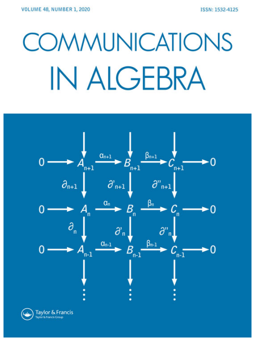

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ABSTRACT

We study the properties of groups that have presentations in which the generating set is a fixed set of involutions and all additional relations are of even length. We consider the parabolic subgroups of such a group and show that every element has a factorization with respect to a given parabolic subgroup. Furthermore, we give a counterexample, using a cluster group presentation, which demonstrates that this factorization is not necessarily unique.

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1. Introduction

A presentation of a group is a concise method of defining a group in terms of generators and relations. In special cases, much information about the corresponding group can be extracted from a given presentation [12]. Coxeter presentations are a classical example of this [2].

Recall that a pair (W, S), where $S = \{s_1, \dots, s_n\}$ is a non-empty finite set and W is a group, is called a Coxeter system if W has a group presentation with generating set S subject to relations of the form $(s_is_j)^{m(i,j)}$, for all $s_i, s_j \in S$, with $m(i,j) = 1$ if $i = j$ and $m(i,j) \geq 2$ otherwise, where no relation occurs on s_i and s_j if $m(i,j) = \infty$ [11, Section 5.1]. Such a group, W, is called a Coxeter group.

An outline of the rich history of research into these groups is given in [3, Historical Note]. In particular, it is well known that the finite Coxeter groups can be classified *via* their Coxeter graphs and the class of finite Coxeter groups is precisely the class of finite reflection groups [3, Chapter VI, Section 4, Theorem 1;5,6]. The applications of Coxeter groups are widespread throughout algebra [3], analysis [9], applied mathematics [4], and geometry [7]. However, the many combinatorial properties of Coxeter groups make them an interesting topic of research in their own right (see [2]).

For example, for any subset $I \subseteq S$, W_I denotes the subgroup of W generated by I. Any subgroup of W which can be obtained in this way is called a parabolic subgroup of W [11, Section 5.4]. It is known that the coset wW_I contains a unique element of minimal length for each $w \in W$, meaning that we can choose a distinguished coset representative of wW_I. It follows that there exists a unique factorization of each element with respect to a given parabolic subgroup [2, Proposition 2.4.4]. In certain cases, the reduced expressions for these distinguished coset

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representatives for a maximal parabolic subgroup can be described explicitly; see for example [13, Corollary 3.3; 15].

In this paper, we will consider group presentations which generalize the Coxeter case by allowing generating sets of infinite size and any relations that have even length. We show that a variation of this property of Coxeter groups holds for any group, G, which has a group presentation of this type.

In particular, we can define a parabolic subgroup of G in an analogous way to the Coxeter case and prove that there exists a (not necessarily unique) factorization of each element of G with respect to a given parabolic subgroup. We also give a counterexample using cluster group presentations (in the sense of [17, Definition 1.2]) showing that, in contrast to the Coxeter case, this factorization is not unique in general.

Some of these results form part of the author's Ph.D. thesis [18], carried out at the University of Leeds. The paper will proceed in the following way. In Section 2, we establish basic properties of the length function on G and provide a more detailed summary of the main result of this paper. In Section 3, we prove our first main result: that there exists a factorization of each element of G with respect to a given parabolic subgroup. In Section 4, we construct an example in which the factorization with respect to a given parabolic subgroup is not unique.

2. Group presentations with even length relations

Let G be a group arising from a group presentation $\langle X|R\rangle$, where X may be finite or infinite. By the definition of the presentation of a group, any element w of G can be written as

$$w = x_1^{a_1} x_2^{a_2} \ldots x_r^{a_r},$$

where $r \in \mathbb{N}, x_j \in X$ and $a_j = \pm 1$, for all $1 \leq j \leq r$. The *length* of $w \in G$, $l(w)$, is the smallest r such that w has an expression of this form and a *reduced expression* of w is any expression of w as a product of $l(w)$ elements of $X \cup X^{-1}$ (where $X^{-1} = \{x^{-1} : x \in X\}$ is a copy of X). We refer to l as the *length function* on G. Taking an index set I, if R is a set of relations of the form $u_i = v_i$ for $i \in I$, with $u_i, v_i \in F(X)$, where $F(X)$ denotes the free group on X, then the *length of the relation* $u_i = v_i$ is given by the length of the word $u_i v_i^{-1}$ in $F(X)$.

