QUANTUM MECHANICS, LOCALITY AND ASYMPTOTIC SEPARABILITY

A thesis submitted for the degree of

Doctor of Philosophy

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by

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St Leonard's College

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DECLARATIONS

I, Timothy D Jackson hereby certify that this thesis, which is approximately 64,000 words in length has been written by me, that it is a record of work carried out by me, and that it has not been submitted in any previous application for a higher degree.

30th September 1985 Timothy D Jackson

I hereby certify that the candidate has fulfilled the conditions of the Resolution and regulations appropriate to the degree of Doctor of Philosophy in the University of St Andrews and that he is qualified to submit this thesis in application for that degree.

30th September 1985 K K Wan

Research Supervisor

DECLARATION

I was admitted as a research student under Ordinance No 12 on 1st October 1981, and as a candidate for the degree of Ph.D on 1st october 1982; the higher study for which this is a record was carried out in the University of St Andrews between 1981 and 1985.

30th September Timothy D Jackson

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ABSTRACT

This thesis investigates the localisation, separation, and separability of quantum systems.

Motivated by the physical limitations of measurement, we define the concept of a local observable for both one and two particle systems, give a precise prescription for the localisation of bounded observables and demonstrate the suitability of a certain family of local observables in describing the measurement of a quantum mechanical system using apparatus of finite size. The localisation of certain unbounded observables (eg momentum) is examined. Localisation in the spectrum is also introduced.

One of the consequences of this approach is that we can provide a sort of short-lived resolution to the EPR paradox for finite times when the particles have separated. Recognising the limitations of this resolution we proceed to formulate a quantum mechanical theory for a two particle system with the property that every state in which the two particles separate into disjoint regions for large times is asymptotically separable. Using this theory, some controversial hidden variable questions are reexamined. The relationship of these results to the experimental evidence for nonlocality in quantum mechanics is discussed.

CONTENTS

CHAPTER 1

INTRODUCTION

There is an enchanting story, related by Heisenberg [1958], in which Max Planck is supposed to have been out walking with his young son, on a winter's day late in 1900, shortly before he presented his historic paper [1900] on the quantisation of radiation. According to the legend, he told his son, excitedly, how he felt that he had found something which was either one of the greatest discoveries since Newton, or else completely wrong. Whatever the truth behind this captivating image, and whether or not Planck realised immediately the profound effects his proposals were to have on the course of physics, by the late 1920's and early 1930's it was slowly becoming evident that the new quantum theory proposed a view of the world which was substantially different from the previously accepted classical view: the quantisation of radiation [Planck 1900], the wave nature of matter [de Broglie 1930], the uncertainty principle [Heisenberg 1925]: all of these underlined the essential novelty of the quantum theory. Perhaps most particularly it was the essentially statistical nature of quantum mechanics which first aroused controversy. The arch proponent of the view that quantum mechanics was wrong, at least in this respect, was Einstein, whose famous remark that "God does not play dice" emerged as the catch-phrase of the opposition to quantum mechanics during the

prolonged Bohr-Einstein debate of the 1930's. Einstein argued that quantum mechanics provided at best an incomplete description of the real world while Bohr and the Copenhagen school replied that the quantum mechanical description was the only possible one: no more complete picture could be obtained. At the centre of Bohr's argument was his "complementarity principle" of which the theorem (2.5.6) is in one sense a manifestation. As a consequence of some of the work in this thesis, we shall have occasion to question the validity of (2.5.6) as anything stronger than an operator relation. In the light of this, it is an interesting question whether or not, and to what extent, complementarity may be maintained in its full philosophical entirety. That is however beyond the scope of our aims in this thesis. Here we are concerned to elucidate the issue of locality arising from the philosophical discussion.

Einstein's argument rested on a famous paper with Podolski and Rosen [1935], in which they proposed a thought experiment designed to show that quantum mechanics was incomplete [cf(2.7)]. One of the principles on which the argument is based is that of (Einstein) locality and it is commonly supposed nowadays that the EPR paradox embodies the underlying non-locality inherent in the quantum mechanical formalism [cf Bell 1964, Clauser and Shimony 1978,

Selleri and Tarozzi 1981]. This view seems to be confirmed by the extensive investigations carried out into the logical foundations of quantum mechanics [cf Hooker 1975,1979, Beltrametti and van Fraasen 1981], and the identification of the "paradoxical" features of quantum mechanics with elements of the underlying logical structure.

In view of the primary importance of locality in the EPR example, it is rather surprising that, almost without exception, discussion of the paradox is concerned only with the spin state of the Bohm version [1951 and 1957] of the EPR experiment. Beltrametti and Cassinelli, for example, claim [1981] that since they "have in mind to consider only physical quantities relative to spin, we only need specify the spin part of the compound system." This limitation in the description of the paradox is even more surprising when we consider remarks of the type made by Pauli as long ago as 1933 concerning the importance to interference effects of the overlap of wavefunctions:

> "From a superficial consideration of the exclusion principle, it might be thought that a sort of action at a distance is being postulated, as a result of which even two widely separated particles are aware of one another.. However,this is not so, because the

exclusion principle is only valid as long as the wave packets of the two particles overlap." [cf Enz (ed) 1973 p168.]

What Pauli suggests (this view is also discussed by Furry [1936(b)] and Hiley [1980], eg) is that nonlocal aspects of the quantum theory arise from the physical interference of wavefunctions. In other words it is being postulated that when the wavefunctions themselves separate, separability is achieved and in this sense locality is preserved. One reason that these comforting words have been largely ignored is because of the well-known phenomenon of the spreading of the wavepacket [cf (7.1)], so that the chances of finding wavefunctions which are not overlapping seem at first sight rather slim. Nevertheless it seems worth investigating, to what extent we might be able to provide a theory which satisfies the Pauli hypothesis by paying closer attention to the localisation of the quantum mechnical systems in question.

We proceed throughout by paying close attention to the physical limitations of measurement. This leads us to formulate [Chapters 3,4,5] local observables appropriate to the measurement of physical systems using apparatus of finite size. In Chapter 7, we investigate a time development for such systems and also outline the ideas of the asymptotic localisation

and separation of particles [Wan and McLean 1983 and 1984] which will be a basis for our investigation of an asymptotically separable quantum mechanics. The correlation of the subsystems in a two particle system is the subject of Chapter 8 and in the following chapter we use these results and an extension of the concept of local observables to two particle systems with spin to provide a resolution of the EPR paradox in a limited sense for finite time. Acknowledging the limitations of this solution we attempt in Chapter 10 an asymptotic resolution of the paradox in the limit of large times, based again on the separation of the particles, this time into disjoint regions at infinity. Chapter 11 investigates some consequences of this asymptotic analysis for the hidden variables questions that have haunted quantum theory from those early days. It turns out that we can provide asymptotically a hidden variable theory that will describe a system of two spin-half particles. Chapter 6 is a little out of the mainstream and entails a neat formulation for providing bounded observables in a slightly different sense to the usual lattice theoretic approach. It is of interest to us partly as an example in which the limitations of measurement lead to significant changes in the formulation, and partly as the basis of a quite general programme of localisation of observables. Before we proceed with any of this it will be convenient to provide some mathematical

background.

CHAPTER 2

MATHEMATICAL BACKGROUND

2.1 Quantum Mechanics in Hilbert Space.

In the conventional Hilbert space formulation of quantum mechanics a quantum mechanical system is associated with a Hilbert space $\mathcal{H}$. For a single particle moving in configuration space $\mathbb{R}^n$ this Hilbert space is $L^2(\mathbb{R}^n)$. The inner product in the space is denoted by $\langle . | . \rangle$. A linear operator T on $\mathcal{H}$ is said to be symmetric if the domain $\mathcal{D}(T)$ of T is dense in $\mathcal{H}$ and for every φ, ψ in $\mathcal{D}(T)$, T satisfies

$$\langle \varphi | T\psi \rangle = \langle T\varphi | \psi \rangle .$$

The adjoint operator T^* of T is a linear operator defined by

$$\langle \varphi | T^*\psi \rangle = \langle T\varphi | \psi \rangle \quad \text{for all} \quad \psi \in \mathcal{D}(T^*)$$

$$\mathcal{D}(T^*) = \{\psi \in \mathcal{H} : \varphi \to \langle T\varphi | \psi \rangle \text{ is continuous on } \mathcal{D}(T)\}$$

An operator is called selfadjoint if $T = T^*$. A symmetric operator T is called essentially selfadjoint if $\overline{T}$, the closure of T, is selfadjoint. $T^* = \overline{T}$ is the unique selfadjoint extension of T.

The states of the quantum mechanical system are described by a certain class of operators known as the density operators $\{\rho\}$. These operators are bounded, positive, selfadjoint linear operators on $\mathcal{H}$ satisfying

$Tr(\rho)=1$ [cf Prugovecki 1971 p383]. The pure states of the system are those operators ρ which satisfy $\rho^2=\rho$ and comprise the set of projections P on $\mathcal{H}$. For each vector φ in $\mathcal{H}$ there exists a projection P_φ mapping $\mathcal{H}$ onto the subspace generated by φ. When $\rho=P_\varphi$ we write ρ_φ to denote the vector state corresponding to φ. Each such vector state is evidently a pure state. We shall sometimes use the Dirac notation for projections [Dirac 1930], namely: $P_\varphi = |\varphi><\varphi|$. States ρ which are not pure are called mixed and consist of convex linear sums $\sum_i \lambda_i P_i$ of pure states P_i, $\sum_i \lambda_i = 1$.

Physical observables of the system are represented by selfadjoint linear operators defined on dense domains in $\mathcal{H}$. The expected value or expectation value $<T;\rho>$ of a physical observable T (by an abuse of language we refer to both the physical quantity and the operator representing it with the same symbol) for a system in the state ρ is defined formally to be:

(2.1.1)

$$<T;\rho>=Tr(\rho T).$$

The time evolution of a free quantum system is described by the time evolution operator

(2.1.2)

$$U_t\varphi = \exp[-iH_o t/\hbar]\varphi$$

for all $\varphi \in \mathcal{H}$, where H_o is the free Hamiltonian

operator. Using the Fourier representation

(2.1.3)

$$\varphi(\underline{x}) = (2\pi\hbar)^{-n/2}\int_{\mathbb{R}^n}\tilde{\varphi}(\underline{p})\exp[i\underline{p}.\underline{x}/\hbar]d\underline{p}$$

where

$$\tilde{\varphi}(p) = (2\pi h)^{-n/2}\int_{\mathbb{R}^n}\varphi(\underline{x})\exp[-i\underline{p}.\underline{x}/\hbar]d\underline{x}$$

the time evolution of the vector φ may be given the integral representation

(2.1.4)

$$\varphi(x,t) = (2\pi\hbar)^{-n/2}\int_{\mathbb{R}^n}\tilde{\varphi}(\underline{p})\exp[-ip^2 t/2m\hbar]\exp[i\underline{p}.\underline{x}/\hbar]d\underline{p}.$$

This is the Schrodinger picture for the time evolution of a system [cf Schiff 1955 p169 eg]. In the Heisenberg picture [op cit p170] the time dependence is thrown onto the observables so that $A_t = U_t^{-1}AU_t$, while the states are regarded as time independent. $\varphi(x)$ is called the coordinate representation of the vector φ while $\tilde{\varphi}(p)$ is the momentum representation of the same vector, the two representations being linked by (2.1.3). We can also write in the Dirac notation $\langle x \mid \varphi \rangle = \varphi(x)$ and $\langle p \mid \varphi \rangle = \tilde{\varphi}(p)$, where $|x\rangle$ and $|p\rangle$ are the generalised eigenfunctions of position and momentum respectively [cf Bohm A 1979].

Problems with domains for unbounded operators and the inconvenience of infinite expectation values have led to the formulation of a Hilbert space quantum mechanics based solely on bounded operators. This

approach is based on the use of projection operators to describe experimental propositions about the system [von Neumann 1955, Mackey 1968, Jauch 1968, Piron 1976]. Let $\mathcal{B}(\mathcal{H})$ denote the set of all bounded operators on $\mathcal{H}$. Selfadjoint elements of $\mathcal{B}(\mathcal{H})$ now correspond to the physical observables. This is of course a smaller set of observables than before. A number of notable observables, including the position operator, the momentum operator and the Hamiltonian operator, have been excluded. However all the projection operators on $\mathcal{H}$ are in $\mathcal{B}(\mathcal{H})$. The projection operators form a lattice of projections $\mathcal{L}(\mathcal{H})$ on $\mathcal{H}$ and Jauch, Mackey and Piron (et al) base their quantum mechanics on this lattice of projections. The link with the conventional theory is made through the spectral theorem.

(2.1.5) The Spectral Theorem

For every selfadjoint operator T in $\mathcal{H}$ there exists exactly one spectral family E(T;t) for which

$$T = \int t\,dE(T;t).$$

Proof: Wiedmann 1981 p191,eg.

The spectral projections $E(T;\Delta)$ of the spectral measure form the basis for the theoretical interpretation in this lattice theoretical approach. The projection $E(T;\Delta)$ corresponds to a physical measurement to test whether the observable T takes a

value which lies in the set Δ. The eigenvalues of $E(T;\Delta)$ are 1 and 0. The value 1 corresponds to the answer yes to the question; does the value of T lie in Δ? The value 0 corresponds to the answer no to the same question. In section (2.6), we expand upon this interpretation in terms of probabilities. Using the spectral projections enables us to dispense with unbounded observables and consider only the projections associated with them via Theorem (2.1.5). The support of the integral in (2.1.5) is the spectrum $\sigma(T)$ of T. We shall use the following definition for the spectrum.

(2.1.6) Definition

The spectrum $\sigma(T)$ of a selfadjoint operator T is defined as the set of all points t in $\mathbb{R}$ such that for every open interval I in $\mathbb{R}$ containing t we have $E(T;I) > 0$ [Prugovecki p253].

For an observable T with discrete spectrum the theorem (2.1.5) reduces to

$$T = \sum_i t_i E_i$$

where $E_i = E(T;\{t_i\})$ is the projection onto the subspace generated by the eigenvector(s) φ_i associated with eigenvalue t_i. Eigenvectors associated with different eigenvalues are orthogonal and each eigenvector is normalised to one so that we have $\langle \varphi_i | \varphi_j \rangle = \delta_{ij}$. For continuous spectra there is an analogous orthogonality

relation in terms of generalised eigenfunctions and δ-function normalisation [Bohm A 1979 p49,eg].

The zero operator has spectral measure given by:

(2.1.7)

$$E(0;\Delta) = \begin{cases} 1 \text{ if } 0 \in \Delta \\ 0 \text{ if } 0 \notin \Delta \end{cases}$$

[cf Weidmann 1980 p195].

2.2 The Algebraic Formulation of Quantum Mechanics.

The algebraic approach to quantum theory is a generalisation from the von Neumann theory in which the algebraic structure of $\mathcal{B}(\mathcal{H})$, namely as a C^*-algebra, is retained as the fundamental building block. One associates to each physical system a C^*-algebra $\mathcal{A}$ .That is, an algebra closed under addition, multiplication and an adjoint operation, possessing a norm $\|\cdot\|$ with respect to which the algebra is complete and such that if A is a member of the algebra the following property is satisfied:

(2.2.1)

$$\|A^* A\| = \|A\|^2$$

[Arveson 1976, Bratteli and Robinson 1979, Dixmier 1977, Emch 1972]. The C^*-property (2.2.1) ensures that the norm on the C^*-algebra is unique [Rickart 1950].

