Equilibrium Stability in a Nonlinear Cobweb Model*

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Abstract

We demonstrate existence, and stability under adaptive learning, of restricted perceptions equilibria in a nonlinear cobweb model.

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Key Words: E-stability, adaptive learning, restricted perceptions equilibrium, cobweb model

1 Introduction

Consider a nonlinear cobweb model in which firms make supply decisions conditional on known cost shocks before observing market price. Closing the model requires specification of how firms make forecasts. The standard *rational expectations equilibrium* (REE) posits that firms have sufficient knowledge and capacity to forecast prices optimally, based on their conditional distributions. However, the model's nonlinear structure, and the fact that the conditional distributions are *endogenous* equilibrium objects that depend on the forecasting models in use, suggest that in this setting reliance on REE is implausibly strong.

To address this concern, we take a restricted perceptions approach in which firms use linear forecasting models to form price expectations. If each firm uses a linear forecasting model that is optimal among those models under consideration then the economy is in a *restricted perceptions equilibrium* (RPE). An RPE is an equilibrium in the Nash sense but it is not an REE because superior forecasting models are in principle available. The RPE concept has been widely used because it is consistent with approaches used in practice

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by econometricians who are faced with complex data structures, partial information, and degrees-of-freedom restrictions – approaches that necessarily result in model misspecification. Most of this literature focuses on misspecification in linear models due to omitted observables or parsimonious lag structures.[1]

The current paper adds to the literature on RPE[2] by focusing squarely on the issue of forecasting in a nonlinear environment using linear forecasting models based on exogenous observables.[3] Because RPE is an equilibrium concept, it is important to assess whether and how it might be attained. Following the adaptive learning literature we assume firms update their forecast models over time, as new data become available. If their estimates converge to models consistent with an RPE then the RPE is said to be *stable under adaptive learning* (or, if the context is clear, just *stable*). We conduct our study using, as a laboratory, a nonlinear version of the linear cobweb model in which RE was introduced by Muth (1961).

This paper is part of a broader research program in which RPE can generate important and sometimes surprising results. Evans and McGough (2020) demonstrate, in a non-linear environment, existence and stability of RPE in which forecast models condition linearly on extrinsic, continuously-measured (sunspot) shocks. Branch, McGough, and Zhu (2017) find that stable "sunspot RPE" exist in linear models for which no sunspot REE exist. Evans and McGough (2018) use linear forecasting rules in nonlinear settings to model optimizing behavior: see also Evans and McGough (2019) for applied examples. The recent focus in macroeconomics on nonlinear models motivates a careful examination of the use of linear forecast models in these environments. The central result of this paper is to show existence, uniqueness and stability of RPE in a nonlinear stochastic cobweb model, using an approach that can be implemented computationally in more general set-ups.

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[1]The term RPE was coined in Evans and Honkapohja (2001), but the conceptual framework – an equilibrium in which agents use optimally misspecified forecast models – is older. Marcet and Sargent (1989) examine RPE in a linear economic environment in which agents' forecasting models are underparameterized in the number of lags (see also Sargent (1991)). Evans and Honkapohja (2001) consider linear models in which agents omit some relevant observables or lags. For a general survey, see Branch (2006).

[2]Adam (2007) presents experimental evidence supporting existence and stability of RPE. Branch and Evans (2006) show the RPE framework is compatible with endogenous forecast-model heterogeneity. Slobodyan and Wouters (2012) estimate a medium-scale DSGE model under the assumption agents use univariate AR(2) forecasting models. Bullard, Evans, and Honkapohja (2008) consider judgement in monetary policy when the forecasting model used by policymakers is a low-order VAR. Hommes and Zhu (2014) show that multiple RPE can arise with underparameterized dynamics.

[3]Several papers have looked at related issues in very specific settings. In a nonlinear OLG model Evans, Honkapohja, and Sargent (1993) introduce a proportion of agents who know the distribution of prices but are unaware of their time-series dependence. Hommes and Sorger (1998) consider a model with multiple steady states and show the existence of "consistent expectations equilibria" (CEE) in which agents' forecasts match equilibrium autocorrelogams of a solution with complex deterministc dynamics; see also Hommes, Sorger, and Wagener (2013). Using a closely related model with multiple steady states and additive white noise shocks, Branch and McGough (2005) establish the existence of stable stochastic CEE.

2 Model and results

We begin by recalling the cobweb model. Time is discrete. There is a unit mass of identical firms indexed by $i \in \mathscr{I}$ which produce a single perishable good sold in a competitive market. Technology is captured by a cost function $c = c(q, w)$, where q is the quantity produced and $w \in \mathbb{R}^n$ is an exogenous vector of cost shocks taken as common to all firms and known when production decisions are made.

