STUDIES IN FORM-THEORY
1. MIXED DETERMINANTS - 2. THE PEDAL CORRESPONDENCE

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S T U D I E S   I N   F O R M - T H E O R Y.

1. MIXED DETERMINANTS.

2. THE PEDAL CORRESPONDENCE.

I qualified for the degree of M.A. with third class honours in mathematics, in the University of Madras, in 1917; this degree has been accepted by the Senate as equivalent to the degree of M.A. or B.Sc. with honours in the university of St. Andrews. I was in attendance as a research student at the University from November 1922 to July 1924.

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MIXED DETERMINANTS.

The idea of extending the determinant-concept to higher dimensions is a natural one. We find as a matter of fact, the first mention of cubic determinants as early as 1861. In the passage from the plane to the cubic determinant, one was however confronted with a difficulty, which necessitated a choice destined to have far-reaching implications.

Two equivalent definitions are possible for the ordinary determinant of order n, namely in the notation usually adopted in this paper,

$$(1)\quad |a_{ij}| = \frac{1}{n!}\sum_{i,j} (-)^{[i,j]} a_{i_1j_1} a_{i_2j_2} \ldots a_{i_nj_n}$$

$$(2)\quad |a_{ij}| = \sum_{j} (-)^{[j]} a_{1j_1} a_{2j_2} \ldots a_{nj_n}.$$

In the first definition the two suffixes i, j are treated impartially. In the second definition the first suffix is given an apparently special footing; but, to shew that this is only apparent and does not affect the interchangeability of the two suffixes, we have only to observe that the first definition is equivalent to the second, since every term of the former occurs $n!$ times with the same sign.

The analogue of the first definition is contra-indicated for the cubic determinant; for, each term of the determinant would occur as many times with the positive, as with the negative sign, so that ~~the~~ it would vanish identically. The second definition does not make the cubic determinant vanish, but when we adopt it, the difference of role belonging to the first suffix, which was only apparent in the case of the determinant $|a_{ij}|$, becomes now a real difference.

Reflecting on this today, one would perhaps be inclined to say that the generalisation of the determinant-concept by the use of suffixes with either of two characters, was writ large on the very initial notion of the cubic determinant, and was only waiting to become a full-fledged theory, for the touch of the first mathematician who chanced to deal with the topic. But from the inception of the idea of the cubic determinant in 1861 to the first publication of the extension by RICE in 1918, the period of time which elapsed was long for a popular subject like determinants. This would seem a little strange, when we consider that among those who worked at the subject in the interval, were some of the greatest contributors to the theory of many-dimensional determinants, CAYLEY, SCOTT, and GEGENBAUER. Two reasons seem to have been mainly responsible for this slowness of emergence of the idea of character in the suffix. The first reason is connected with the general difficulty of sustained mathematical operations in higher dimensions. In most memoirs on many-dimensional determinants one cannot avoid the feeling that the intellect has been kept at full tension by the mere fact of the dimensional extension, and has no spare attention to devote to more delicate matters of theory. There are indeed two well-marked stages undergone by the student of the mathematics of many-dimensional space. The first stage is one of ~~strugg~~ struggle and straining, where all the processes are painfully slow and conscious, and the intellect has to be put through a severe discipline, and taught to function without the visual support; the general feature of this stage is thus the effort to raise thought from a partially perceptual to a purely conceptual level. In the second stage, by dint of practice, the processes have become swift and easy, and the dimensional number is no longer felt to be an obstacle, as the reasoning faculty has learnt to work through concepts in which it figures only as an unessential feature.

The second reason why the possibility of two characters was ~~sh~~ shut out from the eyes of the earlier investigators, is of a more special kind, and is connected with the vanishing of the cubic determinant without non-signant suffixes. On the threshold of the subject of cubic determinants, the mind is naturally not attracted by any hypothesis which necessitates their identical vanishing. There is consequently a complete surrender to the less symmetrical alternative, and the attitude comes to be adopted, that the special role of the first suffix in the cubic determinant is not a matter of choice at all, but belongs to the nature of things. This notion once formed is difficult to eradicate.

Historically, however, it would appear that it was not the feeling of a fundamental unsymmetry in the pre-Rice odd-dimensional determinant, which inspired the extension to mixed determinant, but that the motif was supplied rather by the desire to extend the scope of Scott's law of multiplication. It appears to me to be a misplacement of emphasis to characterise the extension, as Lecat does, as one which admits file-multiplication even when the determinants are of odd dimensions. It is true that the Scott law of multiplication brings out crucially the difference of character in the suffix, and should necessarily be considered in connection with mixed determinants. But logically speaking, it must be insisted that,irrespectiveof any process of multiplication of determinants,we are already fully committed to the extension in having admitted a <u>single</u> suffix of the non-determinantal type in the cubic determinant.

The theory of determinants has been in its first inception and development,and in its general features,inseparable from the theory of forms;this connection has not been lost in the case of the mixed determinant.One aspect in which it appears in the present paper is in the theory of matricular and extensional invariants;there is a connection of mixed determinants with transvectants pointed out by Rice,and there are doubtless other aspects still awaiting discovery.It will be of great use to remember in this connection,that the importance of file-multiplication lies (as has been pointed out in the paper),not so much in its application to the product of two many-dimensional determinants, which is only its secondary or derivative aspect,but in its bearing on the process of transformation of the matrix of the multilinear form by multiplication with <u>two</u>-dimensional determinants. I think it necessary to emphasise this,as the theory as one finds it today,does not possess any conscious method.I think that future research in this subject should guide itself by following the lines of form-theory;only thus will several matters still obscure be cleared up.

There is room for speculation as to the true inward significance of the signant and non-signant characters.The subject of mathematics is full of dualities and reciprocal symmetries of various kinds,for which it would not be easy to find a basis of classification. There is however one particular family of these dualities,which in their rudimentary form appear as the opposition between "even" and "odd", and which, when analysed, seem to depend ultimately on the notion of group and the contrast between the identical operation and the involutoric operation. This even-odd duality, as we may call it, pursues us under different names throughout mathematics, and in the ~~forms~~ form of certain ideas, like "even and odd substitutions", "symmetric and skew", "commutative and alternating", constitutes the foundation of various branches of mathematical science. It is this same duality which appears again in Invariant Theory as the duality of the invariant-process consisting of Polarisation and Convolution; it is this duality again, which is given its rightful place in the foundations of determinant-theory, through the conception of the dual character of the suffix.

A feature of this even-odd duality should be noticed, namely that the relation between its two members is not a symmetric or interchangeable one, that is to say, the difference between them is one of inherent character (like the difference between the identical and the involutoric operation), not one of mutual aspect (like that between right and left). In this respect it differs from certain other familiar dualities, like that of <u>positive</u> and <u>negative</u> (in certain of their significations), and the duality between ~~co~~ contragredient sets of variables, or between the top and bottom suffixes in the tensor notation.

There is no doubt that the subjects in which this duality is developed in full measure, and in faultless balance of contrast (as for instance, the parallel geometrical theory of the quadric polarity and the null system), are among the most elevating and culturally valuable parts of mathematical theory. This balance of contrast is well noticeable in the theory of the mixed determinant, in the interchange of sections, in develop~~e~~ment by linkage, and in skew-symmetry. There is however much scope for further work in this connection, both generally and in regard to skew-symmetry, as a likely result of which, the general features of the two types of suffixes will appear in fuller detail, and in clearer outline.

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XVI.—On Mixed Determinants. By R. Vaidyanathaswamy.

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It is well known that the value of a cubic (or odd-dimensional) determinant $|a_{ijk}|$ $(i, j, k=1, 2, \ldots n)$ will reduce identically to zero, if exactly similar rôles are assigned to all the suffixes in determining the signs of the terms. To escape this difficulty, the usual theory of determinants of higher dimensions assigns a special rôle to the first suffix i, and considers that the class of the permutation of the values of i in any term makes no contribution to the sign of the term. This is, in effect, to make the suffix i *non-determinantal* in character.

It might be reasonably held, that the assigning of a special non-determinantal rôle to one particular suffix is highly objectionable, not only because it is against the spirit of the determinant-concept, but also since it leads to results which are discordant with the Theory of Forms. In this paper we shall, however, take an inclusive view, and investigate a generalised kind of determinant—the *mixed determinant*—which possesses partly suffixes with the familiar determinantal property, and partly non-determinantal suffixes of the type i—which we call *Scott suffixes.* A mixed determinant which possesses no Scott suffixes is a determinant in the strict sense, or a *pure determinant.*

A perusal of this paper will show that Scott suffixes belong *naturally* to the theory of determinants, inasmuch as they arise spontaneously even in the properties of pure determinants. It will appear in fact, that a theory which restricts itself solely to pure determinants is not self-complete, but is on the other hand naturally led therefrom to create Scott suffixes also.

Definition and Fundamental Properties.

§ 1. An array of elements $a_{ij\ldots\alpha\beta\ldots}$ $(i, j, \ldots \alpha, \beta, \ldots=1, 2, \ldots n)$, each of which has p suffixes, is called a p-*dimensional matrix of the* n*th order*. The elements of the matrix may be conceived topographically as arranged in the form of a hyper-cube in space of p dimensions. The totality of the elements of the matrix, in which the value of a particular suffix i is a fixed number, forms a $(p-1)$-dimensional matrix which may be called a *section* of the given matrix, corresponding to the suffix i. Similarly the elements of the matrix, in which r specified suffixes have

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fixed values, form a $(p-r)$-dimensional matrix, which may be called a *subsection* corresponding to the r suffixes. It is clear that as the particular suffix i, or the r specified suffixes, take various fixed values, the corresponding sections and subsections will be parallel to each other in the topographical representation.

The matrix $|a|$ can be associated with a single number called its "determinant."* We recognise, however, that any suffix may have either of two different *characters* which affect the computation of the determinant in different ways. Thus, if $\alpha, \beta, \ldots$ are Scott suffixes (written for distinctness at the top) and $i, j, \ldots$ are determinantal suffixes (written at the bottom), the determinant A of the matrix is defined as follows:

$$\mathrm{A} = \left| a^{\alpha\beta\,\ldots}_{ij\,\ldots} \right| = \frac{1}{n!}\sum (-)^{[i,\,j,\,\ldots]} a^{\alpha_1\beta_1\,\ldots}_{i_1j_1\,\ldots}\; a^{\alpha_2\beta_2\,\ldots}_{i_2j_2\,\ldots} \;\ldots\; a^{\alpha_n\beta_n\,\ldots}_{i_nj_n\,\ldots},$$

summed for all sets of permutations $\alpha_1\alpha_2 \ldots \alpha_n$, $\beta_1\beta_2 \ldots \beta_n, \ldots$, $i_1i_2 \ldots i_n, j_1j_2 \ldots j_n, \ldots$ each of $1, 2, \ldots n$, the sign indicated being the product of the signs associated† with the permutations $i_1i_2 \ldots i_n$, $j_1j_2 \ldots j_n, \ldots$. It should be noticed particularly that the classes of the permutations $\alpha_1\alpha_2 \ldots \alpha_n, \ldots$ of the Scott suffixes, have no effect on the sign of the term. Every term occurs obviously $n!$ times in the summation.

A determinant will be called a *pure determinant* if it has no Scott suffix, and a Scott function‡ if it has no determinantal suffix. To illustrate the definition of determinant, let us evaluate the one-dimensional determinants $|a_i|$ and $|a^i|$, of which the former is a pure determinant, and the latter a Scott function. We have:

$$|a_i| = \frac{1}{n!}\sum (-)^{[i]} a_{i_1}a_{i_2} \ldots a_{i_n} = 0.$$

$$|a^i| = \frac{1}{n!}\sum a^{i_1}a^{i_2} \ldots a^{i_n} = a^1a^2 \ldots a^n.$$

Thus *a pure one-dimensional determinant vanishes identically, and a one-dimensional Scott function is equal to the product of its elements.*§

* The word "determinant" used without any qualification is to be understood to mean "mixed determinant."

† A permutation is of *even* or *odd* class, according as it is equivalent to an even or odd number of interchanges. The sign associated with the permutation is + in the former and − in the latter case.

‡ The term "Scott function" is used by Hedrick, *Annals of Math.*, 2nd series, 1–2, to denote a "signless determinant" $|a_{ij}|$.

§ The question of the validity of this theorem for $n=1$ is one of some interest, though it has little or no practical importance. While, according to this theorem, the pure one-element determinant $|a|$ vanishes, we are assured with equal certainty from the

§ 2. The following fundamental properties are easily seen to follow from the definition of determinant:

(1) *A determinant is not altered in value by interchange of two suffixes of like character.* The determinant is, however, altered by interchanging a Scott and a determinantal suffix.

(2) *The interchange of two values of any suffix* i *leaves the determinant unaltered or changes its sign, according as* i *is a Scott or a determinantal suffix.*

If a section of a matrix corresponding to a suffix be called a *Scott* or a *determinantal* section according to the character of the suffix, this theorem might be stated as follows:

The interchange of two parallel sections of a determinant leaves it unaltered or changes its sign, according as they are Scott or determinantal sections.

As a corollary we have:

If the corresponding elements of two parallel determinantal sections have a constant ratio, the determinant vanishes identically.

(3) *A determinant vanishes identically, unless it has an even number of determinantal suffixes.*

In particular, a pure determinant of odd dimensions vanishes identically.

Development by Linkage of Suffixes.

§ 3. The Scott and the determinantal characters of a suffix have a close analogy with the signs *plus* and *minus.* We define the *product of two or more characters* to mean the character which corresponds to the product of their corresponding signs. Thus the product of *like* (that is, two

theory of elimination from linear equations, that the same pure determinant should have the value a. The contradiction arises from the fact that our theory of determinants is concerned only with determinants of *definite* dimensionality. *A determinant of order one is of indeterminate dimensions*, corresponding to the geometrical fact that a *point* could be considered as the shrinking limit of a piece of continuum of any number of dimensions. Hence if we are given a one-element determinant $|a|$, *and if we are told in addition that it is a* p-*dimensional determinant with* q *Scott suffixes*, we would logically be in a position to evaluate it; namely $|a| = a$ or 0, according as $p - q$ is even or odd.

The following slightly different point of view may perhaps be considered preferable. The concept of determinant as a number associated with an ordered spatial array, is based essentially on the notion of *suffix.* Now a suffix is something which possesses necessarily a *field of variation.* But if the order n is equal to one, there is no field of variation and therefore, strictly speaking, there are no suffixes (unless convention steps in and says that there are). Hence *determinants of order unity are determinants of zero dimensions*, and therefore may be legitimately considered to be meaningless.

As this case does not occur in practice, it is immaterial which of these two views we adopt.

Scott or two determinantal) characters is the Scott character, the product of *unlike* characters is the determinantal character.

The determinant $|a|$ was defined in § 1 as $\frac{1}{n!}$ times the sum of a number of terms, the summation being extended for all independent permutations of the values of the various suffixes. Suppose now that λ, μ are any two suffixes, and that their values for the elements in any term of $|a|$ are $\lambda_1\lambda_2 \ldots \lambda_n$; $\mu_1\mu_2 \ldots \mu_n$. The set $\mu_1\mu_2 \ldots \mu_n$ can be obtained by performing a permutation Ω upon the set $\lambda_1\lambda_2 \ldots \lambda_n$. Here Ω can be any one of the $n!$ permutations of $1, 2, \ldots n$. We can accordingly class the terms in the expansion into $n!$ groups, each group corresponding to a permutation Ω. It should be noticed that the suffix μ has no freedom within any of these groups, since the permutation of its values in a term is determined from the permutation of the values of λ in the same term and the linkage permutation Ω. It is obvious that each of these groups of terms, multiplied by $\frac{1}{n!}$, is the expansion (possibly with the sign changed throughout) of a determinant $|b|$, which has all the suffixes of $|a|$ except the suffix μ.

By interchanging two values of a suffix, it also becomes obvious that any suffix other than λ has the same character in each $|b|$, as it has in $|a|$. To determine the character of λ in $|b|$, suppose that two values of λ are interchanged therein. Then on account of the linkage of μ with λ, it follows that in the corresponding group of terms not only two values of λ, but also two values of μ will be thereby interchanged. Hence *the character of* λ *in each determinant* $|b|$ *is the product of the characters of* λ, μ *in the given determinant* $|a|$.

Finally, to determine whether each group of terms is equal to *plus* or *minus* the corresponding determinant $|b|$, compare the signs with which a term occurs in $|a|$, and in the expansion of a determinant $|b|$. Since suffixes other than λ, μ have the same character in any $|b|$ as in $|a|$, any difference of sign must be due to the fact, that in the former case the sign of the term is affected by the permutations of the λ's and the μ's (according to the character of λ, μ in $|a|$), while in the latter, it is affected by the permutation of the λ's (according to the character of λ in $|b|$).

Supposing first that μ is a Scott suffix, it has no effect on the sign in the former case; further, λ will have like character in $|b|$ and in $|a|$, and will therefore affect the sign similarly in the two cases. Thus there is no difference of sign. Hence, *if* μ *is a Scott suffix, the determinant* $|a|$ *is equal to the sum of the* $n!$ *determinants* $|b|$.

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Suppose next that μ is a determinantal suffix. Then in the former case it introduces the sign pertaining to the permutation of its values. Further, λ will have unlike character in $|b|$ and in $|a|$; therefore it does not affect the sign in one of the two cases, and introduces the sign pertaining to the permutation of its values, in the other. Thus the net difference of sign is the product of the signs associated with the permutations of the λ's and the μ's in the term—that is to say, the sign associated with the corresponding linkage permutation Ω. Hence, *if μ is a determinantal suffix, the determinant* $|\mathrm{a}|$ *is the sum of the* n! *determinants* $|\mathrm{b}|$, *each taken with the sign of its associated linkage permutation* Ω.

Taking all possible characters of λ and μ, we have four types of development which are tabulated below.

$$\mathrm{A} = \left| a_{ijk\ldots}^{\alpha\beta\gamma\ldots} \right|.$$

I. By linking β with α, $\mathrm{A} = \sum \left| b_{ijk\ldots}^{\alpha\gamma\ldots} \right|$, where $b_{ijk\ldots}^{\alpha\gamma\ldots} = a_{ijk\ldots}^{\alpha\beta_\alpha\gamma\ldots}$, and the summation is for all permutations $\beta_1\beta_2 \ldots \beta_n$ of $1, 2, \ldots n$.

II. By linking α with i, $\mathrm{A} = \sum \left| b_{ijk\ldots}^{\beta\gamma\ldots} \right|$, where $b_{ijk\ldots}^{\beta\gamma\ldots} = a_{ijk\ldots}^{\alpha_i\beta\gamma\ldots}$, and the summation is for all permutations $\alpha_1\alpha_2 \ldots \alpha_n$ of $1, 2, \ldots n$.

III. By linking i with α, $\mathrm{A} = \sum \pm \left| b_{\alpha jk\ldots}^{\beta\gamma\ldots} \right|$, where $b_{\alpha jk\ldots}^{\beta\gamma\ldots} = a_{i_\alpha jk\ldots}^{\alpha\beta\gamma\ldots}$, and the summation is for all permutations $i_1i_2 \ldots i_n$ of $1, 2, \ldots n$, the ambiguous sign being that associated with this permutation.

IV. By linking j with i, $\mathrm{A} = \sum \pm \left| b_{k\ldots}^{i\alpha\beta\gamma\ldots} \right|$, where $b_{k\ldots}^{i\alpha\beta\gamma\ldots} = a_{ij_ik\ldots}^{\alpha\beta\gamma\ldots}$, and the summation is for all permutations $j_1j_2 \ldots j_n$ of $1, 2, \ldots n$, the ambiguous sign being that associated with this permutation.

We may observe that the development of an ordinary determinant $|a_{ij}|$ is a particular case of IV. For by IV

$$|a_{ij}| = \Sigma \pm |b^i|, \text{ where } b^i = a_{ij_i}.$$

Since a one-dimensional Scott function $|b^i|$ is equal to the product of its elements, this reduces to the usual development.

§ 4. The process of development by linkage may be applied simultaneously to more than two suffixes. Thus by linking up r suffixes $\mu, \nu, \ldots$ with the suffix λ, a p-dimensional determinant $|a|$ of order n, is expressed as the sum of $(n!)^r(p-r)$-dimensional determinants $|b|$ of the same order, each taken with a certain sign. The suffixes $\mu, \nu, \ldots$ do not appear in $|b|$, suffixes other than λ have the same character in any $|b|$ as in $|a|$, and the character of λ in each $|b|$ is the product of the characters of $\lambda, \mu, \nu, \ldots$ in $|a|$. The sign to be attached to any determinant $|b|$ is

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the product of the signs belonging to the permutations which express the linkage of the *determinantal* suffixes among $\mu, \nu, \ldots$. The proof of all these statements is immediate.

As an instance of development by multiple linkage, we observe, that on developing $|a|$ by linking up *all* its suffixes with one of them, and evaluating the resulting one-dimensional determinants $|b|$, we reach the expansion defining $|a|$.

As a second illustration, let us evaluate a p-dimensional determinant $|a|$ of order n, in which the values of the elements are independent of the values of q of the suffixes $\lambda, \mu, \ldots$ (say). If any of the q suffixes $\lambda, \mu, \ldots$ is determinantal, then parallel determinantal sections of $|a|$ will be identical, so that $|a|$ will vanish identically. We therefore suppose $\lambda, \mu, \ldots$ to be Scott suffixes, and write

$$a^{\alpha\beta\ldots\lambda\mu\ldots}_{ij\ldots} = b^{\alpha\beta\ldots}_{ij\ldots} \text{ for all values of } \lambda, \mu, \ldots.$$

It is easy to see that every subsection of $|a|$ corresponding to $\lambda, \mu, \ldots$ is identical with $|b|$. Each of the $(n\,!)^q$ determinants which arise in the development of $|a|$ by linkage of $\lambda, \mu, \ldots$ with any of the other suffixes, is obviously identical with $|b|$. Hence

$$|a| = (n\,!)^q\,|b|.$$

As a particular case, if in a cubic determinant $|a^i_{jk}|$ we have $a^i_{jk} = b_{jk}$, then $|a^i_{jk}| = n\,!\,|b_{jk}|$.

Skew-symmetry.

§ 5. A determinant is said to possess *skew-symmetry* in two suffixes α, β, if the interchange of the values of α, β in any element merely alters the sign of the element. In particular the elements, in which α, β have equal values, vanish.

A determinant which possesses skew-symmetry in two suffixes of like character, vanishes identically, unless it is of even order.

For, by § 2, the interchange of the two suffixes does not alter the value of the determinant, since they are of like character. But on account of the skew-symmetry, this interchange should multiply the determinant by $(-1)^n$, where n is the order. Whence the result.

A determinant which possesses skew-symmetry in two suffixes α, β of unlike *character, vanishes identically whether the order be even or odd.**

Let β be a Scott and α a determinantal suffix. Develop the given

* This remarkable theorem and its proof are due in substance to Campbell, *Proc. L.M.S.*, vol. xxiv, 1st series.

determinant by linkage of β with α, into the sum of $n!$ determinants $|b|$. Let every linkage permutation Ω be expressed in the form $\omega_1\omega_2 \ldots \omega_r$, where ω_k is a cyclic permutation on n_k of the numbers $1, 2, \ldots n$, so that

$$n_1 + n_2 + \ldots + n_r = n.$$

Let any other linkage permutation Ω' be said to belong to the same *group* as Ω, if it is derived from Ω by changing any number of the cyclic permutations ω into their inverses. Then it is seen easily that any two determinants $|b|$ corresponding to linkage permutations Ω, Ω' of the same group, are such that the terms of either are equal (on taking account of the skew-symmetry) to the terms of the other in absolute value.

To prove that the sum of the determinants $|b|$ corresponding to all linkage permutations of a group is identically zero, suppose that Ω' is derived from Ω by changing k of the cyclic permutations ω—say $\omega_1\omega_2 \ldots \omega_k$—into their inverses. Then the difference of signs attached to a term as it appears in the two determinants $|b|$ corresponding to Ω and Ω', is $(-1)^{n_1+n_2+\ldots+n_k}$ due to the skew-symmetry and $(-1)^{n_1-1+n_2-1\ldots+n_k-1}$ due to the alteration in the class of the α-permutation. The net difference of sign is thus $(-1)^k$. Since the number of ways of choosing an even number of ω's from $\omega_1\omega_2 \ldots \omega_r$ is equal to the number of ways of choosing an odd number, it follows that in each group of determinants $|b|$ every term will occur as often with the positive as with the negative sign. Thus each group of determinants $|b|$, and therefore the given determinant vanishes identically.*

We can also evaluate the determinant for the case in which the skew-symmetry is partial, and does not extend to the vanishing of the elements in which α, β have equal values. It is not difficult to see that the difference of sign which was found to be $(-1)^k$ will now be $(-1)^{k'}$, where k' is the number of numbers $n_1n_2 \ldots n_k$ which are different from unity. The theorem that the sum of the determinants $|b|$ corresponding to all linkage permutations of a group is identically zero, will still be true for all groups, excepting the particular group which is formed by the identical permutation only. Thus the given determinant will in this case be equal to that determinant $|b|$, the linkage permutation of which is the identical permutation.

Multiplication of Determinants.

§ 6. We now go to deal with the three successive stages of the product of two determinants, which we call respectively *Conjunction*, *Identification*,

* This proof is easily seen to be also valid for the case in which some or all of $n_1n_2 \ldots n_r$ are equal to 1 or 2.

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and *Contraction*. These we regard primarily as matricular processes, and discuss separately the question whether they lead to true products of determinants, as the question of their *arithmetical validity*. We shall find that the arithmetical validity of Conjunction is unconditional, but that the others are arithmetically valid only under certain limitations. Multiplication by Contraction is, however, the most important of the three stages, as it reproduces the mode of transformation of the matrix of a multilinear form by linear substitution of the variables.

Product by Conjunction.—Let the determinants $|a_{i\,\ldots}^{a\,\ldots}|$ and $|b_{i'\,\ldots}^{a'\,\ldots}|$ be both of the nth order, and of p and q dimensions respectively. We have then by definition :

$$|a| = \frac{1}{n!}\sum(-)^{[i,\,\ldots]} a_{i_1\,\ldots}^{a_1\,\ldots}\, a_{i_2\,\ldots}^{a_2\,\ldots} \ldots a_{i_n\,\ldots}^{a_n\,\ldots}$$

$$|b| = \frac{1}{n!}\sum(-)^{[i'\,\ldots]} b_{i'_1\,\ldots}^{a'_1\,\ldots} \ldots b_{i'_n\,\ldots}^{a'_n\,\ldots}.$$

Hence by direct multiplication we have :

$$|a|\times|b| = \frac{1}{(n!)^2}\sum(-)^{[i,\,\ldots\,i',\,\ldots]} c_{i_1\,\ldots\,i'_1\,\ldots}^{a_1\,\ldots\,a'_1\,\ldots} \ldots c_{i_n\,\ldots\,i'_n\,\ldots}^{a_n\,\ldots\,a'_n\,\ldots}$$

$$= \frac{1}{n!}\left|c_{i\,\ldots\,i'\,\ldots}^{a\,\ldots\,a'\,\ldots}\right|, \text{ where } c_{i\,\ldots\,i'\,\ldots}^{a\,\ldots\,a'\,\ldots} = a_{i\,\ldots}^{a\,\ldots}\times b_{i'\,\ldots}^{a'\,\ldots}.$$

The determinant $|c|$ is of the nth order and $(p+q)$ dimensions.

§ 7. *Product by Identification.*—To obtain the product of $|a|$ and $|b|$ by Identification, we have to assume at the start that one of them at least —say $|b|$—has an *even* number of determinantal suffixes. In conformity with the notation of § 6 we suppose that the suffixes of $|a|$, $|b|$ are represented by undashed and dashed letters respectively.

We now develop the determinant $|c|$ of § 6 by linking up any dashed with any undashed suffix—say by linking a' with a. Then by expressing the c-elements as products of a- and b-elements, and using the fact that $|b|$ has an even number of determinantal suffixes, it is not difficult to show that *any two of the* n! *determinants* |d| *which appear in the development will be equal when taken with the signs with which they appear.*

It follows from this result that the product of $|a|$ and $|b|$ is equal to any $|d|$ with its sign. Taking the particular determinant $|d|$ in which the linkage is determined by the identical permutation, we see that the product of $|a|$ and $|b|$ is a determinant $|d|$ which is obtained by the following process from $|c|$:

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(1) Replacing a dashed suffix α' by an undashed suffix α, or *identifying* α' and α.

(2) Removing a superfluous α, and giving to the other α the character which is the product of the characters of α' and α.

The suffix α in the product $|d|$ by Identification will be called the *Resultant Suffix*.

