Contradictions and falling bridges: What was Wittgenstein's reply to Turing?

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1 Introduction

Wittgenstein's remarks on inconsistency have attracted much attention in the secondary literature, most of which has been negative in nature, or even scathing[1]—with some recent exceptions (Berto, "Gödel Paradox"; Persichetti "The Later Wittgenstein"). In this paper, I offer a close reading of Wittgenstein's remarks on contradictions, as they appear in the lectures he gave in Cambridge in 1939.[2] I especially focus on an objection Alan Turing, who attended the lectures, gave to Wittgenstein's position, the so-called "falling bridges"-objection.

Wittgenstein's position, or so I will argue, is that contradictions are either purely formal or arise in some practice of using language. In the former case, we can adopt a paraconsistent logic and keep our formal system coherent, and in the latter, contradictions can at most cause confusion, but are not of any *special* concern, and in particular, are not necessarily fatal to the inconsistent practice. An inconsistent practice and the concepts derived from it, even arithmetic or the concept of truth, can be coherent and contentful, on Wittgenstein's view, despite the inconsistency, and the discovery of a contradiction does not necessarily require that any revision of the practice take place.

[1] See e.g. Anderson, "Mathematics and the 'Language Game'"; Kreisel, "Wittgenstein's Remarks"; Bernays, "Comments on Wittgenstein's Remarks"; Chihara, "Wittgenstein's Analysis".

[2] I will refer to the *Lectures on the Foundations of Mathematics* as LFM, the *Remarks on the Foundations of Mathematics* as RFM, *Wittgenstein's Lectures: Cambridge, 1932–1935* as AWL, and the *Philosophical Investigations* as PI.

With this picture in the background, Wittgenstein's answer to Turing is that if we run into trouble building our bridge, it is either because we've made a calculation mistake or our calculus does not actually describe the phenomenon it is intended to model. The possibility of neither kind of error is particular to contradictions nor inconsistency, and hence contradictions have no special status when it comes to the practical application of mathematics.

In order to keep the discussion relatively self-contained, I focus almost exclusively on Wittgenstein's remarks in the *Lectures*, both regarding inconsistency and his philosophy of mathematics more generally.

2 Wittgentein's philosophy of mathematics in the *Lectures*

In this section, I will to discuss two core claims Wittgenstein makes repeatedly in the *Lectures*—claims that will be important for his discussion of contradictions. Neither of these claims should be considered particularly controversial among interpreters of Wittgenstein and my aim here is not to defend them as a reading of his philosophy of mathematics, but rather to provide the background needed to understand his claims about inconsistency. The claims that I have in mind are as follows:

(i) Mathematical statements are not responsible to an external and mind-independent reality.

(ii) Mathematical statements create the form of what we call descriptions.

The first statement can be read as saying that mathematical statements are not propositions at all, and thus are not descriptions of anything, much less an external mind-independent reality (Bangu, "Later Wittgenstein's Philosophy of Mathematics") or a rejection of the view that the ultimate ground for the correctness of our mathematical statements is a mathematical reality, conceived of independently of our mathematical practices (Gerrard, "Wittgenstein's Philosophies", "A Philosophy of Mathematics"). The formulation I wish to focus on here is the latter, although I think it is also quite certain that

Wittgenstein held a version of the former (see e.g. AWL 152, RFM I, §144, RFM I, app. III, §§1–4).

I will start by try to make clearer what Wittgenstein means by this, as it has implications for what he has to say about contradictions and inconsistency.

2.1 Mathematics and correspondance to reality

Wittgenstein does not think, despite his rejection of (i), that mathematical statements are not objective, nor that locutions such as "a reality corresponds to our mathematical propositions" are necessarily false, but that a "wrong picture goes with them" (LFM XIV, 141) and unless we provide an explanation of this correspondance, we have simply said something meaningless or empty. Lecture XXV contains perhaps the clearest expression of this aspect of Wittgenstein's philosophy of mathematics:

> Suppose we said first, "mathematical propositions can be true or false". The only clear thing about this would be that we affirm some mathematical propositions and deny others. If we then translate the words "It is true..." by "A reality corresponds to..."—then to say that a reality corresponds to them would say only that we affirm some mathematical propositions and deny others. We also affirm and deny propositions about physical objects. [...] If that is all that is meant by saying that a reality corresponds to a mathematical propositions, it would come to saying nothing at all, a mere truism: if we leave out the question of how it corresponds, or in what sense it corresponds. (LFM, XXV, 239)

Wittgenstein goes on to say that the words of our language have various different uses and that if we forget where the expression "a reality corresponds to" is "really at home" (LFM XXV, 240) we are liable to be misled (cf. PI, §116). He ends this train of thought by saying:

> What is "reality"? We think of "reality" as something we can *point* to. It is *this*, *that.* (LFM XXV, 240)

It is presumably that picture that invites Wittgenstein's comparison with empirical descriptions, where an analog of the negation of (i) would in fact be apt. The grammatical similarity of empirical statements on the one hand, and mathematical statements on the other, is what lures us into assimilating the way the latter corresponds to a reality to the former—to think that true mathematical statements describe mathematical reality.

Wittgenstein next brings the focus back on to the interpretation of the locution "a reality corresponds to..." which he had declared harmless, namely that mathematics is objective:

> Or to say this [that mathematical statements correspond to a reality] may mean: these propositions are *responsible* to a reality. That is, you cannot just say anything in mathematics, because there is the reality. This comes from saying that propositions of physics are responsible to that apparatus—you can't say any damned thing.
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> It is almost like saying, "Mathematical propositions don't correspond to *moods*; you can't say one thing now and one thing then. Or again: "Please don't think of mathematics as something vague that goes on in the mind." [...] And if you oppose this you are inclined to say "a reality corresponds". (LFM XXV, 240)

Wittgenstein then offers two different ways we could in fact spell out this phrase "a reality corresponds..." and give it content (LFM XXV, 241).

