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Intersections of thick compact sets in $\mathbb{R}^d$

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Abstract
We introduce a definition of thickness in $\mathbb{R}^d$ and obtain a lower bound for the Hausdorff dimension of the intersection of finitely or countably many thick compact sets using a variant of Schmidt's game. As an application we prove that given any compact set in $\mathbb{R}^d$ with thickness τ, there is a number $N(\tau)$ such that the set contains a translate of all sufficiently small similar copies of every set in $\mathbb{R}^d$ with at most $N(\tau)$ elements; indeed the set of such translations has positive Hausdorff dimension. We also prove a gap lemma and bounds relating Hausdorff dimension and thickness.

Keywords Thickness · Intersections · Patterns · Dimension · Schmidt games · Gap Lemma

Mathematics Subject Classification MSC 11B25 · MSC 28A12 · MSC 28A78 · MSC 28A80

Contents

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1 Introduction

The classical co-dimension formula states that if C_1, C_2 are submanifolds of $\mathbb{R}^d$ that intersect transversally then

$$\dim(C_1 \cap C_2) = \dim(C_1) + \dim(C_2) - d \tag{1}$$

provided the right hand side is non-negative, where dim denotes the dimension of the manifolds. There are various versions of (1) that are applicable in other settings, in particular for more general sets using Hausdorff dimension $\dim_H$. For example, for compact sets $C_1, C_2 \subset \mathbb{R}^d$

$$\dim_H(C_1 \cap (C_2 + x)) \leq \max\{0, \dim_H(C_1 \times C_2) - d\} \tag{2}$$

for Lebesgue almost-all $x \in \mathbb{R}^d$; the right-hand side can be replaced by $\max\{0, \dim_H(C_1) + \dim_H(C_2) - d\}$ if, for example, either C_1 or C_2 has equal Hausdorff and upper box-counting dimension, see [17]. On the other hand, for all $\epsilon > 0$,

$$\dim_H(C_1 \cap \sigma(C_2)) \geq \max\{0, \dim_H(C_1) + \dim_H(C_2) - d - \epsilon\} \tag{3}$$

for a set of similarities σ of positive measure with respect to the natural measure on the group of similarities σ on $\mathbb{R}^d$. The similarity group may be replaced by the group of isometries if $\dim_H C_1 > (d+1)/2$ (it is not known if this condition is necessary if $d \geq 2$), see [12, 17]. The disadvantage of these results is that they are measure theoretic, and tell us nothing about which particular similarities or isometries these inequalities are valid for.

At the other extreme, there are classes $\mathcal{C}$ of 'limsup sets' of Hausdorff dimension $0 < s < d$ which are dense in $\mathbb{R}^d$ with the property that the intersection of any countable collection of similar copies of sets in $\mathcal{C}$ still has Hausdorff dimension s, see for example [8].

It is natural to ask for specific conditions on compact sets that are 'close enough' to each other that guarantee non-empty intersection, or even give a lower bound for the dimension of their intersection. For subsets of the real line Newhouse [19] introduced a notion of thickness, see Definition 1, which depends on the relative sizes of the complementary open intervals of the set and showed that two Cantor-like sets, with neither contained in a gap of the other, must intersect if the product of their thickness is greater than 1, see Theorem 2.

In this paper we propose a definition of thickness for compact subsets of $\mathbb{R}^d$ for all $d \geq 1$. We obtain a higher dimensional gap lemma, and show that given several compact sets in $\mathbb{R}^d (d \geq 1)$ that are not too far apart in a sense that will be made precise, if their thicknesses are large enough then they have non-empty intersection, and we obtain a lower bound for the Hausdorff dimension of this intersection.

We first review the definition of thickness for subsets of the real line. Recall that every compact set C on the real line can be constructed by starting with a closed interval $I \equiv I_1$ (the convex hull of C) and successively removing disjoint open complementary intervals (they are the path-connected components of the complement of C). Clearly there are finitely or countably many disjoint open complementary intervals $(G_n)_n$, which we may assume are ordered so that their lengths $|G_n|$ are non-increasing; if several intervals have the same length, we order them arbitrarily. The two unbounded path-connected components of $\mathbb{R} \setminus C$ are not included. For each $n \in \mathbb{N}$ the interval G_n is a subset of some closed path-connected component I_n of $I \setminus (G_1 \cup \cdots \cup G_{n-1})$. We say that such a G_n is *removed* from I_n.

Definition 1 (Thickness in $\mathbb{R}$) Let $C \subset \mathbb{R}$ be compact with convex hull I, and let $(G_n)_n$ be the ordered sequence of open intervals comprising $I \setminus C$. Each G_n is removed from a closed interval I_n, leaving behind two closed intervals L_n and R_n; the left and right intervals of $I_n \setminus G_n$. The *thickness* of $C \subset \mathbb{R}$ is defined as

$$\tau(C) := \inf_{n \in \mathbb{N}} \frac{\min\{|L_n|, |R_n|\}}{|G_n|}.$$

The sequence of complementary intervals $(G_n)_n$ may be finite, in which case the infimum is taken over the finite set of indices.

The thickness of a single point is taken to be 0, and that of a non-degenerate interval to be $+\infty$.

If there are several complementary intervals of equal length, then the ordering of them does not affect the value of $\tau(C)$. See [1, 11, 20, 22] for more information on Newhouse thickness and alternative definitions.

Theorem 2 (Newhouse's Gap Lemma) *Given two compact sets $C_1, C_2 \subset \mathbb{R}$, such that neither set lies in a gap of the other, if $\tau(C_1)\tau(C_2) > 1$ then*

$$C_1 \cap C_2 \neq \emptyset.$$

Theorem 2 was proved only for subsets of $\mathbb{R}$ and it does not guarantee positive Hausdorff dimension of the intersection, nor does it generalise in any simple way to intersections of 3 or more sets.

Here we give a definition of thickness for compact subsets of $\mathbb{R}^d$ that enables us to generalize Theorem 2 to higher dimensions, and also obtain lower bounds for the Hausdorff dimension of the intersection of several sets. For a different definition of thickness for certain dynamically defined subsets of the complex plane see [3].

Our setting throughout the paper is as follows. Given a compact subset C of $\mathbb{R}^d$, we define $(G_n)_{n=1}^{\infty}$ to be the (at most) countably many open bounded path-connected components of C^C and E to be the unbounded open path-connected component of C^C (except when $d = 1$ when E consists of two unbounded intervals). We call E together with G_n ($n \in \mathbb{N}$) the *gaps* of C. We may assume that the sequence of gaps $(G_n)_{n=1}^{\infty}$ is ordered by non-increasing diameter. Note that we make no assumption about the connectedness or simply connectedness of C.

We write dist for the usual distance between points or non-empty subsets of $\mathbb{R}^d$ and diam for the diameter of a non-empty subset of $\mathbb{R}^d$.

Definition 3 (Thickness in $\mathbb{R}^d$) We define the *thickness* of C to be

$$\tau(C) := \inf_{n \in \mathbb{N}} \frac{\operatorname{dist}(G_n, \bigcup_{1 \leq i \leq n-1} G_i \cup E)}{\operatorname{diam}(G_n)},$$

provided that E is not the only path-connected component of C.

When the only complementary path-connected component is E, we define

$$\tau(C) := \begin{cases} +\infty & \text{if } C^\circ \neq \emptyset \\ 0 & \text{if } C^\circ = \emptyset \end{cases} \tag{4}$$

We say C is *thick* if $\tau(C) > 0$.

If the sequence of complementary intervals $(G_n)_n$ is finite then the infimum is taken over the finite set of indices. Moreover, thickness is well-defined in the sense that if two gaps have the same diameter, interchanging their positions in the ordering does not change the definition of thickness.

Note that $\tau \in [0, +\infty]$. Also, τ is invariant under homothetic maps, and agrees with the usual definition of thickness in the real line (recall Definition 1).

Observation 4 *If $C \subset \mathbb{R}^d$ is a thick compact set, then either there are finitely many gaps $(G_n)_n$ or $\lim_{n\to\infty} \operatorname{diam} G_n = 0$. To see this we can assume that E is not the only complementary path-connected component. If $\operatorname{diam} G_n \geq c > 0$ for infinitely many n, taking points $x_n \in G_n$ with $dist(x_n, x_i) \geq c\tau(C)$ for $1 \leq i < n$ contradicts the sequential compactness of E^C.*

In Sect. 2, we obtain a higher dimensional gap lemma, Theorem 10. The gap lemma does not generalize in any simple way to intersections of three or more sets, so we need to use other methods to study such intersections. To achieve this we obtain lower bounds for the Hausdorff dimension of the intersection of several thick compact sets in terms of their thicknessess, which is easy to estimate in many cases.

Our main theorem, Theorem 6 which will follow from Theorem 18 which relates thickness to ‘winning sets’.

The following constants appear in many of our results:

Definition 5 In $\mathbb{R}^d$ $(d \geq 1)$, let

$$K_1 := \frac{2d(24\sqrt{d})^d \log(16\sqrt{d})}{1 - \frac{1}{2^d}} \quad \text{and} \quad K_2 := \left(\frac{(24\sqrt{d})^d (1 + 4^d 2)}{1 - \frac{1}{2^d}} \right)^2. \tag{5}$$

We now state our main theorem which will follow from applying Theorem 18 on ‘winning sets’ to thickness. We write E_i for the unbounded open path-connected component of C_i^C (the union of two unbounded intervals when $d = 1$).

Theorem 6 (Intersection of compact sets in $\mathbb{R}^d$) *Let $(C_i)_i$ be a family of countably many compact sets in $\mathbb{R}^d$, where C_i has thickness $\tau_i > 0$, such that:*

(i) $\sup_i \operatorname{diam}(C_i) < +\infty$,
(ii) *there is a ball B such that $B \cap E_i = \emptyset$ for every i, where E_i is the unbounded component of C_i^C,*
(iii) *there exists $c \in (0, d)$ such that*

$$\sum_i \tau_i^{-c} \leq \frac{1}{K_2} \beta^c (1 - \beta^{d-c})$$

where

$$\beta := \min \left\{ \frac{1}{4}, \frac{\operatorname{diam}(B)}{\sup_i \operatorname{diam}(C_i)} \right\}.$$

Then

$$\dim_H \left(B \cap \bigcap_i C_i \right) \geq d - K_1 \frac{\left(\sum_i \tau_i^{-c}\right)^{d/c}}{\beta^d |\log(\beta)|} > 0.$$

Note that condition (iii) comes from Theorem 18 and is needed both to obtain the lower bound for the dimension of the intersection and to ensure that this bound is positive.

The significance of Theorem 6 is that a condition on thicknesses can give a lower bound for the dimension of intersection of a finite or countable collection of sets in $\mathbb{R}^d$ so ensure that the intersection is non-empty. In practice, the thicknesses needed are rather large as a consequence of the large constants K_1 and K_2.

A very active research area involves finding conditions on a set that guarantees the set contains homothetic copies of a given finite set of points, called a *pattern* in this context. It will follow from Theorem 6 that a set contains homothetic copies of any given pattern in $\mathbb{R}^d$

provided it is sufficiently thick. Patterns and intersections are related: the set C contains a homothetic copy of $A := \{a_1, \ldots, a_n\}$ if and only if there exists $\lambda \neq 0$ such that $\bigcap_{1\le i\le n}(C - \lambda a_i) \neq \emptyset$.