The three forms of $|d|$ corresponding to different characters of α' and α may now be written down:

I. α' and α are Scott suffixes.

$$\left| a^{\alpha\beta\,\cdots}_{i\,\cdots} \right| \times \left| b^{\alpha'\beta'\,\cdots}_{i'\,\cdots} \right| = \left| d^{\alpha\beta\,\cdots\,\beta'\,\cdots}_{i\,\cdots\,i'\,\cdots} \right|,$$

where

$$d^{\alpha\beta\,\cdots\,\beta'\,\cdots}_{i\,\cdots\,i'\,\cdots} = c^{\alpha\beta\,\cdots\,\alpha\beta'\,\cdots}_{i\,\cdots\,i'\,\cdots} = a^{\alpha\beta\,\cdots}_{i\,\cdots} \times b^{\alpha\beta'\,\cdots}_{i'\,\cdots}.$$

II. α' and α are determinantal.

$$\left| a^{\beta\,\cdots}_{\alpha i\,\cdots} \right| \times \left| b^{\beta'\,\cdots}_{\alpha' i'\,\cdots} \right| = \left| d^{\alpha\beta\,\cdots\,\beta'\,\cdots}_{i\,\cdots\,i'\,\cdots} \right|,$$

where

$$d^{\alpha\beta\,\cdots\,\beta'\,\cdots}_{i\,\cdots\,i'\,\cdots} = c^{\beta\,\cdots\,\beta'\,\cdots}_{\alpha i\,\cdots\,\alpha i'\,\cdots} = a^{\beta\,\cdots}_{\alpha i\,\cdots} \times b^{\beta'\,\cdots}_{\alpha i'\,\cdots}.$$

III. α' and α have unlike character.

$$\left| a^{\alpha\beta\,\cdots}_{i\,\cdots} \right| \times \left| b^{\beta'\,\cdots}_{\alpha' i'\,\cdots} \right| = \left| d^{\beta\,\cdots\,\beta'\,\cdots}_{\alpha i\,\cdots\,i'\,\cdots} \right|,$$

where

$$d^{\beta\,\cdots\,\beta'\,\cdots}_{\alpha i\,\cdots\,i'\,\cdots} = c^{\alpha\beta\,\cdots\,\beta'\,\cdots}_{i\,\cdots\,\alpha i'\,\cdots} = a^{\alpha\beta\,\cdots}_{i\,\cdots} \times b^{\beta'\,\cdots}_{\alpha i'\,\cdots}.$$

It is necessary to observe that the product by Identification is *not* valid if each of the determinants $|a|$, $|b|$ has an odd number of determinantal suffixes. The following examples suffice to show this:

(1) $0 = |a_i| \times |b_i| \neq |d^i|$, where $d^i = a_i b_i$.

(2) $0 = |a^i_j| \times |b_k| \neq |d_{jk}|$, where $d_{\ k} = a^k_j \times b_k$.

(3) $0 = |a^i_j| \times |b^k_l| \neq |d^i_{jl}|$, where $d^i_{jl} = a^i_j \times b^i_l$.

The continued product of many determinants by Identification is not valid if more than one of them have an odd number of determinantal suffixes.

Multiplication by Identification may be applied to the continued product of many determinants either by using different pairs of suffixes at each stage, or by using only one suffix from each determinant as the following examples show:

(1) If $d_{ijkl} = a_{il} \times b_{il} \times c_{il}$, then $|d| = |a| \times |b| \times |c|$.

(2) If $d^{\alpha\beta}_{ij} = a^{\alpha\beta} \times b_{i\beta} \times c_{j\beta}$, then $|d| = |a| \times |b| \times |c|$.

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In cases of identification like these, it is important to remember that the character of the resultant suffix (l and β in the examples) in the product is the product of the characters of all the suffixes identified.

§ 8. *Product by Contraction.*—By "contraction" of a suffix α in a matrix $|a|$, we mean the process of deriving from $|a|$ a new matrix of the same order and one dimension less, by summing the elements of $|a|$ over the different values of α. We now go to show that contraction of the resultant suffix α in $|d|$—the product by identification of $|a|$ and $|b|$—is arithmetically permissible in certain cases.

The contraction of α, if it is a *determinantal* suffix in $|d|$, would not in general be valid (except in the trivial case when $|a|$ and $|b|$ have each an odd number of determinantal suffixes), as it leads from an identically vanishing to a non-vanishing determinant, or *vice versa*. We need therefore consider only the case in which α is a Scott suffix in $|d|$.

We assume that one at least of $|a|$, $|b|$ has an even number of determinantal suffixes, so that $|d|$ is a valid product. Let $|e|$ be the contracted form of $|d|$, so that

$$|d| = \left| d_{i\,\ldots\,i'\,\ldots}^{\alpha\beta\,\ldots\,\beta'\,\ldots} \right|, \; |e| = \left| e_{i\,\ldots\,i'\,\ldots}^{\beta\,\ldots\,\beta'\,\ldots} \right|; \; e_{i\,\ldots\,i'\,\ldots}^{\beta\,\ldots\,\beta'\,\ldots} = \sum_{\alpha} d_{i\,\ldots\,i'\,\ldots}^{\alpha\beta\,\ldots\,\beta'\,\ldots}.$$

Since each element e is the sum of n terms, we can express $|e|$ as the sum of n^n determinants of the same order and dimensions, namely:

$$|e| = \sum \left| f_{i\,\ldots\,i'\,\ldots}^{\beta\,\ldots\,\beta'\,\ldots} \right|; \; f_{i\,\ldots\,i'\,\ldots}^{\beta\,\ldots\,\beta'\,\ldots} = d_{i\,\ldots\,i'\,\ldots}^{\alpha_i\beta\,\ldots\,\beta'\,\ldots},$$

the summation extending for all the n^n sets of (not necessarily unequal) numbers $\alpha_1\alpha_2 \ldots \alpha_n$ chosen from $1, 2, \ldots n$. The $n!$ determinants $|f|$ which correspond to sets of *unequal* numbers $\alpha_1\alpha_2 \ldots \alpha_n$ are the terms in the development of $|d|$ by linkage of α with i. Hence to establish the arithmetical validity of the contraction, it suffices to show that each of the $(n^n - n!)$ determinants $|f|$ which correspond to sets of numbers $\alpha_1\alpha_2 \ldots \alpha_n$ *not all different*, vanishes identically.

To prove this, consider a determinant $|f|$ for which $\alpha_1 = \alpha_2$ (say). Develop it by linking up all the undashed suffixes with one of them—say i. Then, by expressing the elements as products of a- and b-elements, it is easy to see that parallel sections corresponding to $i = 1, 2$ will have a constant ratio in each of the determinants $|g|$ which arise in the development, in consequence of $\alpha_1 = \alpha_2$. Hence *if* i *is determinantal in each* $|g|$, each $|g|$ and therefore each of the $(n^n - n!)$ determinants $|f|$ will vanish and contraction will be valid. Now, the character of i in $|g|$ is the product of the characters of all the undashed suffixes in $|f|$. Hence i will be determinantal in $|g|$, only if $|f|$, and therefore $|e|$, and therefore $|d|$ contains an odd number of undashed determinantal suffixes. Thus, finally, contraction is valid only if $|d|$ contains an odd number of either *undashed* or (similarly) *dashed* determinantal suffixes. This will necessarily happen if one of $|a|$, $|b|$ contains an even, and the other an odd number of determinantal suffixes. Thus, for this case, contraction of the resultant Scott suffix α is always valid.

If $|a|$ and $|b|$ each contain an even number of determinantal suffixes, then a determinantal suffix from either could have disappeared in $|d|$ only by alteration of character by identification with a determinantal suffix of the other. Hence *when both* $|a|$ *and* $|b|$ *have an even number of determinantal suffixes, contraction of the resultant Scott suffix is valid, only if it has been derived from the identification of two determinantal suffixes.*

The theorem of multiplication by contraction, in so far as it applies to determinants of higher dimensions usually so called, has been correctly stated by Lloyd Tanner (*Proc. L.M.S.*, 1879), but incorrectly without the necessary qualifications by Scott (Scott and Mathews, *Theory of Determinants*, 1880, para. 17, p. 97).

As regards the validity of contraction in the product of many determinants by simultaneous identification (only one suffix being used from each), it is easily seen from particular cases, that *if among the suffixes identified, an even number (zero excluded) are determinantal, it is permissible to contract the resultant suffix.*

Product of Matrices.

§ 9. In the concept "matrix" as contrasted with "determinant," the order of the suffixes is significant. We define "the product of two matrices of the same order" to mean invariably the product by contraction. The two suffixes identified and contracted will be called *active* and the others *passive*. It is necessary to observe, that in forming the product the order of the passive suffixes should not be disturbed.

The reason of limiting the "product of matrices" to the product by contraction lies (apart from other dominant considerations) in the greater suitability of Contraction for matrices, as compared with *Conjunction* or *Identification*. For Conjunction has the disadvantage of the numerical factor $\frac{1}{n!}$; in Identification there arises the serious question of the *position* of the resultant suffix which can only be settled by artificial assumptions. In Contraction this difficulty is disposed of as the resultant suffix disappears altogether.

The n^r subsections of a matrix $|a|$ of order n, corresponding to all fixed values of r suffixes $\lambda, \mu, \ldots$ may be called a *system of subsections*. Also any subsection of this system may be said to *contain* any suffix of $|a|$ other than $\lambda, \mu \ldots$.

Theorem of Multiplication by subsections.

The product of two matrices with the respective active suffixes α, β may also be built up from the products (with the same active suffixes) of any system of subsections of the first, containing the active suffix α, with any system of subsections of the second, containing β.

The Associative Law.

The product of three matrices A, B, C of the same order is associative, provided the active suffixes in the products $A \,.\, B$ and $B \,.\, C$ are also the active suffixes in the respective products $A \,.\, BC$ and $AB \,.\, C$.

Determinants as Invariants.

§ 10. *The pure determinant of a multilinear form $A = \Sigma a_{ij\ldots l} x_i y_j \ldots w_l$ $(i, j, \ldots l = 1, 2, \ldots n)$ is an invariant of the form for independent linear transformations of the sets of variables.*

The proof of the theorem is straightforward, and the result is obtained by means of the Associative Law, and the conditions of arithmetical validity of contraction.

By writing A symbolically in the shape:

$$A = \alpha_{1_x}\beta_{1_y} \ldots \lambda_{1_w} = \alpha_{2_x}\beta_{2_y} \ldots \lambda_{2_w} = \text{etc.},$$

we see that it has only one invariant $(\alpha_1\alpha_2 \ldots \alpha_n)(\beta_1\beta_2 \ldots \beta_n) \ldots (\lambda_1\lambda_2 \ldots \lambda_n)$ of the nth degree. Hence the pure determinant of the form is a numerical multiple $\left(\text{easily seen to be } \dfrac{1}{n!}\right)$ of this invariant. If the number of sets of variables is odd, the symbolical invariant vanishes identically, verifying the fact that pure odd-dimensional determinants vanish.

By identifying all the sets of variables, we see that the symmetric pure p-dimensional determinant of the n-ary p-ic $\alpha_x^p = \beta_x^p = \ldots$ is $\dfrac{1}{n!}$ times the invariant $(\alpha\beta \ldots)^p$. As a special case the pure p-dimensional determinant of order 2 formed from the coefficients of a binary p-ic is one-half its apolar invariant. A similar remark holds for the apolar invariant of a double-binary form.*

* For the invariant theory of the multilinear form see Study, *Vectorenrechnung* (Braunschweig, 1923), p. 110. For apolar theory of double-binary form see Kasner, *Trans. Am. Math. Soc.*, vol. i.

§ 11. *Matricular Invariants.*—Suppose we have a form (or system of forms) f in certain sets of n variables, and that f is changed into f' by linear transformations of the variables. A matrix $|a|$ of order n, the elements of which are functions of the variables and the coefficients of f, is said to be a *matricular invariant* of f, if the corresponding matrix for f' is the matrix-product of $|a|$ with the matrices of the various linear transformations (each possibly taken more than once). The active suffixes of $|a|$ in these multiplications may be collectively called "the suffixes actively concerned in the transformation of $|a|$," and the remaining suffixes, if any, *the inert suffixes* of $|a|$. (For instance, the matrix of a multilinear form has no inert suffixes.) The manner in which the suffixes actively concerned in the transformation enter into the multiplications with the matrices of the linear transformations may be referred to as "the weight-type of the matricular invariant."

The conditions of arithmetical validity of contraction permit us now to state the general theorem :

The determinant of a matricular invariant $|\mathrm{a}|$ *is also an invariant, provided that in its computation the* determinantal *character has been assigned to every suffix actively concerned in the transformation of* $|\mathrm{a}|$.

It will be noticed that this theorem holds, whatever character we may assign to the inert suffixes (if any) of $|a|$. Hence a matricular invariant $|a|$ possessing q inert suffixes gives rise to 2^q invariant determinants $|a|$.

The theorem of multiplication by subsections gives the following important property of matricular invariants possessing inert suffixes :

If a matricular invariant $|\mathrm{a}|$ *possesses the inert suffixes* $\lambda, \mu, \ldots$ *then every subsection of* $|\mathrm{a}|$ *corresponding to any or all of the suffixes* $\lambda, \mu \ldots$ *is also a matricular invariant of the same weight-type as* $|\mathrm{a}|$.

The converse of this theorem is also important:

If every subsection of a matrix $|\mathrm{a}|$ *corresponding to the suffixes* λ, $\mu \ldots$ *is a matricular invariant of any of the same weight-type, then* $|\mathrm{a}|$ *is also a matricular invariant of the same weight-type, with the inert suffixes* $\lambda, \mu \ldots$.

It appears from these theorems that the presence of inert suffixes in a matricular invariant renders it *reducible* in a sense.

The simplest example of matricular invariants possessing inert suffixes would be furnished by the *extensional invariants.*

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§ 12. *Extensional Invariants of an array of forms.*—Let f be a form in sets of n variables, and $|\alpha|$ a p-dimensional matricular and determinantal invariant of f. Let $|f_{\alpha\beta}\ldots|$ be a q-dimensional array of the n^{th} order, the elements of which are forms of the same order in the same sets of variables as f. By forming the invariant-type $|a|$ for each form of the array, we can build up a $(p+q)$-dimensional matrix $|b|$ possessing in addition to the suffixes of $|a|$, the suffixes $\alpha, \beta \ldots$ pertaining to the array. Every subsection of $|b|$ corresponding to the suffixes $\alpha, \beta \ldots$ is an invariant, of the same invariant-type and therefore of the same weight-type as $|a|$. Hence by § 11 $|b|$ is a simultaneous matricular invariant of the array, of the same weight-type as $|a|$, and the array suffixes $\alpha, \beta, \ldots$ are inert in $|b|$. The matrix $|b|$ may be called *the extensional invariant of the array corresponding to the invariant-type* $|a|$.

Further, by § 11 the *determinant* $|b|$ will also be an invariant of the array, if every suffix of $|a|$ has the same character in $|b|$ as it has in the *determinant* $|a|$. By giving different characters to the q inert suffixes $\alpha, \beta \ldots$ derived from the array, we see that

> *from any type of invariant-determinant* $|a|$, *we can construct* 2^q *extensional invariant-determinants* $|b|$ *of the array.*

Half the total number of the extensional invariants will vanish through having an odd number of determinantal suffixes. It should be noticed also, that extensional invariants have the property of being either *symmetric* or *skew* with respect to the interchange of forms corresponding to two values of any array-suffix.

As the simplest case of extensional invariant, we see that a pure p-dimensional determinant may be regarded either as the invariant of a multilinear form in p sets of variables, or as an extensional invariant of a q-dimensional array of multilinear forms in $(p-q)$ sets of variables. For instance an ordinary determinant may be regarded either as the invariant of a bilinear form, or as a skew combinant of a system of linear forms. In the latter view it would be an extensional invariant constructed from the vanishing type of the one-dimensional determinant of a linear form. Similarly a determinant with p determinantal and p' Scott suffixes may be regarded as an extensional invariant of a $(p'+k)$-dimensional array of multilinear forms in $(p-k)$ sets of variables $(k=0, 1, \ldots p-1)$.

When the elements of the invariant-type $|a|$ are all *linear* in the coefficients of f, the symbolic form of the extensional invariant $|b|$ can be deduced by a simple rule from that of $|a|$. When there is only one

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array-suffix a, this symbolic form results on (1) writing out n! times the symbolic expression of $|a|$ in all possible orders of the n equivalent groups of symbols, attaching in each case the positive sign or the sign pertaining to the order of the symbols, according as a is a Scott or determinantal suffix in $|b|$, (2) considering the equivalent symbols to refer to the forms $f_1 f_2 \ldots f_n$ of the array. The general case where there are many array-suffixes reduces to this by development by linkage. We add two examples:

(1) The one-dimensional determinant $|a_1 a_2 \ldots a_n|$ of the linear form $a_x^{(1)} = a_x^{(2)} = \ldots$ has the symbolic expression $\frac{1}{n!}(a^{(1)} a^{(2)} \ldots a^{(n)})$. Hence the pure determinant of n linear forms $a_x^{(1)}, a_x^{(2)}, \ldots a_x^{(n)}$ (which is our extensional invariant) has the symbolic expression:

$$\frac{1}{n!}((a^{(1)} a^{(2)} \ldots a^{(n)}) - (a^{(2)} a^{(1)} \ldots a^{(n)}) \ldots) = (a^{(1)} a^{(2)} \ldots a^{(n)}).$$

(2) Let $f = a_x^p = \beta_x^p = \ldots$. The matrix

$$\begin{vmatrix} \frac{d^2 f}{dx_1} & \cdots & \frac{d^2 f}{dx_1 dx_n} \\ \vdots & & \\ \frac{d^2 f}{dx_n dx_1} & \cdots & \frac{d^2 f}{dx_n^2} \end{vmatrix}$$

is a matricular invariant in which both suffixes are active. Hence the pure determinant of this matrix is an invariant with the symbolic form $\frac{p^n(p-1)^n}{n!}(a\beta \ldots)^2 a_x^{p-2} \ldots$. By extending this to an array $f_1 f_2 \ldots f_n$, and making the array-suffix a Scott suffix in the extensional invariant, we obtain the simultaneous invariant $p^n(p-1)^n(a\beta \ldots)^2 a_x^{p-2} \ldots$ of $f_1 f_2 \ldots f_n$ as a cubic determinant.*

[*Added* 10*th June*.—Since writing the above, the following work has come to my notice:

Rice, "On p-way determinants with an application to Transvectants." *Amer. Journ. of Math.*, 1918.

This paper treats of generalised determinants, starting like the present one with the postulation of suffixes of two different characters, which are called *signants* and *non-signants*. A process of "decomposition" (which is our "development by linkage") is described, attention being paid to the alteration of character which occurs. But owing to the fact that

* For a few more examples of what we have termed extensional invariants, see Hedrick, *loc. cit.*

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one-dimensional determinants are not admitted, this is not recognised as identical with the process giving the defining expansion of the determinant. The "crossed decomposition" of the paper is a development by simultaneous multiple linkage, in which the suffixes are linked together in several groups. The two usual types of multiplication—*Element-* and *File-*multiplication, which are our *Identification* and *Contraction*—are treated (as is usually done) quite independently of one another; our *serial* theory of multiplication appears preferable from the view-point of unity.

I have also come to learn of the work of M. Lecat on multiplication of determinants of higher dimensions, on determinants and permanents of special form, and on co-determinants (see for instance: *Tohoku Math. Journ.*, vol. xxiii, 1 and 2, Aug. 1923). To him belongs the credit of having been the first to point out the error committed by Scott and Gegenbauer, in applying Cayley's File-multiplication (= Contraction) to odd-dimensional determinants (*Abrégé de la théorie des déterminants à n-dimensions*, Gand, Hoste, 1911, quoted by Rice; this has not, however, proved available for reference).

Lecat has remarked that a one-dimensional permanent is equal to the product of its elements; "—car n'est le plus souvent qu'avec une seule dimension, et sous forme de simples produits—donc, si l'on peut ainsi dire, *incognito*—qu'interviennent alors les permanents" ("Sur les permanents," *Ann. Soc. Sci. Brux.*, t. xxxvii, p. 436).

Lecat also claims to have anticipated Rice in this idea of mixed determinants, as far back as 1915. "Depuis longtemps nous avions songé à cette généralisation. Dès 1915, nous avions établi des propriétés des *péné-déterminants* et comptions les publier dès la fin de la guerre. Nous allions le faire, quand l'important Mémoire de Rice vint à notre connaissance (en février 1919). Nous n'étions du reste pas allé aussi loin que lui. Le grand mérite de l'auteur américain est d'avoir, le premier (puisque pour la question de priorité la date de publication décide seule), bien mis en evidence, la fecondité des nouvelles fonctions" ("Sur une généralisation des déterminants qui permet la multiplication par files, même quand les classes des facteurs sont impaires," *Ann. Soc. Sci. Brux.*, t. xxxix, 1919–1920, pt. 2, p. 10).

The work of both Rice and Lecat excludes completely from the theory the determinants which have an *odd* number of signants, evidently on the ground that they vanish identically. A form-theorist would not naturally approve of this attitude.

The somewhat cumbrous notation of these authors seems to stand in the way, to some extent, of the wide appreciation of their work.

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I add here a comparative list of terms used in this paper, and in the work of these authors, putting an asterisk on those which appear to me to be suitable for future adoption.

* Signant	=	Determinantal suffix.
* Non-signant	=	Scott suffix.
Péné-déterminant	=	* Mixed determinant.
* Permanent	=	Scott function.
		* Pure determinant.
Full-sign determinant	=	Determinant of n-dimensions as defined before Rice.
Per-signant	=	Pure determinant of even dimensions.
* Species	=	Number of determinantal suffixes.
* Conjunctive elements.		
* Transversal = a set of conjunctive elements.	=	Term of the determinant.
* Set of ranges = Locant of a transversal.	=	The permutations of the values of the various suffixes for the elements of a term.
* Co-determinants of a matrix.		
Decomposition	=	* Development by linkage.
		* Conjunction.
* Element-multiplication	=	* Identification.
* File-multiplication	=	* Contraction.

(*Issued separately September* 17, 1924.)

THE PEDAL CORRESPONDENCE.

R.Vaidyanathaswamy.

CONTENTS.

The student of Invariant Theory will be familiar with the kind of work with Double-binary Forms in cogredient variables x, y, which is implied in Gordon's series, and in particular with the special power of these forms of giving rise to covariant forms by a process of what we may call "internal convolution".For the general theory of these forms,their expression as a series of powers of (xy) multiplied by polar forms (in the phraseology of the present work,as a sum of a polar,and of defective polar forms),which is the basis of Gordon's series,is fundamental.This expression,for the case in which the form is of equal order in the two variables,has been

Footnote (1) used very skilfully in a memoir of Waelsch*,and shewn to be inti-

See p120. mately connected with the geometry associated with a norm curve. Double and Multiple binary Forms of various types,in co-gredient variables,~~have~~ also occur~~ed~~ in the more recent researches of Coble in the Geometry of Apolarity.

The present work,in its algebraical aspect,is of a rather different character from these,being the outcome of an attempt to study a particular (3,2) correspondence,which called forth interest by its special character while promising to be fruitful from ~~t~~ its connection with the known geometry of the pedal line. Broadly speaking,the general results which have accrued from this study are of a two-fold character.On the one hand,certain new and important ideas concerning correspondences in general have been suggested,such as the idea of Derivate Forms,and that of Complete and Closed Sets with the properties relating to them. On the other hand,there have resulted contributions of a fundamental character to the Geometry of Apolarity,and to the Binary Geometry of the circle generally.Thus,the covariant specification of the orthocentre,the special properties of orthocentric pencils,and the curious fact that the fundamental rôle in regard to apolarity-properties of inscribed triangles of the circle, is played by a second concentric circle of $\sqrt{3}$ times its radius, are instances in point;we must mention also the valuable concepts of "pedal angle" and "pedo-parallelism" which are extremely useful,and serve as connecting links between many facts of elementary geometry.

The pedal form and its natural generalisations,the semi-pedal and the pedo-similar forms,to which the earlier part of the ~~p~~paper leads up,have a strongly marked individuality of their own, and yield only to methods which follow the lines of that individuality; for instance,the canonical shape as the sum of a polar and of defective polar forms,which is usually so satisfactory,turns out to be totally unprofitable in their case.Their special character comes from their intimate connection with the circular points,as a result of which the processes which lead to significant results are all guided,in their case,by the manner in which the circular points enter into their expression.This indeed constitutes,for the student of Algebraic Form,the distinctive quality of the binary field of the circle,as contrasted with that of the "projective" conic.The conic of projective geometry has no landmarks,its symmetry is too perfect.The difference which the entry of the circular points makes in the course of the algebra,may be appropriately compared with the difference made by the introduction of a powerful centre of disturbance into a uniform field of force.The circular points become the sources of significance, tending to orient every relation towards themselves,sometimes effecting,in the process,algebraic resolutions in the covariant forms which occur.The effect of such resolution is that the general symmetries to which one has become accustomed, tend to disappear or take a subordinate place, while new and unforeseen symmetries spring into existence. The results in apolarity will illustrate this, and will also shew how the significant apolarity-relations in the case of the circle, tend to connect themselves with the orthocentre and the relation of pedo-parallelism. A general fact becomes indeed increasingly apparent in the course of the paper, namely, that the <u>3-dimensional continuum of the inscribed triangles of the circle is itself broken up by the circular points into a (2+1)-dimensional continuum composed of pedo-parallel systems and orthocentric pencils.</u>

Something similar to this would of course be true, though to a much smaller extent, in the case of the "metrical" conic, the parabola or the ellipse; but here, as the circular points, the ~~centr~~ centres of the metrical field of the plane, are outside the ~~cur~~ curve, there would not be the same simplicity and inevitableness, or the same symmetry.

The first section is concerned with general properties of correspondences. The canonical shape for correspondences with given fixed points, and the theory of Multipliers are both extensions from the case of the Homographic Correspondence, and are believed to be new. A limited extension for the simpler case of the rational or (n,1) correspondence, using non-homogenous variables will be found in an earlier paper of the author*. We might mention here that it is possible to give a theory of the multipliers of powers of the (m,n) correspondence, on the same lines as those adopted for the (n,1) correspondence in this earlier paper. In the subsection on Defective Forms, Waelsch's "Wertigkeit" has been rendered by the word "defect".The phrase "defective polar forms" is one which may be recommended for brevity and convenience.

The second section studies Complete sets and closed sets of a correspondence.It is shewn by the geometrical interpretation that the idea of complete set is the counterpart for correspondences,of the geometrical notion of multiple point of a curve.The theorems on the reduction of rank lead to a hitherto unnoticed property of the pedal correspondence in n dimensions.

In the third section there is an investigation of the (2,1) correspondence,and various algebraic results are proved which have geometrical application in the sequel.The (2,1) correspondence has been studied previously in two elaborate memoirs

Footnote (3) of Pitarelli*;in these,Pitarelli develops the notions of what have here been called "apolar pair", "auto-pair" and "apolar cubic",and proves the fundamental result that the auto-pair is the hessian pair of the apolar cubic,though not the other fundamental result that the cubic of fixed points is apolar to the cubicovariant of the apolar cubic.Beyond this there is not much in common,either in method or in the results,between this section and the work of Pitarelli.

The theory of the two-quadratic representation of the pencil of cubics,which is the subject of the fourth section,is the generalised algebraic statement of~~t~~ the properties of Orthocentric pencils.The connection between the two is supplied by the fifth section,which obtains the covariant specification of the orthocentre of an inscribed triangle,as the point which corresponds to the apolar pair of a certain (2,1) correspondence on the circle,defined by the triangle.

The sixth and seventh sections deal with what is directly related to the title of the paper,namely the pedal form and its extensions.These contain new results,as well as new presentations of old results.

The last section is the record of an unsuccessful attempt to Footnote (4) follow up the theorem of Dr.Richmond and Prof.H.F.Baker*,by extending to n dimensions some of the beautiful and varied properties of the pedal correspondence in the plane.The analogy between the two cases does not,as a matter of fact,extend much farther than the characteristic property of the linear relation between the feet of the perpendiculars,which is common to both. The elegant expressions for the pedal form on the circle are replaced,for the case of n dimensions,by an expression too complicated to be set down.The reason why the analogy fails,lies doubtless in the fact,that the pedal correspondence in n dimensions defined in accordance with the Richmond-Baker theorem,is complete but not closed at infinity.It is possibly worth while to remedy this,and restore the closure property by imposing suitable ~~fu~~rther restrictions on the Wallace curve.The nature of these restrictions is indicated by the statement in Subsection (9),regarding the ways in which a complete n-set of a correspondence can be closed with respect to it.The most satisfactory choice of the type of closure for the pedal correspondence,will be that corresponding to the identical permutation for space of odd dimensions,and that corresponding to an interchange of points of the set in pairs,for space of even dimensions.I have not however examined the geometrical consequences which this will entail.

In conclusion,I must express my thanks to Prof.H.W.Turnbull for the interest he has taken in the paper,and for his valuable criticism.

§1. THE GENERAL (m,n) CORRESPONDENCE.

(1). Double-binary Forms in independent and cogredient variables

There is bound to be a great difference in the algebraic treatment of double-binary forms in digredient variables x,y, and those in cogredient variables; for ,though theoretically the only difference is in the introduction of a new invariant type (x,y),yet the possibility of 'fixed elements' peculiar to forms in cogredient variables--or,in other words, to correspondences between the elements of the same binary field,--compels certain lines of procedure,which have nothing similar to them in the case of forms in independent variables,or correspondences between different binary fields. While this is so,it is true on the other hand,that all properties of correspondences between distinct binary fields must continue to hold when the fields ~~bee~~ become identical;for since the change from digredience to cogredience only replaces the group of transformations of the variables by one of its own subgroups,it follows that any general theory of forms in independent variables can not become invalidated by the variables ~~by~~ becoming cogredient. It would thus appear that the forms in cogredient variables while being less general,must also be considerably richer than those in independent variables.