Physical observables of the system are represented by the selfadjoint elements of the algebra. States of the system are represented by normalised, positive linear functionals (NPLF's) w on the algebra: ie w is a map: $\mathcal{A} \to \mathbb{C}$ such that w is linear, $w(A^*A) \geq 0$ for all A, and $w(1) = 1$ if $\mathcal{A}$ possesses the identity or otherwise $\lim_{\alpha \to \infty} w(E_\alpha) = 1$ for every approximate identity E_α. A state w is mixed if it can be written as a convex linear combination of two (or more) other states: $w = \lambda w_1 + (1-\lambda) w_2$, $w_1 \neq w_2$, $0 < \lambda < 1$. A state which is not mixed is called pure.

The C^*-algebras in which we shall be interested are the so-called "concrete" algebras of operators on a Hilbert space. A subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ is a C^*-algebra if it is a *-subalgebra of $\mathcal{B}(\mathcal{H})$, ie it is closed under addition, multiplication and the adjoint operation, and it is also closed in the norm. The C^*-algebraic approach to quantum mechanics is well-established in the theory of infinite systems and in quantum field theory [Bogolubov et al 1975, Bratteli and Robinson 1979, Emch 1972, Haag and Kastler 1964, Streater 1971]. It is also applicable to finite quantum systems however [Segal 1947]. Indeed the simplest case is to consider as C^*-algebra the von Neumann algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on a Hilbert space $\mathcal{H}$. In this case we recover immediately the usual Hilbert space quantum mechanics if we

restrict the physical states to the normal NPLF's on $\mathcal{B}(\mathcal{H})$, ie [Bratteli and Robinson p76] states w_ρ for which there exists a density operator ρ such that

$$w_\rho(A) = \mathrm{Tr}(\rho A)$$

for all $A \in \mathcal{B}(\mathcal{H})$. We shall indulge quite shamelessly in an abuse of notation when considering the conventional theory and refer to both the normal NPLF's and their corresponding density operators (and even the vectors associated with pure states) as states of the system. No confusion is engendered by this so long as we are in the conventional formalism.

One of the advantages of the algebraic approach with its emphasis on observables rather than states is the ability to delimit the set of observables relevant to a particular system. In section (2.4) we shall consider a particular example of this.

We mention some results concerning tensor products and direct sums of C^*-algebras. Firstly, if $\mathcal{A}_1$ and $\mathcal{A}_2$ are C^*-algebras one can define an algebra $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$, the tensor product of $\mathcal{A}_1$ and $\mathcal{A}_2$ [Bratteli and Robinson p142]. This algebra is again a C^*-algebra and its norm is given by the so-called crossnorm [Vowden, 1974] in which the norm $\|A \otimes B\|$ of an element $A \otimes B$ in $\mathcal{A}$ is given by $\|A\|.\|B\|$. In the conventional theory, the algebra $\mathcal{A}$ is the algebra $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2) = \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ of bounded operators on the tensor product Hilbert

space $\mathcal{H}_1 \otimes \mathcal{H}_2$. [For details of the tensor product of Hilbert spaces see (eg) Weidmann 1980 p48, Prugovecki 1971 p144.]

In addition to the tensor product of Hilbert spaces and of operator algebras we shall also have occasion to consider the direct sum of C^*-algebras and their representation spaces. For any family $\{\mathcal{A}_i ; i \in I\}$ of C^*-algebras where I is some index set, possibly uncountable, we can define the direct sum algebra $\mathcal{A}^{\oplus} = \bigoplus_{i \in I} \mathcal{A}_i$ as the set of all functions $i \in I \to A_i \in \mathcal{A}_i$ which satisfy the condition that $\lim_{i \to \infty} \|A_i\| \to 0$ in the sense that for every $\varepsilon > 0$ there is a finite subset $F \subseteq I$ such that $\|A_i\| < \varepsilon$ for all $i \notin F$. We can make $\mathcal{A}^{\oplus}$ into a C^*-algebra by giving the pointwise operations $\{A_i\} + \{B_i\} = \{A_i + B_i\}$, $\{A_i\}.\{B_i\} = \{A_i B_i\}$ and norm $\|\{A_i\}\| = \sup \|A_i\|$. The simplest case is evidently the direct sum of two operator algebras $\mathcal{A}_1 \oplus \mathcal{A}_2$. Operators in this algebra may be given the convenient matrix notation:

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

in anticipation of their representation as operators on a direct sum of Hilbert spaces. For direct sums of C^*-algebras see Arveson p21; for details of the direct sum of Hilbert spaces see Weidmann p32.

2.3 Two particle Systems and Systems with Spin

Consider two one-particle systems I and II described by Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. The two-particle system which we shall denote by I + II is conventionally described by the tensor product space $\mathcal{H}_c = \mathcal{H}_1 \otimes \mathcal{H}_2$. We shall always use the subscript c to denote the composition of systems. Observables of the two-particle system are the selfadjoint elements of the algebra $\mathcal{B}(\mathcal{H}_c)$. States of the system are given by the density operators ρ on $\mathcal{H}_c$. Note that we cannot in general write $\rho = \rho_1 \otimes \rho_2$ where ρ_1, ρ_2 are density operators on $\mathcal{H}_1, \mathcal{H}_2$ respectively. In chapter 8 we shall see how this fact leads to some peculiarly nonclassical features of two-particle quantum systems. The time development that we shall consider in this thesis for two-particle systems will be non-interactive in the sense that we shall use the tensor product $U_t = U_{1t} \otimes U_{2t}$ of the time evolution operators for the separate particles.

When we wish to introduce discussion of spin observables in a system we shall again be concerned with the tensor product. Conventionally one takes the Hilbert space for a single particle with spin to be the tensor product of the Hilbert space corresponding to the configuration space of the particle, eg $L^2(\mathbb{R})$ for

a particle in one dimension, with a 2j+1-dimensional complex space H_j, say, where j is the spin of the particle. Observables of the spin system are then bounded operators on $L^2(\mathbb{R}) \dot{\otimes} H_j$ and a C^*-algebra of such operators is given by $\mathcal{B}(L^2(\mathbb{R})) \dot{\otimes} \mathcal{S}_j$ where $\mathcal{S}_j = \mathcal{B}(H_j)$ is an algebra which has representation in coordinate bases as the algebra of all $(2j+1)\times(2j+1)$ matrices. We have used the symbol $\dot{\otimes}$ to denote the fact that the tensor product is between a space entity and a spin entity rather than system I and system II. This will prove to lend clarity to the discussion of two particle systems with spin although it should be emphasised that the two tensor products are identical in their formal power. In the work presented in the latter part of the thesis we shall restrict our attention to systems of two particles each having spin 1/2 and hence we shall only consider algebras $\mathcal{S}$ of 2×2 matrices and suppress reference to the particular value 1/2. The appropriate Hilbert space for the spin states of a spin-1/2 system is $\mathbb{C}^2$. In deference to the fact that the relevant algebra of operators for such systems is generated by the Pauli matrices:

$$(2.3.1) \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

together with the identity matrix, we shall denote spin

quantities using the superscript σ. Thus $\mathcal{H}^\sigma$ will denote the product space $\mathcal{H}\dot{\otimes}\mathbb{C}^2$ where $\mathcal{H}$ is the Hilbert space for the spatial aspect of the system. Similarly $\mathcal{A}^\sigma$ will denote a product algebra of the form $\mathcal{A}\dot{\otimes}\mathcal{S}$ where $\mathcal{S}$ is an algebra of spin observables. For details of the tensor product representation of spin systems see, in particular, Mackey 1963 & 1968.

When we come to discuss two-particle systems with spin we shall use tensor products in both the $\otimes$ sense and the $\dot{\otimes}$ sense. In the conventional formulation we would therefore be concerned with a Hilbert space $\mathcal{H}_c^\sigma$ given by:

$$\mathcal{H}_c^\sigma = \mathcal{H}_1^\sigma \otimes \mathcal{H}_2^\sigma = (\mathcal{H}_1 \otimes \mathcal{H}_2) \dot{\otimes} (\mathbb{C}^2 \otimes \mathbb{C}^2) = \mathcal{H}_c \dot{\otimes} \mathbb{C}^4 .$$

For system of two particles each moving in configuration space $\mathbb{R}^n$, we have $\mathcal{H}_c^\sigma = L^2(\mathbb{R}^{2n}) \dot{\otimes} \mathbb{C}^4$. The algebra of observables would be:

$$\mathcal{B}(\mathcal{H}_c^\sigma) = \mathcal{B}(L^2(\mathbb{R}^{2n}) \dot{\otimes} \mathbb{C}^4) = \mathcal{B}(L^2(\mathbb{R}^{2n})) \dot{\otimes} (\mathcal{S} \otimes \mathcal{S})$$

where $\mathcal{S}$ is the algebra of 2×2 matrices generated by $\{\underline{\sigma}, 1\}$. For the spin operators in the algebra $\mathcal{S}_c = \mathcal{S} \otimes \mathcal{S}$ an explicit formulation of the tensor product is given by the Kronecker product, or direct product, of matrices [Marcus and Ming 1964, Wedderburn 1934]. Let $S = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ and $S' = \begin{pmatrix} p' & q' \\ r' & s' \end{pmatrix}$ be arbitrary elements of $\mathcal{S}$. The direct product matrix $S \otimes S'$ is given by:

(2.3.2)

$$\begin{pmatrix} p s' & q s' \\ r s' & s s' \end{pmatrix} = \begin{pmatrix} pp' & pq' & qp' & qq' \\ pr' & ps' & qr' & qs' \\ rp' & rq' & sp' & sq' \\ rr' & rs' & sr' & ss' \end{pmatrix}$$

If we choose a vector basis in $\mathbb{C}^2$ such that α_1, the spin-up eigenvector for spin in the z-direction, say, is represented by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and β_1, the spin-down vector, by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and likewise for system II, then the isotropic spin-zero vector for example $(1/\sqrt{2})(\alpha_1 \otimes \beta_2 - \beta_1 \otimes \alpha_2)$ which as we shall see below (2.7) is of paramount importance in discussing the Einstein Podolski and Rosen gedankenexperiment, is represented by the vector:

$$\begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}$$

in $\mathbb{C}^4$.

2.4 The Wan and Mclean Theory

In a recent series of papers Wan and Mclean [1983(a)&(b), 1984(a),(b)&(c)] have proposed a quantum mechanical theory (WM) which is asymptotically separable in a certain sense and for which the EPR

paradox in the momentum formulation receives an asymptotic resolution.

The algebraic theory for a system of one quantum mechanical particle is obtained by considering the algebra of observables:

(2.4.1) $$\mathcal{A}_{WM} = \mathcal{A}_0^S + L^\infty(\underline{p})$$

where $L^\infty(p)$ is the von Neumann algebra of essentially bounded functions of the momentum operator $\hat{\underline{p}}$ and $\mathcal{A}_0^S$ is a C^*-subalgebra of $\mathcal{B}(L^2(\mathbb{R}^n))$ given by:

(2.4.2) $$\mathcal{A}_0^S = \{A \in \mathcal{B}(L^2(\mathbb{R}^n)) : s^*\text{-}\lim_{t\to\infty} A_t = 0\},$$

where s^*-lim denotes the limit in the strong*-topology on operator algebras as time t tends to infinity [for details of the strong -topology see Guenin 1966, Wan and McLean 1984(a)] and A_t denotes the time evolution $U_t^{-1} A U_t$ of A. States of the system are given in WM by the normal NPLF's on $\mathcal{A}_{WM}$ and also by 'normal states at infinity' w^∞ generated from the normal states w in the following way:

(2.4.3)

$$w^\infty(A) = \lim_{t\to\infty} w(A_t) \qquad \text{for all } A \in \mathcal{A}_{WM}.$$

For an arbitrary operator $A \in \mathcal{A}_{WM}$ we can write:

(2.4.4)

$$A=X+Y \qquad X\in\mathcal{A}_0^S\ ,\ Y\in L^\infty(p).$$

In the WM-theory a two particle system is represented by: $$\mathcal{A}_{WMC} = \mathcal{A}_{oc}^S + L^\infty(\underline{p}_1, \underline{p}_2),$$

where $L^\infty(\underline{p}_1, \underline{p}_2)$ is the von Neumann algebra of all essentially bounded functions of $\underline{p}_1$ and $\underline{p}_2$ and

$$\mathcal{A}_{oc}^S = \{A\in\mathcal{B}(L^2(\mathbb{R}^{2n})): s^*\text{-}\lim_{t\to\infty} U_t^{-1}AU_t = 0\}$$

2.5 Uncertainty and Complementarity

It follows from the Hilbert space formulation of quantum mechanics that for any two observables T_1 and T_2:

(2.5.1)

$$\langle T_1;\varphi\rangle\langle T_2;\varphi\rangle \geq 1/4\,|\langle[T_1,T_2];\varphi\rangle|^2$$

for each $\varphi\in\mathcal{D}([T_1,T_2])\cap\mathcal{D}(T_1^2)\cap\mathcal{D}(T_2^2)$

[Prugovecki Ch.4 Lemma 6.1]. If we now set $T_1=Q-\langle Q;\varphi\rangle$, $T_2=P-\langle P;\varphi\rangle$ for some observables Q,P we obtain from (2.5.1):

(2.5.2)

$$\langle(Q-\langle Q;\varphi\rangle)^2\varphi\rangle\langle(P-\langle P;\varphi\rangle)^2\varphi\rangle \geq 1/4\,|\langle[Q,P];\varphi\rangle|^2$$

for all φ in the appropriate domain $\mathcal{D}$.

The quantities on the left of the inequality (2.5.1) are commonly denoted by $(\Delta_\varphi Q)^2$ and $(\Delta_\varphi P)^2$ respectively and $\Delta_\varphi Q$, $\Delta_\varphi P$ are called the variances of the observables Q and P. It can be shown [Lahti 1980] that the variance of an observable is zero if and only if either the observable is constant or the vector state is an eigenvector of the operator, so that the quantity $\Delta_\varphi Q$ gives a measure of the dispersion of values for Q in the state φ. As a particular case we take Q to be the position operator $\hat{\underline{x}}$ defined in (2.1.2) and P to be the momentum operator $\hat{\underline{p}}$ defined in (2.1.3). It is well-known [Prugovecki 1971, eg] that the position and momentum operators satisfy the commutation relations

(2.5.3)

$$[\underline{x}, \hat{\underline{p}}] \subseteq i\hbar 1$$

and it follows that (2.5.2) yields:

(2.5.4)

$$\Delta_\varphi \underline{x} . \Delta_\varphi \hat{\underline{p}} \geq \hbar/2 \quad \text{for all} \quad \varphi \in \mathcal{D} .$$

This is a mathematical derivation of the relations known as the Heisenberg uncertainty relations which received a lot of attention and several attempts at heuristic justification in the early history of the theory. We should point out though that such heuristic justifications for the alleged impossibility of

measuring position and momentum to arbitrary accuracy are strictly independent of the operator relations (2.5.4). The relations (2.5.3) are called the canonical commutation relations. A stronger form of these is given by the Weyl relations:

(2.5.5)

$$\exp[iQt]\exp[iPs] = \exp[-its]\exp[iPs]\exp[iQt]$$

[For a proof that (2.5.5) implies (2.5.3) but not the converse see Prugovecki Chapter 4 Theorem 6.3.] It is clear that any pair of operators satisfying (2.5.5) also satisfies (2.5.4). The restriction in the domain in (2.5.4) is rather important for it is easy enough to find examples for which the inequality is violated if φ is not in $\mathcal{D}$.(Consider "position" and "momentum" operators in a box [-L/2,L/2] and take φ as any eigenfunction of the "momentum" operator.)