The first way he calls "mathematical responsibility", which is how certain mathematical propositions, but not others, can be derived from our axioms by our inference rules, or how certain propositions can be derived given our methods of calculation and not others—where we might say that a theorem is responsible to the axioms from which it was derived, etc.

Another way Wittgenstein considers is how our whole system of mathematics, for example our axioms and inference rules taken as a whole, not merely individual propositions, can be said to be responsible to something. Here, Wittgenstein mentions two constraints, one that seems psychological and one that has to do with the usefulness of our mathematical theories. The first constraint is that if we use a word in a particular way, we are inclined to use it in certain ways in future cases, where some ways to proceed are 'unnatural'. Here Wittgenstein is quite (and perhaps uncharacteristically) explicit:

> Suppose I said, "If you give different logical laws, you are giving the words the wrong meaning." This sounds absurd. What is the wrong meaning? Can a meaning be wrong? There's only one thing that can be wrong with the meaning of a word and that is that it is unnatural. (LFM XXV, 243)

What does Wittgenstein mean by 'unnatural'? One of the examples he takes is of us using the words 'red' and 'green' as we use them now, but also going on and describing things as being "reddish-green"—we do not, Wittgenstein seems to be saying, know how to use such a proposition, given the meanings of the words 'red' and 'green'—it is not natural for us to extend their use in this way.

The second constraint is quite simple and directly concerns contradictions. Wittgenstein says:

> If we allow contradictions in such a way that we accept that *anything* follows, then we no longer get a calculus, or we'd get a useless thing resembling a calculus. (LFM XXV, 243)

The implication seems to be, first of all, that if we do not allow everything to follow from a contradiction, then we can still have a useful calculus and that it is conceivable that we do so, and if we did, then that would be problematic. (This short quote already shows that Wittgenstein wasn't as cavalier about contradictions as many have supposed.)

After discussing these constraints, Wittgenstein then compares the following two propositions:

(1) "There is no reddish-green".

(2) "In this room there is nothing yellowish-green".

If we are tempted to say, Wittgenstein claims, that there is a reality corresponding to the former proposition, the superficial grammatical similarity of the two propositions suggests to us that "the kind of reality corresponding" to each of them is the same (LFM XXV, 243). Wittgenstein goes on to say that the proposition "There is no reddish-green" is more akin to a grammatical rule to the effect that the expression 'red-green' cannot be applied to anything. The correspondance, he says,

> is between this rule and such facts as that we do not normally make a black by mixing a red and a green; that if you mix red and green you get a colour which is "dirty" and dirty colours are difficult to remember. All sorts of facts, psychological and otherwise. (LFM XXV, 244–245)

We could, Wittgenstein seems to be saying, have found a use for the expression 'reddish-green', for instance, by using it for the brown colour that results in mixing the two. But given certains facts about us, we exclude this expression from our language (see also LFM XXIV, 231). Wittgenstein concludes the discussion by saying that when we utter statements like (1), what we are

> saying is not an experiential proposition at all, though it sounds like one; it is a rule. That rule is made important and justified by reality—by a lot of most important things. (LFM XXV, 246)

Early in the next lecture, Wittgenstein brings up the topic of correspondance to reality again, where he elaborates on the two constraints, bringing usefulness to the foreground. He makes a distinction between how we might say that a reality corresponds to a true empirical statement, such as 'it rains' and how we might say the same of individual words, such as 'rain' (LFM XXVI, 247). The first, he seems to say, is the same as the statement being true or assertable. The latter sort, Wittgenstein says, is quite different and amounts to asserting that the word has meaning and showing how such a word corresponds to reality is to *give* the word a meaning.

Accordingly, propositions of that sort, i.e. "this is green" or "'rain' means this" are, as Wittgenstein puts it, "sentence[s] of grammar"—sentences used to explain the use of the word in question (LFM XXVI, 248) and set up a meaning for them in our practice. In the case of 'green' and 'rain', Wittgenstein says, we might point e.g. to green things (or out the window when it is raining, supposedly) but in the case of some words, like 'two' or 'perhaps', there is nothing we can point to at all. The reality that corresponds to these words is, as Cora Diamond puts it, "our having a use for it" (Diamond, "Wittgenstein, Mathematics, and Ethics", 216) and that, as Wittgenstein had said in the previous lecture, itself depends on various facts about us and reality.[3] He goes on:

> What I want to say is this. If one talks of the reality corresponding to a proposition of mathematics or of logic, it is like speaking of a reality corresponding to these *words*—"two" or "perhaps"—more than it is like talking of a reality corresponding to the *sentence* "It rains". Because the structure of a "true" mathematical proposition or a "true" logical proposition is entirely defined in language: it doesn't depend on any external fact at all.[4] (LFM XXVI, 249)

He then concludes that to say that a reality corresponds to a mathematical statement like "2 + 2 = 4" is like saying that a reality corresponds to 'two' in that there is nothing that we can point to directly to give it meaning, and *that* in turn is like saying that a reality corresponds to a rule, which again,

> would come to saying: "It is a useful rule, *most* useful—we couldn't do without it for a thousand reasons, not just *one*". (LFM XXVI, 249, see also LFM XXV, 246)

The idea seems to be, that when we utter statements like "this is two" or ' 'rain' means this" in the contexts Wittgenstein is considering, we are not describing anything, but setting up the meaning of those words. These statements are rule-like in that they tell us how to use the terms in question on future occasions—kind of like constitutive rules for our own practice of using the terms.