As an important instance of a body of general ideas continuing to be valid when the variables become cogredient,we

Footnote(1)

may mention the rank-theory* of the double-binary form. An (m, n) form $F(x,y)$ is said to be of rank r,when it could be expressed in the shape,

$$F(x,y) = f_1(x)\phi_1(y) + f_2(x)\phi_2(y) + \cdots + f_r(x)\phi_r(y),$$

but not in the shape $f_1(x)\phi_1(y)+\cdots+f_{r-1}(x)\phi_{r-1}(y)$ .In particular,F is of rank one,if it is a product of binary forms.The necessary and sufficient condition that F may be of rank r is that G_r be the first of the rank-covariants G_1, G_2, ... which vanishes identically;where,

$$F(x,y) = a_x^m b_y^n = a_{\mu x}^m b_{\mu y}^n \quad (\mu = 0,1,2,\ldots);$$
$$G_r = (a_0 a_1)\cdots(a_0 a_r)(a_1 a_2)\cdots(a_{r-1} a_r)(b_0 b_1)\cdots(b_{r-1} b_r)\, a_{0x}^{m-r}\cdots a_{rx}^{m-r}\, b_{0y}^{n-r}\cdots b_{ry}^{n-r}.$$

In particular,G_1 is the (1,1) transvectant of F.If F is of rank r ,and is expressed in the shape:

$$F(x,y) = f_1(x)\phi_1(y) + f_2(x)\phi_2(y) \cdots + f_r(x)\phi_r(y),$$

it may be shewn that the $(k-1)^{th}$ rank-covariant G_{k-1} is given by:

$$G_{k-1} = \sum J_{12\ldots k}(x)\, J'_{12\ldots k}(y)$$

Footnote(2) where $J_{12\ldots k}$ and $J'_{12\ldots k}$ are the Jacobians* ~~N-et-e~~ of the forms $f_1, f_2 \ldots f_k$ and $\phi_1, \phi_2 \ldots \phi_k$ respectively.In particular,the last non-vanishing rank-covariant G_{r-1} is the product of ~~all of~~ the Jacobians of all the forms f and ϕ respectively,that is, of the Jacobians of the two linear systems

$$L_1:\ \lambda_1 f_1(x) + \lambda_2 f_2(x) + \cdots + \lambda_r f_r(x),$$
$$L_2:\ \mu_1 \phi_1(y) + \mu_2 \phi_2(y) + \cdots + \mu_r \phi_r(y).$$

Here L_1 is the system of forms of the type $(bc)^n a_x^m$,and L_2 ,of those of the type $(ad)^m b_y^n$, c_y^n and d_x^m being arbitrary binary forms.The systems L_1 and L_2 may be called the x- and the y-

Footnote(3) concurrency systems* of F.If $m=n$,and F has its maximum rank $n+1$,then both the concurrency systems are non-existent, that is to say,they comprise all possible forms.If $m>n$, ~~the Jacobian~~ and F has its maximum rank $n+1$,then L_2 is non-existent and the Jacobian of L_1 is the binary form G_n of order $(m-n)(n+1)$.In the particular case $m=n+1$, G_n will be of order $n+1=m$, and therefore L_1 will be the system of (n+1)-ics apolar to G_n. Hence

1·1 the concurrency system of an $(n+1,n)$ form F of maximum rank, is the system of $(n+1)$-ics apolar to its last rank covariant G_n.

All the above holds mutatis mutandis when the variables x, y become cogredient.

We shall be concerned throughout this paper only with forms in cogredient variables. We shall whenever possible make no distinction between forms and the correspondences obtained by equating them to zero; for ~~insten~~ instance, we shall denote the form and the correspondence by the same symbol, and in general speak of them interchangeably, whenever that would cause no error.

(2) CANONICAL SHAPE FOR CORRESPONDENCES WITH GIVEN FIXED POINTS.

An (m,n) correspondence F(x,y) =0 has $m+n$ fixed points given by F(x,x)=0. The form F(x,y) involves $(m+1)(n+1)$ parameters linearly; if its fixed points are known, it will still involve $(m+1)(n+1)-(m+n) = mn+1$ parameters linearly. To express F(x,y) when its fixed points are known, we partition the $m+n$ fixed points (supposed distinct) in any way, into two groups of m and n — $\alpha_1\alpha_2\cdots\alpha_m$ and $\beta_1\beta_2\cdots\beta_n$ say. The form F(x,y) can then be expressed in only one way, in the shape,

$$F(x,y) = \left\{\mu + \Sigma\,\mu_{pq}\frac{(\alpha_p y)(\beta_q x)}{(\alpha_p x)(\beta_q y)}\right\}(\alpha_1 x)\cdots(\alpha_m x)(\beta_1 y)\cdots(\beta_n y); \quad [p=1,2\cdots m;\ q=1,2\cdots n]. \qquad 1{\cdot}2$$

For, this shape involves $mn+1$ parameters μ, μ_{pq}. And when y is put equal to x, it becomes $(\mu+\Sigma\mu_{pq})(\alpha_1 x)\cdots(\alpha_m x)(\beta_1 y)\cdots(\beta_n y)$ shewing that it has the given fixed points. Also, there are no values of μ, μ_{pq} other than zero, for which the right side is identically zero; for, assuming such an identity and putting $x=\alpha_p$, $y=\beta_q$ in it, we have $\mu_{pq}=0$ for all values of p, q, and therefore also $\mu=0$. Thus the $mn+1$ parameters μ, μ_{pq} are effective, and F(x,y) admits of unique expression in this canonical shape. For different partitions of the fixed points, we get in general different canonical shapes; if however m or n is unity, we get the same canonical shape for all partitions.

On putting $x=\alpha_p, y=\beta_q$ in $1\cdot2$, we obtain the value of μ_{pq}, namely,

$$1\cdot3 \qquad \mu_{pq} = \frac{-F(\alpha_p,\beta_q)}{(\alpha_p\beta_q)^2(\alpha_1\alpha_p)\cdots(\alpha_{p-1}\alpha_p)(\alpha_{p+1}\alpha_p)\cdots(\alpha_m\alpha_p)(\beta_1\beta_q)\cdots(\beta_{q-1}\beta_q)(\beta_{q+1}\beta_q)\cdots(\beta_n\beta_q)}.$$

Further if $F(x,x)=M(\alpha_1 x)\cdots(\alpha_m x)(\beta_1 x)\cdots(\beta_n x)$, we have

$$1\cdot4 \qquad \mu+\Sigma\mu_{pq}=M$$

These equations determine all the parameters in terms of the fixed points and the coefficients of F. We shall find it convenient to write

$$1\cdot5 \qquad \begin{aligned} \mu_{1q}+\mu_{2q}+\cdots+\mu_{mq} &= \mu'_q \\ \mu_{p1}+\mu_{p2}\cdots+\mu_{pn} &= \mu_p \end{aligned}$$

so that,

$$\sum_p \mu_p = \sum_q \mu'_q = M-\mu.$$

(3) MULTIPLIERS.

If we consider x to be the independent variable, and y the dependent variablfe, the correspondence F also represents a transformation from x to y. Supposing for the moment that x, y are non-homogenous variables, we define the multiplier of F at a fixed point α to be the value of $\frac{dy}{dx}$ at the place $x=\alpha$, $y=\alpha$, of the associated Riemann surface. To obtain the value of the multiplier at the fixed point α_1 when the variables are homogenous, we put $x=\alpha_1+\delta z, y=\alpha_1+\delta' z$ (z being an arbitrary point and δ, δ' infinitesimal scalars), and suppose that F(x,y) =0, for these values. The limiting value of δ'/δ is then the required multiplier. Thus from 1·2, we havfe

$$\{\mu\delta + \sum_q \mu_{1q}\delta' + \sum_q \mu_{2q}\delta + \cdots + \sum_q \mu_{mq}\delta\}(\alpha_1 z)(\alpha_2\alpha_1)\cdots(\alpha_m\alpha_1)(\beta_1\alpha_1)\cdots(\beta_n\alpha_1)=0.$$

Hence ν_p the multiplier at α_p, and ν'_q the multiplier at β_q are given by:

$$1\cdot6. \qquad \nu_p = \frac{\mu_p - M}{\mu_p}\ ; \quad \nu'_q = \frac{\mu'_q}{\mu'_q - M}. \ (1\cdot4, 1\cdot5)$$

We have thus the fundamental identity between the multipliers:

$$\sum_{p=1}^{m} \frac{1}{1-\nu_p} + \sum_{q=1}^{n} \frac{\nu'_q}{1-\nu'_q} = 0.$$

This identity may be written in the symmetric shape:

$$1\cdot7 \qquad \sum \frac{1}{1-\nu} = n,$$

where the summation on the left is for all the multipliers, and the quantity n on the right is the order of F in the dependent variable.

If the form F is of rank one,---say

$$F(x,y) = (\alpha_1 x)\cdots(\alpha_m x)(\beta_1 y)\cdots(\beta_n y),$$

then its canonical shape 1·2 for the partition $\alpha_1\alpha_2\ldots\alpha_m / \beta_1\beta_2\ldots\beta_n$ has only one term, and 1·6 shews that the multipliers at $\alpha_1\alpha_2 \cdot\cdot \alpha_m$ become infinite, and the remaining multipliers vanish. Conversely, if all the multipliers of F are infinite or zero, then by 1·7 exactly m must be infinite and n zero, and F is necessarily of rank one.

The <u>polar</u> (m,n) forms are those which result on polarising a binary (m+n)-ic in x n times with respect to y. On putting y=x in a polar form, we simply revert to the binary form from which it was derived.

1·8 The multipliers of the polar correspondence are all equal, each being equal by 1·7 to $-m/n$.

For, supposing that the fixed points $\alpha_1\alpha_2 \cdot\cdot \alpha_{m+n}$ are all distinct, let us seek the condition that m+n points $p_1p_2 \cdot\cdot p_{m+n}$ all infinitely near α_1 may form a group apolar to $(\alpha_1\alpha_2 \cdot\cdot \alpha_{m+n})$. Relatively to the scale of the mutual distances of $\alpha_1\, p_1 p_2 \cdot\cdot p_{m+n}$ the points $\alpha_2\alpha_3 \cdot\cdot \alpha_{m+n}$ are all infinitely far away. Hence so far as geometry in the immediate neighbourhood of α_1 is concerned, these latter points may all be considered to be coincident with the point at infinity on the line. Thus, the required condition of apolarity is simply that α_1 should be the centroid of $p_1p_2 \cdot\cdot p_{m+n}$. In particular, if x,y are in the neighbourhood of α_1, and are correspondents in the polar (m,n) correspondence defined by $(\alpha_1 x)(\alpha_2 x) \cdot\cdot (\alpha_{m+n} x)$, then the group composed of m points at x, and n points at y is apolar to $(\alpha_1\alpha_2 \cdot\cdot \alpha_{m+n})$; so that α_1 is the centroid of masses m, n at x,y respectively. Thus the multiplier of the polar correspondence at α_1 (and similarly at any other fixed point) is $-m/n$.

(4)DEFECTIVE FORMS.

8/ If a correspondence F is such that $F(x,x)=0$ identically, (so that $F(x,y)$ has the factor (xy)),thenthe fixed points of F are indeterminate,and so F is an identical---or,as we shall say,a"defective" ---correspondence.Among the (1,1) correspondences there is only one type of identical correspondence, namely $(xy)=0$;but for the (m,n) correspondences F there are different types according to the highest power of (xy) that divides F.

DEF. A form F is said of defect i ,if it has the factor $(xy)^i$,but not the factor $(xy)^{i+1}$.

The condition that F,when expressed in the canonical shape 1·2 may be defective ,is $M=\mu+\Sigma\mu_{pq}=0$;that is to say,a form with given fixed points has to satisfy only one condition for being defective.We see as a matter of fact,that the (mn+1)--parameter-family of (m,n) forms with given fixed points contains the mn-parameter-family consisting of all defective forms.

~~Aøø~~ A polar form $P(x,y)$ is never defective;for,on putting $y=x$ in P,we revert to the binary form from which it was derived.This binary form not being supposed to vanish identically,it follows that ø P is not defective.We can therefore give a new meaning to the term "defective polar form".We accordingly define:

1·9 The product of $(xy)^i$ and a polar form will be called a defective polar form,of defect i .

Just as we have the invariant ~~operator~~ factor (xy) for cogredient variables, we also have the correlative invariant operator $\Omega = \frac{d}{dx_1}\frac{d}{dy_2} - \frac{d}{dx_2}\frac{d}{dy_1}$. The result of operating with Ω on any symbolic product is the sum of terms obtained by convolving every ~~fa~~ factor of the type (αx) with every factor of the type (βy) ~~(th~~ (that is, replacing $(\alpha x)(\beta y)$ by $(\alpha\beta)$).

1·10 The result of operating with Ω on a polar form is zero.

For, a polar form being a symbolic product of the shape $(ax)^m(ay)^n$ every convolution in it leads to zero.

1·11 The result of operating with Ω on a defective polar (m,n) form $(xy)^i P(x,y)$ is $i(m+n-i+1)(xy)^{i-1}P(x,y)$.*

Footnote(4).

Associated with the existence of the invariant factor (xy) and the correlative invariant operator Ω, there is a canonical shape, in which any form F can be expressed uniquely as the sum of a polar form, and defective polar forms; namely,

1·12 $$F(x,y) = (P^{m+n})_{y^n} + (xy)(P^{m+n-2})_{y^{n-1}} + \cdots + (xy)^i (P^{m+n-2i})_{y^{n-i}} + \cdots,$$

where P^r is a binary form of order $\underline{r}$ in $\underline{x}$ and the suffix y^k denotes the k^{th} polar with respect to y. The validity and uniqueness of this shape follow on observing that the number of parameters on both sides is the same. The form F is thus specified by the "ladder"* $(P^{m+n}, P^{m+n-2}, \cdots, P^{m+n-2i}, \cdots)$. The forms of the ladder can all be obtained by means of the operator Ω (1·11). Thus

Footnote(5)

1·13 $$\left[\Omega^i F(x,y)\right]_{y=x} \equiv i!(m+n-i+1)(m+n-i)\cdots(m+n-2i+2)P^{m+n-2i}.$$

(5) DERIVATE FORMS.

If a pencil of (m,n) forms $\lambda F_1 + \mu F_2$ contains a defective member, it follows on putting y=x, thateach member of the pencil has the same fixed points. Conversely,

1·14 A pencil of (m,n) forms determined by two forms with the same fixed points, necessarily contains a defective member.

Now, given an (m,n) form F, there are $\binom{m+n}{m}$ forms of rank one with the same fixed points as F, each corresponding to a partition of the fixed points. [Thus, if $\alpha_1\alpha_2\cdots\alpha_m/\beta_1\beta_2\cdots\beta_n$ be a partition of the fixed points, the corresponding form of rank one is $(\alpha_1 x)\cdots(\alpha_m x)(\beta_1 y)\cdots(\beta_n y)$]. By 1·14 the pencil $F-\lambda(\alpha_1 x)\cdots(\alpha_m x)(\beta_1 y)\cdots(\beta_n y)$ contains a defective member $(xy)\phi(x,y)$ say.

1·15 We define the (m-1,n-1) form $\phi(x,y)$ to be the derivate of F in respect of the partition $\alpha_1\alpha_2\cdots\alpha_m/\beta_1\beta_2\cdots\beta_n$ of its fixed points.

The fundamental property of the derivate is clear from its defintion, namely

1·16 If ϕ be the derivate of F in respect of $\alpha_1\alpha_2\cdots\alpha_m/\beta_1\beta_2\cdots\beta_n$, then the correspondents of $x=\alpha_p$ in ϕ are all the correspondents of $x=\alpha_p$ in F, with the exception of α_p itself ($p=1,2,\ldots m$); a similar statement holds about the cor correspondents of $y=\beta_q$ in ϕ ($q=1,2\cdots n$).

In particular, if $(\alpha_1\alpha_2\cdots\alpha_m)$ or $(\beta_1\beta_2\cdots\beta_n)$ be a complete set of F, it will be a semi-complete set of ϕ (see below §2·(6)).

Of special interest is the derivate of a form of rank one, in respect of a partition not its own. Consider for instance the derivate of $F=(\alpha_1 x)\cdots(\alpha_m x)(\beta_1 y)\cdots(\beta_n y)$ in respect of the partition $\beta_1\beta_2\cdots\beta_k\alpha_{k+1}\cdots\alpha_m/\alpha_1\cdots\alpha_k\beta_{k+1}\cdots\beta_n$, which is obtained from $\alpha_1\alpha_2\cdots\alpha_m/\beta_1\beta_2\cdots\beta_n$ by the interchange of the groups $(\alpha_1\alpha_2..\alpha_k)$ and $(\beta_1\beta_2..\beta_k)$. The required ~~de~~ derivate is

$$\phi(x,y) = (\alpha_{k+1}x)\cdot\cdot(\alpha_m x)(\beta_{k+1}y)\cdot\cdot(\beta_n y)\,\psi(x,y)$$

where $$\psi(x,y)(xy) = (\alpha_1 x)\cdot\cdot(\alpha_k x)(\beta_1 y)\cdot\cdot(\beta_k y) - (\alpha_1 y)\cdot\cdot(\alpha_k y)(\beta_1 x)\cdot\cdot(\beta_k x).$$

The form $\psi(x,y)$ is the symmetric (k-1,k-1) form, in respect of which any two of the points $\alpha_1\alpha_2..\alpha_k$, and also any two of $\beta_1\beta_2..\beta_k$ are correspondents. If $(\gamma_1\gamma_2..\gamma_k)$ is any group of the involution determined by $(\alpha_1\alpha_2..\alpha_k)$ and $(\beta_1\beta_2..\beta_k)$, then it is easy to see that any two of the γ's are also correspondents in ψ. Incidentally we see that the fixed points of ψ are the roots of the Jacobian of the involution. Thus

1·17 The derivate of the form $(\alpha_1 x)\cdot\cdot(\alpha_m x)(\beta_1 y)\cdot\cdot(\beta_n y)$ in respect of the partition $\beta_1..\beta_k\alpha_{k+1}..\alpha_m/\alpha_1..\alpha_k\beta_{k+1}..\beta_n$ is $(\alpha_{k+1}x)\cdot\cdot(\alpha_m x)(\beta_{k+1}y)\cdot\cdot(\beta_n y)\,\psi(x,y)$ where ψ is the unique symmetric (k-1,k-1) form, in respect of which any two of $\alpha_1\alpha_2..\alpha_k$ and any two of $\beta_1\beta_2..\beta_k$ are correspondents.

It follows that if the derivate of F in respect of a particular partition be ϕ, its derivate in respect of any other partition differs from ϕ by an expression of the type obtained in 1·17.

§2. COMPLETE AND CLOSED SETS OF A CORRESPONDENCE.

(6). CANONICAL SHAPE FOR CORRESPONDENCES WITH COMPLETE SETS.

A set of r points $i_1 i_2 \cdots i_r$ will be said to form a "complete set" of a (m,n) correspondence F, if every pair of points i, whether distinct or identical, are correspondents in F. The necessary and sufficient conditions for this are therefore the r^2 equations

$$F(i_p, i_q) = 0\,;\ (p,q = 1,2,\cdots r).$$

The set $(i_1 i_2 \cdots i_r)$ may be said to be a "semi-complete set" of F, if every pair of different points i are correspondents in F; that is, if the $r(r-1)$ equations $F(i_p, i_q) = 0$ are satisfied for $p,q = 1,2,\cdots r\,;\ p \neq q$.

The same set of r points $i_1 i_2 \cdots i_r$ will be said to be a closed set of F, if the m roots of each of the r equations $F(x, i_p) = 0$, and the n roots of each of the r equations $F(i_p, y) = 0$, are all comprised among $i_1 i_2 \cdots i_r$; that is, if no point which is not an i, has a point i for a correspondent in F. In other words, the set $i_1 i_2 \cdots i_r$ is closed in regard to F, if starting from any i, it is impossible to go out of the set by means of F.

It follows from the first definition that a complete set is necessarily a subset of the set of fixed points. This need not be true of a closed set. Aclosed set when it exists need not be complete, nor a complete set, closed. There may however exist sets which are both complete and closed. As an obvious instance, an n-set complete in regard to an (n,n) correspondence, is also closed with respect to it.

A (m,n) form F possessing a complete set $(i_1 i_2 \cdot\cdot i_r)$ can be expressed in the shape:

2·1 $$F(x,y) = (i_1x)\cdots(i_rx)\,[P_1(\overset{m-r}{x},\overset{n}{y})] + (i_1y)\cdots(i_ry)\,[P_2(\overset{m}{x},\overset{n-r}{y})].$$

For, the right hand side vanishes when $x=i_p, y=i_q$, and therefore possesses the complete set $i_1 i_2 \cdot\cdot i_r$. Now the number of available parameters in F is $(m+1)(n+1)-r^2$, while the number of parameters on the right is $(m+1)(n+1)-r^2+(m+1-r)(n+1-r)$. This excess number $(m+1-r)(n+1-r)$ may be accounted for by the existence of identical relations, involving $(m+1-r)(n+1-r)$ parameters, of the form:

$$(i_1x)\cdot\cdot(i_rx)\left[(i_1y)\cdot\cdot(i_ry)\,Q(\overset{m-r}{x},\overset{n-r}{y})\right]-(i_1y)\cdot\cdot(i_ry)\left[(i_1x)\cdot\cdot(i_rx)\,Q(\overset{m-r}{x},\overset{n-r}{y})\right]\equiv 0.$$

Thus the right side of 2·1 involves effectively the same number of parameters as F, and is consequently adequate for representing it.

The representation 2·1 can be rendered unique by imposing either the condition that P_1 should be identically apolar to $(i_1y)(i_2y)\cdot\cdot(i_ry)$, or the condition that P_2 should be identically apolar to $(i_1x)(i_2x)\cdot\cdot(i_rx)$.

(7) REDUCTION OF RANK THROUGH COMPLETE SET.

Supposing for convenience that $m \geqq n$, we go to shew that

2·2 if an (m,n) form F $(m \geqq n)$ possesses a complete r-set, where of course $r \not> n$, and if $r > \frac{m+1}{2}$, $r > m-n$, then F suffers a reduction in rank from $n+1$ to $n+1-(2r-\overline{m+1})$.

For, the first term in the representation 2·1 being the product of a binary form and a $(m-r,n)$ form, its rank is the smaller of $m-r+1, n+1$. Similarly the rank of the second term is the smaller of $n-r+1, m+1$, that is, since $m \geqq n$, it is $n-r+1$. Hence the rank of F cannot exceed

$$(n-r+1) + \text{smaller of } (m-r+1, n+1) .$$

If n is smaller than $m-r$, this statement is trivial. Supposing then, that $m-r<n$ or $r>m-n$, the rank of F cannot exceed $(n+1)-(2r-\overline{m+1})$. This statement again will be trivial unless $2r>m+1$. Thus if $r>m-n$ and $2r>m+1$, there is the actual reduction stated, in the rank of F.

Putting r=n in 2·2, we have the special case,

2·3. A (m,n) form F possessing a complete n-set is of rank $m-n+2$, if $2n-1>m \geqq n$.

A still more special case which has application in the sequel, is:

2·4. A(n+1,n) form possessing a complete n-set is of rank three ($n>1$).

(8) GEOMETRICAL INTERPRETATION IN n DIMENSIONS.

These theorems about the reduction of rank, may be translated into simple geometrical properties pertaining to curves in higher-dimensional space. The (m,n) form F(x,y) ($m \geqq n$) institutes an (m,n) correspondence F between the points of a rational norm curve R_n in a space S_n of n dimensions. To any point x of R_n, F makes correspond n points $y_1 y_2 \cdots y_n$, and therefore the unique prime $y_1 y_2 \cdots y_n$. As x traces R_n, the corresponding prime $y_1 y_2 \cdots y_n$ osculates a curve $\mathbf{R}_m$ of class m, which we may call the associated envelope of F; this associated envelope is evidently in birational correspondence with the norm curve R_n, and is therefore a rational envelope.

If F possesses a complete n-set $i_1 i_2 \dots i_n$,then in this birational correspondence between R_n and $\mathbb{R}_m$,the prime $i_1 i_2 \dots i_n$ corresponds to each of the points $i_1, i_2, \dots i_n$.Hence it is a multiple prime,of multiplicity n ,of the associated envelope. On reciprocating $\mathbb{R}_m$ we obtain a curve of order m,possessing an n-ple point,in S_n .We see therefore that 2·3 asserts that

2·5 a curve of order m in S_n,possessing an n-ple point must be completely contained in an S_{m-n+1} $(m \leqq 2n-1)$.

This is geometrically obvious,since the S_{m-n+1} which joins the n-ple point to any other $m-n+1$ points of the curve,cuts it in $m+1$ points,and therefore contains it completely. Similarly it is seen that 2·2 asserts that,

2·6. if a curve of order $m (\geqq n)$ in S_n ,possesses a space L_{n-r} of $n-r$ dimensions,cutting it in r points,and if $r > m-n, 2r > m+1$,the curve must be completely contained in an $S_{m+n+1-2r}$.

The proof is exactly the same;the $S_{m+n+1-2r}$ which joins the L_{n-r} to any other $m+1-r$ points of the curve,cuts it in $m+1$ points,and therefore contains it completely.

(9).(n+1,n) FORMS POSSESSING A COMPLETE AND CLOSED n-SET.

Let the $(n+1,n)$ form $F(x,y)$ possess the complete and closed n-set given by $f(x)=(i_1x)(i_2x)\dots(i_nx)$.By 2·1 the form F can be written in the shape:

$$F(x,y) = f(x)\cdot(ax)(by)^n + f(y)\cdot(cx)^{n+1},$$

where (a,b) and c are symbolic.If we now use the condition that

$f(x)$ is also a closed set, we see that the n points $y=i_1, i_2, \cdots i_n$ correspond in $(ax)(by)^n$ to the n points $x=i_1, i_2, \cdots i_n$ in some order. Thus the $(1,n)$ correspondence $(ax)(by)^n$ merely effects a substitution ω on the n points $y=i_1, i_2, \cdots i_n$. Hence, there are $n!$ ways in which a complete n-set can be closed in regard to an $(n+1,n)$ correspondence, each of these ways corresponding to a permutation ω on the n points of the set.

Two types of closure appear to be ~~of~~ noteworthy; the first is that in which ω is the identical permutation, the second is that in which n is even, and ω interchanges the n points in pairs. It is the latter type which when $n=2$, occurs in the case of the pedal form.

§3. THE (2,1) CORRESPONDENCE.

(10). QUADRATIC COVARIANTS OF THE (2,1) FORM.

The (2,1) form and its quadratic covariants have a bearing of a fundamental character on the binary geometry of the circle, and their study is indispensable as a preliminary to that of the pedal correspondence.

Let the (2,1) form $F(x,y)$ be written symbolically,

$$F(x,y) = (ax)^2(by) = (a'x)^2(b'y) = (a''x)^2(b''y).$$

The quadratic in x which is apolar to F for all values of y is evidently,

3·1 $$O(x) = (aa')(bb')(ax)(a'x).$$

We call $O(x)$ the apolar quadratic of F. Since $O(x)$ is also the first rank covariant, or the (1,1) transvectant of F, it ~~foll~~ follows that,

3·2 the apolar quadratic of F vanishes identically when and only when F is of rank one.

By iterating the correspondence F, we can see that there exists only one pair of points $i\,j$, each of which corresponds to the other in F. Thus, if $x=p$ correspond in F to $y=q$, and $x=q$ to $y=r$, then the iterated correspondence of F being the (4,1) correspondence (p,r), has five fixed points, among which the three fixed points of F would be included. The two other fixed points i,j, are thus fixed points of the iterated, but not of the original correspondence. This can only mean that each of i,j corresponds to the other in F, or ij is a semi-complete pair of F (§2.(6)). We shall call ij the auto-pair of F, and a constant multiple of $(ix)(jx)$, the <u>auto-quadratic</u> of F.

This iteration-process for determining ij amounts algebraically to the elimination of y between $(ax)^2(by)$ and $(ay)^2(bx)$, and the removal of the factor $f(x)=(ax)^2(bx)$ from the result. The work for the symbolic determination of the auto-quadratic would therefore run as follows:

$$(ax)^2(by)=0\ ; \quad \therefore\ y = b(ax)^2$$

$$\begin{aligned}\therefore\ (a''y)^2(b''x) &= (a''b)(ax)^2(a''b')(a'x)^2(b''x)\\ &= (ax)^2(a'x)^2(a''b')\left[(a''x)(b''b)+(a''b'')(bx)\right]\\ &= f(x)(a'x)^2(a''b')(a''b'') + \tfrac{1}{2}(a'x)^2(ax)(a''x)(b''b)\{(a''b')(ax)-(a''x)(ab')\}\\ &= f(x)\,\{(ax)^2(a'b)(a'b') + \tfrac{1}{2}\,O(x)\}\ (3\cdot 1).\end{aligned}$$

Hence the auto-quadratic $S(x)$ may be taken to be

3·3. $$S(x) = (ax)^2(a'b)(a'b') + \tfrac{1}{2}\,O(x).$$

~~This~~ The first term on the right of this is obtained by substituting $-a'(a'b')$ for y in $(ax)^2(by)$. Since $O(x)$ is apolar to $(ax)^2(by)$ for all values of y, it follows that $O(x)$ is apolar to $(ax)^2(a'b)(a'b')$. Hence from 3·3, we have

3·4. The apolar invariant $(S(x),O(x))^2$ is equal to a half of the discriminant of $O(x)$.

