_{x_2}^{x_3} \mathcal{P}_c(\theta)e^{-\frac{\theta^2}{2}}d\theta\bigg].$$

If $\{n=2\} \cap \{\widetilde{A}_4 > 0\}$, and $\mathcal{P}_c$ has roots $\{x_1, x_2\}$, we have that

$$ZBC(0,t,T,K) = \frac{1}{\sqrt{2\pi}}\bigg[\int_{-\infty}^{x_1} \mathcal{P}_c(\theta)e^{-\frac{\theta^2}{2}}d\theta + \int_{x_2}^{\infty} \mathcal{P}_c(\theta)e^{-\frac{\theta^2}{2}}d\theta\bigg].$$

If $\{n=2\} \cap \{\widetilde{A}_4 < 0\}$, and $\mathcal{P}_c$ has roots $\{x_1, x_2\}$, we have that

$$ZBC(0,t,T,K) = \frac{1}{\sqrt{2\pi}}\int_{x_1}^{x_2} \mathcal{P}_c(\theta)e^{-\frac{\theta^2}{2}}d\theta.$$

If $\{n=2\} \cap \{\widetilde{A}_4 = 0\} \cap \{\widetilde{A}_3 > 0\}$, and $\mathcal{P}_c$ has roots $\{x_1, x_2\}$, we have that

$$ZBC(0,t,T,K) = \frac{1}{\sqrt{2\pi}}\int_{x_1}^{\infty} \mathcal{P}_c(\theta)e^{-\frac{\theta^2}{2}}d\theta.$$

If $\{n=2\} \cap \{\widetilde{A}_4 = 0\} \cap \{\widetilde{A}_3 < 0\}$, and $\mathcal{P}_c$ has roots $\{x_1, x_2\}$, we have that

$$ZBC(0,t,T,K) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x_2} \mathcal{P}_c(\theta) e^{-\frac{\theta^2}{2}} d\theta.$$

If $\{n=2\} \cap \{\widetilde{A}_4 = 0\} \cap \{\widetilde{A}_3 = 0\} \cap \{\widetilde{A}_2 > 0\}$, and $\mathcal{P}_c$ has roots $\{x_1, x_2\}$, we have that

$$ZBC(0,t,T,K) = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{x_1} \mathcal{P}_c(\theta) e^{-\frac{\theta^2}{2}} d\theta + \int_{x_2}^{\infty} \mathcal{P}_c(\theta) e^{-\frac{\theta^2}{2}} d\theta \right].$$

If $\{n=2\} \cap \{\widetilde{A}_4 = 0\} \cap \{\widetilde{A}_3 = 0\} \cap \{\widetilde{A}_2 < 0\}$, and $\mathcal{P}_c$ has roots $\{x_1, x_2\}$, we have that

$$ZBC(0,t,T,K) = \frac{1}{\sqrt{2\pi}} \left[\int_{x_1}^{x_2} \mathcal{P}_c(\theta) e^{-\frac{\theta^2}{2}} d\theta \right].$$

If $\{n = 0 \text{ or } 1\} \cap \{\widetilde{A}_4 > 0\}$, we have that

$$ZBC(0,t,T,K) = (KP_{0t} - P_{0T})^+.$$

If $\{n = 0 \text{ or } 1\} \cap \{\widetilde{A}_4 < 0\}$, we have that

$$ZBC(0,t,T,K) = 0.$$

If $\{n=1\} \cap \{\widetilde{A}_4 = 0\} \cap \{\widetilde{A}_3 > 0\}$, and $\mathcal{P}_c$ has roots $\{x_1, x_2\}$, we have that

$$ZBC(0,t,T,K) = \frac{1}{\sqrt{2\pi}} \int_{x_1}^{\infty} \mathcal{P}_c(\theta) e^{-\frac{\theta^2}{2}} d\theta.$$

If $\{n=1\} \cap \{\widetilde{A}_4 = 0\} \cap \{\widetilde{A}_3 < 0\}$, and $\mathcal{P}_c$ has roots $\{x_1, x_2\}$, we have that

$$ZBC(0,t,T,K) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x_1} \mathcal{P}_c(\theta) e^{-\frac{\theta^2}{2}} d\theta.$$

If $\{n = 0 \text{ or } 1\} \cap \{\widetilde{A}_4 = 0\} \cap \{\widetilde{A}_3 = 0\} \cap \{\widetilde{A}_2 > 0\}$, we have that

$$ZBC(0,t,T,K) = (KP_{0t} - P_{0T})^+.$$

If $\{n = 0 \text{ or } 1\} \cap \{\widetilde{A}_4 = 0\} \cap \{\widetilde{A}_3 = 0\} \cap \{\widetilde{A}_2 < 0\}$, we have that

$$ZBC(0,t,T,K) = 0.$$

If $\{n=1\} \cap \{\widetilde{A}_4 = 0\} \cap \{\widetilde{A}_3 = 0\} \cap \{\widetilde{A}_2 = 0\} \cap \{\widetilde{A}_1 > 0\}$, we have that

$$ZBC(0,t,T,K) = \frac{1}{V_0\sqrt{2\pi}} \int_{x_1}^{\infty} \mathcal{P}_c(\theta) e^{-\frac{\theta^2}{2}} d\theta.$$

If $\{n=1\} \cap \{\widetilde{A}_4 = 0\} \cap \{\widetilde{A}_3 = 0\} \cap \{\widetilde{A}_2 = 0\} \cap \{\widetilde{A}_1 < 0\}$, we have that

$$ZBC(0,t,T,K) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x_1} \mathcal{P}_c(\theta) e^{-\frac{\theta^2}{2}} d\theta.$$

At this point, we wish to describe the possible outcomes for the bond option. In order to do this, we first make the following definitions: For $\{a,b\} \in \mathbb{R}^2$,

$$f_0(a,b) := \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{\theta^2}{2}} d\theta, \quad f_1(a,b) := \frac{1}{\sqrt{2\pi}} \int_a^b \theta e^{-\frac{\theta^2}{2}} d\theta, \quad f_2(a,b) := \frac{1}{\sqrt{2\pi}} \int_a^b \theta^2 e^{-\frac{\theta^2}{2}} d\theta,$$

$$f_3(a,b) := \frac{1}{\sqrt{2\pi}} \int_a^b \theta^3 e^{-\frac{\theta^2}{2}} d\theta \quad \text{and} \quad f_4(a,b) := \frac{1}{\sqrt{2\pi}} \int_a^b \theta^4 e^{-\frac{\theta^2}{2}} d\theta.$$

We can then conclude that the bond option can be described as follows:

If $\{n=4\} \cap \{\widetilde{A}_4 > 0\}$, we have that

$$ZBC(0,t,T,K) = \sum_{i=0}^{4} \widetilde{A}_i \Big(f_i(-\infty, x_1) + f_i(x_2, x_3) + f_i(x_4, \infty) \Big).$$

If $\{n=4\} \cap \{\widetilde{A}_4 < 0\}$, we have that

$$ZBC(0,t,T,K) = \sum_{i=0}^{4} \widetilde{A}_i \Big(f_i(x_1, x_2) + f_i(x_3, x_4) \Big).$$

If $\{n=3\} \cap \{\widetilde{A}_4 > 0\} \cap \{\mathcal{P}_c\big(\frac{x_2+x_3}{2}\big) > 0\}$, and $\mathcal{P}_c$ has roots $\{x_1, x_2, x_3\}$, we have that

$$ZBC(0,t,T,K) = \sum_{i=0}^{4} \widetilde{A}_i \Big(f_i(-\infty, x_1) + f_i(x_2, \infty) \Big).$$

If $\{n=3\} \cap \{\widetilde{A}_4 > 0\} \cap \{\mathcal{P}_c\big(\frac{x_2+x_3}{2}\big) < 0\}$, and $\mathcal{P}_c$ has roots $\{x_1, x_2, x_3\}$, we have that

$$ZBC(0,t,T,K) = \sum_{i=0}^{4} \widetilde{A}_i \Big(f_i(-\infty, x_2) + f_i(x_3, \infty) \Big).$$

If $\{n=3\} \cap \{\widetilde{A}_4 < 0\} \cap \{\mathcal{P}_c\big(\frac{x_2+x_3}{2}\big) > 0\}$, and $\mathcal{P}_c$ has roots $\{x_1, x_2, x_3\}$, we have that

$$ZBC(0,t,T,K) = \sum_{i=0}^{4} \widetilde{A}_i f_i(x_2, x_3).$$

If $\{n=3\} \cap \{\widetilde{A}_4 < 0\} \cap \{\mathcal{P}_c\big(\frac{x_2+x_3}{2}\big) < 0\}$, and $\mathcal{P}_c$ has roots $\{x_1, x_2, x_3\}$, we have that

$$ZBC(0,t,T,K) = \sum_{i=0}^{4} \widetilde{A}_i f_i(x_1, x_2).$$

If $\{n=3\} \cap \{\widetilde{A}_4 = 0\} \cap \{\widetilde{A}_3 > 0\}$, and $\mathcal{P}_c$ has roots $\{x_1, x_2, x_3\}$, we have that

$$ZBC(0,t,T,K) = \sum_{i=0}^{4} \widetilde{A}_i \Big(f_i(x_1, x_2) + f_i(x_3, \infty) \Big).$$

If $\{n=3\} \cap \{\widetilde{A}_4 = 0\} \cap \{\widetilde{A}_3 < 0\}$, and $\mathcal{P}_c$ has roots $\{x_1, x_2, x_3\}$, we have that

$$ZBC(0,t,T,K) = \sum_{i=0}^{4} \widetilde{A}_i \Big(f_i(-\infty, x_1) + f_i(x_2, x_3) \Big).$$

If $\{n=2\} \cap \{\widetilde{A}_4 > 0\}$, having roots $\{x_1, x_2\}$, we have that

$$ZBC(0,t,T,K) = \sum_{i=0}^{4} \widetilde{A}_i \Big(f_i(-\infty, x_1) + f_i(x_2, \infty) \Big).$$

If $\{n=2\} \cap \{\widetilde{A}_4 < 0\}$, having roots $\{x_1, x_2\}$, we have that

$$ZBC(0,t,T,K) = \sum_{i=0}^{4} \widetilde{A}_i f_i(x_1, x_2).$$

If $\{n=2\} \cap \{\widetilde{A}_4 = 0\} \cap \{\widetilde{A}_3 > 0\}$, and $\mathcal{P}_c$ has roots $\{x_1, x_2\}$, we have that

$$ZBC(0,t,T,K) = \sum_{i=0}^{4} \widetilde{A}_i f_i(x_1, \infty).$$

If $\{n=2\} \cap \{\widetilde{A}_4 = 0\} \cap \{\widetilde{A}_3 < 0\}$, and $\mathcal{P}_c$ has roots $\{x_1, x_2\}$, we have that

$$ZBC(0,t,T,K) = \sum_{i=0}^{4} \widetilde{A}_i f_i(-\infty, x_2).$$

If $\{n = 2\} \cap \{\widetilde{A}_4 = 0\} \cap \{\widetilde{A}_3 = 0\} \cap \{\widetilde{A}_2 > 0\}$, and $\mathcal{P}_c$ has roots $\{x_1, x_2\}$, we have that

$$ZBC(0, t, T, K) = \sum_{i=0}^{4} \widetilde{A}_i \Big(f_i(-\infty, x_1) + f_i(x_2, \infty) \Big).$$

If $\{n = 2\} \cap \{\widetilde{A}_4 = 0\} \cap \{\widetilde{A}_3 = 0\} \cap \{\widetilde{A}_2 < 0\}$, and $\mathcal{P}_c$ has roots $\{x_1, x_2\}$, we have that

$$ZBC(0, t, T, K) = \sum_{i=0}^{4} \widetilde{A}_i f_i(x_1, x_2).$$

If $\{n = 0 \text{ or } 1\} \cap \{\widetilde{A}_4 > 0\}$, we have that

$$ZBC(0, t, T, K) = (KP_{0t} - P_{0T})^+.$$

If $\{n = 0 \text{ or } 1\} \cap \{\widetilde{A}_4 < 0\}$, we have that

$$ZBC(0, t, T, K) = 0.$$

If $\{n = 1\} \cap \{\widetilde{A}_4 = 0\} \cap \{\widetilde{A}_3 > 0\}$, we have that

$$ZBC(0, t, T, K) = \sum_{i=0}^{4} \widetilde{A}_i f_i(x_1, \infty).$$

If $\{n = 1\} \cap \{\widetilde{A}_4 = 0\} \cap \{\widetilde{A}_3 < 0\}$, we have that

$$ZBC(0, t, T, K) = \sum_{i=0}^{4} \widetilde{A}_i f_i(-\infty, x_1).$$

If $\{n = 0 \text{ or } 1\} \cap \{\widetilde{A}_4 = 0\} \cap \{\widetilde{A}_3 = 0\} \cap \{\widetilde{A}_2 > 0\}$, we have that

$$ZBC(0, t, T, K) = (KP_{0t} - P_{0T})^+.$$

If $\{n = 0 \text{ or } 1\} \cap \{\widetilde{A}_4 = 0\} \cap \{\widetilde{A}_3 = 0\} \cap \{\widetilde{A}_2 < 0\}$, we have that

$$ZBC(0, t, T, K) = 0.$$

If $\{n = 1\} \cap \{\widetilde{A}_4 = 0\} \cap \{\widetilde{A}_3 = 0\} \cap \{\widetilde{A}_2 = 0\} \cap \{\widetilde{A}_1 > 0\}$, we have that

$$ZBC(0, t, T, K) = \sum_{i=0}^{4} \widetilde{A}_i f_i(x_1, \infty).$$

Finally, if $\{n = 1\} \cap \{\widetilde{A}_4 = 0\} \cap \{\widetilde{A}_3 = 0\} \cap \{\widetilde{A}_2 = 0\} \cap \{\widetilde{A}_1 < 0\}$, we have that

$$ZBC(0, t, T, K) = \sum_{i=0}^{4} \widetilde{A}_i f_i(-\infty, x_1).$$

Swaption

Assuming that $\gamma \equiv \epsilon \equiv \zeta$ in the Factorizable Third Chaos Model, we are able to simplify the method to price swaptions too. In this case, it has a unique normal distribution function:

$$\mathcal{P}_{PS}(x) = A^* + (B^* + C^*)x + (D^* + E^* + F^*)x^2 + G^* x^3 + H^* x^4.$$

Recalling the notations given above, the function $\mathcal{P}_{PS}$ can be expressed in the following way:

$$\begin{aligned}
\mathcal{P}_{PS}(x) =& \Big(P_{0t} - P_{0T_b} - K \sum_{i=a+1}^{b} \tau_i P_{0T_i}\Big) - \frac{1}{A_0}\Big(E_t - E_{T_b} - K \sum_{i=a+1}^{b} \tau_i E_{T_i}\Big) Q_1(t,\gamma) \\
& - \frac{1}{A_0}\Big(F_{tt} - F_{tT_b} - K \sum_{i=a+1}^{b} \tau_i F_{tT_i}\Big) Q_1(t,\zeta) - \frac{1}{A_0}\Big(H_t - H_{T_b} - K \sum_{i=a+1}^{b} \tau_i H_{T_i}\Big) Q_2(t,\epsilon\zeta) \\
& - \frac{1}{2A_0}\Big(D_t - D_{T_b} - K \sum_{i=a+1}^{b} \tau_i D_{T_i}\Big) L(t,\gamma^2) + \frac{1}{4A_0}\Big(H_t - H_{T_b} - K \sum_{i=a+1}^{b} \tau_i H_{T_i}\Big) L^2(t,\gamma^2) \\
& + \frac{1}{A_0}\Big[\Big(B_t - B_{T_b} - K \sum_{i=a+1}^{b} \tau_i B_{T_i}\Big) - \frac{1}{2}\Big(G_t - G_{T_b} - K \sum_{i=a+1}^{b} \tau_i G_{T_i}\Big) L(t,\gamma^2) \\
& + \Big(C_{tt} - C_{tT_b} - K \sum_{i=a+1}^{b} \tau_i C_{tT_i}\Big)\Big] \sqrt{L(t,\gamma^2)}\, x \\
& + \frac{1}{A_0}\Big[\frac{1}{2}\Big(D_t - D_{T_b} - K \sum_{i=a+1}^{b} \tau_i D_{T_i}\Big) - \frac{1}{2}\Big(H_t - H_{T_b} - K \sum_{i=a+1}^{b} \tau_i H_{T_i}\Big) L(t,\gamma^2) \\
& + \Big(E_t - E_{T_b} - K \sum_{i=a+1}^{b} \tau_i E_{T_i}\Big) + \Big(F_{tt} - F_{tT_b} - K \sum_{i=a+1}^{b} \tau_i F_{tT_i}\Big)\Big] L(t,\gamma^2) x^2 \\
& + \frac{1}{2A_0}\Big(G_t - G_{T_b} - K \sum_{i=a+1}^{b} \tau_i G_{T_i}\Big) L^{\frac{3}{2}}(t,\gamma^2) x^3 \\
& + \frac{1}{4A_0}\Big(H_t - H_{T_b} - K \sum_{i=a+1}^{b} \tau_i H_{T_i}\Big) L^2(t,\gamma^2) x^4.
\end{aligned}$$

where we have that

$$\mathbb{E}[\mathcal{P}_{PS}(x)] = P_{0t} - P_{0T_b} - K \sum_{i=a+1}^{b} \tau_i P_{0T_i}.$$

Therefore, we are able to deduce that the swaption pricing form is given by

$$PS(0, \mathcal{T}, \tau, 1, K) = \frac{1}{\sqrt{2\pi}} \int_{\mathcal{P}_{PS}(\theta) \geq 0} \mathcal{P}_{PS}(\theta) e^{-\frac{\theta^2}{2}} d\theta,$$

where the function $\mathcal{P}_{PS}$ has degree-four polynomial form with respect to the normally distributed random variable θ. We are able to investigate this further, again by checking the roots of the function $\mathcal{P}_{PS}$. However we can simply use the classification which we have outlined above for the bond option, applying the coefficients $\{A^*, B^*, C^*, D^*, E^*, F^*, G^*, H^*\}$, instead of using $\{A, B, C, D, E, F, G, H\}$.

4.6 One-variable Chaos Models

In this section, we suggest Chaos Models in which the random variable $X_\infty \in L^2$ is formed from deterministic functions $\hat{\phi}_1, \hat{\phi}_2, \dots$ of only one variable, i.e.,

$$X_\infty = \int_0^\infty \left[\hat{\phi}_1(s) + \int_0^s \hat{\phi}_2(s) dW_{s_1} + \int_0^s \int_0^{s_1} \hat{\phi}_3(s) dW_{s_2} dW_{s_1} + \cdots \right] dW_s.$$

We call this expansion "One-variable Chaos Expansion" and corresponding Chaos Model "One-variable Chaos Model". Applying the expression (4.1.1) we infer that

$$\begin{aligned} \sigma_s =& \hat{R}_1(t,s) + \hat{R}_2(t,s) \int_t^s dW_{s_1} + \hat{R}_3(t,s) \int_t^s \int_t^{s_1} dW s_2 dW s_1 + \cdots \\ =& \hat{R}_1(t,s) + \hat{R}_2(t,s)(W_s - W_t) + \frac{1}{2} \hat{R}_3(t,s) \big[(W_s - W_t)^2 - (s-t) \big] + \cdots, \end{aligned} \tag{4.6.1}$$

where we define $\hat{R}_n(t,s) := R_n(t,s,1,\dots,1)$ for all $n \in \mathbb{N}$, i.e.,

$$\begin{aligned} \hat{R}_n(t,s) =& \hat{\phi}_n(s) + \hat{\phi}_{n+1}(s) \int_0^t dW_{s_n} + \hat{\phi}_{n+2}(s) \int_0^t \int_0^{s_n} dW_{s_{n+1}} dW_{s_n} + \cdots \\ =& \hat{\phi}_n(s) + \hat{\phi}_{n+1}(s) W_t + \hat{\phi}_{n+2}(s) \frac{1}{2}(W_t^2 - t) + \hat{\phi}_{n+3}(s) \Big(\frac{1}{6} W_t^3 - \frac{1}{2} t W_t \Big) \cdots, \end{aligned}$$

because we have that

$$\int_0^t dW s_n = W_t,$$
$$\int_0^t \int_0^{s_n} dW_{s_{n+1}} dW_{s_n} = \int_0^t W_s dW_s = \frac{1}{2}(W_t^2 - t),$$
$$\int_0^t \int_0^{s_n} \int_0^{s_{n+1}} dW_{s_{n+2}} dW_{s_{n+1}} dW_{s_n} = \int_0^t \frac{1}{2}(W_s^2 - s) dW_s = \frac{1}{6}W_t^3 - \frac{1}{2}tW_t.$$

As we already observed above, because each function $\hat{R}_n$ is $\mathcal{F}_t$-measurable, it follows from the Itô isometry and orthogonality that

$$\begin{aligned} \mathbb{E}_t[\sigma_s^2] =& \hat{R}_1^2(t,s) + \hat{R}_2^2(t,s) \int_t^s ds_1 + \hat{R}_3^2(t,s) \int_t^s \int_t^{s_1} ds_2 ds_1 + \cdots \\ =& \hat{R}_1^2(t,s) + (s-t)\hat{R}_2^2(t,s) + \frac{1}{2}(s-t)^2 \hat{R}_3^2(t,s) + \cdots . \end{aligned} \tag{4.6.2}$$

Therefore, we obtain that

$$V_t = \int_t^\infty \Big(\hat{R}_1^2(t,s) + (s-t)\hat{R}_2^2(t,s) + \frac{1}{2}(s-t)^2 \hat{R}_3^2(t,s) + \cdots \Big) ds$$

and

$$Z_{tT} = \int_T^\infty \Big(\hat{R}_1^2(t,s) + (s-t)\hat{R}_2^2(t,s) + \frac{1}{2}(s-t)^2 \hat{R}_3^2(t,s) + \cdots \Big) ds.$$

Consequently, the One-variable Second Chaos Model expresses the state price density via a quadratic form of the Brownian Motion W_t, while the One-variable Third Chaos Model does the same by a degree four polynomial form, and One-variable Fourth Chaos Model does it by a degree six polynomial form. Investigating the models further we find that

(4.6.3)

$$D_t\big[\mathbb{E}_t[\sigma_s^2]\big] = 2\hat{R}_1(t,s)\hat{R}_2(t,s) + 2(s-t)\hat{R}_2(t,s)\hat{R}_3(t,s) + (s-t)^2\hat{R}_3(t,s)\hat{R}_4(t,s) + \cdots ,$$

which forms the risk-adjusted volatility and the market price of risk.

4.6.1 One-variable Second Chaos Model

Let us suppose that $\gamma_t \equiv 1$ for all $t \geq 0$ in the Factorizable Second Chaos Model. Then we have that

$$\hat{R}_t = W_t \quad \text{and} \quad \hat{Q}_t = t,$$

which yields that for $0 \leq t \leq T < \infty$,

$$V_t = A_t + B_t W_t + C_t(W_t^2 - t) \quad \text{and} \quad Z_{tT} = A_T + B_T W_t + C_T(W_t^2 - t), \tag{4.6.4}$$

where

$$A_t = \int_t^\infty (\alpha_s^2 + s\beta_s^2) ds, \quad B_t = 2\int_t^\infty \alpha_s \beta_s ds \quad \text{and} \quad C_t = \int_t^\infty \beta_s^2 ds.$$

Note here using (4.4.3) we may also express these variables in the following way:

$$V_t = P_{0t} A_0 + B_t W_t + C_t(W_t^2 - t) \quad \text{and} \quad Z_{tT} = P_{0T} A_0 + B_T W_t + C_T(W_t^2 - t).$$

Therefore, we may make the interpretation that the state price density V_t and also the variable Z_{tT} under the One-variable Second Chaos Model are non-central chi-squared distributed respectively with mean $P_{0t}A_0$ and $P_{0T}A_0$. The expression (4.1.11) gives, for $0 \leq t \leq T < \infty$, that

$$\hat{V}_{tT} = \frac{2\int_T^\infty \alpha_s \beta_s ds + 2\int_T^\infty \beta_s^2 ds W_t}{\int_T^\infty \left(\alpha_s^2 + s\beta_s^2\right) ds + 2\int_T^\infty \alpha_s \beta_s ds W_t + \int_T^\infty \beta_s^2 ds (W_t^2 - t)}.$$

European call/put option

From expression (4.6.1) we obtain that

$$Z_{tT} - K Z_{tt} = (P_{0T} - K P_{0t}) A_0 + (B_T - K B_t) W_t + (C_T - K C_t)(W_t^2 - t).$$

The call option price is formulated by the expectation rule as follows:

$$ZBC(0,t,T,K) = \frac{1}{\sqrt{2\pi}} \int_{\mathcal{P}_c(\theta) \geq 0} \mathcal{P}_c(\theta) e^{-\frac{\theta^2}{2}} d\theta,$$

where

$$\mathcal{P}_c(\theta) = \hat{A} + \hat{B}\theta + \hat{C}\theta^2, \quad \hat{A} = P_{0T} - KP_{0t} - \frac{1}{A_0}(C_T - KC_t)t,$$
$$\hat{B} = \frac{1}{A_0}(B_T - KB_t)\sqrt{t}, \quad \hat{C} = \frac{1}{A_0}(C_T - KC_t)t \quad \text{and} \quad \theta = \frac{W_t}{\sqrt{t}}.$$

Therefore, we find that the European bond option under the One-variable Second Chaos Model is also non-central chi-squared with mean $P_{0T} - KP_{0t}$. Note that we are able to investigate further by applying the same classification as for the Factorizable Second Chaos Models, that is, considering by the roots of the quadratic functions $\mathcal{P}_c(\theta)$ and $\mathcal{P}_p(\theta)$.

Swaption

The payer swaption price is formulated by the expectation rule as following:

$$PS(0, \mathcal{T}, \tau, 1, K) = \frac{1}{\sqrt{2\pi}} \int_{\mathcal{P}_{PS}(\theta)\geq 0} \mathcal{P}_{PS}(\theta) e^{-\frac{\theta^2}{2}} d\theta,$$

where $\mathcal{P}_{PS}(\theta) = \widetilde{A} + \widetilde{B}\theta + \widetilde{C}\theta^2$,

$$\theta = \frac{W_t}{\sqrt{t}}, \quad \widetilde{A} = \Big(P_{0t} - P_{0T_b} - K\sum_{i=a+1}^{b} \tau_i P_{0T_i}\Big) - \frac{1}{A_0}\Big(C_t - C_{T_b} - K\sum_{i=a+1}^{b} \tau_i C_{T_i}\Big)t,$$

$$\widetilde{B} = \frac{1}{A_0}\Big(B_t - B_{T_b} - K\sum_{i=a+1}^{b} \tau_i B_{T_i}\Big)\sqrt{t} \quad \text{and} \quad \widetilde{C} = \frac{1}{A_0}\Big(C_t - C_{T_b} - K\sum_{i=a+1}^{b} \tau_i C_{T_i}\Big)t.$$

Again, here that we are able to investigate further in the same way as in the Factorizable Second Chaos Models, by considering the roots of the function $\mathcal{P}_{PS}(\theta)$. Therefore we obtain the same distributions for the derivatives in the One-variable Second Chaos Model as in the Factorizable Second Chaos Model without loss of generality.

4.6.2 One-variable Third Chaos Model

Assuming that $\gamma \equiv \zeta \equiv \epsilon \equiv 1$ in the one-distribution Third Chaos Model, we have that

$$\begin{aligned} X_\infty &= \int_0^\infty \Big(\alpha_s + \int_0^s \beta_s dW_{s_1} + \int_0^s \int_0^{s_1} \delta_s dW_{s_2} dW_{s_1}\Big) dW_s \\ &= \int_0^\infty \Big(\alpha_s + \beta_s W_s + \frac{1}{2}\delta_s (W_s^2 - s)\Big) dW_s. \end{aligned}$$

Because from the definitions (4.5.2) we may infer that

$$J_1(t,\gamma) = W_t, \quad J_2(t,\epsilon\zeta) = \frac{1}{2}(W_t^2 - t), \quad Q_1(t,\gamma) = t \quad \text{and} \quad Q_2(t,\epsilon\zeta) = \frac{1}{2}t^2,$$

we can then simplify expression (4.5.1) as

$$\begin{aligned} Z_{tT} = & A_T + (B_T + C_{tT})W_t + \Big(\frac{1}{2}D_T + E_T + F_{tT}\Big)(W_t^2 - t) \\ & + \frac{1}{2}G_T W_t (W_t^2 - t) + H_T\Big[\Big(\frac{1}{2}(W_t^2 - t)\Big)^2 - \frac{1}{2}t^2\Big], \end{aligned} \tag{4.6.5}$$

where we have that

$$\begin{aligned} A_T = & \int_T^\infty \Big(\alpha_s^2 + s\beta_s^2 + \frac{1}{2}s^2\delta_s^2\Big)ds, \quad B_T = 2\int_T^\infty \alpha_s\beta_s ds, \quad C_{tT} = 2\int_T^\infty \beta_s\delta_s(s-t)ds, \\ D_T = & 2\int_T^\infty \alpha_s\delta_s ds, \quad E_T = \int_T^\infty \beta_s^2 ds, \\ F_{tT} = & \int_T^\infty \delta_s^2(s-t)ds, \quad G_T = 2\int_T^\infty \beta_s\delta_s ds \quad \text{and} \quad H_T = \int_T^\infty \delta_s^2 ds. \end{aligned}$$

Note here that we have from (4.5.3) that

$$A_T = P_{0T}A_0, \quad T \geq 0. \tag{4.6.6}$$

This may be simplified as follows:

$$\begin{aligned} Z_{tT} = & P_{0T}A_0 - \Big(\frac{1}{2}D_T + E_T + F_{tT}\Big)t - \frac{1}{4}H_T t^2 \\ & + \Big(B_T + C_{tT} - \frac{1}{2}G_T t\Big)W_t + \Big(\frac{1}{2}D_T + E_T + F_{tT} - \frac{1}{2}H_T t\Big)W_t^2 + \frac{1}{2}G_T W_t^3 + \frac{1}{4}H_T W_t^4. \end{aligned}$$

The state price density is then given by

$$\begin{aligned} V_t = & P_{0t}A_0 - \Big(\frac{1}{2}D_t + E_t + F_{tt}\Big)t - \frac{1}{4}H_t t^2 \\ & + \Big(B_t + C_{tt} - \frac{1}{2}G_t t\Big)W_t + \Big(\frac{1}{2}D_t + E_t + F_{tt} - \frac{1}{2}H_t t\Big)W_t^2 + \frac{1}{2}G_t W_t^3 + \frac{1}{4}H_t W_t^4. \end{aligned}$$

Therefore, we have that both the state price density V_t and the variable Z_{tT} are degree four polynomial forms of Brownian Motion respectively with mean $P_{0t}A_0$ and $P_{0T}A_0$ in the One-variable Third Chaos Model. The One-variable Third Chaos Model gives us that

$$L(t,\gamma^2) = L(t,\zeta^2) = L(t,\epsilon^2) = L(t,\epsilon\zeta) = t, \quad \theta(\gamma) = \theta(\epsilon) = \theta(\zeta) = \frac{W_t}{\sqrt{t}},$$

which means that we have a univariate normal density for the computation of the option price.

European call/put bond option

We are able to express the function $\mathcal{P}_c$ in the following way:

$$\begin{aligned}\mathcal{P}_c(x) =& P_{tT} - KP_{tt} - \frac{1}{A_0}\Big[\frac{1}{2}(D_T - KD_t) + E_T - KE_t + F_{tT} - KF_{tt}\Big]t \\ &- \frac{1}{4A_0}(H_T - KH_t)t^2 \\ &+ \frac{1}{A_0}\Big[B_T - KB_t + C_{tT} - KC_{tt} - \frac{1}{2}(G_T - KG_t)t\Big]\sqrt{t}x \\ &+ \frac{1}{A_0}\Big[\frac{1}{2}(D_T - KD_t) + E_T - KE_t + F_{tT} - KF_{tt} - \frac{1}{2}(H_T - KH_t)t\Big]tx^2 \\ &+ \frac{1}{2A_0}(G_T - KG_t)t^{\frac{3}{2}}x^3 + \frac{1}{4A_0}(H_T - KH_t)t^2x^4,\end{aligned}$$

with

$$\mathbb{E}[\mathcal{P}_c(x)] = P_{tT} - KP_{tt}.$$

Swaption

Similarly, we are able to express the function $\mathcal{P}_{PS}$ as follows:

$$\begin{aligned}\mathcal{P}_{PS}(x) =& P_{0t} - P_{0T_b} - K\sum_{i=a+1}^{b}\tau_i P_{0T_i} - \frac{1}{A_0}\Big[\frac{1}{2}\Big(D_t - D_{T_b} - K\sum_{i=a+1}^{b}\tau_i D_{T_i}\Big)\\ &+ E_t - E_{T_b} - K\sum_{i=a+1}^{b}\tau_i E_{T_i} + F_{tt} - F_{tT_b} - K\sum_{i=a+1}^{b}\tau_i F_{tT_i}\Big]t\\ &- \frac{1}{4A_0}\Big(H_t - H_{T_b} - K\sum_{i=a+1}^{b}\tau_i H_{T_i}\Big)t^2\\ &+ \frac{1}{A_0}\Big[B_t - B_{T_b} - K\sum_{i=a+1}^{b}\tau_i B_{T_i} + C_{tt} - C_{tT_b} - K\sum_{i=a+1}^{b}\tau_i C_{tT_i}\\ &- \frac{1}{2}\Big(G_t - G_{T_b} - K\sum_{i=a+1}^{b}\tau_i G_{T_i}\Big)t\Big]\sqrt{t}x\\ &+ \frac{1}{A_0}\Big[\frac{1}{2}\Big(D_t - D_{T_b} - K\sum_{i=a+1}^{b}\tau_i D_{T_i}\Big) + E_t - E_{T_b} - K\sum_{i=a+1}^{b}\tau_i E_{T_i}\\ &+ F_{tt} - F_{tT_b} - K\sum_{i=a+1}^{b}\tau_i F_{tT_i} - \frac{1}{2}\Big(H_t - H_{T_b} - K\sum_{i=a+1}^{b}\tau_i H_{T_i}\Big)t\Big]tx^2\\ &+ \frac{1}{2A_0}\Big(G_t - G_{T_b} - K\sum_{i=a+1}^{b}\tau_i G_{T_i}\Big)t^{\frac{3}{2}}x^3 + \frac{1}{4A_0}\Big(H_t - H_{T_b} - K\sum_{i=a+1}^{b}\tau_i H_{T_i}\Big)t^2x^4,\end{aligned}$$

with

$$\mathbb{E}[\mathcal{P}_{PS}(x)] = P_{0t} - P_{0T_b} - K\sum_{i=a+1}^{b}\tau_i P_{0T_i}.$$

Therefore we obtain the same distributions for the derivatives in the One-variable Third Chaos Model as in the One-distribution Third Chaos Model without loss of generality.

4.7 Specification of the chaos coefficients

In this section, we apply descriptive forms to the chaos coefficients, which maintain the flexibility of the chaos functions. In addition, these forms enable the corresponding yield curve to have a humped shape.

4.7.1 The descriptive form

The exponential-polynomial family was introduced by Björk and Christensen ([12]) and is given by:

$$f_{0T} = \sum_{i=1}^{n} \Big(\sum_{j=0}^{k_i} b_{ij} T^j \Big) e^{-c_i T} \quad \text{for some constants} \quad b_{ij} \quad \text{and} \quad c_i.$$

The following three families can be considered as special cases of the exponential-polynomial family:

- $f_{0T} = b_0 + [b_1 + b_2 T] e^{-c_1 T}$ (Nelson and Siegel, [64]).
- $f_{0T} = b_0 + [b_1 + b_2 T] e^{-c_1 T} + b_3 T e^{-c_2 T}$ (Svensson, [86]).
- $f_{0T} = b_0 + \sum_{i=1}^{4} b_i e^{-c_i T}$ (Cairns, [21]).

Here, b_0 and c_i for $i = 1, \ldots, 4$ are positive constants, and the exponential parameters are fixed over the calibration dates in the Cairns form. As is claimed in [64] and [86], the asymptote for the instantaneous forward rate is determined by the positive constant b_0, in other words,

$$\lim_{T \to \infty} f_{0T} = b_0$$

for all of the above special cases.