I do not want to understate the difficulties in combining the forgoing into a coherent account. The picture of mathematical statements that arises from this discussion is that mathematical statements are not strictly speaking propositions that can be either true or false in the same way empirical descriptions can be, but should rather be seen as rules, whose usefulness or desirability nevertheless depends on certain facts about reality. When we say

[3] I take my reading of the relevant sections to be in much in agreement with Diamond's reading. She too emphasises that Wittgenstein's rejection of (i) isn't quite as simple as many commentators believe.

[4] It is often claimed that Wittgenstein does not reject mathematical platonism as being false, but only as a truism or a confusion. The preceding discussion should show that this squeamishness is unwarranted.

that mathematical statements are true or false, however, that merely means that we affirm them or reject them, like we affirm and reject empirical statements, but with the crucial difference that the former do not point outside of language, and when we say that mathematical statements are responsible to a reality, we should understand that to mean, on the one hand, that mathematics is not arbitrary, and that the rules that we derive from our mathematical practices are useful, on the other.

2.2 Mathematics as forms of description

This kind of usefulness of certain mathematical statements is the basis for Wittgenstein's claim that mathematical statements have their origin in experience, but have since been made independent of it, and subsequently used to judge experience. He makes this claim in one of the earlier lectures and often repeats it subsequently (a rare instance of an unequivocal philosophical claim by Wittgenstein):

> All the calculi in mathematics have been invented to suit experience and then made independent of experience. (LFM IV, 43)
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> It is the same with 25 × 25 = 625. It was first introduced because of experience. But now we have made it independent of experience; it is a rule of expression for talking about our experiences. We say, "The body must have got heavier" or "It deviates from the calculated weight". (LFM IV, 44)

This leads directly into (ii)—that mathematical statements create the form of description. There are two related strands of this claim that are important for our purposes. The first is that mathematical statements allow us to describe reality in ways that we could not before (see e.g. LFM XXVI, 250) and here Wittgenstein seems to come close to the view that the objects mathematical statements speak of are mere representational aids, as Yablo might put it (see Yablo, "The Myth of Seven"), letting us express facts about the natural world that would otherwise be difficult or impossible.

Perhaps, however, it would be more accurate to say that Wittgenstein holds the view that mathematics gives the very meaning of certain non-mathematical, empirical statements and thereby that mathematics gives us a framework to describe empirical reality (see e.g. RFM VII, §2, §§18–19). Or, as he puts it in one of the first lectures:

> One might also put it crudely by saying that mathematical propositions containing certain symbols are rules for the use of that symbol, and that these symbols can then be used in non-mathematical statements. (LFM III, 33)

The point is, I believe, a simple one, and not particularly deep: empirical statements such as "I have four pounds in my pocket" or empirical generalisations such as "the average airspeed velocity of an unladen European swallow is roughly 11 meters per second" presuppose and in part derive their meaning from our mathematical practices (namely, the number words and the terms "average" and "per second").

This leads us to the second strand of (ii)—which is connected with Wittgenstein's claim that mathematical statements are rules that licence certain inferences from one empirical proposition to another. In the *Lectures*, Wittgenstein uses the metaphor of putting mathematical statements "in the archives"—by which he means that have a special attitude towards mathematical statements of not giving them up when empirical reality contradicts our calculations: we do not say that 25 × 25=625 is false if we only count 624 apples in 25 boxes of 25 apples each, but say that we must have miscounted or one apple got lost.

The statements in the archives are therefore a criterion by which we judge our experience and that explains why we feel entitled to use them as inference rules to move from one empirical statement to another (in the *Remarks*, Wittgenstein is wont to say that mathematical statements are empirical propositions hardened into rules to describe the same phenomenon (see e.g. RFM IV, §22).[5]

[5] There is an extensive literature on Wittgenstein's claim that mathematical propositions are hardened empirical statements (see e.g. Fogelin, *Wittgenstein*; Steiner "Mathematical Intuition", "Empirical Regularities"; Bangu, "Genealogy of Mathematical Necessity", "Wynn's Experiments").

Of course, Wittgenstein realises that not all mathematical statements can be deposited directly in the archives, so to speak, since it is only possible to have this attitude directly towards a finite number of statements, while the number of true mathematical statements is infinite. His examples often include single mathematical statements, but his considered view seems to be that our *calculi* and mathematical techniques have been invented by us for certain practical purposes and that our mathematical statements are moves, so to speak, in the associated language-games and grounded in the techniques used therein (see e.g. RFM VII, §1).

In his discussion of how a theorem is responsible to the axioms and rules of inference, Wittgenstein both explicitly brought up the rule-following considerations of §§185–241 of the *Investigations* and his purported solution to the problems he raises there—a subject prominent in the *Lectures*—and how that might seem to undermine mathematical objectivity. He first says:

> I have constantly stressed that given a set of axioms and rules, we could imagine different ways of using them. You might say, "So, Wittgenstein, you seem to say that there is no such thing as this proposition following necessarily from that". (RFM XXV, 241)

In the short discussion that ensued with Georg Henrik von Wright, one of the students attending the lectures, Wittgenstein goes on to claim that the way the theorems are responsible to the axioms is in fact "based on our peculiar practice of using these rules" (LFM XXV, 242).

Despite this emphasis on practice being the ground of mathematical truth, Wittgenstein does not reject that it is an objective fact that certain things follow from certain rules, and both in the *Lectures*, as well as in *RFM* and the *Investigations*, Wittgenstein goes to great lengths to prevent that interpretation of his position. For instance, he explicitly rejects the idea that we are somehow free to stipulate what the result of an individual calculation is:

> We have learned the rules of multiplication, but we have not learned the result of each multiplication. It is absurd to say that we invent $136 \times 51 = 6936$; we find that this is the result. (LFM X, 101)

The content of our mathematical statements, it seems, is not derived from rules that we lay down and then operate independently from our practice of following those same rules, as a moderate conventionalist might perhaps think (see Dummett, "Wittgenstein's Philosophy of Mathematics", "Wittgenstein on Necessity" for this terminology) but are derived from our actual mathematical practice—what we *actually* do.