A consequence of the Lebesgue density theorem is that any set $E \subset \mathbb{R}^d$ of positive Lebesgue measure contains a homothetic copy of every finite set at all sufficiently small scales, so it is natural to seek conditions on sets of zero Lebesgue measure form which this remains true. Perhaps the most natural notion of size to consider is Hausdorff dimension but there are constructions (see for example [6, 13, 14, 16, 18, 21]) which indicate that Hausdorff dimension cannot, in itself, detect the presence or absence of patterns in sets of Lebesgue measure zero, even in the most basic case of points in arithmetic progressions.

Łaba and Pramanik [15] showed that if, in addition to having large Hausdorff dimension, a subset of $\mathbb{R}$ supports a probability measure with appropriate Fourier decay, then it contains arithmetic progressions of length 3. The hypotheses were relaxed and the family of patterns covered greatly enlarged in subsequent papers [5, 9, 10]. This work uses harmonic analysis, and such methods do not work easily for longer arithmetic progressions. Moreover, the hypotheses may be difficult to check, and are not even known to hold for natural classes of fractals such as central self-similar Cantor sets.

Yavicoli [22] showed that Newhouse thickness, Definition 1, allows the detection of homothetic and more general copies of patterns inside fractal sets in the real line. Newhouse thickness is easy to compute or estimate for many classical fractal sets such as self-similar sets or sets defined in terms of continued fraction coefficients. Our notion of thickness in higher dimensions, Definition 3, enables such results to be extended to $\mathbb{R}^d$.

Theorem 7 *Let $C \subset \mathbb{R}^d$ be a compact set with thickness $\tau := \tau(C)$, such that E^C contains a ball B. Then C contains a homothetic copy of every set A with at most*

$$N(\tau) := \left\lfloor \frac{\beta^d |\log(\beta)|}{eK_2} \frac{\tau^d}{\log(\tau)} \right\rfloor \tag{6}$$

elements, where

$$\beta := \min\left\{\frac{1}{4}, \frac{15\,\mathrm{diam}(B)}{16\,\mathrm{diam}(C)}\right\}.$$

and K_2 is as in (5).

Moreover, for all $\lambda \in \left(0, \frac{\mathrm{diam}(B)}{16\,\mathrm{diam}(A)}\right)$, there exists a set X of positive Hausdorff dimension (depending on A, B, C and λ) such that

$$x + \lambda A \subseteq C \text{ for all } x \in X.$$

We also discuss the relationship between Hausdorff dimension and thickness of a set. It is shown in [20,p.77] that for $C \subset \mathbb{R}$,

$$\dim_H(C) \ge \frac{\log 2}{\log(2 + 1/\tau(C))}, \tag{7}$$

and in Sect. 6 we obtain analogous lower bounds for $C \subset \mathbb{R}^d$.

2 A gap Lemma in $\mathbb{R}^d$

In this section we extend Theorem 2, Newhouse's gap lemma on $\mathbb{R}$, to $\mathbb{R}^d$ for $d \ge 2$. We first study a particular case when the gaps are either linked or do not intersect; in this setting we can use an analogous argument to Newhouse's proof.

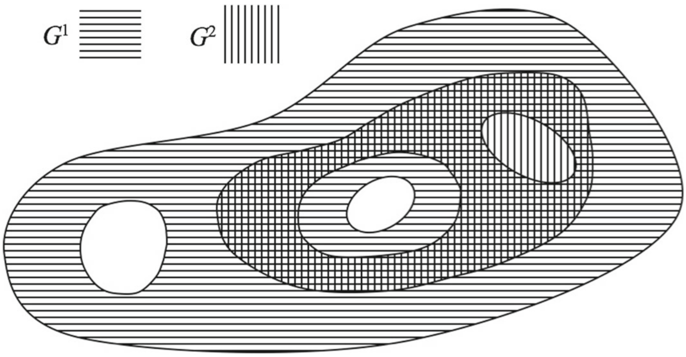

Fig. 1 An example of gaps G^1 and G^2 which intersect but are not linked and which might be parts of the complements of compact sets C_1 and C_2 which satisfy the hypotheses of Theorem 10 but not of Proposition 9

We denote the boundary of $U \subset \mathbb{R}^d$ by ∂U.

Definition 8 We say that $U \subseteq \mathbb{R}^d$ and $V \subseteq \mathbb{R}^d$ are *linked gaps* if $U \cap V \neq \emptyset$, $(\partial U) \setminus V \neq \emptyset$ and $(\partial V) \setminus U \neq \emptyset$.

We say that C_1 and C_2 are *linked compact sets* in $\mathbb{R}^d$ if for every pair of gaps G^1 and G^2 of C_1 and C_2 respectively we have that either their intersection is empty or they are linked gaps.

We first obtain the conclusion when C_1 and C_2 are linked compact sets and then in Theorem 10 we reduce to this case the weaker condition that neither C_1 or C_2 is contained in any gap of the other. Figure 1 illustrates how gaps may satisfy the hypotheses of Theorem 10 but not of Proposition 9.

Proposition 9 *Let C_1 and C_2 be linked compact sets in $\mathbb{R}^d$, with $\tau(C_1)\tau(C_2) > 1$, then $C_1 \cap C_2 \neq \emptyset$.*

Proof By definition of τ,

$$\tau_1 := \tau(C_1) := \inf_m \frac{\operatorname{dist}\big(G_m^1, \bigcup_{1\le i\le m-1} G_i^1 \cup E_1\big)}{\operatorname{diam}(G_m^1)}$$

and

$$\tau_2 := \tau(C_2) := \inf_n \frac{\operatorname{dist}\big(G_n^2, \bigcup_{1\le i\le n-1} G_i^2 \cup E_2\big)}{\operatorname{diam}(G_n^2)}$$

where C_1 and C_2 have gaps G_n^1 and G_n^2 and external path-connected components E_1 and E_1 respectively.

We assume that $C_1 \cap C_2 = \emptyset$ and will obtain a contradiction. Then,

$$C_1 \subseteq C_2^C = \bigcup_i G_i^2 \cup E_2 \text{ and } C_2 \subseteq C_1^C = \bigcup_i G_i^1 \cup E_1.$$

We will construct inductively a sequence $(U_i, V_i)_{i\in\mathbb{N}}$ of pairs of linked bounded gaps that occur in the construction of C_1 and C_2 respectively, such that either $\operatorname{diam} U_i \to 0$ or $\operatorname{diam} V_i \to 0$ (or both).

To start the induction: We will define (U_1, V_1) linked bounded gaps.

Since E_1 and E_2 are linked gaps, there is $x_1 \in \partial E_1 \setminus E_2 \subseteq C_1 \setminus E_2 \subseteq C_2^C \setminus E_2$, so there is a bounded gap $V_1 := G^2_{n_1}$ of C_2 such that $x_1 \in G^2_{n_1}$. Since E_1 and V_1 intersect and C_1 and C_2 are linked compact sets, E_1 and V_1 are linked gaps. Hence, there is $x_2 \in \partial V_1 \setminus E_1 \subseteq C_2 \setminus E_1 \subseteq C_1^C \setminus E_1$, so there is a bounded gap $U_1 := G^1_{m_1}$ of C_1 such that $x_2 \in G^1_{m_1}$. Since U_1 and V_1 are gaps that intersect, and C_1 and C_2 are linked compact sets, U_1 and V_1 are linked.

Inductive step: Given that we have defined a pair of linked gaps (U_k, V_k) of C_1 and C_2 defined, we now define (U_{k+1}, V_{k+1}).

Since (U_k, V_k) is a pair of linked gaps, there is $a_k \in \partial U_k \setminus V_k$. Since $a_k \in \partial U_k$, we have $a_k \in C_1$, hence by assumption $a_k \notin C_2$, so there is a gap $G^2_{n_k}$ of C_2 such that $a_k \in G^2_{n_k}$. Note that $(U_k, G^2_{n_k})$ are linked because they intersect and C_1 and C_2 are linked.

In the same way, since (U_k, V_k) is a pair of linked gaps there is $b_k \in \partial V_k \setminus U_k$. Since $b_k \in \partial V_k$, then $b_k \in C_2$, hence $b_k \notin C_1$, so there is $G^1_{m_k}$ a gap of C_1 such that $b_k \in G^1_{m_k}$ Again $(G^1_{m_k}, V_k)$ are linked.

We will show that we can choose (U_{k+1}, V_{k+1}) to be either $(U_k, G^2_{n_k})$ or $(G^1_{m_k}, V_k)$ in such a way the diameters of either U_k or V_k tends to 0.

We observe that for a fixed pair $n, m \in \mathbb{N}$ the following two inequalities cannot hold simultaneously:

- $\operatorname{dist}(G^1_m, \bigcup_{1\le i\le m-1} G^1_i \cup E_1) \le \operatorname{diam}(G^2_n)$
- $\operatorname{dist}(G^2_n, \bigcup_{1\le i\le n-1} G^2_i \cup E_2) \le \operatorname{diam}(G^1_m)$.

For if both hold, then by definition of thickness,

$$\operatorname{diam}(G^2_n) \ge \tau_1 \operatorname{diam}(G^1_m) \ \text{ and } \ \operatorname{diam}(G^1_m) \ge \tau_2 \operatorname{diam}(G^2_n).$$

Using the hypothesis that $\tau_1\tau_2 > 1$,

$$\operatorname{diam}(G^1_m) \ge \tau_2 \operatorname{diam}(G^2_n) \ge \tau_1\tau_2 \operatorname{diam}(G^1_m) > \operatorname{diam}(G^1_m),$$

which is a contradiction.

The gaps U_k and V_k can be identified as $U_k := G^1_m$ and $V_k := G^2_n$ for some $n, m \in \mathbb{N}$. In the case $\operatorname{dist}(G^1_m, \bigcup_{1\le i\le m-1} G^1_i \cup E_1) > \operatorname{diam}(G^2_n)$, we also know that (U_k, V_k) are linked, so $\overline{V_k}$ does not intersect E_1 or G^1_i for every $1 \le i \le m-1$. Also $b_k \in \partial V_k \setminus U_k \subseteq (\bigcup_{1\le i\le m-1} G^1_i \cup E_1)^C \cap C_1^C$. Then b_k belong to a bounded gap $G^1_{m_k}$ with $m_k > m$, and we take $(U_{k+1}, V_{k+1}) := (G^1_{m_k}, V_k)$.

In the case $\operatorname{dist}(G^1_m, \bigcup_{1\le i\le m-1} G^1_i \cup E_1) \le \operatorname{diam}(G^2_n)$, by the previous observation we have $\operatorname{dist}(G^2_n, \bigcup_{1\le i\le n-1} G^2_i \cup E_2) > \operatorname{diam}(G^1_m)$. Analogously to the previous case a_k belong to a bounded gap $G^2_{n_k}$ with $n_k > n$, and we take $(U_{k+1}, V_{k+1}) := (U_k, G^2_{n_k})$.

Since one or other of these cases occurs infinitely many times, we get a sequence (U_k, V_k) of linked gaps of C_1 and C_2, where at least one of the diameter sequences tends to 0. Assume, by symmetry, that $\operatorname{diam}(U_k) \to 0$. Take $x_k \in \partial U_k \subseteq C_1$, and $y_k \in U_k \cap \partial V_k \subseteq C_2$. Then,

$$\operatorname{dist}(x_k, y_k) \le \operatorname{diam}(U_k) \to 0.$$

Since $(x_k)_k \subseteq C_1$ there exists $(x_{k_j})_j$ a subsequence $(x_{k_j})_j$ convergent to $x \in C_1$. Since $(y_{k_j})_j \subseteq C_2$, we also get $(y_{k_j})_j \to x \in C_2$. So $x \in C_1 \cap C_2$ contradicting the assumption that $C_1 \cap C_2 = \emptyset$. □

We can now relax the hypotheses of Proposition 9.