Analogously to the Coxeter case, we define a parabolic subgroup of G.

Definition 2.1. For $I \subseteq X$, we denote by G_I the subgroup of G generated by I. A (standard) *parabolic subgroup* of G is a subgroup of the form G_I for some $I \subseteq X$.

Moreover, for each $I \subseteq X$ we define the following sets.

$$G^I = \{w \in G : l(wx) > l(w) \forall x \in I\};$$
$${}^I G = \{w \in G : l(xw) > l(w) \forall x \in I\}.$$

In Section 3, we will prove our main result:

Proposition 2.2. *Let G be a group generated by a fixed set, X, of involutions subject only to relations of even length. For $I \subseteq X$, let G_I denote the subgroup of G generated by I and $G^I = \{w \in G : l(wx) > l(w) \forall x \in I\}$. Then every element $w \in G$ has a factorization*

$$w = ab,$$

where $a \in G^I, b \in G_I$ and $l(w) = l(a) + l(b)$.

This result is a strengthening of [18, Proposition 7.2.4]. By comparison, we can see that this result is similar to [2, Proposition 2.4.4] for Coxeter groups. However, unlike the Coxeter case, we will show in Section 4 that the factorization for elements of G with respect to a given parabolic subgroup is not necessarily unique or determined by minimal length elements of the coset wG_I.

3. Proof of main result

To begin, we prove some basic results for the length function on a group with an arbitrary group presentation $\langle X|R\rangle$.

Lemma 3.1. *If $x \in X$ and $w \in G$ then $l(xw) < l(w)$ if and only if there exists a reduced expression of w beginning in x^{-1}.*

Proof. Let $l(xw) < l(w)$. Suppose $xw = x_1^{a_1}x_2^{a_2}\dots x_r^{a_r}$ is a reduced expression, where $x_j \in X$ with $a_j = \pm 1$ for all $1 \leq j \leq r$. Then $w = x^{-1}x_1^{a_1}x_2^{a_2}\dots x_r^{a_r}$ is an expression of w of length $r+1$. Moreover, this expression must be reduced otherwise $l(w) \leq r$, contradicting that $l(xw) < l(w)$.

Conversely, if there exists a reduced expression of w beginning in x^{-1}, say $w = x^{-1}x_2^{a_2}\dots x_r^{a_r}$ where $x_j \in X$ with $a_j = \pm 1$ for all $2 \leq j \leq r$, then $xw = x_2^{a_2}\dots x_r^{a_r}$ and so $l(xw) \leq r-1 < l(w)$. □

For the remaining results, we let G be a group with group presentation $\langle X|R\rangle$ which satisfies the following conditions.

(I) X is a fixed set of involutions. That is, for each $x \in X, x^2 = e$.
(II) Every relation in R has even length.

We establish some properties, analogous to the Coxeter case, of the length function on G. The first result is a consequence of Lemma 3.1.

Lemma 3.2. *Let $I = X \setminus \{x\}$ for some $x \in X$ and take $w \in G$ such that $w \neq e$. Then $w \in {}^IG$ if and only if all reduced expressions of w begin in x.*

Proof. If $w \in {}^IG$ has a reduced expression beginning in y for some $y \in X$ such that $y \neq x$ then $l(yw) < l(w)$, contradicting that $w \in {}^IG$. Conversely, suppose $w \in G$ is such that all reduced expressions of w begin in x. For any $y \in X$ such that $l(yw) < l(w)$ there exists a reduced expression of w beginning in y, by Lemma 3.1. Hence $y = x$ and so $w \in {}^IG$. □

Remark 3.3. It follows from Lemma 3.2 that $w \in G^I$ if and only if all reduced expressions of w end in x.

The following result is analogous to [11, Proposition 5.1] for Coxeter groups.

Proposition 3.4. *There exists a surjective homomorphism $\varepsilon : G \to \{\pm 1\}$ defined by $\varepsilon : x \mapsto -1$ for each $x \in X$. It follows that the order of each generator is 2.*

Remark 3.5. Suppose the group presentation $\langle X|R\rangle$ satisfies only condition (*II*) and each generator has finite order. In this case, the surjective homomorphism ε exists and the order of each generator $x \in X$ of G is even.