4.7.2 Specification of the deterministic function h

Let us first recall the definition of the function h_T:

$$h_T = f_{0T} P_{0T} V_0, \quad T \geq 0.$$

Also recall that this function is always positive in the Chaotic Approach. However, because $\mathbb{E}[\sigma_T^2] = f_{0T} P_{0T} V_0$, we may write

$$f_{0T} = \frac{h_T}{\int_T^\infty h_s ds}, \quad T \geq 0.$$

The denominator is a decreasing function with respect to $T > 0$, because h is a positive function. Hence we would like the numerator to have the same features as the curve of the instantaneous forward rate, i.e., monotonic, humped and S-shaped. Therefore we may also apply the descriptive forms to the function h, for example:

$$h_T = \begin{cases} \left[(b_1 + b_2 T)e^{-c_1 T}\right]^2 \\ \left[(b_1 + b_2 T)e^{-c_1 T} + b_3 s e^{-c_2 T}\right]^2. \end{cases}$$

In light of equation (4.1.3), we have in the Chaotic Approach that

$$h_s = \phi_1^2(s) + \int_0^s \phi_2^2(s, s_1) ds_1 + \int_0^s \int_0^{s_1} \phi_3^2(s, s_1, s_2) ds_2 ds_1 + \cdots, \quad s \geq 0.$$

It follows that

$$f_{0T} = \frac{\phi_1^2(T) + \int_0^T \phi_2^2(T, s_1) ds_1 + \int_0^T \int_0^{s_1} \phi_3^2(T, s_1, s_2) ds_2 ds_1 + \cdots}{\int_T^\infty \left[\phi_1^2(s) + \int_0^s \phi_2^2(s, s_1) ds_1 + \int_0^s \int_0^{s_1} \phi_3^2(s, s_1, s_2) ds_2 ds_1 + \cdots\right] ds}, \quad T \geq 0.$$

In particular, when all of the chaos coefficients are one-variable functions, so that $\hat{\phi}_n(T) := \phi_n(T, s_1, \ldots, s_{n-1})$ for each positive integer n, we find that for $T \geq 0$

$$\begin{aligned} h_T =& \hat{\phi}_1^2(T) + \hat{\phi}_2^2(T)T + \hat{\phi}_3^2(T)\frac{1}{2}T^2 + \hat{\phi}_4^2(T)\frac{1}{6}T^3 + \ldots \\ =& \sum_{i=1}^\infty \hat{\phi}_i^2(T) \frac{1}{(i-1)!} T^{i-1}. \end{aligned}$$

When we take $\hat{\phi}_i(T) = \sum_{j=1}^{m_i} b_{ij} e^{-c_{ij}T}$, where b_{ij} and c_{ij} are some constants, we obtain that for $T \geq 0$

$$\begin{aligned} h_T =& \sum_{i=1}^\infty \Big(\sum_{j=1}^{m_i} b_{ij} e^{-c_{ij}T} \Big)^2 \frac{1}{(i-1)!} T^{i-1} \\ =& \sum_{i=1}^\infty \Big(\sum_{j=1}^{m_i} \widetilde{b}_{ij} e^{-c_{ij}T} \Big)^2 T^{i-1} \quad \text{where} \quad \widetilde{b}_{ij} = \frac{b_{ij}}{\sqrt{(i-1)!}}, \end{aligned}$$

which may be compared with the Björk and Christensen descriptive form, because we have that

$$f_{0T} = \frac{\sum_{i=1}^\infty \Big(\sum_{j=1}^{m_i} \widetilde{b}_{ij} e^{-c_{ij}T} \Big)^2 T^{i-1}}{\int_T^\infty \sum_{i=1}^\infty \Big(\sum_{j=1}^{m_i} \widetilde{b}_{ij} e^{-c_{ij}s} \Big)^2 s^{i-1} ds}, \quad T \geq 0.$$

We observe here that $\lim_{t\to\infty} h_t = 0$. However the instantaneous forward rate is expressed by the quotient form where the function h_t is located in the numerator. So, it is not immediately apparent that

$$\lim_{t\to\infty} f_{0t} = 0.$$

4.7.3 Modelling the chaos coefficients

We now apply the descriptive form to all the chaos coefficients, not only to One-variable Chaos Models. Let us define for the Factorizable Chaos Models that

$$\phi_1(s) = \alpha(s), \quad \phi_2(s, s_1) = \beta(s)\gamma(s_1), \quad \phi_3(s, s_1, s_2) = \delta(s)\epsilon(s_1)\zeta(s_2), \quad \cdots,$$

for $0 \le s_n \cdots \le s_2 \le s_1 \le s < \infty$ where $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$ are deterministic functions. Then we obtain the initial instantaneous forward rate curves can be modelled respectively as follows: For $T \ge 0$

$$f_{0T} = \frac{\phi_1^2(T)}{\int_T^\infty \phi_1^2(s)ds}, \qquad \text{(First Chaos Model).}$$

$$f_{0T} = \frac{\alpha_T^2 + \beta_T^2 T}{\int_T^\infty [\alpha_s^2 + \beta_s^2 s]ds}, \qquad \text{(One-variable Second Chaos Model).}$$

$$f_{0T} = \frac{\alpha_T^2 + \beta_T^2 \int_0^T \gamma_{s_1}^2 ds_1}{\int_T^\infty [\alpha_s^2 + \beta_s^2 \int_0^s \gamma_{s_1}^2 ds_1]ds}, \qquad \text{(Factorizable Second Chaos Model).}$$

$$f_{0T} = \frac{\alpha_T^2 + \beta_T^2 T + \frac{1}{2}\delta_T^2 T^2}{\int_T^\infty [\alpha_s^2 + \beta_s^2 s + \frac{1}{2}\delta_s^2 s^2]ds}, \qquad \text{(One-variable Third Chaos Model).}$$

We list all possible choices of the chaos coefficients by the descriptive form. For the higher order Chaos Models, we investigate all combinations of the forms having six parameters and seven parameters. Note here that it is possible that all of the forms given below belong to the First Chaos Model for the initial yield curve fitting, where we have set $M_{0s} = \phi_1^2(s)$.

First Chaos Models

1. $\phi_1(s) = b_1 e^{-c_1 s}$ (Exponential form, 2 parameters).

2. $\phi_1(s) = (b_1 + b_2 s)e^{-c_1 s}$ (Nelson-Siegel Form, 3 parameters).

3. $\phi_1(s) = (b_1 + b_2 s)e^{-c_1 s} + b_3 s e^{-c_2 s}$ (Svensson Form, 5 parameters).

One variable Second Chaos Models, 6 parameters, 2 functions

4. $\alpha(s) = (b_1 + b_2 s)e^{-c_1 s}, \quad \beta(s) = (b_3 + b_4 s)e^{-c_2 s}.$

One variable Second Chaos Models, 7 parameters, 2 functions

5. $\alpha(s) = b_1 e^{-c_1 s}, \quad \beta(s) = (b_2 + b_3 s)e^{-c_2 s} + b_4 s e^{-c_3 s}.$

6. $\alpha(s) = (b_1 + b_2 s)e^{-c_1 s} + b_3 s e^{-c_2 s}, \quad \beta(s) = b_4 e^{-c_3 s}.$

Factorizable Second Chaos Models, 6 parameters, 3 functions

7. $\alpha(s) = b_1 e^{-c_1 s}, \quad \beta(s) = b_2 e^{-c_2 s}, \quad \gamma(s) = (1 + b_3 s)e^{-c_3 s}.$

8. $\alpha(s) = b_1 e^{-c_1 s}, \quad \beta(s) = (b_2 + b_3 s)e^{-c_2 s}, \quad \gamma(s) = e^{-c_3 s}.$

9. $\alpha(s) = (b_1 + b_2 s)e^{-c_1 s}, \quad \beta(s) = b_3 e^{-c_2 s}, \quad \gamma(s) = e^{-c_3 s}.$

Factorizable Second Chaos Models, 7 parameters, 3 functions

10. $\alpha(s) = b_1 e^{-c_1 s}, \quad \beta(s) = (b_2 + b_3 s)e^{-c_2 s}, \quad \gamma(s) = (1 + b_4 s)e^{-c_3 s}.$

11. $\alpha(s) = (b_1 + b_2 s)e^{-c_1 s}, \quad \beta(s) = b_3 e^{-c_2 s}, \quad \gamma(s) = (1 + b_4 s)e^{-c_3 s}.$

One variable Third Chaos Models, 6 parameters, 3 functions

12. $\alpha(s) = b_1 e^{-c_1 s}, \quad \beta(s) = b_2 e^{-c_2 s}, \quad \delta(s) = b_3 e^{-c_3 s}.$

One variable Third Chaos Models, 7 parameters, 3 functions

13. $\alpha(s) = b_1 e^{-c_1 s}, \quad \beta(s) = b_2 e^{-c_2 s}, \quad \delta(s) = (b_3 + b_4 s)e^{-c_3 s}.$

14. $\alpha(s) = b_1 e^{-c_1 s}, \quad \beta(s) = (b_2 + b_3 s)e^{-c_2 s}, \quad \delta(s) = b_4 e^{-c_3 s}.$

15. $\alpha(s) = (b_1 + b_2 s)e^{-c_1 s}, \quad \beta(s) = b_3 e^{-c_2 s}, \quad \delta(s) = b_4 e^{-c_3 s}.$

One-variable Third Chaos Model, 9 parameters, 3 functions

16. $\alpha(s) = (b_1 + b_2 s)e^{-c_1 s}, \quad \beta(s) = (b_3 + b_4 s)e^{-c_2 s}, \quad \delta(s) = (b_5 + b_6 s)e^{-c_3 s}.$

One-distribution Third Chaos Models, 7 parameters, 4 functions

17. $\alpha(s) = b_1 e^{-c_1 s}, \quad \beta(s) = b_2 e^{-c_2 s}, \quad \delta(s) = b_3 e^{-c_3 s}, \quad \gamma(s) = \epsilon(s) = \eta(s) = e^{-c_4 s}.$

18. $\alpha(s) = b_1 e^{-c_1 s}, \quad \beta(s) = e^{-c_2 s}, \quad \delta(s) = e^{-c_3 s}, \quad \gamma(s) = \epsilon(s) = \eta(s) = (b_2 + b_3 s)e^{-c_4 s}.$

One-distribution Third Chaos Model, 8 parameters, 4 functions

19. $\alpha(s) = (b_1 + b_2 s)e^{-c_1 s}, \quad \beta(s) = b_3 e^{-c_2 s}, \quad \delta(s) = b_4 e^{-c_3 s}, \quad \gamma(s) = \epsilon(s) = \eta(s) = e^{-c_4 s}.$

The other Third Chaos Models, 7 parameters, 4 functions

20. $\alpha(s) = b_1 e^{-c_1 s}, \quad \beta(s) = b_2 e^{-c_2 s}, \quad \gamma(s) = e^{-c_3 s}, \quad \delta(s) = b_3 e^{-c_4 s}.$

21. $\alpha(s) = b_1 e^{-c_1 s}, \quad \beta(s) = b_2 e^{-c_2 s}, \quad \delta(s) = b_3 e^{-c_3 s}, \quad \epsilon(s) = e^{-c_4 s}.$

22. $\alpha(s) = b_1 e^{-c_1 s}, \quad \beta(s) = b_2 e^{-c_2 s}, \quad \delta(s) = b_3 e^{-c_3 s}, \quad \zeta(s) = e^{-c_4 s}.$

One-variable Fourth Chaos Model, 8 parameters, 4 functions

23. $\alpha(s) = b_1 e^{-c_1 s}, \quad \beta(s) = b_2 e^{-c_2 s}, \quad \delta(s) = b_3 e^{-c_3 s}, \quad \eta(s) = b_4 e^{-c_4 s}.$

Chapter 5

Term Structure Calibration

In the previous chapter we specified coefficients of the Chaos Models, without loss of generality, by applying the exponential polynomial family. This specification allows us to compute both initial yield and volatilities at the same time. However, we start our calibration by looking at only initial yield curves. Here, our main concern in this chapter is to check if our specification of the chaos coefficients allows to fit well into the initial yield curves. As seen in [1], the Nelson-Siegel Form ([64]) and the Svensson Form ([86]) are the ones that most central banks apply, with the exception of those in Japan, UK and USA which apply Smoothing splines. These forms may be regarded as a special case of the general parametric form suggested by Björk and Christensen in [12]. Unfortunately this model has the shortcoming that it allows negative interest rates. We compare initial curve fitting ability of the Chaos Models with those parametric forms and also among different chaos orders by using data from the UK bond market. We show that the proposed model attains just as good a fitting to yields as the Svensson Form does, while also keeping the interest rate positivity condition.

5.1 Calibration Data

A set of yield curve data can be extracted either from the government bond market (bond prices) or the money markets (LIBOR and swap rates). In this chapter we use the bond market because it contains more maturities. However, since Caps and Swaptions are underlined respectively on the forward LIBOR rate and the forward swap rate, we extract yield curves from the money markets for the option price calibration in the next chapter.

We find the clean prices of treasury coupon strips in the UK bond market from the United Kingdom Debt Management Office (DMO) [90], and directly apply the zero coupon yield process $(y_{tT})_{0\le t\le T<\infty}$. Here an Actual/Actual day-count convention is applied, i.e.,

$$\text{Factor} = \frac{\text{Days not in leap year}}{365} + \frac{\text{Days in leap year}}{366}.$$

We consider the following two data sets:

- The yield data at 146 dates (every other business day) from January 1998 to January 1999. Each data point has around 49 to 62 maturities,
- The yield data at 157 dates (every Friday) from December 2002 to December 2005. Each data point has around 100 to 130 maturities.

Note that the first data set contains a volatile market including the period of the Long-Term Capital Management (LTCM) crisis, and the second data set is from a more moderate market and holds more maturities of yields.

5.2 Models

For the calibration we consider all possible models of the First Chaos, Factorizable Second Chaos, and One-variable Third Chaos Models, as was specified in Section 4.7.3; i.e., we calibrate the models numbered 2 through 15. We compare our results with the traditional descriptive forms:

Nelson and Siegel Form, four parameters:

(5.2.1) $$f_{0T} = b_0 + [b_1 + b_2 T]e^{-c_1 T}, \quad \text{such that} \quad b_0 \ge 0, c_1 \ge 0,$$

Svensson Form, six parameters:

$$f_{0T} = b_0 + [b_1 + b_2 T]e^{-c_1 T} + b_3 T e^{-c_2 T} \quad \text{such that} \quad b_0 \geq 0, c_1 \geq 0, c_2 \geq 0. \tag{5.2.2}$$

5.3 Calibration Methods

For the calibration there are various methods available already in the literature, such as the weighted least squares method ([20], [21], [73]), the maximum likelihood estimation method ([20], [21], hereafter referred to as MLE), and the Kalman Filtering Method ([54], [82]). We apply the maximum likelihood method for our calibration, and also the global search procedure ([84]) to find the global maximum, that is, we take several random starting points to find the global maximum. Let us now recall the weighted least squares method and the MLE in this section.

5.3.1 Weighted Least Squares Method

Let us first denote by $\overline{y}_{0T_i}$ the real market yield data maturing at time $T_i \geq 0$, and denote by y_{0T_i} the theoretical prices. The weighted least-squared method consists of minimizing the following function with respect to the parameters:

$$\frac{1}{n_1}\sum_{i=1}^{n_1}\left[\frac{y_{0T_i} - \overline{y}_{0T_i}}{w_i}\right]^2, \tag{5.3.1}$$

where $0 \leq T_1 \leq T_2, \ldots, \leq T_{n_1} < \infty$ is a sequence of the maturities in yields and w_i is the weight of the objective function.

5.3.2 Maximum Likelihood Estimation Method

As an alternative to the weighted least-square method, Cairns has suggested in [20], [21] the MLE method. To use this MLE method, we must assume that

$$ln\overline{P}_{0T_i} \sim \mathcal{N}(lnP_{0T_i}, \nu^2(P_{0T_i}, d_i)) \quad \text{for each } T_i \geq 0,$$

where

$$\nu^2(p,d) = \frac{\sigma_0^2(p)[\sigma_\infty^2 d^2 b(p) + 1]}{\sigma_0^2(p) d^2 b(p) + 1}, \quad b(p) = \frac{\sigma_d^2}{\sigma_0^2(p)[\sigma_\infty - \sigma_0^2(p)]},$$

with the standard deviations defined by:

$$\sigma_0(p) := \frac{1}{\lim_{d\to 0}\sqrt{Var(p)}}\frac{1}{p}, \quad \sigma_d := \lim_{d\to 0}\frac{\partial \nu^2(p,d)}{\partial(d^2)}, \quad \sigma_\infty := \lim_{d\to\infty}\nu^2(p,d),$$

and d_i for the Macaulay duration, which is time in years to maturity for strips. We have the following MLE function:

$$\prod_{i=1}^{n_1}\frac{1}{\sqrt{2\pi\nu^2(P_{0T_i},d_i)}}\exp\Big[-\frac{(lnP_{0T_i} - ln\overline{P}_{0T_i})^2}{2\nu^2(P_{0T_i},d_i)}\Big],$$

which leads to the following log-likelihood function:

$$-\frac{1}{2}\sum_{i=1}^{n_1}\Big[ln[2\pi\nu^2(P_{0T_i},d_i)] + \frac{(lnP_{0T_i} - ln\overline{P}_{0T_i})^2}{\nu^2(P_{0T_i},d_i)}\Big].$$

However, because ν is a constant, the Cairns MLE is equivalent to minimizing the following weighted least-squares function:

$$\sum_{i=1}^{n_1}\Big[\frac{lnP_{0T_i} - ln\overline{P}_{0T_i}}{\nu(P_{0T_i},d_i)}\Big]^2. \tag{5.3.2}$$

Cairns has applied the specific choices $(\sigma_0(p), \sigma_d, \sigma_\infty) = (\frac{1}{100p}, 0.0004, 0.001)$ to the German bond market data between 4 January 1996 and 12 April 1997 in the paper [20], and $(\sigma_0(p), \sigma_d, \sigma_\infty) = (\frac{1}{3200p}, 0.0005, 0.001)$ to the UK bond market data between January 1992 and November 1996 in the other paper [21]. Looking at the Cairns paper [21], the assumption there is that the published bond prices have rounding error of around 1/32 per 100 nominal price, and for this reason, the value $\sigma_0(p) = \frac{1}{3200p}$ is applied there. Here Cairns has chosen these values from the historical market data with advice from various practitioners. From our experiment, the form of the function $\sigma_0(p)$ greatly affects the value of the likelihood function value, whereas the other two functions σ_d and σ_∞ do not. Because our calibration dataset is taken from the UK bond market, and is from just after his calibration data set time period, we also apply $(\sigma_0(p), \sigma_d, \sigma_\infty) = (\frac{1}{3200p}, 0.0005, 0.001)$.

5.3.3 Scoring Measures

Setting $w_i \equiv 1$ in the least-squared function (5.3.1), the calibration is equivalent to minimizing the Root-Mean-Squared Error (RMSE). Setting $w_i = \bar{y}_{0T_i}$, it is equivalent to minimizing the Root-Mean-Squared Percentage Error (RMSPE) due to the relations

$$RMSE = \sqrt{\frac{1}{n_1}\sum_{i=1}^{n_1}\left[y_{0T_i} - \overline{y}_{0T_i}\right]^2} \quad \text{and} \quad RMSPE = \sqrt{\frac{1}{n_1}\sum_{i=1}^{n_1}\left[\frac{y_{0T_i} - \overline{y}_{0T_i}}{\overline{y}_{0T_i}}\right]^2}.$$

5.3.4 Diebold-Mariano Statistics

We apply the Diebold-Mariano Statistics ([30], hereafter referred to as DM statistics) with the Newey-West standard errors ([65]) to compare fitting performances as is done in [55] and [88]. Here for the computation we use the program DMARIANO ([4]) in the statistics package STATA, where the lag order is computed from the Schwert criterion to be thirteen in both of our two datasets. The null hypothesis, which is that two models have the same fitting errors, can be rejected at 5% level if the absolute value of the DM statistics is greater than 1.96. The DM statistics is based on RMSPEs which are Squared Percentage Errors in [55] and RMSEs in [88]. We compare the calibration performance of the Chaos Models with the descriptive forms, i.e., Nelson-Siegel Form and Svensson Form. In our computations, the higher number means that the model outperforms the corresponding descriptive form.

5.4 Calibration Results

Lets us first explain the notation used in Tables 5.1 - 5.2. “No.” in the tables stands for the model numbers specified in Section 4.7.3, “N” for the number of the parameters, “L” for the likelihood function, “DM-NS” for DM statistics compared with Nelson-Siegel Form and “DM-Sv” for DM statistics compared with Svensson Form. A higher number of DM statistics means that the model outperforms the descriptive form. Moreover, we represent the RMSEs and RMSPEs by percentage, i.e., 0.73 means 0.73%.

Analyzing the calibration results from Tables 5.1 - 5.2 and Figures 5.1 - 5.2, we first notice that errors by the Nelson-Siegel Form in the volatile market ($1998 - 1999$) are very high, that is, the average RMSPE of all dates is 2.67%, which is much higher than the errors given by the Chaos Models. Indeed, comparing the Chaos Models with the Nelson-Siegel Form by the DM Statistics we are able to show that all higher order Chaos Models work better, as can be seen in Table 5.1. On the contrary, errors by the Nelson-Siegel Form in the moderate market ($2002 - 2005$) are relatively small, that is, the average RMSPE is 0.97% as can be seen in Table 5.2 and Figure 5.2. However, most of the Chaos Models achieved even smaller errors and we show by the DM Statistics in Table 5.2 that around half of the suggested Chaos Models outperform the Nelson-Siegel Form in this data set.

On the other hand, we observe from the tables and figures that the Svensson Form achieved very small RMSPE. Comparing the Chaos Models with the Svensson Form, we are able to accept the null hypothesis in the DM Statistics for most of the Chaos Models as is seen in Tables 5.1 - 5.2. It means that we cannot state that there exists significant difference in the calibration performances between the Svensson Form and the Chaos Models. Here, we are not able to show that the Chaos Models work significantly better in either of the two datasets. However, in addition to ensuring interest rate positivity, the Chaos Models are advantageous for modelling volatilities. Moreover, the calibrated parameters of the yields can be applied directly to the volatility term structure. This saves degrees of freedom in the option price calibration, as we observe in the next chapter.

As is stated in Section 4.7.3, for the initial yield curve fitting, it is possible that all of the Chaos Models belong to the First Chaos Model. Indeed we do not find significant difference in the calibration performances between the One-variable Second Chaos Models and the One-variable Third Chaos Models. However, looking at Figure

5.1 and Figure 5.2, we observe that the Factorizable Second Chaos Models show different results from the One-variable Chaos Models. The stabilities in RMSPE are not maintained over the calibration dates. Therefore having the exponential form in the third function of the Factorizable Second Chaos Model is not desirable.

Table 5.1: Yield Calibration in 1998 − 1999 (Volatile Market)

No.	Model	N	-L	RMSE (%)	RMSPE (%)	DM-NS	DM-Sv
2	1st chaos (a)	3	4420	0.73	4.44	-3.41	-11.46
3	1st chaos (b)	5	250	0.19	0.86	4.09	-3.54
4	one-var 2nd chaos	6	162	0.15	0.82	4.52	-2.26
5	one-var 2nd chaos (a)	7	160	0.15	0.69	4.48	0.22
6	one-var 2nd chaos (b)	7	145	0.14	0.75	4.48	-1.05
7	factorizable 2nd (a)	6	335	0.19	0.88	4.46	-2.54
8	factorizable 2nd (b)	6	245	0.19	0.68	4.20	0.27
9	factorizable 2nd (c)	6	1245	0.37	1.26	3.96	-3.81
10	factorizable 2nd (a)	7	179	0.16	0.63	4.35	1.38
11	factorizable 2nd (b)	7	153	0.14	0.72	4.46	-1.07
12	one-var 3rd chaos	6	168	0.15	0.72	4.40	-1.24
13	one-var 3rd chaos (a)	7	141	0.14	0.76	4.36	-1.16
14	one-var 3rd chaos (b)	7	152	0.14	0.72	4.48	-1.19
15	one-var 3rd chaos (c)	7	149	0.14	0.76	4.42	-1.43
-	Descriptive NS	4	2101	0.49	2.67	-	-4.45
-	Descriptive Sv	6	160	0.15	0.70	4.45	-

Table 5.2: Yield Calibration in 2002 − 2005 (Moderate Market)

No.	Model	N	-L	RMSE (%)	RMSPE (%)	DM-NS	DM-Sv
2	1st chaos (a)	3	8716	0.69	3.96	-3.42	-3.50
3	1st chaos (b)	5	438	0.17	0.99	-0.35	-1.99
4	one-var 2nd chaos	6	388	0.15	0.89	0.75	-1.23
5	one-var 2nd chaos (a)	7	388	0.15	0.80	1.45	-0.38
6	one-var 2nd chaos (b)	7	329	0.14	0.66	5.33	1.26
7	factorizable 2nd (a)	6	437	0.16	1.04	-0.87	-3.33
8	factorizable 2nd (b)	6	495	0.17	0.84	2.16	-0.68
9	factorizable 2nd (c)	6	421	0.16	1.19	-1.70	-2.84
10	factorizable 2nd (a)	7	365	0.15	0.82	1.83	-0.78
11	factorizable 2nd (b)	7	323	0.14	0.72	3.93	0.36
12	one-var 3rd chaos	6	388	0.15	0.87	0.78	-1.06
13	one-var 3rd chaos (a)	7	350	0.15	0.78	2.06	-0.11
14	one-var 3rd chaos (b)	7	367	0.15	0.68	3.31	1.24
15	one-var 3rd chaos (c)	7	325	0.14	0.69	3.46	0.60
-	Descriptive NS	4	541	0.18	0.97	-	-1.76
-	Descriptive Sv	6	442	0.16	0.76	1.76	-

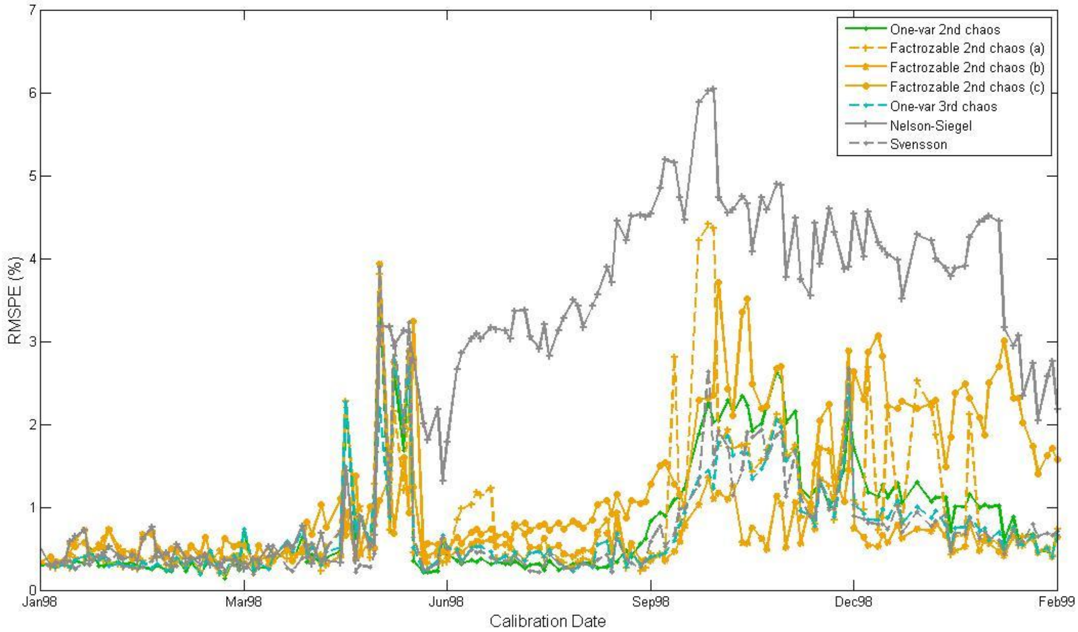

Figure 5.1: RMSPE (Yield Calibration in $1998 - 1999$)

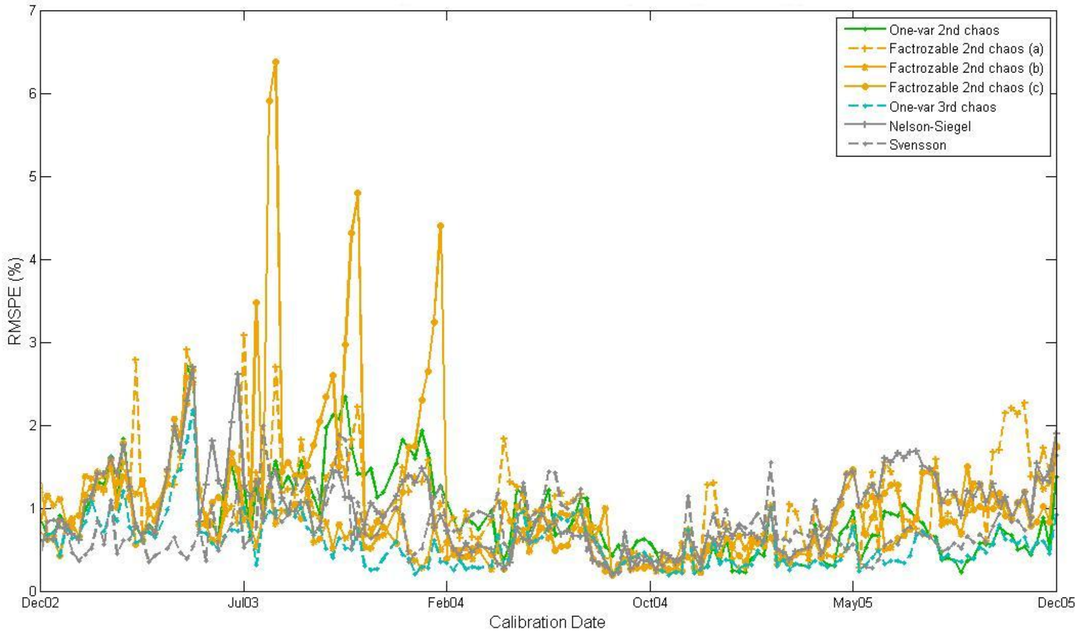

Figure 5.2: RMSPE (Yield Calibration in $2002 - 2005$)

Chapter 6

Option Price Calibration

In the previous chapter we showed that the Chaotic Approach has good fitting ability to the yield curves. In this chapter, we take the ATM European Options, particularly Caps and Swaptions, into account. The issue of the volatility smile will be considered in Chapter 7. We compare the models among different chaos orders, and also some popular and classical interest rate models. The books by Brigo and Mercurio ([14]) and James and Webber ([54]) claim that the term structure of caplet volatility has a humped shape in a moderate market condition. For example, to achieve a good fitting into the humped shape of the implied volatility, Rebonato ([75]) suggested the Nelson-Siegel Form applied to the instantaneous caplet volatility in the LFM, that is, $\sigma_i(t)$. Though the LFM has some crucial problems with the volatility smile, it is able to achieve great fitting ability into the caplet volatility term structure with desirable hump shaped curves, where many other existing models are unable to do this. Nevertheless, our calibration work shows that the Chaos Models also succeed on fitting the humped volatility term structure. The SABR Model holds the stochastic volatility feature used in the current financial practice as a market standard model. It is well known that the SABR Model achieves good fitting to volatility smiles. We observe in this chapter that the calibration errors in the Chaos Models are smaller than the same errors for the SABR Model for the ATM options. This is mainly because the SABR Model does not fit very well to ATM Options across maturity and tenor whereas the Chaos Models do.

For this calibration we apply the day-by-day calibration methodology, which is also called the global approach (for instance, in [51], page 223), by the least squares method, as has been done many times before. For example of this approach, see [35], [55], [62] and [73]. In particular, the literature [35] and [62] mentions the importance of such calibration work rather than time series calibration, claiming that more valuable information about the volatility of forward LIBOR rates is in the present market than in the historical data. Our main motivation here is to replicate the current financial market by as small number of parameters as possible, which may then be used for pricing and hedging exotic options, such as the Chooser flexible cap and Bermudan Swaption. We compare the calibration performances by the DM-statistics, exactly as in the yield calibration.

6.1 Calibration Data

The zero-coupon yields are from the money markets, which are bootstrapped from the LIBOR, Futures and Swap rates (see [83] for the detail of the bootstrapping technique). Interest Rate Option prices are obtained from ICAP (Garban Intercapital - London) and TTKL (Tullett & Tokyo Liberty - London) via the Bloomberg Database.[1] We consider the UK interest rate market for our calibration. The GBP Caps/Floors apply three month frequencies for all caplets with ACT/365 day count convention, where all payments are in arrears. The GBP Swaptions apply Semi/Semi basis and ACT/365 day count convention where all payments are six months in arrears. We particularly consider the following two data sets:

- Data between September 2000 and August 2001 at 53 dates (every Friday closing

[1] Here we would like to acknowledge helpfulness of the Bloomberg help desk staff, who have aided greatly our understanding of the actual market data. We particularly wish to extend thanks David Culshaw, from ICAP, for his assistance.

mid price). In each date we have that:

- 17 zero-coupon yields, maturing in 1M, 2M, 3M, 1Y, 1Y6M, 2Y, 3Y, ..., 10Y, 12Y, 15Y, 20Y[2].
- 37 ATM Caplets implied volatilities maturing in 1Y, 1Y3M, ..., 9Y9M, 10Y.
- 7×6 ATM Swaptions implied volatilities, maturing in 1M, 3M, 6M, 1Y, 2Y, 3Y, 5Y, where the underlying swap contracts are maturing in 1Y, 2Y, 3Y,5Y, 7Y, 10Y, which are lengths called "tenor".

• Data between May 2005 and May 2006 at 53 dates (every Friday closing mid price). In each date we have that:

- 22 zero-coupon yields, maturing in 1M, 2M, 3M, 4M, 7M, 10M, 1Y1M, 1Y4M, 1Y7M, 1Y10M, 2Y1M, 3Y, ..., 10Y, 12Y, 15Y, 20Y[3].
- 77 ATM Caplets implied volatilities maturing in 1Y, 1Y3M, ..., 19Y9M, 20Y.
- 42 ATM Swaptions implied volatilities, maturing in 1M, 3M, 6M, 1Y, 2Y, 3Y, 5Y, where the underlying swap contracts are maturing in 1Y, 2Y, 3Y,5Y, 7Y, 10Y.

Here, M and Y stand for month and year respectively. Note here that the option data corresponds to a part of the data in [88], where data was analyzed between August 1998 and January 2007. The Caplet implied volatilities are bootstrapped from the ATM Caps implied volatilities observed in the market by the technique given in the book [35], where the ATM Caplet implied volatilities maturing at six months and nine months are obtained by constant extrapolation. Though the extrapolation is

[2]Though we observe 23 yields we do not use very short maturities and long maturities yield.
[3]Though we observe 30 yields we do not use very short maturities and long maturities yield.

necessary to bootstrap the other ATM Caplet implied volatilities, when we calibrate the data the extrapolated prices give us great errors. Hence, though we follow the book and implement the extrapolation, we do not use those two short maturities for the calibration. Moreover, we observed some obvious outliers and corrected them accordingly. As Gatheral mentions in his book ([36]), it seems indeed to be difficult to bootstrap the market values without allowing any arbitrage opportunity. The volatility term structures, which we obtained from the bootstrap technique, are not smooth curves as the reader may observe from Figure 6.1 and Figure 6.2. However, as can be seen in [35] (page 78), these are not abnormal feature. The reader might also like to compare it with the smooth curves in the books [14] (page $88-95$) and [54] (page 50).

The book [14] claims the existence of a relationship between the shape of that implied volatility curve and the shape of the instantaneous forward rate volatility curve. It is often observed that both curves have humped shape at the same time. On the other hand, in the paper [88], the authors have applied the Nelson-Siegel Form to the instantaneous forward rate volatility curve and calibrated it, though they have not investigated the implied volatility curve structure.

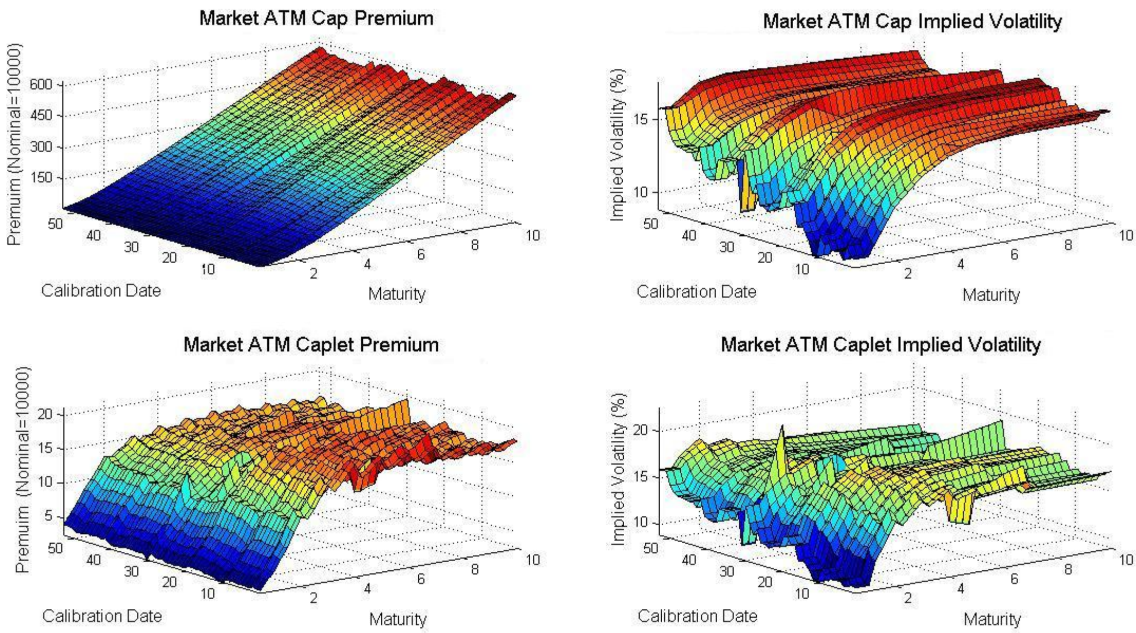

Figure 6.1: Market Quotes of Caps and Caplets in Sep 2000 - Aug 2001

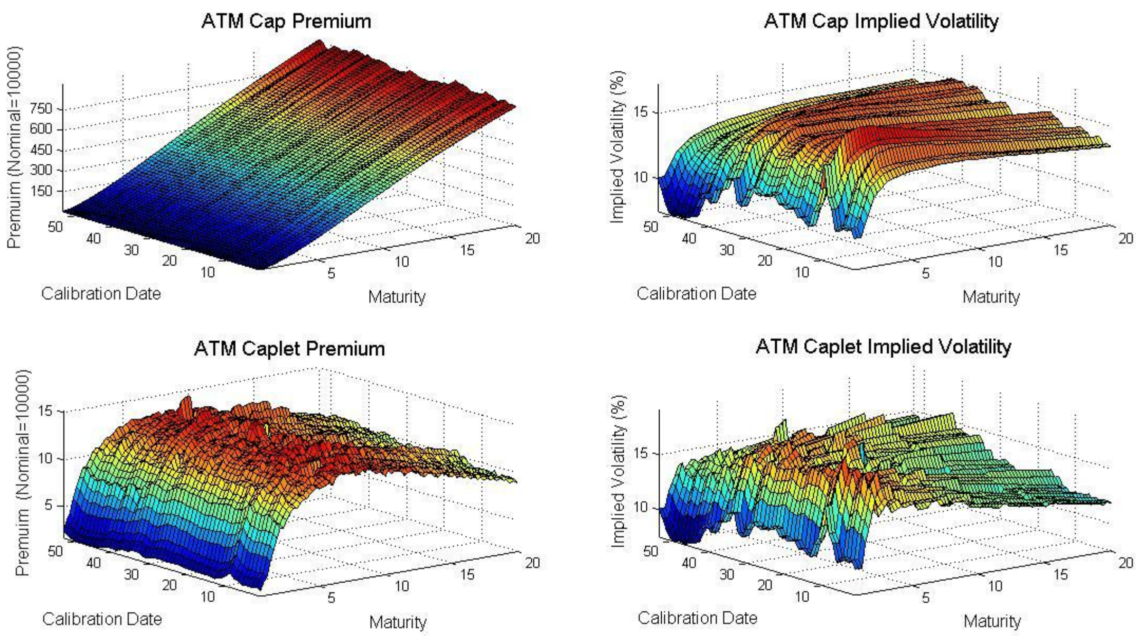

Figure 6.2: Market Quotes of Caps and Caplets in May 2005 - May 2006

6.2 Models

In our calibration we consider the Chaos Models numbered $4, 6, 9, 12, 15, 17, 19, 23$ in Section 4.7.3. We also calibrate the following seven models for the purpose of comparison.

6.2.1 Other models in the Potential Approach

I. Rational Lognormal Model with Nakamura-Yu form and constant $\widetilde{\sigma}$ with Svensson Form, 9 parameters

To implement the Rational Lognormal Model, (see, Section 2.1.4), Nakamura and Yu in [63] choose the following forms of the functions g_1 and g_2:

$$g_1(t) = -\alpha \frac{\partial P_{0t}}{\partial t}(P_{0t})^{\gamma} \quad \text{and} \quad g_2(t) = -\frac{\partial P_{0t}}{\partial t}[1 - \alpha(P_{0t})^{\gamma}], \quad \text{for} \quad t \geq 0,$$

for some constants $\alpha, \gamma \in \mathbb{R}$. This choice gives us that

$$G_1(t) = \frac{\alpha}{\gamma + 1}(P_{0t})^{\gamma+1} \quad \text{and} \quad G_2(t) = P_{0t} - G_1(t), \quad \text{for} \quad t \geq 0.$$

This means that the initial bond price P_{0t} can be modelled independently from the Rational Lognormal Model. We hence apply the Svensson Form for our calibration to the initial forward rate and to express the initial bond prices. A form of the function M_t is not specified in the paper [63]. However in [38] and [73], it is represented as an exponential martingale

$$M_t = \exp\Big[-\frac{1}{2}\int_0^t \widetilde{\sigma}_s^2 ds + \int_0^t \widetilde{\sigma}_s dW_s \Big], \quad \text{for} \quad t \geq 0,$$

for some deterministic function $\widetilde{\sigma}$. In particular, as in [38] we assume that the function $\widetilde{\sigma}$ is just a constant value, i.e. $\widetilde{\sigma} = \beta$ where $\beta \in \mathbb{R}$.