The difficulty with domains for unbounded operators has led to the use of bounded operators exclusively [cf(2.2)]. It is of interest to note that in such a theory the uncertainty relations cannot be formulated. For in such a system there are no pairs of observables A and B which satisfy $\Delta_\varphi A.\Delta_\varphi B \geq c > 0$ for all φ in $\mathcal{D}$. To see this we note that $\mathcal{D} = \mathcal{H}$ and for any bounded A, $(\Delta_\varphi A)^2$ is bounded by $2\|A\|$. Hence we can choose φ to be an eigenfunction of B and obtain $\Delta_\varphi A.\Delta_\varphi B = 0$. This is not in conflict with (2.5.2) but

we observe that if A and B are bounded [A,B] cannot be any constant observable apart from zero [Lahti 1980]. In chapter 6 we shall have occasion to see how this provides a clear indication that heuristic uncertainty and the operator relations (2.5.4) are independent issues.

There is another rather important result which is too often ignored in heuristic discussions of experimental situations. This is the so called complementarity theorem.

(2.5.6) Theorem

Let Q and P be any irreducible, canonically conjugate pair of observables, ie Q and P satisfy the Weyl relations (2.5.5). Then their spectral measures $E(Q;\Lambda)$ and $E(P;\Delta)$ satisfy:

$$E(Q;\Lambda)_{\cap} E(P;\Delta)=0$$

for any finite Borel subsets Λ and Δ of $\mathbb{R}^n$, where $\cap$ is the intersection in the lattice of projections.

Proof: cf Lahti 1980.

The physical consequences of this fact play a large part in the motivation for our consideration of local observables (Chapters 3 and 4).

2.6 Probability and Measurement

In a classical probability space $(\Omega, \mathcal{F}, \mu)$ events are represented by the elements a of the Boolean σ-algebra of subsets $\mathcal{F}$ on the probability space Ω. We define the probability of a particular event a by the measure of the subset representing the event. That is, the probability, $\langle a;;\mu\rangle$, of the event a with respect to the measure μ is given by

(2.6.1)

$$\langle a;;\mu\rangle = \mu(a) = \int \chi_a \, d\mu$$

where χ_a is the characteristic function of the set a and (by abuse of language) we identify events with their representative sets. One also defines the expectation value $\langle A;\mu\rangle$ for a random variable A on Ω measurable with respect to μ by [Moy 1954,eg]:

(2.6.2)

$$\langle A;\mu\rangle = \int A(x) d\mu(x)$$

where A(x) is the value of A at x and $d\mu(x) = \mu((x, x+dx))$. Clearly $\langle .;;\mu\rangle$ is a map from the field of subsets of Ω into the interval [0,1] and $\langle A;\mu\rangle$ is a map from the set $\mathcal{A}$ of random variables on Ω into $\mathbb{R}$. However, there is a certain subset of these random variables which is in one-one correspondence with $\mathcal{F}$, namely the set $\mathcal{X}$ of characteristic functions. The

correspondence is obvious; $a \leftrightarrow \chi_a$, $a \in \mathcal{F}, \chi_a \in \mathcal{X}$. Now the restriction of $\langle . ; \mu \rangle$ to the set $\mathcal{X}$ yields the probabilities by:

(2.6.3)

$$\langle \chi_a ; \mu \rangle = \langle a ; ; \mu \rangle$$

Furthermore, using (2.6.1) we can write $d\mu(x) = \langle (x, x+dx) ; ; \mu \rangle$ and we have:

(2.6.4)

$$\langle A ; \mu \rangle = \int A(x) \langle (x, x+dx) ; ; \mu \rangle$$

Thus expectations may be regarded as weighted averages of the possible values of A, the weights corresponding to the probabilities assigned by the measure to (an infinitesimal set including) the point at which A takes these values. This becomes clearer in the discrete case. Here A can be characterised by its effect on certain "atomic" events $x_i \in \mathcal{F}$, and if $\chi_i = \chi_{\{x_i\}}$, $A_i = A(x_i)$, we have $A = \sum_i A_i \chi_i$. It follows that

(2.6.5)

$$\langle A ; \mu \rangle = \int \sum A_i \chi_i \, d\mu = \sum_i A_i \mu(\chi_i) = \sum_i A_i \langle x_i ; ; \mu \rangle$$

and the expectation value is a sum of the values A taken on each discrete event x_i weighted by the probability $\langle x_i ; \mu \rangle$ of that event. The concepts of expectation and probability are interderivable then, and the duality between them is expressed by (2.6.3) in classical probability theory.

In quantum mechanics one may introduce, axiomatically, a probability measure $\langle .;;\rho\rangle$ on the lattice $\mathcal{L}(\mathcal{H})$ [we use the same notation $\mathcal{L}(\mathcal{H})$ as for the lattice of projections since the two lattices are isomorphic] of subspaces of $\mathcal{H}$ given by:

(2.6.6)

$$\langle M;;\rho\rangle = \mathrm{Tr}(\rho E_M)$$

where ρ is a density operator (and hence defines a normal state on $\mathcal{B}(\mathcal{H})$), $M \in \mathcal{L}(\mathcal{H})$ and E_M is the projection onto the subspace M. Expectations are of course defined by:

(2.6.7)

$$\langle A;\rho\rangle = \mathrm{Tr}(\rho A) \quad \text{for each } A \in \mathcal{B}(\mathcal{H}).$$

Clearly the duality expressed by (2.6.3) is mimicked in the quantum mechanical theory:

(2.6.8)

$$\langle M;;\rho\rangle = \langle E_M;\rho\rangle$$

so that if the map $\langle .;\rho\rangle : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ is restricted to the set of idempotent operators, ie the projections, then the expectation values and the probabilities coincide. Moreover we can derive expectation values from probabilities in the sense that:

(2.6.9)

$$\begin{aligned}\langle A;\rho\rangle &= \mathrm{Tr}(\rho A) \\ &= \mathrm{Tr}\left(\rho \int t\,dE(A;t)\right)\end{aligned}$$

$$= \int td(Tr(\varrho E(A;t)))$$
$$= \int td\langle E(A;t)\mathcal{H};;\varrho\rangle .$$

Now, $E(A;t)\mathcal{H}$ is the subspace of $\mathcal{H}$ on which A takes a value in $(-\infty,t)$ so that we have an analogous interpretation to the classical case, namely that expectation values are weighted averages, with the weights being given by the probability measure. The correspondence between subspaces and projections, and between projections and observables (via the spectral theorem (2.1.5)) allows us to interpret these probability measures on $\mathcal{H}$ in the following way [Mackey 1963, von Neumann 1952, Jauch 1968]: we say that the probability $\langle A;\Delta;\varrho\rangle$ of the observable A taking a value in the set Δ when the system is in state ϱ is given by the probability measure $\langle E(A;\Delta)\mathcal{H};;\varrho\rangle$ of the subspace $E(A;\Delta)\mathcal{H}$ and we have therefore:

(2.6.10)

$$\langle A;\Delta;\varrho\rangle = Tr(\varrho E(A;\Delta)) \text{ for each observable A.}$$

The difference between the classical case and the quantum case arises from the rather less straightforward representation of events in the latter case. This in its turn is a consequence of the fact that an observable A in quantum mechanics does not act as a random observable on $\mathcal{H}$ in the same way as for classical observables with values given by A(x), but as an operator whose values are $\sigma(A)$.

Having made the association (2.6.10) it can be shown [von Neumann 1955] that the probability postulate is derivable from the expectation value postulate and we have already seen that the converse is the case. The duality is expressed by (2.6.8) or equivalently by:

(2.6.11)

$$\langle A;\Delta;\rho\rangle=\langle E(A;\Delta);\rho\rangle$$

The probability postulate plays a significant role in the interpretation of quantum mechanics as a physical theory with predictive statements to make about the results of real physical measurements on quantum systems. Another postulate with an important part to play in enabling quantum mechanics to give an account of the physical world is the controversial von Neumann projection postulate. In answer to the question, what is the state of a system after a measurement of an observable T yields a value in the set Δ, von Neumann postulated that the state must satisfy

(2.6.12)

$$E(T;\Delta)\varphi' = \varphi'.$$

Suppose that an observable T has a discrete, nondegenerate spectrum $\sigma(T)$ and that a system undergoes a measurement in which it is ascertained that the value of T is t, say. Then according to (2.6.12) the state

of the system after the measurement is given by P_{φ_t} where φ_t is the eigenvector associated with the eigenvalue t. We shall call a measurement for which the von Neumann projection postulate is assumed to be valid a "measurement of the first kind" [cf Jauch 1968, Piron 1976].

For a degenerate magnitude, no unique eigenvector is selected on measurement. Von Neumann suggests that the state after measurement should be given by $P_t / Tr(P_t)$ where P_t is the projection onto the subspace spanned by the eigenvectors $\{\varphi_t\}$ associated with t . It is pointed out by Bub [1979],eg, that this cannot in general be correct. His reasoning follows an earlier objection by Lüders [1951] who postulates a different generalisation of the von Neumann projection postulate. The Lüders rule is generally taken to be:

(2.6.13)

$$\rho \rightarrow \rho' = E\rho E/Tr(\rho E)$$

where E is the projection corresponding to the measurement. Now the projection postulate is problematical in various ways and has been the subject of continued and extensive discussion [Bub 1979, Furry 1936 & 1966, Jauch 1968, Lüders 1951, Srinivas 1980,eg]. A careful reading of Lüders' paper, (the original was in German, which may explain why it has been misquoted) reveals that he only treats the case of

degenerate discrete spectra and does not in fact make an explicit formulation of (2.6.13) as a general rule. However Lüders, and many others have, implicitly at least espoused such a rule, and here we shall adopt (2.6.13) as the Luders rule and we shall suppose that at least for a certain class of systems we may regard this rule as appropriate for the change of state through measurement. We shall call a measurement for which the Lüders rule holds an "ideal measurement of the first kind" [following Jauch eg].

2.7 The Einstein, Podolski and Rosen Experiment

In 1935, Einstein, Podolski and Rosen published a now famous paper in which they proposed a thought experiment designed to prove the incompleteness of quantum mechanics. Their argument was based on the fundamental assumption that if it is possible to measure the value of a certain quantity on a system "without in any way disturbing the system", then that quantity constitutes "an element of the physical reality". They then attempted to show that quantum mechanics was incomplete by demonstrating the existence of such an "element of reality" not completely described by quantum mechanics. The essence of their argument was to consider a two-particle system in which

some physical quantity (momentum in the original example) is conserved. By supposing that this quantity is still conserved when the two particles in the system have separated to an arbitrarily large distance EPR maintain that the value of the quantity in question for one of the systems may be measured without disturbing that system by measuring the same quantity for the other system and then using the conservation law. It follows, according to the EPR argument, that this quantity is an element of the physical reality, and as such must have had the measured value before the measurement occurred. This is in sharp contrast to the quantum mechanical analysis in which no definite value for such a quantity exists before measurement. EPR claim that this exhibits the incompleteness of the quantum theory.

The EPR experiment has been formulated by Bohm [1951 & 1957] in terms of the spin of a system of two spin-1/2 particles for which the total spin is conserved as zero. The spin vector for this system which is isotropic (ie invariant under a change of spin axes) is given by:

$$(2.7.1) \qquad \gamma_0 = (1/\sqrt{2})(\alpha_1 \otimes \beta_2 - \beta_1 \otimes \alpha_2)$$

where α_i and β_i are as given in (2.3). The "paradox" that arises from the EPR analysis is that the quantum

mechanical state for which the value of the spin is definite (but unknown) before the measurement occurred (as EPR suggest) is in fact the mixture:

$$(2.7.2) \qquad (1/2)P_{\alpha_1 \otimes \beta_2} + (1/2)P_{\beta_1 \otimes \alpha_2}$$

in contrast to the state (2.7.1) which is a pure state. Now there are two points of interest here. Firstly there is the question: how does the pure state (2.7.1) evolve to a mixed state or if this does not happen which of the two states is the correct description ? Secondly, if the correct state is a mixture, that is a state corresponding to an incomplete knowledge of the system, then is it possible to provide further parameters not as yet belonging to the quantum mechanical description which together with the quantum state determine completely the values for the system? The Copenhagen answer is to deny the validity of the EPR reasoning by attacking the clause "without in any way disturbing the system". A measurement on one subsystem, they claim, necessarily disturbs the other subsystem and hence invalidates the EPR argument. In answer to the hidden variables question they maintain that no further parameters can exist which add anything to the quantum mechanical information about the system. Now we regard this view as extreme for it entails a very high degree of nonlocality for quantum systems, a nonlocality which is not confined to small

spatial regions but extends over the whole configuration space $\mathbb{R}^n$. Although there is quite plainly evidence of quantum mechanical correlation between systems within the confines of a particular laboratory apparatus, for example, this does not seem to us to necessitate the complete denial of locality on a macroscopic level. In this thesis we shall attempt to mitigate the extremity of the Copenhagen position by providing an indication of how and where correlations between subsystems occur and showing that it is possible to provide a quantum mechanics which is separable when the two subsystems are appropriately separated. Having achieved this, we shall then re-examine the controversial hidden variables question.

CHAPTER 3

LOCALISATION OF BOUNDED OBSERVABLES

3.1 Introduction and Motivation

The theory of quantum mechanics is disturbingly nonlocal. This much, as we have already mentioned, was realised rather early on in the history of the subject. Nevertheless there is, generally speaking, something undeniably local about the act of physical measurement using real physical apparatus. An experiment designed to carry out a measurement on an arbitrary quantum mechanical system is performed in a definite physical locality. The apparatus, being essentially classical, is localised in some spatial region and can scan at most a finite region Λ in the finite time taken to perform the measurement. We shall call this region Λ the size of the apparatus. Given these physical limitations of the measurement apparatus, it seems reasonable to make the following assumption concerning such measurement situations.

(3.1.1) Measurement Postulate

A measuring apparatus or device of finite size Λ cannot detect a particle lying outside Λ.

Now let us consider a quantum system moving in configuration space $\mathbb{R}^n$ with Hilbert space given by $\mathcal{H} = L^2(\mathbb{R}^n)$. $\mathcal{B}(\mathcal{H})$ will denote, as above, the algebra of all bounded operators on $\mathcal{H}$. Let $\mathcal{B}(\mathbb{R}^n)$ denote the set

of all Borel subsets on $\mathbb{R}^n$ and let $\mathcal{B}_c(\mathbb{R}^n)$ denote the set of all such Borel sets of compact closure in $\mathbb{R}^n$. Given Λ in $\mathcal{B}(\mathbb{R}^n)$, we shall denote by $\Lambda^\perp$ the complement of Λ in $\mathbb{R}^n$, and by $\mathcal{H}(\Lambda)$ the subspace $L^2(\Lambda)$ of $L^2(\mathbb{R}^n)$. Note that the orthogonal complement $\mathcal{H}(\Lambda)^\perp$ of $\mathcal{H}(\Lambda)$ in $\mathcal{H}$ equals $\mathcal{H}(\Lambda^\perp)$.