This is a point Wittgenstein returns to repeatedly in the *Lectures*—that *we* simply find some ways of following a rule in a new case more *natural* than others and that nothing more needs to be said about the correctness of certain ways of continuing, that those ways are in fact how we all proceed and they are therefore constitutive of that very practice (see e.g. RFM I, §116): somehow it is meant to be *both* true that we discover the objective truth that 136 × 51 = 6936 *and* that the fact that we find it natural to continue our practice of adding in that way at that step, and not in any other way, constitutes the correctness of that step.

But how do we move from facts about all of us agreeing to a certain result—itself an empirical fact—to the corresponding mathematical proposition? On this matter, Wittgenstein says, and it is instructive to quote at some length:

> It has been said: "It's a question of general consensus." There is something true in this. Only—what is it that we agree to? Do we agree to the mathematical proposition, or do we agree in *getting* this result? These are entirely different. [...] Mathematical truth isn't established by their all agreeing that it's true—as if they were witnesses of it. Because they all agree in what they do, we lay it down as a rule, and put it in the archives. Not until we do that have we got to mathematics. One of the main reasons for adopting this as a standard, is that it's the natural way to do it, the natural way to go—for all these people. (LFM XI, 107.)

It is of course nothing new to claim that, for Wittgenstein, following a rule is a practice or a custom (where human agreement plays a pivotal role)— even though few commentators emphasise Wittgenstein's discussion of "naturalness" in this regard. And while I believe that the philosophical literature on Wittgenstein both overestimates our understanding of how exactly he intends practice to ground the objectivity and correctness of mathematics and underestimates the substantial problems in giving such an account, it is clear that he holds a view of roughly this shape and for our purposes here, namely to explicate Wittgentein's claims about inconsistency, it is safe to put those matters to a side.[6]

3 Wittgenstein's discussion of contradictions

In Lecture XXI, Wittgenstein discusses logical propositions on a similar model as that of mathematical propositions, i.e. (i) not corresponding to an external reality and (ii) creating the form of what we call descriptions. He claims that we get convinced of logical laws, of which the law of contradiction is one example, by learning certain practices and techniques where saying one thing and saying the opposite has no use. The most natural way for *us* to extend these practices is by eliminating certain combinations which have no use in the practice, "like contradictions" (LFM XXI, 201):

> How do we get convinced of the law of contradiction?—In this way: We learn a certain practice, a technique of language; then we are all inclined to do away with this form—on which we do not act naturally in any way, unless this particular form is explained afresh to us. (LFM XXI, 206)

We have a certain practice of using the words 'and' and 'not' when describing things, i.e. when we are asserting empirical propositions, and the most natural analogy of extending the use of these words for us makes us exclude the combination where a proposition and its negation are asserted: there is simply nothing we describe as being of that form—just like we exclude the claim "I have 25 boxes of 25 apples, so I have 624 apples" as being necessarily false.

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[6] Naturally, Wittgenstein discusses these matters elsewhere, most prominently in the *Philosophical Investigations*, e.g. PI, §§201–202, §§240–242. Juliet Floyd, in particular, has written on how these Wittgensteinian concepts relate to Turing's work (see e.g. Floyd, "Chains of Life" and Floyd, "Wittgenstein on Ethics", 126–127).

This, Wittgenstein says, has the "queer consequence" that we are puzzled by things such as the liar paradox. This is strange, he says, because

> the thing works like this: if a man says "I'm lying" we say that it follows that he is not lying, from which it follows that he is lying and so. Well, so what? (LFM XXI, 207)

The whole thing is a useless performance, according to Wittgenstein, but he grants that one might ask why does this technique, which serves us so well in general, go wrong in this case. We want to know, why a contradiction "comes with "I am lying" and not with "I am eating'' '' (LFM XXI, 207, see also RFM I, app. III, §12)?

Turing's objection to this line of reasoning is that we usually take a contradiction as a criterion for having done something wrong but in this case we seem not to be able to find what it is. Wittgenstein's reply is that nothing has gone wrong, and if one is worried that the contradiction infects our whole language, we might as well decide not to draw any conclusions from the paradox (RFM XXI, 207–208). We simply abandon the *ex falso*-rule if we are so worried about the contradiction.

In the next lecture, when the matter was taken up again, Wittgenstein makes a distinction between the use of logical expressions *in* a calculus and *outside* a calculus. He does grant that when we use logical expressions outside of the logical calculus, e.g. when used to describe external reality, give orders, etc., we *can* run into trouble:

> This simply means that given a certain training, if I give you a contradiction (which I need not notice myself) you don't know what to do. [...] That is one thing: (a) We do in fact avoid contradictions. (b) Unless we wish to produce confusions (given our training) we have to avoid contradictions.—But it is an entirely different thing to say that we ought to avoid contradictions in *logic*. (LFM XXII, 213)

As I read him, Wittgenstein is *not* saying that descriptions of the form "I am eating and I am not eating" are not false, and necessarily so. That they are simply follows from his view that logic creates the form of our descriptions of the empirical world, and this is how we in fact use them.