Now, ~~it will follow~~ if $(ax)^2(a'b)(a'b')=0$ identically, this would imply either that there is a value of y, namely $y=-a'(a'b')$, which makes $F(x,y)$ vanish identically, or that $(a'x)(a'b')=L(x)$ vanishes identically. If we suppose that F is not of rank one, then there can be no value of y which makes F vanish identically, so that the first alternative becomes impossible. Hence $L(x)\equiv 0$. But $L(x)$ is clearly the second form in the ladder of $F=(ax)^2(by)$ (1·12, 1·13); hence its identical vanishing implies that F is a polar form. Thus,

3·5 if $O(x)\neq 0$ and $(ax)^2(a'b)(a'b')=0$, F must be a polar form.

If $(ax)^2(bx)=f(x)=(\alpha x)(\beta x)(\gamma x)$, and $F(x,y)$ is put in its canonical form 1·2, namely,

$$F(x,y)=\mu_1(\alpha y)(\beta x)(\gamma x)+\mu_2(\beta y)(\gamma x)(\alpha x)+\mu_3(\gamma y)(\alpha x)(\beta x),$$

it is easy to shew that the apolar and auto-quadratics are (except for constant multiples) given by:

$$3\cdot 6.\quad \begin{cases} O(x) = \sum \mu_2\mu_3(\beta\gamma)^2(\alpha x)^2 \\ S(x) = \sum \mu_1(-\mu_1+\mu_2+\mu_3)(\gamma\alpha)(\alpha\beta)(\beta x)(\gamma x). \end{cases}$$

If the auto-pair ij or the apolar pair lm (namely the pair of roots of the apolar quadratic) of F is given, it is easy to see that F could be written in the respective shapes:

$$3\cdot 7\qquad F(x,y)=(ix)(iy)(Ax)+(jx)(jy)(Bx)$$
$$3\cdot 8\qquad F(x,y)=(lx)^2(m'y)+(mx)^2(l'y).$$

Besides the apolar quadratic and auto-quadratic, two other quadratic covariants of F, of secondary importance, may be mentioned. These are firstly, the quadratic $O'(y)$ whose roots $l'm'$* are the values of y which correspond in F to its apolar pair, and secondly, the <u>residual quadratic</u> $R(x)$ whose roots A, B are the values of x other than i, j which correspond in F to its auto-pair ij. $O'(y)$ being clearly the discriminant of F in respect of x, is given by

$$3\cdot 9.\qquad O'(y)=(aa')^2(by)(b'y).$$

The residual pair is composed of the harmonic conjugates of ij

in respect of the apolar pair; hence $R(x) = 4O(x)D_{22} - 2S(x)D_{12}$, where $D_{22} = (S,S)^2$, $D_{12} = (O,S)^2$. Hence, by 3·4,

$$3\cdot 10 \quad R(x) = 4O(x)D_{22} - S(x)D_{11}.$$

(11) RELATION BETWEEN AUTO-PAIR AND APOLAR PAIR FOR GIVEN FIXED POINTS.

The knowledge of either the apolar pair or the auto-pair amounts to two linear conditions. Hence, the (2,1) correspondence F is uniquely determined, when its apolar pair or its auto-pair is known in addition to its fixed points. Therefore for a given triad of fixed points $\alpha\beta\gamma$, the apolar pair and auto-pair determine each other uniquely.

Geometrically, let F be considered as a (2,1) correspondence between the points of a fundamental conic C, the fixed points being $\alpha\beta\gamma$. The apolar and auto-pairs are then represented by two points O and S of the plane. It follows then, that S is a certain cremona (quartic) transformation Γ of O. The transformations Γ, Γ^{-1} have some important properties which we may briefly mention here.

To obtain the equations to Γ, Γ^{-1}, it is best to use a binary coordinate system with $\alpha\beta\gamma$ as the triangle of reference. Any point X is the meet of tangents at two points of C, the binary parameters of which are determined by a quadratic of the form:

$$x_1(\beta\gamma)^2(\alpha x)^2 + x_2(\gamma\alpha)^2(\beta x)^2 + x_3(\alpha\beta)^2(\gamma x)^2.$$

We take $x_1 x_2 x_3$ as the coordinates of X. Similarly if the intersections of C with a line L are determined parametrically by a quadratic of the form

$$L_1(\gamma\alpha)(\alpha\beta)(\beta x)(\gamma x) + L_2(\alpha\beta)(\beta\gamma)(\gamma x)(\alpha x) + L_3(\beta\gamma)(\gamma\alpha)(\alpha x)(\beta x),$$

we take $L_1 L_2 L_3$ to be the coordinates of L. The point-coordinates and line-coordinates thus chosen are mutually compatible. For, the incidence condition of the point X and the line L is, from our definition, the apolarity condition of the corresponding quadratics, which is seen to be $L_1x_1+L_2x_2+L_3x_3=0$. The equation to the conic C in point-coordinates, being the condition that the quadratic $x_1(\beta\gamma)^2(\alpha x)^2+x_2(\gamma\alpha)^2(\beta x)^2+x_3(\alpha\beta)^2(\gamma x)^2$ be a perfect square, is

$$2x_2x_3+2x_3x_1+2x_1x_2=0.$$

If we suppose the correspondence F expressed in the shape $\mu_1(\alpha y)(\beta x)(\gamma x)+\dots$, then the apolar and auto-quadratics are given by 3·6 . Hence the coordinates of the points O and S are

$$O:(\mu_2\mu_3,\mu_3\mu_1,\mu_1\mu_2);\ S:\{(\mu_1^2-\overline{\mu_2-\mu_3}^2),(\mu_2^2-\overline{\mu_3-\mu_1}^2),(\mu_3^2-\overline{\mu_1-\mu_2}^2)\}.$$

Therefore the equations to Γ are

$$x_1'=x_2^2x_3^2-x_1^2(x_2-x_3)^2$$
$$x_2'=x_3^2x_1^2-x_2^2(x_3-x_1)^2$$
$$x_3'=x_1^2x_2^2-x_3^2(x_1-x_2)^2.$$

From these equations it may be shewn that

3·11. a fundamental point of the first order, and one of the second order, of Γ, coalesce at each of α,β,γ, so that these points correspond in Γ to line-pairs; the points $\alpha\beta\gamma$ are also fundamental points of the second order of Γ^{-1}, while the poles of $\beta\gamma,\gamma\alpha,\alpha\beta$ in respect of C are fundamental points of the first order of Γ^{-1}; the conic C is a curve of fixed points for Γ .

These statements also become evident on decomposing Γ into simpler transformations. Thus it may be verified that Γ is the result of performing successively the following operations:

(1)the operation $\mathbb{P}$ of taking the polar line in respect of the triangle $\alpha\beta\gamma$,(2)the operation $\mathbb{C}$ of taking the pole in respect of C,and(3)a certain quadratic involutoric Cremona transformation I (Steinerian or generalised isogonal transformation) with the fundamental points $\alpha\beta\gamma$.

A triangle $\alpha\beta\gamma$ inscribed in a fundamental conic C,determines two noteworthy transformations.The first is the perspectivity V which carries $\alpha\beta\gamma$ respectively into the vertices of its polar triangle in regard to C.The centre and axis of the perspectivity V are pole and polar in regard to C,and correspond, as is well known,to the hessian pair of $\alpha\beta\gamma$.The second transformation is the Cremona involution already called I.Among the ∞^1 quadratic involutions which have the fundamental points $\alpha\beta\gamma$(these points corresponding to the opposite sides), the transformation I is distinguished by the fact,that it has the centre of the perspectivity V for a fixed point (the three other fixed points being the vertices of the polar triangle of $\alpha\beta\gamma$).Between these two transformations,and the transformations $\mathbb{P}$, $\mathbb{C}$,there obtains,as may be easily verified,the relation

$$VI = \mathbb{C}\mathbb{P}.$$

This relation replaces our decomposition of $\nabla\Gamma$ by a more symmetrical decomposition*,namely Footnote(2)

$$\Gamma = I\mathbb{C}\mathbb{P} = IVI; \Gamma^{-1} = IV^{-1}I.$$

(12)APOLAR AND AUTO-PAIRS OF SPECIAL FORMS.

If F is of rank one,say $F=(ix)(jx)(\lambda y)$,the apolar quadratic $O(x)$ vanishes identically $(3\cdot2)$,and therefore the apolar pair is indefinite, and may be any pair apolar to ij. The auto-quadratic reduces now to its first term, which is seen to be a vanishing multiple of $(ix)(jx)$. The auto-pair is accordingly definite and is the pair ij. ---------------------------(3.12).

Consider next the case in which F has a binary linear factor (αx), say $F=(\alpha x).(ax)(by)$. The apolar quadratic is then $\frac{1}{2}(\alpha x)^2(aa')(bb')$ (3.1). By 3.3 the auto-quadratic is seen to be $(\alpha x)(ax)(\alpha b)(a'b')$. Thus if $(ax)(by)$ is a general form, the auto-pair may be any pair containing α; if however $(a'b')=0$, that is, if $(ax)(by)$ is symmetric, the auto-quadratic vanishes, and the auto-pair may be any pair of the involution $(ax)(by)=0$. These results are confirmed by 3.11. ------------------------------(3.13).

We shall frequently meet with problems in which it is required to know the nature of the form F, when certain relations hold between its apolar pair and auto-pair.

If the discriminant of the apolar quadratic is known to vanish, then, either $O(x)$ vanishes identically, so that F is of rank one (3.2), and we have the case 3.12, or the apolar quadratic is a perfect square $(\alpha x)^2$. Since F is identically apolar to $O(x)$, it follows that in the latter case F developes the binary factor (αx), reducing to 3.13.

By 3.4, the apolarity of $S(x)$ and $O(x)$ amounts to the vanishing of the discriminant of $O(x)$. Hence from 3.12, 3.13 we have:

3.14. The only case in which the auto-pair ij $(i \neq j)$ and the apolar pair of F are mutually apolar, without the latter being <u>necessarily</u> a repeated pair, is when F is of the shape $(ix)(jx)(\lambda y)$.

One other case of importance is that in which the given auto-pair $ij\,(i\neq j)$ is known to be also the apolar pair. For this case the first term in the expression $3{\cdot}3$ for $\phi(x)$ must either vanish, or be a multiple of $O(x)$. Since this first term is apolar to $O(x)$, the latter alternative would reduce to the vanishing of the discriminant of $O(x)$, and therefore to $3{\cdot}12, 3{\cdot}13$, in neither of which however the auto-pair and the apolar pair are identical. Hence, the first term must vanish, so that by $3{\cdot}5$, F is a polar form. It is clear that $F(x,x)$ has ij then for its hessian pair. The same results may be reached by using $3{\cdot}7, 3{\cdot}8$. ----------------(3·15)

(13) CASES OF INDETERMINATENESS OF APOLAR PAIR FOR GIVEN AUTO-PAIR.

A form F with a given auto-pair ij can be expressed in only one way in the shape $3{\cdot}7$, namely

$$F(x,y)=(ix)(iy)(Ax)+(jx)(jy)(Bx).$$

The fixed cubic and the apolar quadratic of F are then,

$$3{\cdot}16.\quad \begin{cases} f(x)=F(x,x)=(ix)^2(Ax)+(jx)^2(Bx) \\ O(x)=2(ij)\{(ix)(Ax),(jx)(Bx)\}^1. \end{cases}$$

Hence it follows that $O(x)$ can vanish identically only in three cases, (1) $(Ax)=0$ identically, (2) $(Bx)=0$ identically, (3) $(ix)(Ax)$ and $(jx)(Bx)$ differ only by a constant factor, that is, $(iB)=(jA)=0$. In the first case $f(x)$ is $(jx)^2(Bx)$, and the apolar pair may be any pair apolar to jB $(3{\cdot}12)$. If however we specify a definite mode of approach to $f(x)$, the apolar pair becmes determinate. Thus, let

$$f(x)=(jx)^2(Bx)=\underset{\delta\to 0}{Lt}\left((jx)^2(Bx)+\delta\,\psi(x)\right),$$

where $\psi(x)=(ix)^2(Cx)+(jx)^2(Dx)$ is an arbitrary cubic specifying the mode of approach. Then, from $3\cdot16$,

$$O(x)=\{\delta(ix)(Cx),(jx)[(Bx)+\delta(Dx)]\}'$$
$$=\delta\{(ix)(Cx),(jx)(Bx)\}' \text{ in the limit.}$$

Removing the scalar factor δ, we see that the apolar pair has a definite limiting position. It is also clear that this limiting position depends only on the part $(ix)^2(Cx)$ of $\psi(x)$. Hence,

when the fixed cubic $f(x)$ of a form with the auto-pair ij is of the shape $(jx)^2(Bx)$, the apolar pair may be any pair apolar to jB. If however we specify a direction of approach to $(jx)^2(Bx)$ by means of an arbitrary cubic $\psi(x)$,
3·17 the apolar pair has a definite limiting position which is the same for all cubics ψ of the shape $\psi(x)+(jx)^2(\lambda x)$.

A corresponding result holds for case (2). It is easy to ~~obt~~ obtain the following similar result for case (3):

when the fixed cubic of a form with the auto-pair ij is of the shape $(ix)(jx)(Bx)$, the apolar pair is any pair apolar to ij; if however, we specify a definite direction of approach to $(ix)(jx)(Bx)$ by means of an arbitrary
3·18. cubic $\psi(x)$, the apolar pair has a definite limiting position which is the same for all cubics ψ of the shape $\psi(x)+(ix)(jx)(\lambda x)$.

The case in which the form F has ij for its auto-pair, and a perfect cube $(\alpha x)^3$ for its fixed cubic, is of special interest. Since this does not come under the cases (1), (2), (3), just discussed, it follows that the apolar pair of F is quite definite.

To find it we have the identity:

$$(ij)^3(\alpha x)^3 \equiv (j\alpha)^2(ix)^2\{(ij)(\alpha x)+2(i\alpha)(jx)\}+(i\alpha)^2(jx)^2\{(ij)(\alpha x)-2(j\alpha)(ix)\}.$$

Hence the form F is given by

$$F(x,y)=(ix)(iy)(Ax)+(jx)(jy)(Bx),$$

where
$$(Ax) = (j\alpha)^2\{(ij)(\alpha x)+2(i\alpha)(jx)\}$$
$$(Bx) = (i\alpha)^2\{(ij)(\alpha x)-2(j\alpha)(ix)\}.$$

The apolar quadratic of F is therefore:

3·19. $$O(x) = \{(ix)(Ax),(jx)(Bx)\}' = \tfrac{1}{2}(ij)(i\alpha)^2(j\alpha)^2\{3(ij)^2(\alpha x)^2+4(i\alpha)(j\alpha)(ix)(jx)\}.$$

(14) THE APOLAR CUBIC. SIMILAR FORMS.

Let the values of x which correspond to y in F be $x_1 x_2$. Consider the system Γ of the ∞^1 triads $(y x_1 x_2)$. If one point z of a triad of this type is known, the triad itself is determined in two ways, according as we consider z as an x or a y. Hence Γ is a quadratic system. Also, Γ is a continuous system. Since a continuous curve of the second order in space is necessarily a plane curve, it follows that all triads of Γ are apolar to a certain cubic $f''(x)$, which we call <u>the apolar cubic</u> of F.

Since $f''(x)$ is by definition, identically apolar to $F=(xy)(ax)^2(by)$, it follows that $f''(x)$ is the second rank covariant of the (3,2) form $(xy)(ax)^2(by)$. This would effect the symbolic determination of the apolar cubic.

If ij is the auto-pair of F, and AB the residual pair $(3·10)$, then each of the triads ijA, ijB is of the type yx_1x_2 and therefore apolar to $f''(x)$. Hence

3·20. The auto-pair of F is the hessian pair of its apolar cubic $f''(x)$.

If F is a polar form with the fixed points $\alpha\beta\gamma$,and if $\alpha''\beta''\gamma''$ are the cubicovariant points of $\alpha\beta\gamma$,then $y=\alpha$ corresponds in F to $x=\alpha,\alpha''$.Hence the apolar cubic of F is apolar to the three triads of the type $(\alpha\alpha\alpha'')$,and therefore represents the triad $\alpha''\beta''\gamma''$ .Thus

3·21 The apolar cubic of a polar form F is the cubicovariant of its fixed cubic $f(x)=F(x,x)$.

DEF.Two forms F, F' will be said to be SIMILAR,if their apolar cubics differ only by a constant factor.

It follows that all forms similar to F have the same auto-pair ij as F (3·20).It also follows from the definition, that $\lambda F_1+\mu F_2$ is similar to F,if F_1 and F_2 are similar to F. Hence the family of forms similar to F is a linear family, involving clearly three parameters.Hence there exists a polar form P similar to F.Also,if ij is the auto-pair of F,the apolar cubic of $(ix)(jx)(\lambda y)$ is indeterminate,and may be any cubic apolar to $(ix)(jx)$.Therefore the forms $(ix)(jx)(\lambda y)$ must be reckoned among the forms similar to F.We have therefore finally the result:

3·22 If P is the polar form similar to F,the most general form similar to F is of the shape $\mu P(x,y)+(ix)(jx)(\lambda y)$, where ij is the auto-pair of F and μ,λ_1,λ_2 are arbitrary.

Since F is similar to itself,it follows that F itself must be of this shape.Hence $f(x)=F(x,x)=\mu P(x,x)+(ix)(jx)(\lambda x)$.Let now $f''(x)$ be the apolar cubic of F,and $f'(x)$ the cubicovariant of $f''(x)$.

Since P is similar to F, its apolar cubic is a multiple of $f''(x)$; therefore since P is a polar form, it follows from 3·21 that $P(x,x)$ is a multiple of $f'(x)$. Also, by 3·20, ij is the hessian pair of $f''(x)$, and therefore also of $f(x)$, so that $(ix)(jx)(\lambda x)$ is apolar to $f'(x)$ for any λ. Hence ~~the cubic of fixed~~

3·23. the cubic of fixed points $f(x) = F(x,x)$ is apolar to the cubicovariant of the apolar cubic of F.

From these theorems we can construct the correspondence F, when its apolar triad $f''(x)$ and its apolar pair lm are given. Let x be any point, x' the harmonic conjugate of x in respect of lm, and $x_1 x_2$ the pair harmonic to xx' and ij, where ij is the hessian pair of $f''(x)$, and therefore the auto-pair of F. Let y be the point whose first polar pair in respect of $f''(x)$ is $x_1 x_2$. Then y corresponds to x (and x') in F.

(15) AUTOMORPHIC TRANSFORMATIONS OF THE (2,1) CORRESPONDENCE.

In this and the next subsection we consider correspondences between x, y, as operations which transform x into y; thus the correspondence F(x,y)=0 would be indicated in the operational symbolism, by the equation:

Footnote (3) $$y = Fx \text{ or } x = F^{-1}y. *$$

If F, F' be two correspondence-operations, the product FF' indicates as usual the result of performing F', F in succession.

The involution I whose fixed pair is the apolar pair (lm) of the (2,1) correspondence F, will be called the basic involution of F. Since lm is by definition, apolar to all pairs $F^{-1}y$, it follows that $Fx = FIx$. In other words,

3·24 A (2,1) correspondence F absorbs its basic involution

I ;in symbols $F = FI$.

Two correspondences F, F' will be said to be congruent, if they have the same basic involution. Now, a (2,1) correspondence $F(x,y)$ may be considered as a (1,1) correspondence between y and the pairs of the basic involution of F. Hence it is clear, that by following up F by a homography, we would obtain the most general correspondence congruent to F. Hence

3·25 Any two congruent correspondences F, F' are connected by a relation of the form $F' = HF$, where H is a homography.

We shall now obtain the condition that the correspondence FH, where H is a homography, may be congruent to F. If I is the basic involution of F, then

$$\begin{aligned} FH.H^{-1}IH &= FIH \\ &= FH \ (3\cdot 24). \end{aligned}$$

Hence $H^{-1}IH$ is the basic involution of FH. Thus, for F and FH to be congruent, the involutions $I, H^{-1}IH$ must be the same. Now the fixed pair of I is the apolar pair lm of F, and the fixed pair of $H^{-1}IH$ is the pair $(H^{-1}l, H^{-1}m)$. Hence if $I = H^{-1}IH$, H is either a homography H_1 with the fixed points lm, or an involution I_1 containing the pair lm. The transformations H_1 and I_1 together form the discontinuous automorphic linear group of the point-pair lm, of which the continuous group $[H_1]$ is the maximum selfconjugate sub-group.* Thus

Footnote (4)

3·26 the homographies H which make FH congruent to F, are members of the automorphic linear group $[H_1, I_1]$ of the apolar pair of F.

It should be noticed that $[H_1]$ contains a member, namely the basic involution I, which is permutable with every operation of $[H_1, I_1]$.

Now suppose that H has been so chosen that FH is congruent to F. Then by 3·25, there must be a relation of the form

$$3\cdot27 \qquad F = H'FH,$$

where H' is a homography. It is clear from this relation, that H' is a homography which transforms the pair $l'm'$ into itself, where $l' = Fl$, $m' = Fm$ are the correspondents of the apolar pair in F. Thus the homographies H' generate the automorphic linear group $[H'] = [H_1', I_1']$ of the point-pair $l'm'$. The groups $[H']$, $[H]$ are thus of the same type.

Now in the relation 3·27 H determines H' uniquely; also if H is of the type H_1, H' is also of the type H_1', and if H is of the type I_1, H' is of the type I_1'. Thus 3·27 is composed of two types of relations, $F = H_1'FH_1$, and $F = I_1'FI_1$, which we shall refer to ~~respectively~~, as the direct and skew automorphic transformations of F, respectively.

While H determines H' in 3·27, H' does not however determine H uniquely. The correspondence between H', H is such that when H' corresponds to H and $\overline{H}'$ to $\overline{H}$, then $H'\overline{H}'$ corresponds to $\overline{H}H$, and $\overline{H}'H'$ to $H\overline{H}$. Also, when H' is identity, then H is either identity or the basic involution I. Thus, it follows that the correspondence between H'^{-1} and H is a (1,2) isomorphism. If H' corresponds to H, the other value of H which corresponds to H' is therefore HI=IH. We proceed to determine more precisely the (1,2) correspondence (H', H), for the direct and skew transformations of F.

For the skew transformations $F = I_1' F I_1$, it is easy to see that when I_1' is a singular involution with the fixed pair $(l'l')$, then the two corresponding involutions I_1 both coalesce with the singular involution whose fixed pair is (ll). It is further ~~evidn~~ evident on inspection, that when I_1 contains the auto-pair of F, I_1' also contains the auto-pair. Hence if the involutions I_1', I_1 areparametrically represented by η, ξ, respectively, in such wise that the singular involutions correspond to the values $0, \infty$ of the parameters, and the involutions I_1', I_1 which contain the auto-pair to the values $\eta = 1, \xi = 1$, then the correspondence between I_1', I_1 would clearly be expressed by the relation $\eta = \xi^2$.

Consider next the direct transformations $F = H_1' F H_1$. The operations of the Abelian group $[H_1]$ can be represented by an angular parameter θ, defined modulo π. Thus, if ν is the multiplier of H_1 at the fixed point l (say), we define θ to be $\frac{1}{2\sqrt{-1}} \log \nu$. If the angular parameters θ', θ of H_1', H are reckoned in respect of corresponding fixed points l', l (say), then the (1,2) correspondence (H_1', H_1) is determined by the following theorem:

3·28. If $F = H_1' F H_1$ is any direct transformation of F, then the angular parameters θ', θ of H_1', H_1 are connected by the relation $\theta' = -2\theta$.

For, consider a point $l + \delta z$, where δ is a small scalar, and z is an arbitrary fixed point. Since l is a fixed point of the basic involution of F, it follows that $x = l + \delta z$ corresponds in F to a point which differs from l' by a small quantity of the second order. Thus

$$F\,\overline{l+\delta z} = l' + \mu\,\delta^2 z,$$

where μ is independent of δ. Let now ν, ν' be the multipliers of H_1, H_1' at l, l'. Then

$$\begin{aligned} H_1' F H_1\,\overline{l+\delta z} &= H_1' F\,\overline{l+\nu\delta z} \\ &= H_1'\,\overline{l+\mu\nu^2\delta^2 z} \\ &= l' + \mu\nu'\nu^2\delta^2 z. \end{aligned}$$

Hence, since $F = H_1' F H_1$, $\mu\nu'\nu^2\delta^2 = \mu\delta^2$ or $\nu' = \frac{1}{\nu^2}$.

Thus $\theta' = \frac{1}{2\sqrt{-1}} \log \nu' = \frac{1}{2\sqrt{-1}} \cdot -2 \log \nu = -2\theta.$

(16) CANONICAL RESOLUTION OF THE (2,1) CORRESPONDENCE.

There are ∞^1 polar correspondences which are congruent to F; namely those defined by the ∞^1 triads which have as their hessian pair, the apolar pair $(l\,m)$ of F. Let P_1 be any one of these polar correspondences. Then by $3{\cdot}25$ there is a relation of the form

$$F = H P_1,$$

where H is a homography. Since $P_1 l = m$, $P_1 m = l$, it follows that $m' = Fm = Hl$, $l' = Fl = Hm$. Hence H can be expressed in only one way in the shape $I H_1$, where H_1 is a homography with the fixed points $l\,m$, and I is the involution determined by the pairs $l m'$, $l' m$. Thus $F = I H_1 P_1$; but $P = H_1 P_1$ is only another of the ∞^1 polar correspondences congruent to F (since $H_1 P_1 l = m$, $H_1 P_1 m = l$). Hence

3·29 the correspondence F can be expressed uniquely in the shape IP, where P is a polar correspondence, and I an involution.

We call I and P the <u>involutoric</u> and <u>polar components</u> of F, respectively. These components are related in a simple manner to the correspondence F';

which has the same fixed points as F, and which has for its auto-pair, the apolar pair of F.

DEF. F will be said to be the correlate of F', if F and F' have the same fixed points, and the apolar pair of F is the auto-pair of F'.

With this definition, we have the following theorem which specifies the two components of F:

3·30. If F is the correlate of F', the involutoric component of F is the basic involution of F', and the polar component of F is the polar correspondence defined by the apolar cubic of F'.

To prove this, let I be the basic involution of F', and P the polar correspondence defined by $f''(x)$, the apolar cubic of F'. Let a, b, c be the fixed points of F' and a', b', c' the other points which correspond in F' to $y=a, b, c$ respectively. From the definition of the apolar cubic it follows that each of the triads aa', bbb', ccc' is apolar to $f''(x)$; therefore $Pa=a', Pb=b', Pc=c'$. But since $x=a, a'$ correspond to the same value a of y in F', it follows that aa', bb', cc' are pairs of the basic involution I, so that $IPa=Ia'=a$. Thus the fixed points a, b, c of F' are also the fixed points of IP. Also, from the definition of P, the apolar pair of IP, which is the same as the apolar pair of P, is the hessian pair of $f''(x)$, and therefore the auto-pair of F' (3·20). Hence IP is the correlate of F', as was to be proved.

The correlate F of a correspondence F' with the auto-pair ij may be constructed by a simple rule. Let $x_1 x_2$ correspond in F to y, so that from the definition of correlate, $x_1 x_2$ is a pair apolar to ij. Then if J be the involution whose fixed pair is x_1x_2, two of the fixed points of JF' are clearly ij. Let the third fixed point be y'. The correspondence between y' and the pairs x_1x_2 apolar to ij is clearly (1,1). Also, when x_1x_2 is one of the three pairs aa', bb', cc', $y'=y=a, b,$ or c, respectively. Hence $y'=y$ always; in other words, ~~Hence~~

> if J is a variable involution containing the autopair ij of F', the correlate of F' is simply the
>
> 3·31. correspondence between the fixed pair of J and the fixed point other than ij of JF'.

§ 4 THE TWO-QUADRATIC REPRESENTATION OF THE PENCIL OF CUBICS.

(17) DERIVATION OF THE REPRESENTATION.

It was mentioned that the assigning of either the apolar pair or the auto-pair of a (2,1) correspondence F, amounts to two linear conditions ~~on the conditions~~ on the coefficients of F. Hence it follows that

4·1 the totality of (2,1) forms $F(x,y)$ which have a given auto-pair $P_1(x)$, and a given apolar pair $P_2(x)$, belong to a pencil. Therefore also, the fixed cubics $F(x,x)$ of such forms belong to a pencil, which is completely determined by P_1, P_2.

The pencil to which $F(x,x)$ belongs may therefore be denoted by the symbol $[P_1, P_2]$. Now there are ∞^4 pencils of cubics, and there are also ∞^4 pairs of point-pairs P_1, P_2. Hence any pencil of ~~cu~~ cubics can be represented in a finite number of ways in the form $[P_1, P_2]$. A representation of this kind we shall call "a two-quadratic representation" of the pencil.

4·2. The quadratic P_1 is a factor of the Jacobian of the pencil $[P_1, P_2]$.