We wish to construct a quantum mechanical theory for describing measurements using physical apparatus of finite size Λ, so we must ask ourselves the following question: which observables - selfadjoint linear operators - are the appropriate ones for the description of such a situation? Now, bearing in mind that the physically meaningful values in the quantum mechanical description of the world are the expectation values (and probabilities), the requirement (3.1.1) suggests that we should impose the following limitation on the set of observables measurable using finite apparatus:

(3.1.2) All observables measurable using apparatus of finite size Λ must satisfy:

$$\langle\psi|A\psi\rangle=0 \text{ for all } \psi\in\mathcal{H}(\Lambda^\perp)$$

In this chapter we shall construct a certain class of observables which for obvious reasons we shall call local observables. It will turn out that these observables always satisfy (3.1.2). For the case of

bounded observables we can show additionally (Theorem (3.2.2)) that the local observables exhaust the set of observables which satisfy (3.1.2). By introducing a concept of global relatedness between observables we establish a correspondence between an arbitrary observable in $\mathcal{B}(\mathcal{H})$ and a suitable family of globally related local observables. This enables us to consider using only local observables and hence satisfying the measurement postulate (3.1.1). We investigate the convergence properties of such a family of observables in the limit of large apparatus size and discuss an appropriate measurement scheme from a probabilistic viewpoint.

3.2 Local Operators and Local Observables

(3.2.1) Definition

An operator T on $\mathcal{H}$ is called a local operator if it satisfies:

(L_Λ)

$$T=E(\underline{x};\Lambda)TE(\underline{x};\Lambda) \text{ for some } \Lambda\in\mathcal{B}_c(\mathbb{R}^n)$$

A bounded operator $A\in\mathcal{B}(\mathcal{H})$ satisfying (L_Λ) is called a bounded local operator or L_Λ-operator for short. The corresponding terms for selfadjoint operators are local observable and bounded local observable

(L_Λ-observable), respectively.

(3.2.2) Theorem

Every selfadjoint $A \in \mathcal{B}(\mathcal{H})$ satisfying (3.1.2) is an L_Λ-observable. Conversely, every L_Λ-observable satisfies (3.1.2).

Proof:

Firstly, (3.1.2) implies that $A\psi=0$, $\psi \in \mathcal{H}(\Lambda^\perp)$. hence for any φ in $\mathcal{H}$ we have $AE(\underline{x};\Lambda^\perp)\varphi = 0$ and therefore

$$E(\underline{x};\Lambda)AE(\underline{x};\Lambda^\perp)=E(\underline{x};\Lambda^\perp)AE(\underline{x};\Lambda^\perp)=0.$$

Since A is selfadjoint it follows that:

$$0=(E(\underline{x};\Lambda)AE(\underline{x};\Lambda^\perp)) = E(\underline{x};\Lambda^\perp)AE(\underline{x};\Lambda)$$

Now the result follows easily since:

$$\begin{aligned} A &= (E(\underline{x};\Lambda)+E(\underline{x};\Lambda^\perp))A(E(\underline{x};\Lambda)+E(\underline{x};\Lambda^\perp)) \\ &= E(\underline{x};\Lambda)AE(\underline{x};\Lambda)+E(\underline{x};\Lambda)AE(\underline{x};\Lambda^\perp) \\ &\quad +E(\underline{x};\Lambda^\perp)AE(\underline{x};\Lambda)+E(\underline{x};\Lambda^\perp)AE(\underline{x};\Lambda^\perp) \\ &= E(\underline{x};\Lambda)AE(\underline{x};\Lambda). \end{aligned}$$

The converse is trivial.

3.3 Localisation of Bounded Observables.

We have ascertained that L_Λ-observables are unique in satisfying the physical requirements imposed by the finite size of the measurement apparatus. In quantum mechanics, however, observables (even bounded ones) do not generally satisfy (L_Λ). If we wish to modify the

conventional theory to allow for a local analysis we must try to provide observables which do satisfy (L_Λ). For bounded observables this turns out to be relatively straightforward.

(3.3.1) Definition

Let $A \in \mathcal{B}(\mathcal{H})$. The localisation A_Λ of A in Λ is defined by:

$$A_\Lambda = E(\underline{x};\Lambda) A E(\underline{x};\Lambda), \quad \Lambda \in \mathcal{B}_c(\mathbb{R}^n).$$

It is evident that the localisation in Λ of a bounded operator A is an L_Λ-operator by virtue of the idempotency of $E(\underline{x};\Lambda)$. Moreover, if A is a selfadjoint operator, then A_Λ is also selfadjoint and hence the localisation in Λ of each selfadjoint A in $\mathcal{B}(\mathcal{H})$ is an L_Λ-observable.

(3.3.2) Definition

We shall call an observable global if it is not local.

The term global is not merely a contrast to the concept of the local observable. As we have already mentioned, quantum mechanical observables are not generally local in the sense of (L_Λ) and this is due to the fact that they are influenced by the global characteristics of the configuration space in which the particle moves [Wan and McFarlane 1980, Wan and Viazminsky 1977].

3.4 Globally-Related Families of Local Operators.

Having shown that for each bounded operator on $\mathcal{H}$ we can construct an operator A which is local in Λ in the sense of (L_Λ), we now attempt to carry through our programme of replacing the usual observables with local observables by associating each $A \in \mathcal{B}(\mathcal{H})$ with a certain sort of family $\mathcal{A}$ of local observables.

(3.4.1) Definition

Denote by $\mathcal{A}_\Lambda$ the set:

$$\mathcal{A}_\Lambda = \{A \in \mathcal{B}(\mathcal{H}) : A \text{ is an } L_\Lambda\text{-operator}\}.$$

A map:

$$\mathcal{A} : \mathcal{B}_c(\mathbb{R}^n) \to \bigcup_\Lambda \mathcal{A}_\Lambda \text{ by}$$

$$\Lambda \to \mathcal{A}(\Lambda) \stackrel{(def)}{=} A_\Lambda \in \mathcal{A}_\Lambda$$

is called a bounded globally-related family of local operators if $\mathcal{A}$ satisfies:

(i) the boundedness condition: there exists a number $M \in \mathbb{R}$ such that:

$$\| A_\Lambda \| < M \text{ for all } \Lambda \in \mathcal{B}_c(\mathbb{R}^n),$$

(ii) the isotony condition (global-relatedness property): $A_{\Lambda'}$ is the localisation of $A_{\Lambda''}$ whenever $\Lambda' \subseteq \Lambda''$, or equivalently for any Λ', Λ'' in $\mathcal{B}_c(\mathbb{R}^n)$:

$$E(\underline{x};\Lambda) A_{\Lambda'} E(\underline{x};\Lambda) = E(\underline{x};\Lambda) A_{\Lambda''} E(\underline{x};\Lambda)$$

for all $\Lambda \subseteq \Lambda' \cap \Lambda''$.

The boundedness condition (i) of definition (3.4.1) is a technical point to ground the discussion firmly in the realms of bounded observables. The physical content of the definition is embodied in the condition (ii). Essentially the global-relatedness property will enable us to regard each member of such a family as corresponding to the same physical quantity measured using a measuring device of a certain size. In terms of such a description, the condition ensures that whenever the spatial domains of the two operators overlap the corresponding globally-related observables agree on the common part of the domains. The result linking such families to the usual global observables is:

(3.4.2) Theorem

A map: $\mathcal{A} : \mathcal{B}_c(\mathbb{R}^n) \to \bigcup_{\Lambda} \mathcal{A}_\Lambda$

is a bounded globally-related family of local operators if and only if there is a unique $A \in \mathcal{B}(\mathcal{H})$ such that $\mathcal{A}_\Lambda$ is the localisation A_Λ of A in Λ.

Proof

Firstly, any A in $\mathcal{B}(\mathcal{H})$ obviously defines a unique bounded globally-related family of local operators by $\mathcal{A}_\Lambda = A_\Lambda$. To prove the converse let $\{\Lambda_i\}$ be a monotonically increasing sequence of subsets in $\mathcal{B}_c(\mathbb{R}^n)$ converging to $\mathbb{R}^n$. Given a bounded globally-related family of local operators write $\mathcal{A}_j = \mathcal{A}_{\Lambda_j}$ and E_j =

$E(\underline{x}:\Lambda_j)$. For each pair φ,ψ in $C_o^\infty(\mathbb{R}^n)$ there exists j_o (depending on φ,ψ) such that, for all $j>j_o$ we have $\Lambda_j \supseteq \text{supp}\,\varphi$ and $\Lambda_j \supset \text{supp}\,\psi$. Now consider the sequence $\langle A_j\varphi|\psi\rangle$; we have, for $k \geq j > j_o$:

$$\langle A_j\varphi|\psi\rangle - \langle A_k\varphi|\psi\rangle = \langle E_j A_k E_j\varphi|\psi\rangle - \langle A_k\varphi|\psi\rangle$$
$$= \langle A_k E_j\varphi|E_j\psi\rangle - \langle A_k\varphi|\psi\rangle = 0.$$

This remains true for $j\geq k>j_o$ obviously. We have used (ii) in definition (3.4.1) and the fact that $E_j\varphi = \varphi$ and $E = \psi$ in the above proof. It follows that for each pair φ,ψ in the dense subset $C_o^\infty(\mathbb{R}^n)$ of $\mathcal{H}$ the sequence $\langle A_j\varphi|\psi\rangle$ is a Cauchy sequence. We deduce, by Theorem 4.26 in Weidmann [1980], that A_j is a weak Cauchy sequence and hence converges weakly to some operator A in $\mathcal{B}(\mathcal{H})$. Let us now show that $A_\Lambda = A_\Lambda$ for all $\Lambda\in\mathcal{B}_c(\mathbb{R}^n)$. We have, for any φ,ψ in $\mathcal{H}$:

$$|\langle\varphi|(A_\Lambda - A_\Lambda)\psi\rangle| = |\langle\varphi|E(\underline{x};\Lambda)(A-A_\Lambda)E(\underline{x};\Lambda)\psi\rangle|$$
$$= |\langle E(\underline{x};\Lambda)\varphi|(A-A_\Lambda)E(\underline{x};\Lambda)\psi\rangle|$$
$$= |\langle E(\underline{x};\Lambda)\varphi|(A-A_{\Lambda_k})E(\underline{x};\Lambda)\psi\rangle|$$

for all $\Lambda_k \geq \Lambda$.

We have used the global-relatedness property of A in the last step. Hence:

(1) $\langle\varphi|(A_\Lambda - A_\Lambda)\psi\rangle = 0$ and $A_\Lambda = A_\Lambda$,

since A_{Λ_k} converges weakly to A.

Finally for uniqueness we observe firstly that each weak Cauchy sequence $\{A_{\Lambda_k}\}$ admits a unique limit. Furthermore, suppose a different weak Cauchy sequence A_{Λ_i} with its weak limit A' is chosen, then $A = A'$ since:

$$\langle \varphi | (A'_\Lambda - A_\Lambda)\psi \rangle = \langle \varphi | (\mathcal{A}_\Lambda - \mathcal{A}_\Lambda)\psi \rangle = 0$$

for all Λ by (1).

We can now associate with each bounded operator a unique bounded globally-related family of local operators.

(3.4.3) Theorem

There is a one to one correspondence between the set $\mathcal{B}(\mathcal{H})$ of bounded operators A and the set $\{\mathcal{A}\}$ of all bounded globally-related families of local operators on $\mathcal{H}$ given by:

$$\mathcal{A}_\Lambda = A_\Lambda \quad ; \quad A = \text{s-}\lim_{j\to\infty} \mathcal{A}_{\Lambda_j}$$

where $\{\Lambda_j\}$ is a monotonically-increasing sequence of members of $\mathcal{B}_c(\mathbb{R}^n)$ converging to $\mathbb{R}^n$ as j tends to infinity. [From now on $\{\Lambda_j\}$ will always mean such a sequence.

Proof

By the properties of the spectral measure we have $E(\underline{x};\Lambda_j)$ converges strongly to the identity operator 1 in $\mathcal{H}$ as Λ_j converges monotonically to $\mathbb{R}^n$ [cf (2.1.)]. As in the proof of Theorem (3.4.2), let $\{\mathcal{A}_{\Lambda_j}\}$ be a weak Cauchy sequence with weak limit A. Then $\mathcal{A}_\Lambda = A_\Lambda$ implies that

$$\begin{aligned} \|(A-\mathcal{A}_{\Lambda_j})\varphi\| &= \|(A-A_{\Lambda_j})\varphi\| \\ &\leq \|(A-E(\underline{x};\Lambda_j)A)\varphi + (E(\underline{x};\Lambda_j)A-A_{\Lambda_j})\varphi\| \\ &\leq \|(I-E(\underline{x};\Lambda_j))A\varphi\| + \|A(I-E(\underline{x};\Lambda_j))\varphi\| . \end{aligned}$$

Using the boundedness of A and the result above

concerning $E(\underline{x};\Lambda_j)$ we obtain:

$$\|(A-A_{\Lambda_j})\varphi\| \to 0 \text{ as } \Lambda_j \to \mathbb{R}^n \text{ for all } \varphi \in \mathcal{H},$$

or in other words $\{A_{\Lambda_j}\}$ converges strongly to A.

Finally it is trivial to show that each A in $\mathcal{B}(\mathcal{H})$ defines a $\{\mathcal{A}_\Lambda\}$ by $\mathcal{A}_\Lambda = A_\Lambda$ and that any sequence $\{A_{\Lambda_j}\}$ corresponding to a monotonically increasing sequence $\{\Lambda_j\}$ converging to $\mathbb{R}^n$ has A as its strong limit.

Evidently, Theorems (3.4.2) and (3.4.3) apply in particular to selfadjoint operators. In view of the physical importance of selfadjoint operators as observables of the system, we shall restate the results explicitly in the form of the following theorem.

(3.4.4) Theorem

A map $\mathcal{A} : \mathcal{B}_c(\mathbb{R}^n) \to \bigcup_\Lambda \mathcal{A}_\Lambda$ is a bounded globally-related family of local observables if and only if there is a unique observable $A \in \mathcal{B}(\mathcal{H})$ such that $\mathcal{A}_\Lambda$ is the localisation A_Λ of A in Λ. Further, there is a one to one correspondence between the set of all bounded globally-related families of local observables and the set $\mathcal{B}(\mathcal{H})$ of all bounded observables on $\mathcal{H}$, given by:

$$\mathcal{A}_\Lambda = A_\Lambda \quad ; \quad A = \text{s-}\lim_{j\to\infty} A_{\Lambda_j}$$

Proof

Firstly we observe that the strong limit of a sequence of selfadjoint operators is selfadjoint [Prugovecki

1971 p256]. Also if $A \in \mathcal{B}(\mathcal{H})$ is selfadjoint, so is A_Λ = A_Λ and the result then follows from Theorems (3.4.2) and (3.4.3).

3.5 Convergence

What we have demonstrated is a sort of canonical correspondence between the bounded observables conventionally used to describe a quantum mechanical system and these globally-related families of local observables. In principle we would want to proceed by replacing an arbitrary bounded observable by the relevant local observable for a given situation. By the relevant observable we mean the member of the canonically associated globally- related family which corresponds to the size Λ of the measurement apparatus. In practice of course, we must be able to obtain results which are at least a good approximation to those predicted (usually successfully) by the conventional theory. We demonstrate the following convergence results in the limit of large apparatus size.

(3.5.1) Corollary

Given any $\varphi \in \mathcal{H}$, $A \in \mathcal{B}(\mathcal{H})$ we can choose a $\Lambda \in \mathcal{B}_c(\mathbb{R}^n)$ which renders the difference $\langle\varphi|A\varphi\rangle - \langle\varphi|A_\Lambda\varphi\rangle$ arbitrarily

small. [In other words:expectation values converge.]