His claim is rather that we (1) *could* quite easily invent a meaning for such utterances (because the rules that we lay down do not operate independently of our practice of using them—and if we did stipulate that "I am eating and I am not eating" meant that I was, say nibbling on something, then that contradiction would have a different meaning for us than it actually does (having been "explained afresh" to us, presumably), and thus not necessarily false, but used to describe the situation where I'm nibbling. Secondly, that (2) even though our language allows instances of a contradiction, such as the liar paradox, that is not a cause for concern as long as we do not draw further conclusions from them (and in fact, we don't). That's presumably the case for Wittgenstein because they are not used as descriptions of empirical reality (as "I am eating" is) in the paradoxical case and can therefore not come into conflict with it.[7]

The first point (1) is merely a point about our rules not operating independently of our practice—and not a point about contradictions specifically. Wittgenstein points out that we are very much inclined to say that it cannot be that I'm eating and not eating, that this does not describe any possible action (LFM XIX, 185). He goes on to suggest that if we were to stipulate that whenever something of this form occurs, only the first conjunct should be understood, so that "I am eating and I am not eating" just means that I'm eating, then that, he says, would give us the feeling of being "cheated" since that is not how we naturally use these terms.

Turing then suggests that this feeling of being cheated comes from the fact that what we want is a meaning which is in accord with how we ordinarily use language, not by an arbitrary stipulation (LFM XIX, 185). He then says:

> [Turing:] Could one take as an analogy a person having blocks of wood having two squares on them, like dominoes. If I say to you "White-green", you then have to paint one of the squares on the domino which I give you white and the other green. [...] —Your

[7] This claim is of course reminiscent of Kripke's theory of grounded truth (see Kripke, "Outline of a Theory of Truth").

> suggestion comes to saying that when I say “White-white” you are to paint one of the squares white and the other grey. (LFM XIX, 186)

Wittgenstein accepts this analogy but claims that there is nothing internal to this practice that makes it *wrong*:

> Yes, exactly. And where does the cheating come in? What is wrong with the continuation I have suggested? Why is this continuation in your analogy a wrong continuation? Might it not be the ordinary jargon among painters?
>
> The point is: Is it or is it not a case of one continuation being *natural* for us? (LFM XIX, 186)

He then goes on to say that the picture Turing has in mind when giving this objection relies on seeing rules as operating independently of our practice. The reason contradictions do not make sense to us is that it is natural for us to continue in such a way as excluding them as meaningful—not seeing those forms as continuations of what came before. This does not mean, however, as in the case of the liar paradox, that there is necessarily anything wrong with our language if such inconsistencies arise: we simply do not know what to do in *that* case, but otherwise the language might work well and as long as we do not draw any conclusions from the contradiction, it is nothing to fear: a machine that sometimes jams, we might say, could very well do good work under most circumstances and be very useful.

In a previous lecture, Wittgenstein had made a similar point—bringing naturalness into the discussion yet again. He says that when we say that language ‘jams’ at these points, we don’t mean that people don’t react correctly, but that we do not *tend* to give contradictory orders any meaning. He then says:

> What I’m driving at is that we can’t say, “So-and-so is the logical reason why the contradiction doesn’t work.” Rather: that we exclude the contradiction and don’t normally give it a meaning, is characteristic of our whole use of language, and of a tendency not to regard, say, a hesitating action, or doubtful behaviour, as standing in the same series of actions as those which fulfil orders

> of the form “Do this and don’t do that”—that is of the form ‘*p*.~*p*’
>
> (LFM XVIII, 179)

Presumably, Wittgenstein means that given that our practice and “what we find natural to say” is constitutive of meaning, then it is not *logically* impossible to give meaning to statements of the form ‘$p \wedge \neg p$’. Rather, it would be a *different* logic, if we did so.

Wittgenstein’s overall point, I believe, is that either our logical propositions are used in some practice or not, e.g. to describe reality. If they are not, we do not have to worry about inconsistency if we abandon the *ex falso*-rule, since the calculus might be interesting for other reasons than practical and not necessarily incoherent. This is then merely a formal matter, and the existence of paraconsistent logics shows that we do in fact have a choice here (see e.g. LFM XXII, 213 for an expression of this kind of logical pluralism).

If on the other hand, our logical propositions *are* used in some practice, whether that practice has a practical point or not, and again assuming we do not draw arbitrary conclusions from the contradiction, then we simply either do not know what to do if a contradiction arises or we have assigned it a meaning, as in Turing’s case of the painters. If we have assigned it meaning, there is no reason to say that this meaning is *wrong* (from a logical point of view) and if we do not know what to do, our practice might fall into confusion or we otherwise run into difficulties, but not necessarily—we don’t with the liar, for example, but might with certain orders or descriptions.

If, however, a practice, which is otherwise very useful contains such a contradiction, that contradiction does very little to detract from the usefulness of the practice as a whole, up to the point the contradiction was discovered (and perhaps even not afterwards). An inconsistency is therefore not something we need to avoid in every possible case and at all costs. Consistency is merely one theoretical virtue (we might say) among others and has no special status.

3.1 Turings’s two objections

To these claims, Turing made essentially two objections: (a) even if we abandon the *ex falso*-rule, it is still possible to derive an arbitrary statement from a contradiction (LFM XXIII, 220) and (b) a contradiction might give us trouble in practice, leading to falsehoods accidentally being derived from true premises, leading to disaster (LFM XXII, 211).

I will not have much to say about the first objection, since it has already attracted quite a lot of attention in the literature. The standard response is to point to the possibility of paraconsistent logics that are well-behaved in this respect (see e.g. Berto, "Gödel Paradox"; Persichetti, "The Later Wittgenstein").