Firms make supply decisions before market price is realized, leading to the following problem: $\max_{q_{it}} E_{it-1}\left(p_t q_{it} - c\left(q_{it}, w_{t-1}\right)\right)$, where E_{it-1} denotes the expectations operator of firm i. The first order condition is $p^e_{it-1} = mc_i\left(q_{it}, w_{t-1}\right)$, where $p^e_{it-1} = E_{it-1} p_t$ and mc_i is marginal cost. Assuming marginal cost is invertible in output, we may obtain the supply curve for firm $i \in \mathscr{I}$, which we write as $q_{it} = s_i(p^e_{i,t-1}, w_{t-1})$.

Exogenous demand is subject to an iid shock v that is independent of cost shocks: $q = d(p, v)$. Market equilibrium satisfies $d(p_t, v_t) = \int_{\mathscr{I}} s_i(p^e_{i,t-1}, w_{t-1})di$. Assuming homogeneous forecasts, and that demand is invertible in price we may obtain a function $F : \mathbb{R}_+ \oplus \mathbb{R}^n \oplus \mathbb{R} \to \mathbb{R}_+$ characterizing the equilibrium price path: $p_t = F\left(p^e_{t-1}, w_{t-1}, v_t\right)$, where $\mathbb{R}_+ \equiv (0, \infty)$. Closing the model requires specification of p^e_{t-1}.

2.1 Rational expectations

For fixed w, define $\hat{T} : \mathbb{R}_+ \to \mathbb{R}_+$ by $\hat{T}(p^e) = E_v\left(F\left(p^e, w, v\right)|w\right)$, where E_v is the expectation over v. A fixed point $\bar{p}^e(w)$ of $\hat{T}$ determines the rational forecast, thus

$$p_t = F\left(\bar{p}^e_{t-1}\left(w_{t-1}\right), w_{t-1}, v_t\right) \equiv p\left(w_{t-1}, v_t\right)$$

identifies the associated REE price path. The following assumptions are sufficient to guarantee existence and uniqueness of the REE.

Assumptions A

- **A.1** The cost shocks w_t are determined by $w_t = \rho w_{t-1} + \sigma \varepsilon_t$ where ρ is a stable matrix, ε_t is zero-mean iid with bounded support and $\sigma \geq 0$.
- **A.2** The demand shock v has bounded support $K_v \subset \mathbb{R}$.
- **A.3** The marginal cost function mc is continuous on $\mathrm{cl}\left(\mathbb{R}_+\right) \oplus \mathbb{R}^n$ and continuously differentiable on $\mathbb{R}_+ \oplus \mathbb{R}^n$, with $mc_q > 0$ and $mc_{w_i} \leq 0$. Also, for all $w \in \mathbb{R}$ marginal cost satisfies $\lim_{q\to\infty} mc(q, w) = \infty$.
- **A.4** The demand function $d(\cdot, v) : \mathbb{R}_+ \to \mathbb{R}_+$ is continuously differentiable on $\mathbb{R}_+$ and $d_p(\cdot, v) < 0$ for all $v \in K_v$; and $d(p, \cdot) : K_v \to \mathbb{R}_+$ is measurable for all $p \in \mathbb{R}_+$. Also, for all $v \in K_v$, the following Inada conditions are satisfied: $\lim_{p\to 0} d(p, v) = \infty$ and $\lim_{p\to\infty} d(p, v) = 0$.

Some comments are warranted. Assumption A.1 comprises two parts: the functional form of the recursion identifying the Markov structure of the cost shocks; and the bounded support of the innovations to the cost shocks. The assumption of bounded support simplifies the application of the theory of stochastic recursive algorithms used to prove Theorem 3, and can be weakened if desired – this point also applies to Assumption A.2. The functional form of the recursion characterizing dynamics of w can be modified to include a non-linear dependence on lagged w; however, the additive nature of the innovation is explicitly exploited in the proof of Theorem 2, where the coefficient σ is used as a perturbation device. Assumptions $A.3$ and $A.4$ provide the smoothness needed for our analysis, and include versions of Inada conditions that guarantee a single crossing of supply and demand. Of course there are other constellations of assumptions sufficient to guarantee single crossing, and a natural alternative to $A.3$ and $A.4$ would be to simply assume sufficient smoothness and uniqueness of market equilibrium for all germane expectations.

Theorem 1 *Under the assumptions* **A** *the cobweb model has a unique REE.*

All proofs are in the Appendix.