II. Rational Lognormal Model with Nakamura-Yu form and exponential $\widetilde{\sigma}_t$ with Svensson Form, 9 parameters

We also calibrate the Rational Lognormal Model with an exponential form $\widetilde{\sigma}_t = e^{-\beta t}$ as an experiment, although this example is not found in the literature.

III. Constantinides Model, 5 **parameters**

Let us consider the one-factor Constantinides Model, (see, Section 2.1.5), for our calibration, i.e. we set $i = 1$. Constantinides assumed that $\alpha_1 = 0$. We do not do this, in order to be as general as possible. We find that for $t \geq 0$,

$$V_t = \exp\Big[-\Big(g + \frac{\sigma_0^2}{2}\Big)t + \sigma_0 W_0(t) + (x_1(t) - \alpha_1)^2\Big], \tag{6.2.1}$$

where $x_1(t)$ is the OU process defined by

$$dx_1(t) = -\lambda_1 x_1(t)dt + \sigma_1 dW_1(t).$$

Because the dynamics can be solved by

$$x_1(t) = \sigma_1 e^{-\lambda_1 t} \int_0^t e^{\lambda_1 s} dW_1(s),$$

we may express the state price density as

$$V_t = \exp\Big[-\Big(g + \frac{\sigma_0^2}{2}\Big)t + \sigma_0 W_0(t) + \Big(\sigma_1 e^{-\lambda_1 t} \int_0^t e^{\lambda_1 s} dW_1(s) - \alpha_1\Big)^2\Big].$$

We note here that this model has 6 parameters, these are $g, \sigma_0, \sigma_1, \alpha_1, \lambda_1$ and $x_1(0)$. Then, the discount bond price for $0 \leq t \leq T < \infty$ is given by

$$P_{tT} = H_1^{-\frac{1}{2}}(T-t)\exp\Big[(-g+\lambda_1)(T-t) + H_1^{-1}(T-t)\Big(x_1(t) - \alpha_1 e^{\lambda_1(T-t)}\Big)^2 - (x_1(t) - \alpha_1)^2\Big].$$

In particular, at the initial time $t = 0$ we have that

$$P_{0T} = H_1^{-\frac{1}{2}}(T)\exp\Big[(-g + \lambda_1)T + H_1^{-1}(T)\Big(x_1(0) - \alpha_1 e^{\lambda_1 T}\Big)^2 - (x_1(0) - \alpha_1)^2\Big].$$

We leave the option pricing forms of the Constantinides Model to the Appendix.

6.2.2 Short Rate Models

IV. Hull-White Model with Svensson Form, 8 **parameters**

$$dr_t = (\theta_t - ar_t)dt + \sigma dW_t, \quad f_{0t} = b_0 + [b_1 + b_2 t]e^{-c_1 t} + b_3 t e^{-c_2 t}.$$

V. Cox-Ingersoll-Ross Model, 3 **parameters**

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t.$$

6.2.3 Market Models

VI. Lognormal Forward LIBOR Model with Svensson Form, 13 **parameters**

In our calibration of the LFM (see Section 2.2.3) we apply formulation 6 from [14] (page 224), i.e., the Rebonato Form:

$$\sigma_i(t) = b_1 + (b_2 + b_3(T_{i-1} - t))e^{-c_1(T_{i-1}-t)},$$

for some parameters $b_1, b_2, b_3, c_1 \in \mathbb{R}$. Considering a Swaption maturing at T_a with tenor $T_b - T_a$, its swaption implied volatility may be modelled in the LFM by the Rebonato Approximation ([75]):

$$v_{a,b} = \sqrt{\frac{1}{T_b}\sum_{i,j=a+1}^{b}\frac{w_i(0)w_j(0)F_i(0)F_j(0)}{S_{a,b}^2(0)}\rho_{ij}\int_0^{T_a}\sigma_i(t)\sigma_j(t)dt},$$

where the forward swap rates are assumed to be expressed by the linear combination of forward swap rates as follows:

$$S_{a,b}(t) = \sum_{i=a+1}^{b} w_i(t)F_i(t) \approx \sum_{i=a+1}^{b} w_i(0)F_i(t) \quad \text{where} \quad w_i(t) = \frac{\tau_i P_{tT_i}}{\sum_{k=a+1}^{b}\tau_k P_{tT_k}}.$$

We apply the Schoenmakers and Coffey Form ([85]) for the correlation ρ_{ij}:

$$\rho_{ij} = \exp\Big[-\frac{|j-i|}{m-1}\Big(-ln\rho_\infty + \eta_1\frac{i^2+j^2+ij-3mi-3mj+3i+3j+2m^2-m-4}{(m-2)(m-3)} - \eta_2\frac{i^2+j^2+ij-mi-mj-3i-3j+3m+2}{(n-2)(m-3)}\Big)\Big],$$

for $i,j = 1,2,\ldots,m$ where $m = b-a$ and parameters $\eta_1, \eta_2, \rho_\infty \in \mathbb{R}$ such that $3\eta_2 \geq \eta_2 \geq 0, 0 \leq \eta_1 + \eta_2 \leq -ln\rho_\infty$. Therefore, we need 5 parameters to model the forward rate volatility, 3 parameters to model the correlation. Considering the Svensson Form for the initial curve we use 10 parameters totally to compute a caplet, 13 parameters totally to computer a swaption.

VII. SABR Model with Svensson Form, 12 parameters

The reader might like to read Section 2.2.4 for recalling the SABR Model. As is stated before, there is no link between Caplet pricing and Swaption pricing in the SABR Model. If we compute either only Caps or only Swaptions, the required number of parameters is 9. However we need 12 parameters to compute both Caps and Swaptions.

6.3 Calibration Methods

We implement following three types of calibrations:

1. ATM Swaption with yields calibration (Three dimensional).
2. ATM Caplet with yields calibration (Two dimensional).
3. ATM Swaption and ATM Caplet with yields calibration (Four dimensional).

This last calibration is called "Joint calibration", see [14] (page 539 − 544) and also [88]. Note here that the ATM Swaptions contain both the maturity and tenor dimensions while the ATM Caplets contain only the maturity dimension, since the tenor is usually fixed in the market. Hence, we will show six result tables in total, containing the results of ATM Swaption calibration, ATM Caplet calibration and Joint calibration for each data set. In addition, we compare the fitting performances of the proposed models with the LFM and the SABR Model by the DM statistics. The null hypothesis, which is that two models have the same fitting errors, may be rejected at 5% level if the absolute value of the DM statistics is greater than 1.96. In our computations in the tables in Section 6.4, a higher number in the DM statistics means that the corresponding model displays a better fitting than the benchmarked model, i.e., LFM and SABR.

6.3.1 Objective Function

For ATM option price calibrations we apply the weighted least-square method, that is, we minimize the following function:

$$\frac{1}{n_1}\sum_{i=1}^{n_1}\left[\frac{y_{0T_i}-\overline{y}_{0T_i}}{\overline{y}_{0T_i}}\right]^2+\frac{1}{n_2}\sum_{i=1}^{n_2}\left[\frac{OP(T_i,K_{ATM})-\overline{OP}(T_i,K_{ATM})}{\overline{OP}(T_i,K_{ATM})}\right]^2$$

where $\overline{y}_{0T_i}$ and $\overline{OP}(T_i, K_{ATM})$ are defined for the real market data of yields and at the money interest rate option maturing at $T_i \geq 0$, while y and OP are defined for theoretical prices. We also consider the fitting of both Caps (Floors) and Swaptions, i.e., Joint calibration. Therefore, in addition to the calibration with one type of option, we also consider minimizing the following functions:

$$\frac{1}{n_1}\sum_{i=1}^{n_1}\left[\frac{y_{0T_i}-\overline{y}_{0T_i}}{\overline{y}_{0T_i}}\right]^2+\frac{1}{n_2}\sum_{i=1}^{n_2}\left[\frac{Cpl(T_i,K_{ATM})-\overline{Cpl}(T_i,K_{ATM})}{\overline{Cpl}(T_i,K_{ATM})}\right]^2 \\ +\frac{1}{n_2}\sum_{i=1}^{n_2}\left[\frac{SW(T_i,K_{ATM})-\overline{SW}(T_i,K_{ATM})}{\overline{SW}(T_i,K_{ATM})}\right]^2,$$

where $\bar{Cpl}$ and $\bar{SW}$ denote the real market data of caplets and swaptions respectively, while Cpl and SW denote theoretical prices. We apply the parameters obtained from the calibration for pricing purpose. For example, after calibrating the ATM Swaptions, we price the ATM Caplets which are then compared with the market ATM Caplet prices. Pricing errors are denoted by CplP-PE and CplV-PE for pricing errors in terms of premium and implied volatility respectively (we will see detail of those scoring measures in a later section). Similarly, after calibrating the ATM Caps, we price the ATM Swaptions and compute pricing errors which are denoted by SWP-PE and SWV-PE.

In some models, such as the Hull-White model and the SABR Model, we are able to model initial yield curves and options separately. For these we apply the Svensson Form for the initial curves and minimize ATM options errors only. Moreover, we minimize the least square sum of implied volatilities when analytical implied volatility forms are available, such as in FLM and SABR Model. As we know, we have a one-to-one correspondence between an option premium and its implied volatility via the Black formula. Though we are able to compute analytically the premium from the implied volatility by the Black formula, it is not straightforward to do this in the converse direction, as it requires some approximations. Many models, such as Hull-White, CIR, Constantinides, Chaos Models, and others, do not have analytical implied volatility forms. Indeed, the calibrations for those models are usually implemented by minimizing the least squares sum of the premiums. Because financial markets show the implied volatilities instead of the premiums, and the shapes of the implied volatility curve and surface are very much of interest, it may be better if we could minimize the error of the implied volatilities. However, sensitivity between the implied volatility and premium is high, especially for options away from the money. A small error in implied volatility fitting may cause a great error in premium. Therefore, it is sometimes claimed that calibration by premiums is important, because that is what traders pay.

6.3.2 Simulation

After the calibration we obtain all parameters, which gives us yield and option prices as close as possible to the market values. We keep these artificial prices, but leave our parameters aside. We calibrate our model again but to these artificial prices. This work allows us to see whether we have really achieved the global minimum. For this work we focus on the One-variable Third chaos 6 parameter model, i.e.,

$$\alpha(s) = b_1 e^{-c_1 s}, \quad \beta(s) = b_2 e^{-c_2 s}, \quad \delta(s) = b_3 e^{-c_3 s},$$

and apply the artificial data set obtained from the former calibration work in the second data set, i.e., $2005 - 2006$.

6.3.3 Scoring Measure

We apply RMSPE for our scoring measure. Let us first define

$$\text{Total-E} = \sqrt{\frac{1}{n_1}\sum_{i=1}^{n_1}\left[\frac{y_{0T_i} - \overline{y}_{0T_i}}{\overline{y}_{0T_i}}\right]^2 + \frac{1}{n_2}\sum_{i=1}^{n_2}\left[\frac{OP(T_i, K_{ATM}) - \overline{OP}(T_i, K_{ATM})}{\overline{OP}(T_i, K_{ATM})}\right]^2},$$

where OP is the corresponding option for the calibration,

$$\text{Yield-E} = \sqrt{\frac{1}{n_1}\sum_{i=1}^{n_1}\left[\frac{y_{0T_i} - \overline{y}_{0T_i}}{\overline{y}_{0T_i}}\right]^2},$$

$$\text{SWP-E} = \sqrt{\frac{1}{n_2}\sum_{i=1}^{n_2}\left[\frac{SW(T_i, K_{ATM}) - \overline{SW}(T_i, K_{ATM})}{\overline{SW}(T_i, K_{ATM})}\right]^2},$$

$$\text{SWV-E} = \sqrt{\frac{1}{n_2}\sum_{i=1}^{n_2}\left[\frac{SW^{vol}(T_i, K_{ATM}) - \overline{SW}^{vol}(T_i, K_{ATM})}{\overline{SW}^{vol}(T_i, K_{ATM})}\right]^2},$$

where SW is swaption premium and SW^{vol} is swaption implied volatility,

$$\text{CplP-E} = \sqrt{\frac{1}{n_2}\sum_{i=1}^{n_2}\left[\frac{Cpl(T_i, K_{ATM}) - \overline{Cpl}(T_i, K_{ATM})}{\overline{Cpl}(T_i, K_{ATM})}\right]^2},$$

$$\text{CplV-E} = \sqrt{\frac{1}{n_2}\sum_{i=1}^{n_2}\left[\frac{Cpl^{vol}(T_i, K_{ATM}) - \overline{Cpl}^{vol}(T_i, K_{ATM})}{\overline{Cpl}^{vol}(T_i, K_{ATM})}\right]^2},$$

where Cpl is caplet premium and Cpl^{vol} is caplet implied volatility.

6.4 Calibration Results

We analyze the calibration results using Tables 6.1 - 6.8 and Figures 6.3 - 6.17. Let us first explain the notation used in the tables. As was mentioned before, the model numbers are specified in Section 4.7.3 and Section 6.2 and we apply the Svensson Form in the models I,II,IV,VI,VII. CplP-PE and CplV-PE denotes pricing errors of respectively Caplet premium and Caplet implied volatility from the Swaption calibration. Similarly, SWP-PE and SWV-PE pricing denotes errors of respectively Swaption premium and Swaption implied volatility from the Caplet calibration. The symbol "#" denotes a value which is greater than 100%. In the LFM, we need to calibrate the correlation to compute the Swaptions. Hence we are unable to compute the pricing errors of the Swaptions from the Caplet calibration. Moreover, since Caplet and Swaption formulas are inconsistent in the SABR Model, we are unable to compute the pricing errors in the SABR Model.

In our calibration we found the parameters of the models by minimizing the errors between the market values and the theoretical values so that these parameters replicate the interest rate market as closely as possible. Let us start our observation by looking at Figures 6.9 - 6.12. These plots shows the comparisons between the market data and the replicated data by the interest rate models. From there we can see how well the models simulate the market data. For instance, we observe that the Rational Lognormal Model, Hull-White Model, CIR Model and SABR Model all fail to fit into the implied volatilities across the maturity and the tenor. In particular these models do not succeed in the humped shape curve of the caplet volatility term structure.

The comparison among the chaos orders are analyzed by Figures 6.3 - 6.8. Looking at the plots on the left side, we can see the green lines are below the red lines in most of the cases where the green lines represent the One-variable Third Chaos Models and the red lines represent the Second Chaos Models. This feature is obvious particularly in the Swaption calibrations and the Joint calibrations. Furthermore, Figures 6.13 - 6.16 show the same results in pricing performance. The One-variable Third Chaos Model, numbered 15, would be a particularly ideal model.

Let us further look at the calibration results using Tables 6.7 - 6.8 which show the DM Statistics compared respectively with the LFM and the SABR Model. Because the LFM is formed particularly for fitting well into the volatility term structure, it outperforms the Chaos Models. However, looking at the plots on the right side in

Figures 6.3 - 6.8, we observe the RMSPEs over the calibration time are not greatly different between the LFM and One-variable Third Chaos Models. Furthermore, the LFM has a crucial problem in fitting volatility smile which we consider in the next chapter. On the other hand, the calibration results show that most of the Chaos Models are able to fit the ATM Options better than the SABR Model. One of the remarkable points is that the Chaos Models have a smaller number of parameters, even while incorporating the initial yield curve calibration at the same time.

After the calibration we obtain the best parameters to replicate the market data. In other words, we are able to simulate the market data by the interest rate model. We implement the calibration again but on the replicated artificial data, while putting our parameters aside. We obtain very small average errors in percentage, as is seen in Table 6.9. In this calibration, Figure 6.17 compares the parameters which we obtained from the first calibration and the second calibration. The linear parameters b_1, b_2, b_3 are different between the calibrations. This is because the yield and the options are formulated by the quotient forms in the Chaos Models as seen in Section 4.5.1. However, we observe that the exactly same exponential parameters c_1, c_2, c_3 are obtained. These results convince us that the the global minimization is achieved in our calibrations.

Table 6.1: The average RMSPE (%) of ATM Swaption in 2000 – 2001

No.	Model	N	Total-E	Yield-E	SWP-E	SWV-E	CplP-PE	CplV-PE
4	one-var 2nd chaos	6	7.1	1.8	6.8	7.0	14.5	14.2
6	one-var 2nd chaos	7	7.1	2.0	6.7	7.1	14.6	13.6
9	factorizable 2nd	6	7.1	2.1	6.8	6.8	14.3	13.3
12	one-var 3rd chaos	6	5.3	2.9	4.1	5.2	10.2	11.1
15	one-var 3rd chaos	7	3.8	1.5	3.4	3.6	8.6	8.1
17	one-dist 3rd chaos	7	4.9	2.5	4.1	4.7	13.8	13.8
19	one-dist 3rd chaos	8	3.9	1.7	3.5	3.8	12.8	12.5
23	one-var 4th chaos	8	4.8	2.7	3.9	4.7	9.0	9.9
I	Rational-log (a)	9	8.4	0.6	8.4	8.4	15.3	14.4
II	Rational-log (b)	9	5.9	0.6	5.9	6.0	24.8	24.3
III	Constantinides	5	7.0	2.9	6.3	6.5	99.9	99.9
IV	Hull-White	8	10.2	0.6	10.2	10.3	17.6	16.7
V	CIR	3	8.5	5.1	6.5	8.7	14.2	14.2
VI	LFM	13	5.0	0.6	5.0	5.0	8.1	7.9
VII	SABR	9	7.5	0.6	7.5	7.5	-	-

Table 6.2: The average RMSPE (%) of ATM Swaption in 2005 – 2006

No.	Model	N	Total-E	Yield-E	SWP-E	SWV-E	CplP-PE	CplV-PE
4	one-var 2nd chaos	6	6.5	3.2	5.5	6.4	32.2	35.1
6	one-var 2nd chaos	7	5.0	1.5	4.8	5.1	11.9	13.7
9	factorizable 2nd	6	6.8	2.3	6.4	6.7	13.7	17.1
12	one-var 3rd chaos	6	4.5	2.2	3.8	4.4	21.2	23.3
15	one-var 3rd chaos	7	4.2	1.6	3.8	4.3	13.4	14.5
17	one-dist 3rd chaos	7	4.5	1.9	4.0	4.5	23.7	25.4
19	one-dist 3rd chaos	8	4.1	1.4	3.8	4.0	28.1	26.4
23	one-var 4th chaos	8	4.3	2.0	3.8	4.5	15.1	19.3
I	Rational-log (a)	9	8.2	0.4	8.2	8.0	10.9	10.3
II	Rational-log (b)	9	5.8	0.4	5.8	5.7	35.1	35.0
III	Constantinides	5	6.3	1.9	5.9	6.3	99.8	99.8
IV	Hull-White	8	9.5	0.4	9.5	9.5	11.2	10.7
V	CIR	3	5.9	3.5	4.6	6.0	10.1	12.5
VI	LFM	13	3.5	0.4	3.5	3.5	14.8	14.9
VII	SABR	9	7.5	0.4	7.5	7.5	-	-

Table 6.3: The average RMSPE (%) of ATM Caplet in 2000 − 2001

No.	Model	N	Total-E	Yield-E	CplP-E	CplV-E	SWP-PE	SWV-PE
4	one-var 2nd chaos	6	5.1	2.0	4.6	4.3	14.9	14.4
6	one-var 2nd chaos	7	3.3	1.7	2.7	3.5	16.3	16.7
9	factorizable 2nd	6	3.8	2.1	3.1	3.6	26.5	26.6
12	one-var 3rd chaos	6	4.2	2.0	3.5	4.6	15.5	16.2
15	one-var 3rd chaos	7	3.2	1.3	2.9	3.2	15.7	15.6
17	one-dist 3rd chaos	7	3.4	1.7	2.9	3.2	41.9	41.5
19	one-dist 3rd chaos	8	3.0	1.6	2.4	2.9	41.6	41.3
23	one-var 4th chaos	8	3.7	1.9	3.1	4.3	32.5	32.5
I	Rational-log (a)	9	9.2	0.6	9.2	9.2	13.9	13.9
II	Rational-log (b)	9	4.7	0.6	4.7	5.0	28.9	29.1
III	Constantinides	5	3.8	1.9	3.2	4.2	#	#
IV	Hull-White	8	9.4	0.6	9.4	9.4	#	#
V	CIR	3	10.2	2.8	9.5	9.3	36.0	34.8
VI	LFM	10	3.0	0.6	3.0	1.9	-	-
VII	SABR	9	8.0	0.6	8.0	7.7	-	-

Table 6.4: The average RMSPE (%) of ATM Caplet in 2005 − 2006

No.	Model	N	Total-E	Yield-E	CplP-E	CplV-E	SWP-PE	SWV-PE
4	one-var 2nd chaos	6	6.3	1.6	6.1	7.4	9.4	9.2
6	one-var 2nd chaos	7	3.4	1.5	3.0	4.7	14.0	14.5
9	factorizable 2nd	6	4.3	2.4	3.4	5.1	20.0	19.9
12	one-var 3rd chaos	6	4.9	1.9	4.4	5.7	26.2	25.7
15	one-var 3rd chaos	7	3.6	1.4	3.2	6.1	14.2	14.5
17	one-dist 3rd chaos	7	3.6	1.3	3.3	4.9	35.3	35.3
19	one-dist 3rd chaos	8	3.4	1.3	3.1	5.5	34.1	34.0
23	one-var 4th chaos	8	4.3	1.9	3.7	5.6	35.9	35.4
I	Rational-log (a)	9	9.4	0.4	9.4	9.1	10.5	10.4
II	Rational-log (b)	9	7.4	0.4	7.4	7.0	14.2	14.1
III	Constantinides	5	4.6	1.8	3.4	5.3	#	#
IV	Hull-White	8	8.4	0.4	8.4	8.4	16.3	16.3
V	CIR	3	8.4	2.4	7.8	9.5	28.7	27.3
VI	LFM	10	3.5	0.4	3.5	2.8	-	-
VII	SABR	9	7.8	0.4	7.8	7.5	-	-

Table 6.5: The average RMSPE (%) from Joint Calibration in $2000 - 2001$

No.	Model	N	Total-E	Yield-E	SWP-E	SWV-E	CplP-E	CplV-E
4	one-var 2nd chaos	6	12.5	2.2	9.3	8.6	7.9	8.1
6	one-var 2nd chaos	7	12.1	2.4	9.3	8.7	7.3	7.9
9	factorizable 2nd	6	12.1	2.6	8.4	9.2	8.2	7.9
12	one-var 3rd chaos	6	8.2	4.3	4.4	5.1	5.2	7.2
15	one-var 3rd chaos	7	7.1	1.6	4.4	4.5	5.2	4.9
17	one-dist 3rd chaos	7	8.2	4.4	4.5	5.1	5.1	7.2
19	one-dist 3rd chaos	8	8.0	2.2	4.8	4.8	5.9	5.9
23	one-var 4th chaos	8	8.1	4.3	4.4	5.1	5.2	7.2
I	Rational-log (a)	9	14.6	0.6	10.0	10.0	10.6	9.9
II	Rational-log (b)	9	16.8	0.6	12.3	12.3	11.4	10.3
III	Constantinides	5	25.8	9.2	22.5	24.0	8.1	14.2
IV	Hull-White	8	18.4	0.6	12.2	12.3	13.7	13.0
V	CIR	3	15.3	5.1	8.3	10.2	11.3	12.0
VI	LFM	13	6.5	0.6	5.5	5.5	3.1	3.1
VII	SABR	12	11.1	0.6	7.5	7.5	8.0	7.8

Table 6.6: The average RMSPE (%) from Joint Calibration in $2005 - 2006$

No.	Model	N	Total-E	Yield-E	SWP-E	SWV-E	CplP-E	CplV-E
4	one-var 2nd chaos	6	10.4	2.5	7.3	7.8	6.9	10.6
6	one-var 2nd chaos	7	8.6	1.5	6.3	6.7	5.6	5.8
9	factorizable 2nd	6	10.3	1.9	7.9	8.1	6.2	7.0
12	one-var 3rd chaos	6	9.1	3.3	5.8	6.3	6.1	9.1
15	one-var 3rd chaos	7	7.8	1.8	5.0	5.1	5.5	7.4
17	one-dist 3rd chaos	7	8.7	3.0	5.8	6.6	5.5	7.5
19	one-dist 3rd chaos	8	8.3	1.9	5.7	5.6	5.5	7.5
23	one-var 4th chaos	8	8.5	2.9	5.3	5.8	5.8	8.8
I	Rational-log (a)	9	13.0	0.4	8.4	8.3	9.9	9.4
II	Rational-log (b)	9	13.8	0.4	10.5	10.4	8.8	7.9
III	Constantinides	5	24.1	6.5	19.9	20.2	11.7	14.6
IV	Hull-White	8	14.0	0.4	10.1	10.1	9.5	9.2
V	CIR	3	10.5	3.5	5.3	6.0	8.0	9.5
VI	LFM	13	6.2	0.4	4.8	4.8	3.8	3.8
VII	SABR	12	10.8	0.4	7.5	7.5	7.8	7.5

Table 6.7: Comparison with the LFM by DM-Statistics

No.	Model	N	SW 00'-01'	SW 05'-06'	Cpl 00'-01'	Cpl 05'-06'	JT 00'-01'	JT 05'-06'
4	one-var 2nd chaos	6	-25.18	-21.76	-24.93	-17.28	-35.86	-33.71
6	one-var 2nd chaos	7	-23.97	-9.11	-3.34	1.20	-35.57	-9.95
9	factorizable 2nd	6	-8.88	-13.52	-7.89	-5.37	-18.96	-20.93
12	one-var 3rd chaos	6	-1.32	-4.05	-7.41	-9.14	-8.32	-21.86
15	one-var 3rd chaos	7	4.67	-3.13	-3.60	-1.09	-2.19	-9.21
17	one-dist 3rd chaos	7	0.50	-4.73	-5.41	-0.94	-8.44	-10.41
19	one-dist 3rd chaos	8	3.87	-3.52	1.26	1.12	-5.36	-9.13
23	one-var 4th chaos	8	0.63	-4.30	-5.17	-6.33	-8.05	-13.50
I	Rational-log (a)	9	-16.40	-21.30	-13.80	-17.31	-22.59	-20.83
II	Rational-log (b)	9	-7.38	-22.53	-6.66	-14.65	-45.01	-33.69
III	Constantinides	5	-5.25	-12.33	-5.22	-7.45	-64.33	-49.43
IV	Hull-White	8	-10.53	-12.55	-13.71	-15.49	-14.22	-12.36
V	CIR	3	-12.99	-10.04	-17.60	-17.89	-33.88	-15.07
VII	SABR	12	-13.50	-26.12	-12.35	-13.13	-22.81	-15.27

Table 6.8: Comparison with the SABR Model by DM-Statistics

No.	Model	N	SW 00'-01'	SW 05'-06'	Cpl 00'-01'	Cpl 05'-06'	JT 00'-01'	JT 05'-06'
4	one-var 2nd chaos	6	2.35	4.39	8.21	5.80	-8.77	1.27
6	one-var 2nd chaos	7	2.38	9.97	10.88	14.60	-6.34	5.47
9	factorizable 2nd	6	1.79	1.99	11.91	11.63	-3.41	1.40
12	one-var 3rd chaos	6	9.04	9.71	7.46	8.98	12.18	4.97
15	one-var 3rd chaos	7	14.70	11.70	12.20	13.44	12.77	7.65
17	one-dist 3rd chaos	7	11.38	10.54	10.50	14.90	11.78	13.03
19	one-dist 3rd chaos	8	16.08	14.25	12.04	14.12	11.42	17.61
23	one-var 4th chaos	8	9.90	11.85	8.67	11.82	12.90	7.96
I	Rational-log (a)	9	-3.14	-5.19	-22.85	-40.87	-14.52	-21.65
II	Rational-log (b)	9	14.04	14.72	19.10	4.64	-49.22	-9.04
III	Constantinides	5	1.94	6.80	8.29	8.69	-43.73	-50.95
IV	Hull-White	8	-4.87	-4.85	-11.92	-13.99	-10.03	-7.85
V	CIR	3	-3.75	5.39	-15.18	-3.96	-23.42	1.46
VI	LFM	13	13.50	26.12	12.35	13.13	22.81	15.27

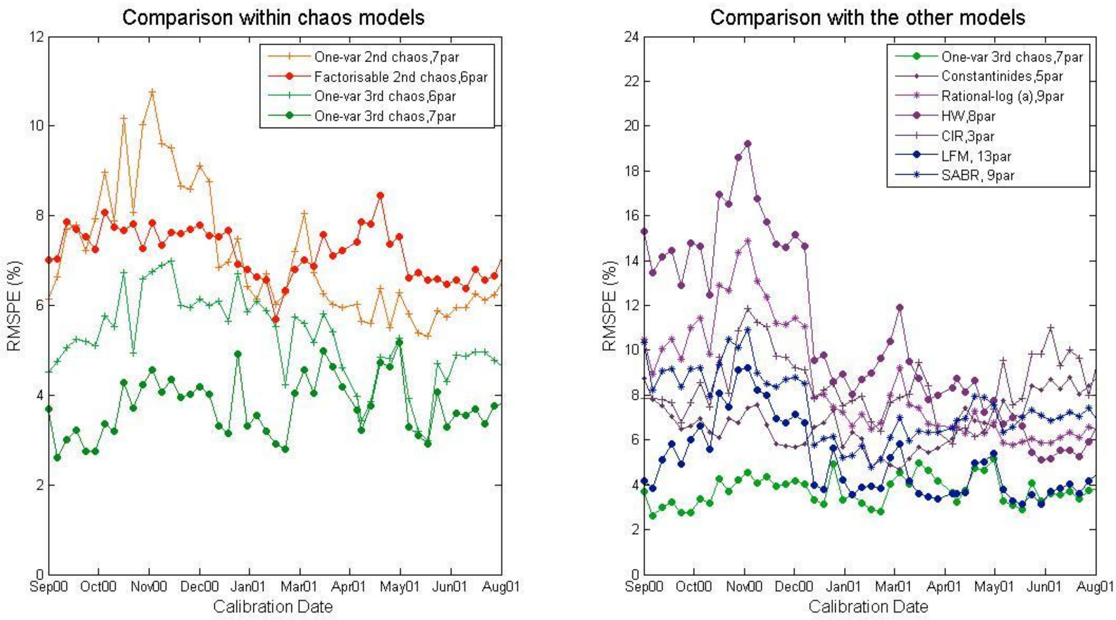

Figure 6.3: Total RMSPE (ATM Swaption Calibration in Sep 2000 - Aug 2001)

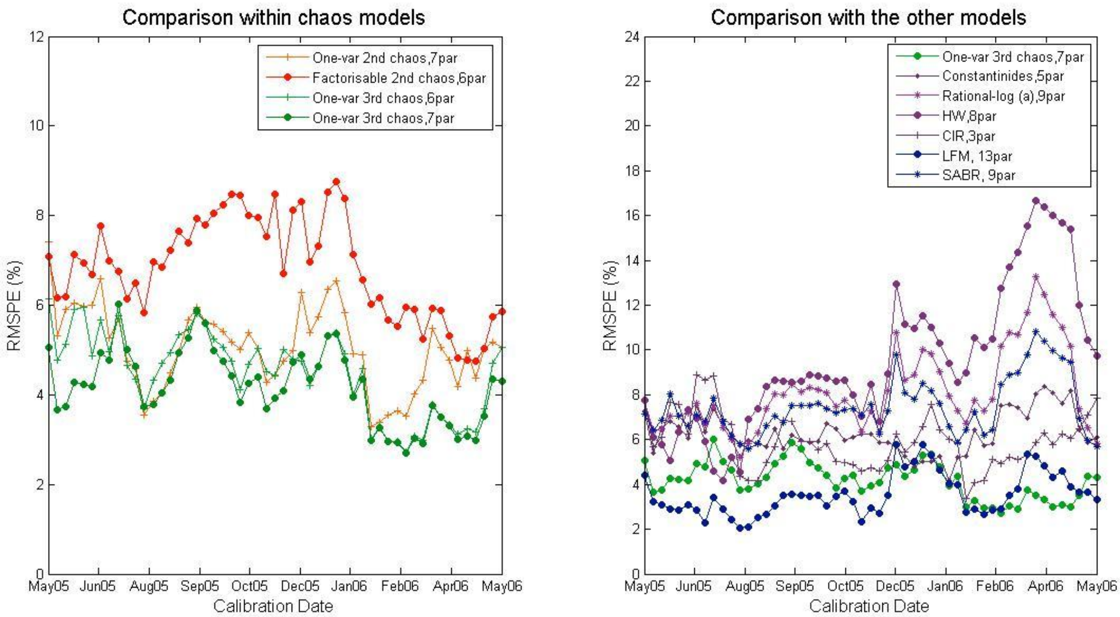

Figure 6.4: Total RMSPE (ATM Swaption Calibration in Sep 2005 - Aug 2006)

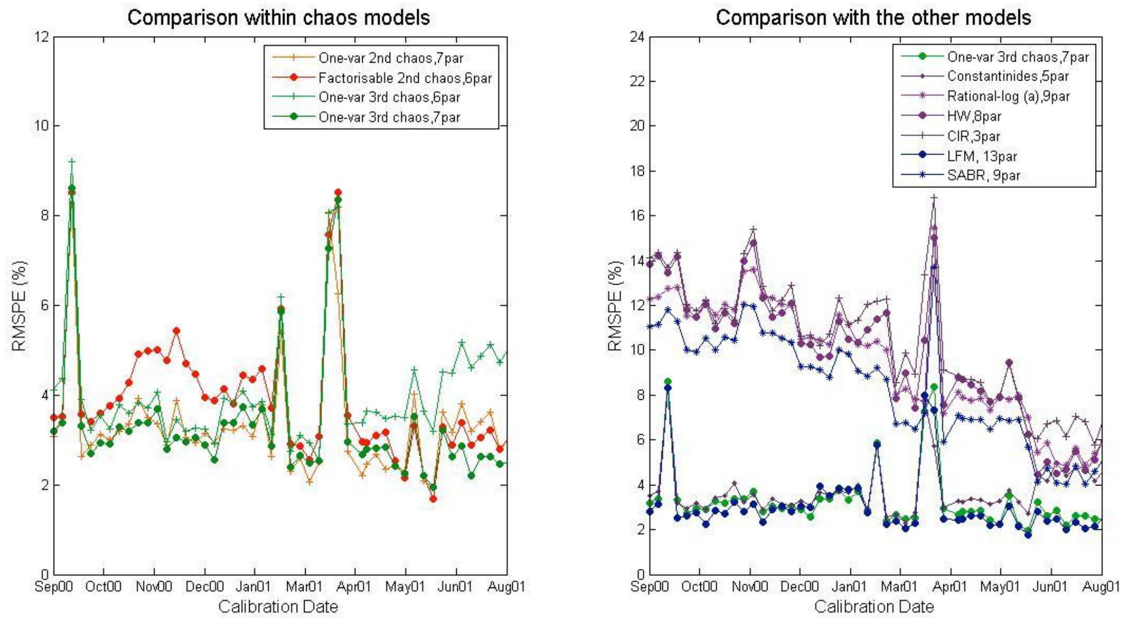

Figure 6.5: Total RMSPE (ATM Caplet Calibration in Sep 2000 - Aug 2001)

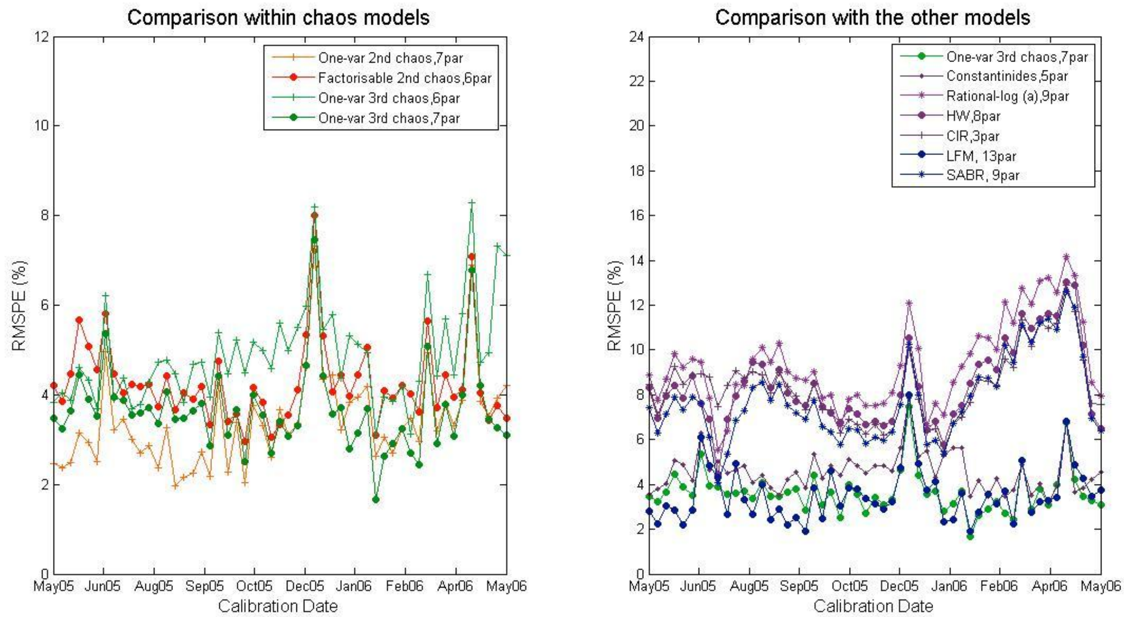

Figure 6.6: Total RMSPE (ATM Caplet Calibration in Sep 2005 - Aug 2006)