For, let $P_1(x)=(ix)(jx)$. The family of correspondences which have the auto-pair ij can be expressed in the shape 3·7, namely

$$(ix)(iy)(Ax)+(jx)(jy)(Bx).$$

There is a unique member of this family, say F, which has P_2 for its apolar pair, and a fixed point at i. For i to be a ~~fi~~ fixed point of $F(x,y)$, B must be equal to ki, where k is numerical. If now $F(x,y)$ were of full rank two, its apolar quadratic would be a multiple of $k(ix)^2$, and therefore its apolar pair would not be that given by $P_2(x)$. Hence k must be zero, so that F reduces to $(ix)(iy)(Ax)$, where for P_2 to be the apolar pair, iA must be apolar to P_2. Thus $F(x,x)=(ix)^2(Ax)$, shewing that i and similarly j are roots of the jacobian of the pencil denoted by $[P_1,P_2]$. We have also obtained here a rule for constructing the pencil $[P_1,P_2]$, namely

4·3 Let $P_1(x)=(ix)(jx)$, and let the harmonic conjugates of i,j with respect to P_2 be A, B respectively. Then the pencil $[P_1,P_2]$ is simply the pencil determined by the two cubics $(ix)^2(Ax), (jx)^2(Bx)$.

Therefore, to obtain a two-quadratic representation of any pencil of cubics, we can take any quadratic factor of its Jacobian for P_1, and then construct P_2 by means of 4·3. Thus

4·4 there are exactly six two-quadratic representations of a pencil of cubics.

These six representations fall into three conjugate pairs; namely the representations $[P_1,P_2]$, $[P_1',P_2']$ are conjugate, if the Jacobian of the pencil is $P_1 P_1'$.

Footnote(1)
(18) REPRESENTATION OF APOLAR PENCILS.*

To see how the two-quadratic representations of two apolar pencils are mutually connected, we have first to specify the relation of the pencil $[P_1,P_2]$ to the quadratic apolar to P_1,P_2. For this purpose we begin by recalling some fundamental properties of ~~the~~ a pencil Γ of cubics.

The pencil Γ possesses several quartic combinants, those which concern us here being; the apolar quartic, which is the unique quartic apolar to every member of Γ, the Jacobian quartic, the roots of which are the repeated roots of members of Γ, and the residual quartic, the roots of which ~~it~~ are the non-repeated roots of those members of Γ which possess repeated roots ~~of those~~. Since Γ determines, and is determined ~~by~~ uniquely by, its apolar quartic, it follows that all the quartic combinants must be covariants of the apolar quartic, and must Footnote(2) as,* such, belong to its syzygetic pencil*. The sextic covariant of this associated syzygetic pencil, which is, as is well known, the product of three mutually apolar quadratics, may be called the sextic combinant of Γ.

The three involutions determined by the three mutually apolar quadratic factors of the sextic combinant of Γ, form with identity a group G of order four, which transforms every quartette of the asso~~s~~ciated syzygetic pencil into itself. Hence ~~T~~ G must transform Γ into itself, and therefore every member of Γ into another member. In particular, G transforms a member with a repeated root into another member with a repeated root. Thus if $i_1 i_2 i_3 i_4$ be the Jacobian quartette, and $A_1 A_2 A_3 A_4$ the residual quartette, so that $(i_r x)^2 (A_r x)$ $(r=1,2,3,4)$ are the members of Γ with a repeated root, it follows that

> any involution of G effects the SAME permutation on the two quartettes $i_1 i_2 i_3 i_4$, $A_1 A_2 A_3 A_4$ --- which are the
> 4·5 Jacobian and residual quartettes, written in the order in which they are associated in Γ.

Now any two quartettes $i_1 i_2 i_3 i_4$, $A_1 A_2 A_3 A_4$, of a syzygetic pencil, when written down in arbitrary orders, may either be iso-

<u>morphic</u> or <u>antimorphic</u> in regard to the group G of the pencil; they would be isomorphic if any involution of G effects the <u>same</u> permutation on them both, antimorphic otherwise. The theorem $4{\cdot}5$ amounts therefore to the isomorphism of the Jacobian and residual quartettes, when in the orders in which they are associated in Γ. This isomorphism clearly implies that the quadratic apolar to $(i_r x)(i_s x)$ and $(A_r x)(A_s x)$ is one of the three mutually apolar quadratic factors of the sextic covariant, for $r,s = 1,2,3,4;\ r \neq s$.

Now take any one of the six two-quadratic representations of Γ, say, the representation $[P_1, P_2]$, where $P_1(x) = (i_1 x)(i_2 x)$. BY $4{\cdot}3$ the quadratic P_2 is then the quadratic apolar to $(i_1 x)(A_1 x)$ and $(i_2 x)(A_2 x)$. The quadratic apolar to P_1 and P_2 is then seen to be simply the quadratic apolar to $(i_1 x)(i_2 x)$ and $(A_1 x)(A_2 x)$. Hence by the theorem of isomorphism of order $(4{\cdot}5)$, it is one of the quadratic factors of the sextic combinant of Γ. We therefore have the result:

> $4{\cdot}6$ the quadratic apolar to P_1 and P_2 is one of the three mutually apolar quadratic factors of the sextic combinant of the pencil $[P_1, P_2]$.

It is well known that apolar pencils of cubics have the same Jacobian, and therefore the same sextic combinant. Therefore they could be simultaneously represented in the shapes $[P_1, P_2]$, $[P_1, P_3]$, P_1 being any factor of their common Jacobian. Now, there is only one quadratic factor of the sextic combinant, which is apolar to P_1. This quadratic factor is by $4{\cdot}6$ apolar to P_2, as well as to P_3. Thus there is a linear relation between P_1, P_2, P_3 .Hence:

4·7 Apolar pencils of cubics admit of simultaneous two-quadratic representations in the shapes $[P_1, P_2], [P_1, P_1+\lambda P_2]$.

(19) REPRESENTATION OF THE NULL PENCIL.

The last theorem 4·7 points to the necessity of studying the family of pencils whose two-quadratic representations are of the shape $[P_1, P_1+\mu P_2]$ .Let us call the pencil $[P_1, P_1+\mu P_2]$, the pencil Γ_μ .There are three fundamental μ-involutions which have to be considered; namely,

(i) the involution I_1 between μ_1, μ_2 ,such that the quadratics $P_1+\mu_1 P_2, P_1+\mu_2 P_2$ are apolar.The fixed elements of this involution correspond to the perfect squares of the pencil $P_1+\mu P_2$ and are therefore given by

$$\mu^2 d_{22} + 2\mu d_{12} + d_{11} = 0$$

where the d's are the usual invariants of P_1, P_2 .

(ii) It is seen from 4·7 that the apolar pencil of a pencil Γ_{μ_1} is a pencil Γ_{μ_2} .We call the involution between $\mu_1 \mu_2$,the involution I_2.The fixed elements of I_2 are the values of μ for which Γ_μ is a null pencil; these values have yet to be determined.

(iii) WE define <u>the absolute invariant</u> of two quadratics P_1, P_2 to be the number $\frac{d_{11}d_{22}}{d_{12}^2}$; thus the absolute invariant vanishes when either of the quadratics is a perfect square, becomes infinite when they are apolar, and takes the value unity when Footnote (3) they have a common factor.* Since the absolute invariant is of the second degree in the coefficients of either quadratic, there must be two values of μ for which the absolute invariant of $P_1, P_1+\mu P_2$ is a given number k. We define I_3 to be the involution of pairs of such values of μ. It is immediately verified that the two values of μ coalesce only when $k=\infty$ or 1. Now the absolute invariant of $P_1, P_1+\mu P_2$ is equal to 1 only when P_1 and $P_1+\mu P_2$ have a common factor, that is, only when $\mu=0$; and the same absolute invariant is infinite only when $P_1+\mu P_2$ is the quadratic P_1' of the pencil, which is apolar to P_1. Thus the fixed elements of I_3 correspond to the quadratics P_1, P_1'.

But since the quadratics P_1, P_1' are apolar, they correspond to pair of the involution I_1; thus the fixed elements of I_3 belong to I_1. The two involutions I_1, I_3 are therefore symmetrically related, and are permutable with each other.

We next shew that the fixed elements of I_3 belong also to I_2. For this we have to shew that the pencils, whose two-quadratic representations are $[P_1, P_1]$, $[P_1, P_1']$, are apolar pencils. By 3·15 the (2,1) correspondences whose auto-pair and apolar pair are identical with $P_1(x)$, are the polar correspondences whose fixed triads are apolar to P_1. Hence the pencil $[P_1, P_1]$ is the pencil of triads apolar to P_1. By 3·14 the correspondences whose auto-pair is $P_1(x)$, and whose apolar pair is apolar to $P_1(x)$, are of the shape $P_1(x)(\lambda y)$, where λ is arbitrary. Hence the pencil $[P_1, P_1']$ is the singular pencil $P_1(x)(\lambda x)$. Thus the pencils $[P_1, P_1]$, $[P_1, P_1']$ are apolar pencils, shewing that the fixed elements of I_3 belong also to I_2. The involution I_3 may therefore be specified as that which contains the fixed pairs of I_1 and I_2.

To specify I_2 itself, we write every quadratic $P_1+\mu P_2$ in the shape $P_1+\lambda P_1'$; the perfect squares of the pencil $P_1+\mu P_2$ are then of the shape $P_1 \pm tP_1'$, where t is given by $t^2 d' + d = 0$, d', d being the respective discriminants of P_1', P_1 . Now write $P_1 + tP_1' = (\alpha x)^2$, and consider the apolar pencil of the pencil $[P_1, (\alpha x)^2]$. The $(2,1)$ form which has $\overset{(\alpha x)^2}{\cancel{P(x)}}$ for apolar pair, and $\overset{P_1(x)}{\cancel{(\alpha x)^2}}$ for its auto-pair must possess the binary factor (αx) , as was seen in (12) . Thus the members of the pencil $[P_1, (\alpha x)^2]$ have the common factor (αx), so that its apolar pencil contains the member $(\alpha x)^3$. We see then that the apolar pencil of $[P_1, (\alpha x)^2]$ is that pencil of the family $[P_1, P_1+\mu P_2] = \Gamma_\mu$, which contains the member $(\alpha x)^3$. Now the apolar pair of the (2,1) form which has the auto-pair $P_1(x) = (ix)(jx)$, and the fixed cubic $(\alpha x)^3$, has been determined in $3{\cdot}19$ to be

$$3(ij)^2(\alpha x)^2 + 4(i\alpha)(j\alpha)(ix)(jx) = -6d(P_1 + tP_1') + 4dP_1$$
$$= -2d(P_1 + 3tP_1').$$

Thus the two pencils $[P_1, P_1+tP_1'], [P_1, P_1+3tP_1']$ are apolar pencils; so also are the two pencils $[P_1, P_1-tP_1'], [P_1, P_1-3tP_1']$. The involution I_2 therefore contains the two pairs $\lambda = t, 3t$ and $\lambda = -t, -3t$; hence its fixed pair is given by $\lambda = \pm\sqrt{3}\,t$. This determines the null pencils of our family, namely

$4{\cdot}8$ if the perfect squares of the pencil $P_1+\mu P_2$ are $P_1 \pm tP_1'$, the null pencils of our family are ~~given by~~ the pencils $[P_1, P_1 \pm \sqrt{3}\,tP_1']$.

Since $t^2 d' + d = 0$, the absolute invariant of $P_1, P_1 \pm \sqrt{3}\,tP_1'$ is equal to $\dfrac{d(d+3t^2d')}{d^2} = -2$. Hence

$4{\cdot}9$ If $[P_1, P_2]$ is a null pencil, the absolute invariant of P_1, P_2 is equal to -2 .

The results of this subsection give the fundamental properties of the two-quadratic representation. It is easy to derive from them any formulae or invariant expressions that may be required.

(20) THE ABSOLUTE PARAMETERS OF A PENCIL.

If the pencil Γ has the two-quadratic representation $[P_1, P_2]$, then it is clear that the absolute invariant of P_1, P_2 is an irrational absolute invariant of the pencil Γ; we shall call it an absolute parameter of Γ.

Suppose the Jacobian of Γ to be the product of the quadratics P_1, P_1', and let the corresponding conjugate representations of Γ be $[P_1, P_2], [P_1', P_2']$. It is clear from (18), that two involutions of the group G associated with Γ transform P_1, P_2 simultaneously into P_1', P_2'. Thus the absolute invariants $(P_1, P_2), (P_1', P_2')$ are equal, so that the six two-quadratic representations of Γ give rise only to three distinct absolute parameters.

It is clear that two pencils Γ, Γ' are linearly transformable into one another, if one absolute parameter of either is equal to one absolute parameter of the other. It follows from this, that the absolute invariant $t \left(= \frac{i^3}{6j^2}\right)$ of the apolar quartic of the pencil Γ must be a rational function of the absolute parameter s of Γ. We proceed to determine the exact form of this function.

If t is given, the pencil Γ is determined apart from an arbitrary linear transformation, and therefore its absolute parameter s is determined in three ways. Hence the relation between t and s is of the form

$$t = \frac{f(s)}{\phi(s)},$$

where $f(s), \phi(s)$ are cubics. Now if $t = 0$, the apolar quartic of Γ is self-apolar and therefore Γ is a null pencil. But 4·9 shews that the absolute parameters of a null pencil are all equal to -2 ;hence $f(s)=(s+2)^3$.Again when $t=1$,the apolar quartic of Γ has a squared factor $(\alpha x)^2$,and therefore the pencil Γ contains the perfect cube $(\alpha x)^3$.In this case it is easy to see that for four of the six representations $[P_1, P_2]$ of Γ, P_1 and P_2 have the common factor (αx).Thus when $t=1$,two of the values of s are equal to 1 .Lastly when $t=\infty$,the apolar quartic has vanishing j and possesses an apolar quadratic $P_1(x)$.The pencil Γ is then the singular pencil $P_1(x)(\lambda x)$ and has therefore the representation $[P_1, P_1']$ (where P_1' is apolar to P_1),which counts four times among its six representations.Hence when $t=\infty$,two of the values of s are infinite. These conditions determine $\phi(s)$ as being equal to $27s$ .We have therefore finally the relation:

$$4\cdot10. \qquad t=\frac{(s+2)^3}{27s},$$

between the absolute parameter of Γ and the absolute invariant of its apolar quartic.* Footnote (4).

§5. THE BINARY GEOMETRY OF THE CIRCLE.

(21) THE SPECIAL (2,1) CORRESPONDENCES ON THE CIRCLE.

Henceforth we are concerned with the geometry in respect of a fundamental circle, of centre S and radius ρ. The points on the circle will be represented by binary symbols, the circular being invariably represented by i, j. The types of (2,1) correspondence on the circle which have a metrical significance are those which are specially related to ij. We consider two such types, namely

(i) Auto-correspondences, or the (2,1) correspondences which have the auto-pair ij.

(ii) Diametral correspondences, or the (2,1) correspondences which have the apolar pair ij.

An auto-correspondence (or diametral correspondence) is completely determined by its triangle of fixed points $\alpha\beta\gamma$, and may be called "the auto-correspondence (or diametral correspondence) of $\alpha\beta\gamma$."

From the definition of correlate (16), it follows that the diametral correspondence of $\alpha\beta\gamma$ is the correlate of its auto-correspondence. The theorem 3·31 might therefore be utilised to derive the former from the latter.

A pair of points on the circle may be named indifferently by the chord joining them, or by the pole of this chord in respect of the circle. If two pairs on the circle be harmonic, the line which names either is incident with the point which names the other. Thus if the apolar pair of a (2,1) correspondence F on the circle be named by the point O, then the pairs of the basic involution of F are the extremities of chords passing through O. F itself may accordingly be considered as a (1,1) correspondence $\{F\}$ between points of the circle and lines through O. In particular, if F is a diametral correspondence, then O coincides with the centre S of the circle; thus the general diametral correspondence is a (1,1) correspondence between diameters and points on the circle.

If F is the auto-correspondence of $\alpha\beta\gamma$, and if $y=p$ corresponds in F to $x=q_1,q_2$, the chord Oq_1q_2 may be called for brevity <u>the auto-chord</u> of p in respect of $\alpha\beta\gamma$. The (1,1) correspondence $\{F\}$ is then the correspondence between points on the circle and their auto-chords. Since i, j correspond to each other in the auto-correspondence F, the auto-chords of each of $i\,j$ passes through the other. If now Oq, Oq' be the respective auto-chords of p, p' it must therefore follow from the property of a (1,1) correspondence, that the cross ratios $O\{qq'ji\}$ and $\{pp'ij\}$ are equal. Hence follows the fundamental angle-property of the auto-correspondence:

5·1 If the point p moves on the circle through an angle ω about the centre, its auto-chord turns through an angle $-\frac{\omega}{2}$.

If the triangle $\alpha\beta\gamma$ is equilateral, then ij would be the hessian pair of $\alpha\beta\gamma$, and the polar correspondence which has the fixed points $\alpha\beta\gamma$ would have ij for its apolar as well as its auto-pair (Cf 3·15). Thus

5·2 The diametral correspondence and the auto-correspondence of an equilateral triangle are both identical with its polar correspondence.

We shall hereafter concern ourselfves only with the auto-correspondence, the diametral correspondence being of little importance for our purpose.

The apolar cubic $f''(x)$ of the auto-correspondence F of $\alpha\beta\gamma$ has ij for its hessian pair (3·20), and therefore represents the vertices $\alpha''\beta''\gamma''$ of a certain equilateral triangle. If $\alpha'\beta'\gamma'$ is the unique equilateral triangle apolar to $\alpha\beta\gamma$, then by 3·23, $\alpha'\beta'\gamma', \alpha''\beta''\gamma''$ are cubicovariant equilateral triangles, so that $\alpha'\alpha'', \beta'\beta'', \gamma'\gamma''$ are diameters. Hence if $\alpha''\beta''\gamma''$ is given, the possible triangles $\alpha\beta\gamma$ are apolar to a fixed equilateral triangle $\alpha'\beta'\gamma'$, which is the cubicovariant triangle of $\alpha''\beta''\gamma''$. Also, since similar correspondences have the same auto-pair (14), the correspondences similar to an auto-correspondence must also be auto-correspondences. It is therefore clear that

5·3 the auto-correspondences of the ∞^2 triangles apolar to an equilateral triangle, constitute a family of similar correspondences.

But if $F(x,y)$ is an auto-form (that is, a form which determines an auto-correspondence), the family of forms similar to $F(x,y)$ is given by $\mu F(x,y) + (ix)(jx)(\lambda y)$ (3·22). Thus the auto-chords of a point y in regard to two similar auto-correspondences, are represented by two quadratics which differ by a multiple of $(ix)(jx)$, and are therefore parallel. Hence, using 5·3, we have the result:

5·4. The auto-chords of a fixed point y in respect of all triangles apolar to an equilateral triangle, are parallel straight lines.

(22) THE PEDAL ANGLE.

The fundamental angle-property 5·1 may be stated in the following symmetrical form:

~~5.4~~ If the auto-chord in respect of $\alpha\beta\gamma$, of <u>one</u> vertex of any inscribed triangle abc is parallel to its opposite

5·5 side, the same is true of every vertex; and the triangles abc with this property are all the triangles apolar to the equilateral triangle $\alpha''\beta''\gamma''$ of (2·1) .

The first part follows from 5·1 ; to prove the latter part of the theorem, we notice that the triangles abc are determined by any two of their vertices, and constitute therefore a linear ∞^2-system, apolar to a certain triangle. Now, if $p_1 p_2$ is the auto-chord of a point p in respect of $\alpha\beta\gamma$, the triangles pp_1p_2 are of the type abc, and are apolar to $\alpha''\beta''\gamma''$ from the definition of the apolar triad. Hence the system of triangles abc is the system apolar to $\alpha''\beta''\gamma''$.

Let $\alpha\beta\gamma$, abc be two inscribed triangles, and let the auto-chords of a point p in respect ~~in~~ of them be inclined at an angle θ with one another. Then, by 5·1, the auto-chords of any other point p' in respect of the same triangles will also make the same angle θ with one another. Thus the angle θ depends only on the two triangles, and may be called their <u>pedal angle</u>. The triangles $\alpha\beta\gamma$, abc will also be said to be <u>pedo-parallel</u> when their pedal angle $\theta = 0 \pmod{\pi}$, and <u>pedo-perpendicular</u> when the pedal angle $= \frac{\pi}{2} \pmod{\pi}$. The ∞^2-system of all triangles pedo-parallel to $\alpha\beta\gamma$ may be called <u>the pedo-parallel system</u> of $\alpha\beta\gamma$. A pedo-parallel system is clearly a linear system, and it follows from 5·4 that

5·6. a pedo-parallel system may also be defined as the ~~s~~system of triangles apolar to an equilateral triangle.

Thus the ∞^3 inscribed triangles of the circle fall into ∞^1 pedo-parallel systems, each of which is specified by its equilateral member, or by any one of its members. The triangles of the type $ij\lambda$ being apolar to every equilateral triangle, belong to every pedo-parallel system.

5·7 Cubicovariant equilateral triangles are pedo-perpendicular.

For, let $abc, a'b'c'$ be cubicovariant equilateral triangles, so that aa', bb', cc' are diameters. By 5·2, the auto-chord of a in respect of abc and of a' in respect of $a'b'c'$, is in each case the diameter aa'. Hence from the fundamental angle-property 5·1, it follows that $abc, a'b'c'$ are pedo-perpendicular.

We can conclude from this theorem that the triangle $\alpha''\beta''\gamma''$ which represents the apolar triad of the auto-correspondence of $\alpha\beta\gamma$, is pedo-perpendicular to $\alpha\beta\gamma$. For, the equilateral triangle $\alpha'\beta'\gamma'$ apolar to $\alpha\beta\gamma$ is also the pedo-parallel equilateral triangle of $\alpha\beta\gamma$ (5·6); and from the last section (21), $\alpha''\beta''\gamma''$ is the cubicovariant equilateral triangle of $\alpha'\beta'\gamma'$. Hence $\alpha''\beta''\gamma''$ is the equilateral triangle pedo-perpendicular to $\alpha'\beta'\gamma'$, and therefore to $\alpha\beta\gamma$. It also follows that the triangles abc in theorem 5·5 are all the triangles pedo-perpendicular to $\alpha\beta\gamma$. This theorem might also be extended in the following manner:

5·8 The auto-chords in respect of $\alpha\beta\gamma$, of the vertices of any other inscribed triangle abc, make the same angle θ with their opposite sides; this θ is the complement of the pedal angle between $\alpha\beta\gamma$ and abc.

The first part follows from 5·1. To prove the second part, we have only to take a triangle $\alpha_1\beta_1\gamma_1$, which makes the pedal angle $-\theta$ with $\alpha\beta\gamma$. Then abc is such that the auto-chords of its vertices, in respect of $\alpha_1\beta_1\gamma_1$, are parallel to their opposite sides; hence in virtue of our previous remark, abc is pedo-perpendicular to $\alpha_1\beta_1\gamma_1$, and therefore makes the pedal angle $\frac{\pi}{2}-\theta$ with $\alpha\beta\gamma$.

A special case of 5·8 arises if abc is identical with $\alpha\beta\gamma$. Since $\alpha\beta\gamma$ is pedo-parallel to itself, it follows that the auto-chords of α, β, γ in respect of $\alpha\beta\gamma$ are perpendicular to their opposite sides. But these auto-chords should pass through α, β, γ, respectively, and also through the point O which represents the apolar pair of the auto-correspondence of $\alpha\beta\gamma$. Hence we have the fundamental result:

5·9 The perpendiculars of the triangle $\alpha\beta\gamma$ are concurrent; and the point of concurrence, namely the orthocentre, is none other than the point O which names the apolar pair of the auto-correspondence of $\alpha\beta\gamma$ *.

Footnote (1)

If now p be any point of the circle, and pq the chord perpendicular to $\beta\gamma$, then in the triangle αpq the auto-chord of α being perpendicular to $\beta\gamma$, is parallel to pq. Therefore by 5·5 the auto-chord of p is parallel to αq. But this is a well-known property of the pedal line of p. Hence

5·10 the auto-chord of p and the pedal line of p, in respect of the same triangle, are parallel to one another.

This parallelism justifies the name "pedal angle". The following direct expression, which is easily obtained for the pedal angle between two triangles, might have also been made the starting point of a treatment of pedo-parallelism;

the pedal angle between two inscribed triangles $\alpha\beta\gamma, abc$
is equal to $\frac{1}{2}\{a+b+c-\alpha-\beta-\gamma\}$, where $\alpha,\beta,\gamma,a,b,c$ also repre-
5·11 sent the angles which the central radii vectores to the
corresponding points, make with a fixed direction.

We may now set down the geometrical implications, which certain of our previous results possess in virtue of 5·9. Thus the ~~theo~~ theorems 3·17, 3·18 of (13) give the following:

the orthocentres of all inscribed triangles are definite
with the exception of triangles of the types $iiA, jjB, ij\lambda$;
the orthocentre of iiA is an indefinite point on iA, but
assumes a definite limiting position if iiA is approached
along any pencil Γ, this limiting position depending
5·12 only on the linear system containing Γ and the singular
pencil (iiA); the orthocentre of $ij\lambda$ is an indefinite point
at infinity, but has a definite limiting position for any
linear approach to $ij\lambda$, and the limiting position is the
same for all approaches in the same pedo-parallel system.

It is easy to shew that 3·19 reduces to the following:

if $\alpha\alpha'$ be a diameter of the circle, the orthocentre of the
5·13 inscribed point-triangle $\alpha\alpha\alpha$ is the reflection of α' in α.

We shall close this subsection with some invariant expressions relating to the triangle $\alpha\beta\gamma$ and the circular points ij. Writing

$$f(x)=(\alpha x)(\beta x)(\gamma x);\quad \phi(x)=(ix)(jx),$$

it is clear that the cubic $f'(x)$ which gives the pedo-parallel equilateral triangle of $\alpha\beta\gamma$ is the cubic apolar to f and ϕ. It is easy to shew that

5·14 $$f'(x)=(ij)^2 f(x)+3\,\phi(x)\{\phi(x),f(x)\}^2.$$

The pedo-perpendicular equilateral triangle of $\alpha\beta\gamma$ being the cubicovariant of $f'(x)$, is given by $f''(x)$, the Jacobian of f' and ϕ. ~~No~~ Since

$$\{\phi(\phi,f)^2,\phi\}^1 = \tfrac{1}{3}\phi\{(\phi,f)^2,\phi\}^1 = -\tfrac{1}{3}\phi(\phi^2,f)^3 \text{ by Gordon's series,}$$

we have the following expression for $f''(x)$:

5.15 $$f''(x) = (f',\phi)^1 = (ij)^2(f,\phi)^1 - \phi(\phi^2,f)^3.$$

The linear covariant $(\phi,f)^2$ represents a point p (which we shall call <u>the Euler point</u> of $\alpha\beta\gamma$), such that ijp is apolar to $\alpha\beta\gamma$. Now the condition of apolarity of two triads $a_1b_1c_1, a_2b_2c_2$ of a conic, can be shewn to be, that the conic inscribed to $a_1b_1c_1, a_2b_2c_2$ should ~~be~~ touch the polar lines of each of these triangles in regard to the conic. Hence the apolarity of $\alpha\beta\gamma$ and ijp implies that the inscribed parabola of $\alpha\beta\gamma$ with focus at p, touches the polar line of ijp in regard to the circle. But this polar line is seen to be the perpendicular-bisector of Sp . Thus the mid-point of Sp lies on the tangent at the vertex of the inscribed parabola, which is also the pedal line of p . Since this pedal line also bisects the join of p to the orthocentre O , it must be parallel to SO. Thus

5.16. The Euler point $(\phi,f)^2$ is the point whose pedal line in respect of $\alpha\beta\gamma$ is parallel to the Euler line of $\alpha\beta\gamma$ (namely the line SO); the point $(\phi^2,f)^3$ is the diametrically opposite point of $(\phi,f)^2$. *

Footnote(2)

(23) A POLAR CORRESPONDENCE RELATED TO THE AUTO-CORRESPONDENCE.

Let p_1p_2 be the diameter perpendicular to the auto-chord of p in respect of $\alpha\beta\gamma$. Then by 5.2 , p_1p_2 is the auto-chord of p in respect of the pedo-perpendicular equilateral triangle of $\alpha\beta\gamma$. On applying 5.8 to the triangle pp_1p_2 , it appears that the pedal lines of p_1, p_2

are parallel to pp_1, pp_2 respectively. Hence

5·17 If through each point p_1 we draw the chord p_1p parallel to the pedal line of p_1 in respect of $\alpha\beta\gamma$, the correspondence (p_1, p) is the (2,1) polar correspondence of the pedo-perpendicular equilateral triangle of $\alpha\beta\gamma$.

Let O' be an arbitrary point in the plane, r_1r_2 the auto-chord of p in respect of $\alpha\beta\gamma$, and q_1q_2 the chord through O' parallel to r_1r_2. We see immediately that the correspondence (p, q_1q_2) is an auto-correspondence. If $\alpha'\beta'\gamma'$ be its fixed points (so that O' is the orthocentre of $\alpha'\beta'\gamma'$), then the auto-chords of p in respect of $\alpha\beta\gamma, \alpha'\beta'\gamma'$ are by construction parallel, or $\alpha'\beta'\gamma'$ is pedo-parallel to $\alpha\beta\gamma$. Thus,

5·18 The lines through α', β', γ' parallel to their respective auto-chords in respect of $\alpha\beta\gamma$, are concurrent only if $\alpha'\beta'\gamma'$ is pedo-parallel to $\alpha\beta\gamma$; the point of concurrence is therefore the orthocentre of $\alpha'\beta'\gamma'$.