Proof

This is a corollary to Theorems (3.4.2) and (3.4.3) for by Theorem (3.4.2) we can construct a bounded globally related family of local operators $\mathcal{A}$ such that $\mathcal{A}_\Lambda = A_\Lambda$. By Theorem (3.4.3) we know that A_Λ converges strongly to A and hence [Prugovecki 1971 p230] A_Λ converges weakly to A and it follows that $\langle\varphi|A\varphi\rangle - \langle\varphi|A_\Lambda\varphi\rangle$ can be made arbitrarily small.

(3.5.2) Theorem

(i) For each value λ in the spectrum of A and for any monotone sequence $\{\Lambda_j\}$ converging to $\mathbb{R}^n$ there exists a sequence $\{\lambda_j\}$ of members of the spectra $\sigma(A_{\Lambda_j})$ of A_{Λ_j} converging to λ, ie there exist $\lambda_j \in \sigma(A_{\Lambda_j})$ such that $\lambda_j \to \lambda$ as $j \to \infty$.

(ii) For every value λ not in the discrete part of the spectrum $\sigma(A)$ of A, we have:

$$\underset{j\to\infty}{s\text{-}\lim}\ E(A_{\Lambda_j};\lambda) = E(A;\lambda).$$

Proof

(i) It can be seen from Weidmann 1980 p282 that the sequences $\{A_{\Lambda_j}\}$ satisfy the conditions specified there for "strong resolvent convergence". The result then follows from Theorem 1.14 p431 in Kato 1966.

(ii) This follows from Theorem 1.15 p432 in Kato's book.

(3.5.4) Theorem

If A_{Λ_j} admits an eigenfunction $\varphi_j \in \mathcal{H}$ with eigenvalue λ_j for each j and if the sequence $\{\lambda_j\}$ of eigenvalues converges to some (real) value λ and the sequence $\{\varphi_j\}$ of eigenfunctions converges uniformly to $\varphi \in \mathcal{H}$ then φ is an eigenfunction of A with eigenvalue λ.

Proof

$$\begin{aligned}\|A_{\Lambda_j}\varphi_j - A\varphi\| &= \|A_{\Lambda_j}\varphi_j - A_{\Lambda_j}\varphi + A_{\Lambda_j}\varphi - A\varphi\| \\ &\leq \|A_{\Lambda_j}(\varphi_j - \varphi)\| + \|A_{\Lambda_j}\varphi - A\varphi\| \\ &\leq M\|\varphi_j - \varphi\| + \|A_{\Lambda_j}\varphi - A\varphi\| \to 0 \text{ as } j \to \infty.\end{aligned}$$

Hence: $\underset{j\to\infty}{\text{s-lim}}\, A_{\Lambda_j}\varphi_j = A\varphi$.

$$\begin{aligned}\text{Now } \|A_{\Lambda_j}\varphi_j - \lambda\varphi\| &= \|\lambda_j\varphi_j - \lambda\varphi\| \\ &= \|\lambda_j\varphi_j - \lambda\varphi_j + \lambda\varphi_j - \lambda\varphi\| \\ &\leq \|(\lambda_j - \lambda)\varphi_j\| + |\lambda|\|\varphi_j - \varphi\| \to 0 \text{ as } j \to \infty.\end{aligned}$$

Therefore: $\underset{j\to\infty}{\text{s-lim}}\, A_{\Lambda_j}\varphi_j = \lambda\varphi$ and hence $A\varphi = \lambda\varphi$.

We observe that actually the convergence requirement on λ_j is not crucial in this theorem since the λ_j are bounded and therefore there always exists a convergent subsequence.

The results of the preceding theorems and the corollary indicate how a bounded (global) observable A can be approximated by its localisations in the sense that the spectrum and the expectation values can be approximated by those of its localisations.

3.6 Local Observables and Measurement.

At the beginning of this chapter we postulated (3.1.1) that measuring devices of finite size cannot detect particles lying outside the spatial domain of the apparatus. A mathematical formulation of this requirement enabled us to deduce the necessity of local observables for describing this finite measurement situation.

Another aspect of the measurement situation is of interest when considering the finite size of the measurement device and here again we shall find that local observables play a crucial role. Suppose that an experimenter wishes to study the properties of a quantum mechanical system or prepare a quantum mechanical system in a particular state, perhaps an eigenstate of some bounded observable A, for future study or interaction. He is physically limited in his design by the fact that he has at his disposal only measuring apparatus of some finite size Λ, say. Nevertheless he wants to do the following rather natural kind of things: firstly, he wishes to make a measurement (we restrict our attention here to "measurements of the first kind" [cf (2.6)]) of the observable A which will, according to the von Neumann projection postulate, enable him to say that after the

measurement the system is in a certain eigenstate φ_λ, say, corresponding to the measured value λ of the observable; and secondly, he will often want to be able to say that after the above operation the system remains inside the apparatus; for if it does not his projected future study or interaction will be fruitless, whereas if it does so remain the experimenter can proceed to enact further processes on the prepared system.

Now the question is: does such an observable A exist in quantum mechanics, on which the process outlined above may be fulfilled ? Well firstly we can show that except for certain special cases measurements of local observables do admit this possibility.

(3.6.1) Lemma

Let $A_\Lambda = A(\Lambda) \oplus 0$ be a local observable in Λ, where $A(\Lambda)$ is an observable on $\mathcal{H}(\Lambda)$. Let $b \subseteq \sigma(A_\Lambda)$, $b \in \mathcal{B}(\mathbb{R})$ and suppose that b does not contain zero. Then after measurement of the proposition $E(A_\Lambda; b)$ the system is in Λ.

Proof

The spectral projector $E(A_\Lambda; b)$ can be written as:

$$E(A_\Lambda; b) = E(A(\Lambda); b) \oplus E(0_{\Lambda^\perp}; b)$$

where $0_{\Lambda^\perp}$ signifies the zero operator on $\mathcal{H}(\Lambda^\perp)$. Since $b \not\ni 0$, $E(0_{\Lambda^\perp}; b) = 0$ (2.1.7) and so $E(A_\Lambda; b) = E(A(\Lambda); b)$. According to von Neumann the state φ' after measurement

satisfies $\varphi' = E(A_\Lambda;b)\varphi' = E(A(\Lambda);b)\varphi'$ and now it is clear that $\varphi' = E(\underline{x};\Lambda)\varphi'$ since $A(\Lambda)$ is an operator on $\mathcal{H}(\Lambda)$.

In the event that b contains zero however, $E(0_{\Lambda^\perp};b)$ takes the value unity in $\mathcal{H}(\Lambda^\perp)$ and hence there may be a nonzero part of the wavefunction φ' which lies in $\mathcal{H}(\Lambda^\perp)$. In terms of eigenfunctions we may say that this is because of the existence of trivial eigenfunctions $0\oplus\psi$, $\psi \in \mathcal{H}(\Lambda^\perp)$, of $E(A_\Lambda;b)$. To proceed further with our investigation let us assume for definiteness that the Lüders rule holds [cf (2.6.13)]. Then we have:

(3.6.2) Lemma

A system which is initially in Λ and which is subject to a measurement of any local observable A_Λ remains in Λ after the measurement.

Proof

It suffices to note that $E(\underline{x};\Lambda)$ commutes with $E(A_\Lambda;b)$.

In fact of course for any operator A_Λ which commutes with $E(\underline{x};\Lambda)$ the above result holds.

(3.6.3) Lemma

A system which is initially in Λ and which is subject to any measurement of an observable $A_\Lambda = A(\Lambda)\oplus A(\Lambda^\perp)$ remains in Λ after the measurement.

We can also prove the following with regard to observables of the direct sum form $A(\Lambda)\oplus A(\Lambda^{\perp})$.

(3.6.4) Lemma

An observable A is of the form $A(\Lambda)\oplus A(\Lambda^{\perp})$ if and only if for every initial vector state φ localised in Λ and for every $b\in\mathcal{B}(\mathbb{R})$ measurement corresponding to the set b leaves the system in Λ.

Proof

Let $A = A(\Lambda)\oplus A(\Lambda^{\perp})$; then the necessity follows from (3.6.3) which follows in turn from the preceding discussion.

For the sufficiency we notice that for each $\varphi\in\mathcal{H}(\Lambda)$ and for every $b\in\mathcal{B}(\mathbb{R})$ we have by the Lüders rule:

$$\varphi' = E(A;b)\varphi/\|E(A;b)\varphi\|$$

Then $\varphi' = E(\underline{x};\Lambda)\varphi'$ implies that

$$E(A;b)E(\underline{x};\Lambda)\varphi = E(\underline{x};\Lambda)E(A;b)\varphi$$

for all $\varphi\in\mathcal{H}(\Lambda), b\in\mathcal{B}(\mathbb{R})$.

Therefore for each $\psi\in\mathcal{H}$ it follows that: $E(A;b)E(\underline{x};\Lambda)\psi = E(\underline{x};\Lambda)E(A;b)E(\underline{x};\Lambda)\psi$ for all $b\in\mathcal{B}(\mathbb{R})$.

By taking adjoints we see that:

$$[E(A;b),E(\underline{x};\Lambda)] = 0 \text{ for all } b.$$

Hence A commutes with $E(\underline{x};\Lambda)$ and we have $A = A(\Lambda)\oplus A(\Lambda^{\perp})$.

The relevance of these last results to the discussion on local observables is not purely academic. It is of interest that the general form $A(\Lambda)\oplus A(\Lambda^{\perp})$ arises as a less stringent alternative to local observables. For example, we could have allowed local observables to be quite generally of the form $A(\Lambda)\oplus\lambda 1$ in our formulation without seriously offending the physical requirement (3.1.1). The constant term λ merely reflects in this case the meter setting or zero reading for a particular apparatus, so that the expectation value is given by $\lambda+\langle A(\Lambda);\varphi\rangle$.

Now despite the previous lemmas we have still not succeeded in fulfilling quite generally the experimenter's requirement that the system remain in Λ after measurement. Even restricting our discussion purely to local observables we see from the remarks following Lemma (3.6.1) that certain measurements of local observables appear to offend the requirement in question. As a solution to this problem we propose the following definition and postulate.

(3.6.5) Definition

An L_{Λ}-measurement or local measurement in Λ of a local observable A_{Λ} (or more generally an observable $A(\Lambda)\oplus A(\Lambda^{\perp})$) is a measurement which entails the following sequence of events: first an ideal

measurement of the first kind is carried out in which it is ascertained that the particle is in Λ and then a measurement of the L_Λ-observable is made.

We see that an L_Λ-measurement is a measurement which involves first "catching" the particle in Λ and then performing the measurement. According to the Lüders rule (2.6.13) this is equivalent to performing the measurement on the state $E(\underline{x};\Lambda)\varphi/\|E(\underline{x};\Lambda)\varphi\|$ when the initial state is φ.

We now boldly cut the Gordian knot and make the following assumption concerning finite measurement situations.

(3.6.6) Postulate
A measuring apparatus of finite size Λ makes L_Λ-measurements on L_Λ-observables.

It follows from this postulate and Lemma (3.6.4) that the requirement expressed at the beginning of this section has now been fulfilled.

One way of regarding the concept of L_Λ-measurement is to see it in terms of keeping track of the statistics when a quantum system becomes confined in a finite spatial region. In the next section we examine those statistics in more detail.

3.7 L_Λ-Probabilities and L_Λ-Expectation Values.

The concept of L_Λ-measurement induces concepts of L_Λ-probabilities and L_Λ-expectation values which differ from the usual probabilities and expectation values. We define the L_Λ-probability of obtaining an answer yes for a particular question as the probability of obtaining the answer yes to that question on L_Λ-measurement and we denote by $\langle A_\Lambda ; b ; \varrho \rangle_\Lambda$ the L_Λ-probability of obtaining a value for A_Λ in the set b when the state is ϱ. Consideration of the definition (3.6.5) and the remarks following it leads us to formulate L_Λ-probability by:

$$(3.7.1) \qquad \langle A_\Lambda ; b ; \varrho \rangle_\Lambda = \langle A_\Lambda ; b ; \varrho_\Lambda \rangle$$

where $\varrho_\Lambda = E(\underline{x};\Lambda)\varrho E(\underline{x};\Lambda)/Tr(E(\underline{x};\Lambda)\varrho)$. If we denote L_Λ-expectation values by $\langle A_\Lambda ; \varrho \rangle_\Lambda$ and derive them from the probabilities in an analogous fashion to the derivation (2.6.4) we obtain:

$$\begin{aligned} (3.7.2) \qquad \langle A_\Lambda ; \varrho \rangle_\Lambda &= \int \lambda \, d\langle A_\Lambda ; (-\infty, \lambda) ; \varrho \rangle_\Lambda \\ &= \int \lambda \, d\langle A_\Lambda ; (-\infty, \lambda) ; \varrho_\Lambda \rangle \\ &= \langle A_\Lambda ; \varrho_\Lambda \rangle \\ &= \langle A_\Lambda ; \varrho \rangle / Tr(E(\underline{x};\Lambda)\varrho) \end{aligned}$$

It is to be noted that the L_Λ-expectation values are not the same as the expectation values in the usual formulation for A_Λ. This rather highlights the physical effect arising from the spatial limitations of the measuring device. Using the results of (3.4) and (3.5) however, we observe that L_Λ-expectation values converge to the usual ones in the limit of large apparatus size as we would hope. Notice also that ρ_Λ is precisely the state predicted by the Luders rule for catching the particle in Λ so that these formulations agree with definition (3.6.5) and the remarks which follow it.

In the probabilistic interpretation of quantum mechanics probabilities and expectations are defined as maps from the lattice of subspaces of a Hilbert space into [0,1] and from the observables on the Hilbert space into the reals, respectively. A duality exists between probability and expectations which is given by (2.6.8). Now the first thing to notice about L_Λ-probability is that it may (most generally) be defined for observables A which commute with $E(\underline{x};\Lambda)$; that is, observables of the form $A(\Lambda)\oplus A(\Lambda^\perp)$; and therefore it does not define a probability measure on the whole lattice of subspaces $\mathcal{L}(\mathcal{H})$. L_Λ-probability does however define a map on the lattice $\mathcal{L}(\mathcal{H}(\Lambda)\oplus\mathcal{H}(\Lambda^\perp))$ of all subspaces $M_\Lambda \oplus M_{\Lambda^\perp}$, $M_\Lambda \subseteq \mathcal{H}(\Lambda)$,

$M_{\Lambda^\perp} \subseteq \mathcal{H}(\Lambda^\perp)$, corresponding to the lattice of projections of observables A_Λ with $[A_\Lambda, E(\underline{x};\Lambda)] = 0$. Notice that if we try to restrict the formalism for the special case of local observables by restricting the lattice of projections to $\mathcal{L}(\mathcal{H}(\Lambda))$ we run into trouble because the projections of a local observable contain the term $E(0_{\Lambda^\perp};b)$ [cf(3.6.1)] which lies in $\mathcal{L}(\mathcal{H}(\Lambda^\perp))$. The more general description seems to be formally the more appropriate and we have:

(3.7.3)

$$\langle .;;\rho\rangle_\Lambda : \mathcal{L}(\mathcal{H}(\Lambda)\oplus\mathcal{H}(\Lambda^\perp)) \to [0,1]$$

$$\langle M;;\rho\rangle_\Lambda = \mathrm{Tr}(M\rho_\Lambda),\quad M\in\mathcal{L}(\mathcal{H}(\Lambda)\oplus\mathcal{H}(\Lambda^\perp))$$

and

$$\langle .;\rho\rangle_\Lambda : \mathcal{B}(\mathcal{H}(\Lambda)\oplus\mathcal{H}(\Lambda^\perp)) \to \mathbb{C}$$

$$\langle A_\Lambda;\rho\rangle_\Lambda = \mathrm{Tr}(A_\Lambda\rho_\Lambda),\quad A_\Lambda\in\mathcal{B}(\mathcal{H}(\Lambda)\oplus\mathcal{H}(\Lambda^\perp))$$

with the duality of (2.6.8) echoed here by:

(3.7.4)

$$\langle M;;\rho\rangle_\Lambda = \langle E_M;\rho\rangle_\Lambda .$$

The point is that the duality (3.7.4) cannot be maintained unless the domains of the formal maps are as given in (3.7.3).