Theorem 10 (Gap Lemma in $\mathbb{R}^d$) *Let C_1 and C_2 be compact sets in $\mathbb{R}^d$ such that neither of them is contained in a gap of the other and $\tau(C_1)\tau(C_2) > 1$. Then $C_1 \cap C_2 \neq \emptyset$.*

Proof We write $\tau_1 := \tau(C_1)$ and $\tau_2 := \tau(C_2)$. By hypothesis, C_1 and C_2 are thick compact sets.

We will show that if the theorem is not trivially true then there are sets $\widetilde{C}_1$ and $\widetilde{C}_2$ with thicknesses $\tilde{\tau}_1 \geq \tau_1$ and $\tilde{\tau}_2 \geq \tau_2$ that satisfy the conditions of Proposition 9 such that $\widetilde{C}_1 \cap \widetilde{C}_2 = C_1 \cap C_2$, from which the theorem follows immediately. We do this using a sequence of steps to modify the sets so that we can assume that the sets satisfy such stronger conditions.

Note that in these steps G^1 will always be a gap of C_1 and G^2 will be a gap of C_2; such gaps may be unbounded unless stated otherwise.

(0) **We may assume that there is at least one bounded gap in the construction of C_1, and similarly for C_2.** Otherwise $C_1 = E_1^C$. But by hypothesis C_2 is not contained in E_1, so $C_1 \cap C_2 \neq \emptyset$ and the theorem is trivially true.

(1) **We may assume that $\partial G^1 \cap \partial G^2 = \emptyset$ for all gaps G^1 and G^2 of C_1 and C_2 respectively.** Otherwise there exist gaps G^1 and G^2 of C_1 and C_2 such that $\partial G^1 \cap \partial G^2 \neq \emptyset$, so $C_1 \cap C_2 \neq \emptyset$ and the theorem is trivially true.

(2) **We may assume that $\partial E_1 \nsubseteq E_2$ and $\partial E_2 \nsubseteq E_1$.** Otherwise $\partial E_1 \subseteq E_2$ (or vice-versa). Since C_1 and C_2 are compact, there exists a closed ball $B_R(x)$ such that $C_1 \cup C_2 \subseteq B_R(x)$. We define $r := R/(2\tau_2 + 1) \in (0, R)$, $\tilde{x} \in \mathbb{R}^d$ such that $\text{dist}(x, \tilde{x}) > 2R$ and $\widetilde{C}_2 := C_2 \cup \overline{B_R(\tilde{x}) \setminus B_r(\tilde{x})}$. Thus the external path-connected component of $\widetilde{C}_2$ is $\widetilde{E}_2 = E_2 \setminus \overline{B}_R(\tilde{x})$, and there is a new gap $G_2 := B_r(\tilde{x})$ that was not in the construction of C_2. Then $\tau_2 = \tilde{\tau}_2$ by definition of r.
We take $\tilde{r} \in (0, r)$ and define $\tilde{C}_1 := C_1 \cup \overline{B}_{\tilde{r}}(\tilde{x})$. Then the external path-connected component of $\tilde{C}_1$ is $\tilde{E}_1 := E_1 \setminus B_{\tilde{r}}(\tilde{x})$ and $\tilde{\tau}_1 = \tau_1$.
By construction $\widetilde{C}_1$ and $\widetilde{C}_2$ are compact sets, with the same thicknesses as C_1 and C_2, such that $\widetilde{C}_1 \cap \widetilde{C}_2 = C_1 \cap C_2$, and $\partial \widetilde{E}_1 \nsubseteq \widetilde{E}_2$ and $\partial \widetilde{E}_2 \nsubseteq \widetilde{E}_1$.

(3) **We may assume that no bounded gap of C_1 is contained in a gap of C_2, and vice-versa.** If there are bounded gaps G_i^1 of C_1 contained in bounded gaps G_j^2 of C_2, we set

$$\widetilde{C}_1 := C_1 \cup \bigcup_j \bigcup_{G_i^1 \subseteq G_j^2} G_i^1;$$

thus $\widetilde{C}_1$ is obtained from C_1 by 'filling in' the gaps that are contained in a gap of C_2. Then $\widetilde{C}_1$ is compact with $\tilde{\tau}_1 \geq \tau_1$ and $\widetilde{C}_1 \cap C_2 = C_1 \cap C_2$ and no gaps of $\widetilde{C}_1$ are contained in gaps of C_2.
Now we can apply the same argument to C_2 and $\widetilde{C}_1$ (filling in certain gaps of C_2) to obtain a set $\widetilde{C}_2$. Hence, $\widetilde{C}_1$ and $\widetilde{C}_2$ are compact sets such that $\widetilde{\tau}_1\widetilde{\tau}_2 > 1$ and $\widetilde{C}_1 \cap \widetilde{C}_2 = C_1 \cap C_2$.

(4) **We may assume that there are no bounded gaps G^1 of C_1 such that $\partial G^1 \subseteq G^2$ and $G^1 \nsubseteq G^2$ for any gap G^2 of C_2, and vice-versa.** If this is not the case, we can inductively replace each gap G^2 of C_2 by $\widetilde{G}^2 := G^2 \cup \bigcup_{\partial G_j^1 \subseteq G^2} G_j^1$ (since $\partial G^1 \subseteq G^2$ this is intuitively G^2 with some holes filled in). Then $\text{diam}(G^2) = \text{diam}(\widetilde{G}^2)$, possibly infinity. Note that a priori in this new sequence of gaps we could have gaps contained in another gap, but that can be easily fixed by removing (in order of the sequence) the gaps that are contained in a previous gap. In this way we obtained a compact set $\widetilde{C}_2$ that satisfies $\tilde{\tau}_2 \geq \tau_2$ and $C_1 \cap C_2 = C_1 \cap \widetilde{C}_2$.
In a symmetric manner, we may repeat this procedure with C_1 and $\widetilde{C}_2$ to obtain $\widetilde{C}_1$ and $\widetilde{C}_2$ with $\tilde{\tau}_1 \geq \tau_1$ and $\tilde{\tau}_2 \geq \tau_2$ and $C_1 \cap C_2 = \widetilde{C}_1 \cap \widetilde{C}_2$, and such that all gaps satisfy condition (4). (There may remain gaps of $\widetilde{C}_1$ fully contained in gaps of $\widetilde{C}_2$ or vive-versa, and these may be removed by Step (3).)

(5) **We may assume that $\partial G^1 \not\subseteq G^2$ for every bounded gap G^1 of C_1 and every gap G^2 of C_2, and vice-versa.** This combines Steps 3 and 4.
(6) **We may assume that $\partial G^1 \not\subseteq G^2$ and $\partial G^2 \not\subseteq G^1$ for all gaps G^1 of C_1 and all gaps G^2 of C_2.** This means that we can assume that C_1 and C_2 satisfy the hypothesis of Theorem 9.
We consider in turn the cases when G^1 and G^2 are unbounded and bounded gaps.

- Case $G^1 = E_1$ and $G^2 = E_2$: was proved in Step 2.
- Case G^1 bounded and $G^2 = E_2$: By Step 5 we have that $\partial G^1 \not\subseteq E_2$. To check that $\partial E_2 \not\subseteq G^1$, note that if $\partial E_2 \subseteq G^1$ with G^1 a bounded gap of C_1, then $C_2 \subseteq G^1$, contradicting the gap containment hypothesis of this Theorem.
- Case G^2 bounded and $G^1 = E_1$: as in the previous case.
- Case G^1 and G^2 bounded: was proved in Step 5.

Thus we can replace C_1 and C_2 by a pair of sets with the same intersection and at least the same thicknesses which satisfy the hypotheses of Proposition 9, and applying it completes this proof. □

3 Thickness and winning sets

Schmidt's game and its variants are a powerful tool for investigating properties of intersections of sequences of sets, see [2] for a survey. We will define a game and prove that every set with positive thickness can be seen as a winning set with certain parameters for the game. We will show that game has good properties, for example monotonicity in its parameters, invariance under similarities, and that the intersection of winning sets is again a winning set with different parameters. Theorem 18, proved in the Appendix, gives a lower bound for the Hausdorff dimension of winning sets for this game and this leads to Theorem 6 on the dimension of intersections.

Definition of the Game

We define a game in $\mathbb{R}^d$ similar to the potential game from [4] but adapted to our purposes:

Definition 11 Given $\alpha, \beta, \rho > 0$ and $c \geq 0$, Alice and Bob play the (α, β, c, ρ)-game in $\mathbb{R}^d$ under the following rules:

- For each $m \in \mathbb{N}_0$ Bob plays first, and then Alice plays.
- On the m-th turn, Bob plays a closed ball $B_m := B[x_m, \rho_m]$, satisfying $\rho_0 \geq \rho$, and $\rho_m \geq \beta\rho_{m-1}$ and $B_m \subseteq B_{m-1}$ for every $m \in \mathbb{N}$.
- On the m-th turn Alice responds by choosing and erasing a finite or countably infinite collection $\mathcal{A}_m$ of open sets. Alice's collection must satisfy $\sum_i (\operatorname{diam} A_{i,m})^c \leq (\alpha\rho_m)^c$ if $c > 0$, or $\operatorname{diam} A_{1,m} \leq \alpha\rho_m$ if $c = 0$ (in the case $c = 0$ Alice can erase just one set).
- $\lim_{m\to\infty} \rho_m = 0$ (Note that this is a non-local rule for Bob. One can define the game without this rule, adding that Alice wins if $\lim_{m\to\infty} \rho_m \neq 0$. But to make the definitions simpler we added this condition as a rule for Bob.)

Alice is allowed not to erase any set, or equivalently to pass her turn.

There exists a single point $x_\infty = \bigcap_{m\in\mathbb{N}_0} B_m$ called the *outcome of the game*. We say a set $S \subset \mathbb{R}^d$ is an (α, β, c, ρ)-*winning set*, or just a *winning set* when the game is clear, if Alice has a strategy guaranteeing that if $x_\infty \notin \bigcup_{m\in\mathbb{N}_0} \bigcup_i A_{i,m}$, then $x_\infty \in S$.

Note that the conditions $B_0 \supseteq B_1 \supseteq \cdots$ and $\lim_{m\to\infty} \rho_m = 0$ imply $\beta < 1$.

Good properties of the game

Proposition 12 (Countable intersection property) *Let J be a countable index set, and for each $j \in J$ let S_j be an $(\alpha_j, \beta, c, \rho)$-winning set, where $c > 0$. Then, the set $S := \bigcap_{j\in J} S_j$ is (α, β, c, ρ)-winning where $\alpha^c = \sum_{j\in J} \alpha_j^c$ (assuming that the series converges).*

To see this, it is enough to consider the following strategy for Alice: in the turn k she plays the union over j of all the strategies of turn k.

Proposition 13 (Monotonicity) *If S is (α, β, c, ρ)-winning and $\tilde{\alpha} \geq \alpha$, $\tilde{\beta} \geq \beta$, $\tilde{c} \geq c$ and $\tilde{\rho} \geq \rho$, then S is $(\tilde{\alpha}, \tilde{\beta}, \tilde{c}, \tilde{\rho})$-winning.*

This holds because

$$\Big(\sum_i \alpha_i^{\tilde{c}}\Big)^{1/\tilde{c}} \leq \Big(\sum_i \alpha_i^{c}\Big)^{1/c} \text{ when } c \leq \tilde{c},$$

so Alice can answer in the $(\tilde{\alpha}, \tilde{\beta}, \tilde{c}, \tilde{\rho})$-game using her strategy to answer from the (α, β, c, ρ)-game.