The triangles f', f'' of 5·14, 5·15, occur in connection with the study of the inscribed parabolas of $\alpha\beta\gamma$. It is easy to see that,

5·19 if p_1p_2, q_1q_2 be the diameters parallel and perpendicular to the axis of the inscribed parabola whose focus is p, the correspondences $(p, p_1p_2), (p, q_1q_2)$ are the respective polar correspondences of f'' and f'. The triangles f', f'' therefore give the foci of inscribed parabolas, whose axes ~~pa~~ touch the circle, and pass through its centre respectively.

We know from 5·12 that the orthocentre of the triangle ijp has a definite limiting position P for all approaches in the pedo-parallel system of $\alpha\beta\gamma$; it is clear from the above that P is the point at infinity on the directrix of the inscribed para-

bola

bola whose focus is p .

(24)PEDO-PARALLEL AND ORTHOCENTRIC PENCILS.

A pedo-parallel ~~system~~ pencil of inscribed triangles is a pencil any two members of which are pedo-parallel, that is, a pencil wholly contained in a pedo-parallel system.

Let O be any point of the plane, and let $\mathcal{S}, O$ ~~s~~stand also for the quadratics which represent parametrically the point-pairs on the circle which are named by $\mathcal{S}$, O respectively (so that $\mathcal{S} = (ix)(jx)$). Then from the definition of the two-quadratic representation, and from 5·9, it follows that the pencil $[\mathcal{S}, O]$ consists of triangles having their orthocentre at O . Thus the inscribed triangles with a given orthocentre O form a pencil, which we may call the orthocentric pencil of O .

It is~~s~~ clear from 4·2 that any pencil, among the four Jacobian points of which i and j are included, is an orthocentric pencil. Since the singular pencil $ij\lambda$ belongs to every pedo-parallel system, a pedo-parallel pencil must contain a member $ij\lambda$. Hence

5·20 an orthocentric pencil may also be defined as a pencil which contains two members of the form iiA, jjB , the common orthocentre being the intersection of iA, jB (4·3); a pedo-parallel pencil may be defined as one which has a member of the form $ij\lambda$.

Hence

5·21 the necessary and sufficient condition for pedo-parallelism of two inscribed triangles, is that the conic inscribed to them be a parabola; the locus of orthocentres of the pedo-parallel pencil determined by them is then the di- [handwritten above: apolar] rectrix of the parabola.

By supposing that the quadratic P_1 represents the circular points on the circle, the theorem 4·7 reduces to:

5·22 the apolar pencil of the orthocentric pencil of O is the orthocentric pencil of another point in the diameter through O.

By the same supposition the family of pencils studied in (19) becomes the family of orthocentric pencils of points on a diameter A_1A_2 (say). The involution I_1 then corresponds to the involution of pairs of points in A_1A_2 which are inverse in regard to the fundamental circle; by Footnote (3) Section 4 the involution I_3 corresponds to the involution of points on A_1A_2 equidistant from the centre. Further if the two points determining the null orthocentric pencils are at a distance $\mu\rho$ from ς, then by 4·9 and the same footnote $\mu^2=3$. Thus the fixed points of I_2 are at a distance $\sqrt{3}\rho$ from ς, a result which is also directly evident from 4·8. We have thus

5·23 the locus of points determining the null orthocentric pencils is a concentric circle of radius $\sqrt{3}\rho$; inverse points in respect of this circle determine apolar orthocentric pencils.

The result may be verified in a particular case. If α is a point on the circle, the orthocentric pencil of α is the pencil of right-angled triangles which have their right angle at α. The apolar pencil therefore contains the member $(\alpha x)^3$, and is further an orthocentric pencil (5·22). Now by 5·13 the orthocentre of $\alpha\alpha\alpha$ is the point β on $\varsigma\alpha$ produced, which is distant $3\,\varsigma\alpha$ from ς. So the orthocentric pencils of α, β are ~~orthocentric~~ apolar pencils. The points α, β being evidently inverse in the concentric circle of radius $\sqrt{3}\rho$, the theorem $5\cdot 23$ is verified.

(25) ORTHOCENTRIC LOCI.

A genral linear ∞^2-system $[M]$ of inscribed triangles consists of triangles apolar to a fixed triangle $\alpha\beta\gamma$. If O be the orthocentre of $\alpha\beta\gamma$, and O' the inverse of O in respect of the concentric circle of radius $\sqrt{3}\rho$, then the orthocentric pencil of O' being apolar to $\alpha\beta\gamma$ $(5\cdot 23)$, must be completely contained in $[M]$. It is clear that this is the only orthocentric pencil contained in $[M]$. Also, the system $[M]$ has in general only one member in common with the pencil of equilateral triangles. A very convenient way of specifying $[M]$ is by means of its equilateral member and its unique orthocentric pencil.

Let M_1 be the member of $[M]$, of the form $ij\lambda$, and M_2, M_3 the members of the form iiA, jjB. If iA, jB intersect in O, then the pencil (M_2, M_3) is the orthocentric pencil of O, and is the unique orthocentric pencil contained in $[M]$. Let M be the general member of $[M]$, and m its orthocentre. The correspondence between M, m is in general $(1,1)$, the exceptions being covered by $5\cdot 12$. Since $[M]$ is a general linear system, the approaches to M_1 contained in $[M]$ are not pedo-parallel, so that the orthocentre of M_1 is indefinite, and may be anywhere at infinity $(5\cdot 12)$. Similarly the orthocentres of M_2, M_3 are indefinite, and may be anywhere on Oi, Oj respectively. Hence the relation between M and m is a cremona quadratic transformation with the fundamental elements $M=M_1, M_2, M_3$; $m=O, j, i$.

Now in a cremona quadratic transformation between two planes, a linear locus in either is transformed into a quadratic locus passing through the three fundamental points in the other. Hence, 7 The locus of orthocentres of any pencil of $[M]$ is a conic through O_{ij}, that is, a circle (diametrically opposite points of this circle may be verified to be the orthocentres of pedo-perpendicular members of the pencil); the circle degenerates into a point-circle, when the pencil is

5·24 orthocentric, and into a line when it is a pedo-parallel pencil (5·21). The locus of the orthocentres of a conic-system of triangles containing three members of the type M_1, M_2, M_3 is a straight line.

It should be observed however, that if $[M]$ is a pedo-parallel system (or a system containing either of the singular pencils $ii A, jj B$), the correspondence between M, m is no longer a cremona, but a linear correspondence.

Suppose now that the circles C, C' are the orthocentric loci of two apolar pencils Γ, Γ'. Let $\alpha\beta\gamma$ be any member of Γ, and O its orthocentre, so that O is a point on C. Let O' be the point inverse to O in regard to the concentric circle of radius $\sqrt{3}\rho$. Then the apolar system of $\alpha\beta\gamma$ contains Γ' and also the orthocentric pencil of O'. Hence Γ' contains a member whose orthocentre is O', so that O' is a point on C'. It follows that C and C' are inverse circles in regard to the concentric circle of radius $\sqrt{3}\rho$. Thus,

> The orthocentric loci of two apolar pencils are two circles mutually inverse in the second fundamental circle
>
> 5·25 (by which we mean the concentric circle of radius $\sqrt{3}\rho$); therefore the orthocentric locus of a null pencil is a circle orthogonal to the second fundamental circle.

The latter part of this theorem may be verified for a particular case.The simplest null pencils are those of the type $pp\lambda$,where p is a fixed point.Now it is easy to see that the orthocentre of the inscribed line-triangle $pp\lambda$ is the reflection in p of the extremity of the chord through p perpendicular to $p\lambda$ .Thus the orthocentric locus of the null pencil $(pp\lambda)$ is the reflection of the fundamental circle in p .It is an easy verification to shew that the second fundamental circle of radius $\sqrt{3}\rho$ is orthogonal to the reflection of the fundamental circle in any of its points.

As a particular case of 5·25,we see that the orthocentric locus of a null pedo-parallel pencil must be a diameter A_1A_2.Now there are two remarkable harmonic point-pairs on a diameter A_1A_2, namely the pair of extremities A_1A_2,and the pair $\varsigma\infty$ .Hence there are four remarkable members of the pedo-parallel null pencil, namely those whose orthocentres are these four points.The members with their orthocentres at ς,∞ are the equilateral member and the member of the form $ij\lambda$,respectively;those which have their orthocentres at A_1,A_2 are the two right-angled members which have their right angles at A_1,A_2,and which (since they are pedo-parallel) have perpendicular diameters for their hypoteneuses. Hence

5·26. a null pedo-parallel pencil contains two harmonic pairs of remarkable members;the first pair consists of the two right-angled triangles of the pencil,the second of the equilateral triangle,and the triangle of the form $ij\lambda$. The former have their right angles at diametrically opposite points,and have perpendicular diameters for their hypoteneuses.

(26) OTHER SYSTEMS OF INSCRIBED TRIANGLES. THE ANTIPEDAL ANGLE.

Let every inscribed triangle $\alpha\beta\gamma$ be represented by the point in three-dimensional space whose homogenous coordinates are proportional to the coefficients of the binary cubic $(\alpha x)(\beta x)(\gamma x)$. The coalesced triangles of the type ttt will then correspond to the point T of a certain fundamental twisted cubic C; in particular, the two triads iii, jjj will correspond to two points I, J on C. The equilateral triangles will correspond to points on the chord IJ of C, and the singular pencil $ij\lambda$ to the axis IJ (that is, the intersection of the osculating planes at I, J to C). Pedo-parallel systems are represented by planes through the axis IJ, pedo-parallel pencils by lines meeting the axis IJ, and orthocentric pencils by lines meeting the tangents at I, J.

The system of inscribed triangles congruent to a given inscribed triangle $\alpha\beta\gamma$ may be called the congruent system of $\alpha\beta\gamma$; it is clear that it is represented by a twisted cubic C' passing through the point P which corresponds to $\alpha\beta\gamma$. It is easy to shew that C' passes through the points I, J of C, and has also the same osculating tetrahedron at IJ as C (so that C, C' have the same tangents and the same osculating planes at I, J). One and only one twisted cubic of the type C' passes through each point of space. It will be noticed also that C itself is a particular case of C'.

We define equilateral systems to be the linear ∞^2-systems which contain all equilateral triangles. Thus the equilateral systems correspond to the planes through the chord IJ of C, and are therefore, in a sense, correlative to the pedo-parallel systems. The triangle apolar to all members of an equilateral systems.

tem is clearly of the form $\iota j\lambda$. Hence, from the definition of the Euler point (22), it follows that

5·27 an equilateral system is the totality of triangles which have a given Euler point.

Just as the pedal angle between two triangles is the measure of separation between the pedo-parallel systems containing them, so there is a second angle, namely the antipedal angle, which is the measure of separation between the equilateral systems containing them. We define the antipedal angle between two triangles to be half the angle subtended at the centre by their Euler points.

It was shewn in 5·16 that the pedal line of the Euler point is parallel to the Euler line of the triangle. If we call the angle between the Euler lines of two inscribed triangles, their Eulerian angle, then from 5·1 it easily follows that the pedal, antipedal, and Eulerian angles are connected by the following relation:

5·28 (pedal angle)—(antipedal angle)=Eulerian angle.

Since the Eulerian angle between two co-orthocentric triangles is zero, we see, as a particular case, that

5·29 the pedal angle between two co-orthocentric triangles is also their antipedal angle.

The condition that the pencil determined by two triangles may contain an equilateral member, is that their antipedal angle be equal to zero (mod π).

§6. THE PEDAL AND THE SEMI-PEDAL FORMS.

(27) THE SEMI-PEDAL PENCIL AND THE PEDAL FORM.

If $\alpha\beta\gamma$ is an inscribed triangle, the feet of the perpendiculars on its sides from any point x on the circle are collinear; if the line of collinearity cuts the circle in y, the correspondence (x,y) is determined by a (3,2) form $\Pi(x,y)$, which we call <u>the pedal form</u> of $\alpha\beta\gamma$.

We shall prove the collinearity property by means of the idea of Derivate Forms, introduced in (5). It was shewn in $5{\cdot}17$, that if through each p_1 we draw the chord p_1p parallel to the pedal line of p_1 in regard to $\alpha\beta\gamma$, the correspondence (p_1,p) is the $(2,1)$ polar correspondence $\{f''(x)\}_y$, where $f''(x)$ is given by $5{\cdot}15$. Hence the $(3,2)$ correspondence (p_1,p_1p) is represented by the defective polar form $(xy)\{f''(x)\}_y$ $(1{\cdot}9)$. Thus the two values of y corresponding to any x in the $(3,2)$ correspondence $(xy)\{f''(x)\}_y$ are the extremities of a chord parallel to the pedal line of x. Hence the pedal form $\Pi(x,y)$ considered as a quadratic in y, can only differ by a multiple of $(iy)(jy)$ from $(xy)\{f''(x)\}_y$. Combining with this the fact (which we may be supposed to know from elementary geometry) that α,β,γ are finite fixed points of the pedal correspondence, it is clearly suggested that we should look for the pedal form $\Pi(x,y)$ among the members of the pencil:

$$6{\cdot}1 \qquad \Pi'(x,y) = \lambda(ij)^2(\alpha x)(\beta x)(\gamma x)(iy)(jy) - 4\mu(xy)\{f''(x)\}_y.$$

We shall indeed find that $\Pi'(x,y)$ becomes the pedal form $\Pi(x,y)$ for the values $\lambda=1, \mu=1$ of the parameters. Before we do this, it is well to notice some obvious properties of the pencil of forms $\Pi'(x,y)$, which we call <u>the semi-pedal forms</u> of $\alpha\beta\gamma$.

The semi-pedal pencil Π' belongs to the special type of pencil which contains a defective member; hence every semi-pedal form Π' has the same fixed points $\alpha\beta\gamma i j$. From the definition of the derivate form, we see also that the derivate of any semi-pedal form in respect of the partition $\alpha\beta\gamma/ij$, is a multiple of the polar form $\{f''(x)\}_y$. Of particular interest is the behaviour of a semipedal correspondence Π' at i, j. We see by actual substitution that $y = i, j$ correspond in any Π' to $x = (i,j);(j,i)$ respectively. Hence

(ij) is a closed and complete pair of any semi-pedal cor-
6.2 respondence Π' (including by anticipation, the pedal correspondence Π also); or briefly, any Π' is complete and closed at infinity.

Our method of further procedure is the investigation of the $(2,1)$ forms which are the derivates of the semi-pedal forms in respect of ~~the~~ a partition of the type $\alpha ij/\beta\gamma$. Denoting by Π_α' the derivate of Π' in regard to this partition, we see that Π_α' is the sum of two terms which are the respective derivates of the two terms in the expression 6.1 for $\Pi'(x,y)$. Now the derivate of $-4\mu(xy)\{f''(x)\}_y$ is simply $-4\mu\{f''(x)\}_y$; and the derivate of $(\alpha x)(\beta x)(\gamma x)(iy)(jy)$, being the derivate of a form of rank one in respect of a partition not its own, considered in 1.17, is the product of (αx) and the symmetric bilinear form $(\gamma x)(jy)(\beta i)+(\beta y)(ix)(\gamma j)$, which corresponds to the involut~~oric~~ion determined by the pairs $\beta\gamma, ij$. Thus

$$\Pi_\alpha'(x,y) = \lambda(ij)^2(\alpha x)\{(\gamma x)(jy)(\beta i)+(\beta y)(ix)(\gamma j)\} - 4\mu\{f''(x)\}_y \qquad 6.3.$$

Consider the first term in this expression. It is a $(2,1)$ correspondence in which i, j correspond to each other, that is to say, an auto-correspondence; its fixed points consist of α, and the fixed points of the involution $(\beta\gamma, ij)$,that is,the extremities of the diameter $\alpha_1\alpha_2$ perpendicular to $\beta\gamma$ .Thus the first term in $6\cdot3$ is the auto-form of the right-angled triangle $\alpha\alpha_1\alpha_2$;the second term is the polar form,and therefore the auto-form, of the equilateral triangle $f''(x)(5\cdot2)$.Thus the forms Π_α' for different values of λ,μ are the auto-forms of triangles of the pencil $\{(\alpha x)(\alpha_1 x)(\alpha_2 x), f''(x)\}$.Now by $5\cdot5$ the right-angled triangle $\alpha\alpha_1\alpha_2$ is pedo-perpendicular to $\alpha\beta\gamma$,and therefore pedo-parallel to the triangle $f''(x)$ which is the pedo-perpendicular equilateral triangle of $\alpha\beta\gamma$ .Thus the pencil $\{(\alpha x)(\alpha_1 x)(\alpha_2 x), f''(x)\}$ is a pedo-parallel pencil containing an equilateral member $f''(x)$,and therefore a null pedo-parallel pencil.Hence

$6\cdot4$ the pencil Π_α' (which is the derivate of the semi-pedal pencil Π' in regard to the partition $\alpha ij/\beta\gamma$) is the pencil of auto-forms of triangles of the null pedo-parallel pencil,whose orthocentric locus is the diameter through α.

We recall now the four remarkable members of a null pedo-parallel pencil,spoken of in $5\cdot26$ .Two of the remarkable members in the present case evidently correspond to the two terms in the expression $6\cdot3$ for Π_α' .A third remarkable member is the <u>other</u> right-angled triangle in the null pedo-parallel pencil of $6\cdot4$, which by $5\cdot26$ must have its right angle at the diametrically opposite point of α ,and the diameter parallel to $\beta\gamma$ for its hypoteneuse.Now the diametrically opposite point of α corresponds to the linear form $(ix)(j\alpha)+(jx)(i\alpha)$ and the involution of chords perpendicular to $\beta\gamma$,to the symmetric bilinear form $(ix)(iy)(j\beta)(j\gamma)+(jx)(jy)(i\beta)(i\gamma)$.Hence the auto-form of this second right-angled triangle is of the shape,

$$\{(ix)(j\alpha)+(jx)(i\alpha)\}\{(ix)(iy)(j\beta)(j\gamma)+(jx)(jy)(i\beta)(i\gamma)\}.$$

Accordingly, by 6·4, there must exist values of λ, μ which make $\Pi'_\alpha(x,y)$ equal to this auto-form. Using the value 5·15 for $f''(x)$, we can verify that these values are $\lambda=1, \mu=1$.* Supposing that $\Pi'(x,y)$ becomes $\Pi(x,y)$ for $\lambda=\mu=1$, we see then by symmetry that the derivates $\Pi_\alpha, \Pi_\beta, \Pi_\gamma$ of $\Pi(x,y)$ in regard to the respective partitions $\alpha ij/\beta\gamma, \beta ij/\gamma\alpha, \gamma ij/\alpha\beta$, are given by:

Footnote(1)

6·5
$$\begin{aligned}
\Pi_\alpha(x,y) &= \{(ix)(j\alpha)+(jx)(i\alpha)\}\{(ix)(iy)(j\beta)(j\gamma)+(jx)(jy)(i\beta)(i\gamma)\}\\
\Pi_\beta(x,y) &= \{(ix)(j\beta)+(jx)(i\beta)\}\{(ix)(iy)(j\gamma)(j\alpha)+(jx)(jy)(i\gamma)(i\alpha)\}\\
\Pi_\gamma(x,y) &= \text{similar expression.}
\end{aligned}$$

To shew that $\Pi(x,y)$ is the pedal form, we have to shew that it arises from the collinearity of the feet of the perpendiculars. Let x be a given point, y', y'', y''' the values of y which correspond to x in $\Pi_\alpha, \Pi_\beta, \Pi_\gamma$ respectively; also let y_1, y_2 correspond to x in $\Pi(x,y)$. Since Π_α is the derivate of Π we have a relation of the form

$$\Pi(x,y) = (ij)^2(\alpha x)(ix)(jx)(\beta y)(\gamma y) + (xy)\,\Pi_\alpha(x,y).$$

This relation, interpreted as a linear relation between three binary quadratics in y, clearly implies that the three chords y_1y_2, $\beta\gamma, xy'$ are concurrent. Similarly $(y_1y_2, \gamma\alpha, xy'')$, $(y_1y_2, \alpha\beta, xy''')$ are concurrent. But, as has been seen already, xy', xy'', xy''' are perpendicular to $\beta\gamma, \gamma\alpha, \alpha\beta$ respectively. Therefore the feet of the perpendiculars from x on the sides of $\alpha\beta\gamma$ are collinear, the line of collinearity being y_1y_2, so that $\Pi(x,y)$ is indeed the pedal form.

The different shapes obtained for the pedal form may be set down here for future reference:

6·6
$$\Pi(x,y) = (ij)^2(\alpha x)(\beta x)(\gamma x)(iy)(jy) - 4(xy)\{f''(x)\}_y.$$

6·7
$$\begin{aligned}
\Pi(x,y) &= (ij)^2(\alpha x)(ix)(jx)(\beta y)(\gamma y) + (xy)\,\Pi_\alpha(x,y),\\
&= (ij)^2(\beta x)(ix)(jx)(\gamma y)(\alpha y) + (xy)\,\Pi_\beta(x,y),\\
&= (ij)^2(\gamma x)(ix)(jx)(\alpha y)(\beta y) + (xy)\,\Pi_\gamma(x,y),
\end{aligned}$$

where $f''(x)$ is given by 5·15, and $\Pi_\alpha, \Pi_\beta, \Pi_\gamma$ by 6·5.

It was seen in 6·2 that the pedal correspondence has a particular type of closure at infinity; namely $y=i,j$ correspond in it to $x=(ijj),(jii)$ respectively. This type of closure does not characterise exclusively, the pedal or even the semi-pedal, correspondences; it belongs in fact to the family of $\infty^{11-6}=\infty^{5}$ "pedo-similar" correspondences studied in the next section. The exclusive property of the pedal correspondences Π among these latter, is that if αa be the diameter through α, $x=a$ corresponds in Π to $y=\beta,\gamma$.

(28) THE PEDAL FORM AND THE AUTO-FORM.

From 3·6, the $(1,2)$ auto-form of $\alpha\beta\gamma$, namely the form which has ij for auto-pair, and $\alpha\beta\gamma$ for its fixed points, is shewn to be of the shape:

$$\sum(\alpha i)(\alpha j)(\beta\gamma)(\beta\gamma,ij)(\alpha x)(\beta y)(\gamma y),\ \text{where}\ (\beta\gamma,ij)=(\beta i)(\gamma j)+(\beta j)(\gamma i).$$

We shall find it convenient to denote by $A(x,y)$ a certain multiple of this auto-form; namely

$$6{\cdot}8\quad A(x,y)=\frac{2}{(\beta\gamma)(\gamma\alpha)(\alpha\beta)}\sum(\alpha i)(\alpha j)(\beta\gamma)(\beta\gamma,ij)(\alpha x)(\beta y)(\gamma y).$$

We notice that $A(x,x)=2(ij)^2(\alpha x)(\beta x)(\gamma x)$.

Putting now $x=i$ in 6·7, we have $\Pi(i,y)=(i\alpha)(i\beta)(i\gamma)(ij)^2(iy)(jy)$. Hence $\Pi(x,y)+(ij)^2(\alpha x)(\beta x)(\gamma x)(iy)(jy)$ vanishes when $x=i$ or j. Therefore there is an identity of the form

$$6{\cdot}9\quad \Pi(x,y)+(ij)^2(\alpha x)(\beta x)(\gamma x)(iy)(jy)=k(ix)(jx)A(x,y),$$

where k is numerical, and from the closure property of the pedal form, $A(x,y)$ is simply the auto-form 6·8 of $\alpha\beta\gamma$. Putting $y=x$, we see that k must be equal to 1.

This relation 6·9 may be reached in a more instructive way. It was seen in 6·4 that the derivate pencil Π'_α of the semi-pedal pencil Π' consisted of the auto-forms of members of a null pedo-parallel pencil, and three semi-pedal forms (namely, the pedal form and the two terms in the expression 6·1) were found, which corresponded to three of the four remarkable members of the null pedo-parallel pencil. The fourth remarkable member is that of the shape $(ix)(jx)(\lambda y)$, the auto-form of which is $(ix)(jx)(\lambda y)$. Now Π' can have the derivate $(ix)(jx)(\lambda y)=\Pi'_\alpha$, only if Π' has the factor $(ix)(jx)$. Thus there <u>is</u> a semi-pedal form $(ix)(jx)A(x,y)$, having the factor $(ix)(jx)$; ~~an~~ and from the closure property of Π' , $A(x,y)$ must be an autoform. The harmonic property of the four remarkable members of the null pedo-parallel pencil (5·26) easily shews that the values of λ,μ in 6·1, which give this semi-pedal form are $\mu:\lambda=\cancel{2:1}\ 1:2$. We may now verify that

6·9 $$2(ij)^2(\alpha x)(\beta x)(\gamma x)(iy)(jy)-4(xy)\{f''(x)\}_y=\Pi(x,y)+(ij)^2(\alpha x)(\beta x)(\gamma x)(iy)(jy)=(ix)(jx)A(x,y),$$
where $A(x,y)$ is given by 6·8.

We have in (27), (28) reached a fundamental theorem of our subject, which we may set forth as follows:

6·10 The semi-pedal pencil of $\alpha\beta\gamma$ contains, like the null pedo-parallel pencil, two harmonic pairs of remarkable members; one of these pairs is composed of the pedal form $\Pi(x,y)$ and the form of rank one $(\alpha x)(\beta x)(\gamma x)(iy)(jy)$, the other, of the defective polar form of the pedo-perpendicular equilateral triangle of $\alpha\beta\gamma$, and a form of rank two, namely the product of $(ix)(jx)$ and the (1,2) auto-form of $\alpha\beta\gamma$.

The geometrical meaning of the harmonic property of these four members is obviously, that the pedal line of x bisects the join of x with the orthocentre of $\alpha\beta\gamma$.

(29) THE PEDAL AND THE SEMI-PEDAL ENVELOPES.

From (8) it follows that the pedal envelope (namely, the envelope of the pedal lines of $\alpha\beta\gamma$) is a rational envelope of the third class, with the line at infinity for its bitangent. From the closure property, the points of contact of the bitangent are i,j ;

hence the envelope is a three-cusped hypocycloid. It may also be seen that the pedal line of a point near i is the tangent to the envelope at a point near j. All this is equally true of any ~~semi~~ semi-pedal correspondence.

The concurrency system of the pedal correspondence of $\alpha\beta\gamma$ is the system of triads of points, whose pedal lines are concurrent. It is clearly a linear ∞^2-system. The pedal lines of i,j being each the line at infinity, it follows that the concurrency system contains the singular pencil $ij\lambda$, and is therefore a pedo-parallel system. Now the chords which correspond to α,β,γ in the pedal, or in any semi-pedal, correspondence of $\alpha\beta\gamma$, are simply the perpendiculars of the triangle $\alpha\beta\gamma$, which are concurrent at the orthocentre. Thus $\alpha\beta\gamma$ belongs to the concurrency system of every semi-pedal form defined by it. We have therefore a theorem which enlarges still further the content of "pedo-parallelism":

the concurrency system of the pedal or any semi-pedal

6·11. Footnote(2) form of $\alpha\beta\gamma$ is the pedo-parallel system of $\alpha\beta\gamma$*.

From 6·1, 6·9 a semi-pedal form $\Pi'(x,y)$ can be written

$$6{\cdot}12 \quad \Pi'(x,y) = \lambda(ix)(jx)A(x,y) - k\lambda(ij)^2(\alpha x)(\beta x)(\gamma x)(iy)(jy).$$

We shall term k the measure of Π'; we shall also call the chord y_1y_2 which corresponds to any x in Π', the k-pedal line of x in respect of $\alpha\beta\gamma$, and its envelope the k-pedal envelope. By comparing 6·9, 6·12, it follows that the k-pedal line of x is the line parallel to, and k times as far from the orthocentre as, the pedal line of x. Hence

the k-pedal envelope results on uniform dilatation of the

6·13. pedal envelope from the orthocentre, the constant of dilatation being k.

In particular for $k=\infty$,we have the form $(\alpha x)(\beta x)(\gamma x)(iy)(jy)$, whose envelope lies wholly at infinity. For $k=0$,we have the form $(ix)(jx)A(x,y)$ whose envelope is the orthocentre and the circular points. For $k=2$ we have the envelope E of the defective polar form, defined by the pedo-perpendicular equilateral triangle $\alpha''\beta''\gamma''$, of $\alpha\beta\gamma$. The three tangents to E from any point x on the circle are the chords joining x to its second polar point x' and to its first polar points $x_1 x_2$, in respect of $\alpha''\beta''\gamma''$. Hence <u>one</u> of the three tangents to E from any point on the circle is algebraically distinct from the other two. It can be shewn from general reasoning that this necessitates that E should have triple contact with the circle. As a matter of fact we see directly that it has triple contact at $\alpha''\beta''\gamma''$. From 6·11 it follows that the cuspidal ~~tan-~~ tangents of E are the diameters $\alpha'\alpha''$, $\beta'\beta''$, $\gamma'\gamma''$, where $\alpha'\beta'\gamma'$ is the pedo-parallel equilateral triangle of $\alpha\beta\gamma$.

The envelope E is also the locus of orthocentres of

6·14 line-triangles pedo-parallel to $\alpha\beta\gamma$.

For, let $x_1 x_2$ be a diameter, and let $x x x_1$ be a line-triangle pedo-parallel to $\alpha\beta\gamma$. The pedal line of x is then perpendicular to $x x_1$ (5·8), and therefore parallel to $x x_2$. Hence by 5·17, $x x_2$ touches E ; from 5·1 it may be shewn that the point of contact is the reflection of x_2 in x . But the reflection of x_2 in x is the orthocentre of the line-triangle $x x x_1$. Hence the theorem.