2.2 Restricted perceptions

The REE of our cobweb model takes the form $p_t = p(w_{t-1}, v_t)$. Rational expectations requires firms compute $E_{t-1}p(w_{t-1}, v_t)$. We view this requirement as unrealistic; instead, we assume firms use a linear forecasting rule, or *perceived law of motion* (PLM) of the form $p_t = a + b'w_{t-1} + \zeta_t$ to compute price expectations, where ζ_t is an error term.[4] It follows that $p^e_{t-1} = a + b'w_{t-1}$, thus yielding the *actual law of motion* (ALM) implied by firms' beliefs:

$$p_t = F\left(a + b'w_{t-1}, w_{t-1}, v_t\right) \equiv \hat{F}\left(a, b, w_{t-1}, v_t\right). \tag{1}$$

By projecting the p_t onto the span of $(1, w_{t-1})$ we obtain a map from firms' perceived regression coefficients (a, b') to the implied regression coefficients $T(a, b)$:

$$\begin{array}{rcl} a & \xrightarrow{T^a} & E_{w^\sigma,v}\hat{F}\left(a, b, w^\sigma, v\right) \\ b & \xrightarrow{T^b} & \Sigma_{w^\sigma}^{-1} \cdot E_{w^\sigma,v}\left(w^\sigma \cdot \hat{F}\left(a, b, w^\sigma, v\right)\right). \end{array}$$

Here the presence of σ emphasizes the dependence of w^σ on the variance of the cost-shock innovations $\sigma\varepsilon_t$. The positive definite matrix Σ_{w^σ} is the covariance of the cost shocks and $E_{w^\sigma,v}$ is the expectation taken with respect to the distribution of v and the stationary distribution of w^σ.

At a fixed point $(\bar{a}^\sigma, \bar{b}^\sigma)$ of the T-map agents' beliefs minimize mean-square forecast error. These beliefs, together with the implied process for prices, characterize the RPE.

Theorem 2 *Under the assumptions* **A***, and for all* σ *sufficiently small, the cobweb model has a unique RPE.*

[4] Our approach is consistent with more general forecast rules. Least-squares updating requires forecast models that are linear in its parameters, but the forecast models could condition, for example, on finite-degree polynomials of observables. A version of our results would extend to this setting. In principle one could also envisage firms updating their forecast rule using kernel estimators. However, in practice degrees of freedom considerations in time-series forecasting favor parsimonious models. Here we focus on the simplest restricted perceptions formulations with expectations depending on w_{t-1}.

2.3 Adaptive learning

To assess whether the RPE is stable under adaptive learning, denote by (a_t, b_t) the firms' estimates obtained using data (p_s, w^{σ}_{s-1}) for $s \leq t$. Assume, without loss of generality, that firms know $\Sigma_{w^{\sigma}}$. The state at the beginning of time t is $(w^{\sigma}_{t-1}, a_{t-1}, b_{t-1})$. The following recursion provides the economy's dynamics:

$$\begin{aligned} p_t &= \hat{F}\left(a_{t-1}, b_{t-1}, w_{t-1}, v_t\right) \\ a_t &= a_{t-1} + \kappa_t\left(p_t - a_{t-1} - b'_{t-1} \cdot w^{\sigma}_{t-1}\right) \\ b_t &= b_{t-1} + \kappa_t \Sigma^{-1}_{w^{\sigma}} \cdot w^{\sigma}_{t-1} \cdot \left(p_t - a_{t-1} - b'_{t-1} \cdot w^{\sigma}_{t-1}\right) \\ w^{\sigma}_t &= w^{\sigma}_{t-1} + \sigma \varepsilon_t \end{aligned} \tag{2}$$

Here $\kappa_t > 0$ is the *gain* sequence, used to weight the forecast error.[5] If there is an open neighborhood U of $(\bar{a}^{\sigma}, \bar{b}^{\sigma})$ such that $(a_t, b_t) \xrightarrow{a.s.} (\bar{a}^{\sigma}, \bar{b}^{\sigma})$ whenever $(a_0, b_0) \in U$ then we say that the associated RPE is *stable under adaptive learning*.

Theorem 3 *Assume $\sum \kappa_t = \infty$ and $\sum \kappa_t^2 < \infty$. Under the assumptions* **A**, *and for all σ sufficiently small, the unique RPE of the cobweb model is stable under adaptive learning.*

2.4 Example

The cost function is quadratic, resulting in the supply curve $s(p^e, w) = cp^e - \gamma' w$. The demand curve is $d(p, v) = vp^{-\alpha}$. Here, $\alpha > 0$ and v is iid and distributed uniformly on the interval $[\bar{v} - \delta, \bar{v} + \delta]$, with $0 < \delta < \bar{v}$.

The temporary equilibrium map is $p = v^{1/\alpha}\left(cp^e - \gamma w\right)^{-1/\alpha}$. Integrating each side of this equation with respect to v determines REE price expectations $\bar{p}^e(w)$. The non-stochastic steady state is $\bar{p} = \bar{p}^e(0) = \bar{v}^{1/1+\alpha} c^{-1/1+\alpha}$.