The other objection raised by Turing we might call the "falling bridges"-objection. The idea is roughly that if we were to try to apply an inconsistent calculus to some practical matter, such as building a bridge, we might go from a true statement to a false one without noticing, leading to catastrophe. More concretely, we might suppose that some engineers are building a bridge and use for this purpose an arithmetical calculus. The building of this bridge is a fairly complicated affair and the calculations of the engineers are very long. Now it happens that the calculus is inconsistent, and on one line of their calculations the engineers derive the proposition $12\times12 = 144$ and on another line they derive a proposition equivalent to $12\times12 \neq 144$. They have tried to follow Wittgenstein's advice of not deriving anything from a contradiction, but their calculations were just so complicated that they didn't notice that they had derived one. They accidentally derive an arbitrary proposition from their contradiction, resulting in a mistake in building the bridge, and it falls down.[8]

This objection, unlike Turing's first, has not been given much attention in the literature and when it has, it has either been seen as obviously fatal to Wittgenstein's conception (Chihara, "Wittgenstein's Analysis") or a mere extension of the former (Persichetti, "The Later Wittgenstein"). For Persichetti, Wittgenstein's tolerant attitude to contradictions is indeed rooted in his rejection of the correspondance of mathematical statements to any sort of external reality and since mathematical propositions are statements of

[8] It might seem that I've stacked the deck in Wittgenstein's favour by describing the problem in this way. But since we assume that we have adopted a well-behaved paraconsistent logic, any derivation of an arbitrary sentence from a contradiction counts as a miscalculation.

grammar, they will never conflict with reality—what matters is how we use them. Consequently, Wittgenstein's debate with Turing is mostly an exercise in philosophical therapy.

Persichetti's starting point in their debate is the following remark by Turing:

> [Turing:] If one takes Frege's symbolism and gives someone the technique of multiplying in it, then by using a Russell paradox he could get a wrong multiplication. (LFM XXII, 218)

Persichetti describes Turing's objection as follows:

> Turing notices a conflict between the Wittgensteinian account and the common practices of applied mathematics. He defends implicitly this thesis: if there is an applied calculation, it is in virtue of some isomorphism between the system employed and the reality (cf. LFM, XII p. 118, XIV pp. 138–139, XV p. 150). Consequently, if there is an isomorphism as a condition of possibility for the effectiveness of a calculation, then every contradiction in mathematics or logic will have an effect on my calculations in practice (cf. LFM, XXII p. 211). Hence, contradictions cannot be amended arbitrarily with the introduction of a rule because they must respect this isomorphism with reality. (Persichetti, "The Later Wittgenstein")

And since Wittgenstein rejects that mathematical statements are responsible to an external reality, this argument is supposed to have no force against him.

And indeed, Persichetti claims that Wittgenstein reply is precisely to remind Turing that mathematical statements are grammatical rules, but not empirical statements. Hence, if we simply change "the grammar governing the calculation" (Persichetti, "The Later Wittgenstein") and thereby our practice, the contradiction does not necessarily lead to trouble. It is only if such a change is ill-considered that the contradiction becomes problematic. Persichetti concludes:

> So, it is the fault of the technique employed and not of the contradiction alone. The collapse of the bridge concerns which

> technique we have picked up for that goal. Indeed, a grammar regulates a particular technique, which in turn is constituted by a network of rules; thence, it concerns which rule we have chosen in our grammar to deal with that contradiction. (Persichetti, "The Later Wittgenstein")

It is first and foremost how we *use* the contradiction, Persichetti thinks, that can cause havoc and if we just establish a use for the contradiction which is in good order, the problem is dissolved.

This is on the right track, but cannot be the full story, however. Nobody would ever claim that contradictions cause bridges to collapse without anyone using them to build bridges, and if that was Wittgenstein's claim, he hasn't made a very substantial point. For instance, if we had a consistent calculus that either was ill-suited to the task at hand or we ourselves misused (maybe by having made a calculation error), we wouldn't say that neither the bad calculus or our calculation error were irrelevant to our task of building a bridge and only the way we use the bad calculus or the mistaken calculation was the cause of the bridge collapsing. It is trivial that only the actual use of our tools (even abstract ones like a calculus) can be a causal factor in what subsequently happens. If we have a bad calculus and use it in such a way that no disaster results, then of course things work out well. Turing's point is precisely that if we *use* Frege's calculus as we would naturally do, then we are liable to run into trouble.

Pointing out that (i) logical statements do not correspond to a reality, however, is not a good reply to this argument, exactly because they are supposed to (ii) create the form of what we call descriptions. The point is this: if mathematical statements create the form of our empirical descriptions, the notion of correspondance comes back in a different way (and it better, because otherwise mathematics would always be useless!). If a general receives the (true) report that the enemy has 10,000 infantry and 3,000 cavalry, one way of explaining what is defective about his belief that the force of the enemy is 15,000 men is that it does not correspond to reality. This is not a point about realism or anti-realism in mathematics, but simply follows from the idea that mathematical statements create the form of what we call facts: the meaning of the statements in question is partially derived from our mathematical practices and are themselves a criterion for the general's belief corresponding to reality or not, and since 10,000 + 3,000 = 13,000, and not 15,000, the general has a false belief. The practice of using mathematical statements in this way is what it is for the corresponding empirical statements to correspond to reality, or so Wittgenstein seems to be saying. And hence, Turing would be right to be unconvinced by Wittgensteins's reasoning as Persichetti presents it, even by Wittgenstein's own lights.

So what was Wittgenstein's point? Wittgenstein gave a somewhat puzzling reply to Turing's point, posing the following dilemma:

> Now it does not sound quite right to say that a bridge might fall down because of a contradiction. We have an idea of the sort of mistake which would lead to a bridge falling.
>
> (a) We've got hold of a wrong natural law—a wrong coefficient.
>
> (b) There has been a mistake in calculation—someone has multiplied wrongly. (LFM XXII, 212)

The first case has obviously nothing to do with having a contradiction; and the second is not quite clear. After some discussion, Turing repeats his objection. He says:

> [Turing:] The sort of case I had in mind was the case where you have a logical system, a system of calculations, which you use in order to build bridges. You give this system to your clerks and they build a bridge with and the bridge falls down. You then find a contradiction in the system. (LFM XXII, 212)

To this Wittgenstein essentially makes the point that we do in fact avoid contradictions and indeed have to do so, given our training, but that is quite another matter than the conclusion that we *ought* to avoid contradictions in logic.