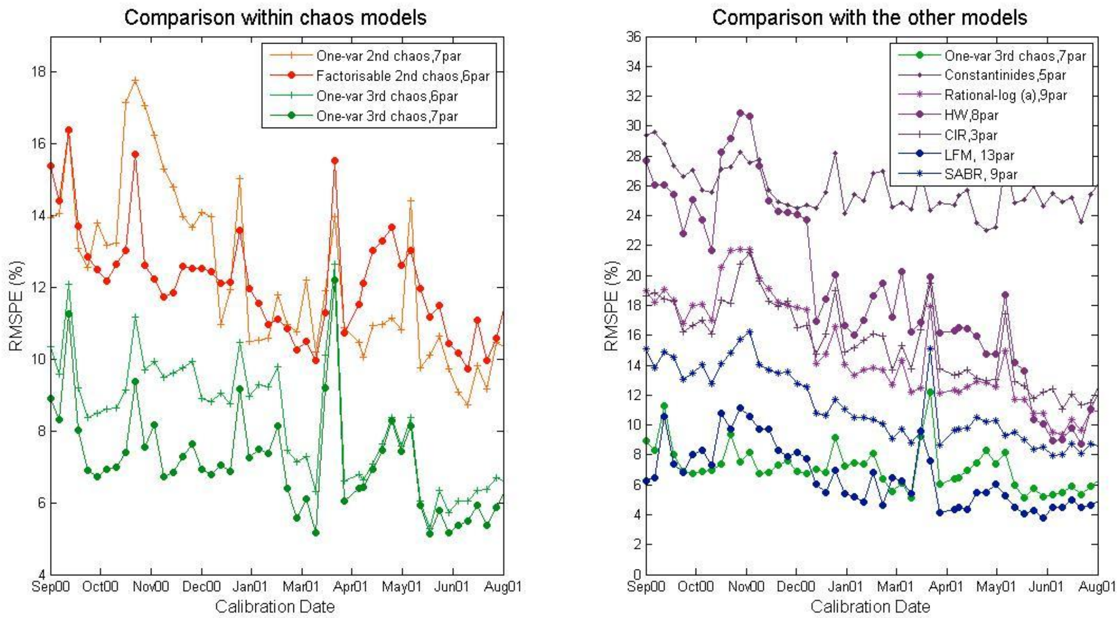

Figure 6.7: Total RMSPE (Joint Calibration in Sep 2000 - Aug 2001)

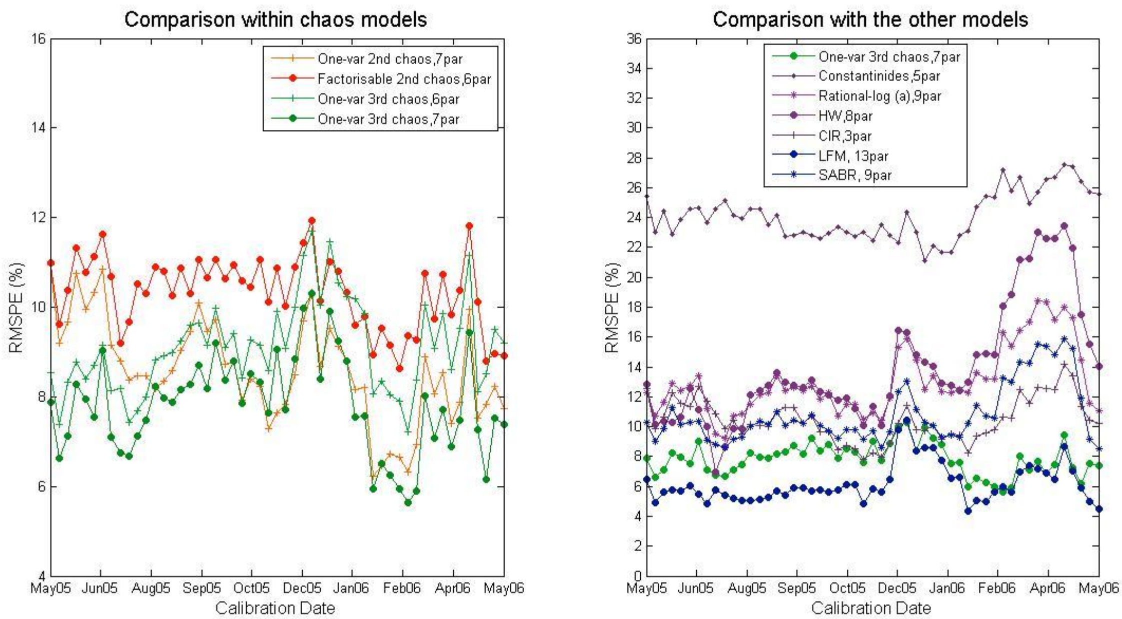

Figure 6.8: Total RMSPE (Joint Calibration in May 2005 - May 2006)

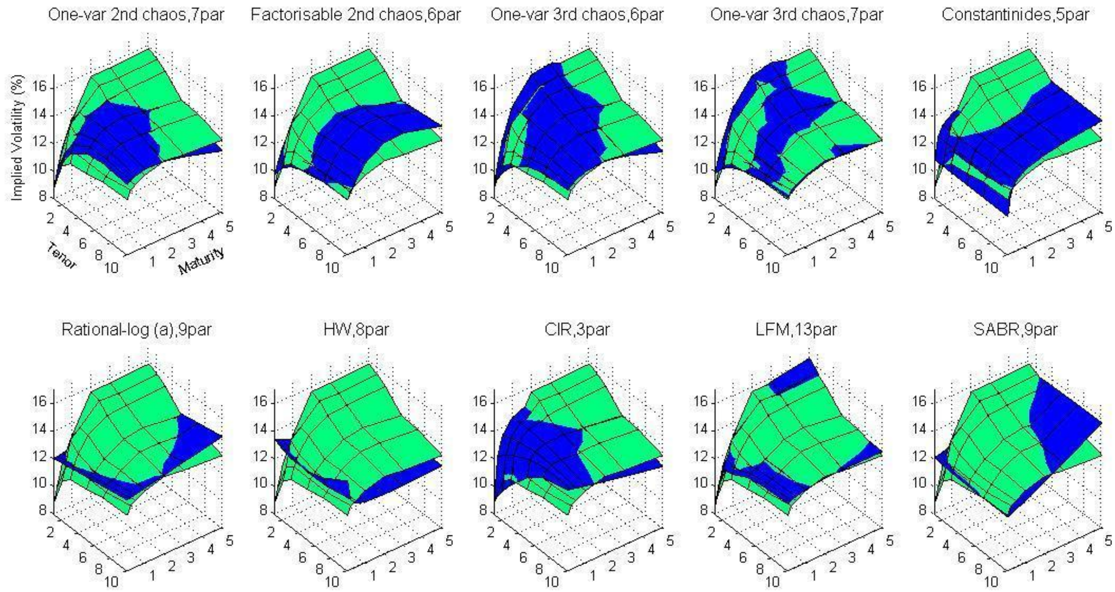

Figure 6.9: ATM Swaption Implied Volatility in 1st Sep 2000 (Blue: Market Quotes, Green: Theoretical Values)

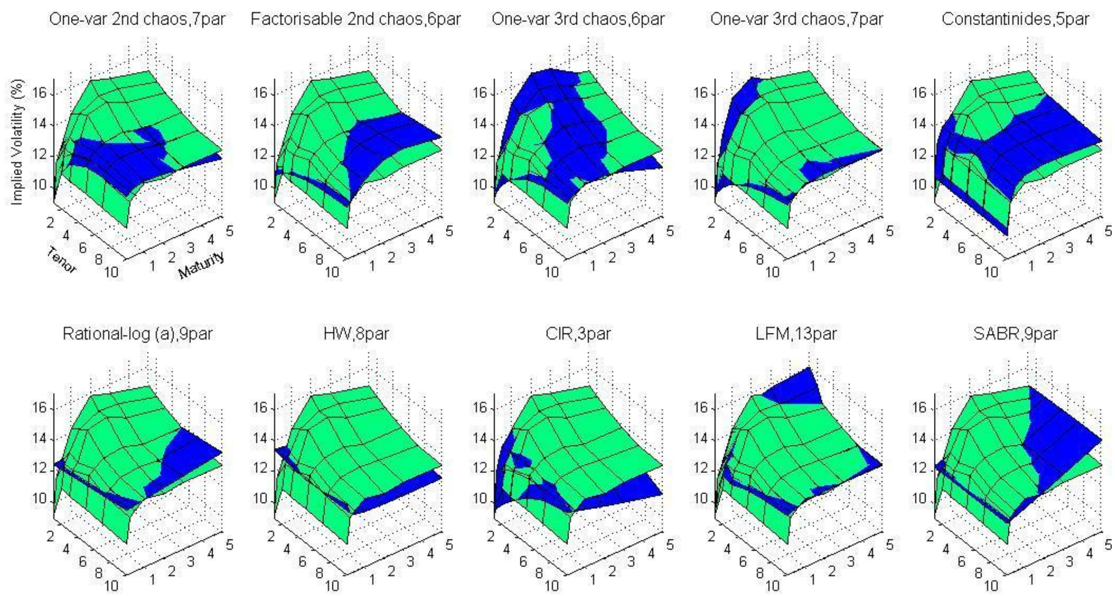

Figure 6.10: ATM Swaption Implied Volatility in 2nd Dec 2005 (Blue: Market Quotes, Green: Theoretical Values)

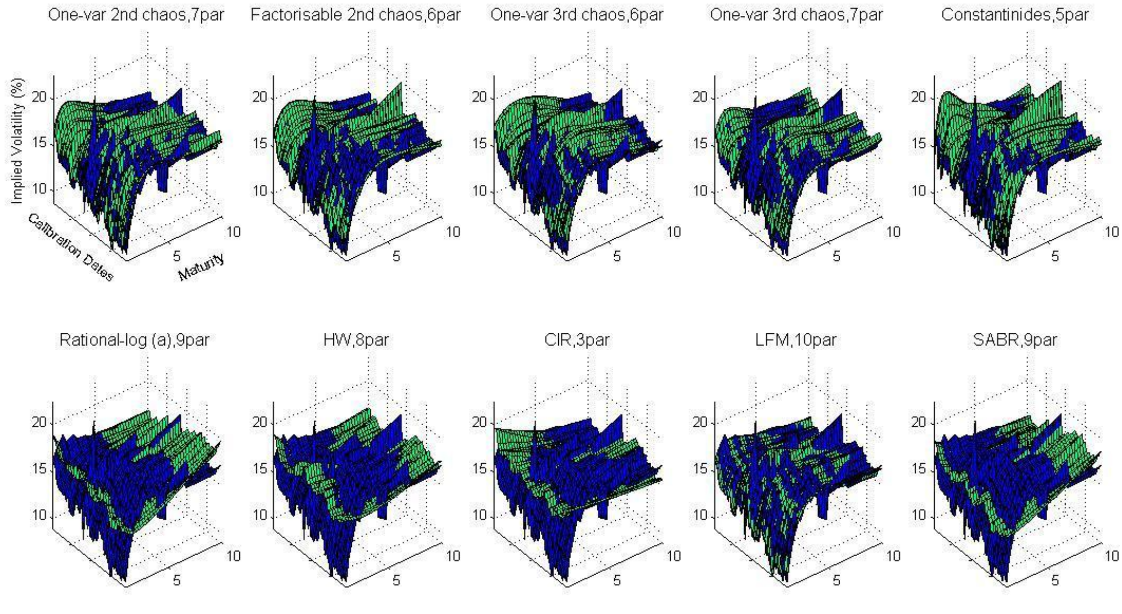

Figure 6.11: ATM Caplet Implied Volatility in Sep 2000 - Aug 2001 (Blue: Market Quotes, Green: Theoretical Values)

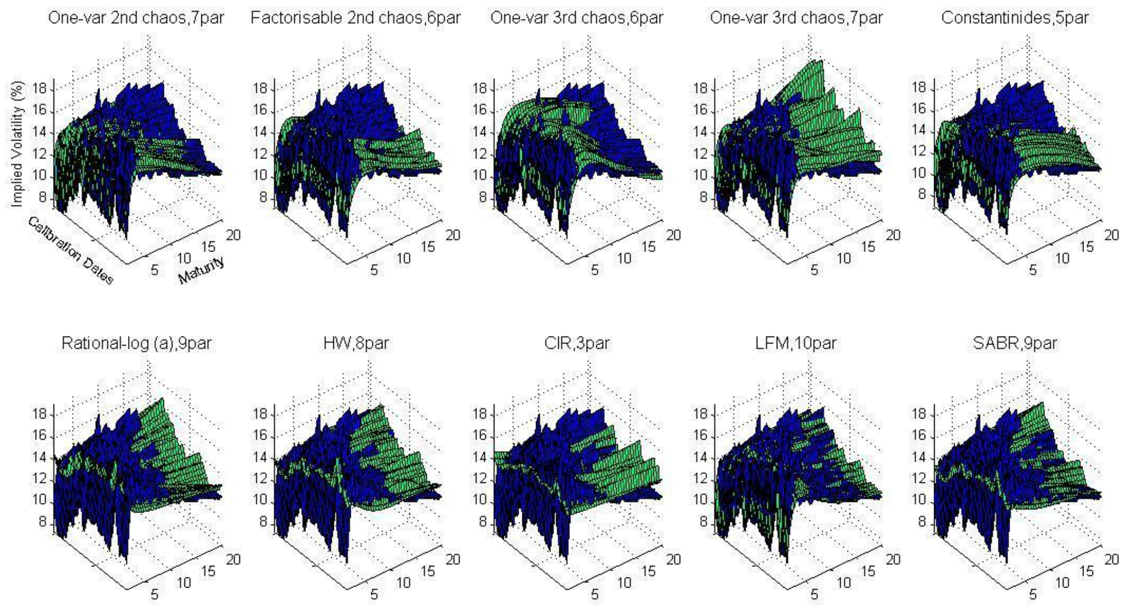

Figure 6.12: ATM Caplet Implied Volatility in May 2005 - May 2006 (Blue: Market Quotes, Green: Theoretical Values)

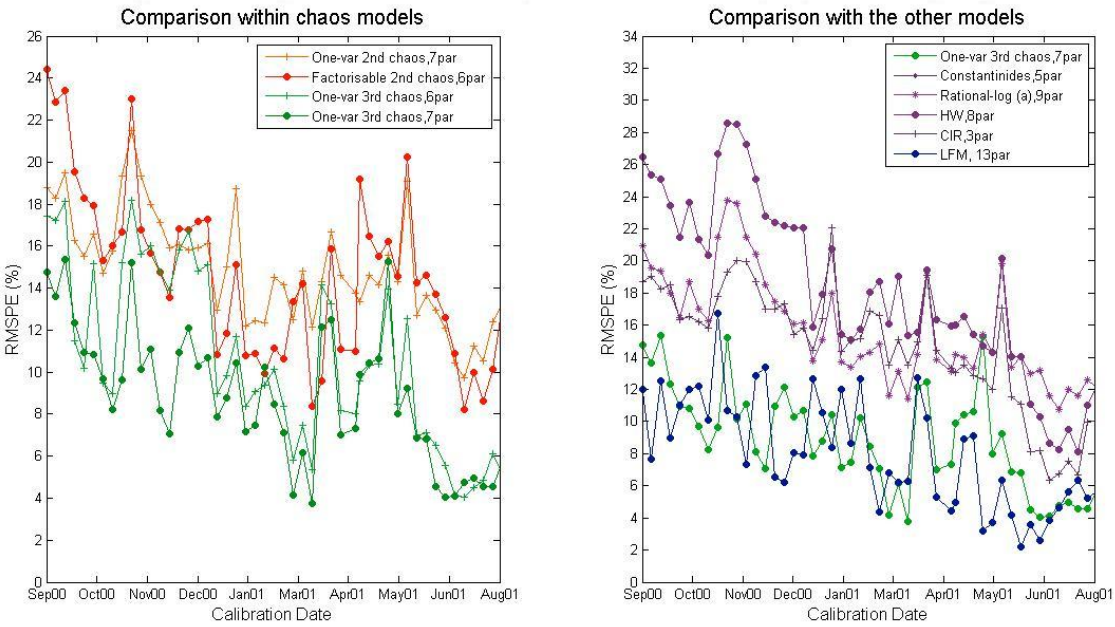

Figure 6.13: Caplet Pricing Errors from Swaption Calibration in Sep 2000 - Aug 2001

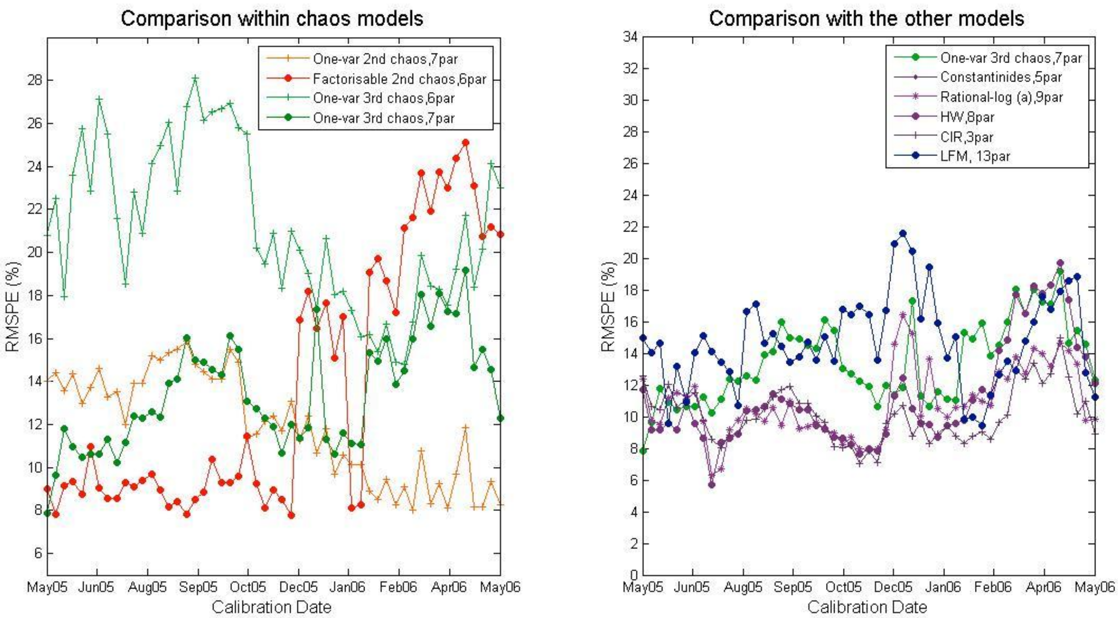

Figure 6.14: Caplet Pricing Errors from Swaption Calibration in May 2005 - May 2006

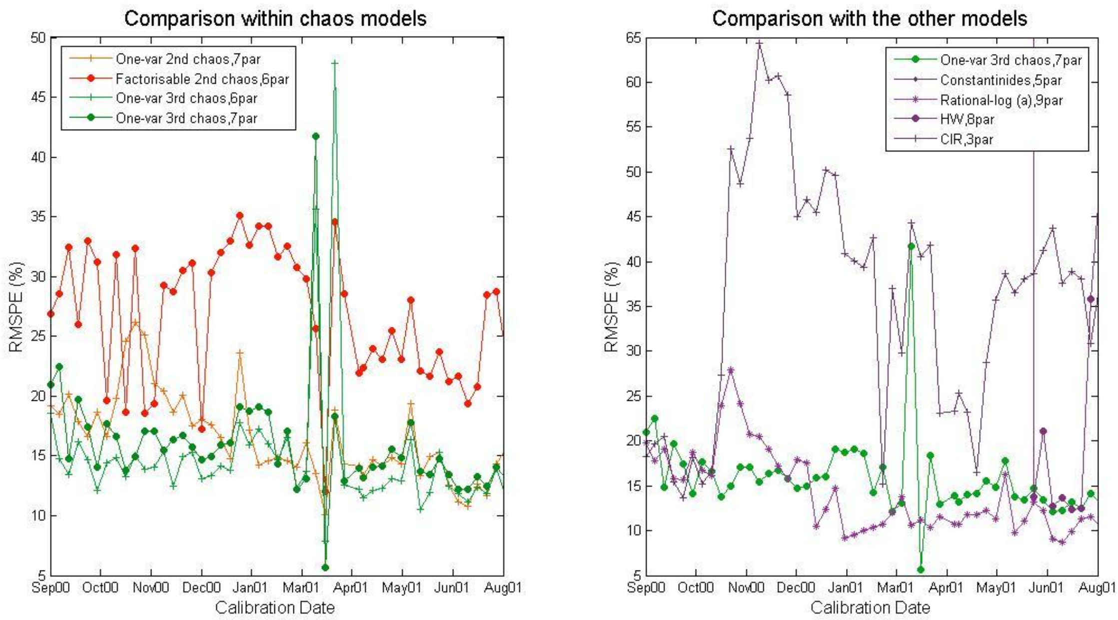

Figure 6.15: Swaption Pricing Errors from Caplet Calibration in Sep 2000 - Aug 2001

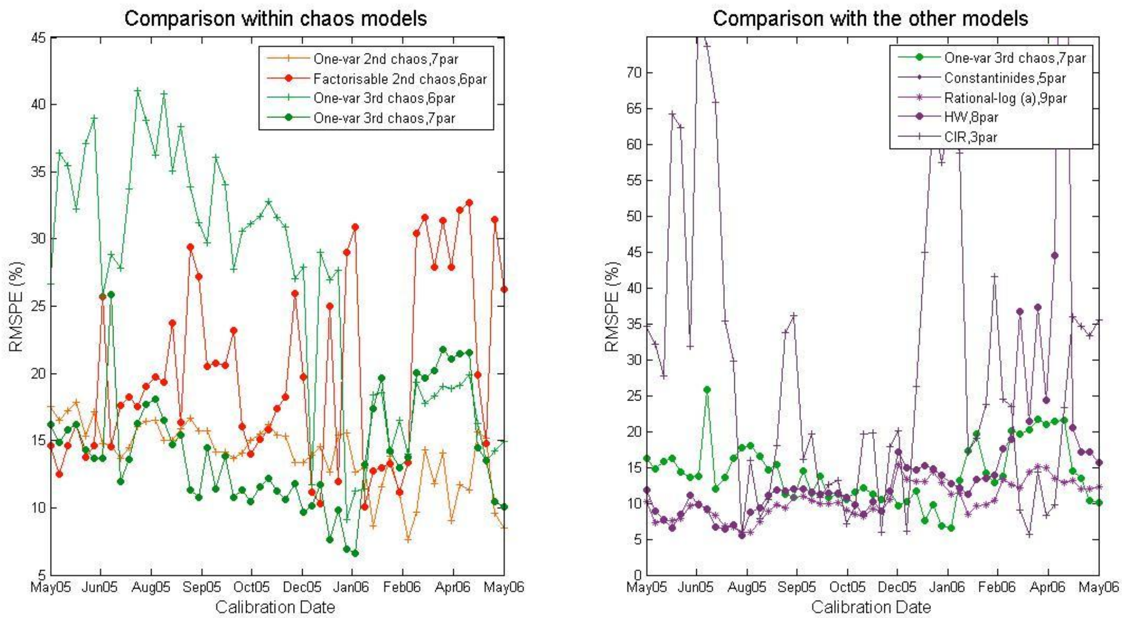

Figure 6.16: Swaption Pricing Errors from Caplet Calibration in May 2005 - May 2006

Table 6.9: Simulation Errors

No.	Model	N	Total-E	Yield-E	SWP-E
12.	one-var 3rd chaos	6	0.01	1.98E-07	1.22E-06

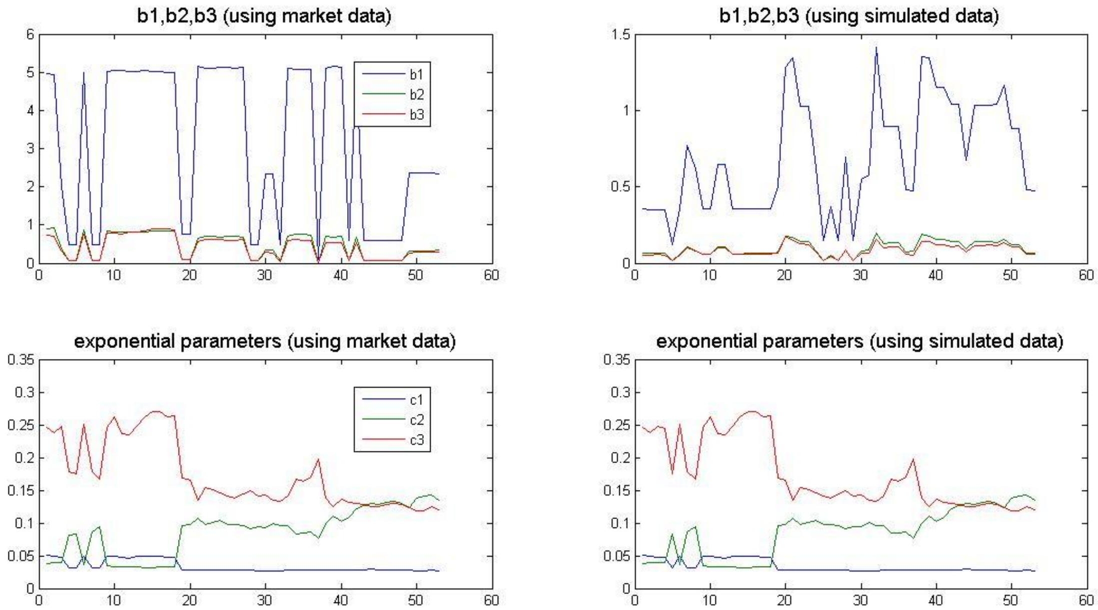

Figure 6.17: Parameter comparisons

Chapter 7

Smile Calibration

Our calibration works in the previous chapter focused on ATM European Options. To distinguish the models further we consider the volatility smile concept. In other words we incorporate the ITM and OTM Options. We have shown that the Chaos Models have stochastic volatility, which property produces volatility smile curves, as can be seen in [35]. The Chaos Models outperform the LFM in this sense. In this chapter, we compare the calibration performance with the LIBOR stochastic volatility models, particularly the SABR Model which we consider to be the most popular model in the current market. We here notice that we have used a one-factor model of the Chaos Model while the SABR Model belongs to two-factor model. Following the literature [14], we implement in our calibration:

- Yield and Caplet smile/skew Calibration for fixed maturity (Two dimensional).
- Yield and Swaption smile/skew Calibration for fixed tenor and maturity (Two dimensional).

Though it is possible to also consider the following greater dimensional data for our calibration, we have not found this in the literature about these calibration:

- Yield and Caplet Vol surface Calibration (Three dimensional).
- Yield and Swaption smile/skew Calibration (Three dimensional).

- Yield and Swaption surface Calibration (Four dimensional),
- Yield, Caplet and Swaption surfaces Calibration (Six dimensional).

We have observed in the previous chapter that the SABR Model cannot fit well into the data across the maturity. Indeed, in our brief experiment we find that it is difficult to achieve good fitting of these high dimensional data and so we do not discuss them here.

7.1 Calibration Data

We analyze data from the UK interest rate markets between May 2005 and May 2006 at 53 dates (every Friday, closing mid price). The data is obtained from ICAP and TTKL via the Bloomberg Database. For each strike K_j we compute the log moneyness ratio (hereafter, referred to as LMR), that is,

$$LMR_j = \ln\left(\frac{K_{ATM}}{K_j}\right).$$

Then, we obtain the following data set for each date using a cubic spline :

- 22 zero-coupon yields, maturing in 1M, 2M, 3M, 4M, 7M, 10M, 1Y1M, 1Y4M, 1Y7M, 1Y10M, 2Y1M, 3Y,...,10Y, 12Y, 15Y, 20Y[1].
- 20×7 Caplet implied volatilities maturing in 1Y, 2Y,...,20Y with strikes which LMR are from -0.3 to 0.3 with 0.1 interval[2].
- $7 \times 6 \times 7$ Swaption implied volatility, maturing in 1M, 3M, 6M, 1Y, 2Y, 3Y, 5Y, where underlying swap contracts are maturing in 1Y, 2Y, 3Y,5Y, 7Y, 10Y with strikes which LMR are from -0.3 to 0.3 with 0.1 interval[3].

[1] Though we observe 30 yields we do not use very short maturity and long maturity yields.

[2] This corresponds to moneyness from 0.74 to 1.35 which may be comparable with the works [14], [55], [88]. Our raw data contains strikes, $2.5, 3.0, 3.5, 4.0, 4.5, 5.0, 6.0, 7.0$, and 8.0%.

[3] Data is available between May and July in 2005 at 11 dates. Our raw data contains strikes, $-200, -100, -50, -25, 0, +25, +50, +100, +200$ basis points away from the money.

7.2 Models

We calibrate the One-variable Third Chaos Models numbered 15 in Section 4.7.3 and the SABR Model. Both models have nine parameters, but we recall here that the Chaos Model is a one-factor model whereas the SABR Model is a two-factor model.

7.3 Calibration Methods

In our calibrations we apply the weighted least-squares method; that is, we minimize the following function for the One-variable Third Chaos Model:

$$\frac{1}{n_1}\sum_{i=1}^{n_1}\left[\frac{y_{0T_i}-\overline{y}_{0T_i}}{\overline{y}_{0T_i}}\right]^2+\frac{1}{n_2}\sum_{j=1}^{n_2}\left[\frac{OP(T_i,K_j)-\overline{OP}(T_i,K_j)}{\overline{OP}(T_i,K_j)}\right]^2,$$

where we fix the option maturity at $T_i \geq 0$ but consider in the money and out of the money strikes, $K_1, K_2, \cdots, K_{n_2}$. However we minimize only by the implied volatilities in the SABR Model:

$$\frac{1}{n_2}\sum_{j=1}^{n_2}\left[\frac{OP^{vol}(T_i,K_j)-\overline{OP}^{vol}(T_i,K_j)}{\overline{OP}^{vol}(T_i,K_j)}\right]^2,$$

because we may choose the yield curve without constraints, and analytical expressions for the implied volatilities are available in the SABR Model.

We apply the DM Statistics to compare the fitting performance of the One-variable Third Chaos Model with the SABR Model. The null hypothesis, which is that two models have the same fitting errors, can be rejected at 5% level if the absolute value of the DM statistics is greater than 1.96. In our computations, a higher number means that the One-variable Third Chaos Model works better.

7.4 Calibration Results

We analyze the Swaption smile/skew calibration results using Tables 7.1 - 7.5 and Figures 7.1 while we do the Caplet smile/skew calibration results using Tables 7.6 - 7.7

and Figures 7.2 - 7.9.

Let us first analyze the result of the Swaption smile/skew calibration. We calibrate the Chaos Model using the Swaption premiums and the SABR Model using the Swaption implied volatilities for fixed maturity and tenor. The average RMSPE (%) in all calibration dates (11 dates) in 2005 are shown in Tables 7.1 - 7.4. The short maturity out of the money swaptions are very sensitive, by which we mean that a small error in premiums can cause a big error in implied volatilities when we convert. The other direction is also true, as we can observe this from Tables 7.3 - 7.4. Therefore we should compare the calibration performance using the objective functions. As seen from Table 7.3 and Table 7.4 the average RMSPEs of the swaption premiums in the One-variable Third Chaos Model are smaller than the average RMSPEs of the swaption implied volatilities in the SABR Model. However, looking at the RMSPEs of the yield fitting in Table 7.2, the Svensson form outperforms the One-variable Third Chaos Model. However, Table 7.1 compares RMSPEs of yields and swaption premiums in the Chaos Model and RMSPEs of yields and implied volatilities in the SABR Model and shows smaller RMSPEs in the Chaos Model. Indeed, in Figure 7.1 we observe the green lines representing the Chaos Model are below the red lines representing the SABR Model in most of the maturities and the tenors. It means that the One-variable Third Chaos Model outperforms the SABR Model for most of the maturities and tenors. The DM Statistics in Table 7.5 confirm this positive result.

Similarly, for the caplets away from the maturity we calibrate the Chaos Models on the caplet premiums and the SABR Model by the implied volatilities. The average RMSPE (%) of all calibration dates (53 dates) in $2005-2006$ are shown in Table 7.6. We observe again that a small error in premiums of the short maturity caplets can cause a big error in implied volatilities when we convert. We are here computing the total errors from the RMSPEs of yields and premiums for both interest rate models. Although the calibration performance in the Chaos Model is outperformed by the

SABR Model as seen from the DM Statistics in Table 7.7, we also observe in Figure 7.2 that the Chaos Model works as well as the SABR Model on the short maturity caplets. In particular we observe that the Chaos Model produces a smile/skew curve as was expected from the stochasticity of the volatility drift. For instance, Figure 7.4 and Figure 7.5 show the volatility skews by the Chaos Model. On the contrary, looking at Figures 7.6 - 7.8, we observe that the Chaos Model does not fit well on the long maturity Caplets data. Here, we have not found the reason of this.

Though we need the additional three parameters to compute the caplets and the swaptions at the same time in the SABR Model, we can compute them in a straightforward way in the Chaos Models. It seems reasonable to consider more parameters in the Chaos Model for the further improvement. However, since our brief experiment has shown that One-variable Third chaos with 12 parameters model does not improve the calibration result very much, we should perhaps consider a two-factor Chaos Model for further investigations.

Swaption Smile/Skew Calibration

Table 7.1: Total Errors in (maturity * tenor) Swaption Smile Calibration

3rd chaos, 9par (Objective Function)						
	1Y	2Y	3Y	5Y	7Y	10Y
1M	4.6	3.9	8.8	3.5	5.8	4.9
3M	2.4	1.1	1.3	1.9	2.3	2.4
6M	1.3	1.3	2.0	2.2	2.2	2.9
1Y	1.1	1.3	1.7	2.5	2.5	3.0
2Y	1.6	1.6	1.5	2.1	2.2	2.4
3Y	2.0	2.1	2.1	2.2	2.3	2.1
5Y	1.2	0.9	1.0	1.0	0.8	1.8

SABR						
	1Y	2Y	3Y	5Y	7Y	10Y
1M	4.8	4.7	4.7	4.7	4.7	4.7
3M	4.8	4.8	4.7	4.7	4.7	4.8
6M	4.8	4.7	4.7	4.7	4.8	4.7
1Y	2.9	2.9	2.9	2.9	2.9	2.9
2Y	2.4	2.4	2.4	2.4	2.4	2.4
3Y	2.4	2.4	2.4	2.4	2.1	2.4
5Y	0.7	3.6	3.9	4.1	3.5	3.7

Table 7.2: Errors in Yields

3rd chaos, 9par						
	1Y	2Y	3Y	5Y	7Y	10Y
1M	2.3	2.5	2.2	2.6	4.5	3.0
3M	1.7	0.9	1.0	1.5	2.0	2.1
6M	1.1	1.1	1.8	2.0	1.8	2.3
1Y	0.9	1.0	1.3	2.1	2.2	2.5
2Y	1.5	1.5	0.9	1.8	1.9	2.2
3Y	1.8	1.8	1.8	1.9	2.0	1.8
5Y	1.1	0.6	0.6	0.7	0.4	1.4

Svensson						
	1Y	2Y	3Y	5Y	7Y	10Y
1M	0.6	0.6	0.6	0.6	0.6	0.6
3M	0.6	0.6	0.6	0.6	0.6	0.6
6M	0.6	0.6	0.6	0.6	0.6	0.6
1Y	0.6	0.6	0.6	0.6	0.6	0.6
2Y	0.6	0.6	0.6	0.6	0.6	0.6
3Y	0.6	0.6	0.6	0.6	0.6	0.6
5Y	0.6	0.6	0.6	0.6	0.6	0.6

Table 7.3: Errors in (maturity * tenor) Swaption Premiums

3rd chaos, 9par						
	1Y	2Y	3Y	5Y	7Y	10Y
1M	3.9	2.8	7.9	2.3	3.4	3.6
3M	1.6	0.5	0.7	0.9	0.9	1.0
6M	0.6	0.6	0.7	0.9	1.1	1.6
1Y	0.5	0.6	0.9	1.3	1.2	1.6
2Y	0.7	0.5	1.2	1.0	1.1	1.0
3Y	0.9	1.1	1.1	1.0	1.1	1.1
5Y	0.2	0.6	0.7	0.6	0.6	1.1

SABR						
	1Y	2Y	3Y	5Y	7Y	10Y
1M	51.8	44.3	44.2	45.8	47.7	48.2
3M	31.8	29.9	30.1	31.4	32.7	33.0
6M	22.1	21.4	21.9	23.2	23.9	24.9
1Y	10.0	10.2	10.3	11.0	11.8	12.4
2Y	6.6	6.7	6.8	7.4	7.7	8.0
3Y	5.8	5.9	6.1	6.5	4.4	6.9
5Y	0.4	6.0	6.6	7.4	6.2	6.8

Table 7.4: Error in (maturity * tenor) Swaption Implied volatilities

3rd chaos, 9par						
	1Y	2Y	3Y	5Y	7Y	10Y
1M	44.8	89.7	81.5	33.9	10.0	2.3
3M	49.6	29.8	32.1	37.4	43.9	42.4
6M	8.4	11.1	12.8	15.1	20.0	24.3
1Y	3.7	5.8	9.3	11.8	11.7	13.8
2Y	1.9	1.2	4.6	3.5	4.1	4.7
3Y	1.8	2.1	2.5	2.7	3.1	4.4
5Y	0.6	1.4	1.6	1.1	1.4	3.2

SABR (Objective Function)						
	1Y	2Y	3Y	5Y	7Y	10Y
1M	4.7	4.7	4.7	4.7	4.7	4.7
3M	4.7	4.7	4.7	4.7	4.7	4.7
6M	4.7	4.7	4.7	4.7	4.7	4.7
1Y	2.8	2.8	2.8	2.8	2.8	2.8
2Y	2.4	2.4	2.4	2.4	2.4	2.3
3Y	2.3	2.3	2.3	2.3	2.0	2.3
5Y	0.3	3.6	3.9	4.0	3.4	3.7

Table 7.5: DM-Statistics for (maturity * tenor) Swaption Smile Calibration between One-variable Third chaos and the SABR Model

	1Y	2Y	3Y	5Y	7Y	10Y
1M	0.19	1.80	-0.96	5.75	-0.55	-0.31
3M	14.81	13.33	11.65	8.36	7.41	7.56
6M	11.44	11.70	7.87	7.74	9.53	9.68
1Y	16.90	13.41	7.77	2.20	1.89	-0.53
2Y	5.55	4.71	9.53	2.14	1.63	0.34
3Y	2.49	1.78	1.82	1.82	1.82	1.82
5Y	-10.96	37.15	33.05	39.51	62.14	14.25

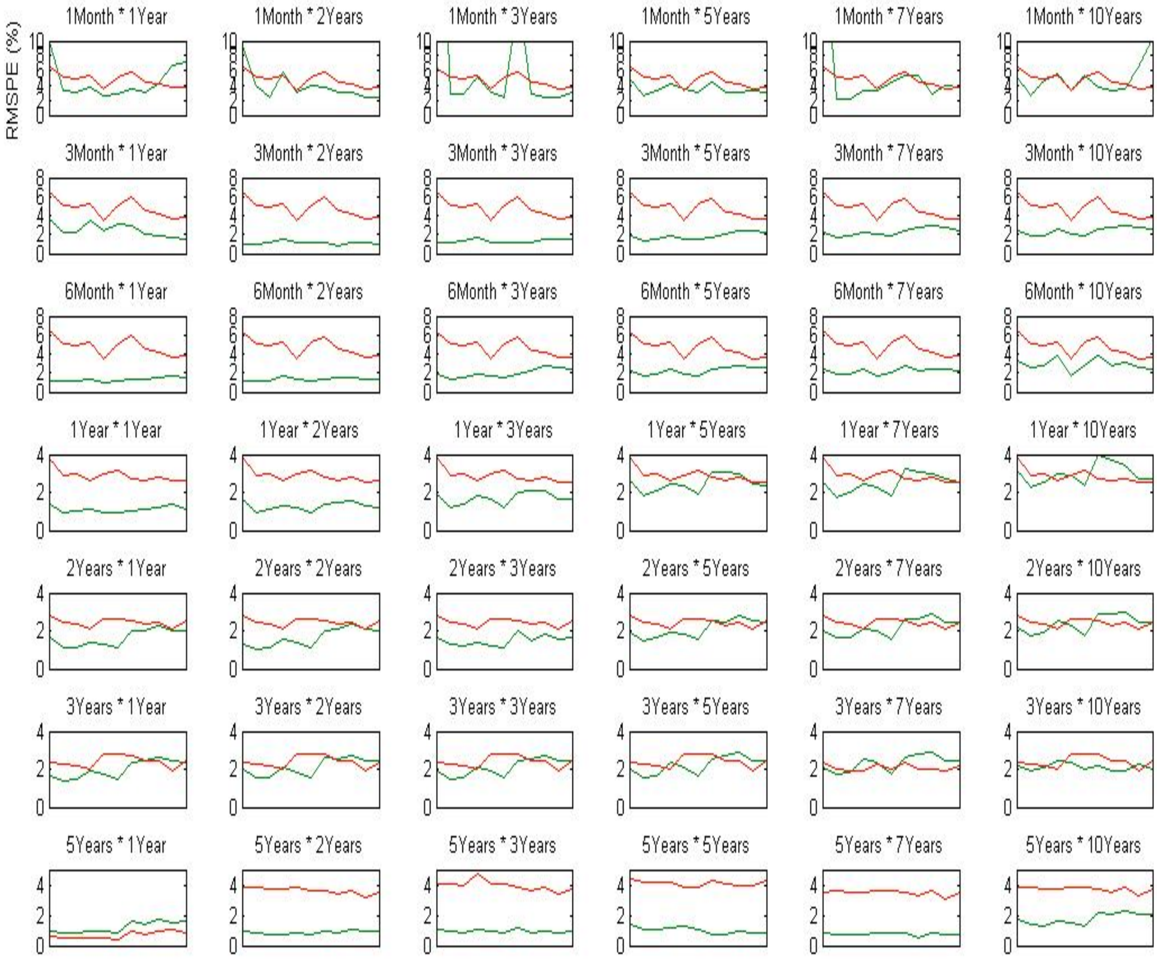

Figure 7.1: (maturity * tenor) Swaption volatility smile/skew Calibration, Total RMSPE in May 2005 - Jul 2005 (Green: One-variable Third chaos, Red: SABR)

Caplet Smile/Skew Calibration

Table 7.6: RMSPE (%) of Caplet Smile Calibration in 2005 − 2006

	one-var 3rd chaos, 9par				SABR with $\beta = \frac{1}{2}$, 9par			
Maturity	Total-E	Yield-E	CplP-E	CplV-E	Total-E[5]	Yield-E[6]	CplP-E	CplV-E
1Y	10.5	5.0	9.1	40.8	12.7	0.4	12.7	2.6
2Y	3.9	2.5	2.9	11.4	2.6	0.4	2.6	1.4
3Y	3.1	1.8	2.4	5.1	1.4	0.4	1.4	0.9
4Y	2.6	1.8	1.7	2.5	1.5	0.4	1.4	1.0
5Y	2.5	1.6	1.7	2.2	1.6	0.4	1.5	1.2
6Y	2.0	1.3	1.4	1.9	1.5	0.4	1.4	1.2
7Y	4.4	3.1	3.1	4.0	1.9	0.4	1.8	1.6
8Y	4.8	3.2	3.5	3.0	2.6	0.4	2.5	2.1
9Y	3.2	1.8	2.5	3.0	2.6	0.4	2.6	2.3
10Y	6.7	4.2	5.1	6.6	2.9	0.4	2.9	2.5
11Y	5.5	3.7	4.1	5.2	1.6	0.4	1.5	1.3
12Y	5.0	2.9	4.0	4.2	1.9	0.4	1.9	1.7
13Y	5.5	3.4	4.3	5.6	1.4	0.4	1.3	1.2
14Y	5.4	3.2	4.4	5.7	1.7	0.4	1.6	1.5
15Y	5.4	3.0	4.5	5.9	1.9	0.4	1.9	1.7
16Y	7.0	3.1	6.2	8.0	3.4	0.4	3.4	3.4
17Y	7.3	3.0	6.6	8.4	3.9	0.4	3.8	3.8
18Y	8.1	3.9	7.1	8.6	4.3	0.4	4.2	4.3
19Y	8.3	3.1	7.6	7.5	4.8	0.4	4.7	4.8
20Y	8.1	3.9	6.9	7.7	5.3	0.4	5.3	5.3

[5] The total errors are computed by yield errors and caplet premium errors.
[6] We use the Svensson Form for the yield fitting.