In [11, Section 5.2], the length function on a Coxeter group is defined along with five basic properties. Below, we consider the length function on G.

As for the Coxeter case, each element of the generating set X of G is an involution and so any element w of G can be written in the form $w = x_1x_2\dots x_r$ for $x_j \in X$. So the length, $l(w)$, of $w \in G$, will be the smallest r such that $w = x_1x_2\dots x_r$.

Lemma 3.6. *For all $w_1, w_2 \in G$ and $x \in X$ the following properties hold.*

(1) $l(w_1) = l(w_1^{-1})$.
(2) $l(w_1) = 1$ *if and only if* $w_1 \in X$.
(3) $l(w_1w_2) \leq l(w_1) + l(w_2)$.
(4) $l(w_1w_2) \geq l(w_1) - l(w_2)$.

(5) $l(w_1) - 1 \leq l(w_1x) \leq l(w_1) + 1$.
(6) $l(w_1x) = l(w_1) \pm 1$ *and* $l(xw_1) \neq l(w_1)$

Proof. The proof is analogous to the Coxeter case, see [11, Section 5.2]. □

Remark 3.7. The properties (1), (3) and (4) in Lemma 3.6 hold for a group with arbitrary presentation.

It is easy to see that (2) and (6) fail for the trivial group given by the presentation $\langle x|x = e\rangle$. However, the property

(2^*) $w_1 \in X \cup X^{-1}$ implies $l(w_1) \leq 1$

holds for any group with group presentation $\langle X|R\rangle$. We can then apply (2^*), (3) and (4) to prove that (5) also holds for a group with arbitrary presentation.

We are now ready to prove Proposition 2.2.

Proof of Proposition 2.2. We proceed by induction on $l(w)$. If $l(w) = 1$ then $w = x$, for some $x \in X$. If $x \in I$, then we choose $a = e, b = x$ and, by Lemma 3.6(2), this is the desired factorization. If $x \notin I$, then we claim that $l(xy) > l(x)$ for all $y \in I$ and so we choose $a = x, b = e$. Taking any $y \in I$, if $l(xy) < l(x)$ then $l(xy) = 0$ as $l(x) = 1$, by Lemma 3.6(2). Thus, $xy = e$. It follows that $x = y \in I$, contradicting that $x \notin I$. As $l(xy) \neq l(x)$ by Lemma 3.6(6), it must be that $l(xy) > l(x)$.

Suppose $l(w) = r \geq 1$ and that the statement holds for every element of G of shorter length. If $w \in G^I$ then we choose $b = e$ and $a = w$. Similarly, if $w \in G_I$ then we choose $a = e$ and $b = w$. So we need only consider the case when $w \notin G^I$ and $w \notin G_I$.

As $w \notin G^I$, there exists $x \in I$ such that $l(wx) < l(w)$. By Lemma 3.6(5), $l(wx) = l(w) - 1 < r$. By induction, there exists $a' \in G^I$ and $b' \in G_I$ such that $wx = a'b'$ and

$$l(wx) = l(w) - 1 = l(a') + l(b').$$

Let $a = a'$ and $b = b'x$. Then $ab = a'b'x = (wx)x = wx^2 = w$. It remains to show that $l(b'x) = l(b') + 1$, giving $l(a) + l(b) = l(a') + l(b'x) = l(a') + l(b') + 1 = l(wx) + 1 = l(w)$.

We assume, for a contradiction, that $l(b'x) < l(b')$. That is, by Lemma 3.6(5), $l(b'x) = l(b') - 1$. By the above, $wx = a'b'$, so $w = a'b'x$, and $l(wx) = l(a') + l(b')$. Thus

$$\begin{aligned} l(w) = l(a'b'x) &\leq l(a') + l(b'x) \\ &= l(a') + (l(b') - 1) \\ &= (l(a') + l(b')) - 1 \\ &= l(wx) - 1 \\ &< l(wx), \end{aligned}$$

contradicting the fact that $l(wx) < l(w)$. Since $l(b'x) \neq l(b')$ by Lemma 3.6(6), we have $l(b'x) > l(b')$ and so $l(b'x) = l(b') + 1$ by Lemma 3.6(5). Therefore, $l(w) = l(a) + l(b)$. Finally, we note that $a = a' \in G^I$ and, as $x, b' \in G_I$, we have that $b \in G_I$. Thus, we have obtained the required factorization of w. □

Remark 3.8. By applying Proposition 2.2 to w^{-1}, it can be shown that, for any $I \subseteq X$, every element $w \in G$ has a factorization $w = ab$ for some $a \in G_I, b \in {}^IG$ such that $l(w) = l(a) + l(b)$.