Proposition 14 (Invariance under similarities) *Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a similarity satisfying*

$$dist(f(x), f(y)) = \lambda dist(x, y) \;\; \text{for all} \;\; x, y \in \mathbb{R}^d.$$

Then a set S is (α, β, c, ρ)-winning if and only if the set $f(S)$ is $(\alpha, \beta, c, \lambda\rho)$-winning.

This holds by "translating" the strategies being played through f.

Remark 15 (Relationship with the potential game in [4]) Let $\mathcal{P}$ be the set of singletons in $\mathbb{R}^d$. Since every set A is contained in a ball of radius $\mathrm{diam}(A)$, if $S \subseteq \mathbb{R}^d$ is an (α, β, c, ρ)-winning set, then it is an $(\alpha, \beta, c, \rho, \mathcal{P})$-potential winning set in the game defined in [4].

Relationship between thickness and winning sets

We now establish the key property that relates winning sets to thickness.

Proposition 16 *Let $C \subset \mathbb{R}^d$ be compact with unbounded complement E and write $S := C \cup E$. If $\tau := \tau(C) > 0$, then S is $\big(\frac{1}{\tau\beta}, \beta, 0, \frac{\beta\,\mathrm{diam}(C)}{2}\big)$-winning for every $\beta \in (0, 1)$.*

Proof We first describe a strategy for Alice. Given a move B by Bob, how does Alice respond? If there exists $n \in \mathbb{N}$ such that B intersects G_n and $\mathrm{diam}(B) < \mathrm{dist}(G_n, \bigcup_{1\leq i\leq n-1} G_i \cup E)$, then $B \cap G_n \neq \emptyset$ and $B \cap G_k = \emptyset$ for all $1 \leq k < n$ and $B \cap E = \emptyset$. Alice erases G_n if it is a legal move, otherwise Alice does not erase anything.

To show that this strategy is winning, suppose that $x_\infty \notin \bigcup_m A_m$. We want to show that $x_\infty \in S$. Otherwise $x_\infty \notin S$ so there exists n such that $x_\infty \in G_n$. We will show that Alice erases G_n at some stage of the game. By definition $x_\infty \in B_m$ for all $m \in \mathbb{N}_0$, and we assumed $x_\infty \in G_n$, so $x_\infty \in B_m \cap G_n$ for all $m \in \mathbb{N}_0$. Since $\tau > 0$, then $\mathrm{dist}(G_n, \bigcup_{1\leq i\leq n-1} G_i \cup E) > 0$. Also $\lim_{m\to\infty} \mathrm{diam}(B_m) = 0$, so taking $m_n \in \mathbb{N}_0$ to be the smallest integer such that $\mathrm{dist}(G_n, \bigcup_{1\leq i\leq n-1} G_i \cup E) > \mathrm{diam}(B_{m_n})$, we know that $B_{m_n} \cap G_n \neq \emptyset$ and $B_{m_n} \cap G_k = \emptyset$ for all $1 \leq k < n$. If $m_n = 0$, then

$$\mathrm{diam}(B_0) = 2\rho_0 \geq 2\rho = \beta\,\mathrm{diam}(C) \geq \beta \mathrm{dist}\Big(G_n, \bigcup_{1\leq i\leq n-1} G_i \cup E\Big).$$

If $m_n > 0$, then

$$\operatorname{diam}(B_{m_n}) \geq \beta \operatorname{diam}(B_{m_n-1}) \geq \beta \operatorname{dist}\Big(G_n, \bigcup_{1\leq i\leq n-1} G_i \cup E\Big).$$

So $\operatorname{diam}(B_{m_n}) \geq \beta \operatorname{dist}\big(G_n, \bigcup_{1\leq i\leq n-1} G_i \bigcup E\big)$. Hence,

$$\operatorname{diam}(G_n) \leq \frac{1}{\tau}\operatorname{dist}\Big(G_n, \bigcup_{1\leq i\leq n-1} G_i \cup E\Big) \leq \frac{1}{\tau\beta}\operatorname{diam}(B_{m_n}).$$

This means that it is legal for Alice to erase G_n in the m_n-th turn, and her strategy specifies that she does so. Finally, if $m_i = m_j$ then the first gap intersecting $B_{m_i} = B_{m_j}$ is G_j and also G_i, so $i = j$; thus the elements of $\{m_n : n \in \mathbb{N}\}$ are all different. □

Observation 17 *Let C be a compact set in $\mathbb{R}^d$ and $\tau := \tau(C) > 0$. Then, by Proposition 16 and monotonicity, $S := C \cup E$ is a $\big(\frac{1}{\tau\beta}, \beta, c, \frac{\beta}{2}\operatorname{diam}(C)\big)$-winning set for all $\beta \in (0, 1)$ and all $c \geq 0$.*

4 A lower bound for the dimension of intersections of thick compact sets in $\mathbb{R}^d$

Whilst the gap lemma, Theorem 10, concerns the intersection of just two sets, it is of interest to obtain conditions that ensure that finitely many, or even countably many compact subsets of $\mathbb{R}^d$ have non-empty intersection. Using the game introduced in Definition 11 we not only obtain conditions involving thickness that ensure that such collection of sets in has non-empty intersection, but also get a lower bound for the Hausdorff dimension of this intersection, as stated in Theorem 6.

To achieve this we use the following technical theorem that gives a lower bound for the dimension of winning sets, based on [4,Theorem 5.5] and [22,Theorem 4] and proved in the Appendix A. The parameters of a winning set provide a measure of its size and we translate this in terms of thickness which is a single number that is easy to compute and work with.

Theorem 18 *Let $S \subseteq \mathbb{R}^d$ be an (α, β, c, ρ)-winning set with $c < d$ and $\beta \leq \frac{1}{4}$. Then for every ball B_0 of radius larger than ρ,*

$$\dim_H(S \cap B_0) \geq d - K_1 \frac{\alpha^d}{|\log(\beta)|} > 0 \text{ if } \alpha^c \leq \frac{1}{K_2}(1 - \beta^{d-c}),$$

where K_1 and K_2 are as in (5).

We now prove Theorem 6 by combining Theorem 18 with the fact that sets of positive thickness can be regarded as winning sets.

Proof of Theorem 6 By Observation 17, for each i

$$S_i := E_i \cup C_i \text{ is a } \left(\frac{1}{\tau_i\beta}, \beta, c, \frac{\beta}{2}\operatorname{diam}(C_i)\right)\text{-winning set}$$

for all $\beta \in (0, 1)$ and all $c \geq 0$. We fix $c \in (0, d)$ and $\beta \in (0, \frac{1}{4}]$ from the hypothesis (iii). We define $\rho := \frac{\beta}{2}\sup_i \operatorname{diam}(C_i)$ which is a finite number by hypothesis (i). By monotonicity, Proposition 13, S_i a $\big(\frac{1}{\tau_i\beta}, \beta, c, \rho\big)$-winning set. Hence, by Proposition 12,

$$S := \bigcap_i S_i \text{ is a } (\alpha, \beta, c, \rho)\text{-winning set,}$$

where

$$\alpha := \Big(\sum_i (\tau_i \beta)^{-c}\Big)^{1/c} = \frac{1}{\beta}\Big(\sum_i \tau_i^{-c}\Big)^{1/c}. \tag{8}$$

By hypothesis (ii) there exists a ball B such that $B \cap E_i = \emptyset$ for all i and we take r to be the radius of B. By definition of ρ and β, we have $r \geq \rho$. By hypothesis (iii) and equation (8) we have $\alpha^c \leq \frac{1}{K_2}(1-\beta^{d-c})$, hence we can apply Theorem 18 to get

$$\dim_H(S \cap B) \geq d - K_1 \frac{\alpha^d}{|\log(\beta)|} > 0,$$

and we know by definition of α that $d - K_1 \frac{\alpha^d}{|\log(\beta)|} = d - K_1 \frac{(\sum_i \tau_i^{-c})^{d/c}}{\beta^d |\log(\beta)|}$.

Since B does not intersect any E_i,

$$S_i \cap B \subseteq S_i \cap E_i^C \cap B = C_i \cap B \text{ for every } i,$$

so $S \cap B \subseteq B \cap \bigcap_i C_i$. The conclusion follows. □

5 Application: patterns in thick compact sets of $\mathbb{R}^d$

In this section we deduce Theorem 7 on the existence of small copies of pattens in sufficiently thick sets from Theorem 6 and illustrate this in the case of Sierpiński carpets.

Proof of Theorem 7 We write $B_0 := \frac{1}{8}B$ for the ball with the same centre as B but with radius $\frac{1}{8}\,\mathrm{rad}(B)$. Given a finite set A and $\lambda \in \big(0, \frac{\mathrm{diam}(B)}{16\,\mathrm{diam}(A)}\big)$ we seek translates of $\lambda A := \{b_1, \cdots, b_n\}$ with $b_i \in \mathbb{R}^d$ where we can assume $b_1 = 0$. As $\mathrm{diam}(\lambda A) < \frac{\mathrm{diam}(B)}{16}$ then $\lambda A \subseteq B(0, \frac{\mathrm{diam}(B)}{16})$.

We define $C_i := C - b_i$ which is a compact set with thickness τ for every $1 \leq i \leq n$. By hypothesis there is a ball $B \subseteq E^C$, so there is a ball $\widetilde{B} \subseteq \bigcap_{1\leq i\leq n}(B - b_i) \subseteq \bigcap_{1\leq i \leq n} E_i^C$ of diameter $\mathrm{diam}(B)(1-\frac{1}{16}) = \frac{15}{16}\,\mathrm{diam}(B)$.

We take $\beta := \min\{\frac{1}{4}, \frac{\mathrm{diam}(\widetilde{B})}{\mathrm{diam}(C)}\}$, $\alpha := 1/\tau\beta$ and $c := d - 1/\log(\tau\beta)$. Then $\alpha^c = e\alpha^d$ and $d - c = 1/\log(\tau\beta)$.

By Theorem 6, if

$$n\alpha^c \leq \frac{1}{K_2}(1-\beta^{d-c}) \quad \text{or equivalently} \quad n \leq \frac{1}{K_2}\alpha^{-c}(1-\beta^{d-c}), \tag{9}$$

then $\dim_H(\widetilde{B} \cap \bigcap_{1\leq i\leq n} C_i) > 0$.

By definition of α, β and c, and using that $f(\tau) := \log(\tau)(1-\beta^{1/\log(\tau\beta)})$ is a decreasing function with $\lim_{\tau\to\infty} f(\tau) = |\log(\beta)|$,

$$\begin{aligned}\frac{1}{K_2}\alpha^{-c}(1-\beta^{d-c}) &= \frac{1}{eK_2}(\tau\beta)^d(1-\beta^{1/\log(\tau\beta)})\\ &= \frac{1}{eK_2}\frac{\tau^d}{\log(\tau)}\beta^d \log(\tau)(1-\beta^{1/\log(\tau\beta)})\\ &\geq \frac{1}{eK_2}\beta^d|\log(\beta)|\frac{\tau^d}{\log(\tau)}\end{aligned}$$

Setting

$$N(\tau) := \left\lfloor \frac{\beta^d |\log(\beta)|}{eK_2} \frac{\tau^d}{\log(\tau)} \right\rfloor$$

it follows from if (9) that if $n \leq N(\tau)$ then $\dim_H(\widetilde{B} \cap \bigcap_{1\leq i\leq n} C_i) > 0$. If $x \in X := \widetilde{B} \cap \bigcap_{1\leq i\leq n} C_i$, then $x+b_i \in C_i+b_i = C$ for every $1 \leq i \leq n$, so $C \supseteq x+\{b_1, \cdots, b_n\} = x+\lambda A$ as required. □

Sierpiński carpets and sponges

Sierpiński carpets and sponges provide examples of sets for which thickness is easily found and which satisfy Theorem 7 .