Table 7.7: DM-Statistics for Caplet Smile Calibration between One-variable Third chaos and the SABR Model

Maturity	DM Statistics
1Y	1.78
2Y	-4.99
3Y	-6.40
4Y	-8.80
5Y	-7.13
6Y	-3.66
7Y	-13.27
8Y	-10.06
9Y	-4.70
10Y	-14.51
11Y	-26.23
12Y	-12.69
13Y	-30.35
14Y	-24.67
15Y	-21.20
16Y	-21.69
17Y	-20.66
18Y	-19.16
19Y	-16.18
20Y	-11.35

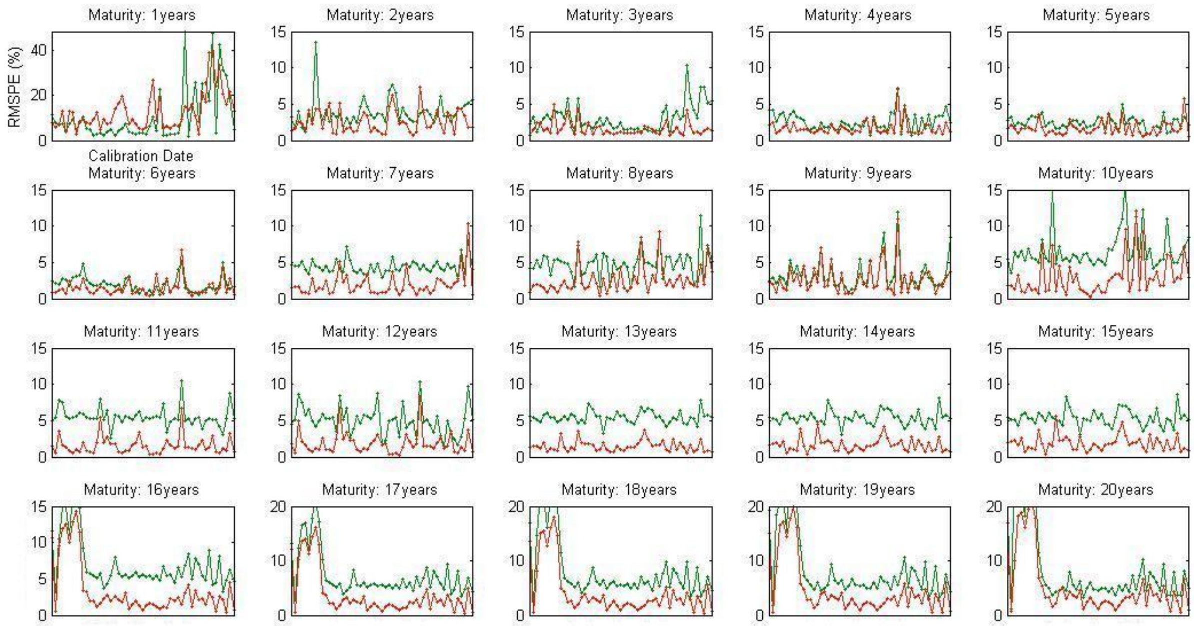

Figure 7.2: Caplet volatility smile/skew RMSPE in May 2005 - May 2006 (Green: One-variable Third chaos, Red: SABR)

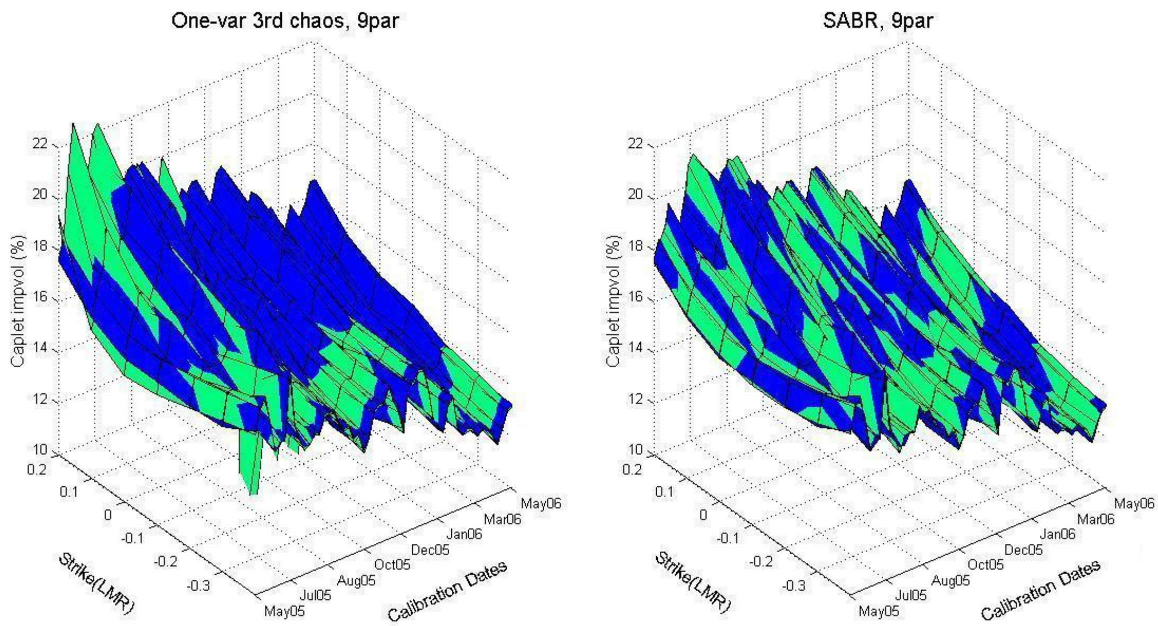

Figure 7.3: Caplet volatility smile/skew, Maturity: 2 years (Blue: Market Quotes, Green: Theoretical Values)

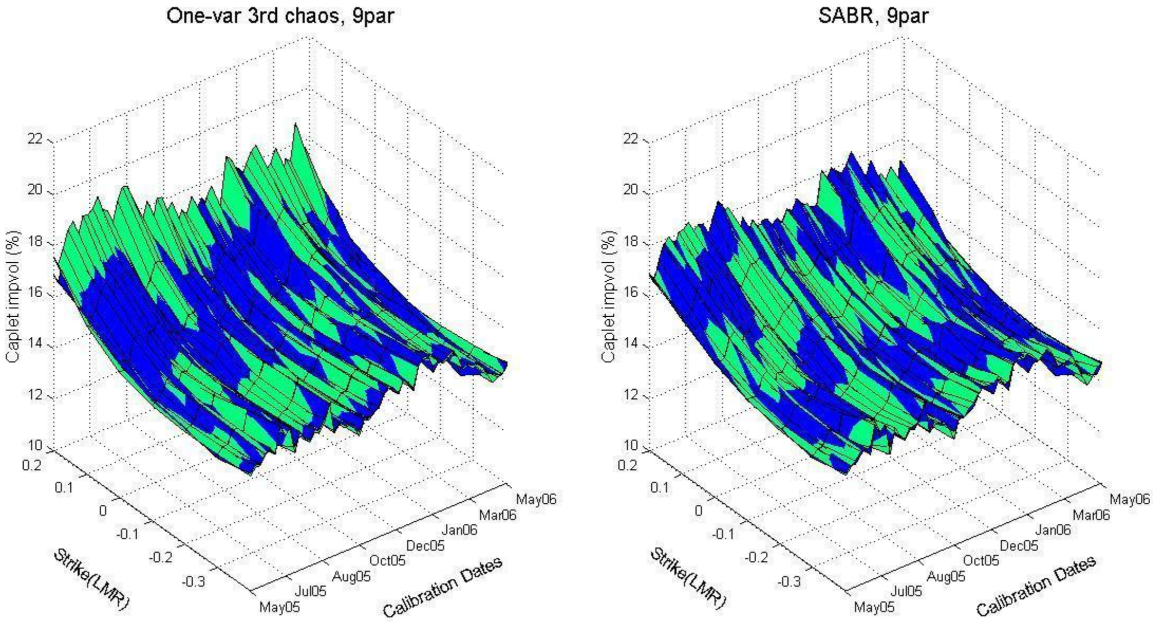

Figure 7.4: Caplet volatility smile/skew, Maturity: 6 years (Blue: Market Quotes, Green: Theoretical Values)

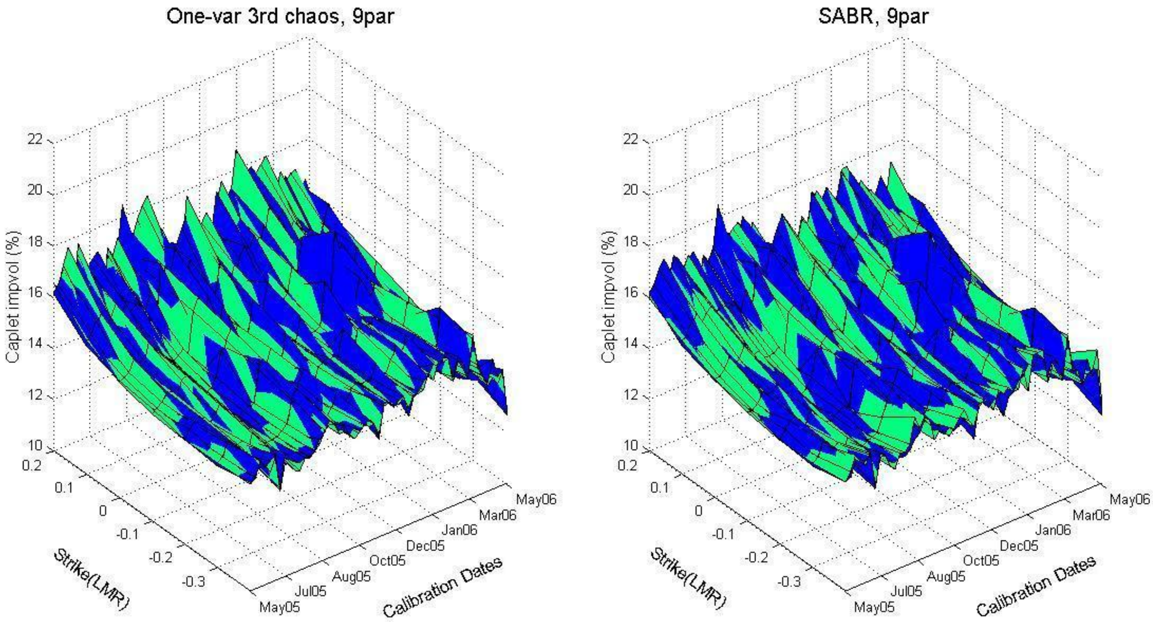

Figure 7.5: Caplet volatility smile/skew, Maturity: 8 years (Blue: Market Quotes, Green: Theoretical Values)

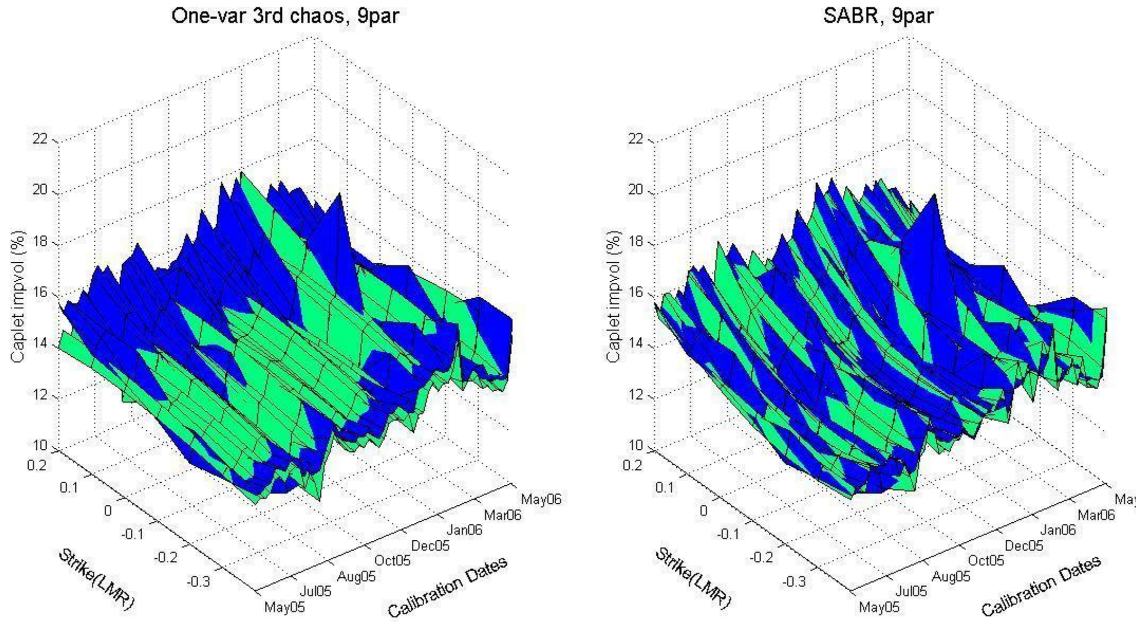

Figure 7.6: Caplet volatility smile/skew, Maturity: 10 years (Blue: Market Quotes, Green: Theoretical Values)

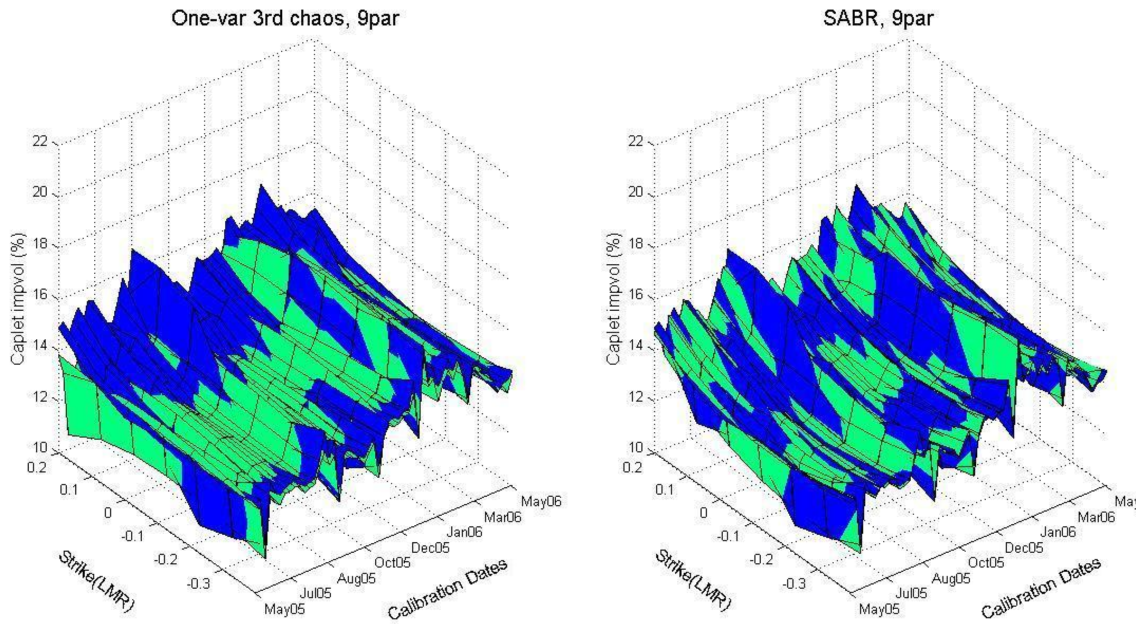

Figure 7.7: Caplet volatility smile/skew, Maturity: 12 years (Blue: Market Quotes, Green: Theoretical Values)

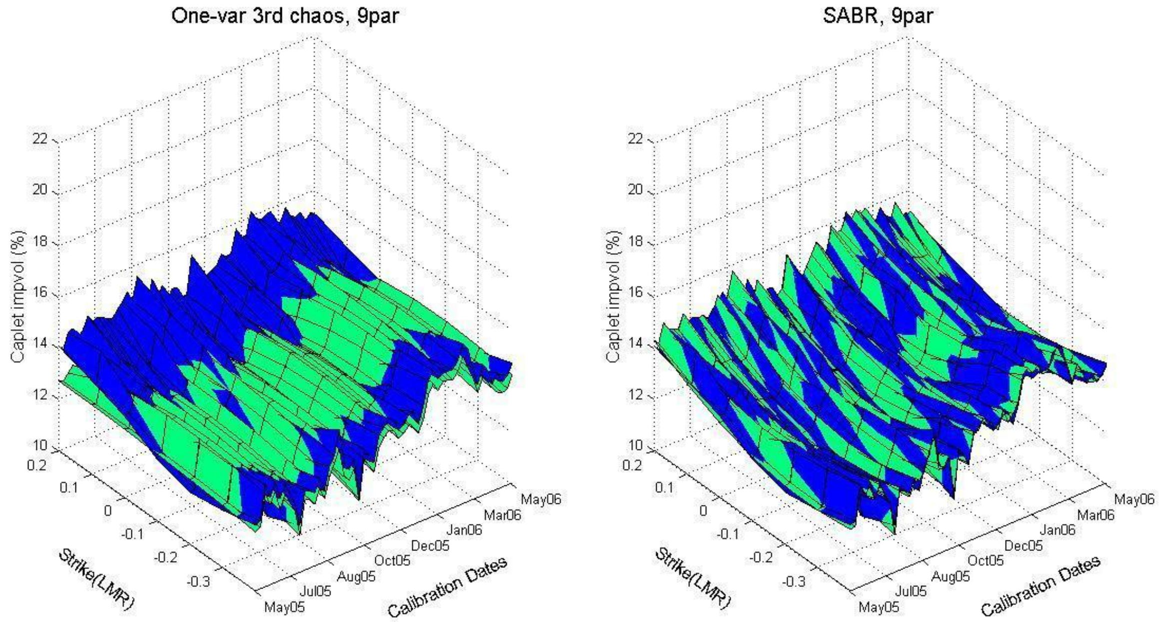

Figure 7.8: Caplet volatility smile/skew, Maturity: 14 years (Blue: Market Quotes, Green: Theoretical Values)

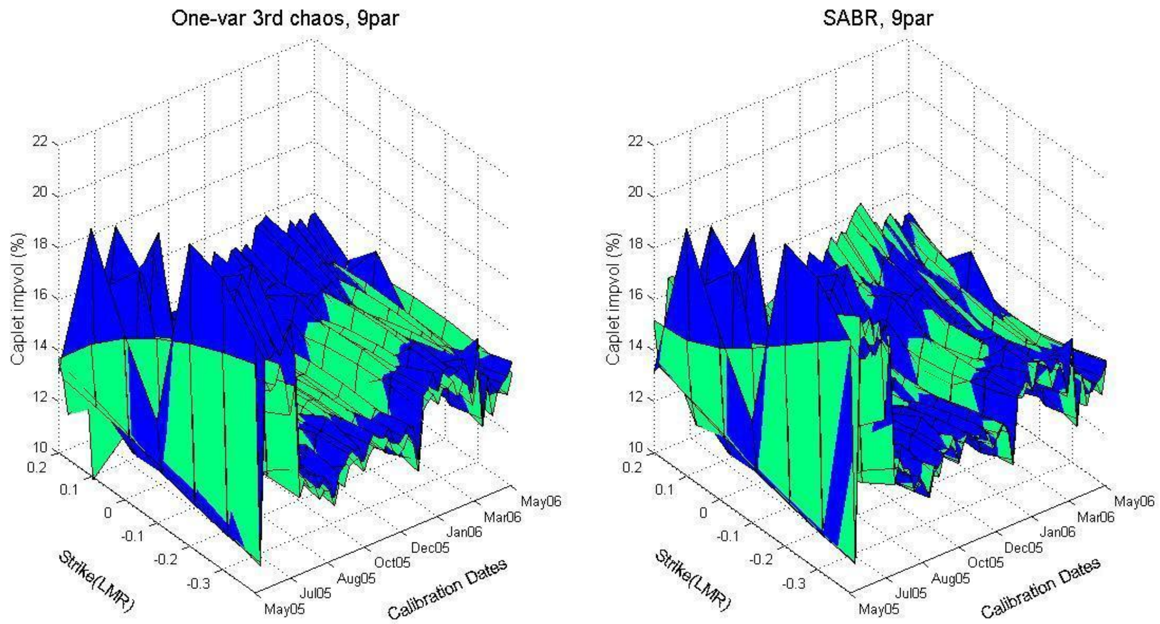

Figure 7.9: Caplet volatility smile/skew, Maturity: 18 years (Blue: Market Quotes, Green: Theoretical Values)

Chapter 8

Alternative Models

In this chapter, we introduce several new interest rate models under the Potential Approach. Firstly, we construct a new framework where we start our argument from the Short Rate Model so that we keep the state price density potential. On the one hand, this expression of the state price density works for pricing options. On the other hand, it allows the LIBOR rate and swap rate volatilities to be explicitly expressed only by the short rate. Secondly, we investigate the FH framework further, and introduce n^{th}-order FH Model, which is comparable with the One-variable Chaos Model. Thirdly, we investigate the Chaotic Approach using the FH framework. Here, we compute the chaos coefficients using the Malliavin derivative. Moreover, fourthly, we specify the stochastic differential equation of the random variable σ_t and compare the corresponding model with the Chaos Models. Lastly, we model the term structure from the variable Z_{tT}. Since our main concern in this thesis resides with Chaos Models, we leave calibration of these models for future works.

8.1 Modelling the volatility drifts from the Short Rate Models

As we have observed in Chapter 4, due to the fact that we check roots of the distribution function to compute an option premium in the Chaos Models, we are unable to compute many options at the same time. This causes relatively slow computational speed. However, this can be fixed when we form the state price density as exponential as in the Constantinides Model. As we usually apply the bank account process for the market numeraire and the discount bond for the T-forward adjusted measure, that is respectively,

$$B_t = B_0 e^{\int_0^t r_s ds} \quad \text{and} \quad P_{tT} = e^{-\int_t^T f_{ts} ds}, \quad \text{for} \quad 0 \leq t \leq T < \infty,$$

the natural numeraire can also be expected to have an exponential form. Indeed, as observed in (2.1.13), we may express the state price density, that is the inverse of the natural numeraire, in the following way:

$$V_t = V_0 e^{-\int_0^t (r_s + \frac{1}{2}\lambda_s^2) ds - \int_0^t \lambda_s dW_s}, \quad \text{for} \quad t \geq 0.$$

As stated in Section 2.1.1, the state price density is a potential when the short rate is a positive process. Looking into this further, as observed in (2.1.15) and (2.1.17), we have for each $0 \leq t \leq T < \infty$ the following expressions:

$$Z_{tT} = \mathbb{E}_t\big[V_0 e^{-\int_0^T (r_s + \frac{1}{2}\lambda_s^2) ds - \int_0^T \lambda_s dW_s}\big]$$

and

$$Z_{tT} = V_0 e^{-\int_0^t (\frac{1}{2}\lambda_s^2 + r_s) ds - \int_0^t \lambda_s dW_s} \mathbb{E}_t^{\mathbb{Q}}\big[e^{-\int_t^T r_s ds}\big].$$

This conditional expectation in the last equation can be explicitly solved under the Affine Term Structure Model ([32]) as we will investigate it later in this section. Moreover, because the risk-adjusted volatility has the following quotient form:

$$\hat{V}_{tT} = \frac{D_t[Z_{tT}]}{Z_{tT}}, \quad 0 \leq t \leq T < \infty,$$

in light of the chain rule of the Malliavin derivative, we can expect the risk-adjusted volatility to be expressed in a simple form if we have the state price density in an exponential form. Indeed, by expression (2.1.17) we may infer using the chain rule that

(8.1.1)

$$
\begin{aligned}
D_t[Z_{tT}] =& D_t[V_0 e^{-\int_0^t(\frac{1}{2}\lambda_s^2+r_s)ds-\int_0^t\lambda_s dW_s}\mathbb{E}_t^{\mathbb{Q}}[e^{-\int_t^T r_s ds}]] \\
=& D_t[V_0 e^{-\int_0^t(\frac{1}{2}\lambda_s^2+r_s)ds-\int_0^t\lambda_s dW_s}]\mathbb{E}_t^{\mathbb{Q}}[e^{-\int_t^T r_s ds}] + V_0 e^{-\int_0^t(\frac{1}{2}\lambda_s^2+r_s)ds-\int_0^t\lambda_s dW_s} D_t[\mathbb{E}_t^{\mathbb{Q}}[e^{-\int_t^T r_s ds}]] \\
=& \big(-\lambda_t \mathbb{E}_t^{\mathbb{Q}}[e^{-\int_t^T r_s ds}] + D_t[\mathbb{E}_t^{\mathbb{Q}}[e^{-\int_t^T r_s ds}]]\big) V_0 e^{-\int_0^t(\frac{1}{2}\lambda_s^2+r_s)ds-\int_0^t\lambda_s dW_s} \\
=& \Big(-\lambda_t + \frac{D_t[\mathbb{E}_t^{\mathbb{Q}}[e^{-\int_t^T r_s ds}]]}{\mathbb{E}_t^{\mathbb{Q}}[e^{-\int_t^T r_s ds}]}\Big) Z_{tT}.
\end{aligned}
$$

Therefore, the risk-adjusted volatility may be expressed in the following way:

$$
\hat{V}_{tT} = -\lambda_t + \frac{D_t\big[\mathbb{E}_t^{\mathbb{Q}}[e^{-\int_t^T r_s ds}]\big]}{\mathbb{E}_t^{\mathbb{Q}}[e^{-\int_t^T r_s ds}]}, \quad 0 \le t \le T < \infty. \tag{8.1.2}
$$

This can be also shown from (3.2.7). By definitions (3.1.2) and (3.1.3), i.e., $F_{tTS} = \frac{1}{S-T}\left(\frac{Z_{tT}}{Z_{tS}} - 1\right)$ and $S_{a,b}(t) = \frac{Z_{tT_a}-Z_{tT_b}}{\sum_{i=a+1}^b \tau_i Z_{tT_i}}$, we obtain that for $0 \le t \le T \le S < \infty$

$$
F_{tTS} = \frac{1}{S-T}\left(\frac{\mathbb{E}_t^{\mathbb{Q}}\big[e^{-\int_t^T r_s ds}\big]}{\mathbb{E}_t^{\mathbb{Q}}\big[e^{-\int_t^S r_s ds}\big]} - 1\right) \quad \text{and} \quad S_{a,b}(t) = \frac{\mathbb{E}_t^{\mathbb{Q}}\big[e^{-\int_t^{T_a} r_s ds} - e^{-\int_t^{T_b} r_s ds}\big]}{\mathbb{E}_t^{\mathbb{Q}}\big[\sum_{i=a+1}^b \tau_i e^{-\int_t^{T_i} r_s ds}\big]}.
$$

Now, we should be able to obtain the volatility drifts in the forward LIBOR rate and forward swap rate dynamics in terms of the short rate and the market price of risk. Let us recall the LIBOR rate volatility γ_{tTS} from (3.3.19) and the swap rate volatility $\widetilde{\gamma}_{a,b}(t)$ from (3.3.22):

$$
\gamma_{tTS} = \frac{D_t[Z_{tT} - Z_{tS}]}{Z_{tT} - Z_{tS}} - \frac{D_t[Z_{tS}]}{Z_{tS}} \quad \text{and} \quad \widetilde{\gamma}_{a,b}(t) = \frac{D_t[Z_{tT_a} - Z_{tT_b}]}{Z_{tT_a} - Z_{tT_b}} - \frac{D_t\big[\sum_{i=a+1}^b \tau_i Z_{tT_i}\big]}{\sum_{i=a+1}^b \tau_i Z_{tT_i}}.
$$

Thus, using (8.1.1) we may deduce that

$$\begin{aligned}\gamma_{tTS} =& \frac{D_t[Z_{tT} - Z_{tS}]}{Z_{tT} - Z_{tS}} + \lambda_t - \frac{D_t\big[\mathbb{E}_t^{\mathbb{Q}}\big[e^{-\int_t^S r_s ds}\big]\big]}{\mathbb{E}_t^{\mathbb{Q}}\big[e^{-\int_t^S r_s ds}\big]} \\
=& \frac{D_t[e^{-\int_0^t(\frac{1}{2}\lambda_s^2+r_s)ds-\int_0^t \lambda_s dW_s}\mathbb{E}_t^{\mathbb{Q}}[e^{-\int_t^T r_s ds} - e^{-\int_t^S r_s ds}]]}{e^{-\int_0^t(\frac{1}{2}\lambda_s^2+r_s)ds-\int_0^t \lambda_s dW_s}\mathbb{E}_t^{\mathbb{Q}}[e^{-\int_t^T r_s ds} - e^{-\int_t^S r_s ds}]} + \lambda_t - \frac{D_t\big[\mathbb{E}_t^{\mathbb{Q}}\big[e^{-\int_t^S r_s ds}\big]\big]}{\mathbb{E}_t^{\mathbb{Q}}\big[e^{-\int_t^S r_s ds}\big]} \\
=& \frac{-\lambda_t\mathbb{E}_t^{\mathbb{Q}}[e^{-\int_t^T r_s ds} - e^{-\int_t^S r_s ds}] + D_t\big[\mathbb{E}_t^{\mathbb{Q}}[e^{-\int_t^T r_s ds} - e^{-\int_t^S r_s ds}]\big]}{\mathbb{E}_t^{\mathbb{Q}}[e^{-\int_t^T r_s ds} - e^{-\int_t^S r_s ds}]} + \lambda_t - \frac{D_t\big[\mathbb{E}_t^{\mathbb{Q}}\big[e^{-\int_t^S r_s ds}\big]\big]}{\mathbb{E}_t^{\mathbb{Q}}\big[e^{-\int_t^S r_s ds}\big]} \\
=& \frac{D_t\big[\mathbb{E}_t^{\mathbb{Q}}\big[e^{-\int_t^T r_s ds} - e^{-\int_t^S r_s ds}\big]\big]}{\mathbb{E}_t^{\mathbb{Q}}\big[e^{-\int_t^T r_s ds} - e^{-\int_t^S r_s ds}\big]} - \frac{D_t\big[\mathbb{E}_t^{\mathbb{Q}}\big[e^{-\int_t^S r_s ds}\big]\big]}{\mathbb{E}_t^{\mathbb{Q}}\big[e^{-\int_t^S r_s ds}\big]}.\end{aligned}$$

Similarly, we obtain for the swap rate volatility that

$$\widetilde{\gamma}_{a,b}(t) = \frac{D_t\big[\mathbb{E}_t^{\mathbb{Q}}\big[e^{-\int_t^{T_a} r_s ds} - e^{-\int_t^{T_b} r_s ds}\big]\big]}{\mathbb{E}_t^{\mathbb{Q}}\big[e^{-\int_t^{T_a} r_s ds} - e^{-\int_t^{T_b} r_s ds}\big]} - \frac{D_t\big[\mathbb{E}_t^{\mathbb{Q}}\big[\sum_{i=a+1}^b \tau_i e^{-\int_t^{T_i} r_s ds}\big]\big]}{\mathbb{E}_t^{\mathbb{Q}}\big[\sum_{i=a+1}^b \tau_i e^{-\int_t^{T_i} r_s ds}\big]}.$$

Therefore, the forward LIBOR rate dynamics can be expressed in the following way:

$$(8.1.3)\quad dF_{tTS} = [\cdots]dt + \left(\frac{D_t\big[\mathbb{E}_t^{\mathbb{Q}}[e^{-\int_t^T r_s ds} - e^{-\int_t^S r_s ds}]\big]}{\mathbb{E}_t^{\mathbb{Q}}[e^{-\int_t^T r_s ds} - e^{-\int_t^S r_s ds}]} - \frac{D_t\big[\mathbb{E}_t^{\mathbb{Q}}[e^{-\int_t^S r_s ds}]\big]}{\mathbb{E}_t^{\mathbb{Q}}[e^{-\int_t^S r_s ds}]}\right)F_{tTS}dW_t$$

and the forward swap rate dynamics can be expressed in the following way:

(8.1.4)

$$dS_{a,b}(t) = [\cdots]dt + \left(\frac{D_t\big[\mathbb{E}_t^{\mathbb{Q}}\big[e^{-\int_t^{T_a} r_s ds} - e^{-\int_t^{T_b} r_s ds}\big]\big]}{\mathbb{E}_t^{\mathbb{Q}}\big[e^{-\int_t^{T_a} r_s ds} - e^{-\int_t^{T_b} r_s ds}\big]} - \frac{D_t\big[\mathbb{E}_t^{\mathbb{Q}}\big[\sum_{i=a+1}^b \tau_i e^{-\int_t^{T_i} r_s ds}\big]\big]}{\mathbb{E}_t^{\mathbb{Q}}\big[\sum_{i=a+1}^b \tau_i e^{-\int_t^{T_i} r_s ds}\big]}\right)S_{a,b}(t)dW_t.$$

This means that modelling the short rate process $(r_t)_{t\geq 0}$ is equivalent to modelling the forward LIBOR rate and swap rate volatilities. Although we here consider the same problem as in the short rate models, we gain some advantages in our framework, i.e., explicit specification of the volatility terms and analytical option pricing via the state price density.