In order to get a slightly different perspective on the L_Λ-probability measure and L_Λ-expectation map, let us consider the classical probability theory again. The conditional expectation value $\langle A;\mathcal{M}\rangle|_a$ of A

with respect to a particular event a, say, is given by [cf Kolmogorov 1950, Moy 1954, Bub 1979]

(3.7.5)

$$\langle A;\mu\rangle|_a = \int_a A d\mu / \int_a d\mu,$$

An alternative expression for (3.7.5) is:

(3.7.6)

$$\langle A;\mu\rangle|_a = \int A \chi_a d\mu / \int \chi_a d\mu,$$

or again:

$$\langle A;\mu\rangle|_a = \int A d\mu_a$$

where $\mu_a(f) = \int \chi_a \chi_f d\mu / \int \chi_a d\mu = \mu(f \wedge a)/\mu(a)$ for all $f \in \mathcal{F}$.

Clearly the measure μ_a corresponds to the conditional probability measure in the conventional sense [Kolmogorov 1950] and again we may derive conditional probabilities from conditional expectations (analogously to (2.6.8)) by:

(3.7.7)

$$\langle f;;\mu\rangle|_a = \langle \chi_f;\mu\rangle|_a , \quad f \in \mathcal{F}.$$

Conditional expectation with respect to the event Ω (or with respect to the field generated by the partition $\{\Omega, \phi\}$) is the usual expectation. From (3.7.6) we observe that:

(3.7.8)

$$\langle A;\mu\rangle|_a = \langle A;\mu_a\rangle$$

and hence we see that conditionalisation induces a map

from the set $\{\mu\}$ of probability measures into itself given by: $\mu \to \mu_a$. Correspondingly we have:

(3.7.9)

$$\langle M;;\mu\rangle|_a = \langle\ ;;\mu_a\rangle.$$

There have been various attempts to extend the concepts of conditional expectation to the quantum case [Bub 1979, Davies and Lewis 1970, Nakamura and Umegaki 1961, Umegaki 1954 & 1956]. The whole thing is complicated by the noncommutativity of the algebra of quantum mechanical magnitudes: not all magnitudes (observables) will commute with the particular event (proposition) with respect to which conditionalisation occurs and the straightforward procedure $\mu \to \mu_a$ of the classical case does not in general work. For the case of local observables however, or more generally the observables of the form $A(\Lambda)\oplus A(\Lambda^{\perp})$, we see that it is possible to conditionalise with respect to the proposition $E(\underline{x};\Lambda)$ since all these observables commute with that proposition and the difficulties outlined above are circumvented. We proceed then by defining the conditional probability with respect to $E(\underline{x};\Lambda)$ on the lattice $\mathcal{L}(\mathcal{H}(\Lambda)\oplus\mathcal{H}(\Lambda^{\perp}))$ by:

(3.7.10)

$$\langle M;;\rho\rangle|_{E(\underline{x};\Lambda)} = \langle E(\underline{x};\Lambda)\mathcal{H}_{\Lambda}\, M;;\rho\rangle / \langle E(\underline{x};\Lambda)\mathcal{H};;\rho\rangle$$

in analogy with (3.7.6). It is then straightforward, given $M = E(A_{\Lambda};b)\mathcal{H}$, say, to show that

$$\langle A_\Lambda; b; \rho \rangle|_{E(\underline{x};\Lambda)} = (\text{by definition}) \langle E(A_\Lambda; b)\mathcal{H}; ; \rho \rangle|_{E(\underline{x};\Lambda)}$$

$$= \langle A_\Lambda; b; \rho_\Lambda \rangle$$

where ρ_Λ is as given by (3.7.1). Now we see immediately that the conditional probabilities deduced by analogy to their classical counterparts are none other than the L_Λ-probabilities defined above. Similarly we can deduce that the conditional expectation values are given by

(3.7.11)

$$\langle A_\Lambda ; \rho \rangle|_{E(\underline{x};\Lambda)} = \langle A_\Lambda ; \rho_\Lambda \rangle$$

and these are exactly the same as the L_Λ-expectation values (3.7.2). Hence we see that L_Λ-probability and L_Λ-expectations may be regarded as conditionalisations with respect to the event that the particle lies in Λ.

3.8 Summary and Remarks

We have defined a concept of local observables appropriate to the physical limitations of the measurement apparatus and a localisation for bounded observables, demonstrated a correspondence between certain families of such observables and the usual bounded observables and shown some important convergence properties of the local observables. Further physical considerations suggest the concepts of

L_Λ-probability and L_Λ-expectation values which turn out to correspond to a conditionalisation with respect to the particle being inside the apparatus. Some algebraic properties of local observables may be found in McLean 1984. Some of the results of this chapter have been published in Wan and Jackson [1984]. In a recent paper de Muynck [1984] has proposed a scheme for local observables which appears on the surface to differ from the one proposed here. Happily it turns out (de Muynck - private communication) that his definition and ours coincide for the case of a single particle as considered here.

CHAPTER 4

LOCALISATION OF MOMENTUM OBSERVABLES

4.1 Introduction

It is a well known consequence of Fourier transform theory [Carrier,Krook and Pearson 1966,eg] that, if Λ and Δ are both bounded Borel sets in $\mathcal{B}(\mathbb{R}^3)$,then the spectral projections $E(\underline{x};\Lambda)$ and $E(\hat{\underline{p}};\Delta)$ for the quantum mechanical position and momentum (respectively) admit no nontrivial solution to the simultaneous equations:

(4.1.1)

$$E(\underline{x};\Lambda)\varphi = \varphi \; ; \; E(\hat{\underline{p}};\Delta)\varphi = \varphi \, .$$

This is just an expression of the complementarity theorem (2.5.6). An immediate physical consequence of this fact is that we cannot simultaneously confine the position and momentum of a particle. Let us consider an ideal measurement of the first kind [cf(2.6)] where the change of state obeys the Lüders rule (2.6.13), and let us suppose that an experimenter wishing to measure the momentum has at his disposal an apparatus of finite size Λ. The state of the system after a measurement yielding a value in the bounded set Δ is given by the Lüders rule as:

(4.1.2)

$$\varphi' = E(\hat{\underline{p}};\Delta)\varphi / \| E(\hat{\underline{p}};\Delta)\varphi \| ,$$

where φ is the initial vector state. Evidently the

state φ' satisfies $\varphi' = E(\hat{\underline{p}}; \Delta)\varphi'$ in agreement with the von Neumann projection postulate (2.6.12) and, by virtue of (4.1.1), the state after measurement φ' can never satisfy $\varphi' = E(\underline{x};\Lambda)\varphi'$. That is, even when the initial wavefunction lies in the spatial domain of the apparatus Λ, the wavefunction φ' after the measurement will no longer be contained in Λ. Indeed $\varphi'(x)$ is non-zero almost everywhere [Carrier, Krook and Pearson 1966, Lahti 1980]. In other words the wavefunction after measurement is spread over the whole of $\hat{\mathbb{R}}$. By the act of measuring the momentum, the experimenter no longer necessarily retains the system he is measuring within the apparatus. Here then is a specific example underlining the motivation of the previous chapter for the introduction of "local observables". There we were able to overcome the problem of an infinitely spreading wavefunction in the measurement situation by introducing a localisation procedure for each bounded observable A given by:

(4.1.3)

$$A_\Lambda = E(\underline{x};\Lambda)AE(\underline{x};\Lambda).$$

This procedure makes sense for bounded observables since A_Λ, so-defined, is selfadjoint and unique. If A is an unbounded observable however we cannot in general proceed in the same fashion because the operator A defined by (4.1.3) would be symmetric but not in general selfadjoint. One may try to define the

localisation of A by looking for selfadjoint extensions of A_Λ. But again, in general this does not work since A_Λ may admit no selfadjoint extensions, or if it does there may be many such extensions, depending on boundary conditions [Bratteli and Robinson 1979 p187, McFarlane and Wan 1981]. In chapter 5 we shall discuss more generally the problem of localising unbounded observables. In this chapter we confine our attention to the momentum observable for which a localisation has already been proposed [McFarlane and Wan 1981, McKenna 1982] which is selfadjoint-preserving. We shall briefly describe this procedure and a similar one for the radial momentum observable and then show how the physical situation may be interpreted in terms of these local observables. We obtain an explicit convergence calculation and illustrate with a diagram how the experimenters requirements are satisfied and the results still agree with the usual quantum mechanical ones within a certain domain.

4.2 Local Momentum Observables

We shall confine our attention here to the linear momentum $\hat{p}$ for a particle moving in the one-dimensional configuration space $\mathbb{R}$. Let $\xi(x)$ be a C^∞- function which vanishes outside the interval $\Lambda = (a,b)$ and which

equals one on a closed interval $\Lambda_o = [a_o, b_o]$ inside (a,b). Notice that such a function $\xi(x)$ exists even if $[a_o, b_o]$ is arbitrarily close to (a,b) [Matsushima 1972 p69,93]. The classical observable $\xi(x)p$ generates the Hamiltonian vector field $X = \xi(\partial/\partial x)$ [Abraham and Marsden p187]. The obvious fact that the support of the vector field X is compact implies that X is a complete vector field [op.cit p70]. Hence $\xi(x)p$ is quantisable in a straightforward manner to give a unique selfadjoint operator [Mackey 1963, Wan and McFarlane 1980, Wan and Viazminsky 1977]:

(4.2.1)

$$\widehat{\xi p} = -i\hbar(\xi d/dx + \tfrac{1}{2} d\xi/dx)$$

with domain:

$$\{\psi \in L^2(\mathbb{R}) : \psi \in AC(X,\mathbb{R}), (\widehat{\xi p})\psi \in L^2(\mathbb{R})\}$$

where $AC(X,\mathbb{R})$ is the set of functions on $\mathbb{R}$ differentiable with respect to X almost everywhere [McFarlane and Wan 1983]. Note that $\widehat{\xi p}$ is the unique selfadjoint extension of the essentially selfadjoint operator:

$$(\widehat{\xi p})_o = -i\hbar(\xi d/dx + \tfrac{1}{2} d\xi/dx)$$

on the domain $C_o^\infty(\mathbb{R})$.

Now we can introduce the following definition.

(4.2.2) Definition

The observable $\hat{\xi}_p$ is called a local momentum observable in Λ with centre of localisation Λ_o and boundaries of localisation $\Lambda - \Lambda_o$ (the two disjoint regions in $\Lambda - \Lambda_o$ comprise the boundaries of localisation).

We now demonstrate formally how the local momentum observables satisfy the definition (3.2.1) to be local observables.

(4.2.3) Lemma

Let $\hat{\hat{\xi}}_p$ denote the operator defined by

$$\hat{\xi}_p \varphi = -i\hbar \xi^{1/2} (d/dx)(\xi^{1/2}\varphi)$$

on the domain $\mathcal{D}(\hat{\xi}_p)$. $\hat{\hat{\xi}}_p$ is a well-defined selfadjoint operator and indeed

$$\hat{\hat{\xi}}_p = \hat{\xi}_p .$$

Proof

Using the differential expression $-i\hbar \xi^{1/2} d/dx \xi^{1/2}$ we have for each φ in $C_o^\infty(\mathbb{R})$

$$\begin{aligned} (\hat{\hat{\xi}}_p)_o \varphi &= -i\hbar \xi^{1/2} d/dx(\xi^{1/2}\varphi) \\ &= -i\hbar \xi^{1/2}(\varphi d(\xi^{1/2})/dx + \xi^{1/2} d\varphi/dx) \\ &= -ih(\xi d/dx + \tfrac{1}{2} d\xi/dx)\varphi \\ &= (\hat{\xi}_p)_o \varphi . \end{aligned}$$

Hence $(\hat{\hat{\xi}}_p)_o = (\hat{\xi}_p)_o$. But $(\hat{\xi}_p)_o$ is an essentially selfadjoint operator on $C_o^\infty(\mathbb{R})$ with a unique

selfadjoint extension given by (4.2.1). It follows that $(\hat{\hat{\xi}}p)_o$ has the same unique extension and hence $\hat{\hat{\xi}}p = \hat{\xi}p$ is the unique selfadjoint extension of $(\hat{\hat{\xi}}p)_o$.

(4.2.4) Theorem

$\hat{\xi}p$ is a local observable in Λ as defined in (3.2.1).

Proof

$\hat{\xi}p$ is formally equivalent to $\hat{\hat{\xi}}p$ by lemma (4.2.3) and since $\xi^{1/2}E(\underline{x};\Lambda) = E(\underline{x};\Lambda)\xi^{1/2} = \xi^{1/2}$ we have :

$$\begin{aligned} E(\underline{x};\Lambda)\hat{\xi}pE(\underline{x};\Lambda) &= E(\underline{x};\Lambda)\hat{\hat{\xi}}pE(\underline{x};\Lambda) \\ &= E(\underline{x};\Lambda)(-i\hbar\xi^{1/2}(d/dx)\xi^{1/2})E(\underline{x};\Lambda) \\ &= \hat{\hat{\xi}}p = \hat{\xi}p. \end{aligned}$$

Given Λ and Λ_o there are infinitely many different local momentum observables $\hat{\xi}p$ in Λ with centre Λ_o. But these local observables differ only in the boundaries $\Lambda-\Lambda_o$, which can be made as small as we please. Although our localisation of momentum is not, strictly speaking, unique therefore, this procedure does produce local momentum observables which possess the essential physical characteristics of local observables.