To conduct our numerical exercise, we set $c = \alpha = \bar{v} = \gamma = 1$, yielding $\bar{p} = \bar{p}^e = 1$. The cost shock is univariate and its innovation ε is standard normal.[6] Finally, $\rho = 0.5, \sigma = 0.25$ and $\delta = 0.02$. With this calibration the unique RPE beliefs are $(\bar{a}^{\sigma}, \bar{b}^{\sigma}) \approx (1.03, 0.534)$.

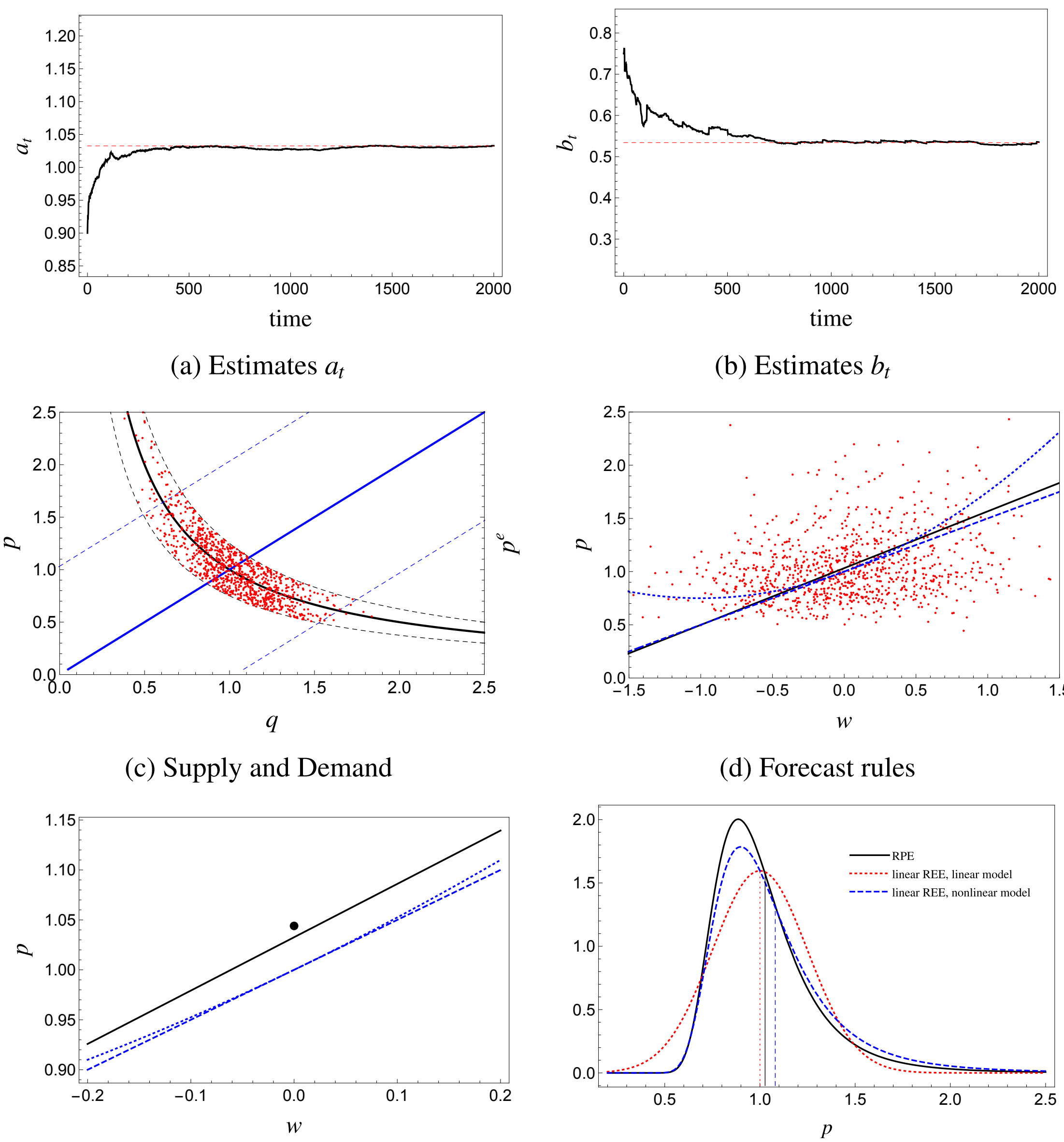
Panels 1(a) and 1(b) of Figure 1 provide the result of a simulation initialized with $(a_0, b_0) = (0.9, 0.75)$.[7]. The dashed red lines identify the RPE beliefs and the solid black lines provide the real-time estimates. As expected, convergence to RPE obtains.