8.1.1 From the Affine Term Structure Model to the Market Model

Let us now apply the one-factor Affine Term Structure Model, i.e., we assume that

$$dr_t = (\kappa_t r_t + \eta_t)dt + \sqrt{\gamma_t r_t + \delta_t}dW_t,$$

for some deterministic functions $\kappa, \eta, \gamma, \delta$. In this model we are able to obtain by the Feynman-Kac Formula, (see for example, [57]), that

$$P_{tT} = \mathbb{E}_t[e^{-\int_t^T r_s ds}] = e^{A_{tT}+B_{tT}r_t}, \quad \text{for} \quad 0 \leq t \leq T < \infty,$$

for deterministic functions A and B such that $A_{TT} = 0, B_{TT} = 0$ which satisfy the following Riccati equations:

$$\begin{cases} \frac{\partial}{\partial t} B_{tT} = -\kappa_t B_{tT} - \frac{1}{2}\gamma_t B_{tT}^2 + 1 \\ \frac{\partial}{\partial t} A_{tT} = -\eta_t B_{tT} - \frac{1}{2}\delta_t B_{tT}^2 \end{cases}.$$

It gives the discount bond volatility expressed in the following way:

$$\Omega_{tT} = \frac{D_t[e^{A_{tT}+B_{tT}r_t}]}{e^{A_{tT}+B_{tT}r_t}} = B_{tT}D_t[r_t], \quad 0 \leq t \leq T < \infty.$$

Recalling the expression (8.1.2) we obtain the risk-adjusted volatility as follows:

$$\hat{V}_{tT} = -\lambda_t + B_{tT}D_t[r_t], \quad 0 \leq t \leq T < \infty.$$

Inserting this expression into equation (3.3.2) we obtain that

$$dF_{tTS} = (\lambda_t - B_{tT}D_t[r_t])(B_{tT}-B_{tS})D_t[r_t]\Big(F_{tTS}+\frac{1}{S-T}\Big)dt+(B_{tT}-B_{tS})D_t[r_t]\Big(F_{tTS}+\frac{1}{S-T}\Big)dW_t.$$

Here, we observe the market price of risk has disappeared in the volatility drift. Using the expression (3.3.5) the forward LIBOR rate volatility is expressed under the Affine Term Structure in the following way:

$$\gamma_{tTS} = \frac{1}{1-e^{(A_{tS}-A_{tT})+(B_{tS}-B_{tT})r_t}}(B_{tT}-B_{tS})D_t[r_t].$$

Similarly, using the expression (3.3.9) the swap rate volatility is expressed in the following way:

$$\widetilde{\gamma}_{a,b}(t) = \Big[\frac{e^{A_{tT_a}+B_{tT_a}r_t}B_{tT_a} - e^{A_{tT_b}+B_{tT_b}r_t}B_{tT_b}}{e^{A_{tT_a}+B_{tT_a}r_t} - e^{A_{tT_b}+B_{tT_b}r_t}} - \frac{\sum_{i=a+1}^b \tau_i e^{A_{tT_i}+B_{tT_i}r_t}B_{tT_i}}{\sum_{i=a+1}^b \tau_i e^{A_{tT_i}+B_{tT_i}r_t}}\Big]D_t[r_t].$$

From the Vasicek Model to the Market Model

For instance, in the Vasicek Model, it is assumed that

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t,$$

for some constants $\kappa, \theta, \sigma \in \mathbb{R}$. Because this Gaussian process gives $D_t[r_t] = \sigma$, we obtain in the Vasicek Model that

$$dF_{tTS} = (\lambda_t - B_{tT}\sigma)(B_{tT} - B_{tS})\sigma\Big(F_{tTS} + \frac{1}{S-T}\Big)dt + (B_{tT} - B_{tS})\sigma\Big(F_{tTS} + \frac{1}{S-T}\Big)dW_t.$$

which corresponds to the Shifted-Lognormal Market Model, see (3.3.6). Taking some function to express the market price of risk to be $\lambda_t = g(t, r_t)$, we express the super-martingale process in the following way:

$$V_t = V_0 e^{-\int_0^t (\frac{1}{2}g^2(s,r_s)+r_s)ds - \int_0^t g(s,r_s)dW_s}, \quad \text{for} \quad t \geq 0.$$

In the case of the Vasicek Model we take $g(t, r_t) = kr_t$ for some constant $k \in \mathbb{R}$ so that we obtain that

$$V_t = V_0 e^{-\int_0^t (\frac{1}{2}k^2 r_s^2 + r_s)ds - \int_0^t kr_s dW_s}, \quad \text{for} \quad t \geq 0,$$

However, this model is not in our interest, since we know that the Shifted-Lognormal Market Model performs badly for hedging derivatives. Moreover, because in the Vasicek Model the process is not guaranteed to be positive, the state price density is not potential.

From the CIR Model to the Market Model

For other example, the CIR Model assumes that

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t$$

for some positive constants κ, θ, σ such that $2\kappa\theta \geq \sigma^2$, and $g(t, r_t) = k\sqrt{r_t}$, that is,

$$V_t = V_0 e^{-\int_0^t (\frac{1}{2}k^2+1)r_s ds - \int_0^t k\sqrt{r_s}dW_s}, \quad \text{for} \quad t \geq 0.$$

Application of the CIR Model seems desirable for the positivity condition, which makes the state price density a potential. However, we find that it is difficult to compute the Malliavin derivative $D_t[r_t]$ in the CIR Model.

8.1.2 From the Squared Gaussian Model to the Market Model

Let us now consider the short rate process taking a powered form of a state variable, i.e., $r_t = \hat{r}_t^n$, $t \geq 0$, $n \in \mathbb{N}$, for a continuous adapted process $(\hat{r}_t)_{t\geq 0}$. In particular, taking a squared form, i.e., $r_t = \hat{r}_t^2$, $t \geq 0$ would be ideal to secure the interest rate positivity condition. For example, in the Squared Gaussian Model ([69]), we assume that

$$d\hat{r}_t = \kappa(\theta - \hat{r}_t)dt + \sigma dW_t,$$

for some constants $\kappa, \theta, \sigma \in \mathbb{R}$, which gives the discount bond expressed by some deterministic function $\widetilde{A}, \widetilde{B}, \widetilde{C}$ in the following way

$$P_{tT} = e^{\widetilde{A}_{tT} + \widetilde{B}_{tT}\hat{r}_t + \widetilde{C}_{tT}\hat{r}_t^2}, \quad 0 \leq t \leq T < \infty.$$

Therefore, the Squared Gaussian Model gives the risk adjusted volatility expressed by the market price of risk and the Gaussian process, that is,

$$\begin{aligned}\hat{V}_{tT} =& -\lambda_t + D_t[\widetilde{A}_{tT} + \widetilde{B}_{tT}\hat{r}_t + \widetilde{C}_{tT}\hat{r}_t^2] = -\lambda_t + \widetilde{B}_{tT}D_t[\hat{r}_t] + \widetilde{C}_{tT}D_t[\hat{r}_t^2] \\ =& -\lambda_t + \widetilde{B}_{tT}\sigma + 2\widetilde{C}_{tT}\sigma\hat{r}_t\end{aligned}$$

Inserting this expression in the equation (3.3.2) we obtain that

(8.1.5)

$$\begin{aligned}dF_{tTS} =& (-\lambda_t + \widetilde{B}_{tS}\sigma + 2\widetilde{C}_{tS}\sigma\hat{r}_t)(\widetilde{B}_{tS} - \widetilde{B}_{tT} + (\widetilde{C}_{tS} - \widetilde{C}_{tT})\hat{r}_t)\Big(F_{tTS} + \frac{1}{S-T}\Big)dt \\ & - (\widetilde{B}_{tS} - \widetilde{B}_{tT} + (\widetilde{C}_{tS} - \widetilde{C}_{tT})\hat{r}_t)\Big(F_{tTS} + \frac{1}{S-T}\Big)dW_t.\end{aligned}$$

Because $\hat{r}_t$ is normally distributed, we obtain stochasticity in the LIBOR rate volatility. We here suggest that it would be useful to model the process $(\hat{r}_t)_{\geq 0}$ by the two-factor Affine Model so that the LIBOR rate volatility and the swap rate volatility have the desirable distributions. However, it would not be good idea to apply a Log-r Model (taking the exponential of a state variable as $r_t = e^{\hat{r}_t}$, such as Black-Derman-Toy Model ([8]) and Black-Karasinski Model ([9]), see Section 9.3.3 in [54]), because in these model we do not have analytical expression of the discount bond, which means that we can not express the forward LIBOR rate volatility explicitly. We leave the remaining works open.

8.2 Modelling the term structure from $(\eta_{tT})_{0\le t\le T<\infty}$ in the FH Framework

In this section, we make further investigations of the FH Framework. The reader might like to recall that in the FH Framework the variable Z_{tT} for $0 \le t \le T < \infty$ is expressed as follows:

$$Z_{tT} = \int_T^\infty h_s \hat{M}_{ts} ds, \quad 0 \le t \le T < \infty, \tag{8.2.1}$$

where $h_s = -\frac{d}{ds}P_{0s}$ and $\hat{M}_{ts}$ is a strictly positive martingale for each $t \in [0,\infty)$ such that $\hat{M}_{0s} = 1$ and $\lim_{s\to\infty} \hat{M}_{ts} = 1$. The Martingale Representation Theorem implies that

$$d\hat{M}_{ts} = \eta_{ts}\hat{M}_{ts}dW_t, \quad 0 \le t \le s < \infty,$$

for some adapted process $(\eta_{ts})_{0\le t\le s<\infty}$. Solving the stochastic differential equation we obtain the following expression:

$$\hat{M}_{ts} = \exp\left[\int_0^t \eta_{us}dW_u - \frac{1}{2}\int_0^t \eta_{us}^2 du\right], \quad 0 \le t \le s < \infty. \tag{8.2.2}$$

From this, it follows that

$$P_{tT} = \frac{\int_T^\infty h_s \exp\left[-\frac{1}{2}\int_0^t \eta_{us}^2 du + \int_0^t \eta_{us}dW_u\right]ds}{\int_t^\infty h_s \exp\left[-\frac{1}{2}\int_0^t \eta_{us}^2 du + \int_0^t \eta_{us}dW_u\right]ds}.$$

Therefore, in the FH Framework, we apply the same framework as we observe in the Chaotic Approach, but specify the function η. Note that in particular we have the following form for the initial curve:

$$f_{0T} = \frac{h_T}{\int_T^\infty h_s ds}, \quad \text{for} \quad T \geq 0,$$

which corresponds to the curves in the Chaos Models.

8.2.1 Deterministic η

Let us suppose the function η is deterministic. Then, because the generating function for the Hermite polynomials is given by

$$\exp\left[tx - \frac{1}{2}t^2\right] = \sum_{n=0}^{\infty} t^n H_n(x), \quad \text{where} \quad H_n(x) = \frac{1}{n!}(-1)^n e^{\frac{1}{2}x^2} \frac{d^n}{dx^n}(e^{-\frac{1}{2}x^2}),$$

we infer that

$$\exp\left[\int_0^t \eta_{us} dW_u - \frac{1}{2}\int_0^t \eta_{us}^2 du\right] = \sum_{n=0}^{\infty} \xi_{ts}^{\frac{n}{2}} H_n(\theta_t).$$

Here, we have defined

$$\xi_{ts} := \int_0^t \eta_{us}^2 du \quad \text{and} \qquad \theta_t := \frac{\int_0^t \eta_{us} dW_u}{\sqrt{\int_0^t \eta_{us}^2 du}} \sim \mathcal{N}(0,1).$$

Therefore, by (8.2.1) and (8.2.2) we can write

$$Z_{tT} = \int_T^\infty h_s \sum_{n=0}^{\infty} \xi_{ts}^{\frac{n}{2}} H_n(\theta_t) ds.$$

Continuing further, it follows by the linearity of the Riemann integral that

$$Z_{tT} = \sum_{n=0}^{\infty} \int_T^\infty h_s \xi_{ts}^{\frac{n}{2}} ds H_n(\theta_t). \tag{8.2.3}$$

Since the first few terms of the Hermite polynomials are given by $H_0(x) = 1, H_1(x) = x, H_2(x) = \frac{1}{2}(x^2 - 1), H_3(x) = \frac{1}{6}(x^3 - 3x), H_4(x) = \frac{1}{24}(x^4 - 6x^2 + 3)$ and so on, we can see that

$$\begin{aligned} Z_{tT} = &\int_T^\infty h_s ds + \int_T^\infty h_s (\xi_{ts})^{\frac{1}{2}} ds \theta + \frac{1}{2}\int_T^\infty h_s \xi_{ts} ds(\theta^2 - 1) \\ &+ \frac{1}{6}\int_T^\infty h_s (\xi_{ts})^{\frac{3}{2}} ds(\theta^3 - 3\theta) + \frac{1}{24}\int_T^\infty h_s \xi_{ts}^2 ds(\theta^4 - 6\theta^2 + 3) + \cdots, \end{aligned}$$

which implies that the whole term structure can be modelled by a unique Gaussian process. Here, we truncate the polynomial at n-th term and then call this the "n^{th}-order FH Model". Therefore, the n^{th}-order FH Model gives that both the state price density V_t and the variable Z_{tT} is distributed by a degree n polynomial of the Gaussian distribution. It gives us analytical tractability for all main processes and derivatives, as we will see in Section 8.2.2.

First FH Models

When we truncate the expansion at the first term, we obtain the following deterministic term structure models:

$$Z_{tT} = \int_T^\infty h_s ds, \quad V_t = \int_t^\infty h_s ds, \quad P_{tT} = \frac{\int_T^\infty h_s ds}{\int_t^\infty h_s ds} \quad \text{and} \quad f_{tT} = \frac{h_T}{\int_T^\infty h_s ds}.$$

This corresponds to the First Chaos Model.

Second FH Models

When we truncate the expansion at the second term, we obtain that

$$Z_{tT} = \int_T^\infty h_s ds + \int_T^\infty h_s \sqrt{\xi_{ts}} ds \theta_t \quad \text{and} \quad P_{tT} = \frac{\int_T^\infty h_s ds + \int_T^\infty h_s \sqrt{\xi_{ts}} ds \theta_t}{\int_t^\infty h_s ds + \int_t^\infty h_s \sqrt{\xi_{ts}} ds \theta_t}.$$

This allows us to model the swaption and caplet normally distributed as in (2.3.4) and (2.3.10). This corresponds to the Factorizable Second Chaos Model. Note here that even in this simple case we achieve the construction of a forward LIBOR rate dynamics with a stochastic volatility.

Exponential example

Cairns ([23]) suggests the exponential form $\eta_{us} = \alpha e^{-\beta(s-u)}$ for some constants α and β in the FH framework, that is,

$$M_{tT} = f_{0T} P_{0T} V_0 \exp\Big[-\frac{1}{2}\int_0^t (\alpha e^{-\beta(T-u)})^2 du + \int_0^t \alpha e^{-\beta(T-u)} dW_u \Big],$$

$$V_t = \int_t^\infty f_{0s} P_{0s} V_0 \exp\Big[-\frac{1}{2}\int_0^t \alpha^2 e^{-2\beta(s-u)} du + \int_0^t \alpha e^{-\beta(s-u)} dW_u \Big] ds,$$

$$Z_{tT} = \int_T^\infty f_{0s} P_{0s} V_0 \exp\Big[-\frac{1}{2}\int_0^t \alpha^2 e^{-2\beta(s-u)} du + \int_0^t \alpha e^{-\beta(s-u)} dW_u \Big] ds.$$

If we also take the exponential form in our argument we obtain that

$$\xi_{ts} = \alpha^2 e^{-2\beta s} \int_0^t e^{2\beta u} du, \quad \theta_t = \frac{\int_0^t e^{\beta u} dW_u}{\sqrt{\int_0^t e^{2\beta u} du}} \sim \mathcal{N}(0,1).$$

Remark

Note that if we consider the multi-dimensional case, we obtain that

$$\hat{M}_{ts} = \exp\Big[\sum_{j=1}^m \int_0^t \eta_j(u,s) dW_j(u) - \sum_{j=1}^m \frac{1}{2} \int_0^t \eta_j^2(u,s) du \Big].$$

Therefore we obtain in the multi-dimensional case that

$$Z_{tT} = \int_T^\infty h_s \prod_{j=1}^m \Big[\sum_{n=0}^\infty \Big(\int_0^t \eta_j^2(u,s) du \Big)^{\frac{n}{2}} H_n\Big(\frac{\int_0^t \eta_j(u,s) dW_j(u)}{\sqrt{\int_0^t \eta_j^2(u,s) du}} \Big) \Big] ds.$$

8.2.2 Pricing the European Call/Put Bond Options within the FH framework

Applying the form (8.2.3), we find that

$$\begin{aligned} Z_{tT} - K Z_{tt} =& \sum_{n=0}^\infty \int_T^\infty h_s \xi_{ts}^{\frac{n}{2}} ds H_n(\theta_t) - K \sum_{n=0}^\infty \int_t^\infty h_s \xi_{ts}^{\frac{n}{2}} ds H_n(\theta_t) \\ =& \sum_{n=0}^\infty \Big[\int_T^\infty h_s \xi_{ts}^{\frac{n}{2}} ds - K \int_t^\infty h_s (\xi_{ts})^{\frac{n}{2}} ds \Big] H_n(\theta_t). \end{aligned}$$

Applying the expectation rule, we obtain that

$$ZBC(0,t,T,K) = \frac{1}{V_0 \sqrt{2\pi}} \int_{\mathcal{P}_c(\theta) \geq 0} \mathcal{P}_c(\theta) e^{-\frac{\theta^2}{2}} d\theta,$$

where

$$V_0 = \int_0^\infty h_s ds \quad \text{and} \quad \mathcal{P}_c(\theta) := \sum_{n=0}^\infty \Big[\int_T^\infty h_s \xi_{ts}^{\frac{n}{2}} ds - K \int_t^\infty h_s \xi_{ts}^{\frac{n}{2}} ds \Big] H_n(\theta).$$

We can solve the integral by checking the roots of the function $\mathcal{P}_c(\theta)$. A similar argument may be applied for pricing swaptions.

8.3 Modelling the term structure from $(\eta_{tT})_{0\le t\le T<\infty}$ in the Chaotic Approach

Recall that the process $(\eta_{tT})_{0\le t\le T<\infty}$ was specified as follows:

$$\sigma_s^2 = h_s \exp\Big[-\frac{1}{2}\int_0^s \eta_{us}^2 du + \int_0^s \eta_{us} dW_u\Big] \quad \text{where} \quad h_s = f_{0s}P_{0s}V_0.$$

In the Chaotic Approach, we implement the chaos expansion on the variable σ_s and obtain that:

$$\sigma_s = \mathbb{E}[\sigma_s] + \int_0^s \mathbb{E}\big[D_{s_1}[\sigma_s]\big] dW_{s_1} + \int_0^s \int_0^{s_1} \mathbb{E}\big[D_{s_2}[D_{s_1}[\sigma_s]]\big] dW_{s_2} dW_{s_1} + \cdots.$$

Therefore the chaos coefficients may be computed by specifying the process $(\eta_{tT})_{0\le t\le T<\infty}$.

8.3.1 Deterministic η

For simplicity, we first assume that the function η is deterministic and find chaos coefficients. Let us investigate the first chaos coefficient $\phi_1(s)$:

$$\phi_1(s) = \mathbb{E}[\sigma_s] = \mathbb{E}\Big[\sqrt{h_s}\exp\Big[-\frac{1}{4}\int_0^s \eta_{us}^2 du + \frac{1}{2}\int_0^s \eta_{us} dW_u\Big]\Big].$$

Since we have that

$$\frac{1}{2}\int_0^s \eta_{us} dW_u \sim \mathcal{N}(0, \hat{\sigma}_s^2) \quad \text{where} \quad \hat{\sigma}_s^2 = \frac{1}{4}\int_0^s \eta_{us}^2 du,$$

we infer that

$$\begin{aligned}
\mathbb{E}[\sqrt{\hat{M}_{ss}}] =& \mathbb{E}\Big[\exp\Big[\frac{1}{2}\int_0^s \eta_{us} dW_u\Big]\Big] \exp\Big[-\frac{1}{4}\int_0^s \eta_{us}^2 du\Big] \\
=& \frac{1}{\sqrt{2\pi\hat{\sigma}_s^2}}\int_{-\infty}^{\infty} e^x e^{-\frac{x^2}{2\hat{\sigma}_s^2}} dx e^{-\hat{\sigma}_s^2} \\
=& \frac{1}{\sqrt{2\pi\hat{\sigma}_s^2}}\int_{-\infty}^{\infty} \exp\Big[-\frac{1}{2\hat{\sigma}_s^2}(x-\hat{\sigma}_s^2)^2 + \frac{\hat{\sigma}_s^2}{2}\Big] dx e^{-\hat{\sigma}_s^2} \\
=& e^{\frac{\hat{\sigma}_s^2}{2}} e^{-\hat{\sigma}_s^2} = \exp\Big[-\frac{1}{8}\int_0^s \eta_{us}^2 du\Big].
\end{aligned}$$

Therefore we obtain that the first chaos coefficient is as follows:

$$\phi_1(s) = \mathbb{E}[\sigma_s] = \sqrt{h_s}\mathbb{E}[\sqrt{\hat{M}_{ss}}] = \sqrt{h_s}\exp\left[-\frac{1}{8}\int_0^s \eta_{us}^2 du\right].$$

We now compute the second chaos coefficient, applying the Malliavin derivative:

$$\begin{aligned}
\phi_2(s, s_1) =& \mathbb{E}[D_{s_1}\sigma_s] \\
=& \sqrt{h_s}\mathbb{E}[D_{s_1}\sqrt{\hat{M}_{ss}}] \\
=& \sqrt{h_s}\mathbb{E}\left[\frac{1}{2}\eta_{s_1 s}\sqrt{\hat{M}_{ss}}\right] \\
=& \frac{1}{2}\sqrt{h_s}\mathbb{E}\left[\sqrt{\hat{M}_{ss}}\right]\eta_{s_1 s} \\
=& \frac{1}{2}\sqrt{h_s}\exp\left[-\frac{1}{8}\int_0^s \eta_{us}^2 du\right]\eta_{s_1 s}.
\end{aligned}$$

Similarly, the third chaos coefficient is derived as follows:

$$\begin{aligned}
\phi_3(s, s_1, s_2) =& \mathbb{E}[D_{s_2}[D_{s_1}\sigma_s]] \\
=& \sqrt{h_s}\mathbb{E}[D_{s_2}[D_{s_1}\sqrt{\hat{M}_{ss}}]] \\
=& \sqrt{h_s}\mathbb{E}\left[D_{s_2}\left[\frac{1}{2}\eta_{s_1 s}\sqrt{\hat{M}_{ss}}\right]\right] \\
=& \sqrt{h_s}\mathbb{E}\left[\frac{1}{4}\eta_{s_1 s}\eta_{s_2 s}\sqrt{\hat{M}_{ss}}\right] \\
=& \frac{1}{4}\sqrt{h_s}\mathbb{E}\left[\sqrt{\hat{M}_{ss}}\right]\eta_{s_1 s}\eta_{s_2 s} \\
=& \frac{1}{4}\sqrt{h_s}\exp\left[-\frac{1}{8}\int_0^s \eta_{us}^2 du\right]\eta_{s_1 s}\eta_{s_2 s}
\end{aligned}$$

Continuing in this manner, we find the chaos coefficients are as follows:

$$\begin{aligned}
&\phi_1(s) = \sqrt{h_s}\exp\left[-\frac{1}{8}\int_0^s \eta_{us}^2 du\right], \quad \phi_2(s, s_1) = \frac{1}{2}\sqrt{h_s}\exp\left[-\frac{1}{8}\int_0^s \eta_{us}^2 du\right]\eta_{s_1 s}, \\
&\phi_3(s, s_1, s_2) = \frac{1}{4}\sqrt{h_s}\exp\left[-\frac{1}{8}\int_0^s \eta_{us}^2 du\right]\eta_{s_1 s}\eta_{s_2 s}, \\
&\phi_4(s, s_1, s_2, s_3) = \frac{1}{8}\sqrt{h_s}\exp\left[-\frac{1}{8}\int_0^s \eta_{us}^2 du\right]\eta_{s_1 s}\eta_{s_2 s}\eta_{s_3 s}, \quad \cdots.
\end{aligned}$$

First Chaos Model

In the first Chaos Model we have that

$$Z_{tT} = \int_T^\infty h_s \exp\left[-\frac{1}{4}\int_0^s \eta_{us}^2 du\right] ds.$$

Second Chaos Model

In the Second Chaos Model we have that

$$M_{ts} = R_1^2(t,s) + \int_t^s R_2^2(t,s,s_1) ds_1, \quad 0 \le t \le s < \infty$$

where

$$R_1(t,s) = \sqrt{h_s} \exp\left[-\frac{1}{8}\int_0^s \eta_{us}^2 du\right] + \int_0^t \frac{1}{2}\sqrt{h_s} \exp\left[-\frac{1}{8}\int_0^s \eta_{us}^2 du\right] \eta_{s_1 s} dW s_1,$$

$$R_2(t,s,s_1) = \frac{1}{2}\sqrt{h_s} \exp\left[-\frac{1}{8}\int_0^s \eta_{us}^2 du\right] \eta_{s_1 s}.$$

Given that we thus have

$$R_1^2(t,s) = h_s \exp\left[-\frac{1}{4}\int_0^s \eta_{us}^2 du\right]\left(1 + \frac{1}{2}\int_0^t \eta_{s_1 s} dW s_1\right)^2,$$

$$R_2^2(t,s,s_1) = \frac{1}{4} h_s \exp\left[-\frac{1}{4}\int_0^s \eta_{us}^2 du\right] \eta_{s_1 s}^2,$$

we infer that

$$M_{ts} = h_s \exp\left[-\frac{1}{4}\int_0^s \eta_{us}^2 du\right]\left(\left(1+\frac{1}{2}\int_0^t \eta_{s_1 s} dW s_1\right)^2 + \frac{1}{4}\int_t^s \eta_{s_1 s}^2 ds_1\right), \quad 0 \le t \le s < \infty.$$

From this, it follows that

$$Z_{ts} = \int_T^\infty h_s \exp\left[-\frac{1}{4}\int_0^s \eta_{us}^2 du\right]\left(\left(1+\frac{1}{2}\int_0^t \eta_{s_1 s} dW s_1\right)^2 + \frac{1}{4}\int_t^s \eta_{s_1 s}^2 ds_1\right) ds, \quad 0 \le t \le T < \infty.$$

Exponential example

As a specific example, when we suppose the deterministic function to be $\eta_{us} = \alpha e^{-\beta(s-u)}$ for some constants α and β, the main processes can be approximated using the chaos coefficients:

$$\phi_1(s) = \sqrt{h_s}\exp\Big[-\frac{\alpha^2}{16\beta}(1-e^{-2\beta s})\Big], \quad \phi_2(s,s_1) = \phi_1(s)\frac{\alpha}{2}e^{-\beta s}e^{\beta s_1},$$
$$\phi_3(s,s_1,s_2) = \phi_1(s)\Big(\frac{\alpha}{2}e^{-\beta s}\Big)^2 e^{\beta s_1}e^{\beta s_2}, \quad \phi_4(s,s_1,s_2,s_3) = \phi_1(s)\Big(\frac{\alpha}{2}e^{-\beta s}\Big)^3 e^{\beta s_1}e^{\beta s_2}e^{\beta s_3}, \quad \cdots.$$

Therefore we obtain the Factorizable Chaos Models. For example, we may construct a Factorizable Second Chaos Model as follows:

$$Z_{tT} = A_T + B_T\hat{R}_t + C_T(\hat{R}_t^2 - \hat{Q}_t),$$

where

$$A_t = \int_t^\infty \phi_1^2(s)\Big[1+\Big(\frac{\alpha}{2}e^{-\beta s}\Big)^2\Big]ds, \quad B_t = 2\int_t^\infty \phi_1^2(s)\Big(\frac{\alpha}{2}e^{-\beta s}\Big)ds, \quad C_t = 2\int_t^\infty \phi_1^2(s)\Big(\frac{\alpha}{2}e^{-\beta s}\Big)^2 ds,$$
$$\phi_1(s) = \sqrt{h_s}\exp\Big[-\frac{\alpha^2}{16\beta}(1-e^{-2\beta s})\Big], \quad \hat{R}_t = \int_0^t e^{\beta s}dW_s \quad \text{and} \quad \hat{Q}_t = \int_0^t e^{2\beta s}ds.$$

8.4 Modelling the primitive process from its SDE

Let us now model the stochastic differential equation of the primitive process $(\sigma_t)_{t\geq 0}$. Starting our argument with this process, we do not need to be careful about the positivity problem, but we need to model the process such that $\sigma_t \in L^2$, that is,

$$\sup_{t\in\mathbb{R}^+} \mathbb{E}[\sigma_t^2] < \infty.$$

The reader might like to recall here that $\mathbb{E}[\sigma_t^2] = V_0 f_{0t} P_{0t}$. We observe some relationship with the Chaotic Approach, recalling the following form of the variable σ_t from (2.1.23) in the Chaotic Approach:

$$d\sigma_t = \Big(\frac{d}{dt}\phi_1(t)\Big)dt + \Big(\phi_2(t,t) + \int_0^t \phi_3(t,t,s_2)dW_{s_2} + \int_0^t\int_0^{s_1} \phi_4(t,t,s_2,s_3)dW_{s_3}dW_{s_2} + \cdots\Big)dW_t.$$

Therefore the First Chaos Model presents a deterministic dynamics, the Second Chaos Model presents local volatility dynamics, and the higher Chaos Models offer stochastic volatility dynamics.

8.4.1 One Factor Deterministic Volatility Case

We first consider deterministic term structure and the local volatility case. In other words, up to the Second Chaos Models are investigated in this section. The higher order models are researched in the next section.

Deterministic Form

Suppose that we have some integrable deterministic process $(\varphi_t)_{t\geq 0}$ with the property that

$$d\sigma_t = \varphi_t dt.$$

Then this corresponds to the First Chaos Model, as can be seen by recalling expression (3.1.8) for the short rate:

$$r_t = \frac{\sigma_t^2}{\int_t^\infty \mathbb{E}_t[\sigma_s^2]ds},$$

which shows a deterministic term structure.

Zero Drift Form

Suppose that we have some integrable deterministic process $(\upsilon_t)_{t\geq 0}$ with the property that

$$d\sigma_t = \upsilon_t dW_t.$$

In this case we have that

$$\sigma_t = \sigma_0 + \int_0^t \upsilon_s dW_s \quad \text{and} \quad \mathbb{E}[\sigma_t^2] = \sigma_0^2 + \int_0^t \upsilon_s^2 ds.$$

Therefore, in this case, we observe that $\sigma_t \notin L^2$.

Geometric Brownian Motion Form

Suppose that for some constants φ and υ, we have that

$$d\sigma_t = \varphi\sigma_t dt + \upsilon\sigma_t dW_t.$$

It follows that

$$\sigma_t = \sigma_0 e^{\left(\varphi - \frac{\upsilon^2}{2}\right)t + \upsilon W_t}, \quad \mathbb{E}[\sigma_t] = \sigma_0 e^{\varphi t} \quad \text{and} \quad \mathbb{E}[\sigma_t^2] = \sigma_0^2 e^{(2\varphi + \upsilon^2)t}.$$

Therefore we need to restrict the parameters so that $2\varphi + \upsilon^2 < 0$, in order to ensure that $\sigma_t \in L^2$. However, this form is too simple for initial curve fitting, that is $\mathbb{E}[\sigma_t^2] = V_0 f_{0t} P_{0t}$, and so is not desirable in practice.

Geometric Brownian Motion++

To have a better initial curve fitting we next consider the following extensional form:

$$\sigma_t = x_t + z_t,$$

where

$$dx_t = \varphi x_t dt + \upsilon x_t dW_t,$$

φ and υ are some constants and z_t is some deterministic function. This implies that

$$\sigma_t = x_0 e^{\left(\varphi - \frac{\upsilon^2}{2}\right)t + \upsilon W_t} + z_t,$$

from which it follows that

$$\mathbb{E}[\sigma_t] = x_0 e^{\varphi t} + z_t \quad \text{and} \quad \mathbb{E}[\sigma_t^2] = x_0^2 e^{(2\varphi + \upsilon^2)t} + 2x_0 e^{\varphi t} z_t + z_t^2.$$

Here the conditions $2\varphi + \upsilon^2 < 0$ and $\varphi < 0$ must be imposed. Though this form does not correspond to the Chaotic Approach, it would satisfy the initial curve condition. We particularly suggest to model the variable z_t by the descriptive form.

Log-Normal Form

We now consider a more general form. Suppose that

$$d\sigma_t = \varphi_t \sigma_t dt + \upsilon_t \sigma_t dW_t,$$

for some integrable deterministic processes $(\varphi_t)_{t\geq 0}$ and $(\upsilon_t)_{t\geq 0}$. It follows that

$$\sigma_t = \sigma_0 e^{\int_0^t (\varphi_s - \frac{\upsilon_s^2}{2})ds + \int_0^t \upsilon_s dW_s}, \quad \mathbb{E}[\sigma_t] = \sigma_0 e^{\int_0^t \varphi_s ds},$$

$$\begin{aligned}\mathbb{E}[\sigma_t^2] =& \mathbb{E}[\sigma_0^2 e^{\int_0^t (2\varphi_s - \upsilon_s^2)ds + \int_0^t 2\upsilon_s dW_s}] \\ =& \sigma_0^2 e^{\int_0^t (2\varphi_s - \upsilon_s^2)ds} \mathbb{E}[e^{\int_0^t 2\upsilon_s dW_s}] \\ =& \sigma_0^2 e^{\int_0^t (2\varphi_s + \upsilon_s^2)ds}.\end{aligned}$$

Log-Normal++

To obtain a better initial curve fitting than in the log-normal case, we next consider the following form:

$$\sigma_t = x_t + z_t,$$

where

$$dx_t = \varphi_t x_t dt + \upsilon_t x_t dW_t,$$

for some integrable deterministic processes $(\varphi_t)_{t\geq 0}$ and $(\upsilon_t)_{t\geq 0}$ and some deterministic function z_t. From this we infer that

$$\sigma_t = x_0 e^{\int_0^t (\varphi_s - \frac{\upsilon_s^2}{2})ds + \int_0^t \upsilon_s dW_s} + z_t,$$

and further that

$$\mathbb{E}[\sigma_t] = x_0 e^{\int_0^t \varphi_s ds} + z_t \quad \text{and} \quad \mathbb{E}[\sigma_t^2] = x_0^2 e^{\int_0^t (2\varphi_s + \upsilon_s^2)ds} + 2x_0 e^{\int_0^t \varphi_s ds} z_t + z_t^2.$$

This is a more general form. Although this form does not correspond to the Second Chaos Model, it would give us good initial curve fitting.

Gaussian Form

We now consider the dynamics of the primitive process, by assuming that

$$d\sigma_t = -\varphi\sigma_t dt + \upsilon dW_t$$

for some constants φ and υ. It follows that

$$\sigma_t = \sigma_0 e^{-\varphi t} + \upsilon \int_0^t e^{-\varphi(t-u)} dW_u \quad \text{and} \quad \mathbb{E}[\sigma_t^2] = \sigma_0^2 e^{-2\varphi t} + \frac{\upsilon}{2\varphi}\Big[1 - e^{-2\varphi t}\Big],$$

which belongs to the Factorizable Second Chaos Model. The constant φ needs to be positive so that $\sigma_t \in L^2$.

Gaussian++

Next we consider a more general case of the Gaussian form. Suppose that

$$\sigma_t = x_t + z_t,$$

where

$$dx_t = -\varphi x_t dt + \upsilon_t dW_t \quad \text{such that} \quad \varphi > 0,$$

where υ_t and z_t are some deterministic functions. This implies that

$$\sigma_t = \sigma_0 e^{-\varphi t} + z_t + \int_0^t e^{-\varphi(t-u)} \upsilon_u dW_u,$$

from which it follows that

$$\mathbb{E}[\sigma_t] = \sigma_0 e^{-\varphi t} + z_t \quad \text{and} \quad \mathbb{E}[\sigma_t^2] = (x_0 e^{-\varphi t} + z_t)^2 + \int_0^t e^{-2\varphi(t-u)} \upsilon_u^2 du.$$

Vasicek Form, OU process

Recalling the short rate formula, which is given by

$$r_t = \frac{\sigma_t^2}{\int_t^\infty \mathbb{E}_t[\sigma_s^2] ds},$$

it would be natural to consider the primitive process with mean reverting property. Suppose that

$$d\sigma_t = \kappa(\varphi - \sigma_t)dt + \upsilon dW_t,$$

for some constants κ, φ and υ. It follows that

$$\sigma_t = \sigma_0 e^{-\kappa t} + \varphi\Big(1 - e^{-\kappa t}\Big) + \upsilon \int_0^t e^{-\kappa(t-u)} dW_u,$$

and also that

$$\mathbb{E}[\sigma_t] = \sigma_0 e^{-\kappa t} + \varphi(1 - e^{-\kappa t}) \quad \text{and} \quad \mathbb{E}[\sigma_t^2] = \Big(\sigma_0 e^{-\kappa t} + \varphi(1 - e^{-\kappa t})\Big)^2 + \frac{\upsilon^2}{2\kappa}\Big(1 - e^{-2\kappa t}\Big).$$

The constant κ must be positive to ensure $\sigma_t \in L^2$.

Vasicek++ Form

Let us suppose that

$$\sigma_t = x_t + z_t,$$

where

$$dx_t = \kappa(\varphi - x_t)dt + \upsilon_t dW_t \quad \text{such that} \quad \kappa > 0,$$

with some deterministic functions υ_t and z_t. It follows that

$$\sigma_t = x_0 e^{-\kappa t} + \varphi\Big(1 - e^{-\kappa t}\Big) + z_t + \int_0^t e^{-\kappa(t-u)} \upsilon_u dW_u,$$

and

$$\mathbb{E}[\sigma_t] = x_0 e^{-\kappa t} + \varphi(1 - e^{-\kappa t}) + z_t, \quad \mathbb{E}[\sigma_t^2] = \Big(x_0 e^{-\kappa t} + \varphi(1 - e^{-\kappa t}) + z_t\Big)^2 + \int_0^t e^{-2\kappa(t-u)} \upsilon_u^2 du.$$

Hull-White Form

Let us also consider the following Hull-White Form:

$$d\sigma_t = (\varphi_t - \kappa\sigma_t)dt + \upsilon_t dW_t,$$

with some positive constant κ and some deterministic functions φ_t and υ_t. Integrating the dynamics in this case we obtain that

$$\sigma_t = \sigma_0 e^{-\kappa t} + \int_0^t e^{-\kappa(t-u)}\varphi_u du + \int_0^t e^{-\kappa(t-u)}\upsilon_u dW_u.$$

Thus, we have that

$$\mathbb{E}[\sigma_t] = \sigma_0 e^{-\kappa t} + \int_0^t e^{-\kappa(t-u)}\varphi_t du, \quad \mathbb{E}[\sigma_t^2] = \Big(\sigma_0 e^{-\kappa t} + \int_0^t e^{-\kappa(t-u)}\varphi_u du\Big)^2 + \int_0^t e^{-2\kappa(t-u)}\upsilon_u^2 du.$$

Hull-White++ Form

We now consider a more general case of the Hull-White form:

$$\sigma_t = x_t + z_t,$$

where

$$dx_t = (\varphi_t - \kappa x_t)dt + \upsilon_t dW_t, \quad \text{such that} \quad \kappa > 0,$$

with some deterministic functions φ_t, υ_t, and z_t. It follows that

$$\sigma_t = x_0 e^{-\kappa t} + \int_0^t e^{-\kappa(t-u)}\varphi_u du + z_t + \int_0^t e^{-\kappa(t-u)}\upsilon_u dW_u.$$

In this case, we obtain that

$$\mathbb{E}[\sigma_t] = x_0 e^{-\kappa t} + \int_0^t e^{-\kappa(t-u)}\varphi_t du + z_t$$

and

$$\mathbb{E}[\sigma_t^2] = \Big(x_0 e^{-\kappa t} + \int_0^t e^{-\kappa(t-u)}\varphi_u du + z_t\Big)^2 + \int_0^t e^{-2\kappa(t-u)}\upsilon_u^2 du.$$

This is the most general Gaussian one-factor form that we will be concerned with. It allows a plain option pricing formula with the same framework as the Factorizable Second Chaos Model. For example, we may take the functions

$$\varphi_t = b_1 e^{-c_1 t}, \quad \upsilon_t = b_2 e^{-c_2 t} \quad \text{and} \quad z_t = b_3 t e^{-c_3 t}$$

for some constants b_1, b_2, b_3 and some positive constants c_1, c_2, c_3. Note here that in this case we have that $\sigma_0 = x_0$. This choice of functions satisfies the required conditions. We then have that

$$\sigma_t = \Big(\sigma_0 - \frac{b_1}{\kappa - c_1}\Big)e^{-\kappa t} + \frac{b_1}{\kappa - c_1}e^{-c_1 t} + b_3 t e^{-c_3 t} + \int_0^t b_2 e^{-\kappa t} e^{(\kappa - c_2)u} dW_u.$$

Therefore, we can set

$$\phi_1(t) = \Big(\sigma_0 - \frac{b_1}{\kappa - c_1}\Big)e^{-\kappa t} + \frac{b_1}{\kappa - c_1}e^{-c_1 t} + b_3 t e^{-c_3 t}, \quad \phi_2(t,u) = b_2 e^{-\kappa t} e^{(\kappa - c_2)u},$$

so that the Second Chaos Framework can be applied. Note here that if we take $\kappa = c_2$, the One-variable Second Chaos Model Framework can be applied.