Remark 3.9. Proposition 2.2 does not hold in general for groups with an arbitrary group presentation. A counterexample is given by the Klein four-group, $\mathcal{V} = \{e, i, j, k\}$ [19, Section 44.5], which has group presentation:

$$\mathcal{V} = \langle i, j, k | i^2 = j^2 = k^2 = ijk = e \rangle.$$

Taking $w = j$, this is a unique reduced expression of w. We have $\mathcal{V}_{\{i\}} = \{e, i\}$ and so $j \notin \mathcal{V}_{\{i\}}$. However, $ji = k$ and so $l(ji) = l(j)$, meaning $w \notin \mathcal{V}^{\{i\}}$. Thus, j has no reduced factorization with respect to $\mathcal{V}_{\{i\}}$.

As stated in [2, Proposition 2.4.4], in the Coxeter case this factorization exists and is furthermore unique. The element in the factorization lying in the set W^I can be shown to be the unique element of wW_I of minimal length [11, Proposition 1.10]. The uniqueness of these minimal length coset elements distinguish them as coset representatives and they are referred to as the *minimal coset representatives* [11, Section 1.10]. Thus, the set $wW_I \cap W^I$ contains only one element, namely the minimal coset representative of wW_I. The uniqueness of the factorization for elements of Coxeter groups is a consequence of the Deletion Condition and is not a property that is transferable to the factorizations of elements in G with respect to given parabolic subgroup, G_I. A counterexample proving this will be given in the next section.

However, in the cases when $I = X$ and $|I| = 1$ the factorization of all elements of the group with respect to the corresponding parabolic subgroup will be unique.

Lemma 3.10. *If $I = X$ or $I = \{x\}$ for some $x \in X$, then for all $w \in G, wG_I \cap G^I$ contains a unique element. Hence the factorization of each element of G with respect to G_I is unique.*

Proof. If $I = X$ then $G_I = G$ and $G^I = \{e\}$, thus $wG_I \cap G^I = \{e\}$.

If $I = \{x\}$ for some $x \in X$, then $G_I = \{e, x\}$, as X is a fixed set of involutions generating G, and $wG_I = \{w, wx\}$ for any $w \in G$. By Lemma 3.6(6), $l(wx) = l(w) \pm 1$ meaning that exactly one of w or wx is an element of G^I and thus the unique element in $wG_I \cap G^I$.

We conclude that, in both cases, the factorization of any $w \in G$ with respect to G_I is unique as if $w = ab = a'b'$ where $a, a' \in G^I$ and $b, b' \in G_I$, then $a, a' \in wG_I \cap G^I$. Thus, $a = a'$ and consequently $b = b'$. □

4. Non-uniqueness of factorizations

In this section, we present a counterexample demonstrating that the factorizations, shown to exist by Proposition 2.2, for elements of a group with a presentation whose generators are involutions and whose relations are of even length are not necessarily unique.

Specifically, we consider the finite group, G, arising from the group presentation $\langle t_1, t_2, t_3 | R \rangle$, where R is the following set of relations:

(a) $t_1^2 = t_2^2 = t_3^2 = e$ *(each generator is an involution).*
(b) $t_1t_2t_1 = t_2t_1t_2, t_1t_3t_1 = t_3t_1t_3, t_2t_3t_2 = t_3t_2t_3$ *(the braid relations).*
(c) $t_1t_2t_3t_1 = t_2t_3t_1t_2 = t_3t_1t_2t_3$ *(the cycle relation).*

We note that the above group presentation of G satisfies conditions (I) and (II), thus Proposition 2.2 holds for elements of G with respect to a given parabolic subgroup.