Let $n_1, \cdots, n_d \in \mathbb{N}_{\geq 3}$ be odd natural numbers. Let

$$D = \left\{\mathbf{i} := (i_1, \ldots, i_d) : 1 \leq i_k \leq n_k, \text{ with } (i_1, \ldots, i_d) \neq \left(\tfrac{1}{2}(n_1+1), \ldots, \tfrac{1}{2}(n_d+1)\right)\right\}.$$

The family of affine maps

$$\left\{f_{\mathbf{i}} : \mathbb{R}^d \to \mathbb{R}^d : \mathbf{i} \in D\right\},$$

where

$$f_{\mathbf{i}}(x_1, \ldots, x_d) = \left(\frac{x_1+i_1-1}{n_1}, \ldots, \frac{x_d+i_d-1}{n_d}\right),$$

forms an iterated function system, which defines a unique non-empty compact set $C \subset \mathbb{R}^d$ such that $C = \bigcup_{\mathbf{i}\in D} f_{\mathbf{i}}(C)$, see [7,Chapter 9]. Then C is a self-affine Sierpiński sponge (carpet if $d = 2$) which can also be realised iteratively by repeatedly substituting the coordinate parallelepipeds obtained by dividing the unit cube $[0, 1]^d$ into $n_1 \times \cdots \times n_d$ smaller parallelepipeds, with the central one removed, into themselves. In other words

$$C = \bigcap_{k=0}^{\infty} \bigcup_{\mathbf{i}_1,\ldots,\mathbf{i}_k\in D} f_{\mathbf{i}_1} \circ \cdots \circ f_{\mathbf{i}_k}([0, 1]^d).$$

We will find the thickness of C. Each parallelepiped at the kth step of the iterative construction has side-lengths $1/n_i^k$ $(1 \leq i \leq d)$. Thus the central parallelepipeds that are removed and which form gaps at the kth step have diameter

$$\mathrm{diam}_k := \sqrt{\sum_{1\leq k\leq d} \frac{1}{n_i^{2k}}}.$$

The minimal distance of a gap removed at the kth step from the previous gaps and the external complementary component E is

$$\mathrm{dist}_k := \min_{1\leq i\leq d} \frac{1}{n_i^k} \frac{n_i-1}{2}.$$

Hence, the thickness of C is

$$\tau := \tau(C) = \inf_{k\in\mathbb{N}} \frac{\mathrm{dist}_k}{\mathrm{diam}_k} = \inf_{k\in\mathbb{N}} \frac{\min_{1\leq i\leq d} \frac{1}{n_i^k} \frac{n_i-1}{2}}{\sqrt{\sum_{1\leq k\leq d} \frac{1}{n_i^{2k}}}}.$$

Thus, with $\beta := \min\{\frac{1}{4}, \frac{15}{16\sqrt{d}}\}$, Theorem 7 gives that C contains homothetic copies of every pattern with at most $N(\tau)$ points where $N(\tau)$ is given by (6).

For example, the self-similar Sierpiński carpet C_n in $\mathbb{R}^2$, taking $n_1 = n_2 = n$ above, has thickness $\tau = (n-1)\big/2\sqrt{2}$, so there is a homothetic copy in C_n of every configuration of up to $N(\tau)$ points. Because K_2 is large, n needs to be large to guarantee even that similar copies of all triangles can be found in C_n. On the other hand, for C_n to contain copies of all k-point configurations, $n = O((k \log k)^{1/2})$ which does not increase too rapidly for large k.

6 Thickness and Hausdorff dimension

In this section we obtain two different lower bounds for the Hausdorff dimension of sets in $\mathbb{R}^d$ in terms of their thickness.

Firstly, Theorem 6 yields a lower bound by taking a single set C.

Corollary 19 *Let C be a compact set in $\mathbb{R}^d$ with positive thickness τ (so $\operatorname{diam}(C) < +\infty$ and there is a ball B such that $B \cap E = \emptyset$). If there exists $c \in (0, d)$ such that $\tau^{-c} \leq \frac{1}{K_2}\beta^c(1-\beta^{d-c})$ for $\beta := \min\{\frac{1}{4}, \frac{\operatorname{diam}(B)}{\operatorname{diam}(C)}\}$ then*

$$\dim_H(C) \geq \dim_H(B \cap C) \geq d - K_1 \frac{\tau^{-d}}{\beta^d |\log(\beta)|} > 0.$$

Secondly, we can get a lower bound in the case of convex sets with convex gaps by considering 1-dimensional sections.

Proposition 20 *Let C_0 be a proper compact convex set in $\mathbb{R}^d$ where $d \geq 2$, and let $C = C_0 \setminus \bigcup_{k=1}^{\infty} G_k$, where $\{G_k\}_k$ are open convex gaps ordered by decreasing diameters. Then $\tau(C \cap L) \geq \tau(C)$ for every straight line L that properly intersects C_0.*

Proof Let L be a straight line that properly intersects C_0. Let $\{I_i\}_{i=1}^{\infty}$ be the (countable or finite) set of open intervals $I_i := G_{k(i)} \cap L$ in L ordered so that $|I_i| \leq |I_j|$ if $i \geq j$, where $|\ |$ denotes the length of an interval. Let $1 \leq i \leq j-1$. There are two cases:

(a) if $k(i) < k(j)$ then

$$\operatorname{dist}(I_i, I_j) \ \geq \ \operatorname{dist}(G_{k(i)}, G_{k(j)}) \ \geq \ \tau(C)\operatorname{diam}(G_{k(j)}) \ \geq \ \tau(C)|I_j|;$$

(b) if $k(j) < k(i)$ then

$$\operatorname{dist}(I_i, I_j) \ \geq \ \operatorname{dist}(G_{k(i)}, G_{k(j)}) \ \geq \ \tau(C)\operatorname{diam}(G_{k(i)}) \ \geq \ \tau(C)|I_i| \ \geq \ \tau(C)|I_j|.$$

In both cases $\operatorname{dist}(I_i, I_j) \geq |I_j|$ for all $1 \leq i \leq j-1$ so $\tau(C \cap L) \geq \tau(C)$ from the definition of thickness.

□

We can now obtain a lower bound for the Hausdorff dimension for these sets in terms of thickness using the bound (7) for sets in $\mathbb{R}$.

Proposition 21 *Let $C_0 \subseteq \mathbb{R}^d$ be a proper compact convex set, and let $C = C_0 \setminus \bigcup_{i=1}^{\infty} G_i$ where $\{G_i\}_i$ are open convex gaps. Then*

$$\dim_H(C) \ \geq \ d - 1 + \frac{\log 2}{\log(2 + 1/\tau(C))} \tag{10}$$

where $\tau(C)$ is the thickness of C.

Proof Let L be a straight line that properly intersects C_0. Combining the relationship between thickness and Hausdorff dimension for subsets of $\mathbb{R}$ stated in (7) with Proposition 20,

$$\dim_H(C \cap L) \geq \frac{\log 2}{\log(2 + 1/\tau(C \cap L))} \geq \frac{\log 2}{\log(2 + 1/\tau(C))}.$$

This is true for all lines L in a given direction that properly intersect C_0, so by a standard result relating the Hausdorff dimension of a set to the Hausdorff dimensions of parallel sections, see for example, [7,Corollary 7.10], inequality (10) follows. □

Observation 22 *When $d = 1$ Proposition 21 is better than Corollary 19. For $d \geq 2$, Corollary 19 gives a better bound than Proposition 21 when τ is large but when τ is small Proposition 21 is better.*

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7 Proof of Theorem 18

The proof of Theorem 18 is based on [4,Theorem 5.5] and [22,Theorem 4] and adapted to our particular setting.

Proof We can assume without loss of generality that the radius of B_0 is ρ. We let x_0 be the center of B_0,

$$\rho_n := \beta^n \rho \text{ radii of balls, and}$$

$$E_n := \frac{\rho_n}{2}\mathbb{Z}^d + x_0 \text{ centers of balls of the family}$$

$$\mathcal{E}_n := \left\{ B(\frac{\rho_n}{2} z + x_0, \rho_n) : \ z \in \mathbb{Z}^d \right\}.$$

We will take Bob's move of the nth turn from $\mathcal{E}_n$.

We also define

$$D_n := 3\rho_n \mathbb{Z}^d + x_0 \subset E_n,$$

$$\mathcal{D}_n := \{B(3\rho_n z + x_0, \rho_n) : \ z \in \mathbb{Z}^d\} \subset \mathcal{E}_n.$$

Note that the elements of $\mathcal{D}_n$ are disjoint (moreover they are at distance ρ_n).

We fix $\gamma \in (0, 1)$, a small number to be determined later (independent of α, β, c and ρ). Let $N := \lfloor \frac{\gamma^d}{\alpha^d} \rfloor$.

We define the function $\pi_n : \mathcal{E}_{n+1} \to \mathcal{E}_n$, $B \mapsto \pi_n(B)$ in the following way:

- When $n \neq jN$ for all j: we define $\pi_n(B)$ as the element of $\mathcal{E}_n$ that contains B such that B is as centered as possible inside that element.
- When $n = jN$ for some j: If there exists $B' \in \mathcal{D}_{jN}$ containing B, we define $\pi_n(B) := B'$ (it is well defined because in that case there is only one element belonging to $\mathcal{D}_{jN}$). If not, we define the function as before.

Intuitively the function π_n carries the elements of level $n+1$ to its ancestor of level n.

We use the following notation: for $m < n$ and $B \in \mathcal{E}_n$, $\pi_m(B) := \pi_m \circ \pi_{m+1} \circ \cdots \circ \pi_{n-1}(B) \in \mathcal{E}_m$. This is to say, we carry B to its ancestor of level m via the functions π. If Bob plays $B \in \mathcal{E}_n$ in the turn n, we consider that in the previous turns $m \in \{0, \cdots, n-1\}$ Bob has played $\pi_m(B)$. Then, we have the following inclusions of movements from the turn n to the turn 0:

$$B \subset \pi_{n-1}(B) \subset \cdots \subset \pi_0(B).$$

We defined the function in this way to guarantee that Bob's moves are legal. Alice responds under her winning strategy. If in the turn n Bob plays $B \in \mathcal{E}_n$, we define $\mathcal{A}(B)$ as Alice's answer (each $A \in \mathcal{A}(B)$ is a countable collection of sets $A := \{A_{i,n}\}_i$, and a legal movement as an answer for B, i.e.: $\sum_i \operatorname{diam}(A_{i,n})^c \leq (\alpha\rho_n)^c$). Let

$$\mathcal{A}^*_m(B) := \{A \in \mathcal{A}(\pi_m(B)) : B \cap A \neq \emptyset\}$$

be Alice's answer (this is a list of sets) to the ancestor of B of level $m < n$.

Given any ball B, we denote by $\frac{1}{2}B$ the ball with the same center as B and the half of the radius.

Note that as $\beta \leq \frac{1}{4}$, if $B \in \mathcal{D}_{jN}$ and $B' \in \mathcal{E}_{jN+1}$ satisfy $B' \cap \frac{1}{2}B \neq \emptyset$, then $B' \subset B$, so $\pi_{jN}(B') = B$. It follows that

$$\text{if } n > jN,\ B' \subset \frac{1}{2}B \text{ with } B' \in \mathcal{E}_n \text{ and } B \in \mathcal{D}_{jN},\ \text{then } \pi_{jN}(B') = B. \tag{11}$$

This is true because if we look at the ancestor of B' of level $jN+1$, since π_n chooses the element belonging to $\mathcal{E}_n$ that contains B such that B is as centered as possible, that element must intersect $\frac{1}{2}B$.