[5] We assume the algorithm (2) includes a projection facility that guards against unusual patterns of shocks leading to unrealistic estimates. Use of projection facilities is standard when specifying real-time learning algorithms. See Evans and Honkapohja (2001) for further discussion.

[6] That ε is taken as normally distributed violates assumption $A.2$ requiring that the innovation ε have bounded support. Because the support of ε can be of arbitrarily large finite diameter, we do not view this violation as particularly egregious. Alternative, but more cumbersome methods can dispense with A.2.

[7] While it is common to assume the gain is proportional to t^{-1}, in practice it is often convenient to use a gain that converges to zero more slowly; this allows the data to move the estimates by larger magnitudes, particularly earlier on in the simulation, thus facilitating timely convergence. Note that any gain converging to zero more rapidly than $t^{-.5}$ meets the hypothesis of the theorem. For the simulation at hand, the gain is set as $\kappa_t = .1(1 + t^{0.6})^{-1}$

Figure 1: Cobweb Example



(a) Estimates a_t

(b) Estimates b_t

(c) Supply and Demand

(d) Forecast rules

(e) Forecast rules near steady state

(f) Price densities

To better understand how data in this model are generated, we plot, in q, p-space and w, p-space, the last 1000 realizations of the simulation underlying the upper panels of Figure 1: see panels 1(c) and 1(d). Panel 1(c) provides the q, p-plot, together with the mean (solid) and extreme (dashed) supply and demand curves, and nicely illustrates the impact of the model's inherent non-linearity. Note that while demand is a function of price (left axis), supply is a function of expected price (right axis, same scale).[8] Panel 1(d) provides

[8]The extreme demand curves correspond to demand curves with the taste shocks at $\bar{v} \pm \delta$; the extreme

the w, p-plot, the RPE forecasting model (solid black), and first- (dashed blue) and second- (dotted blue) order approximations to the REE beliefs function. Importantly, the first-order approximation to the REE is distinct from the RPE, reflecting that the RPE retains the model's non-linearity.

The asymmetry in the data – specifically, that there are more data points "above" the high-cost supply curve than "below" the low-cost one – reflects that supply is in fact a function of price expectation, not price.

To explore this point more fully, panel 1(e) of Figure 1 provides the same plot as panel 1(d) of Figure 1, but without the data; and also, the scale has been altered to provide a magnified view near the steady state. Naturally, at the non-stochastic steady state $(w, p) = (0, 1)$ the first- and second-order approximations to the REE are coincident. The graph of the RPE forecasting model is above each of these approximations to the REE: the RPE coefficients adjust to compensate for the non-linearity in the data-generating process. The large black dot, located just above the RPE graph at $w = 0$ identifies the sample mean price level, a feature of the data that is well-captured by the RPE.

Finally, panel 1(f) of Figure 1 provides the analytically computed asymptotic densities of prices, for alternative modeling assumptions, with the demand shock set to its mean. The associated mean prices are identified by vertical lines. The solid, black curve is the density for prices in an RPE, with its right skewness reflecting the model's non-linearity: the symmetric cost shock, together with constant elasticity of demand, yields a price distribution symmetric in logs but skewed in levels. The red, dotted curve provides the density for prices using the fully linearized model: naturally, this approach is symmetric in levels, entirely missing the skewness induced by the non-linearity. Finally, the blue, dashed curve provides the density that obtains if firms form forecasts using linear REE beliefs, but prices are determined in temporary equilibrium using the non-linear specification of demand. This density qualitatively lies part way between the solid, black RPE density and the red, dotted linear REE density, and provides a measure of the skewness induced by the model's nonlinear demand (i.e. the "difference" between the red, dotted graph and the blue, dashed graph), and the skewness induced by optimal linear beliefs (i.e. the "difference" between the blue, dashed graph and the solid, black graph).

3 Conclusion

A significant issue with RE in non-linear models is the apparent implausibility that agents would know and use the correct functional form when making forecasts. We show how this issue can be overcome by making the realistic assumption that agents adopt linear forecast rules that are optimally chosen.

supply curves take the cost shocks to be at plus/minus two standard deviations.

Appendix

Proof of Theorem 1. We begin by observing a consequence of the implicit function theorem: informally, if an equation $f(x,y)=0$ has a unique solution for each x in some open set U, and if the conditions of the implicit function theorem hold at each of these solutions, then the solutions (x,y) with $x \in U$ may be parametrized by one function on all U, rather than by a collection of functions each defined in an open neighborhood of a given point in U. More formally, let $U \subset \mathbb{R}^m$ be open and suppose $f : U \oplus \mathbb{R}^n \to \mathbb{R}^n$ is C^k, for $k \geq 1$. Assume

1. $\forall x \in U \, \exists! y(x) \in \mathbb{R}^n$ with $f(x,y(x)) = 0$
2. $\forall x \in U \, \det f_y(x,y(x)) \neq 0$

Then $y : U \to \mathbb{R}^n$ is C^{k-1} and $y_x(x) = f_y(x,y(x))^{-1} f_x(x,y(x))$.