4.3 Local Radial Momentum Observables

As a further promotion of the idea of local observables, we will show now by means of an explicit example that classical momentum observables hitherto regarded as unquantisable (globally) and consequently discarded [Mackey 1963, Abraham and Marsden 1978 p434] can be quantised locally to obtain meaningful quantum local observables. The best-known classical observable of this kind is perhaps the radial angular momentum p_r [Mackey 1963, Messiah 1961 p346, Dicke and Wittke 1963 p143]. The reason p_r is not quantisable is that its associated vector field $\partial/\partial r$ is not complete. While it may be reasonable for a quantum radial momentum not to exist in a small region containing the origin since p_r is not defined at the origin, it is difficult to see why we should not have a quantum radial momentum in a region far removed from the origin. To be definite, let us consider a two-dimensional configuration space $\mathbb{R}^2$. Let Λ be an open rectangular region in $\mathbb{R}^2$ defined by:

$$\Lambda = \{(x,y): x\in(a,b)\subset(0,\infty), y\in(-c,c), c>0\}.$$

When x is sufficiently larger than $|y|$, the radial momentum p_r and the linear momentum along the x-axis, p_x, are indistinguishable classically. To reflect this classical situation, one would like to introduce a

quantum radial momentum observable $\hat{p}_{r\Lambda}$ localised in Λ such that for wave functions lying entirely within Λ the observables $\hat{p}_{r\Lambda}$ and $\hat{p}_x$ approximate each other. To this end let us consider a closed rectangular region Λ_o within Λ and let $\xi(x,y)$ be a C_o^∞-function on $\mathbb{R}^2$ which equals unity on Λ_o and vanishes outside Λ. Then the classical observable ξp_r generates the vector field $\xi\partial/\partial r$ which is complete. Consequently we can quantise ξp_r to obtain an essentially selfadjoint operator:

(4.3.1)

$$(\widehat{\xi p_r})_o = -i\hbar(\xi\partial/\partial r + (1/2r)\partial(r\xi)/\partial r)$$

on the domain $C_o^\infty(\mathbb{R})$. In the centre of localisation Λ_o, where $\xi = 1$, we have:

(4.3.2)

$$(\widehat{\xi p_r})_o = -i\hbar(\partial/\partial r + 1/2r)$$
$$= \frac{x}{(x^2+y^2)^{1/2}}\hat{p}_x + \frac{y}{(x^2+y^2)^{1/2}}\hat{p}_y - \frac{i\hbar}{(x^2+y^2)^{1/2}}.$$

We can now regard $\widehat{\xi p_r} = (\widehat{\xi p_r})^*$ as a (quantum) local radial momentum observable in Λ with centre of localisation Λ_o. For a region Λ such that a is much bigger than c, we see from expression (4.3.2) that in the centre of localisation $\widehat{\xi p_r}$ can be approximated by $\hat{p}_x$ in that for a C^∞-wave function φ with support in Λ_o we have $\langle\varphi|\widehat{\xi p_r}\varphi\rangle \simeq \langle\varphi|\hat{p}_x\varphi\rangle$.

In the three dimensional case the expression for $(\widehat{\xi p_r})_o$ is given by:

(4.3.3)

$$(\widehat{\xi p_r})_o = -i\hbar(\xi\partial/\partial r + \frac{1}{2r^2\sin\theta}\partial(\xi r^2\sin\theta)/\partial r)$$
$$= -i\hbar(\xi\partial/\partial r + (1/2r^2)\partial(\xi r^2)/\partial r$$

In the localisation centre, therefore, we have:

(4.3.4)

$$(\widehat{\xi p_r})_o = -i\hbar(\partial/\partial r + 1/r)$$
$$= \frac{x}{(x^2+y^2+z^2)^{1/2}}\hat{P}_x + \frac{y}{(x^2+y^2+z^2)^{1/2}}\hat{P}_y + \frac{z}{(x^2+y^2+z^2)^{1/2}}\hat{P}_z - \frac{i\hbar}{(x^2+y^2+z^2)^{1/2}}.$$

The approximation by the linear momentum $\hat{p}_x$ is a straightforward extension of the two dimensional case.

4.4 The Physics of Local Momentum Observables

For simplicity let us confine our attention to one-dimensional systems and consider a local momentum observable $\widehat{\xi p}$ in an open interval $\Lambda = (a,b)$ with centre of localisation $\Lambda_o = [a_o, b_o]$. Intuitively we can see that $\widehat{\xi p}$ should behave like $\hat{p}$ in Λ_o. Such an intuition can be given a precise meaning. To begin with, we can verify without undue difficulty that the following hold:

(i) Given $\varphi \in \mathcal{H}$ in the domain of $\hat{p}$, we have $\widehat{\xi p}\varphi(x) = \hat{p}\varphi(x)$, $x \in \Lambda_0$.

(ii) The generalised eigenfunction of $\widehat{\xi p}$ corresponding to the generalised eigenvalue λ is [McFarlane 1980]:

(4.4.1)

$$F(\lambda,x) = \begin{cases} (2\pi\hbar\xi)^{-\frac{1}{2}} \exp[i\lambda/\hbar)\int_{x_0}^{x} \xi(x')dx'], & x \in \Lambda, \\ 0 & , \ x \notin \Lambda, \end{cases}$$

where $x_0 \in \Lambda$. These functions $F(\lambda,x)$ are not normalisable since they diverge as x tends to a and to b. They satisfy the usual δ-function normalisation for generalised eigenfunctions [cf(2.1)]:

(4.4.2)

$$\int_a^b F^*(\lambda,x)F(\lambda',x)dx = \delta(\lambda-\lambda').$$

Choosing x_0 to lie in Λ_0, we observe that:

$$F(\lambda,x) = (2\pi\hbar)^{-\frac{1}{2}} \exp\{i\lambda(x-x_0)/\hbar\}, \quad x \in \Lambda_0.$$

In other words the function $F(\lambda,x)$ is a plane wave in Λ_0 and hence corresponds in this region to a generalised eigenfunction of (global) momentum $\hat{p}$.

(iii) The spectral function $E(\widehat{\xi p};\lambda)$ of $\widehat{\xi p}$ is given by [McFarlane 1980]:

$$(4.4.3) \quad (E(\widehat{\xi p};\lambda)\varphi)(x) = \int_{-\infty}^{\lambda} d\lambda F(\lambda,x)\int_a^b F^*(\lambda,x')\varphi(x')dx'.$$

Using these results we can prove the following theorem.

(4.4.4) Theorem

Let φ be a member of $\mathcal{H}$ such that $\varphi(x) = 0$, $x \notin \Lambda_0$, and let $\Delta = (\lambda_1, \lambda_2)$ be an interval in $\mathbb{R}$. Then:

(i) $(E(\hat{\xi}p; \Delta)\varphi)(x) = (E(\hat{p}; \Delta)\varphi)(x)$

$= (E(x; \Lambda_0)E(\hat{p}; \Delta)E(x; \Lambda_0)\varphi)(x)$

for all $x \in \Lambda_0$.

(ii) $\|E(\hat{\xi}p; \Delta)\varphi\| = \|E(\hat{p}; \Delta)\varphi\|$.

Proof

Let $\varphi \in L^2(\mathbb{R})$, $\varphi(x) = 0$, $x \notin \Lambda_0$ and let $f(x)$, $F(x)$ be any two vectors in $L^2(\mathbb{R})$ differing by at most a phase factor, say

$$f(x) = \exp[-i\alpha]F(x), \quad x \in \Lambda_0.$$

It follows that

(1) $\langle f|\varphi\rangle = \exp[-i\alpha]\langle F|\varphi\rangle$, $|\langle f|\varphi\rangle|^2 = |\langle F|\varphi\rangle|^2$.

Now for each unbounded selfadjoint operator T, say, in $L^2(\mathbb{R})$ with generalised eigenfunctions $g(\lambda, x)$ satisfying the usual δ-function normalisation we have, for each Borel set Δ of $\mathbb{R}$:

(2)
$$(E(T; \Delta)\varphi)(x) = \int_\Delta g(\lambda, x)\langle g|\varphi\rangle d\lambda,$$

$$\|E(T; \Delta)\varphi\|^2 = \int |\langle g|\varphi\rangle|^2 \, d\lambda.$$

We can apply (1) and (2) to the comparison between

momentum and local momentum. Let $f(\lambda,x) = (2\pi h)^{-\frac{1}{2}}\exp[i\lambda x/\hbar]$ be the generalised momentum eigenfunctions [Byron and Fuller 1969 p286]. It follows that $f(\lambda,x)$ and $F(\lambda,x)$ (as given in (4.4.1)) differ on Λ_o only be the phase factor $\exp[i\lambda x_o/\hbar]$:

$$f(\lambda,x) = \exp[i\lambda x_o/\hbar]F(\lambda,x), \quad x\in\Lambda_o.$$

We deduce from (1) and (2) that:

$$\text{(i)} \quad (E(\widehat{\S p};\Delta)\varphi)(x) = \int_\Delta F(\lambda,x)\langle F|\varphi\rangle d\lambda$$

$$= \int_\Delta f(\lambda,x)\langle f|\varphi\rangle d\lambda, \quad x\in\Lambda_o,$$

$$= (E(p;\Delta)\varphi)(x), \quad x\in\Lambda_o.$$

The equality:

$$(E(\widehat{\S p};\Delta)\varphi)(x) = (E(x;\Lambda_o)E(\hat{p};\Delta)E(x;\Lambda_o)\varphi)(x), x\in\Lambda_o,$$

follows trivially.

$$\text{(ii)} \quad \|E(\widehat{\S p};\Delta)\varphi\|^2 = \int_\Delta |\langle F|\varphi\rangle|^2 \, d\lambda$$

$$= \int_\Delta |\langle f|\varphi\rangle|^2 \, d\lambda$$

$$= \|E(\hat{p};\Delta)\varphi\|^2 .$$

The physical significance of this theorem becomes obvious when we recall that measurements (of the first kind) of the observables $\widehat{\S p}$ and $\hat{p}$ can be reduced to measurements of propositions associated with $\widehat{\S p}$ and $\hat{p}$,

and that these propositions correspond to the projectors $E(\widehat{\xi p}; \Delta)$ and $E(\hat{p}; \Delta)$.

We can carry out some further analysis. Let us see what would happen to the physical system in an initial vector state given by a normalised wave packet φ which vanishes outside Λ_o if a measurement is carried out to obtain a value λ in the range Δ for the global or local momentum.

Firstly, the normalised state functions after measurement of $\widehat{\xi p}$ and $\hat{p}$ are, according to the Lüders rule (2.6.13):

$$E(\widehat{\xi p}; \Delta)\varphi / \| E(\widehat{\xi p}; \Delta)\varphi \| \text{ and } E(\hat{p}; \Delta)\varphi / \| E(\hat{p}; \Delta)\varphi \|,$$

respectively. According to Theorem (4.4.4) these two functions coincide in the centre of localisation Λ_o.

Secondly, it follows from this that the probability $w(\widehat{\xi p}, \Delta, \varphi, \Lambda_o)$ that the particle is in the centre of localisation Λ_o after a local measurement is equal to the corresponding probability $w(\hat{p}, \Delta, \varphi, \Lambda_o)$ after a global measurement. For in the local case the probability is:

$$w(\widehat{\xi p}, \Delta, \varphi, \Lambda_o) = \int_{\Lambda_o} |(E(\widehat{\xi p}; \Delta)\varphi)(x)|^2 \, dx \Big/ \| E(\widehat{\xi p}; \Delta)\varphi \|^2$$

and in the global case it is:

$$w(\hat{p}, \Delta, \varphi, \Lambda_o) = \int_{\Lambda_o} |(E(\hat{p}; \Delta)\varphi)(x)|^2 \, dx \Big/ \| E(\hat{p}; \Delta)\varphi \|^2.$$

We have used the w notation for the probabilities denoted here for ease of handling although of course we could have used the conditional probability notation of (3.7). The probability $w(\hat{\xi}p, \Delta, \varphi, \Lambda_o)$, it is to be noted, is independent of the particular behaviour of the function ξ in the boundaries of localisation $\Lambda - \Lambda_o$ and, moreover, $w(\hat{\xi}p, \Delta, \varphi, \Lambda_o)$ can be as close to unity as we require by enlarging Λ and Λ_o accordingly, since

$$w(\hat{\xi}p, \Delta, \varphi, \Lambda_o) = w(\hat{p}, \Delta, \varphi, \Lambda_o)$$

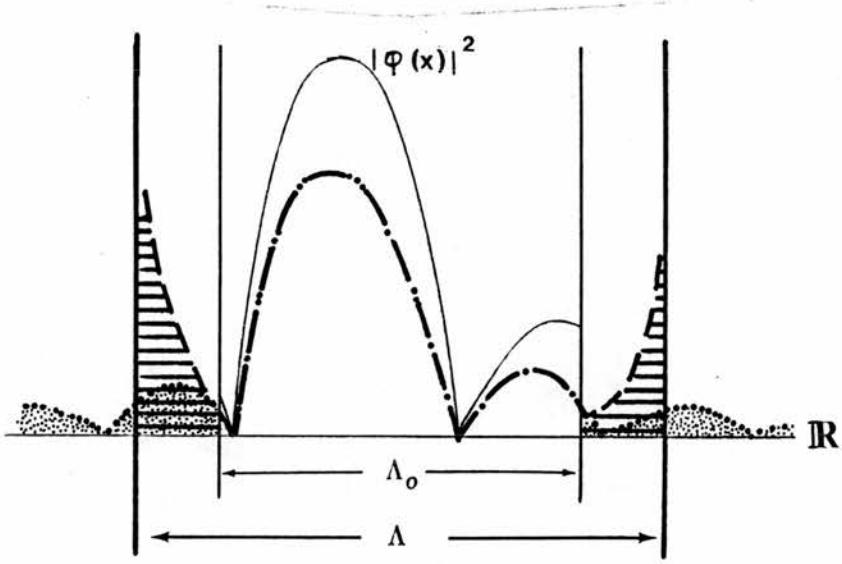
and $w(\hat{p}, \Delta, \varphi, \Lambda_o)$ tends to unity as Λ_o tends to $\mathbb{R}$.

Thus, by taking the measuring device large enough, we can always ensure that there is a probability as close to unity as we please that the particle is in the interior of the apparatus (ie the centre of localisation) after a measurement of the local momentum $\hat{\xi}p$. In other words the probability of finding the particle in the boundary region $\Lambda - \Lambda_o$ (where global and local momentum eigenfunctions differ) can be made as small as we please despite the fact that generalised eigenfunctions of $\hat{\xi}p$ diverge in $\Lambda - \Lambda_o$.

Now it is evident that one may localise an unbounded observable in a certain sense by localising its spectral projectors according to the procedure (3.2.1) for the localisation of bounded observables. The localisation procedure described here is not unrelated to the previous idea as may be seen from Theorem (4.4.4) above. Acting on wave functions localised in Λ_o, the spectral projector $E(\hat{\xi}p;\Delta)$ of the local momentum $\hat{\xi}p$ agrees on Λ_o with the localisation:

$$(E(\hat{p};\Delta))_{\Lambda_o} = E(x;\Lambda_o)E(\hat{p};\Delta)E(x;\Lambda_o)$$

of the spectral projector $E(\hat{p};\Delta)$ of the global momentum $\hat{p}$. The convergence of $w(\hat{\xi}p,\Delta,\varphi,\Lambda_o)$ to unity is, therefore, hardly surprising since this result is equivalent to the convergence of $(E(\hat{p};\Delta))_{\Lambda_o}$ to $E(\hat{p};\Delta)$ in the strong operator topology as Λ_o converges to $\mathbb{R}$. The results discussed above are illustrated in the figure.



(4.4.5) Figure: Comparative effects of local and global momentum measurements on an initial state φ which vanishes outside Λ_o. Notice that the curve $|(E(\hat{\zeta}p;\Delta)\varphi)(x)|^2$ (- - -) for local momentum and the curve $|(E(\hat{p};\Delta)\varphi)(x)|^2$ (...) for global momentum overlap in Λ_o and that the shaded area (corresponding to $1-w(\hat{\zeta}p,\Delta,\varphi,\Lambda_o)$) and the dotted area (corresponding to $1-w(\hat{p},\Delta,\varphi,\Lambda_o)$) are equal. -•-•- is the overlap region.

4.5 Maximisation of Probabilities $w(\widehat{\xi p}, \Delta, \varphi, \Lambda_o)$

Let us carry our analysis a little further to see what kind of magnitude of Λ_o is required to render $w(\widehat{\xi p}, \Delta, \varphi, \Lambda_o)$ close to unity. We can achieve this by considering $w(\hat{p}, \Delta, \varphi, \Lambda_o)$ and examining its relationship to Λ_o and Δ. Specifically for a given "accuracy" Δ in momentum $\hat{p}$ and given Λ_o, we wish to know the maximum possible value w_{max} for the probability w and the kind of wave function φ_{max} which gives rise to this maximum value. The mathematics involved here is well-known in signals analysis [Landau and Pollak 1961 & 1962, Papoulis 1962, Slepian 1964, Slepian and Pollak 1961] in which one is concerned to find the energy of a band-limited signal (momentum-limited here) lying within a certain time (position,here) range.