We define for every $B \in \mathcal{D}_j$

$$\phi_j(B) := \sum_{n<j} \sum_{A \in \mathcal{A}^*_n(B)} \operatorname{diam}(A_{i,n})^c.$$

This is a measure of all of Alice's answers to the ancestors of B. Note that $\phi_0(B) = 0$.

Let

$$\mathcal{D}'_j := \{B \in \mathcal{D}_j : \ \phi_j(B) \leq (\gamma\rho_j)^c\}.$$

We define

$$\mathcal{D}_j(B) := \{B' \in \mathcal{D}_j : \ B' \subset \tfrac{1}{2}B\}.$$

Some useful bounds.

We denote by $\operatorname{rad}(B)$ the radius of the ball B.

Observation 23 *If* $B \in \mathcal{D}'_{jN}$, *we have that* $\mathrm{rad}(\frac{1}{2}\mathrm{B}) = \frac{1}{2}fi^{\mathrm{jN}}$æ, *and* $\mathrm{rad}(\mathrm{B}') =$ fi$^{(\mathrm{j}+1)\mathrm{N}}$æ *for every* $B' \in \mathcal{D}_{(j+1)N}$. *We can cover* $\frac{1}{2}B$ *with enlarged balls from* $\mathcal{D}_{(j+1)N}(B)$ *(with radii* $4\rho_{(j+1)N}\sqrt{d}$*). This gives us a lower bound for* $\#\mathcal{D}_{(j+1)N}(B)$*:*

$$\mathcal{L}^d(\tfrac{1}{2}B) \leq \#\mathcal{D}_{(j+1)N}(B)\mathcal{L}^d(B_{4\rho_{(j+1)N}\sqrt{d}}),$$

so

$$\beta^{-Nd}\frac{1}{2^d 4^d \sqrt{d}^d} \leq \#\mathcal{D}_{(j+1)N}(B).$$

Proposition 24 *If* $\alpha^c \leq \frac{1}{K_2}(1-\beta^{d-c})$ *where* $K_2 := \max\{\gamma^{-2d}, 2\gamma^{-d}\log(\gamma^{-d})\}$, *we have that*

$$\#(\mathcal{D}_{(j+1)N}(B) \cap \mathcal{D}'_{(j+1)N}) \geq \beta^{-Nd}\left(\frac{1}{2^d 4^d \sqrt{d}^d} - 3^d\gamma^d(1+4^d 2)\right) \textit{ for all } B \in \mathcal{D}'_{jN}.$$

We start by proving two preliminary lemmas:

Lemma 25 (a) *For all* $n \in \mathbb{N}$ *and* $B' \in \mathcal{E}_n$ *we have that*

$$\sum_{A \in \mathcal{A}(B')} \min\left\{1, \frac{\mathrm{diam}(A_{i,n})^c}{(\gamma\rho_{(j+1)N})^c}\right\}\left(\frac{\mathrm{diam}(A_{i,n}) + 2\rho_{(j+1)N}}{\mathrm{rad}(B')}\right)^d$$
$$\leq 3^d\alpha^c \max\left\{\alpha^{d-c}, \gamma^{-c}\left(\frac{\rho_{(j+1)N}}{\mathrm{rad}(B')}\right)^{d-c}\right\}.$$

(b) *If* $B \in \mathcal{D}'_{jN}$ *then*

$$\sum_{n<jN}\sum_{A \in \mathcal{A}^*_n(B)} \min\left\{1, \frac{\mathrm{diam}(A_{i,n})^c}{(\gamma\rho_{(j+1)N})^c}\right\}\left(\frac{\mathrm{diam}(A_{i,n}) + 2\rho_{(j+1)N}}{\mathrm{rad}(B)}\right)^d$$
$$\leq 3^d\gamma^c \max\left\{\gamma^{d-c}, \gamma^{-c}\left(\frac{\rho_{(j+1)N}}{\mathrm{rad}(B)}\right)^{d-c}\right\}.$$

Proof of Lemma 25 Firstly, splitting into the cases $x \leq y$ and $y \leq x$, it is easy to see that

$$\min\left\{1, \frac{x^c}{(\gamma y)^c}\right\}(x+2y)^d \leq 3^d x^c \max\left\{x^{d-c}, \frac{y^{d-c}}{\gamma^c}\right\} \text{ for all } x, y > 0. \tag{12}$$

Secondly, we will prove that if $n \in \mathbb{N}$ and $B' \in \mathcal{E}_n$ then the claim a) holds. By applying the inequality (12) to $x := \frac{\mathrm{diam}(A_{i,n})}{\mathrm{rad}(B')}$ and $y := \frac{\rho_{(j+1)N}}{\mathrm{rad}(B')}$, summing over all $A_{i,n} \in \mathcal{A}(B')$ and using that Alice is playing legally, we have that

$$\sum_{A_{i,n} \in \mathcal{A}(B')} \min\left\{1, \frac{\mathrm{diam}(A_{i,n})^c}{(\gamma\rho_{(j+1)N})^c}\right\}\left(\frac{\mathrm{diam}(A_{i,n}) + 2\rho_{(j+1)N}}{\mathrm{rad}(B')}\right)^d$$
$$\leq 3^d \sum_{A_{i,n} \in \mathcal{A}(B')}\left(\frac{\mathrm{diam}(A_{i,n})}{\mathrm{rad}(B')}\right)^c \max\left\{\left(\frac{\mathrm{diam}(A_{i,n})}{\mathrm{rad}(B')}\right)^{d-c}, \gamma^{-c}\left(\frac{\rho_{(j+1)N}}{\mathrm{rad}(B')}\right)^{d-c}\right\}$$
$$\leq 3^d \max\left\{\left(\max_{A_{i,n} \in \mathcal{A}(B')}\frac{\mathrm{diam}(A_{i,n})}{\mathrm{rad}(B')}\right)^{d-c}, \gamma^{-c}\left(\frac{\rho_{(j+1)N}}{\mathrm{rad}(B')}\right)^{d-c}\right\}\sum_{A_{i,n} \in \mathcal{A}(B')}\left(\frac{\mathrm{diam}(A_{i,n})}{\mathrm{rad}(B')}\right)^c$$

$$\le 3^d \alpha^c \max\left\{\alpha^{d-c}, \gamma^{-c}\left(\frac{\rho_{(j+1)N}}{\mathrm{rad}(B')}\right)^{d-c}\right\}.$$

Finally, we prove the claim b). By applying the inequality (12) to $x := \frac{\mathrm{diam}(A_{i,n})}{\mathrm{rad}(B)}$ and $y := \frac{\rho_{(j+1)N}}{\mathrm{rad}(B)}$, summing over all elements of $\bigcup_{n<jN} \mathcal{A}_n^*(B)$, and using that, since $B \in \mathcal{D}'_{jN}$, we have

$$\sum_{n<jN} \sum_{A_{i,n}\in\mathcal{A}_n^*(B)} \left(\frac{\mathrm{diam}(A_{i,n})}{\mathrm{rad}(B)}\right)^c \le \gamma^c,$$

and in particular $\frac{\mathrm{diam}(A_{i,n})}{\mathrm{rad}(B)} \le \gamma$ for every i and every $n < jN$, we obtain that:

$$\begin{aligned}
&\sum_{n<jN} \sum_{A_{i,n}\in\mathcal{A}_n^*(B)} \min\left\{1, \frac{\mathrm{diam}(A_{i,n})^c}{(\gamma\rho_{(j+1)N})^c}\right\}\left(\frac{\mathrm{diam}(A_{i,n}) + 2\rho_{(j+1)N}}{\mathrm{rad}(B)}\right)^d \\
&\le 3^d \sum_{n<jN} \sum_{A_{i,n}\in\mathcal{A}_n^*(B)} \left(\frac{\mathrm{diam}(A_{i,n})}{\mathrm{rad}(B)}\right)^c \max\left\{\left(\frac{\mathrm{diam}(A_{i,n})}{\mathrm{rad}(B)}\right)^{d-c}, \gamma^{-c}\left(\frac{\rho_{(j+1)N}}{\mathrm{rad}(B)}\right)^{d-c}\right\} \\
&\le 3^d \left(\sum_{n<jN} \sum_{A_{i,n}\in\mathcal{A}_n^*(B)} \left(\frac{\mathrm{diam}(A_{i,n})}{\mathrm{rad}(B)}\right)^c\right) \max\left\{\gamma^{d-c}, \gamma^{-c}\left(\frac{\rho_{(j+1)N}}{\mathrm{rad}(B)}\right)^{d-c}\right\} \\
&\le 3^d \gamma^c \max\left\{\gamma^{d-c}, \gamma^{-c}\left(\frac{\rho_{(j+1)N}}{\mathrm{rad}(B)}\right)^{d-c}\right\}.
\end{aligned}$$

□

Now we are ready to prove Proposition 24.

Proof of Proposition 24

$$\begin{aligned}
&\#(\mathcal{D}_{(j+1)N}(B) \setminus \mathcal{D}'_{(j+1)N}) \\
&\quad \le \#\left\{B' \in \mathcal{D}_{(j+1)N}(B) : \frac{\phi_{(j+1)N}(B')}{(\gamma\rho_{(j+1)N})^c} > 1\right\} \\
&\quad \le \sum_{B'\in\mathcal{D}_{(j+1)N}(B)} \min\left\{1, \frac{\phi_{(j+1)N}(B')}{(\gamma\rho_{(j+1)N})^c}\right\} \\
&\quad \le \sum_{B'\in\mathcal{D}_{(j+1)N}(B)} \min\left\{1, \sum_{n<(j+1)N} \sum_{A_{i,n}\in\mathcal{A}_n^*(B')} \frac{\mathrm{diam}(A_{i,n})^c}{(\gamma\rho_{(j+1)N})^c}\right\} \\
&\quad \le \sum_{B'\in\mathcal{D}_{(j+1)N}(B)} \sum_{n<(j+1)N} \sum_{A_{i,n}\in\mathcal{A}_n^*(B')} \min\left\{1, \frac{\mathrm{diam}(A_{i,n})^c}{(\gamma\rho_{(j+1)N})^c}\right\} \\
&\quad \le \sum_{B'\in\mathcal{D}_{(j+1)N}(B)} \sum_{n<jN} \sum_{A_{i,n}\in\mathcal{A}_n^*(B')} \min\left\{1, \frac{\mathrm{diam}(A_{i,n})^c}{(\gamma\rho_{(j+1)N})^c}\right\} \\
&\qquad + \sum_{B'\in\mathcal{D}_{(j+1)N}(B)} \sum_{jN\le n<(j+1)N} \sum_{A_{i,n}\in\mathcal{A}_n^*(B')} \min\left\{1, \frac{\mathrm{diam}(A_{i,n})^c}{(\gamma\rho_{(j+1)N})^c}\right\}. \qquad (13)
\end{aligned}$$

We have split the sum in (13) into two parts, depending on whether $n < jN$ or $jN \le n < (j+1)N$.