The debate then moves on to the case where we would never in fact notice that we were essentially multiplying in two different ways because the contradiction was 'hidden' or latent in the calculus. Wittgenstein says here that he understands why people should fear contradictions *outside* mathematics “in orders, descriptions, etc.” (see above) (LFM XXII, 217) but if something would go wrong, that would be akin to using a wrong natural law. He says,

> There seems to me to be an enormous mistake there. For your calculus gives certain results and you want the bridge not to break down. I’d say things can go wrong in two ways: either the bridge breaks down or you have made a mistake in your calculation—for example, you multiplied wrongly. But you seem to think there may be a third thing wrong: the calculus is wrong. (LFM XXII, 218)

This reply is slightly odd, because Wittgenstein seems to be saying that a bridge might break down because the bridge might break down. However, given the reply he gave to Turing before, he probably meant that either the bridge breaks down *because* of a wrong natural law being used, or a mistake being made. In either case, the answer isn’t very clear at all.

One commentator who is unhappy with Wittgenstein’s presentation of this dilemma is Charles Chihara, who sides with Turing. For him, Wittgenstein has indeed forgotten the third possibility: “(3) the logical system they used was unsound and led them to make invalid inferences (that is, they followed the rules of derivation correctly, but their calculus was wrong)” (Chihara, “Wittgenstein’s Analysis”, 377). He then claims that in this case, the collapse of the bridge was not due to wrong calculations nor faulty data. He goes on:

> In fact, as I have described the situation, if the engineers were to recheck their data and retest their empirical theories, they would find everything in order. Hopefully, there would be some nonWittgensteinian logicians around to discover the unsoundness of their logical system. (Chihara, “Wittgenstein’s Analysis”, 379)

But this is in fact not the case. The way the example is set up is exactly so that using a contradiction falls under a miscalculation (that is what Wittgenstein’s abolishment of the ex *falso-rule* means) and this comes out even stronger in Chiara’s own example, which is somewhat different than mine. In his example a computer is used to check the work of the engineers, and if that were the case, the computer would obviously have to be programmed to use a paraconsistent logic if it were to function according to Wittgenstein's recommendation at all. The computer would simply never make such a mistake as to derive something from a contradiction, were it properly programmed and in good working order.

That is to say, *if* what Chihara describes were to happen, and the engineers had made it a rule in their calculus that one should not draw conclusions from a contradiction (and made it in an appropriate way), they would find their mistake if they rechecked their calculations rigorously enough—there is no essential difference between checking whether one rule of inference was broken or another, and if the engineers had made some other calculation mistake (say, accidentally written $13 \times 13 = 144$ on one line), they would also find their data and empirical theories in order. Yet Chihara wouldn't say that this showed that their usual way of multiplying was wrong. Making a mistake in the calculation is what led them into trouble, not the contradiction. Chihara doesn't consider this objection by Wittgenstein, possibly because he phrased it somewhat obscurely, and he didn't have time to elaborate on it.

One might of course object that if the case is as we have described, the engineers might easily derive something like $F = ma$ on one line of their calculations and something like $F \neq ma$ on another. Isn't that enough for them to run into trouble in building the bridge?

That is of course true, but just brings us to the other strand of Wittgenstein's dichotomy—that the bridge might fall down because of the engineers applying a wrong natural law. On this point, the following exchange is of particular interest:

> *Wittgenstein*: If a contradiction may lead you into trouble, so may anything. It is no more likely to do so than anything else.
> *Turing*: You seem to be saying that if one uses a little common sense, one will not get into trouble.
> *Wittgenstein*: No, that is NOT what I mean at all.—The trouble described is something you get into in a way that leads to something breaking. This you can do with *any* calculation, contradiction or no contradiction. (LFM, XXII 219)

Wittgenstein then goes on to say that this cannot be a matter of common sense, unless *physics* is a question of common sense and if one gets the physics right, the bridge will not collapse.[9]

Wittgenstein's point here is, I believe, that it is not the contradiction that gave our engineers trouble, it is the fact that their calculus derived a false statement, $F \neq ma$, which doesn't accurately model the reality they were trying to describe with their calculus (in this case, the physical world as it pertains to bridge building). The problem here is then not that a contradiction was derived, but that something *false* was derived, namely $F \neq ma$. This is not to say that deriving a contradiction isn't a mistake, but that it is not a *special* kind of mistake that we should distinguish from any other kind of wrong modeling of reality by a formal system.

This is what Wittgenstein meant when he said in the lectures that one way the bridge could fall down was by getting "hold of a wrong natural law": deriving a contradiction isn't a special *kind* of mistake in a class of its own. The problem here isn't that the physics is inconsistent (although it would be if they insisted to draw any conclusion from that!) but that in fact, $F \neq ma$ is false! It's not that that they can derive both $F = ma$ *and* $F \neq ma$, which is problematic, but that they can derive $F \neq ma$, since after all, they would *want* to derive $F = ma$ as that correctly describes the relation between force, acceleration and mass they require in building their bridge, and not $F \neq ma$, as that incorrectly describes it. To put the point bluntly: if it is possible to derive a contradiction in some calculus and what it is supposed to apply to or describe (e.g. physical reality) is in fact not inconsistent, then the problem has to do with an incorrect description, i.e. the wrong axioms, relative to the practical application at hand, not some special mistake of deriving a contradiction—and in fact, the contradiction is parasitic on the falsehood, because there could never such a thing as deriving a contradiction unless at least one conjunct of it is false. There

[9] Juliet Floyd has connected Turing's explicit recognition of Wittgenstein's influence on him in the *Lectures* with this discussion (see Turing, "The Reform of Mathematical Notation", 245–249, Floyd, "Turing, Wittgenstein and Types", 250–253, "Chains of Life", and "Turing on 'Common Sense' ").

is no chance that the contradiction itself could lead us into trouble, only the falsehood.