As an immediate application, the cusps of E being the orthocentres of the point-triangles $\alpha\alpha'\alpha'$, $\beta\beta'\beta'$, $\gamma\gamma'\gamma'$, lie on $s\alpha'$, $s\beta'$, $s\gamma'$, at distances 3ρ from s (5·13) . By 6·13, the pedal envelope is obtained on contracting E from O , to half its linear dimensions. Hence:

the nine-point circle of $\alpha\beta\gamma$ is the in-circle of its pedal envelope, the points of contact being the midpoints of $o\alpha'', o\beta'', o\gamma''$. The cusps of the pedal envelope are situated on radii vectores from the nine-point centre N parallel to $\delta\alpha', \delta\beta', \delta\gamma'$, at a distance $\frac{3}{2}\rho$ from it. The pedal envelope when shifted through a distance $\frac{1}{2}O\delta$ parallel to $O\delta$ becomes

6·15 the locus of nine-point centres of line-triangles pedo-parallel to $\alpha\beta\gamma$. The point of contact of the pedal line of x with its envelope, lies therefore on the join of O with the orthocentre of the line-triangle xxx' pedo-parallel to $\alpha\beta\gamma$.

The algebraic distinctness of one of the tangents to E from any point on the circle, from the others, reappears after contraction, as the distinctness of one of the three pedal lines through any point on the nine-point circle, from the other two; the distinctness arises from the proposition, that the pedal lines of the extremities of a diameter meet at right angles on the nine-point circle, the third pedal line through their intersection being consequently, of an algebraically different character from them.

From 6·13, 6·15 we now easily obtain:

the centre of the k-pedal envelope is in $O\delta$ at a distance $\frac{k}{2}.O\delta$ from O. If the pedal angle between two co-orthocentric triangles is θ, the k-pedal envelope of one is

6·16. obtained by turning that of the other through an angle $2\theta/3$ about its centre. Thus for a given k, the k-pedal envelope has always the same dimensions, but takes its ∞^3 different positions in the plane for the ∞^3 inscribed triangles $\alpha\beta\gamma$. The position of the centre of the envelope indicates the particular orthocentric pencil to which $\alpha\beta\gamma$ belongs, while the disposition of the envelope about its centre indicates the pedo-parallel system to which $\alpha\beta\gamma$ belongs.

§7. THE PEDO-similar forms.

(30) DOUBLE SPECIFICATION BY MEASURE-DEVIATION.

The most general $(3,2)$ forms which have closure at infinity in the manner of the pedal form (that is, in which $y=i,j$ correspond to $x=ijj, jji$ respectively), we call "the pedo-similar forms". The pedo-similar forms involve $12-6=6$ parameters, and include as special cases the semi-pedal and the pedal forms. Thus, there are ∞^3 pedal correspondences, ∞^4 semi-pedal correspondences, and ∞^5 pedo-similar correspondences. Since the angle-property 5·1 is simply a consequence of the behaviour at $i.j$, we see that

7·1 the angle-property 5·1 holds for the general pedo-similar form.

This theorem suggests a specification of any pedo-similar form $\pi''(x,y)$ by means of a basal triangle $\alpha\beta\gamma$ (which, be it noted, is not in general the fixed triangle of π''), a parent semi-pedal form $\pi'(x,y)$ of measure k, defined by $\alpha\beta\gamma$, and a "deviation" θ. Let x be any point on the circle, x' the point such that $x\hat{s}x'$ is equal to the angle of deviation θ. If y_1y_2 be the k-pedal line of x' in respect of $\alpha\beta\gamma$, then the correspondence (x, y_1y_2) is clearly a pedo-similar correspondence; it is defined in respect of the basal triangle $\alpha\beta\gamma$ and is said to be of measure-deviation (k,θ). We shall also call y_1y_2 the (k,θ)-pedal line of x in regard to $\alpha\beta\gamma$. Since there are two parameters k,θ in addition to the three in the choice of the basal triangle, it follows that all pedo-similar correspondences will be covered by this specification. Also, from our construction the set of (k,θ)-pedal lines of $\alpha\beta\gamma$ is composed simply of the k-pedal lines of $\alpha\beta\gamma$. Hence the envelope associated with the pedo-similar form, with the

7·2 the basal triangle $\alpha\beta\gamma$, and the measure-deviation (k,θ), is

simply the k-pedal envelope of $\alpha\beta\gamma$.

Hence, as in 6·11, the concurrency system of the pedo-similar form is a pedo-parallel system; if we now rotate $\alpha\beta\gamma$ through an angle $-\theta$ about the centre, into the position $\alpha'\beta'\gamma'$, then the (k,θ)-pedal lines of α',β',γ' are clearly the k-pedal lines of α,β,γ, which, being the perpendiculars of the triangle $\alpha\beta\gamma$, are concurrent. Thus $\alpha'\beta'\gamma'$ belongs to the concurrency system. Since the pedal angle between $\alpha\beta\gamma, \alpha'\beta'\gamma'$ is $-\frac{3\theta}{2}$,

the concurrency system of a pedo-similar form of devia-

7·3 tion θ is the pedo-parallel system which makes the pedal

angle $-\frac{3\theta}{2}$ with its basal triangle.

We have now to examine in how many ways a given pedo-similar correspondence can be specified in this manner by a basal triangle Δ, a measure k, and a deviation θ. We shall find there are precisely two specifications $(\Delta_1,k_1,\theta_1),(\Delta_2,k_2,\theta_2)$ where $k_2=-k_1$, $\theta_2=\theta_1+\pi$; the two possible basal triangles Δ_1,Δ_2 are what we are going to call <u>associated triangles</u>.

The theory of associated triangles depends on the following important property of the null pedo-parallel pencil:

If $\alpha\beta\gamma$ is an inscribed triangle, there are precisely three

points a,b,c, which have the property that their k-pedal

7·4 lines in respect of $\alpha\beta\gamma$ pass through their diametrically

opposite points a',b',c'. For different values of k, the trian-

gle abc belongs to the null pedo-parallel pencil which con-

tains $\alpha\beta\gamma$. Conversely, any two members $\alpha\beta\gamma, abc$ of a null pedo-

parallel pencil are such that there is a number k, such that the κ-pedal lines of a,b,c in regard to $\alpha\beta\gamma$ pass through their diametrically opposite points.

That there are exactly three points a,b,c is clear from the fact that a',b',c' are the finite fixed points of the (κ,π)-pedo-similar correspondence with the basal triangle $\alpha\beta\gamma$. To prove that the pencil $\{\alpha\beta\gamma, abc\}$ is a null pedo-parallel pencil, let O be the orthocentre of $\alpha\beta\gamma$, and let our fundamental circle be called C. Let x be any point on C, and let Ox ~~Sx~~ be divided at ξ in the ratio $\kappa:(2-\kappa)$. The locus of ξ is then a circle C'; O is clearly one of the two centres of similitude of C, C', let O_1 be the other, so that if S' be the centre of C', O and O_1 divides SS' in the ratios $-\frac{\kappa}{2}, \frac{\kappa}{2}$. From the property of the centre of similitude, it then follows that the join of ξ to the diametrically opposite point of x passes through O_1 for all positions of x. But the join of ξ and the diametrically opposite point of x is the κ-pedal line of x, when x is at $a, b,$ or c. Thus the κ-pedal lines of a,b,c are concurrent at O_1, and therefore by 6·11, abc is pedo-parallel to $\alpha\beta\gamma$.

It was seen in (25) that the members of the pedo-parallel system of $\alpha\beta\gamma$ are in linear correspondence $\mathcal{A}$ with their orthocentres. It is also easy to see that these members are also in linear correspondence $\mathcal{A}_\kappa$ with the point of concurrence of the κ-pedal lines of their vertices in regard to $\alpha\beta\gamma$. It is immediately verified that ~~the~~ any member~~s~~ $ij\lambda$ ~~ijj, jii~~ corresponds ~~respectively~~ to ~~ij~~ the point at infinity on the pedal line of λ, in $\mathcal{A}$ as well as in $\mathcal{A}_\kappa$. From this it easily follows that the point of concurrence of the κ-pedal lines of a,b,c is obtained from the orthocentre of abc by a uniform radial dilatation from the orthocentre of $\alpha\beta\gamma$. Thus we have the theorem

the join of the orthocentres of two pedo-parallel trian-

gles $\alpha\beta\gamma, abc$ contains the points of concurrence of the
7.5 k-pedal lines of the vertices of either in regard to the other.

Hence the orthocentre of abc lies in the line OO_1, that is, in the diameter through O. The orthocentric locus of the pencil $\{\alpha\beta\gamma, abc\}$, which must be a line (5.24), is thus shewn to be a <u>diameter</u>, so that the pencil $\{\alpha\beta\gamma, abc\}$ is a null pedo-parallel pencil (25).

We shall say that <u>the triangle $a'b'c'$</u> (where a', b', c' are the diametrically opposite points of a, b, c) is <u>the triangle associated with $\alpha\beta\gamma$ in respect of the modulus k</u>. Now the pedal angle as well as the antipedal angle, between two diametrically opposite triangles abc, $a'b'c'$ is $\pi/2$; that is, abc and $a'b'c'$ are pedo-perpendicular, as well as, anti-pedo-perpendicular. But, since $\alpha\beta\gamma, abc$ belong to a null pedo-parallel pencil, their pedal, as well as their anti-pedal angle vanishes. Hence

the triangles associated mod k with $\alpha\beta\gamma$ are, for different values of k, the triangles diametrically opposite to the
7.6 members of the null pedo-parallel pencil containing $\alpha\beta\gamma$;
thus, associated triangles are pedo-perpendicular, as well as anti-pedo-perpendicular.

This indicates that the relation between associated triangles is a symmetrical one. More precisely, we have the result:

if $a'b'c'$ is associated modulus k with $\alpha\beta\gamma$, then $\alpha\beta\gamma$ is asso-
7.7 ciated modulus $-k$ with $a'b'c'$.

For, it is clear from the symmetry of 7.6 that $a'b'c'$ must be associated in respect of some modulus k' with $\alpha\beta\gamma$. Here k' must be a rational function $R(k)$ of k, such that $R\{R(k)\}=k$ identically. Hence R must be a homographic involution. Now the 0-pedal line is the auto-chord, and therefore the triangle associated modulus 0 with $\alpha\beta\gamma$ is the co-orthocentric pedo-perpendicular triangle of $\alpha\beta\gamma$. Again the 2-pedal line of x in regard to $\alpha\beta\gamma$ is the join of x with its second polar point in respect of its pedo-perpendicular equilateral triangle $\alpha''\beta''\gamma''$ (5·17, 6·10). It follows that the triangle associated mod 2 with $\alpha\beta\gamma$ is $\alpha''\beta''\gamma''$. Now the pedal line of α in regard to $\alpha''\beta''\gamma''$ bisects $S\alpha$, and is parallel to $\beta\gamma$ (5·8). Hence the k'-pedal line of α will pass through its diametrically opposite point, if $k'=-2$. Thus $R(0)=0$, and $R(2)=-2$. Hence $R(k)\equiv -k$.

We may now state the fundamental property of associated triangles:

7·8 If x, x' be variable points on the circle, such that $x\hat{S}x'=\theta$, the k-pedal line of x in regard to Δ_1, can be, for all positions of x, the k'-pedal line of x' in regard to a second inscribed triangle Δ_2, only if $\theta=\pi$; then k' must be equal to $-k$, and Δ_2 is the triangle associated mod k with Δ_1.

We easily see that there are only three positions x_1, x_2, x_3 of x, whose k-pedal lines pass through the corresponding positions x_1', x_2', x_3' of x'. Hence if a second triangle Δ_2 with the required property exists at all, it must be $x_1'x_2'x_3'$. Now the k'-pedal lines of x_1', x_2', x_3' in regard to $x_1'x_2'x_3'$ are the perpendiculars of the triangle $x_1'x_2'x_3'$, and are concurrent; hence the k-pedals of x_1, x_2, x_3 in regard to $\cancel{\alpha\beta\gamma}$ Δ_1 are concurrent, so that $x_1x_2x_3$ is pedo-parallel to $\cancel{\alpha\beta\gamma}$ Δ_1. This involves that the k-pedal line of x_1 should be ~~parallel~~ perpendicular to x_2x_3 (5·8), so that the triangles $x_1x_2x_3, x_1'x_2'x_3'$ have their corresponding sides parallel. This can not be unless $\theta=\pi$.

If $\theta=\pi$, then from our definition $\Delta_2=x_1'x_2'x_3'$ is the triangle associated mod k with Δ_1, and by 7·7, Δ_1 is associated mod $-k$ with Δ_2.

Now supposing that x' is the diametrically opposite point of x, if $(x \to y)$ is the k-pedal correspondence of Δ_1, then the correspondence $(x' \to y)$ must be a semi-pedal correspondence of Δ_2 (since the chords corresponding to x_1', x_2', x_3' are perpendicular to their opposite sides). Suppose then, that it is the k'-pedal correspondence of Δ_2. From the reciprocity of the triangles Δ_1, Δ_2 we can shew, as in 7·7, that $k' = -k$. We might also shew the same thing directly by considering the k-pedal line of either extremity of the diameter containing the orthocentres of Δ_1, Δ_2. Hence it follows that

7·9 the (k, θ_1)-pedo-similar correspondence of the basal triangle Δ_1 can be expressed in only one other way as the (k', θ_2) pedo-similar correspondence of Δ_2; $k' = -k$, $\theta_2 = \theta_1 + \pi$, and Δ_2 is the triangle associated mod k with Δ_1.

It is seen from this that the deviation θ of a given pedo-similar correspondence is determinate to the modulus 2π, when the corresponding basal triangle is given, but determinate only to the modulus π when it is not given.

(31) THE CO-BASAL FAMILY OF PEDO-SIMILAR FORMS.

If x, x' are points on the circle such that $x \hat{S} x' = \theta$, then since $(ij)x = (ix)j - (jx)i$, we could take $x' = e^{\sqrt{-1}\frac{\theta}{2}}(ix)j - e^{-\sqrt{-1}\frac{\theta}{2}}(jx)i = (ij)x + \sqrt{-1}\tan\frac{\theta}{2}((ix)j + (jx)i)$, discarding the scalar factor $\cos\frac{\theta}{2}$. Hence the (k, θ)-pedo-similar form defined by the basal triangle $\alpha\beta\gamma$ can be taken to be

$$7{\cdot}10 \quad \pi''(x, y) = \pi'(x', y) = \pi(x', y) - (k-1)(ij)^2(\alpha x')(\beta x')(\gamma x')(iy)(jy),$$

where $x' = (ij)x + \sqrt{-1}\tan\frac{\theta}{2}(\cancel{(ij)x+}(ix)j + (jx)i)$.

We notice that π'' involves the parameter k linearly, and the parameter $\tan\frac{\theta}{2}$ to the third degree. Hence

the fixed triangles of co-basal pedo-similar forms π'', of given deviation θ, belong to a pencil F_θ; the fixed tri-

~~angles~~ angles of cobasal pedo-similar forms of given measure κ constitute a cubic family G_κ. Since the semi-pedal ~~pedo-similar~~ form

7·11 of ~~given~~ measure κ has $\alpha\beta\gamma$ for its fixed triangle, it follows that G_κ contains the basal triangle for all values of κ.

The system G_∞ is the set of fixed triangles of forms of the shape $(\alpha x')(\beta x')(\gamma x')(iy)(jy)$, that is, it is the congruent system of $\alpha\beta\gamma$ studied in (2·6). There are two values of κ for which G_κ apparently becomes a pencil. It is clear first of all, that the pedo-similar forms of zero measure are, from the definition, simply the products of $(ix)(jx)$ with auto-forms of triangles co-orthocentric with $\alpha\beta\gamma$. To be more precise, we see from the angle-property 5·1 that the fixed triangle of the $(0,\theta)$ pedo-similar form is that co-orthocentric triangle which makes the pedal angle $-\frac{\theta}{2}$ with the basal triangle. Thus:

G_0 is apparently the orthocentric pencil of the basal

7·12 triangle $\alpha\beta\gamma$. The fixed triangle of the $(0,\theta)$ pedo-similar form is that co-orthocentric triangle which makes the pedal angle $-\frac{\theta}{2}$ with $\alpha\beta\gamma$.

Again the semi-pedal form of measure 2 is the defective polar form; consider the pedo-similar form of measure 2 and deviation θ. The fixed points of the form are clearly the points ϵ such that the pedal of ϵ' is parallel to $\epsilon\epsilon'$, where $\epsilon \hat{S} \epsilon' = \theta$, from which it may be easily shewn that the fixed triangle is an equilateral triangle which makes the pedal angle $\frac{\pi}{2}-\theta$ with $\alpha\beta\gamma$. Hence:

The system G_2 is apparently the pencil of equilateral triangles; the fixed triangle of the $(2,\theta)$ pedo-similar form is

7·13 that equilateral triangle which makes the pedal angle $\frac{\pi}{2}-\theta$ with the basal triangle.

Any pencil F_θ has a member in common with each G_k; in particular, from 7·12, 7·13, we see that F_θ is the pencil determined by the two triangles $\Delta_{-\frac{\theta}{2}}, E_{\frac{\pi}{2}-\theta}$, which are respectively the co-orthocentric and equilateral triangles making the pedal angles indicated in the suffix, with $\alpha\beta\gamma$. Now since F_θ contains an equilateral triangle, its members make a constant anti-pedal angle with the basal triangle (26). By 5·29 we see that this constant anti-pedal angle is simply the pedal angle between the co-orthocentric triangles $\Delta_{-\frac{\theta}{2}}$ and $\alpha\beta\gamma$. Hence we have the theorem:

7·14 the anti-pedal angle between the basal triangle, and the fixed triangle of a pedo-similar form of deviation θ, is $-\frac{\theta}{2}$.

To obtain the orthocentric locus of F_θ, let O be the orthocentre of $\alpha\beta\gamma$, O' the orthocentre of the triangle $\alpha'\beta'\gamma'$ obtained by turning $\alpha\beta\gamma$ through an angle $-\theta$ about S. Then $SO = SO'$, and $OSO' = -\theta$. By 7·12, 7·13, F_θ contains an equilateral triangle and a co-orthocentric triangle of $\alpha\beta\gamma$, so that its orthocentric locus is a circle (5·24), passing through SO. Also the (∞,θ) pedo-similar form being $(\alpha x')(\beta x')(\gamma x')(ij)(jj)$, its fixed triangle is $\alpha'\beta'\gamma'$. Therefore

7·15 the orthocentric locus of the pencil F_θ is the circle SOO'.

The pedal angle is determinate to the modulus π, while, since the basal triangle is known, the deviation is determinate to the modulus 2π. Hence, if a member of G_0 is given as the fixed triangle, then θ is determined modulo 2π by 7·12, and therefore the pedo-similar form and the corresponding pencil F_θ are determined uniquely. But if a member of G_2 is given as the fixed triangle, then 7·13 determines θ only to the modulus π, and therefore there are two pencils F_θ of the form $F_{\theta_1}, F_{\theta_1+\pi}$. Hence

In general the pencils F_θ establish a (1,1) correspondence between any two systems G_k, in particular, between G_0 and G_k. The case $k=2$ is however exceptional; the pencils F_θ establish a (1,2) correspondence between G_2 and any G_k, in

7·16 particular between G_2 and G_0. Hence the totality of fixed triangles of a co-basal family of pedo-similar forms constitute a ruled-cubic system.

These results can all be directly verified by a geometrical representation of the type indicated in (26). For this purpose we write the (k,θ) form in the shape 7·10, namely

$$\pi''(x,y)=\pi'(x',y)=\pi(x',y)-(k-1)(ij)^2(\alpha x')(\beta x')(\gamma x')(iy)(jy),$$

where $x'=\lambda(jx)i+\mu(ix)j$, so that $\lambda+e^{\sqrt{-1}\,\theta}\mu=0$.

Using 6·7 we find that the fixed cubic $\pi''(x,x)=\pi'(x',x)$ reduces to:

$$7{\cdot}17 \quad x_0(ix)^3(\alpha j)(\beta j)(\gamma j)+x_1(ix)^2(jx)\Sigma(\alpha j)(\beta j)(\gamma i)+x_2(ix)(jx)^2\Sigma(\alpha i)(\beta i)(\gamma j)+x_3(jx)^3(\alpha i)(\beta i)(\gamma i),$$

where

$$7{\cdot}18 \quad \begin{aligned} x_0&=-\mu^2(2\lambda+k\mu)\\ x_1&=(2-k)\lambda\mu^2\\ x_2&=(2-k)\lambda^2\mu\\ x_3&=-\lambda^2(2\mu+k\lambda).\end{aligned}$$

If we represent the cubic 7·17 by the point $(x_0x_1x_2x_3)$ of three-space, the tetrahedron of reference would clearly be the osculating tetrahedron at I,J to the fundamental cubic C of (26), namely the tetrahedron whose edges are the chord IJ, the axis IJ, and the tangents at I,J to C. The equations 7·18 are then the parametric equations of the cubic scroll:

$$7{\cdot}19 \quad (x_0+x_1)x_2^2=(x_2+x_3)x_1^2.$$

Supposing λ,μ to be fixed, the equations 7·18 give the generators F_θ of this scroll. Supposing k to be fixed, the same equations are the parametric equations of a twisted cubic G_k. We easily verify that G_k passes through the point corresponding to the basal triangle, namely the point $(1,-1,1,-1)$, and that it passes through the points I, J (which are the points x_0, x_3 of reference), and has the same tangents at IJ as the fundamental cubic C (namely x_0x_1, x_3x_2). Thus the cubics G_k pass through five points, so that any two of them constitute the complete intersection of the cubic scroll with a quadric. One particular G_k, namely G_∞ represents the congruent system of the basal triangle, and therefore has not only the same tangents, but also the same osculating planes at I, J as C.

For $k=2$, G_k becomes the chord IJ (hitherto called G_2 (7.13)) counted twice, and the line F_θ; for $k=0$, G_k becomes the line through $(1,-1,1,-1)$ meeting the tangents x_0x_1, x_3x_2 at IJ (hitherto called G_0), and the tangents at I, J. Thus G_2 (that is, the chord IJ) is the double line of the scroll, and as already proved, the generators F_θ establish a (1,2) correspondence between G_2 and G_0.

(32) THE NET OF PEDO-SIMILAR FORMS WITH A GIVEN FIXED TRIANGLE.

Since the pedo-similar correspondences with a given fixed triangle $\alpha\beta\gamma$ constitute a linear ∞^2-system, it follows that the totality of defective pedo-similar forms must constitute a pencil contained in every such system (4). But we know that the defective polar forms of equilateral triangles are semi-pedal forms, and do constitute a pencil. We conclude therefore that the only defective pedo-similar forms are the defective polar forms of equilateral triangles. Since $(\alpha x)(\beta x)(\gamma x)(iy)(jy)$ is a particular semi-pedal form with the fixed triangle $\alpha\beta\gamma$, it follows that the general pedo-similar form with the same fixed triangle differs from $(\alpha x)(\beta x)(\gamma x)(iy)(jy)$ by the defective polar form of an arbitrary equilateral triangle; otherwise expressed, the derivates, in respect of the partition $\alpha\beta\gamma/ij$ of forms of the pedo-similar net determined by the fixed triangle $\alpha\beta\gamma$, are the polar forms of equilateral triangles.

7·20 If $\alpha\alpha'$ be the chord perpendicular to $\beta\gamma$, the derivates in respect of the partition $\alpha ij/\beta\gamma$ of forms of the same pedo-similar net, are the auto-forms of triangles of the equilateral system determined by the Euler point α'.

This theorem is the natural extension of 6·4; for, the semi-pedal pencil of $\alpha\beta\gamma$ is obviously contained in the pedo-similar net determined by the fixed triangle $\alpha\beta\gamma$, and we can easily verify that the null pedo-parallel pencil of 6·4 is contained in the equilateral system of 7·20. To prove 7·20, we see from 1·17 et seq that the derivates in respect of $\alpha ij/\beta\gamma$ differ from the derivates in respect of $\alpha\beta\gamma/ij$, namely the polar forms of equilateral triangles, by a multiple of the derivate of $(\alpha x)(\beta x)(\gamma x)(ix)(jx)$ in respect of $\alpha ij/\beta\gamma$. This last derivate was seen in (27) to be the auto-form of a right-angled triangle, whose right angle is at α, and whose hypoteneuse is the diameter perpendicular to $\beta\gamma$. Hence the derivates in respect of $\alpha ij/\beta\gamma$ of forms of the net are auto-forms of triangles of the equilateral system containing this right-angled triangle. Since the Euler point of a right-angled triangle is the extremity of the chord through the right angle parallel to the hypoteneuse, we have the result stated, from 5·27.

The problem of finding the basal triangle when the fixed triangle $\alpha\beta\gamma$ and the measure-deviation (k,θ) are known, can be solved analytically by adapting equations ~~to~~ 7·18 to a slightly different co-ordinate system. Let the point $(x_0 x_1 x_2 x_3)$ now represent the cubic $\sum x_r (\xi x)^{3-r} (\eta x)^r$; let $X_0 X_1 X_2 X_3$ correspond to the fixed triangle and $A_0 A_1 A_2 A_3$ to the basal triangle. It may be shewn that 7·19 now becomes

$$7{\cdot}21 \qquad (A_1 X_0 - A_0 X_1) X_2^2 A_1 A_3 = (A_3 X_2 - A_2 X_3) X_1^2 A_0 A_2,$$

and the equations 7·18 could now be written

$$7{\cdot}22 \qquad \begin{aligned} A_0 &= -\lambda^2 X_0 (2-k)(2\mu + k\lambda) \\ A_1 &= -\lambda X_1 (2\lambda + k\mu)(2\mu + k\lambda) \\ A_2 &= \mu X_2 (2\lambda + k\mu)(2\mu + k\lambda) \\ A_3 &= \mu^2 X_3 (2-k)(2\lambda + k\mu). \end{aligned}$$

These determine the basal triangle. When the fixed triangle and the deviation θ are known, 7·22 shews that the locus of $(A_0 A_1 A_2 A_3)$ is a conic through the points of reference X_0, X_3 and through a point in $X_1 X_2$. Thus the system of basal triangles $A_0 A_1 A_2 A_3$ is a conic-system of the type mentioned in 5·24. Hence by 5·24,

7·23 the orthocentric locus of the basal triangles of pedo-similar forms of deviation θ and fixed triangle $\alpha\beta\gamma$ is the line OO', where O is the orthocentre of $\alpha\beta\gamma$ and O' is such that $SO' = SO$, $O\hat{S}O' = \theta$.*

Footnote(1)

We may also find the basal triangle in a more instructive way. Consider first the following problem:

7·24 The K-pedal line of a given point p in respect of an unknown inscribed triangle $\alpha'\beta'\gamma'$ passes through a given point Q. It is required to specify $\alpha'\beta'\gamma'$.

It is clear that the possible triangles $\alpha'\beta'\gamma'$ belong to a linear ∞^2-system L, and are therefore apolar to a certain triangle $\alpha\beta\gamma$. We specify L by means of its equilateral member and its orthocentric pencil (25). Join pQ and produce it to O, so that $OQ:Op = K:2$. The orthocentric pencil of O clearly belongs to L. Also join Sp and divide it at Q', so that $SQ':Sp = K:2$. Then the particular equilateral triangle in respect of which the pedal line of p is parallel to QQ' is clearly the required equilateral member of L. Thus L is known. Now the assigning of the fixed triangle abc of a (k,θ) pedo-similar form π'' is equivalent to the statement that the k-pedal lines of a',b',c' in regard to the basal triangle of π'' pass respectively through a,b,c, where $a'b'c'$ is obtained by turning abc through an angle θ about S. Thus the basal triangle of π'' is determined as the triangle common to three systems of the type L.

We may notice one other result in this connection. The k-pedal lines of p in regard to members of any pedo-parallel pencil contained in L must both pass through Q and be parallel, and must therefore be identical. Now it is clear from 5·21, that the only point which has the same k-pedal line in regard to members of a pedo-parallel pencil is the focus of the parabola of 5·21, that is, the point λ such that $ij\lambda$ is a member of the pencil. Hence ijp belongs to every pedo-parallel pencil of L, so that ijp is apolar to the apolar triangle $\alpha\beta\gamma$ of L, or p is the Euler point of $\alpha\beta\gamma$. We have therefore a property of the Euler point:

7·25 The k-pedal line of the Euler point p of $\alpha\beta\gamma$, in regard to all triangles apolar to $\alpha\beta\gamma$ passes through a fixed point Q; Q is the point which divides pO' in the ratio $2-k:k$, where O' is the inverse of the orthocentre of $\alpha\beta\gamma$ in the second fundamental circle (5·25).

(33) THE STRUCTURE OF THE MANIFOLD OF PEDO-SIMILAR FORMS.