To get a bound for the left-hand sum of (13) we will use that if $n < jN$ then

$$\left\{(B', A):\ B' \in \mathcal{D}_{(j+1)N}(B),\, A \in \mathcal{A}_n^*(B')\right\} \subset \left\{(B', A):\ B' \in \mathcal{D}_{(j+1)N}(B),\, A \in \mathcal{A}_n^*(B),\, A \cap B' \neq \emptyset\right\}.$$

Since $B \in \mathcal{D}'_{jN}$ the set $\mathcal{A}_n^*(B)$ only makes sense for $n < jN$. This inclusion holds because of (11), as $\mathcal{A}(\pi_n(B')) \subset \mathcal{A}(\pi_n(B))$ since $B' \subset B$.

So,

$$\begin{aligned}
&\sum_{B' \in \mathcal{D}_{(j+1)N}(B)} \sum_{A_{i,n} \in \mathcal{A}_n^*(B')} \min\left\{1, \frac{\operatorname{diam}(A_{i,n})^c}{(\gamma\rho_{(j+1)N})^c}\right\} \\
&\le \sum_{A_{i,n} \in \mathcal{A}_n^*(B)} \sum_{\substack{B' \in \mathcal{D}_{(j+1)N}(B) \\ B' \cap A \neq \emptyset}} \min\left\{1, \frac{\operatorname{diam}(A_{i,n})^c}{(\gamma\rho_{(j+1)N})^c}\right\} \\
&= \sum_{A_{i,n} \in \mathcal{A}_n^*(B)} \min\left\{1, \frac{\operatorname{diam}(A_{i,n})^c}{(\gamma\rho_{(j+1)N})^c}\right\} \#\{B' \in \mathcal{D}_{(j+1)N}(B),\, B' \cap A \neq \emptyset\}.
\end{aligned}$$

Now, we will get a bound for the right-hand sum in (13), when $jN \le n < (j+1)N$. Recall that $B \in \mathcal{D}'_{jN}$. First, note that if $B'' \in \mathcal{D}_{(j+1)N}(B)$, then $B'' \in \mathcal{E}_{(j+1)N}$, $B'' \subset \frac{1}{2}B$ where $B \in \mathcal{D}_{jN}$. If we take $B' := \pi_{jN}(B'') \in \mathcal{E}_{jN}$, by (11) we have that $B' = B$.

For all $B'' \in \mathcal{D}_{(j+1)N}(B)$, there exists $B' \in \mathcal{E}_n$ with $B' \subset B$ and $B'' \subset \frac{1}{2}B'$ ($B' = B$ if $n = jN$). Hence

$$\begin{aligned}
&\sum_{B'' \in \mathcal{D}_{(j+1)N}(B)} \sum_{A_{i,n} \in \mathcal{A}_n^*(B'')} \min\left\{1, \frac{\operatorname{diam}(A_{i,n})^c}{(\gamma\rho_{(j+1)N})^c}\right\} \\
&\le \sum_{\substack{B' \in \mathcal{E}_n \\ B' \subset B}} \sum_{B'' \in \mathcal{D}_{(j+1)N}(B')} \sum_{\substack{A \in \mathcal{A}(B') \\ A \cap B'' \neq \emptyset}} \min\left\{1, \frac{\operatorname{diam}(A_{i,n})^c}{(\gamma\rho_{(j+1)N})^c}\right\} \\
&= \sum_{\substack{B' \in \mathcal{E}_n \\ B' \subset B}} \sum_{A \in \mathcal{A}(B')} \sum_{\substack{B'' \in \mathcal{D}_{(j+1)N}(B') \\ A \cap B'' \neq \emptyset}} \min\left\{1, \frac{\operatorname{diam}(A_{i,n})^c}{(\gamma\rho_{(j+1)N})^c}\right\} \\
&= \sum_{\substack{B' \in \mathcal{E}_n \\ B' \subset B}} \sum_{A \in \mathcal{A}(B')} \min\left\{1, \frac{\operatorname{diam}(A_{i,n})^c}{(\gamma\rho_{(j+1)N})^c}\right\} \#\{B'' \in \mathcal{D}_{(j+1)N}(B') :\ A \cap B'' \neq \emptyset\},
\end{aligned}$$

where the inequality holds by considering in particular $B' := \pi_n(B'') \subset B$.

By inequality (13), and what we have noted before,

$$\begin{aligned}
&\#(\mathcal{D}_{(j+1)N}(B) \setminus \mathcal{D}'_{(j+1)N}) \\
&\le \sum_{B' \in \mathcal{D}_{(j+1)N}(B)} \sum_{n < jN} \sum_{A_{i,n} \in \mathcal{A}_n^*(B')} \min\left\{1, \frac{\operatorname{diam}(A_{i,n})^c}{(\gamma\rho_{(j+1)N})^c}\right\} \\
&\quad + \sum_{B' \in \mathcal{D}_{(j+1)N}(B)} \sum_{n=jN}^{(j+1)N-1} \sum_{A_{i,n} \in \mathcal{A}_n^*(B')} \min\left\{1, \frac{\operatorname{diam}(A_{i,n})^c}{(\gamma\rho_{(j+1)N})^c}\right\}
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{n<jN} \sum_{A_{i,n}\in\mathcal{A}_n^*(B)} \min\left\{1, \frac{\operatorname{diam}(A_{i,n})^c}{(\gamma\rho_{(j+1)N})^c}\right\} \#\{B'\in\mathcal{D}_{(j+1)N}(B):\ B'\cap A\neq\emptyset\}\\
&\quad+\sum_{n=jN}^{(j+1)N-1} \sum_{\substack{B'\in\mathcal{E}_n\\ B'\subset B}} \sum_{A_{i,n}\in\mathcal{A}_n^*(B')} \min\left\{1, \frac{\operatorname{diam}(A_{i,n})^c}{(\gamma\rho_{(j+1)N})^c}\right\}\\
&\quad \#\{B''\in\mathcal{D}_{(j+1)N}(B'):\ B''\cap A\neq\emptyset\}\\
&\le \left(\frac{\operatorname{rad}(B)}{\rho_{(j+1)N}}\right)^d \sum_{n<jN} \sum_{A_{i,n}\in\mathcal{A}_n^*(B)}\\
&\quad \min\left\{1, \frac{\operatorname{diam}(A_{i,n})^c}{(\gamma\rho_{(j+1)N})^c}\right\}\left(\frac{\operatorname{diam}(A_{i,n})+2\rho_{(j+1)N}}{\operatorname{rad}(B)}\right)^d\\
&\quad+\sum_{n=jN}^{(j+1)N-1} \sum_{\substack{B'\in\mathcal{E}_n\\ B'\subset B}} \left(\frac{\operatorname{rad}(B')}{\rho_{(j+1)N}}\right)^d \sum_{A_{i,n}\in\mathcal{A}_n^*(B')}\\
&\quad \min\left\{1, \frac{\operatorname{diam}(A_{i,n})^c}{(\gamma\rho_{(j+1)N})^c}\right\}\left(\frac{\operatorname{diam}(A_{i,n})+2\rho_{(j+1)N}}{\operatorname{rad}(B')}\right)^d, \qquad (14)
\end{aligned}
$$

where in the first term of the last inequality we use that $B\in\mathcal{D}'_{jN}$ (so $\phi_{jN}(B)\le(\gamma\rho_{jN})^c$), in the second term that Alice is playing legally (i.e.: $\sum_i \operatorname{diam}(A_{i,m})^c\le(\alpha\rho_m)^c$), and in both terms that: for every $B'\in\bigcup_n\mathcal{E}_n$ and every $A_{i,n}$, since the elements of $\mathcal{D}_{(j+1)N}$ are disjoint, $\mathcal{L}^d(B'')=C_d\rho^d_{(j+1)N}$, and if moreover $B''\cap A_{i,n}\neq\emptyset$ then $B''\subset\mathcal{N}(A_{i,n},2\rho_{(j+1)N})$ (the $2\rho_{(j+1)N}$-neighborhood of $A_{i,n}$), which is contained in a ball of radius $\operatorname{diam}(A_{i,n})+2\rho_{(j+1)N}$. Therefore,

$$
\begin{aligned}
&\#\{B''\in\mathcal{D}_{(j+1)N}(B'):\ B''\cap A_{i,n}\neq\emptyset\}C_d\rho^d_{(j+1)N}=\mathcal{L}^d\Bigg(\bigcup_{\substack{B''\in\mathcal{D}_{(j+1)N}(B')\\ B''\cap A_{i,n}\neq\emptyset}} B''\Bigg)\\
&\quad\le\mathcal{L}^d\left(\mathcal{N}(A_{i,n},2\rho_{(j+1)N})\right)\le C_d(\operatorname{diam}(A_{i,n})+2\rho_{(j+1)N})^d,
\end{aligned}
$$

in other words,

$$
\#\{B''\in\mathcal{D}_{(j+1)N}(B'):\ B''\cap A_{i,n}\neq\emptyset\}\le\frac{(\operatorname{diam}(A_{i,n})+2\rho_{(j+1)N})^d}{\rho^d_{(j+1)N}}.
$$

By inequality (14), using claim b) from Lemma 25 to bound the first term, and claim a) from Lemma 25 to bound the second one, we obtain:

$$
\begin{aligned}
&\#(\mathcal{D}_{(j+1)N}(B)\setminus\mathcal{D}'_{(j+1)N})\\
&\quad\le\left(\frac{\operatorname{rad}(B)}{\rho_{(j+1)N}}\right)^d\sum_{n<jN}\sum_{A_{i,n}\in\mathcal{A}_n^*(B)}\min\left\{1, \frac{\operatorname{diam}(A_{i,n})^c}{(\gamma\rho_{(j+1)N})^c}\right\}\left(\frac{\operatorname{diam}(A_{i,n})+2\rho_{(j+1)N}}{\operatorname{rad}(B)}\right)^d\\
&\quad+\sum_{jN\le n<(j+1)N}\sum_{\substack{B'\in\mathcal{E}_n\\ B'\subset B}}\left(\frac{\operatorname{rad}(B')}{\rho_{(j+1)N}}\right)^d\sum_{A_{i,n}\in\mathcal{A}_n^*(B')}\\
&\quad\min\left\{1, \frac{\operatorname{diam}(A_{i,n})^c}{(\gamma\rho_{(j+1)N})^c}\right\}\left(\frac{\operatorname{diam}(A_{i,n})+2\rho_{(j+1)N}}{\operatorname{rad}(B')}\right)^d
\end{aligned}
$$

$$\leq \left(\frac{\text{rad}(B)}{\rho_{(j+1)N}}\right)^d 3^d \gamma^c \max\left\{\gamma^{d-c}, \gamma^{-c}\left(\frac{\rho_{(j+1)N}}{\text{rad}(B)}\right)^{d-c}\right\}$$
$$+ \sum_{jN \leq n < (j+1)N} \sum_{\substack{B' \in \mathcal{E}_n \\ B' \subset B}} \left(\frac{\text{rad}(B')}{\rho_{(j+1)N}}\right)^d 3^d \alpha^c \max\left\{\alpha^{d-c}, \gamma^{-c}(\frac{\rho_{(j+1)N}}{\text{rad}(B')})^{d-c}\right\} \tag{15}$$

To continue the estimates, we will use that

$$\frac{\text{rad}(B)}{\rho_{(j+1)N}} = \frac{\beta^{jN}\rho}{\beta^{(j+1)N}\rho} = \beta^{-N}.$$