Turning now to the theorem we begin by establishing existence and properties of supply. Define

$$X = \left\{(p^e,w) \in \mathbb{R}_+ \oplus \mathbb{R}^n : p^e > mc(0,w)\right\}.$$

We claim that X is open in $\mathbb{R}^{n+1}$. To see this, let $(\tilde{p}^e,\tilde{w}) \in X$ and let $\varepsilon = 1/2(\tilde{p}^e - mc(0,\tilde{w}))$. Then $mc(0,\tilde{w}) < \tilde{p}^e - \varepsilon$, thus, since $mc(0,\cdot) : \mathbb{R}^n \to \mathbb{R}_+$ is continuous, there is an open neighborhood U of $\tilde{w}$ so that $w \in U \implies mc(0,w) < \tilde{p}^e - \varepsilon$. We conclude that $(\tilde{p}^e - \varepsilon, \infty) \times U$ is an open subset of X containing $(\tilde{p}^e,\tilde{w})$, whence X is open.

By assumption A.3, given $(p^e,w) \in X$ there is a unique $q > 0$ such that $p^e = mc(q,w)$. It follows from the implicit function theorem that s is continuously differentiable on X, with $s_{p^e} > 0$ and $s_{w_i} \leq 0$.

Having demonstrated the existence and properties of supply we turn to temporary equilibrium. Observe that $p^e > mc(0,w)$ implies $s(p^e,w) > 0$ and that $s(p^e,w) \to 0$ as $p^e \to mc(0,w)^+$. Thus, for fixed w and v, the Inada conditions on demand and the positive slope of supply show that provided $p^e > mc(0,w)$ for each $v \in K_v$ there exists a unique $p \equiv F(p^e,w,v)$ solving $d(p,v) = s(p^e,w)$. The implicit function theorem then implies that for each $v \in K_v$, F is continuously differentiable on X. Direct computation may then be used to show that on $X \oplus K_v$ we have that $F_{p^e} < 0$ and $\lim_{p^e \to mc(0,w)^+} F(p^e,w,v) = \infty$.

Now define $\hat{T} : X \to \mathbb{R}_+$ by $\hat{T}(p^e,w) = \int_{K_v} F(p^e,w,v)\mu(dv)$ with μ the measure associated to v. The properties of F then imply $\hat{T}_{p^e}(p^e,w) < 0$ and $\lim_{p^e \to mc(0,w)^+} \hat{T}(p^e,w) = \infty$. It follows that for each w there is a unique $\bar{p}^e(w) > mc(0,w)$ such that $\bar{p}^e(w) = \hat{T}(p^e(w),w)$. This completes the proof of the theorem, though we note that the implicit function theorem may then be used to show that $\bar{p}^e$ is continuously differentiable in w and that $\bar{p}^e_w < 0$. ■

Proof of Theorem 2. Let $\tilde{w}_t = \rho \tilde{w}_{t-1} + \varepsilon_t$ and define $\tilde{T}(a,b,\sigma)$, where $\tilde{T} : \mathbb{R} \oplus \mathbb{R}^n \oplus \mathbb{R} \to \mathbb{R} \oplus \mathbb{R}^n$, by

$$\begin{aligned} a &\xrightarrow{\tilde{T}^a} E_{\tilde{w},v} F\left(a + b'\tilde{w}, \sigma\tilde{w}, v\right) \\ b &\xrightarrow{\tilde{T}^b} \Sigma_{\tilde{w}}^{-1} \cdot E_{\tilde{w},v}\left(\tilde{w} \cdot F\left(a + b'\tilde{w}, \sigma\tilde{w}, v\right)\right). \end{aligned}$$

We make explicit the dependence of $\tilde{T}$ on σ because perturbations of σ will be required to obtain our result. It follows that $\tilde{T}$ is continuously differentiable, and a generalized version of Leibniz's theorem allows for the computation of derivatives.