8.4.2 Two Factor Deterministic Volatility Case

Two-Additive-Factor Gaussian Form G$2++$

We now consider the dynamics of the primitive process assuming that

$$\sigma_t = x_t + y_t + z_t,$$

where we suppose that x_t and y_t satisfy the following conditions:

$$dx_t = -\varphi x_t dt + \upsilon dW_t^1, \quad dy_t = -\hat{\varphi} y_t dt + \hat{\upsilon} dW_t^2,$$

such that $\varphi \geq 0$, $\hat{\varphi} \geq 0$ and such that for some $0 \leq \rho \leq 1$

$$dW_t^1 dW_t^2 = \rho dt,$$

and where z_t is some deterministic function. It follows that

$$\sigma_t = x_0 e^{-\varphi t} + y_0 e^{-\hat{\varphi} t} + z_t + \upsilon \int_0^t e^{-\varphi(t-u)} dW_u^1 + \hat{\upsilon} \int_0^t e^{-\hat{\varphi}(t-u)} dW_u^2,$$

and

$$\mathbb{E}[\sigma_t^2] = (x_0 e^{-\varphi t} + y_0 e^{-\hat{\varphi} t} + z_t)^2 + \frac{\upsilon^2}{2\varphi}\Big[1 - e^{-2\varphi t}\Big] + \frac{\hat{\upsilon}^2}{2\hat{\varphi}}\Big[1 - e^{-2\hat{\varphi} t}\Big] + 2\rho \frac{\upsilon \hat{\upsilon}}{\varphi + \hat{\varphi}}\Big[1 - e^{-(\varphi + \hat{\varphi})t}\Big].$$

This can be shown, for instance, by arguments contained in [14]. In addition we are freely able to specify the deterministic function z_t to resolve the initial curve fitting issue. As an example, for some constants $\alpha, \beta \in \mathbb{R}$ such that $\alpha > 0$, we could choose the function z_t in the following way:

$$z_t = \beta t e^{-\alpha t}.$$

Therefore this two factor stochastic differential equation is also ideal for our purpose.

8.4.3 Stochastic Volatility Form

We now investigate the primitive process $(\sigma_t)_{\geq 0}$ where this is assumed to have a stochastic volatility. We first recall the Third Chaos Model:

$$d\sigma_t = \Big(\frac{d}{dt}\phi_1(t)\Big)dt + \Big[\phi_2(t,t) + \int_0^t \phi_2(t,t,s_2)dW_{s_2}\Big]dW_t.$$

It is clear that the Third Chaos Model belongs to the following one factor stochastic volatility model:

$$d\sigma_t = \mu_1(t)dt + \sigma_1(t)dW_t,$$

$$d\sigma_1(t) = \mu_2(t)dt + \sigma_2(t)dW_t,$$

where $\mu_1(t)$, $\mu_2(t)$, and $\sigma_2(t)$ are deterministic functions. Similarly, we interpret the Fourth Chaos Model as belonging to the one factor stochastic volatility model with three stochastic differential equations, and so on. In this section we focus on the two factor stochastic volatility model, and leave the higher multi-factor stochastic volatility models for Section 9.2.2.

Two Factor Stochastic Volatility Hull-White++ Form

We now consider the following form:

$$\sigma_t = x_t + z_t,$$

where

$$(8.4.1) \qquad dx_t = (\varphi_t - \kappa x_t)dt + \upsilon_t dW_t^1, \quad \text{such that} \quad \kappa > 0,$$

with some deterministic functions φ_t and z_t and some positive constant κ where the volatility drift is modelled as

$$(8.4.2) \qquad d\upsilon_t = (\hat{\varphi}_t - \hat{\kappa}\upsilon_t)dt + \hat{\upsilon}_t dW_t^2, \quad \text{such that} \quad \hat{\kappa} > 0,$$

with some deterministic functions $\hat{\varphi}_t$ and $\hat{\upsilon}_t$ and some positive constant $\hat{\kappa}$ such that for some $0 \leq \rho \leq 1$

$$dW_t^1 dW_t^2 = \rho dt.$$

Equation (8.4.2) can be solved explicitly as follows:

$$\upsilon_t = \upsilon_0 e^{-\hat{\kappa}t} + \int_0^t e^{-\hat{\kappa}(t-u)}\hat{\varphi}_u du + \int_0^t e^{-\hat{\kappa}(t-u)}\hat{\upsilon}_u dW_u^2.$$

Hence, equation (8.4.1) can be expressed in the following way:

$$dx_t = (\varphi_t - \kappa x_t)dt + \left[\upsilon_0 e^{-\hat{\kappa}t} + \int_0^t e^{-\hat{\kappa}(t-u)}\hat{\varphi}_u du + \int_0^t e^{-\hat{\kappa}(t-u)}\hat{\upsilon}_u dW_u^2\right] dW_t^1.$$

Integrating this yields that

$$\begin{aligned}\sigma_t =& x_0 e^{-\kappa t} + \int_0^t e^{-\kappa(t-s_1)}\varphi_{s_1} ds_1 + z_t + \int_0^t e^{-\kappa t}\left(\upsilon_0 e^{\kappa s_1 - \hat{\kappa}s_1} + \int_0^{s_1} e^{\kappa s_1 - \hat{\kappa}(s_1-s_2)}\hat{\varphi}_{s_2} ds_2\right) dW_{s_1}^1 \\ &+ \int_0^t \int_0^{s_1} e^{-\kappa t} e^{\kappa s_1 - \hat{\kappa}(s_1-s_2)}\hat{\upsilon}_{s_2} dW_{s_2}^2 dW_{s_1}^1.\end{aligned}$$

Also, we have

$$\mathbb{E}[\sigma_t] = x_0 e^{-\kappa t} + \int_0^t e^{-\kappa(t-u)}\varphi_t du + z_t,$$

and

$$\begin{aligned}\mathbb{E}[\sigma_t^2] =& \left(x_0 e^{-\kappa t} + \int_0^t e^{-\kappa(t-u)}\varphi_u du + z_t\right)^2 \\ &+ \int_0^t e^{-2\kappa t}\left(\upsilon_0 e^{\kappa s_1 - \hat{\kappa}s_1} + \int_0^{s_1} e^{\kappa s_1 - \hat{\kappa}(s_1-s_2)}\hat{\varphi}_{s_2} ds_2\right)^2 ds_1 \\ &+ \int_0^t \int_0^{s_1} e^{-2\kappa t} e^{2\kappa s_1 - 2\hat{\kappa}(s_1-s_2)}\hat{\upsilon}_{s_2}^2 ds_2 ds_1.\end{aligned}$$

We observe here that when we set $\rho = 1$, we are able to apply the Factorizable Third Chaos Model system and find the pricing option formula explicitly. For example, we may set the functions $\varphi_t, \hat{\varphi}_t, \hat{\upsilon}_t$ and z_t to be given by

$$\varphi_t = b_1 e^{-c_1 t}, \quad \hat{\varphi}_t = b_2 e^{-c_2 t}, \quad \hat{\upsilon}_t = b_3 e^{-c_3 t} \quad \text{and} \quad z_t = b_4 t e^{-c_4 t},$$

for some constants b_1, b_2, b_3, b_4 and some positive constants c_1, c_2, c_3, c_4. Note here that in this case we have that $\sigma_0 = x_0$. We then find that

$$\begin{aligned}\sigma_t =& \Big(\sigma_0 - \frac{b_1}{\kappa - c_1}\Big) e^{-\kappa t} + \frac{b_1}{\kappa - c_1} e^{-c_1 t} + b_4 t e^{-c_4 t} \\ &+ \int_0^t e^{-\kappa t} e^{(\kappa - \hat{\kappa}) s_1} \Big(\upsilon_0 - \frac{b_2}{\hat{\kappa} - c_2} + \frac{b_2}{\hat{\kappa} - c_2} e^{(\hat{\kappa} - c_2) s_1}\Big) dW_{s_1} \\ &+ \int_0^t \int_0^{s_1} b_3 e^{-\kappa t} e^{(\kappa - \hat{\kappa}) s_1} e^{(\hat{\kappa} - c_3) s_2} dW_{s_2} dW_{s_1}.\end{aligned}$$

Therefore, we can choose the functions ϕ_1, ϕ_2 and ϕ_3 to be given by

$$\begin{aligned}\phi_1(t) &= \Big(\sigma_0 - \frac{b_1}{\kappa - c_1}\Big) e^{-\kappa t} + \frac{b_1}{\kappa - c_1} e^{-c_1 t} + b_4 t e^{-c_4 t}, \\ \phi_2(t, s_1) &= e^{-\kappa t} e^{(\kappa - \hat{\kappa}) s_1} \Big(\upsilon_0 - \frac{b_2}{\hat{\kappa} - c_2} + \frac{b_2}{\hat{\kappa} - c_2} e^{(\hat{\kappa} - c_2) s_1}\Big), \\ \phi_3(t, s_1, s_2) &= b_3 e^{-\kappa t} e^{(\kappa - \hat{\kappa}) s_1} e^{(\hat{\kappa} - c_3) s_2},\end{aligned}$$

so that the Factorizable Third Chaos Framework can be applied. Note here that if we take $\hat{\kappa} = c_3$, the Two-variable Third Chaos Framework can be applied. Moreover, if we take $\kappa = \hat{\kappa} = c_2 = c_3$, the One-variable Third Chaos Framework can be applied.

8.5 Modelling the forward LIBOR rate dynamics from $(Z_{tT})_{0 \le t \le T < \infty}$

We have observed the forward LIBOR rate dynamics expressed only by the risk-adjusted volatilities, which is computed in the following way:

$$\hat{V}_{tT} = \frac{D_t[Z_{tT}]}{Z_{tT}}, \quad 0 \le t \le T < \infty.$$

We now consider whether modelling the process $(Z_{tT})_{0\le t\le T<\infty}$ may allow us to integrate the advantages of the SVM and the Potential Approach. It would yield a desirable forward rate dynamics, satisfying the arbitrage-free and positive interest rate conditions.

8.5.1 Arbitrage free condition from Z_{tT}

Let us first recall the potential property of the state price density $(V_t)_{t\ge 0}$, that is, V_t is a supermartingale with respect to $\mathcal{F}_t$ such that the following asymptotic condition is satisfied:

$$\lim_{T\to\infty} \mathbb{E}[V_T] = 0. \tag{8.5.1}$$

Because we defined

$$Z_{tT} := \mathbb{E}_t[V_T], \quad \text{for} \quad 0 \le t \le T < \infty,$$

the supermartingale property of V_t, that is, that $V_t \ge \mathbb{E}_t[V_T]$ for any $0 \le t \le T < \infty$, is equivalent to the following condition

$$Z_{tt} \ge Z_{tT}, \quad \text{for any} \quad 0 \le t \le T < \infty.$$

Because the tower property gives us that

$$\mathbb{E}[Z_{tT}] = \mathbb{E}[V_T],$$

the asymptotic condition (8.5.1) is equivalent to another asymptotic condition

$$\lim_{T\to\infty} \mathbb{E}[Z_{tT}] = 0.$$

Therefore we find that the decreasing condition with respect to $T \ge 0$ for the martingale process $(Z_{tT})_{0\le t\le T<\infty}$ with the asymptotic condition means that the non-arbitrage and positive interest rate conditions are satisfied. We recall the following form of the process $(Z_{tT})_{0\le t\le T<\infty}$ observed in the Chaotic Approach:

$$Z_{tT} = \mathbb{E}_t\Big[\int_T^\infty \sigma_s^2 ds\Big],$$

which is indeed a martingale with respect to $\mathcal{F}_t$, decreasing with respect to $T \geq 0$ for any $\sigma_s \in L^2$.

8.5.2 Application of the SABR dynamics

Let us assume the random variable Z_{tT} satisfies the SABR dynamics, that is,

$$dZ_{tT} = -\widetilde{Y}_{tT} Z_{tT}^{\beta} dW_t,$$

$$d\widetilde{Y}_{tT} = \epsilon \widetilde{Y}_{tT} d\widetilde{W}_t,$$

where $\beta \in (0,1]$, ϵ and α are some positive constants, and W_t and $\widetilde{W}_t$ are Brownian Motions with a correlation

$$dW_t d\widetilde{W}_t = \rho dt, \quad \text{and} \quad \rho \in [-1,1].$$

This does not already guarantee that the process $(Z_{tT})_{0\leq t \leq T<\infty}$ is decreasing with respect to $T \geq 0$. In light of Itô's Lemma, for $0 \leq t \leq T \leq S < \infty$, we have that

$$\begin{aligned} d\Big(\frac{1}{Z_{tS}}\Big) =& \frac{1}{Z_{tS}^3}(dZ_{tS})^2 - \frac{1}{Z_{tS}^2} dZ_{tS} \\ =& \frac{\widetilde{Y}_{tS}^2 Z_{tS}^{2\beta}}{Z_{tS}^3} dt + \frac{\widetilde{Y}_{tS} Z_{tS}^{\beta}}{Z_{tS}^2} dW_t \\ =& \widetilde{Y}_{tS}^2 Z_{tS}^{2\beta-3} dt + \widetilde{Y}_{tS} Z_{tS}^{\beta-2} dW_t. \end{aligned}$$

Consequently, we may further infer that

$$\begin{aligned} d\Big(\frac{Z_{tT}}{Z_{tS}}\Big) =& Z_{tT} d\Big(\frac{1}{Z_{tS}}\Big) + \frac{1}{Z_{tS}} dZ_{tT} + d\Big(\frac{1}{Z_{tS}}\Big) dZ_{tT} \\ =& \widetilde{Y}_{tS}^2 Z_{tS}^{2\beta-3} Z_{tT} dt + \widetilde{Y}_{tS} Z_{tS}^{\beta-2} Z_{tT} dW_t^1 - \widetilde{Y}_{tT} \frac{Z_{tT}^{\beta}}{Z_{tS}} dW_t - \widetilde{Y}_{tS} \widetilde{Y}_{tT} Z_{tS}^{\beta-2} Z_{tT}^{\beta} dt \\ =& \Big(\widetilde{Y}_{tS}^2 Z_{tS}^{2\beta-3} Z_{tT} - \widetilde{Y}_{tS} \widetilde{Y}_{tT} Z_{tS}^{\beta-2} Z_{tT}^{\beta}\Big) dt + \Big(\widetilde{Y}_{tS} Z_{tS}^{\beta-2} Z_{tT} - \widetilde{Y}_{tT} \frac{Z_{tT}^{\beta}}{Z_{tS}}\Big) dW_t \\ =& \widetilde{Y}_{tS} \Big(\frac{1}{Z_{tS}}\Big)^{1-\beta} \frac{Z_{tT}}{Z_{tS}} \Big[\widetilde{Y}_{tS} \Big(\frac{1}{Z_{tS}}\Big)^{1-\beta} - \widetilde{Y}_{tT} \Big(\frac{1}{Z_{tT}}\Big)^{1-\beta}\Big] dt \\ &+ \frac{Z_{tT}}{Z_{tS}} \Big[\widetilde{Y}_{tS} \Big(\frac{1}{Z_{tS}}\Big)^{1-\beta} - \widetilde{Y}_{tT} \Big(\frac{1}{Z_{tT}}\Big)^{1-\beta}\Big] dW_t. \end{aligned}$$

This then implies

$$\begin{aligned}dF_{tTS} =&\frac{1}{S-T}d\Big(\frac{Z_{tT}}{Z_{tS}}\Big)\\ =&\frac{1}{S-T}\widetilde{Y}_{tS}\Big(\frac{1}{Z_{tS}}\Big)^{1-\beta}\frac{Z_{tT}}{Z_{tS}}\Big[\widetilde{Y}_{tS}\Big(\frac{1}{Z_{tS}}\Big)^{1-\beta}-\widetilde{Y}_{tT}\Big(\frac{1}{Z_{tT}}\Big)^{1-\beta}\Big]dt\\ &+\frac{1}{S-T}\frac{Z_{tT}}{Z_{tS}}\Big[\widetilde{Y}_{tS}\Big(\frac{1}{Z_{tS}}\Big)^{1-\beta}-\widetilde{Y}_{tT}\Big(\frac{1}{Z_{tT}}\Big)^{1-\beta}\Big]dW_t.\end{aligned}$$

Because the relationship $F_{tTS} = \frac{1}{S-T}\big(\frac{Z_{tT}}{Z_{tS}} - 1\big)$ may be expressed as follows:

$$\frac{1}{S-T}\Big(\frac{Z_{tT}}{Z_{tS}}\Big) = F_{tTS} + \frac{1}{S-T},$$

we conclude that

$$\begin{aligned}dF_{tTS} =&\widetilde{Y}_{tS}\Big(\frac{1}{Z_{tS}}\Big)^{1-\beta}\Big[\widetilde{Y}_{tS}\Big(\frac{1}{Z_{tS}}\Big)^{1-\beta}-\widetilde{Y}_{tT}\Big(\frac{1}{Z_{tT}}\Big)^{1-\beta}\Big]\Big(F_{tTS}+\frac{1}{S-T}\Big)dt\\ &+\Big[\widetilde{Y}_{tS}\Big(\frac{1}{Z_{tS}}\Big)^{1-\beta}-\widetilde{Y}_{tT}\Big(\frac{1}{Z_{tT}}\Big)^{1-\beta}\Big]\Big(F_{tTS}+\frac{1}{S-T}\Big)dW_t.\end{aligned}$$

where

$$\widetilde{Y}_{tT} = \widetilde{Y}_{0T}e^{-\frac{1}{2}\epsilon^2 t+\epsilon\widetilde{W}_t^2}.$$

We now distinguish three cases.

<u>Case 1</u>. $\beta = 1$

When $\beta = 1$, we find that

$$dZ_{tT} = -\widetilde{Y}_{tT}Z_{tT}dW_t.$$

Therefore we have that $\widetilde{Y}_{tT} = -\hat{V}_{tT}$ and we obtain the same dynamics

$$dF_{tTS} = [\cdots]dt - (\hat{V}_{tS} - \hat{V}_{tT})\Big(F_{tTS} + \frac{1}{S-T}\Big)dW_t.$$

Because $\hat{V}_{tT}$ is lognormally distributed, $(\hat{V}_{tS} - \hat{V}_{tT})$ has the same distribution. Therefore it corresponds to the SABR Model with $\beta = 1$.

Case 2. $\beta = 0$

When $\beta = 0$, we find that

$$dZ_{tT} = -\widetilde{Y}_{tT} dW_t, \quad \text{or, in other words, that} \quad Z_{tT} = -\int_0^t \widetilde{Y}_{uT} dW_u.$$

This gives us that

$$dF_{tTS} = [\cdots]dt + \Big(\frac{\widetilde{Y}_{tS}}{Z_{tS}} - \frac{\widetilde{Y}_{tT}}{Z_{tT}}\Big)\Big(F_{tTS} + \frac{1}{S-T}\Big)dW_t.$$

Case 3. $\beta = \frac{1}{2}$

When $\beta = \frac{1}{2}$ we find that

$$dZ_{tT} = -\widetilde{Y}_{tT}\sqrt{Z_{tT}}dW_t,$$

which implies that

$$dF_{tTS} = [\cdots]dt + \Big(\frac{\widetilde{Y}_{tS}}{\sqrt{Z_{tS}}} - \frac{\widetilde{Y}_{tT}}{\sqrt{Z_{tT}}}\Big)\Big(F_{tTS} + \frac{1}{S-T}\Big)dW_t.$$

We have not so far found a condition to guarantee that the decreasing property is satisfied. However we make a note of it here, in case we get further ideas in the future.

Chapter 9

Conclusion and further works

9.1 Summary of the thesis

This project was started with the aim of calibrating the Chaotic Approach. Knowing that the Chaotic Approach was constructed under the Potential Approach, we first investigated the Potential Approach and noticed that SVM may be produced from the state price density. This implies that the Potential Approach may be regarded as a framework to model the forward LIBOR rate and forward swap rate dynamics under which the arbitrage-free and interest rate positivity conditions are satisfied. Let us recall here the corresponding dynamics:

$$dF_{tTS} = -\frac{D_t[Z_{tS}]}{Z_{tS}}\Big(\frac{D_t[Z_{tT}-Z_{tS}]}{Z_{tT}-Z_{tS}}-\frac{D_t[Z_{tS}]}{Z_{tS}}\Big)F_{tTS}dt+\Big(\frac{D_t[Z_{tT}-Z_{tS}]}{Z_{tT}-Z_{tS}}-\frac{D_t[Z_{tS}]}{Z_{tS}}\Big)F_{tTS}dW_t$$

and

$$\begin{aligned} dS_{a,b}(t) = &-\frac{D_t\big[\sum_{i=a+1}^{b}\tau_i Z_{tT_i}\big]}{\sum_{i=a+1}^{b}\tau_i Z_{tT_i}}\Big(\frac{D_t[Z_{tT_a}-Z_{tT_b}]}{Z_{tT_a}-Z_{tT_b}} - \frac{D_t\big[\sum_{i=a+1}^{b}\tau_i Z_{tT_i}\big]}{\sum_{i=a+1}^{b}\tau_i Z_{tT_i}}\Big)S_{a,b}(t)dt \\ &+\Big(\frac{D_t[Z_{tT_a}-Z_{tT_b}]}{Z_{tT_a}-Z_{tT_b}} - \frac{D_t\big[\sum_{i=a+1}^{b}\tau_i Z_{tT_i}\big]}{\sum_{i=a+1}^{b}\tau_i Z_{tT_i}}\Big)S_{a,b}(t)dW_t, \end{aligned}$$

where the variable Z_{tT} is conditional expectation of the state price density V_T with respect to $\mathcal{F}_t$ for $0 \le t \le T < \infty$ and D_t is the Malliavin derivative with respect to time t. The SABR type equations are expressed in (3.3.7) and (3.3.10). In addition, we observed in (8.1.3) and (8.1.4) that these dynamics may be modelled from the short rate. In particular, the state price density is potential when the short rate is a positive process.

For the calibration of the Chaotic Approach we started our argument from initial yield curves without options. We proposed a family of chaos coefficients expressed by functions of one-variable and called the corresponding model the "One-variable Chaos Model". Specification of these one-variable functions by exponential polynomial forms allows the deduction of initial forward rates compatible with the Björk and Christensen descriptive form as follows:

$$f_{0T} = \frac{\sum_{i=1}^{\infty} \left(\sum_{j=1}^{m_i} \widetilde{b}_{ij} e^{-c_{ij}T} \right)^2 T^{i-1}}{\int_T^{\infty} \sum_{i=1}^{\infty} \left(\sum_{j=1}^{m_i} \widetilde{b}_{ij} e^{-c_{ij}s} \right)^2 s^{i-1} ds}, \quad \text{for} \quad T \geq 0.$$

From this polynomial family we suggested some specific models to compare with Nelson-Siegel Form and Svensson Form, which are special cases of the Björk and Christensen descriptive form. Our calibration gave successful results. Most of the Chaos Models outperform the Nelson-Siegel Form and have as good a fitting ability as Svensson Form, while also satisfying the interest rate positivity condition.

In [48], it is shown that the variable Z_{tT} is formed as a squared polynomial in a Gaussian process in the Factorizable Second Chaos Model. This also holds for the class of One-variable Second Chaos Models, which can be seen for each $0 \leq t \leq T < \infty$ as follows:

$$Z_{tT} = P_{0T} \int_0^{\infty} (\alpha_s^2 + s\beta_s^2) ds + 2 \int_T^{\infty} \alpha_s \beta_s ds W_t + \int_T^{\infty} \beta_s^2 ds (W_t^2 - t),$$

where the initial curve may also be expressed in the following way:

$$P_{0T} = \frac{\int_T^{\infty} (\alpha_s^2 + s\beta_s^2) ds}{\int_0^{\infty} (\alpha_s^2 + s\beta_s^2) ds}.$$

When we take the third chaos coefficient into account, One-variable Third Chaos Models form the variable Z_{tT} by degree four polynomials in a Gaussian process, while

One-variable Fourth Chaos Models form degree six polynomials. Those properties not only secure stochasticity in the volatility term for volatility skew and smile, but also allow enough flexibility to model the distribution of derivatives.

We observe that there exist two types of interest rate models. The first group gives us freedom to model initial curves separately from the volatility dynamics. In the market the Svensson Form is often used to attain a reasonable fit into the initial curves. Moreover, calibrating options separately, the global minimum may be found faster. For example the HW, LFM and SABR Model belong that group. Models in the other group, such as the CIR Model do not give us this freedom. As can be seen in [14], the CIR++ Model is proposed to correct the nature of the CIR Model to obtain the tractability for the initial curves. However it costs six parameters to apply the Svensson Form. For example the SABR Model spends three parameters to model the forward LIBOR rate dynamics, and the other three parameters to model the forward swap rate dynamics. Therefore we need twelve parameters in total. On the other hand, we can choose the group in the Chaotic Approach, that is, we are also able to model the initial curves from the chaos coefficients. The One-variable Chaos Models achieve reasonable fit into initial curve and volatilities at the same time without increasing the number of parameters. We observe that even seven parameter Chaos Models may generate reasonable option prices with good fit into initial curve.

The calibrations were implemented with two goals in mind. Firstly, we wanted to compare the calibration performances within the Chaos Models. Secondly, we wanted to compare the performances with the popular models and other traditional models. We found that One-variable Third Chaos Models are outstanding among all Chaos Models, particularly regarding fast computational speed. This model is comparable with the LFM and the SABR Model. For example, the One-variable Third Chaos Model with seven parameters works better than the SABR Model for calibrating the ATM options. We observe that the application of the descriptive form in the Chaos

Models forms enough flexibility to reproduce the humped shape of the caplet volatility term structure, and also smile/skew shape for in the money and out of the money options. Although it does not generate great fit into Caplet smiles, the one-variable nine parameters model gives us very small errors for fitting into Swaption smiles, even smaller than those we observed in the SABR Model.

We noticed from the literature that the Stochastic Volatility Market Models are one of the most successful and popular models amongst practitioners in recent years, and many researchers focus on modelling a stochastic volatility. Our research described here suggests indirect methodology to model the forward LIBOR rate and forward swap rate dynamics via the state price density. We hope it will be a framework for the next generation of interest rate modelling. We list possible further works in the next section and finish the argument.

9.2 Further work

We believe that there are a lot of exciting avenues open for further research. Here we suggest four possible topics.

9.2.1 Improvements of the Model

Though we focused our calibration on the Chaotic Approach, it is exciting work to model the state price density paying particular attention to those volatility drifts of the underlying dynamics for improving fitting ability into the volatility surface. Our final goal is to establish a model which enables to fit well into a volatility surface across the maturity and the strike. We here list shortcomings of the One-variable Chaos Models so that possible improvement may be discussed.

- Option premiums can be computed only one at a time.
- The parameters in the model do not have an intuitive real-world meaning.

- An explicit implied volatility form is not available.

- Analytical forward LIBOR rate correlation form is not available.

The first shortcoming is discussed in Section 8.1. We proposed to take an exponential form for the state price density. In particular, we suggested modelling the LIBOR rate and swap rate volatility by the application of the short rate models. In other words, we suggested to incorporate the Short Rate Model, the Market Model and the Potential Approach. Since the Vasicek Model in 1977 there have been many interest rate models developed by various researchers. However, as is suggested in Chapter 8, it is possible to combine all the previous described techniques of interest rate theory, in order to make the best advantage of existing work. We observed that the Affine Term Structure Model gives an analytical stochastic differential equations of the underlying assets. As examples, we showed that the Vasicek Model belongs to the Shifted-lognormal Market Model, and the Squared Gaussian Model belongs to the SVM. Here, it would be optimistic to generate the market by only one factor, we would need multiple factors. For example the SABR Model applies two correlated factors. Therefore we should consider two-factor Affine Term Structure Model so that we indirectly model the distribution of the volatility drift terms in the underlying assets. It is also advantageous to have an intuitive meaning for each parameter. A question now is about the market price of risk, which is not present in the volatility drift terms. However, the state price density is expressed by the short rate and the market price of risk. Hence, it is important to model the market price of risk for the propose of pricing options. To model the state price density we should also incorporate the discussions in Economics, see for example [25] and [31]. Some other ideas are also proposed in Chapter 8, but we leave the remaining questions open.

Finally, having explicit implied volatility and forward rate correlation forms is a desirable feature in interest rate modelling. Though it is straightforward to compute premium from implied volatility, the other way around is not simple and is often approximated. We understand that not many practitioners apply the lognormal distribution for underlying assets any more, but they still apply the Black formula to measure volatilities, using implied volatility as a benchmark. We observe that the SABR Model outperforms the Wu-Zhang Model ([93]) in this sense. However, Chaos Models do not have that capacity either. Moreover, as is stated in [62] we should extract the forward LIBOR rate correlation information from the market in the calibration work, not only the volatility information. Therefore, we should also derive the correlation form in the Potential Approach for future work.

9.2.2 Improvements of the calibration

Though we understand that the market does not apply historical data but estimates volatilities only from the current data, the model assumption claims that the parameters are time independent. For example, Rogers ([80]) suggests time series calibration methodologies. Kalman Filtering, General Method of Moments, or Maximum Likelihood method may be applied where the bid-ask spreads or liquidity would work to estimate volatility for the Maximum Likelihood function. Moreover we should also calibrate the models proposed in Chapter 8 and the other popular SVM such as Wu-Zhang Model and Piterbarg Model ([70]), not only the SABR Model.

Calibration is implemented for pricing and hedging purpose. Here we take exotic options into account. Particularly, the chooser flexible cap and the Bermudan Swaption are liquidly traded in the market. However, as is the case for the SABR Model, it is often not easy to price those options, we must rely on either Monte Carlo Simulation or the trinomial tree algorithm, (see, for example [87]). One of the most appealing parts of the Potential Approach is its tractability to price options, because we are modelling the stochastic discount factor itself. We could investigate pricing method for those exotic options and also check pricing errors, using calibrated parameters by European options.

In addition to checking the pricing error, model performance may be evaluated by its hedging performance. Although we observe some literature about the hedging the delta and vega risks under the SABR Model, (see for example [3] and [41]), we do not find it under the Potential Approach. As stated in Rebonato's book ([76]), we may not say a model is perfectly good unless hedging ability is checked. A desirable model has to have a stable and non-erratic feature of prediction in the future time. For example, as stated in the book [76], the Local Volatility Market Models do not have great ability in this sense, since there the dynamics move the other way around, even though it satisfies fitting ability to the volatility smiles. We notice that nobody has investigated evolution in time of the term structure of volatility in the Potential Approach. Here again the SABR Model would work as a benchmark of the performance.

9.2.3 Further investigations in Mathematics

In this thesis we proposed the One-variable Wiener-Chaos expansion. Although we did not find loss of generality under the One-variable Chaos Models, it is still an open topic to compare the convergence speed between the usual Wiener-Chaos expansion and the One-variable Wiener-Chaos expansion. As an alternative direction, may we suggest applying the Winker-Askey Polynomial Chaos Expansion (or Generalized Polynomial Wiener-Chaos expansion, sometimes written as GPCE), which has been used recently in Physics and Engineering fields to estimate a square integrable random variable as an alternative to the Monte Carlo Method, see for example [94]. This method is appropriate to estimate a non-Gaussian variable.

9.2.4 Application to other products

The expressions of the stock price process and FX system are derived in [48]. Hence, it is straightforward work to price stock options under the Potential Approach in particular the One-variable Chaos Models. Moreover, it gives an easy access to Hybrid products. As also claimed by Rogers ([79]), it is advantageous under the Potential Approach that we are able to model the interest rate markets in several countries at the same time with those exchange rate. Here, the Market Models encounter computational difficulty on the multi-currency products as discussed in Appendix H from [14]. A possible extension of the Potential Approach is also for Credit Risk.

Chapter 10

Appendix

10.1 Pricing Options in the Constantinides Model

We formulate a method to price the European bond option and Swaption in the Constantinides Model. Though Swaptions are not considered in the original paper, we believe that it is straightforward to extend to this.

European call/put bond option

Because the initial values of the European bond options are formulated by

$$ZBC(0,t,T,K) = \frac{1}{V_0}\mathbb{E}[V_t(P_{tT}-K)^+] \quad \text{and} \quad ZBP(0,t,T,K) = \frac{1}{V_0}\mathbb{E}[V_t(K-P_{tT})^+],$$

we find for the call option that

$$\begin{aligned}
&ZBC(0,t,T,K)\\
=&\mathbb{E}\left[\frac{\exp\big[-\big(g+\frac{\sigma_0^2}{2}\big)t+\sigma_0 W_0(t)+(x_1(t)-\alpha_1)^2\big]}{\exp\big[\sigma_0 W_0(0)+(x_1(0)-\alpha_1)^2\big]}(P_{tT}-K)^+\right]\\
=&\mathbb{E}\left[\exp\big[-\frac{\sigma_0^2}{2}t+\sigma_0 W_0(t)\big]\exp\big[-gt+(x_1(t)-\alpha_1)^2-(x_1(0)-\alpha_1)^2\big](P_{tT}-K)^+\right].
\end{aligned}$$

Because the Wiener processes W_0 and W_1 are independent it follows that

$$ZBC(0,t,T,K) = \mathbb{E}\left[\exp\big[-gt+(x_1(t)-\alpha_1)^2-(x_1(0)-\alpha_1)^2\big](P_{tT}-K)^+\right].$$

Recalling the discount bond formula, it follows that

$$P_{tT}-K = H_1^{-\frac{1}{2}}(T-t)\exp\Big[(-g+\lambda_1)(T-t)+H_1^{-1}(T-t)\Big(x_1(t)-\alpha_1 e^{\lambda_1(T-t)}\Big)^2-(x_1(t)-\alpha_1)^2\Big]-K.$$

Therefore, the condition $P_{tT} - K \geq 0$ is equivalent to the following inequality:

$$I_c(x) := Cx^2 + Bx + A \geq 0,$$

where

$$C := -(1 - H_1^{-1}(T-t)), \quad B := 2\alpha_1(1 - H_1^{-1}(T-t)e^{\lambda_1(T-t)})$$

and

$$A := -(1 - H_1^{-1}(T-t)e^{2\lambda_1(T-t)})\alpha_1^2 + (-g+\lambda_1)(T-t) - \ln(KH_1^{\frac{1}{2}}(T-t)).$$

Similarly, the condition $K - P_{tT} \geq 0$ is equivalent to the following inequality:

$$I_p(x) := -Cx^2 - Bx - A \geq 0.$$

However, because we know that $H_1(T-t) > 1$ for $T > t$, we notice here that $C < 0$. Therefore both functions $I_c(x)$ and $I_p(x)$ have a quadratic form. If $\Delta := B^2 - 4AC \leq 0$ we obtain that

$$ZBC(0,t,T,K) = 0 \quad \text{and} \quad ZBP(0,t,T,K) = 0.$$

Now let us consider the case $\Delta > 0$. Because we know that

$$ZBC(0,t,T,K) = \mathbb{E}\big[\big(\mathcal{P}_c(x)\big)^+\big],$$

where

$$\begin{aligned}\mathcal{P}_c(x) =& H_1^{-\frac{1}{2}}(T-t)\exp\Big[\lambda_1(T-t) - gT + H_1^{-1}(T-t)\Big(x - \alpha_1 e^{\lambda_1(T-t)}\Big)^2 - (x_1(0)-\alpha_1)^2\Big] \\ & - K\exp\big[-gt + (x-\alpha_1)^2 - (x_1(0)-\alpha_1)^2\big] \\ =& H_1^{-\frac{1}{2}}(T-t)\exp\big[\lambda_1(T-t) - gT - (x_1(0)-\alpha_1)^2\big]\exp\Big[H_1^{-1}(T-t)\Big(x-\alpha_1 e^{\lambda_1(T-t)}\Big)^2\Big] \\ & - K\exp\big[-gt - (x_1(0)-\alpha_1)^2\big]\exp\big[(x-\alpha_1)^2\big],\end{aligned}$$

denoting the roots of the function $I_c(x)$ by $z_1 \leq z_2$, we find that, when $\Delta > 0$,

$$ZBC(0,t,T,K) = \int_{z_1}^{z_2} \mathcal{P}_c(x) f_x(x) dx$$

where f_x is the probability density function of the random variable $x_1(t)$. Similarly for the put option we obtain that

$$ZBP(0,t,T,K) = \int_{-\infty}^{z_1} [-\mathcal{P}_c(x)] f_x(x) dx + \int_{z_2}^{\infty} [-\mathcal{P}_c(x)] f_x(x) dx.$$

Because the dynamics of $x_1(t)$ can be solved explicitly to give

$$x_1(t) = x_1(0) e^{-\lambda_1 t} + \sigma_1 \int_0^t e^{-\lambda_1 (t-u)} dW_1(u),$$

we find that $x_1(t)$ is normally distributed with mean $\mu := x_1(0)e^{-\lambda_1 t}$ and variance $s^2 := \frac{\sigma_1^2}{2\lambda_1}(1 - e^{-2\lambda_1 t})$. Therefore we infer that, when $\Delta > 0$,

$$\begin{aligned}
ZBC(0,t,T,K) =& \frac{1}{s\sqrt{2\pi}} \int_{z_1}^{z_2} \mathcal{P}_c(x) e^{-\frac{(x-\mu)^2}{2s^2}} dx \\
=& H_1^{-\frac{1}{2}}(T-t) \exp\Big[\lambda_1(T-t) - gT - (x_1(0) - \alpha_1)^2\Big] \\
& \times \frac{1}{s\sqrt{2\pi}} \int_{z_1}^{z_2} \exp\Big[H_1^{-1}(T-t)\Big(x - \alpha_1 e^{\lambda_1(T-t)}\Big)^2\Big] e^{-\frac{(x-\mu)^2}{2s^2}} dx \\
& - K \exp\Big[-gt - (x_1(0) - \alpha_1)^2\Big] \frac{1}{s\sqrt{2\pi}} \int_{z_1}^{z_2} \exp\Big[(x - \alpha_1)^2\Big] e^{-\frac{(x-\mu)^2}{2s^2}} dx.
\end{aligned}$$