One might however object, as Chiara seems to do with his suggestion of a third possibility, that the empirical part of the theory and the logical or arithmetical part must be kept strictly separate, and there can therefore be no question of whether or not *it* aptly describes the empirical phenomena or not.

But it doesn't seem *a priori* that *any* theoretical entity that engineers use in the empirical part of the theory should be correctly described by even our best arithmetic. Sure, 12 times 12 oranges are 144 oranges, etc. but is it *necessarily* the case that e.g. two forces obey the laws of vector multiplication, themselves derived from ordinary arithmetic? Would it not be *possible* to discover that the empirical phenomenon we describe by means of vectors, namely forces, are actually not aptly described by such means? If so, it is also an *a posteriori* matter that our arithmetical theories are apt for practical matters, such as bridge building.

This leads to a further objection. One might say that this is tantamount to giving empirical content to our arithmetical theories, in contradiction to claims (i) and (ii). I do not think that this follows, however. For Wittgenstein, or so I have claimed, our mathematical practices create the form of what we call descriptions, but that does not mean that the arithmetical statements themselves *are* those descriptions: the arithmetical proposition that 5+3=8 is not the same proposition as the empirical proposition, which relies on it, that five oranges plus three apples are eight pieces of fruit.

It is therefore true that the discovery that vector arithmetic did not in fact describe physical forces would most likely force us to abandon the practice of *using* vector arithmetic to describe forces in physics, but that does not mean that any particular proposition of vector arithmetic has thereby been falsified, just as Euclidian geometry would not be falsified were it discovered that the local geometry of space is not Euclidian. Hence, it is not our practice of vector arithmetic that has any empirical content, and no empirical investigation can falsify any of its propositions, but merely suggest an adoption of a new practice, with its own internal correctness conditions and its own set of true propositions.

It is in the light of the above arguments we should understand Wittgenstein's otherwise puzzling remarks about a hidden contradiction not being a matter to worry about until we discover it, since if the trouble that a contradiction can give us either amounts to a mistake in calculation or a wrong kind of natural law, and we grant, with Wittgenstein that we don't necessarily fall into trouble when we encounter paradoxes, then an inconsistent practice is perfectly useful, insofar as we know what to do (when we encounter the contradiction, we might be at a loss, however). The contradiction doesn't, as Wittgenstein puts it, vitate what came before (LFM XXI, 210).

This view is open to Wittgenstein, because on his view, arithmetic is not merely a body of truths, but a practice that has a *point*. We use it to weigh, mesure, count, play games and puzzles and so forth, and our arithmetical rules are derived from these practices and inform them (derived from experience and made independent from experience, as he puts it).

The fact that we presumable wouldn't stop counting and weighing in fear of the contradiction (and certainly wouldn't say that all our past instances of counting were not actually so) shows that such a contradiction wouldn't render our practice useless. From Wittgenstein's perspective, there is nothing odd or irrational about just continuing on as before, as we have no reason to think that the contradiction would thereby infect our whole language, or even other parts of that selfsame practice. It wouldn't then be an obvious mistake or puzzling in any way to simply ignore the contradiction as a mere mathematical curiosity, completely isolated from our own practice of arithmetic and happily go on with our life, counting, weighing and measuring (as most of us apparently do with the liar paradox).

Why do mathematicians then try to avoid contradictions when they are doing mathematics? Wittgenstein doesn't say much about that, apart from describing that as one rule of our mathematical practice (LFM XXI, 208) but from the picture of mathematics that he develops in the *Lectures*, I believe the answer is something as follows: Our most basic mathematical calculi have their origin in experience and made independent of it. They are not responsible to an external reality directly, but their rules derive from our experience and their theorems are used by us to judge that experience. From the way we use language and how it connects with reality through our training, we have no use for descriptions of reality that include contradictions, and so we make up our calculi so that they are excluded—we naturally do not find a use for them.

But if we accept (i) there is no *a priori* reason for mathematicians to always avoid contradictions on Wittgenstein's view, and indeed they do not, as work on inconsistent mathematics shows. There is, for instance, work on inconsistent models of arithmetic (see Priest, "Inconsistent Models for Arithmetic: I", "Inconsistent Models for Arithmetic: II") and set theory (see Weber "Paraconsistent Set Theory"), analysis (see McKubre-Jordens and Weber, "Real Analysis in Paraconsistent Logic") and even geometry (see Mortensen, *Inconsistent Geometry*).[10] These examples show that there is no reason to think that incoherence and inconsistency amount to the same thing, as the concepts involved in this work seem just as clear and useful for their purposes as those employed in classical mathematics.

The insistence of mathematicians for their theories to always be consistent would then, for Wittgenstein, either be (a) a requirement for those branches that lie close to experience, for practical reasons, (b) a natural outgrowth of the branches in (a) where the avoidance of contradiction acquires an aesthetic interest, (see RFM I, §167) or (c) at worst, a mere sociological norm that established itself from the practice of the two other cases, ending in what he calls a "superstitious dread" of contradictions by mathematicians (RFM I, app. III, §17).[11] At the very least, inconsistent mathematics cannot be said to be meaningless or incoherent *unless* one also ascribes to the negation of (i) where a consistent, external reality is the final judge of the value of mathematical theorems and proofs. Otherwise, inconsistent mathematics is also mathematics, however uninterested mathematicians in general are in it.

[10] For a general work, see Mortensen, *Inconsistent Mathematics*.

[11] See also RFM IV, §55: It is one thing to use the technique of avoiding contradictions, another to philosophise against them.

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