We will use the implicit function theorem to determine a parameterized family of fixed points $\left(\tilde{a}^\sigma, \tilde{b}^\sigma\right)$ for the map $\tilde{T}$. Recall that $\bar{p}^e(w)$ is the REE expectation and note that $\tilde{T}(\bar{p}^e(0),0,0) = (\bar{p}^e(0),0) \equiv \left(\tilde{a}^0, \tilde{b}^0\right)$, and that, by Theorem 1, this is the unique such fixed point. We now compute derivatives, which are all taken as evaluated at $\left(\tilde{a}^0, \tilde{b}^0, 0\right)$; these arguments will be omitted for convenience, but we will retain the dependence on v as needed.

$$
\begin{aligned}
\tilde{T}^a_a &= E_{\tilde{w},v} F_{p^e}(v) = E_v d_p(v)^{-1} s_{p^e} < 0 \\
\tilde{T}^a_b &= E_{\tilde{w},v} F_{p^e}(v) \tilde{w}' = 0 \\
\tilde{T}^a_\sigma &= E_{\tilde{w},v} F_w(v) \tilde{w} = 0 \\
\tilde{T}^b_a &= \Sigma_{\tilde{w}}^{-1} \cdot E_{\tilde{w},v} \tilde{w} \cdot F_{p^e}(v) = \Sigma_{\tilde{w}}^{-1} \cdot E_{\tilde{w},v} \tilde{w} \cdot d_p(v)^{-1} s_{p^e} = 0 \\
\tilde{T}^b_b &= \Sigma_{\tilde{w}}^{-1} \cdot E_{\tilde{w},v} \tilde{w} F_{p^e}(v) \tilde{w}' = I_n \cdot E_v d_p(v)^{-1} s_{p^e} \\
\tilde{T}^b_\sigma &= \Sigma_{\tilde{w}}^{-1} \cdot E_{\tilde{w},v} \tilde{w} F_w(v) \tilde{w} = E_v F_w(v)'
\end{aligned}
$$

It follows that $\det D_{(a,b)} \tilde{T} \neq 0$, and so, by the implicit function theorem, there exists $\bar{\sigma} > 0$ so for $|\sigma| < \bar{\sigma}$ there is a C^1 collection $\left(\tilde{a}^\sigma, \tilde{b}^\sigma\right)$ with $\tilde{T}\left(\tilde{a}^\sigma, \tilde{b}^\sigma, \sigma\right) = \left(\tilde{a}^\sigma, \tilde{b}^\sigma\right)$. It's worth noticing that $\tilde{b}^\sigma_\sigma = \left(E_v F_{p^e}(v)\right)^{-1} E_v F_w(v)'$, which is generically non-zero, as expected. Also, $\tilde{a}^\sigma_\sigma = 0$, which aligns with the standard finding that stochasticity has no first order impact on the steady-state.

The proof is complete by verifying that when $0 < \sigma < \bar{\sigma}$ it follows that $(\tilde{a}^\sigma, \sigma^{-1}\tilde{b}^\sigma)$ is a fixed point of the T-map, and thus identifies an RPE. We proceed with direct computation:

$$
\begin{aligned}
T^a\left(\tilde{a}^\sigma, \sigma^{-1}\tilde{b}^\sigma, \sigma\right) &= E_{w^\sigma,v} F\left(\tilde{a}^\sigma + \sigma^{-1}\tilde{b}^\sigma w^\sigma, w^\sigma, v\right) \\
&= E_{\tilde{w},v} F\left(\tilde{a}^\sigma + \tilde{b}^\sigma \tilde{w}, \sigma\tilde{w}, v\right) = \tilde{a}^\sigma \\
T^b\left(\tilde{a}^\sigma, \sigma^{-1}\tilde{b}^\sigma, \sigma\right) &= \Sigma_{w^\sigma}^{-1} \cdot E_{w^\sigma,v}\left(w^\sigma \cdot \hat{F}\left(\tilde{a}^\sigma + \sigma^{-1}\tilde{b}^\sigma w^\sigma, w^\sigma, v\right)\right) \\
&= \Sigma_{w^\sigma}^{-1} \cdot E_{\tilde{w},v}\left(\sigma\tilde{w} \cdot \hat{F}\left(\tilde{a}^\sigma + \tilde{b}^\sigma \tilde{w}, \sigma\tilde{w}, v\right)\right) \\
&= \sigma^{-1}\Sigma_{\tilde{w}}^{-1} \cdot E_{\tilde{w},v}\left(\tilde{w} \cdot \hat{F}\left(\tilde{a}^\sigma + \tilde{b}^\sigma \tilde{w}, \sigma\tilde{w}, v\right)\right) = \sigma^{-1}\tilde{b}^\sigma. \blacksquare
\end{aligned}
$$