Remark 4.1. This group presentation is a *cluster group presentation*. Cluster groups are groups defined by presentations arising from cluster algebras. It was first shown in [1] and then in [10] that a group presentation could be associated to a quiver appearing in a seed of a cluster algebra of finite type and that the corresponding group is invariant under mutation of the quiver [1, Theorem 5.4; 10, Lemma 2.5]. For detailed definitions of cluster algebras and quiver mutation, see [14, Chapter 2]. It is the group presentation based on the work done in [10], that was considered more generally in [17], where due to the context, the corresponding groups were labeled

$$\begin{tikzcd} 1 \arrow[rr, bend left] & 2 \arrow[l] & 3 \arrow[l] \end{tikzcd}$$

Figure 1. G is the cluster group corresponding to the above quiver of mutation-Dynkin type A_3.

cluster groups. Each quiver appearing in a cluster algebra of finite type is mutation-equivalent to an oriented Dynkin diagram [8, Theorem 1.4] and the corresponding cluster group presentation is precisely a Coxeter presentation. Consequently, a cluster group associated to a mutation-Dynkin quiver is isomorphic to the finite reflection group of the same Dynkin type. The presentation defined above is the cluster group presentation associated to the quiver of mutation-Dynkin type A_3 in Figure 1.

It was shown in [17, Lemma 3.10] that an isomorphism between a cluster group associated to a mutation-Dynkin quiver of type A_n and the symmetric group on $n+1$ elements, denoted by Σ_{n+1}, can be constructed explicitly from the quiver. It follows from [17, Lemma 3.10] that the following map defines an isomorphism between G and Σ_4.

Lemma 4.2. [16, Proposition 3.4; 17, Lemma 3.10]. *There exists an isomorphism* $\pi : G \to \Sigma_4$ *given by*

$$\pi : t_1 \mapsto (1,2),$$
$$\pi : t_2 \mapsto (2,3),$$
$$\pi : t_3 \mapsto (2,4).$$

From Lemma 4.2, we conclude that G contains 24 distinct elements. Moreover, we can use this isomorphism to determine that $t_it_jt_it_k = t_kt_it_jt_i$ for all combinations of pairwise distinct i, j, k. Using this together with the group relations, we construct the Cayley graph of G with respect to this presentation, which is displayed in Figure 2.

Our goal is to choose a subset, I, of the generating set of G such that we can find an element of G with two distinct factorizations with respect to the corresponding parabolic subgroup, thus proving the factorizations shown to exist in Proposition 2.2 are not necessarily unique. Due to Lemma 3.10, I must contain two distinct elements.

Take $I = \{t_1, t_2\}$. From the Cayley graph of G (Figure 2) we have

$$G^I = \{e, t_3, t_1t_3, t_2t_3, t_1t_2t_3, t_2t_1t_3\}.$$

For $w = t_2t_3t_1t_2$, we have $a = t_2t_3 \in G^I$ and $b = t_1t_2 \in G_I$. Using the Cayley graph of G in Figure 2, we observe that each of these are all reduced expressions and so a and b yield a factorization of w as given in Proposition 2.2. Moreover, we can apply the cycle relation to w to obtain another reduced expression of w, $w = t_1t_2t_3t_1$. Let $a' = t_1t_2t_3 \in G^I$ and $b' = t_1 \in G_I$. As this expression for a' is a subexpression of a reduced expression of w, it must be reduced. From Lemma 4.2 we conclude that $t_i \neq e$ for $i = 1$, 2, 3, thus t_1 is also reduced. Clearly, as a, a' and b, b' are of different lengths, these are distinct pairs of elements in G^I and G_I, respectively. Thus, we obtain distinct factorizations $w = ab = a'b'$ where $l(w) = l(a) + l(b) = l(a') + l(b')$ for $a, a' \in G^I$ and $b, b' \in G_I$.

This counterexample demonstrates that for a group G with presentation $\langle X|R\rangle$ satisfying conditions (I) and (II), unlike in the Coxeter case, it is possible for the set $wG_I \cap G^I$ to have more than one element for some $I \subseteq X$ and $w \in G$. In the counterexample above, two distinct elements in this set are given by $a = t_2t_3$ and $a' = t_1t_2t_3$ for $w = t_2t_3t_1t_2$.

Furthermore, for a Coxeter group, W, the unique factorization of an element $w \in W$ with respect to a parabolic subgroup, W_I, is determined by the unique minimal length element of the coset wW_I [2, Corollary 2.4.5]. In the more general case, the same example used to demonstrate that the factorization is not necessarily unique can also be used to show that minimal length elements of wG_I do not necessarily yield factorizations of w.