A pedo-similar form being one which is complete and closed at ij in the manner of the pedal form, it can by 2·1 be written in the form:

$$7{\cdot}26 \cdot\ \pi''(x,y) = (iy)(jy)(cx)^3 + (ix)(jx)(ax)(by)^2,$$

Wwhere from the closure property of π'' it follows that $(ax)(by)^2$ is an auto-form. It was seen in (6) that the shape 7·26 for π'' was not unique, but that $(cx)^3$ and $(ax)(by)^2$ could be simultaneously replaced by $(cx)^3-(ix)(jx)(\lambda x)$ and $(ax)(by)^2+(iy)(jy)(\lambda x)$ respectively. Thus for a given π'', the possible forms $(cx)^3$ are members of a pedo-parallel system L_1, and the possible forms $(ax)(by)^2$ are auto-forms of members of a second pedo-parallel system L_2. Now it is clear from the form 7·26 that L_1 is the concurrency system of π''; hence by 7·3, if θ be the deviation of π'', L_1 makes the pedal angle $-\frac{3\theta}{2}$ with the basal triangle of π''. Also from the form 7·26, the auto-chord of x in regard to $(ax)(bx)^2$ is clearly parallel to the chord which corresponds to x in π''. It follows therefore that $(ax)(bx)^2$ makes the pedal angle $-\frac{\theta}{2}$ with the basal triangle. Hence we deduce that

7·27 if θ be the pedal angle between the pedo-parallel systems L_1, L_2, the deviation of π'' is either θ or $\theta+\pi$. In particular, the necessary and sufficient condition that the form π'' given by 7·26 be semi-pedal, is the pedo-parallelism of $(cx)^3$ and $(ax)(bx)^2$.

It was seen in (6) that the representation 7·26 could be rendered unique by choosing $(cx)^3$ to be apolar to $(ix)(jx)$, that is, to be of the form $p_1(ix)^3+p_2(jx)^3$ representing an equilateral triangle. If we also write the auto-form $(ax)(by)^2$ in the shape $(ix)(iy)(Ay)+(jx)(jy)(By)$ (3·7), we have the unique representation of π'' in the form:

$$7{\cdot}28 \quad \pi''(x,y) = (iy)(jy)\{p_1(ix)^3+p_2(jx)^3\}+(ix)(jx)\{(ix)(iy)(Ay)+(jx)(jy)(By)\}.$$

We shall term $p_1 p_2 A_1 A_2 B_1 B_2$ the six co-ordinates of the form π''.

To find the measure-deviation of a form in terms of its co-ordinates, we first find the co-ordinates of a form $\pi''(x,y)$ given by its basal triangle $\alpha\beta\gamma$ and its measure-deviation (κ,θ). Writing as before $x' = \lambda(jx)i + \mu(ix)j$, so that $\lambda + e^{\sqrt{-1}\theta}\mu = 0$, and $(\alpha x)(\beta x)(\gamma x) \equiv (ix)^2(Px) + (jx)^2(Qx)$, we easily find from 6·12 that

$$\pi''(x,y) = \pi'(x',y) = (ij)^2\{(ix')(jx')[(ix')(iy)(Py)+(jx')(jy)(Qy)] - \kappa(iy)(jy)[(ix')^3(Px')+(jx')^3(Qx')]\}$$

$$= (ij)^4\{(iy)(jy)[p_1(ix)^3 + p_2(jx)^3] + (ix)(jx)[(ix)(iy)(Ay)+(jx)(jy)(By)]\}$$

where

$$\begin{aligned} p_1 &= -\kappa\mu^3(Pj) \\ p_2 &= -\kappa\lambda^3(Qi) \\ A &= -\lambda\mu^2\{2(ij)P + \kappa(Pi)j\} \\ B &= \lambda^2\mu\{2(ij)Q - \kappa(Qj)i\}. \end{aligned} \tag{7·29}$$

On eliminating P, Q from these, we obtain the following equations for the measure and deviation of the form π'' with co-ordinates $p_1 p_2 A_1 A_2 B_1 B_2$:

$$7{\cdot}30 \begin{cases} U_\theta \equiv \lambda^2 p_1(Bi) + \mu^2 p_2(Aj) = 0, \text{ where } \lambda + \mu e^{\sqrt{-1}\theta} = 0, \\ V_\kappa \equiv \kappa^2(Aj)(Bi) + 4(ij)^2 p_1 p_2 = 0. \end{cases}$$

We now represent the form π'' with the co-ordinates $p_1 p_2 A_1 A_2 B_1 B_2$ by a point π'' with the same co-ordinates in a five-dimensional space Σ_5 which corresponds to the pedo-similar manifold. The forms with a given measure κ or a given deviation θ will then correspond to two families of quadrics U, V, whose equations are given by 7·30. We see that the general quadric U_θ, as well as the general quadric V_κ is singular, with the same line of vertices L, given by the equations $p_1 = p_2 = (Aj) = (Bi) = 0$. The points of L correspond to what we may call the <u>singular pedo-similar forms</u>, namely those of the shape $(ix)(jx)(iy)(jy)(tx)$ where t may be arbitrary.

Since U_θ and V_κ are cones with the line-vertex L, they have two distinct systems of generating three-dimensional regions passing through L. The family of quadrics U have clearly the four common generating regions:

$$\gamma_1 : p_1 = p_2 = 0 \qquad \gamma_3 : p_1 = (Aj) = 0$$

$$\gamma_2 : (Aj) = (Bi) = 0 \qquad \gamma_4 : p_2 = (Bi) = 0.$$

The quadrics V have similarly the four common generating regions:

$$\gamma_5 : \ p_1 = (Bi) = 0 \qquad \gamma_3 : \ p_1 = (Aj) = 0$$
$$\gamma_6 : \ p_2 = (Aj) = 0 \qquad \gamma_4 : \ p_2 = (Bi) = 0.$$

The two families U, V have thus in common the two generating regions γ_3, γ_4; the system of generating regions of U or V, to which γ_3, γ_4 belong will be called the secondary system, and the other system, the primary system.

The general primary generating regions of U_θ, V_κ are easily seen from 7·30 to be given by the respective equations:

$$p_1 = t\mu^2 (Aj), \ p_2 = -t\lambda^2 (Bi);$$
$$p_1 = t\kappa^2 \frac{(Bi)}{(Aj)}, \ \frac{(Aj)}{(Bi)} = -4(ij)^2 t p_2.$$

From the expressions 7·30 for the measure and deviation, it follows that κ and θ are respectively constant on these generating regions. Hence we see that

7·31 the primary generating regions of U_θ or V_κ represent families of forms with constant measure-deviation.

In particular, the primary generating regions γ_1, γ_2 of U represent respectively the products of $(ix)(jx)$ with auto-forms, and the forms of rank one. Since U_θ and V_κ have the secondary generating regions γ_3, γ_4 in common, they must also have two primary generating regions in common; remembering the ambiguity in the measure-deviation of a pedo-similar form (7·9), it clearly follows from 7·31 that these represent the two families of forms whose measure-deviations are of the shape $\{(\kappa,\theta) \text{ or } (-\kappa, \theta+\pi)\}, \{(\kappa, \theta+\pi) \text{ or } (-\kappa, \theta)\}$.

If in obtaining equations 7·29, we had replaced the basal triangle by any other triangle pedo-parallel to it, this would have meant that P, Q in 7·29 are increased by arbitrary scalar multiples of j, i respectively. It is clear that the following relations hold between the co-ordinates in 7·29 ,independently of such increase:

$$7{\cdot}32 \qquad \frac{p_1}{p_2} = \frac{\mu^3}{\lambda^3}\cdot\frac{(Pj)}{(Qi)} \; ; \; \frac{(Aj)}{(Bi)} = -\frac{\mu}{\lambda}\cdot\frac{(Pj)}{(Qi)}$$

If θ is given, 7·32 is satisfied by all forms of deviation θ, whose basal triangles belong to a pedo-parallel system. ~~similarly if is given, is satisfied by all forms of measure ,whose basal triangles belong to a pedo-parallel system~~.But obviously, equations of the form 7·32 represent secondary generating regions of U_θ ~~respectively~~.Hence

the general secondary generating region of U_θ ( ) represents the family of pedo-similar forms of deviation θ (~~measure~~),whose basal triangles are members of a pedo-parallel system.

The pencil of defective polar forms of equilateral triangles corresponds to a line M ;the quadric U_θ which contains M is none other than the semi-pedal quadric U_0 .The~~s~~ semi-pedal pencils, namely those of the type 6'1 correspond to those lines on U_0 which meet γ_1, V_2 and M .

8. EXTENSION TO SPACE OF n DIMENSIONS.

(34) THE WALLACE CURVE.

A norm curve in Euclidean space of n dimensions, which has the property, that the feet of the perpendiculars from any point on
Footnote (1) it to the faces of any inscribed simplex lie on a prime*, may be called a <u>Wallace curve</u>. It has been shewn by Dr. Richmond that the Wallace curves are those norm curves, which intersect the prime at infinity in the vertices of a circumscribed simplex of
Footnote (2) the Absolute*. The following is a sketch of a modification of Richmond's proof of the ~~same~~ result ~~on somewhat different lines.~~

Let $\alpha_0\alpha_1 \cdots \alpha_n$ be an inscribed simplex of a given norm curve C which cuts the prime at infinity in $i_1 i_2 \cdots i_n$ (all points on C being named by binary symbols). Using the symbol $[p_1 p_2 \cdots p_k]$ to denote $(p_1p_2)(p_1p_3)\cdots(p_1p_k)(p_2p_3)\cdots(p_{k-1}p_k)$, we have a preliminary identity:

$$8.1 \quad \frac{(\alpha_1 x)^{n-1}}{(\alpha_1\beta)}[\alpha_2\alpha_3\cdots\alpha_n] - \frac{(\alpha_2 x)^{n-1}}{(\alpha_2\beta)}[\alpha_1\alpha_3\cdots\alpha_n] + \cdots + (-1)^{n-1}\frac{(\alpha_n x)^{n-1}}{(\alpha_n\beta)}[\alpha_1\alpha_2\cdots\alpha_{n-1}] \equiv [\alpha_1\alpha_2\cdots\alpha_n]\frac{(x\beta)^{n-1}}{(\alpha_1\beta)\cdots(\alpha_n\beta)}.$$

We may prove this identity by shewing that when regarded as an equation in β, it possesses the n roots $\beta=\alpha_1,\alpha_2,\cdots\alpha_n$, though only of the $(n-1)^{th}$ degree in β. The identity may be stated in a slightly more general form by polarising each side in respect of the variable x.

If we denote by the symbol $\Delta(i_1i_2\cdots i_n, \alpha_1\alpha_2\cdots\alpha_n)$ the determinant:

$$\begin{vmatrix} (i_1\alpha_2)(i_1\alpha_3)\cdots(i_1\alpha_n) & (i_1\alpha_1)(i_1\alpha_3)\cdots(i_1\alpha_n) & \cdots & (i_1\alpha_1)\cdots(i_1\alpha_{n-1}) \\ (i_2\alpha_2)(i_2\alpha_3)\cdots(i_2\alpha_n) & (i_2\alpha_1)(i_2\alpha_3)\cdots(i_2\alpha_n) & & \vdots \\ \vdots & & & \\ (i_n\alpha_2)(i_n\alpha_3)\cdots(i_n\alpha_n) & (i_n\alpha_1)(i_n\alpha_3)\cdots(i_n\alpha_n) & & (i_n\alpha_1)\cdots(i_n\alpha_{n-1}) \end{vmatrix},$$

we see easily that

$$8.2 \quad \Delta(i_1i_2\cdots i_n, \alpha_1\alpha_2\cdots\alpha_n) = (-1)^{\frac{n(n-1)}{2}}[i_1i_2\cdots i_n][\alpha_1\alpha_2\cdots\alpha_n].$$

Now, if we consider that the point t on the norm curve C corresponds to the binary form $(tx)^n$, then a point on the prime at infinity corresponds to a binary form of the shape $\sum_{r=1}^{n} X_r(i_rx)^n$.

We take $x_1 x_2 \cdots x_n$ to be the co-ordinates of the point, and take $L_1 L_2 \cdots L_n$ to be the dual co-ordinates, so that the Absolute will have a tangential equation of the form:

$$\sum_{r,s=1,2\cdots n} p_{rs} L_r L_s = 0$$

Now, the intersection of the prime at infinity with $\alpha_1\alpha_2\cdots\alpha_n$ is easily seen to have the co-ordinates

$$L_r = (i_r\alpha_1)(i_r\alpha_2)\cdots(i_r\alpha_n) \quad (r=1,2,\cdots n).$$

Hence the pole of $\alpha_1\alpha_2\cdots\alpha_n$ in regard to the Absolute corresponds to the binary form

$$P_0(x) = \sum_{r=1}^{n} P_{0r}(i_r x)^n, \text{ where } P_{kr} = \sum_{s=1}^{n} p_{rs}\frac{(i_s\alpha_0)\cdots(i_s\alpha_n)}{(i_s\alpha_k)}.$$

Thus the foot of the perpendicular from the point t of the norm curve C to the face $\alpha_1\alpha_2\cdots\alpha_n$ corresponds to the binary form $Q_0(x) = P_0(x) + \epsilon(tx)^n$, where ϵ is determined by the condition that this is apolar to $(\alpha_1 x)(\alpha_2 x)\cdots(\alpha_n x)$. Thus

$$Q_0(x) = P_0(x) - \frac{R_0}{(t\alpha_1)\cdots(t\alpha_n)}(tx)^n = \sum_{r=1}^{n} P_{0r}(i_r x)^n - \frac{R_0}{(t\alpha_1)\cdots(t\alpha_n)}(tx)^n,$$

where

$$8{\cdot}3 \quad R_k = \sum_{r,s=1,2\cdots n} p_{rs}\frac{(i_r\alpha_0)\cdots(i_r\alpha_n)(i_s\alpha_0)\cdots(i_s\alpha_n)}{(i_r\alpha_k)(i_s\alpha_k)}.$$

For C to be a Wallace curve, the feet of the perpendiculars on ~~th~~ the faces must lie on a prime for all values of $t, \alpha_0, \alpha_1, \cdots \alpha_n$, so that the forms $Q_0(x), Q_1(x), \cdots Q_n(x)$ must be linearly related. The condition for this is the vanishing of the determinant

$$8{\cdot}4 \quad \begin{vmatrix} P_{01} & P_{02} & \cdots & P_{0n} & R_0(t\alpha_0) \\ P_{11} & P_{12} & \cdots & P_{1n} & R_1(t\alpha_1) \\ \vdots & & & & \\ P_{n1} & & \cdots & P_{nn} & R_n(t\alpha_n) \end{vmatrix}$$

If $\Delta_0, \Delta_1, \cdots \Delta_n$ be the minors of the elements of the last column of this determinant, we easily that

$$\Delta_r = |p_{rs}| \times (i_1\alpha_r)(i_2\alpha_r)\cdots(i_n\alpha_r) \times \Delta(i_1 i_2\cdots i_n, \alpha_0\alpha_1\cdots\alpha_{r-1}\alpha_{r+1}\cdots\alpha_n)$$

$$= (-1)^{\frac{n(n-1)}{2}} |p_{rs}| \times (i_1\alpha_r)\cdots(i_n\alpha_r) \times [i_1 i_2\cdots i_n] \times [\alpha_0\cdots\alpha_{r-1}\alpha_{r+1}\cdots\alpha_n] \text{ (by 8·2)}$$

Hence the determinant 8·4 will vanish if

8·5 $$\sum_{k=0}^{n} (-1)^k R_k (t\alpha_k)(i_1\alpha_k)\cdots(i_n\alpha_k) \times [\alpha_0\alpha_1\cdots\alpha_{k-1}\alpha_{k+1}\cdots\alpha_n] = 0.$$

Since this equation is linear in t, we conclude that in any case there is always one point t, the feet of the perpendiculars from which on the faces of the simplex $\alpha_0\alpha_1\cdots\alpha_n$ lie on a prime. If C is a Wallace curve, then 8·5 must be identically true. Replacing R_k by its expression 8·3, the coefficient of p_{rs} on the left of 8·5 is:

$$(i_r\alpha_0)\cdots(i_r\alpha_n)(i_s\alpha_0)\cdots(i_s\alpha_n)\sum_{k=0}^{n}\left\{(-1)^k\frac{(t\alpha_k)(i_1\alpha_k)\cdots(i_n\alpha_k)}{(i_r\alpha_k)(i_s\alpha_k)}[\alpha_0\alpha_1\cdots\alpha_{k-1}\alpha_{k+1}\cdots\alpha_n]\right\},$$

which, by 8·1, vanishes if $s \neq r$, and is equal to

$$(i_r\alpha_0)\cdots(i_r\alpha_n)[\alpha_0\alpha_1\cdots\alpha_n](ti_r)(i_1i_r)\cdots(i_{r-1}i_r)(i_{r+1}i_r)\cdots(i_ni_r),$$

if $s = r$. Hence 8·5 reduces to

$$\sum_{r=1}^{n} p_{rr}(i_r\alpha_0)\cdots(i_r\alpha_n)(ti_r)(i_1i_r)\cdots(i_{r-1}i_r)(i_{r+1}i_r)\cdots(i_ni_r) = 0.$$

It is obvious that, if this is true for all values of $\alpha_0\alpha_1\cdots\alpha_n$, then p_{rr} must be zero for $r = 1, 2, \cdots n$. Thus for C to be a Wallace curve the Absolute must be inscribed in $i_1i_2\cdots i_n$.

(35) THE PEDAL CORRESPONDENCE ON THE WALLACE CURVE.

Let C be a Wallace curve cutting the prime at infinity in $i_1i_2\cdots i_n$, and $\alpha_0\alpha_1\cdots\alpha_n$, a simplex inscribed in C. Let the pedal prime of a point x on C —— namely, the prime which contains the feet of the perpendiculars from x on the faces of $\alpha_0\alpha_1\cdots\alpha_n$ —— cut C in y. It is easily seen that the pedal correspondence (x,y) is an $(n+1, n)$ correspondence Π. Also, the perpendicular from any of the points i on any face of the simplex lies wholly at infinity, so that the pedal prime of any i is the prime $i_1i_2\cdots i_n$. Thus $i_1i_2\cdots i_n$ is a complete (but not a closed) set of the pedal correspondence. Thus

8·6 The pedal correspondence Π in n dimensions, as defined thus, is —— in contrast to the two-dimensional case 6·2 —— complete, but not closed at infinity.

By 2·4, the existence of the complete n-set reduces the rank of Π to three. Hence

8·7 the pedal correspondence obtains no increase in rank through the increase in the number of dimensions.

From the fact that the rank is three, it follows that all pedal primes could be expressed as linear combinations of three of them. This shews that

8·8 every pedal prime passes through a fixed region L_{n-3} of $n-3$ dimensions situated wholly at infinity. In particular for the three-dimensional case, every pedal plane is parallel to a fixed direction.

The pedal envelope has the prime at infinity for an n-ple prime, the corresponding regions of contact being the $(n-2)$-dimensional regions which join L_{n-3} to $i_1, i_2, \dots i_n$. Since the envelope is of class n+1 (cf (8) and has the prime at infinity for an n-ple prime), it follows that if any pedal prime P cuts the prime at infinity in a region L_1 (containing L), there is a homographic correspondence between P and L_1. Since there are two regions L_1 which touch the Absolute, it follows that there are two points $j_1 j_2$ on the Wallace curve, whose pedal primes touch the Absolute. There is a faint analogy between $j_1 j_2$ and the circular points of the two-dimensional case. Thus the analogue of the angle-property would be:

8·9 If a homography on the curve with the fixed points $j_1 j_2$ carries p to p', then the pedal primes of p, p' make a constant angle with one another for all positions of p.

It is easy to see also that the concurrency system of the pedal form Π is a linear ∞^2— system containing the singular pencil $(i_1x)(i_2x)\dots(i_nx)(\lambda x)$. Such systems are the analogues of "pedo-parallel" systems. Every such system contains a unique "equilateral" member, namely a member which is apolar to $(\iota_1 x)(\iota_2 x)\cdots(\iota_n x)$. If two such systems contain the respective "equilateral" members

$$\lambda_1(\iota_1 x)^{n+1} + \lambda_2(\iota_2 x)^{n+1} \cdots + \lambda_n(\iota_n x)^{n+1},$$
$$\mu_1(\iota_1 x)^{n+1} + \mu_2(\iota_2 x)^{n+1} \cdots + \mu_n(\iota_n x)^{n+1},$$

the analogue of the "pedal angle" between the two systems will be a series of angles ϕ_{rs}, which are defined in terms of the ratios

$$\frac{\mu_1}{\lambda_1}, \frac{\mu_2}{\lambda_2}, \cdots \frac{\mu_n}{\lambda_n}$$

by means of the equations

$$\phi_{rs} = \frac{1}{2\sqrt{-1}} \log \frac{\mu_r \lambda_s}{\mu_s \lambda_r}$$

so that $\phi_{rs} = -\phi_{sr}$, $\phi_{rr} = 0 \pmod{\pi}$.

FOOT-NOTES.

INTRODUCTION.

(1)Waelsch: "Uber Binaren Formen und Correlationen mehrer-dimensionaler Raume" Monatsheft fur Math. und Physik.Wien (6) ~~X~~1895.
(2)"The theory of the Rational Transformation", Jour.of the Ind. Math.Soc. April 1921.
(3)Pittarelli,Atti. della R.Acad. Lincei.1885-1886 Vol.3.I must thank Mr.F.P.White (St.John's College) for having drawn my attention to these memoirs of Pittarelli.
(4) Richmond 'On an extension of the Wallace Property of the Circumcircle' Proc. C.P.S. Vol 22. Page 34; Baker 'on the generalisation of a theorem of Steiner' ibid p. 33.

Section 1.

(1)See "On the Rank of the Double-binary Form" Proc.L.M.S.Ser.2 Vol 24.p.83.
(2)The Jacobian of $r+1$ binary m-ics $f_K(x)=a_{k_x}^{m}\ (K=0,1,\dots r)$ is the covariant $(a_0a_1)(a_0a_2)\cdots(a_0a_r)(a_1a_2)\cdots(a_{r-1}a_r)\,a_{0x}^{m-r}\cdots a_{rx}^{m-r}$. It is a combinant of the linear system $\lambda_0 f_0+\lambda_1 f_1+\cdots\lambda_r f_r$. The Jacobian may be verified to be a constant multiple of the determinant

$$\left|\frac{d^r f_k}{dx_1^r}\quad \frac{d^r f_k}{dx_1^{r-1}dx_2}\quad\cdots\quad \frac{d^r f_k}{dx_2^r}\right|.$$

(3)The name "concurrency system" is suggested by a geometrical interpretation similar to that described in subsection (8).Let R_m, R_n be rational norm curves in two independent spaces S_m, S_n, of m and n dimensions.To any point x of R_m F makes correspond n points y on R_n, or in other words, a unique prime (namely, an S_{n-1}) in S_n; and vice versa.The x-groups on R_m whose corresponding primes in S_n pass through an arbitrary assigned point, and the y-groups on R_n whose corresponding primes in S_m pass through an arbitrary assigned point, form the two concurrency systems L_1, L_2 of F. ~~A/similar/~~The ~~ne~~modifications which are necessary when x, y are cogredient, are obvious.
(4)See Grace and Young "Algebra of Invariants" pp53,54.
(5)The term "ladder" is taken from Waelsch.loc.cit.

Section 3.

(1)Some properties related to the points $l'm'$ (under the name of "punti di diramazione") will be found in Pittarelli.loc.cit.
(2)I owe this symmetrical decomposition to a suggestion of Prof.H.F.Baker.
(3)The brackets are omitted after the operational symbols F, F^{-1} in order that there may be no confusion with the functional notation.
(4)These are characteristic features of the automorphic linear group of a non-singular quadric in space of any odd number of dimensions, and are consequences of the existence of two algebraically distinct systems of generating regions on the quadric. In space of one dimension, the quadric is the point-pair, and the two points which constitute the pair answer to the two algebrai-

cally distinct systems of generating regions.

Section 4.

(1) Two pencils of cubics Γ, Γ' are <u>apolar pencils</u>, if any member of either is apolar to any member of the other. Γ is a <u>null pencil</u>, if any two of its members are apolar.

If binary cubic forms are represented by points in three-space (as in subsection 26, infra), the forms which are perfect cubes will correspond to a certain twisted cubic C. The null pencils will then correspond to the lines which belong to the tangent linear complex T of C; apolar pencils will correspond to polar lines of the complex T. It is well known that two polar lines of T are met by the same four tangents of C; the meaning of this is easily seen to be that apolar pencils have the same Jacobian.

(2) For properties of the syzygetic pencil of quartics, see Grace and Young, Algebra of Invariants pp197-208. For the study of the syzygetic pencil in relation to pencils of cubics, Meyer's Apolaritat may be consulted; Study ="On the irrational Covariants of certain Binary Forms" Am. Jour. of Math. 17(1895). will also be found useful.

(3) Let p' be one cross ratio of the point-pairs P_1, P_2, so that the other is $\frac{1}{p'}$. Then it is easily shewn that the absolute invariant of (P_1, P_2) is $\frac{p-2}{p+2}$, where $p = p' + \frac{1}{p'}$. Again, if q' be a cross ratio of (P_1, P_2) and the pair of perfect squares in their pencil, and $q = q' + \frac{1}{q'}$, the absolute invariant may be shewn to be $\frac{4}{q+2}$. If we represent quadratics as points in the plane of a fundamental circle of centre S and radius ρ, the points on the circle representing the perfect squares, then the absolute invariant of the quadratics represented by S and K may be shewn to be $1-\mu^2$, where SK$=\mu\rho$.

(4) $4{\cdot}10$ shews that the <u>third</u> absolute parameters of the singular pencil $[P_1, P_1']$, and the pencil containing a perfect cube, are 0 and 8 respectively. It will be an excellent verification of $4{\cdot}10$, to obtain these values directly. The former value can be obtained by the method of limits, the latter by considering the pencil $\lambda\{(\beta x)^3 + 3(\beta x)^2(\gamma x)\} + \mu\{3(\beta x)(\gamma x)^2 + (\gamma x)^3\}$ in respect of the representation $[P_1, P_2]$ where $P_1 = (\beta x)(\gamma x)$, $P_2 = 3(\beta x)^2 + 2(\beta x)(\gamma x) + 3(\gamma x)^2$.

Section 5.

(1) This might be verified algebraically. The canonical form $1{\cdot}2$ of the auto-form of $\alpha\beta\gamma$ may be shewn to be

$$\sum (\alpha i)(\alpha j)(\beta\gamma)(\beta\gamma, ij)(\beta x)(\gamma x)(\alpha y),$$

where $(\beta\gamma, ij)$ is short for $(\beta i)(\gamma j) + (\beta j)(\gamma i)$.

By $3{\cdot}6$, the apolar quadratic of this is

$$\sum \frac{(\beta\gamma)(\alpha x)^2}{(\beta\gamma, ij)(\alpha i)(\alpha j)}.$$

Now, the condition of perpendicularity of two chords $\beta\gamma, xy$ is:

$$(i\beta)(i\gamma)(jx)(jy) + (ix)(iy)(j\beta)(j\gamma) = 0.$$

Hence the extremity of the chord through α perpendicular to $\beta\gamma$ is $(i\beta)(i\gamma)(j\alpha)j + (j\beta)(j\gamma)(i\alpha)i$, so that we have only to shew that the above apolar quadratic of the auto-form is apolar to $(\alpha x)\{(i\beta)(i\gamma)(j\alpha)(jx) + (j\beta)(j\gamma)(i\alpha)(ix)\}$. This is easily seen to be the case.

(2) For these two linear covariants and a related property of

the Euler line, see F. Morley "Some Polar Constructions" Math. Ann. Bd. 51.

Section 6.

(1) For the verification we may begin by using the following lemma:

If $L_1(x), L_2(x)$ be linear, and $P_1(x); P_2(x)$ quadratic forms, the necessary and sufficient condition that

$$\cancel{L_1[P_1(x)]_y + L_2 P_2 x},\quad L_1(x)[P_1(x)]_y + L_2(x)[P_2(x)]_y$$

be a polar form, is $(L_1, P_1)' + (L_2, P_2)' \equiv 0$.

This follows at once by using the development in ladder-shape 1·12, 1·13.

By the application of the lemma it follows easily that

$$\{(ix)(j\alpha)+(jx)(i\alpha)\}\{(ix)(iy)(j\beta)(j\gamma)+(jx)(jy)(i\beta)(i\gamma)\} - \lambda(ij)^2(\alpha x)\{(\gamma x)(jy)(\beta i)+(\beta y)(ix)(\gamma j)\}$$

is a polar form only for $\lambda = 1$; it is then clearly the polar form of:

$$\{(ix)(j\alpha)+(jx)(i\alpha)\}\{(ix)^2(j\beta)(j\gamma)+(jx)^2(i\beta)(i\gamma)\} - (ij)^2(\alpha x)\{(\gamma x)(jx)(\beta i)+(\beta x)(ix)(\gamma j)\}.$$

This expression may be verified to be equal to $-4f''(x)$, where $f''(x)$ is given by 5·15.

(2) Thus the concurrency-systems of all semi-pedal forms of $\alpha\beta\gamma$ are the same. 5·18 shews the sameness of the concurrency-systems for two particular semi-pedal forms, namely the pedal form and the defective polar form. The sameness of the concurrency-systems is also a direct consequence of 6·13.

It follows from 1·1 and 6·11 that the second rank covariant of any semi-pedal form of $\alpha\beta\gamma$ is a binary cubic whose roots correspond to the vertices of the pedo-perpendicular equilateral triangle of $\alpha\beta\gamma$.

Section 7.

7(1) The theorem 7·23 may also be obtained by elementary geometry as a necessary consequence of 7·15.

Section 8.

(1) The word "prime" has been suggested by Professor Baker for denoting a flat $(n-1)$ dimensional space situated in a flat space of n dimensions.

(2) A geometrical proof of the result is given by Baker. loc. cit.