Proof of Theorem 3. The learning algorithm may be written

$$
\begin{pmatrix} a_t \\ b_t \end{pmatrix} = \begin{pmatrix} a_{t-1} \\ b_{t-1} \end{pmatrix} + \kappa_t \mathscr{R}^{-1} \begin{pmatrix} 1 \\ w^\sigma_{t-1} \end{pmatrix} \left(\hat{F}\left(a_{t-1}, b_{t-1}, w^\sigma_{t-1}, v_t\right) - \left(a_{t-1} + b'_{t-1} w^\sigma_{t-1}\right)\right) \qquad (3)
$$

where $\mathscr{R} = 1 \oplus \Sigma_{w^\sigma}$. Stability analysis involves the study of the differential system $(\dot{a}, \dot{b}) = h(a,b)$, where

$$
h(a,b) = E_{w^\sigma,v}\left[\mathscr{R}^{-1} \begin{pmatrix} 1 \\ w^\sigma \end{pmatrix} \left(\hat{F}\left(a, b, w^\sigma, v\right) - \left(a + b' w^\sigma\right)\right)\right].
$$

Now observe

$$
\begin{aligned}
h(a,b) = &= \mathscr{R}^{-1} E_{w^\sigma,v} \left[\begin{pmatrix} 1 \\ w^\sigma \end{pmatrix} \hat{F}(a,b,w^\sigma,v) - \begin{pmatrix} 1 & (w^\sigma)' \\ w^\sigma & (w^\sigma)' \otimes w^\sigma \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \right] \\
&= \begin{pmatrix} E_{w^\sigma,v} \hat{F}(a,b,w^\sigma,v) - a \\ \Sigma_{w^\sigma}^{-1} E_{w^\sigma,v} w^\sigma \hat{F}(a,b,w^\sigma,v) - b \end{pmatrix} \\
&= \begin{pmatrix} T^a(a,b,\sigma) \\ T^b(a,b,\sigma) \end{pmatrix} - \begin{pmatrix} a \\ b \end{pmatrix}.
\end{aligned}
$$

It follows that the RPE (a^σ, b^σ) is a fixed point of the differential system. Ljung's theory tells us that if the system is Lyapunov stable, and if the learning algorithm is augmented with a projection facility, then the theorem is proved.

To establish Lyapunov stability, we show that the derivative of h evaluated at (a^σ, b^σ) has eigenvalues with negative real parts; and to establish this, we show that the derivative of the T-map has eigenvalues with real parts less than one when evaluated at $(a^0, b^0, 0)$, and then we appeal to the continuity of eigenvalues in σ. We compute

$$
\begin{aligned}
&T_a^a = E_{w^\sigma,v} F_{p^e}(v) = E_v d_p(v)^{-1} s_{p^e} < 0 \\
&T_b^a = E_{w^\sigma,v} F_{p^e}(v)(w^\sigma)' = 0 \\
&T_a^b = \Sigma_{w^\sigma}^{-1} \cdot E_{w^\sigma,v} w^\sigma F_{p^e}(v) = \Sigma_{w^\sigma}^{-1} \cdot E_{w^\sigma,v} w^\sigma \cdot d_p(v)^{-1} s_{p^e} = 0 \\
&T_b^b = \Sigma_{w^\sigma}^{-1} \cdot E_{w^\sigma,v} w^\sigma F_{p^e}(v)(w^\sigma)' = I_n \cdot E_v d_p(v)^{-1} s_{p^e} \qquad .
\end{aligned}
$$

It follows that DT has $n+1$ eigenvalues, all equal to $E_v d_p(v)^{-1} s_{p^e} < 0$, so that (a^σ, b^σ) is indeed a Lyapunov stable fixed point of the system $(\dot{a}, \dot{b}) = h(a,b)$.

It remains to specify the projection facility, which amounts to choosing two sets $K_1 \subset K_2$ containing the steady state, and then requiring, at each time t, that if the right-hand-side of the recursion would place (a_t, b_t) outside of K_2 then instead the value for (a_t, b_t) is selected arbitrarily (randomly or deterministically) from within K_1. To specify these two sets, we require a Lyapunov function, which is guaranteed by the stability of (a^σ, b^σ). Specifically, there exists a twice continuously differentiable Lyapunov function V on the basin of attraction of (a^σ, b^σ), which we denote by $D \subset \mathbb{R}^{n+1}$. Thus $V : D \to \mathbb{R}_+$ satisfies $V(a^\sigma, b^\sigma) = 0$ and $V(x) > 0$ for $x \neq (a^\sigma, b^\sigma)$, as well as other important properties: see page 132 of Evans and Honkapohja (2001) for details. For $c > 0$ let

$$
K(c) = \{x \in D : V(x) \leq c\}.
$$

Choose $0 < c_1 < c_2 < c$ so that $K(c_2) \subset \operatorname{int}(K(c))$. Then $K_i = K(c_i)$ determine the required sets. That the various shocks are bounded allows for the technical conditions on pages 124 and 125 of Evans and Honkapohja (2001) to be easily verified, and the result follows from Corollary 6.8 on page 136 of Evans and Honkapohja (2001). ■

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