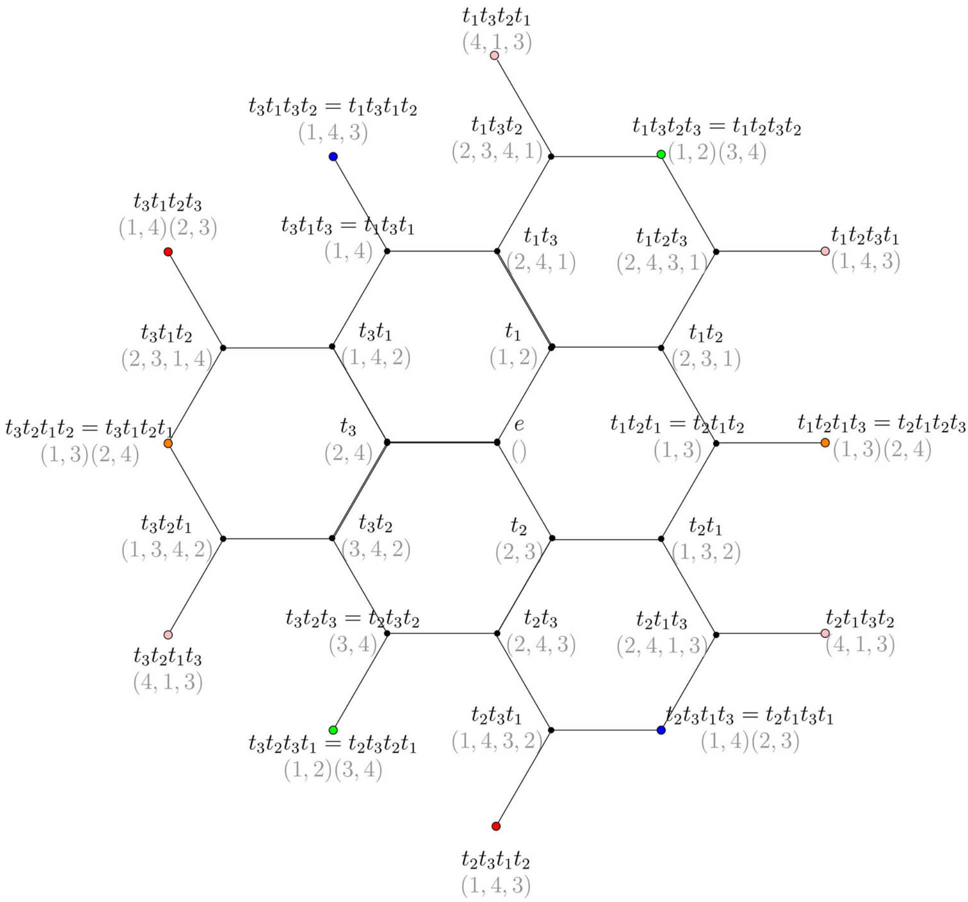

Figure 2. The Cayley graph of $(G, \{t_1, t_2, t_3\})$ and the corresponding permutations of each element under π. The colored nodes denote an identification of vertices.

Indeed, from the Cayley graph in Figure 2, we have

$$G_I = \{e, t_1, t_2, t_1t_2, t_2t_1, t_1t_2t_1 = t_2t_1t_2\}.$$

Choosing $w = t_1t_2t_3 \in G^I$, we observe from the Cayley graph that this is a unique reduced expression of w and so the only factorization with respect to G_I is obtained by taking $a = w$ and $b = e$. However,

$$wG_I = \{t_1t_2t_3, t_1t_2t_3t_1, t_1t_2t_3t_2, t_2t_3t_1, t_2t_3t_2, t_2t_3\}.$$

Thus, no factorization of w can be obtained from the minimal length element, t_2t_3, of wG_I.

The property that this factorization is unique for elements of W, and determined by the unique minimal length element of wW_I, is a result of the Deletion Condition [11, Theorem 1.7]. The Deletion Condition is a characterizing result for Coxeter groups. That is, if W is a group and S a set of involutions generating W, then (W, S) has the Deletion Condition if and only if (W, S) is a Coxeter system [2, Theorem 1.5.1]. Considering the proof of [11, Proposition 1.10(c)], we see that the factorizations of w with respect to G_I, as taken in the above counterexample, are no longer unique or determined by the minimal length elements of wG_I because, without the Deletion Condition, we can no longer omit certain factors from a non-reduced expression of w and leave w unchanged.

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