To bound the second term in (15) we write $n = (j+1)N - k$ for some $k \in \{1, \cdots, N\}$. We know that $B := B(3\rho_{jN} z + x_0, \rho_{jN})$ for some $z \in \mathbb{Z}^d$, and recall that $\mathcal{E}_n := \{B(\frac{\rho_n}{2} z' + x_0, \rho_n) : \ z' \in \mathbb{Z}^d\}$. So,

$$\frac{\text{rad}(B')}{\rho_{(j+1)N}} = \beta^{-k}$$

and

$$\begin{aligned}
\#\{B' \in \mathcal{E}_n : \ B' \subset B\} &\leq \#\{\frac{\rho_n}{2} z' + x_0 \in B : \ z' \in \mathbb{Z}^d\} \\
&= \#\left\{z' \in \mathbb{Z}^d \cap B\left(\frac{6}{\beta^{N-k}} z, 2\Big(\frac{1}{\beta^{N-k}} - 1\Big)\right)\right\} \\
&\leq \left(4\Big(\frac{1}{\beta^{N-k}} - 1\Big) + 1\right)^d \leq 4^d \frac{1}{\beta^{d(N-k)}}.
\end{aligned}$$

Combining with (15),

$$\begin{aligned}
&\#(\mathcal{D}_{(j+1)N}(B) \setminus \mathcal{D}'_{(j+1)N}) \\
&\quad \leq \left(\frac{\text{rad}(B)}{\rho_{(j+1)N}}\right)^d 3^d \gamma^c \max\left\{\gamma^{d-c}, \gamma^{-c}\Big(\frac{\rho_{(j+1)N}}{\text{rad}(B)}\Big)^{d-c}\right\} \\
&\qquad + \sum_{jN \leq n < (j+1)N} \sum_{\substack{B' \in \mathcal{E}_n \\ B' \subset B}} \left(\frac{\text{rad}(B')}{\rho_{(j+1)N}}\right)^d 3^d \alpha^c \max\left\{\alpha^{d-c}, \gamma^{-c}\Big(\frac{\rho_{(j+1)N}}{\text{rad}(B')}\Big)^{d-c}\right\} \\
&\quad \leq \beta^{-Nd} 3^d \gamma^c \max\{\gamma^{d-c}, \gamma^{-c}\beta^{N(d-c)}\} \\
&\qquad + \sum_{1 \leq k \leq N} 4^d \beta^{-d(N-k)} \beta^{dk} 3^d \alpha^c \max\{\alpha^{d-c}, \gamma^{-c}\beta^{k(d-c)}\} \\
&\quad \leq \beta^{-Nd} 3^d \left(\max\{\gamma^d, \beta^{N(d-c)}\} + 4^d \Big(N\alpha^d + \alpha^c \gamma^{-c} \sum_{1 \leq k \leq N} \beta^{k(d-c)}\Big)\right),
\end{aligned} \tag{16}$$

where in the last inequality we have used that if $a_n, b_n \geq 0$ then $\sum_n \max\{a_n, b_n\} \leq \sum_n a_n + \sum_n b_n$.

Provided we can establish the following claims:

(i) $N\alpha^d \leq \gamma^d$,
(ii) $\alpha^c \gamma^{-c} \sum_{k \in N_0} \beta^{k(d-c)} \leq \gamma^d$,
(iii) $\beta^{N(d-c)} \leq \gamma^d$,

then

$$\#(\mathcal{D}_{(j+1)N}(B) \setminus \mathcal{D}'_{(j+1)N}) \leq \beta^{-Nd} 3^d \gamma^d (1 + 4^d 2);$$

hence, by Observation 23,

$$\#(\mathcal{D}_{(j+1)N}(B) \cap \mathcal{D}'_{(j+1)N}) \geq \beta^{-Nd} \left(\frac{1}{2^d 4^d \sqrt{d}^d} - 3^d \gamma^d (1 + 4^d 2) \right),$$

as required.

Let us prove (i)–(iii):

(i) This holds by the definition of N.

(ii) Take $K_2 := \max\{\gamma^{-2d}, 2\gamma^{-d} \log(\gamma^{-d})\}$. By hypothesis and by using $c \in (0, d)$, $\beta \in (0, \frac{1}{4}]$, we have

$$\frac{\alpha^c}{\gamma^c (1 - \beta^{d-c})} \leq \frac{1}{\gamma^c K_2} \leq \gamma^{2d-c} < \gamma^d,$$

for the second claim.

(iii) Continuing, since $\alpha^c \leq \alpha^c \frac{1}{1-\beta^{d-c}} \leq \frac{1}{K_2} \leq \gamma^{2d}$, then $1 \leq \gamma^{-(2d-c)} \leq (\frac{\gamma}{\alpha})^c$, so $\gamma/\alpha \geq 1$, and thus

$$N \geq \frac{1}{2} \gamma^d \alpha^{-d}. \tag{17}$$

On the other hand, using the hypotheses, $c \in (0, d)$ and $\alpha \in (0, 1)$,

$$\alpha^d \leq \alpha^c \leq \frac{1}{K_2}(1 - \beta^{d-c}) \leq \frac{1}{K_2} |\log(\beta^{d-c})| = \frac{1}{K_2}(d - c)|\log(\beta)|, \tag{18}$$

where in the last inequality we have used that $d - c \in (0, 1)$, $\beta \in (0, \frac{1}{4}]$, $z := \beta^{d-c} \in (0, 1)$, and $f(z) := \log(\frac{1}{z}) + z + 1$ is a positive function on $(0, 1)$, so $1 - z \leq \log(\frac{1}{z})$. Then,

$$N\alpha^d K_2 \leq N(d - c)|\log(\beta)|. \tag{19}$$

By inequalities (17) and (19) and the definition of K_2,

$$N(d - c)|\log(\beta)| \geq N\alpha^d K_2 \geq \frac{\gamma^d}{2} K_2 \geq |\log(\gamma^d)|$$

which is equivalent to claim (iii).

This concludes the proof of Proposition 24. □

Conclusion of the proof.

For each $\gamma \in (0, 1)$ we proceed as follows:

By definition, $B_0 \in \mathcal{D}_0$. Moreover, $\phi_0(B_0) := 0 < (\gamma\rho)^c$, so $B_0 \in \mathcal{D}'_0$. We will construct a Cantor set F as the intersection of a sequence of unions of closed sets:

- $\mathcal{B}_0 := \{B_0\} \subset \mathcal{D}'_0$.
- Given a collection $\mathcal{B}_j \subset \mathcal{D}'_{jN}$ we construct the next level of sets $\mathcal{B}_{j+1} \subset \mathcal{D}'_{(j+1)N}$ by replacing each element of $B \in \mathcal{B}_j$ by $M := \left\lceil \beta^{-Nd} \left(\frac{1}{2^d 4^d \sqrt{d}^d} - 3^d \gamma^d (1 + 4^d 2) \right) \right\rceil$ elements of $\mathcal{D}_{(j+1)N}(B) \cap \mathcal{D}'_{(j+1)N}$; this is possible by Proposition 24.

We define

$$F := \bigcap_{j\in\mathbb{N}_0} \bigcup_{B\in\mathcal{B}_j} B.$$

By a standard argument (see e.g. [7,Example 4.6]),

$$\begin{aligned}
\dim_{\mathrm{H}}(F) &\geq \frac{\log(M)}{|\log(\beta^N)|} \geq \frac{\log(\beta^{-Nd}) + \log\left(\frac{1}{2^d 4^d \sqrt{d}^d} - 3^d\gamma^d(1+4^d 2)\right)}{N|\log(\beta)|} \\
&= d + \frac{\log\left(\frac{1}{2^d 4^d \sqrt{d}^d} - 3^d\gamma^d(1+4^d 2)\right)}{N|\log(\beta)|} \\
&\geq d - \frac{2\alpha^d \log\left(\left(\frac{1}{2^d 4^d \sqrt{d}^d} - 3^d\gamma^d(1+4^d 2)\right)^{-1}\right)}{\gamma^d|\log(\beta)|},
\end{aligned}$$

where we have used (17).

This last inequality holds for every $\gamma \in (0, 1)$. We can take, for example, $\gamma \in (0, 1)$ such that $3^d\gamma^d(1+4^d 2) = \left(1 - \frac{1}{2^d}\right)\frac{1}{(8\sqrt{d})^d}$ (this is not sharp, but it is close enough). For this γ we get

$$\dim_{\mathrm{H}}(F) \geq d - K_1 \frac{\alpha^d}{|\log(\beta)|},$$

where

$$K_1 := \frac{2d(24\sqrt{d})^d\left(\log(2) + \log(8\sqrt{d})\right)}{1 - \frac{1}{2^d}} \quad \text{making} \quad K_2 := \left(\frac{(24\sqrt{d})^d(1+4^d 2)}{1 - \frac{1}{2^d}}\right)^2.$$

It remains to prove that $F \subset S \cap B_0$, since then

$$\dim_{\mathrm{H}}(S) \geq \dim_{\mathrm{H}}(S \cap B_0) \geq \dim_{\mathrm{H}}(F) \geq d - K_1 \frac{\alpha^d}{|\log(\beta)|}.$$

Clearly $F \subset B_0$, by definition of F. We need to show that $F \subset S$. Let $x \in F$. For every $j \in \mathbb{N}$ there exists a unique $B_{jN} \in \mathcal{B}_j$ containing x. By definition of $\mathcal{B}_{j+1}$ we have that $B_{(j+1)} \subset \frac{1}{2}B_{jN}$. By (11), $\pi_{jN}(B_{(j+1)N}) = B_{jN}$. The sequence $(B_{jN})_j$ can be extended in a unique way to a sequence $(B_n)_n$ satisfying $B_n \in \mathcal{E}_n$ and $B_n := \pi_n(B_{n+1})$ for all n. We interpret this sequence as Bob's moves, to which Alice responds according to her winning strategy.

Thus, for each $x \in F$ we construct a sequence $(B_n)_n$ as before, where x is the only element of $\bigcap_n B_n$ (so $x = x_\infty$ is the outcome of the game). We will show that $x \in S$ by contradiction. Otherwise, suppose that $x \notin S$ where S is an (α, β, c, ρ)-winning set. Then, $x \in \bigcup_{m\in\mathbb{N}_0}\bigcup_i A_{i,m}$, where $\sum_i(\operatorname{diam} A_{i,m})^c \leq (\alpha\rho_m)^c = (\alpha\beta^m\rho)^c$ (since it is a legal move for Alice we know that $\bigcup_i A_{i,m} \in \mathcal{A}(B_m)$). So $x \in A \in \mathcal{A}(B_m)$ for some m, and as $x \in B_m$ we have $x \in A \cap B_m$. Since $\mathcal{A}_m^*(B_n) = \mathcal{A}(B_m)$ for every $n > m$ (because $\pi_m(B_n) = B_m$), then $\phi_j(B_{jN}) \geq (\operatorname{diam} A)^c$ for every j such that $jN > m$ (because $(\operatorname{diam} A)^c$ is just one term in the sum of the definition of $\phi_j(B_{jN})$ when $A \in \mathcal{A}_m^*(B_n)$).

On the other hand, since $B_{jN} \in \mathcal{D}'_j$, then $\phi_j(B_{jN}) \leq (\gamma\rho_{jN})^c$. Putting everything together, diam $A \leq \gamma\rho_{jN}$ for all j such that $jN > m$. Letting $j \to \infty$, we get diam $A = 0$, a contradiction. So $x \in S$, that is $F \subset S$.

Finally, using (18), and that $K_1/K_2 < 1$,

$$K_1 \frac{\alpha^d}{|\log(\beta)|} \leq \frac{(d-c)K_1}{K_2} < d,$$

so

$$d - K_1 \frac{\alpha^d}{|\log(\beta)|} > 0 \text{ if } \alpha^c \leq \frac{1}{K_2}(1-\beta^{d-c}).$$

This concludes the proof. □

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