As in [26], the following is satisfied for $\beta, \gamma \in \mathbb{R}$:

$$\begin{aligned}
\frac{1}{s\sqrt{2\pi}} \int_{-\infty}^{\beta} e^{-\gamma x^2} e^{-\frac{(x-\mu)^2}{2s^2}} dx =& e^{\frac{-\gamma\mu^2}{1+2\gamma s^2}} \frac{1}{s\sqrt{2\pi}} \int_{-\infty}^{\beta} e^{-\frac{(x-a)^2}{2b^2}} dx \\
=& (1+2\gamma s^2)^{-\frac{1}{2}} e^{\frac{-\gamma\mu^2}{1+2\gamma s^2}} \frac{1}{b\sqrt{2\pi}} \int_{-\infty}^{\beta} e^{-\frac{(x-a)^2}{2b^2}} dx \\
=& (1+2\gamma s^2)^{-\frac{1}{2}} e^{\frac{-\gamma\mu^2}{1+2\gamma s^2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(\beta-a)/b} e^{-z^2/2} dz \\
=& (1+2\gamma s^2)^{-\frac{1}{2}} e^{\frac{-\gamma\mu^2}{1+2\gamma s^2}} \Phi\Big(\frac{\beta - a}{b}\Big),
\end{aligned}$$

where $a = \frac{\mu}{1+2\gamma s^2}$ and $b = s(1+2\gamma s^2)^{-\frac{1}{2}}$, provided that $1 + 2\gamma s^2 > 0$ and

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{X^2}{2}} dX.$$

Similarly we find that for $\alpha, \beta, \gamma \in \mathbb{R}$

$$\begin{aligned}\frac{1}{s\sqrt{2\pi}} \int_{-\infty}^{\beta} e^{-\gamma(x-\alpha)^2} e^{-\frac{(x-\mu)^2}{2s^2}} dx =& e^{-\frac{\gamma(\alpha-\mu)^2}{1+2\gamma s^2}} \frac{1}{s\sqrt{2\pi}} \int_{-\infty}^{\beta} e^{-\frac{(x-\hat{a})^2}{2b^2}} dx \\ =& (1+2\gamma s^2)^{-\frac{1}{2}} e^{-\frac{\gamma(\alpha-\mu)^2}{1+2\gamma s^2}} \frac{1}{b\sqrt{2\pi}} \int_{-\infty}^{\beta} e^{-\frac{(x-\hat{a})^2}{2b^2}} dx \\ =& (1+2\gamma s^2)^{-\frac{1}{2}} e^{-\frac{\gamma(\alpha-\mu)^2}{1+2\gamma s^2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(\beta-\hat{a})/b} e^{-z^2/2} dz \\ =& (1+2\gamma s^2)^{-\frac{1}{2}} e^{-\frac{\gamma(\alpha-\mu)^2}{1+2\gamma s^2}} \Phi\Big(\frac{\beta-\hat{a}}{b}\Big),\end{aligned}$$

where $\hat{a} = \frac{2s^2\gamma\alpha+\mu}{1+2\gamma s^2}$. Therefore we find that for the first term of the option pricing formula, setting $\gamma := -H_1^{-1}(T-t)$, we obtain

$$\begin{aligned}& H_1^{-\frac{1}{2}}(T-t) \exp\Big[\lambda_1(T-t) - gT - (x_1(0)-\alpha_1)^2\Big] \\ & \times \frac{1}{s\sqrt{2\pi}} \int_{z_1}^{z_2} \exp\Big[H_1^{-1}(T-t)\Big(x - \alpha_1 e^{\lambda_1(T-t)}\Big)^2\Big] e^{-\frac{(x-\mu)^2}{2s^2}} dx \\ =& H_1^{-\frac{1}{2}}(T-t) \exp\Big[\lambda_1(T-t) - gT - (x_1(0)-\alpha_1)^2\Big] \\ & \times \Bigg[\frac{1}{s\sqrt{2\pi}} \int_{-\infty}^{z_2} \exp\Big[H_1^{-1}(T-t)\Big(x - \alpha_1 e^{\lambda_1(T-t)}\Big)^2\Big] e^{-\frac{(x-\mu)^2}{2s^2}} dx \\ & - \frac{1}{s\sqrt{2\pi}} \int_{-\infty}^{z_1} \exp\Big[H_1^{-1}(T-t)\Big(x - \alpha_1 e^{\lambda_1(T-t)}\Big)^2\Big] e^{-\frac{(x-\mu)^2}{2s^2}} dx\Bigg] \\ =& H_1^{-\frac{1}{2}}(T-t) \exp\Big[\lambda_1(T-t) - gT - (x_1(0)-\alpha_1)^2\Big] \\ & \times (1-2H_1^{-1}(T-t)s^2)^{-\frac{1}{2}} \exp\Big[\frac{H_1^{-1}(T-t)(\alpha_1 e^{\lambda_1(T-t)}-\mu)^2}{1-2H_1^{-1}(T-t)s^2}\Big] \\ & \times \Big[\Phi\Big(\frac{z_2-\hat{a}}{b}\Big) - \Phi\Big(\frac{z_1-\hat{a}}{b}\Big)\Big] \\ =& H_1^{-\frac{1}{2}}(T-t)(1-2H_1^{-1}(T-t)s^2)^{-\frac{1}{2}} \\ & \times \exp\Big[\lambda_1(T-t) - gT - (x_1(0)-\alpha_1)^2 + \frac{H_1^{-1}(T-t)(\alpha_1 e^{\lambda_1(T-t)}-\mu)^2}{1-2H_1^{-1}(T-t)s^2}\Big] \\ & \times \Big[\Phi\Big(\frac{z_2-\hat{a}}{b}\Big) - \Phi\Big(\frac{z_1-\hat{a}}{b}\Big)\Big],\end{aligned}$$

where

$$
\begin{aligned}
\hat{a} =& \frac{2s^2\gamma\alpha_1 e^{\lambda_1(T-t)} + \mu}{1+2\gamma s^2} = \frac{-2s^2 H_1^{-1}(T-t)\alpha_1 e^{\lambda_1(T-t)} + \mu}{1-2H_1^{-1}(T-t)s^2} \\
=& -\frac{2\alpha_1 e^{\lambda_1(T-t)} H_1^{-1}(T-t)s^2}{1-2H_1^{-1}(T-t)s^2} + \frac{\mu}{1-2H_1^{-1}(T-t)s^2} \\
=& -\frac{\alpha_1 e^{\lambda_1(T-t)} H_1^{-1}(T-t)\frac{\sigma_1^2}{\lambda_1}(1-e^{-2\lambda_1 t})}{1-H_1^{-1}(T-t)\frac{\sigma_1^2}{\lambda_1}(1-e^{-2\lambda_1 t})} + \frac{x_1(0)e^{-\lambda_1 t}}{1-H_1^{-1}(T-t)\frac{\sigma_1^2}{\lambda_1}(1-e^{-2\lambda_1 t})}
\end{aligned}
$$

and

$$
\begin{aligned}
b =& s(1+2\gamma s^2)^{-\frac{1}{2}} = s(1-2H_1^{-1}(T-t)s^2)^{-\frac{1}{2}} \\
=& \Big[\frac{s^2}{1-2H_1^{-1}(T-t)s^2}\Big]^{\frac{1}{2}} = \Big[\frac{\frac{\sigma_1^2}{2\lambda_1}(1-e^{-2\lambda_1 t})}{1-H_1^{-1}(T-t)\frac{\sigma_1^2}{\lambda_1}(1-e^{-2\lambda_1 t})}\Big]^{\frac{1}{2}}.
\end{aligned}
$$

Recalling the definitions:

$$
H_1(\tau) := \frac{\sigma_1^2}{\lambda_1} + \Big(1-\frac{\sigma_1^2}{\lambda_1}\Big)e^{2\lambda_1\tau} \quad \text{and} \quad s^2 := \frac{\sigma_1^2}{2\lambda_1}(1-e^{-2\lambda_1 t}),
$$

we observe that

$$
\begin{aligned}
H_1(T-t)(1-2H_1^{-1}(T-t)s^2) =& H_1(T-t) - 2s^2 \\
=& \frac{\sigma_1^2}{\lambda_1} + \Big(1-\frac{\sigma_1^2}{\lambda_1}\Big)e^{2\lambda_1(T-t)} - \frac{\sigma_1^2}{\lambda_1}(1-e^{-2\lambda_1 t}) \\
=& \Big(1-\frac{\sigma_1^2}{\lambda_1}\Big)e^{2\lambda_1(T-t)} + \frac{\sigma_1^2}{\lambda_1}e^{-2\lambda_1 t} \\
=& e^{-2\lambda_1 t}\Big(\Big(1-\frac{\sigma_1^2}{\lambda_1}\Big)e^{2\lambda_1 T} + \frac{\sigma_1^2}{\lambda_1}\Big) \\
=& e^{-2\lambda_1 t}H_1(T).
\end{aligned}
$$

Therefore, recalling also that $\mu := x_1(0)e^{-\lambda_1 t}$, we find that

$$\begin{aligned}
&H_1^{-\frac{1}{2}}(T-t)(1-2H_1^{-1}(T-t)s^2)^{-\frac{1}{2}}\\
&\times \exp\Big[\lambda_1(T-t) - gT - (x_1(0)-\alpha_1)^2 + \frac{H_1^{-1}(T-t)(\alpha_1 e^{\lambda_1(T-t)}-\mu)^2}{1-2H_1^{-1}(T-t)s^2}\Big]\\
&\times \Big[\Phi\Big(\frac{z_2-\hat{a}}{b}\Big) - \Phi\Big(\frac{z_1-\hat{a}}{b}\Big)\Big]\\
=&e^{\lambda_1 t}H_1^{-\frac{1}{2}}(T)\exp\Big[\lambda_1(T-t) - gT - (x_1(0)-\alpha_1)^2 + H_1^{-1}(T)e^{2\lambda t}(\alpha_1 e^{\lambda_1(T-t)} - x_1(0)e^{-\lambda_1 t})^2\Big]\\
&\times \Big[\Phi\Big(\frac{z_2-\hat{a}}{b}\Big) - \Phi\Big(\frac{z_1-\hat{a}}{b}\Big)\Big]\\
=&H_1^{-\frac{1}{2}}(T)\exp\Big[(-g+\lambda_1)T + H_1^{-1}(T)(x_1(0)-\alpha_1 e^{\lambda T})^2 - (x_1(0)-\alpha_1)^2\Big]\\
&\times \Big[\Phi\Big(\frac{z_2-\hat{a}}{b}\Big) - \Phi\Big(\frac{z_1-\hat{a}}{b}\Big)\Big]\\
=&P_{0T}\Big[\Phi\Big(\frac{z_2-\hat{a}}{b}\Big) - \Phi\Big(\frac{z_1-\hat{a}}{b}\Big)\Big].
\end{aligned}$$

Similarly, the second term follows as, setting $\gamma := -1$, we have that

$$\begin{aligned}
&K\exp\Big[-gt-(x_1(0)-\alpha_1)^2\Big]\frac{1}{s\sqrt{2\pi}}\int_{z_1}^{z_2}\exp\Big[(x-\alpha_1)^2\Big]e^{-\frac{(x-\mu)^2}{2s^2}}dx\\
=&K\exp\Big[-gt-(x_1(0)-\alpha_1)^2\Big](1-2s^2)^{-\frac{1}{2}}\exp\Big[\frac{(\alpha_1-\mu)^2}{1-2s^2}\Big]\Big[\Phi\Big(\frac{z_2-a'}{b'}\Big) - \Phi\Big(\frac{z_1-a'}{b'}\Big)\Big]\\
=&(1-2s^2)^{-\frac{1}{2}}\exp\Big[-gt-(x_1(0)-\alpha_1)^2+\frac{(\alpha_1-\mu)^2}{1-2s^2}\Big]K\Big[\Phi\Big(\frac{z_2-a'}{b'}\Big) - \Phi\Big(\frac{z_1-a'}{b'}\Big)\Big],
\end{aligned}$$

where

$$a' := \frac{2s^2\gamma\alpha_1+\mu}{1+2\gamma s^2} = \frac{-2s^2\alpha_1+\mu}{1-2s^2} = -\frac{\alpha_1\frac{\sigma_1^2}{\lambda_1}(1-e^{-2\lambda_1 t})}{1-\frac{\sigma_1^2}{\lambda_1}(1-e^{-2\lambda_1 t})} + \frac{x_1(0)e^{-\lambda_1 t}}{1-\frac{\sigma_1^2}{\lambda_1}(1-e^{-2\lambda_1 t})}$$

and

$$b' := s(1+2\gamma s^2)^{-\frac{1}{2}} = s(1-2s^2)^{-\frac{1}{2}} = \Big[\frac{s^2}{1-2s^2}\Big]^{\frac{1}{2}} = \Big[\frac{\frac{\sigma_1^2}{2\lambda_1}(1-e^{-2\lambda_1 t})}{1-\frac{\sigma_1^2}{\lambda_1}(1-e^{-2\lambda_1 t})}\Big]^{\frac{1}{2}}.$$

However, because

$$e^{-2\lambda_1 t}H_1(t) = \frac{\sigma_1^2}{\lambda_1}e^{-2\lambda_1 t} + 1 - \frac{\sigma_1^2}{\lambda_1} = 1 - \frac{\sigma_1^2}{\lambda_1}(1-e^{-2\lambda_1 t}) = 1-2s^2,$$

we find that

$$
\begin{aligned}
&(1-2s^2)^{-\frac{1}{2}}\exp\Big[-gt-(x_1(0)-\alpha_1)^2+\frac{(\alpha_1-\mu)^2}{1-2s^2}\Big]K\Big[\Phi\Big(\frac{z_2-a'}{b'}\Big)-\Phi\Big(\frac{z_1-a'}{b'}\Big)\Big]\\
=&(e^{-2\lambda_1 t}H_1(t))^{-\frac{1}{2}}\exp\Big[-gt-(x_1(0)-\alpha_1)^2+\frac{(\alpha_1-\mu)^2}{e^{-2\lambda_1 t}H_1(t)}\Big]K\Big[\Phi\Big(\frac{z_2-a'}{b'}\Big)-\Phi\Big(\frac{z_1-a'}{b'}\Big)\Big]\\
=&H_1^{-\frac{1}{2}}(t)\exp\Big[(-g+\lambda_1)t+H_1^{-1}(t)(x_1(0)-\alpha_1 e^{\lambda_1 t})^2-(x_1(0)-\alpha_1)^2\Big]K\Big[\Phi\Big(\frac{z_2-a'}{b'}\Big)-\Phi\Big(\frac{z_1-a'}{b'}\Big)\Big]\\
=&P_{0t}K\Big[\Phi\Big(\frac{z_2-a'}{b'}\Big)-\Phi\Big(\frac{z_1-a'}{b'}\Big)\Big].
\end{aligned}
$$

Therefore, we conclude that

$$
ZBC(0,t,T,K)=\begin{cases} 0 & \text{if } \Delta\le 0\\ P_{0T}\big[\Phi\big(\frac{z_2-\hat{a}}{b}\big)-\Phi\big(\frac{z_1-\hat{a}}{b}\big)\big]-P_{0t}K\big[\Phi\big(\frac{z_2-a'}{b'}\big)-\Phi\big(\frac{z_1-a'}{b'}\big)\big] & \text{if } \Delta>0\end{cases}.
$$

Considering now the put option, when $\Delta>0$, we find that

$$
\begin{aligned}
ZBP(0,t,T,K)&=\int_{-\infty}^{z_1}[-\mathcal{P}_c(x)]f_x(x)dx+\int_{z_2}^{\infty}[-\mathcal{P}_c(x)]f_x(x)dx\\
&=-\frac{1}{s\sqrt{2\pi}}\int_{-\infty}^{z_1}\mathcal{P}_c(x)e^{-\frac{(x-\mu)^2}{2s^2}}dx-\frac{1}{s\sqrt{2\pi}}\int_{z_2}^{\infty}\mathcal{P}_c(x)e^{-\frac{(x-\mu)^2}{2s^2}}dx
\end{aligned}
$$

where we remind the reader that $\mu:=x_1(0)e^{-\lambda_1 t}$ and $s^2:=\frac{\sigma_1^2}{2\lambda_1}(1-e^{-2\lambda_1 t})$. Therefore we obtain that

$$
\begin{aligned}
&ZBP(0,t,T,K)\\
=&-P_{0T}\Big[\Phi\Big(\frac{z_1-\hat{a}}{b}\Big)-\Phi\Big(\frac{-\infty-\hat{a}}{b}\Big)\Big]+P_{0t}K\Big[\Phi\Big(\frac{z_1-a'}{b'}\Big)-\Phi\Big(\frac{-\infty-a'}{b'}\Big)\Big]\\
&-P_{0T}\Big[\Phi\Big(\frac{\infty-\hat{a}}{b}\Big)-\Phi\Big(\frac{z_2-\hat{a}}{b}\Big)\Big]+P_{0t}K\Big[\Phi\Big(\frac{\infty-a'}{b'}\Big)-\Phi\Big(\frac{z_2-a'}{b'}\Big)\Big]\\
=&-P_{0T}\Big[\Phi\Big(\frac{z_1-\hat{a}}{b}\Big)-\Phi\Big(\frac{z_2-\hat{a}}{b}\Big)+1\Big]+P_{0t}K\Big[\Phi\Big(\frac{z_1-a'}{b'}\Big)-\Phi\Big(\frac{z_2-a'}{b'}\Big)+1\Big].
\end{aligned}
$$

This then gives us that

$$
ZBP(0,t,T,K)=\begin{cases} 0 & \text{if } \Delta\le 0\\ -P_{0T}\big[\Phi\big(\frac{z_1-\hat{a}}{b}\big)-\Phi\big(\frac{z_2-\hat{a}}{b}\big)+1\big] & \\ \qquad +P_{0t}K\big[\Phi\big(\frac{z_1-a'}{b'}\big)-\Phi\big(\frac{z_2-a'}{b'}\big)+1\big] & \text{if } \Delta>0\end{cases}.
$$

Swaption

Let us consider the payer Swaption maturing at $t > 0$, that is,

$$PS(0, \mathcal{T}, \tau, N, K) = \frac{N}{V_0}\mathbb{E}\Big[V_t\Big(1 - P_{tt_n} - K\sum_{i=1}^{n}\tau_i P_{tt_i}\Big)^+\Big].$$

As we have already seen, it follows that for $N = 1$, we have that

$$PS(0, \mathcal{T}, \tau, N, K) = \mathbb{E}\Big[\exp\Big[-gt+(x_1(t)-\alpha_1)^2-(x_1(0)-\alpha_1)^2\Big]\Big(1-P_{tt_n}-K\sum_{i=1}^{n}\tau_i P_{tt_i}\Big)^+\Big].$$

Recalling the discount bond formula, we find that

$$\begin{aligned}
&1 - P_{tt_n} - K\sum_{i=1}^{n}\tau_i P_{tt_i}\\
=&1 - H_1^{-\frac{1}{2}}(t_n - t)\exp\Big[(-g+\lambda_1)(t_n - t) + H_1^{-1}(t_n - t)\Big(x_1(t) - \alpha_1 e^{\lambda_1(t_n - t)}\Big)^2 - (x_1(t) - \alpha_1)^2\Big]\\
&- K\sum_{i=1}^{n}\tau_i H_1^{-\frac{1}{2}}(t_i - t)\exp\Big[(-g+\lambda_1)(t_i - t) + H_1^{-1}(t_i - t)\Big(x_1(t) - \alpha_1 e^{\lambda_1(t_i - t)}\Big)^2 - (x_1(t) - \alpha_1)^2\Big].
\end{aligned}$$

Therefore we obtain the following expression for the initial price of the payers Swaption:

$$PS(0, \mathcal{T}, \tau, N, K) = \mathbb{E}\big[\big(\mathcal{P}_{PS}(x_1(t))\big)^+\big],$$

where

$$\begin{aligned}
&\mathcal{P}_{PS}(x) := \exp\Big[-gt - (x_1(0) - \alpha_1)^2\Big]\exp\Big[(x - \alpha_1)^2\Big]\\
&- H_1^{-\frac{1}{2}}(t_n - t)\exp\Big[\lambda_1(t_n - t) - gt_n - (x_1(0) - \alpha_1)^2\Big]\exp\Big[H_1^{-1}(t_n - t)\Big(x - \alpha_1 e^{\lambda_1(t_n - t)}\Big)^2\Big]\\
&- K\sum_{i=1}^{n}\tau_i H_1^{-\frac{1}{2}}(t_i - t)\exp\Big[\lambda_1(t_i - t) - gt_i - (x_1(0) - \alpha_1)^2\Big]\exp\Big[H_1^{-1}(t_i - t)\Big(x - \alpha_1 e^{\lambda_1(t_i - t)}\Big)^2\Big].
\end{aligned}$$

Since for any $\alpha, \gamma \in \mathbb{R}$, we have that

$$e^{-\gamma(x-\alpha)^2}\frac{1}{s\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2s^2}} = (1+2\gamma s^2)^{-\frac{1}{2}}e^{-\frac{\gamma(\alpha-\mu)^2}{1+2\gamma s^2}}\frac{1}{b\sqrt{2\pi}}e^{-\frac{(x-\hat{a})^2}{2b^2}},$$

where $\hat{a} := \frac{2s^2\gamma\alpha+\mu}{1+2\gamma s^2}$ and $b := s(1+2\gamma s^2)^{-\frac{1}{2}}$, with $1+2\gamma s^2 > 0$, we infer that

$$e^{-\gamma(x_1(t)-\alpha)^2} \sim (1+2\gamma s^2)^{-\frac{1}{2}}e^{-\frac{\gamma(\alpha-\mu)^2}{1+2\gamma s^2}}\mathcal{N}(\hat{a}, b^2).$$

It follows that

$$Z := \frac{1}{b}\Big[(1+2\gamma s^2)^{\frac{1}{2}} e^{\frac{\gamma(\alpha-\mu)^2}{1+2\gamma s^2}} e^{-\gamma(x_1(t)-\alpha)^2} - \hat{a}\Big] \sim \mathcal{N}(0,1).$$

Therefore, if in order to simplify notation, we first define

$$\bar{\alpha}_0 := \alpha_1, \quad \bar{\gamma}_0 := -1,$$

$$\bar{\alpha}_i := \alpha_1 e^{\lambda_1(t_i-t)}, \quad \bar{\gamma}_i := -H_1^{-1}(t_i-t), \quad \text{for} \quad i=1,\ldots,n,$$
$$\hat{a}_i := \frac{2s^2\bar{\gamma}_i\bar{\alpha}_i+\mu}{1+2\bar{\gamma}_i s^2}, \quad b_i := s(1+2\bar{\gamma}_i s^2)^{-\frac{1}{2}}, \quad \text{for} \quad i=0,1,\ldots,n,$$

we infer that

$$\begin{aligned}
&\bar{\mathcal{P}}_{PS}(Z) := \exp\Big[-gt-(x_1(0)-\alpha_1)^2\Big]\Big[b_0(1+2\bar{\gamma}_0 s^2)^{-\frac{1}{2}} e^{-\frac{\bar{\gamma}_0(\bar{\alpha}_0-\mu)^2}{1+2\bar{\gamma}_0 s^2}}(Z+\frac{\hat{a}_0}{b_0})\Big] \\
&-H_1^{-\frac{1}{2}}(t_n-t)\exp\Big[\lambda_1(t_n-t)-gt_n-(x_1(0)-\alpha_1)^2\Big]\Big[b_n(1+2\bar{\gamma}_n s^2)^{-\frac{1}{2}} e^{-\frac{\bar{\gamma}_n(\bar{\alpha}_n-\mu)^2}{1+2\bar{\gamma}_n s^2}}(Z+\frac{\hat{a}_n}{b_n})\Big] \\
&-K\sum_{i=1}^{n}\tau_i H_1^{-\frac{1}{2}}(t_i-t)\exp\Big[\lambda_1(t_i-t)-gt_i-(x_1(0)-\alpha_1)^2\Big]\Big[b_i(1+2\bar{\gamma}_i s^2)^{-\frac{1}{2}} e^{-\frac{\bar{\gamma}_i(\bar{\alpha}_i-\mu)^2}{1+2\bar{\gamma}_i s^2}}(Z+\frac{\hat{a}_i}{b_i})\Big],
\end{aligned}$$

such that

$$PS(0,\mathcal{T},\tau,N,K) = \frac{1}{\sqrt{2\pi}}\int_{\bar{\mathcal{P}}_{PS}\geq 0} \bar{\mathcal{P}}_{PS}(z) e^{-\frac{z^2}{2}}\,dz.$$

Because the function $\bar{\mathcal{P}}_{PS}$ can be expressed as

$$\bar{\mathcal{P}}_{PS}(Z) = BZ + A,$$

where

$$\begin{aligned}
B := &\exp\Big[-gt-(x_1(0)-\alpha_1)^2\Big](1+2\bar{\gamma}_0 s^2)^{-\frac{1}{2}} e^{-\frac{\bar{\gamma}_0(\bar{\alpha}_0-\mu)^2}{1+2\bar{\gamma}_0 s^2}} b_0 \\
&-H_1^{-\frac{1}{2}}(t_n-t)\exp\Big[\lambda_1(t_n-t)-gt_n-(x_1(0)-\alpha_1)^2\Big](1+2\bar{\gamma}_n s^2)^{-\frac{1}{2}} e^{-\frac{\bar{\gamma}_n(\bar{\alpha}_n-\mu)^2}{1+2\bar{\gamma}_n s^2}} b_n \\
&-K\sum_{i=1}^{n}\tau_i H_1^{-\frac{1}{2}}(t_i-t)\exp\Big[\lambda_1(t_i-t)-gt_i-(x_1(0)-\alpha_1)^2\Big](1+2\bar{\gamma}_i s^2)^{-\frac{1}{2}} e^{-\frac{\bar{\gamma}_i(\bar{\alpha}_i-\mu)^2}{1+2\bar{\gamma}_i s^2}} b_i
\end{aligned}$$

and

$$A := \exp\Big[-gt - (x_1(0)-\alpha_1)^2\Big](1+2\bar{\gamma}_0 s^2)^{-\frac{1}{2}} e^{-\frac{\bar{\gamma}_0(\bar{\alpha}_0-\mu)^2}{1+2\bar{\gamma}_0 s^2}}\hat{a}_0$$
$$- H_1^{-\frac{1}{2}}(t_n-t)\exp\Big[\lambda_1(t_n-t) - gt_n - (x_1(0)-\alpha_1)^2\Big](1+2\bar{\gamma}_n s^2)^{-\frac{1}{2}} e^{-\frac{\bar{\gamma}_n(\bar{\alpha}_n-\mu)^2}{1+2\bar{\gamma}_n s^2}}\hat{a}_n$$
$$- K\sum_{i=1}^{n}\tau_i H_1^{-\frac{1}{2}}(t_i-t)\exp\Big[\lambda_1(t_i-t) - gt_i - (x_1(0)-\alpha_1)^2\Big](1+2\bar{\gamma}_i s^2)^{-\frac{1}{2}} e^{-\frac{\bar{\gamma}_i(\bar{\alpha}_i-\mu)^2}{1+2\bar{\gamma}_i s^2}}\hat{a}_i,$$

we infer that

$$PS(0,\mathcal{T},\tau,N,K) = \begin{cases} \frac{1}{\sqrt{2\pi}}\int_{-\frac{A}{B}}^{\infty}(Bz+A)e^{-\frac{z^2}{2}}dz & \text{if } B>0 \\ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\frac{A}{B}}(Bz+A)e^{-\frac{z^2}{2}}dz & \text{if } B<0 \end{cases}.$$

Therefore, we conclude that

$$PS(0,\mathcal{T},\tau,N,K) = \begin{cases} B\rho(-\frac{A}{B}) + A\Phi(\frac{A}{B}) & \text{if } B>0 \\ -B\rho(-\frac{A}{B}) + A\Phi(-\frac{A}{B}) & \text{if } B<0 \end{cases}.$$

where

$$\rho(x) := \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}.$$

Note here that

$$\frac{1}{\sqrt{2\pi}}\int_x^{\infty} e^{-\frac{X^2}{2}}dX = \Phi(-x), \quad \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} Xe^{-\frac{X^2}{2}}dX = -\rho(x)$$

and

$$\frac{1}{\sqrt{2\pi}}\int_x^{\infty} Xe^{-\frac{X^2}{2}}dX = \rho(x).$$

10.2 Proof of Proposition 20.5 in [11]

Let us first recall the dynamics of the instantaneous forward rate expressed in (3.2.16), that is,

$$df_{tT} = \hat{\alpha}_{tT}dt + \hat{\sigma}_{tT}dW_t.$$

From this, Proposition 20.5 gives the dynamics of the short rate expressed as (3.2.17). Because we have that

$$f_{tT} = f_{0T} + \int_0^t \hat{\alpha}_{uT}du + \int_0^t \hat{\sigma}_{uT}dW_u, \quad 0 \le t \le T < \infty.$$

it follows that

$$
\begin{aligned}
dr_t =& d\Big(f_{0t} + \int_0^t \hat{\alpha}_{ut} du + \int_0^t \hat{\sigma}_{ut} dW_u\Big) \\
=& d(f_{0t}) + d\Big(\int_0^t \hat{\alpha}_{ut} du\Big) + d\Big(\int_0^t \hat{\sigma}_{ut} dW_u\Big).
\end{aligned} \tag{10.2.1}
$$

By the Leibniz integral rule, (see, for example [71]), we have that

$$
d\Big(\int_0^t \hat{\alpha}_{ut} du\Big) = \hat{\alpha}_{tt} dt + \Big(\int_0^t \frac{\partial}{\partial t}\hat{\alpha}_{ut} du\Big) dt. \tag{10.2.2}
$$

On the other hand, because, we also have for $0 \leq u \leq t < \infty$ that

$$
\hat{\sigma}_{ut} = \hat{\sigma}_{uu} + \int_u^t \frac{\partial}{\partial s}\hat{\sigma}_{us} ds,
$$

we infer that

$$
\begin{aligned}
d\Big(\int_0^t \hat{\sigma}_{ut} dW_u\Big) =& d\Big(\int_0^t \hat{\sigma}_{uu} dW_u\Big) + d\Big(\int_0^t \int_u^t \frac{\partial}{\partial u}\hat{\sigma}_{us} ds dW_u\Big) \\
=& \hat{\sigma}_{tt} dW_t + d\Big(\int_0^t \int_u^t \frac{\partial}{\partial s}\hat{\sigma}_{us} ds dW_u\Big).
\end{aligned} \tag{10.2.3}
$$

However, because the equality

$$
\int_0^t \int_u^t \frac{\partial}{\partial s}\hat{\sigma}_{us} ds dW_u = \int_0^t \int_0^s \frac{\partial}{\partial s}\hat{\sigma}_{us} dW_u ds
$$

is satisfied, the expression (10.2.3) may be simplified in the following way:

$$
d\Big(\int_0^t \hat{\sigma}_{ut} dW_u\Big) = \hat{\sigma}_{tt} dW_t + \Big(\int_0^t \frac{\partial}{\partial t}\hat{\sigma}_{ut} dW_u\Big) dt. \tag{10.2.4}
$$

Combining the expressions (10.2.1), (10.2.2) and (10.2.4) we may conclude that

$$
\begin{aligned}
dr_t =& \Big(\frac{\partial}{\partial t} f_{0t} + \int_0^t \frac{\partial}{\partial t}\hat{\alpha}_{ut} du + \int_0^t \frac{\partial}{\partial t}\hat{\sigma}_{ut} dW_u\Big) dt + \hat{\alpha}_{tt} dt + \hat{\sigma}_{tt} dW_t \\
=& \Big(\hat{\alpha}_{tt} + \frac{\partial}{\partial T} f_{tT}\Big|_{T=t^+}\Big) dt + \hat{\sigma}_{tt} dW_t.
\end{aligned}
$$

10.3 Appendix to the Chaotic Approach

The variable $M_{ts} := \mathbb{E}_t[\sigma_s^2]$ is formulated in the Chaotic Approach as the expression (4.6.2). In this section, we deduce the same form by another method. Recalling the chaos expansion from (2.1.23), we obtain that for $0 \le t \le s < \infty$,

$$\begin{aligned} M_{ts} =& \mathbb{E}\left[\left(\phi_1(s) + \int_0^s \phi_2(s,s_1)dW_{s_1} + \int_0^s \int_0^{s_1} \phi_3(s,s_1,s_2)dW_{s_2}dW_{s_1} + \cdots\right)^2\right] \\ =& \mathbb{E}_t\left[\left(R_1(t,s) + \int_t^s \phi_2(s,s_1)dWs_1 + \int_t^s \int_0^{s_1} \phi_3(s,s_1,s_2)dWs_2dWs_1 + \cdots\right)^2\right]. \end{aligned}$$

where

$$R_1(t,s) = \phi_1(s) + \int_0^t \phi_2(s,s_1)dWs_1 + \int_0^t \int_0^{s_1} \phi_3(s,s_1,s_2)dWs_2dWs_1 + \cdots .$$

Notice here that the random variable $R_1(t,s)$ is $\mathcal{F}_t$-measurable and $R_1(t,t) = \sigma_t$. Therefore it follows that

$$\begin{aligned} M_{ts} =& R_1^2(t,s) + R_1(t,s)\mathbb{E}_t\left[\left(\int_t^s \phi_2(s,s_1)dWs_1 + \int_t^s \int_0^{s_1} \phi_3(s,s_1,s_2)dWs_2dWs_1 + \cdots\right)\right] \\ &+ \mathbb{E}_t\left[\left(\int_t^s \phi_2(s,s_1)dWs_1 + \int_t^s \int_0^{s_1} \phi_3(s,s_1,s_2)dWs_2dWs_1 + \cdots\right)^2\right] \\ =& R_1^2(t,s) + \mathbb{E}_t\left[\left(\int_t^s \phi_2(s,s_1)dWs_1 + \int_t^s \int_0^{s_1} \phi_3(s,s_1,s_2)dWs_2dWs_1 + \cdots\right)^2\right]. \end{aligned}$$

The conditional Itô isometry gives us that

$$\begin{aligned} M_{ts} =& R_1^2(t,s) + \int_t^s \mathbb{E}_t\left[\left(\phi_2(s,s_1) + \int_0^{s_1} \phi_3(s,s_1,s_2)dWs_2 + \cdots\right)^2\right]ds_1 \\ =& R_1^2(t,s) + \int_t^s \mathbb{E}_t\Bigg[\Bigg(R_2(t,s,s_1) + \int_t^{s_1} \phi_3(s,s_1,s_2)dWs_2 \\ &+ \int_t^{s_1}\int_0^{s_2} \phi_4(s,s_1,s_2,s_3)dWs_3dWs_2 + \cdots\Bigg)^2\Bigg]ds_1, \end{aligned}$$

where

$$R_2(t,s,s_1) = \phi_2(s,s_1) + \int_0^t \phi_3(s,s_1,s_2)dWs_2 + \int_0^t \int_0^{s_2} \phi_4(s,s_1,s_2,s_3)dWs_3dWs_2 + \cdots .$$

Because $R_2^2(t,s,s_1)$ is $\mathcal{F}_t$-measurable, we infer that

$$\begin{aligned}
M_{ts} =& R_1^2(t,s) + \int_t^s \Bigg\{ R_2^2(t,s,s_1) \\
&+ R_2(t,s,s_1)\mathbb{E}_t\Bigg[\Bigg(\int_t^{s_1} \phi_3(s,s_1,s_2)dWs_2 + \int_t^{s_1}\int_0^{s_2} \phi_4(s,s_1,s_2,s_3)dWs_3dWs_2 + \cdots\Bigg)\Bigg] \\
&+ \mathbb{E}_t\Bigg[\Bigg(\int_t^{s_1} \phi_3(s,s_1,s_2)dWs_2 + \int_t^{s_1}\int_0^{s_2} \phi_4(s,s_1,s_2,s_3)dWs_3dWs_2 + \cdots\Bigg)^2\Bigg]\Bigg\}ds_1 \\
=& R_1^2(t,s) + \int_t^s R_2^2(t,s,s_1)ds_1 \\
&+ \int_t^s \mathbb{E}_t\Bigg[\Bigg(\int_t^{s_1} \phi_3(s,s_1,s_2)dWs_2 + \int_t^{s_1}\int_0^{s_2} \phi_4(s,s_1,s_2,s_3)dWs_3dWs_2 + \cdots\Bigg)^2\Bigg]ds_1.
\end{aligned}$$

Applying the Itô isometry again, we find similarly that

$$\begin{aligned}
M_{ts} =& R_1^2(t,s) + \int_t^s R_2^2(t,s,s_1)ds_1 \\
&+ \int_t^s\int_t^{s_1} \mathbb{E}_t\Bigg[\Bigg(\phi_3(s,s_1,s_2) + \int_0^{s_2} \phi_4(s,s_1,s_2,s_3)dWs_3 + \cdots\Bigg)^2\Bigg]ds_2ds_1 \\
=& R_1^2(t,s) + \int_t^s R_2^2(t,s,s_1)ds_1 \\
&+ \int_t^s\int_t^{s_1} \mathbb{E}_t\Bigg[\Bigg(R_3(t,s,s_1,s_2) + \int_t^{s_2} \phi_4(s,s_1,s_2,s_3)dWs_3 + \cdots\Bigg)^2\Bigg]ds_2ds_1,
\end{aligned}$$

where

$$\begin{aligned}
R_3(t,s,s_1,s_2) =& \phi_3(s,s_1,s_2) + \int_0^t \phi_4(s,s_1,s_2,s_3)dWs_3 \\
&+ \int_0^t\int_0^{s_3} \phi_5(s,s_1,s_2,s_3,s_4)dWs_4dWs_3 + \cdots .
\end{aligned}$$

As before, it follows that

$$\begin{aligned}
M_{ts} =& R_1^2(t,s) + \int_t^s R_2^2(t,s,s_1)ds_1 + \int_t^s\int_t^{s_1} R_3^2(t,s,s_1,s_2)ds_2ds_1 \\
&+ \int_t^s\int_t^{s_1} \mathbb{E}_t\Bigg[\Bigg(\int_t^{s_2} \phi_4(s,s_1,s_2,s_3)dWs_3 + \int_t^{s_2}\int_0^{s_3} \phi_5(s,s_1,s_2,s_3,s_4)dWs_4dWs_3 + \cdots\Bigg)^2\Bigg]ds_2ds_1.
\end{aligned}$$

Therefore iterating the expression gives us that

$$
\begin{aligned}
M_{ts} =& R_1^2(t,s) + \int_t^s R_2^2(t,s,s_1)ds_1 + \int_t^s \int_t^{s_1} R_3^2(t,s,s_1,s_2)ds_2ds_1 \\
&+ \int_t^s \int_t^{s_1} \int_t^{s_2} R_4^2(t,s,s_1,s_2,s_3)ds_3ds_2ds_1 \\
&+ \int_t^s \int_t^{s_1} \int_t^{s_2} \int_t^{s_3} R_5^2(t,s,s_1,s_2,s_3,s_4)ds_4ds_3ds_2ds_1 + \cdots ,
\end{aligned}
$$

where

$$
R_1(t,s) = \phi_1(s) + \int_0^t \phi_2(s,s_1)dWs_1 + \int_0^t \int_0^{s_1} \phi_3(s,s_1,s_2)dWs_2dWs_1 + \cdots ,
$$

$$
\begin{aligned}
R_n(t,s,s_1,\ldots,s_{n-1}) =& \phi_n(s,s_1,\ldots,s_{n-1}) + \int_0^t \phi_{n+1}(s,s_1,\ldots,s_n)dWs_n \\
&+ \int_0^t \int_0^{s_n} \phi_{n+2}(s,s_1,\ldots,s_{n+1})dWs_{n+1}dWs_n + \cdots , \quad \text{for} \quad n = 2,3,\ldots .
\end{aligned}
$$

Therefore we obtain the same result here.

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