Let us for convenience choose our position coordinate so that $\Lambda_o = [-T, T]$, $T>0$, and we shall suppose that our momentum range Δ is of the form $(-\Omega + \Omega_o, \Omega_o + \Omega), \Omega > 0$. Our problem is to find the maximum value w_{max}, say, of:

$$w = \int_{\Lambda_o} |(E(\hat{p}; \Delta)\varphi)(x)|^2 \, dx / \|E(\hat{p}; \Delta)\varphi\|^2 .$$

This maximum value is given (Appendix A.1) by the largest eigenvalue μ_o of the integral equation:

$$\mu \Psi(x) = \int_{\Lambda_0} \frac{\sin[\Omega(x-x')/\hbar]}{\pi(x-x')} \Psi(x')dx'$$

The solutions to this equation are a set $\{\Psi_0, \Psi_1, ..\}$ of wave functions complete in the subspace of $\mathcal{H}$ associated with the projector $E(\hat{p};\Delta)$ with corresponding eigenvalues (in descending order of magnitude) $\mu_0, \mu_1, ..$ which are all positive.

These solutions are known as the prolate spheroidal wave functions and it is of particular interest to note that the maximum eigenvalue μ_0 and its corresponding eigenfunction Ψ_0 are dependent on the product $\Omega T/\hbar$. Indeed, as we would expect, $\mu_0 = \mu_0(\Omega T/\hbar)$ is an increasing function of that product. For some idea of the behaviour of μ_0 against $\Omega T/\hbar$ we refer the reader to the works cited above. Here we observe only that, for $\Omega T/\hbar = 4$, say, $w_{max} = \mu_0$ is as large as 0.996. In order to achieve this maximum value, our initial wave function φ_{max} must satisfy, apart from a multiplicative constant (see Appendix A.1):

$$\varphi_{max}(x) = E(x;\Lambda_0)\exp[i\Omega_0 x/\hbar]\Psi_0.$$

Now suppose that we have a measuring device of size Λ which contains and is slightly bigger than the centre of localisation $\Lambda_0 = [-T, T]$. Let us consider a particle of mass m approximately equal to that of an

electron $m = 10^{-27}$ gm, say, with a typical velocity v of about 10^{9} cm s^{-1}. Then $\Delta v/v = \Omega/mv$ represents the (fractional) accuracy in velocity or momentum measurement and we can relate this accuracy to the size of the measuring apparatus when $\Omega T/\hbar = 4$ as follows (take $\hbar \simeq 10^{-27}$ erg s^{-1}):

(4.5.1)

T(cms)	= 40	1	10^{-1}	10^{-4}	10^{-6}
Ω(g cm s^{-1})	= 10^{-28}	4.10^{-27}	4.10^{-26}	4.10^{-23}	4.10^{-21}
Δv(cm s^{-1})	= 10^{-1}	4	40	4.10^{4}	4.10^{6}
$\Delta v/v$	= 10^{-9}	4.10^{-8}	4.10^{-7}	4.10^{-4}	4.10^{-2}.

4.6 Summary and Comments

Local momentum observables have been shown to possess rather pleasing similarities with the global momentum observables within the centre of localisation. For states initially localised in the centre of localisation, measurement of local momentum yields probability densities which are identical in to those given by measurement of the global momentum and the probability that the particle remains in the centre of localisation can be as close to unity as we please by taking the the apparatus large enough. Explicit calculation gives a numerical idea of how large the apparatus needs to be for this to occur. The

results are expressed in table (4.5.1). Two points here are worth commenting on. Firstly, we have solved the experimenter's dilemma concerning simultaneously confining position and momentum, since the simultaneous equations:

$$E(x;\Lambda)\varphi = \varphi \ , \ E(\hat{3}p;\Delta)\varphi = \varphi$$

have many nontrivial solutions. Secondly, we see from the table of values (4.5.1) that with a localisation in a region as small as 10^{-1} cm we can still achieve a very high accuracy in velocity, namely $\Delta v/v$ is of the order 10^{-7}.

The results of this chapter have been published in Wan, Jackson and McKenna [1984].

CHAPTER 5

FURTHER LOCALISATION PROCEDURES

5.1 Introduction

In the two preceding chapters we have discussed the localisation of bounded operators and a localisation for momentum operators. As mentioned in chapter 4, the problem of localising an arbitrary quantum mechanical operator is complicated by the failure of unbounded operators generally to admit a selfadjoint localisation. Explicitly, the localisation $T_\Lambda = E(\underline{x};\Lambda)TE(\underline{x};\Lambda)$ is not always permissible without restricting the domain because $\varphi \in \mathfrak{D}(T)$ does not necessarily imply that $E(\underline{x};\Lambda)\varphi$ belongs to $\mathfrak{D}(T)$. If we restrict the domain to those vectors φ for which $E(\underline{x};\Lambda)\varphi$ does belong to $\mathfrak{D}(T)$ then there is no guarantee that the restricted operator is selfadjoint.

We mention here that when the operator in question is "reduced" by the subspace $\mathfrak{H}(\Lambda)$ in the sense that [Akhiezer and Glasmann 1961, Naimark 1968, Riesz and Nagy 1956]

(5.1.1)

$\mathfrak{H}(\Lambda)$ is an invariant subspace of T, ie

$$\varphi \in \mathfrak{D}(T) \wedge \mathfrak{H}(\Lambda) \Rightarrow T\varphi \in \mathfrak{H}(\Lambda)$$

and

$$E(\underline{x};\Lambda)\mathfrak{D}(T) \subseteq \mathfrak{D}(T),$$

then there does exist a local operator T_Λ in Λ given by

the restriction of T to $\mathcal{H}(\Lambda)$. This operator is selfadjoint (op.cit) and obviously satisfies the definition (3.2.1).

In general however, and this includes the cases of many important unbounded operators, the reduction (5.1.1) does not hold and we must look at other methods of localising unbounded operators. We shall not attempt here anything so extensive as a complete, coherent scheme for such localisation. Rather we shall content ourselves with discussion of one or two areas of the problem suggested by the previous analyses. Particularly, we shall consider a localisation scheme for some quantum mechanical operators in differential form (including Hamiltonian operators) suggested by the localisation $\hat{\mathfrak{z}}_p$ of momentum and we shall examine in more detail the localisation of spectral measures and the convergence of the localised probability measures.

5.2 Localisation of Some Differential Operators

We recall that the local momentum operator $\hat{\mathfrak{z}}_p$ may be written in the form $\hat{\mathfrak{z}}_p = \mathfrak{z}^{1/2}\hat{p}\mathfrak{z}^{1/2}$ [Lemma (4.2.3)]. Now $\mathfrak{z}^2$ is just another C^∞-function which takes the value unity on the same spatial domain Λ_o as does and vanishes smoothly outside the same interval Λ.

Therefore we could equally have considered as momentum observable the operator given by $\chi\hat{p}\chi$ which differs from $\hat{\chi p}$ only in the boundaries $\Lambda-\Lambda_o$ of localisation and obviously corresponds to the operator $\hat{\chi^2 p}$. This suggests that we adopt a localisation scheme of the form $T_\Lambda = \chi T \chi$ for an arbitrary observable T. To make this rigorous we have to show the existence of selfadjoint operators T_Λ when appropriate account is taken of domain constraints. Let us first consider operators given by the formal differential expression

(5.2.1)

$$T_m = (-i)^m d^m/dx^m$$

and the localisation defined by

(5.2.2)

$$(\widehat{\chi T_m})_o \varphi = \chi T_m(\chi\varphi) \; ; \; \mathfrak{D}((\widehat{\chi T_m})_o) = C^\infty(\mathbb{R}).$$

Then we can show the following result.

(5.2.3) Lemma

The operator $(\widehat{\chi T_m})_o$ defined by (5.2.2) is symmetric in $L^2(\mathbb{R})$.

Proof

Firstly we observe that the domain $C^\infty(\mathbb{R})$ of $(\widehat{\chi T_m})_o$ is dense in $L^2(\mathbb{R})$ and that for each φ in the domain, $\chi\varphi$ is in $C_o^\infty(\mathbb{R})$ since χ is a C^∞-function of compact support. χ is real-valued so we have for each φ, ψ in $C^\infty(\mathbb{R})$

$$\langle\varphi|(\widehat{\chi T_m})_o\psi\rangle = \langle\chi\varphi|T_m(\chi\psi)\rangle$$

and symmetry follows from the symmetry of T_m on the domain $C_o^\infty(\mathbb{R})$ [Weidmann 1980 p161].

We now want to demonstrate the existence of selfadjoint extensions of $(\widehat{\S T_m})_o$. We start by proving this for even powers of m.

(5.2.4) Theorem

There exist selfadjoint operators $\widehat{\S T_{2m}}$ corresponding to the formal differential operator $\S T_{2m} \S$.

Proof

We have already demonstrated symmetry for the general case. We show that $(\widehat{\S T_{2m}})_o$ is non-negative. Let $\varphi \in C^\infty(\mathbb{R})$, then $\S\varphi \in C_o^\infty(\mathbb{R})$ and $\langle \varphi | (\widehat{\S T_{2m}})_o \varphi \rangle = \langle \S\varphi | T_{2m} \S\varphi \rangle \geq 0$ since T_{2m} is a non-negative differential form on $C_o^\infty(\mathbb{R})$ [Weidmann Theorem 6.32]. It follows from a general result on semi-bounded operators [Weidmann Theorem 5.38] that there exists a positive selfadjoint extension of $(\widehat{\S T_{2m}})_o$.

For the general case we proceed as follows.

(5.2.5) Definitions [Weidmann 1980 p235 eg]

(a) An operator $K: \mathcal{H} \to \mathcal{H}$ is called a conjugation if for every $\varphi, \psi \in \mathcal{H}$

(i) $K(a\varphi + b\psi) = a^* K(\varphi) + b^* K(\psi)$, $a, b \in \mathbb{C}$

(ii) $K^2 = I$

(iii) $\langle K(\varphi) | K(\psi) \rangle = \langle \varphi | \psi \rangle$.

(b) An operator T is said to be K-real if

(i) $K\mathcal{D}(T) \subseteq \mathcal{D}(T)$

(ii) $TK\varphi = KT\varphi$, $\varphi \in \mathcal{D}(T)$.

(5.2.6) Lemma

Let T be a symmetric operator in $L^2(M)$, $M \subseteq \mathbb{R}$, and suppose that T is K-real for some conjugation K on $L^2(M)$. Then T possesses selfadjoint extensions.

Proof

Weidmann Theorem 8.9 p235.

Hence we prove the result we need.

(5.2.7) Theorem

The operators $(\widehat{\mathfrak{s}T_m})_0$ of (5.2.2) all possess selfadjoint extensions.

Proof

We have already demonstrated the symmetry of the operators $(\widehat{\mathfrak{s}T_m})_0$. In order to show that they possess selfadjoint extensions we shall proceed by finding conjugations K_m with respect to which the $(\widehat{\mathfrak{s}T_m})_0$ are K_m-real. Let us write $T_{m0} = (\widehat{\mathfrak{s}T_m})_0$. Consider the operation

$$K_m\varphi(x) = \varphi((-1)^m x)^* \quad , \quad \varphi \in \mathcal{H}.$$

It is a simple matter to show that K_m is a conjugation. Moreover the domain constraint (5.2.5)(b)(i) is satisfied trivially on $C^\infty(\mathbb{R})$. We now demonstrate that for each φ in $C^\infty(\mathbb{R})$

$$T_{mo} K_m = K_m T_{mo}$$

First notice that by choosing our origin so that $\zeta(x) = \zeta((-1)^m x)$ we immediately deduce:

$$(d\zeta/dx)(x) = (-1)^m (d\zeta/dx)((-1)^m x)$$

$$(d^2\zeta/dx^2)(x) = (d^2\zeta/dx^2)((-1)^m x)$$

and indeed, since ζ is real,

(1)

$$(d^r\zeta/dx^r)(x) = ((-1)^m)^r (d^r\zeta/dx^r)((-1)^m x)$$

$$= ((-1)^m)^r K_m (d^r\zeta/dx^r)(x)$$

or in an abbreviated notation:

$$\zeta^{(r)}(x) = ((-1)^m)^r \zeta^{(r)}((-1)^m x) = ((-1)^m)^r K_m \zeta^{(r)}(x).$$

Now using (5.2.5)(a)(i) we have

(2)

$$i^m K_m \varphi = K_m (-i)^m \varphi$$

We shall also use the result

(3)

$$((-1)^m)^m = (-1)^m \quad , m \text{ an integer} \geq 0,$$

which is easily verifiable by considering even and odd cases separately. Now for each φ in $C^\infty(\mathbb{R})$ we have

$$T_{mo} K_m \varphi(x) = T_{mo} \varphi((-1)^m x)^*$$

$$= \zeta(-i)^m (d^m/dx^m)(\zeta(x)\varphi((-1)^m x)^*)$$

$$= \zeta(-i)^m [\, \zeta^{(m)}(x)\varphi((-1)^m x)^* + m(-1)^m \zeta^{(m-1)}(x)\varphi'((-1)^m x)$$

$$+ \ldots + (m!/(m-r)!r!)((-1)^m)^r \zeta^{(m-r)}(x)\varphi^{(r)}((-1)^m x)^* + \ldots$$

$$+ m((-1)^m)^{m-1}\zeta'(x)\varphi^{(m-1)}((-1)^m x)^* + ((-1)^m)^m \zeta(x)\varphi^{(m)}((-1)^m x)^*]$$

$$= \zeta(-i)^m [((-1)^m)^m \{ \zeta^{(n)}((-1)^n x)\varphi((-1)^{(n)}x) + \ldots$$

$$+ (m!/(m-r)!r!)\zeta^{(m-r)}((-1)^m x)\varphi^{(r)}((-1)^m x) + \ldots$$

$$\ldots + \zeta((-1)^m x)\varphi^{(m)}((-1)^m x) \} \quad \text{(using (1) above)}$$

$$= \zeta i^m K_m [\, \zeta^{(m)}\varphi(x) + m\zeta^{(m-1)}\varphi'(x) + \ldots .$$

$$+(m!/(m-r)!r!)\xi^{(m-r)}\varphi^{(r)}(x)+..$$

$$...+m\xi'\varphi^{(m-1)}(x)+\xi\varphi^{(m)}(x)]$$

(using (3) and the definition of K)

$$= \xi i^m K_m d^m[\xi\varphi]/dx^m$$

$$= K_m \xi(-i)^m d^m[\xi\varphi]/dx$$

(using (2) and the reality of ξ)

$$= K_m T_{m0}\varphi$$

The result now follows from Lemma (5.2.6).

The analysis of the preceding theorem is much simplified when we consider that in fact only two conjugations need be considered: for even powers we can employ the conjugation $K_{2m}\varphi(x) = \varphi^*(x)$ and for odd powers we have $K_{2m-1}\varphi(x) = \varphi^*(-x)$. Also, we only need to bother about the origin relative to ξ in the latter case. The simplest such case is:

(5.2.8) Example

$$T_{10}\, K_1\varphi(x) = \xi(-id/dx)\xi\,\varphi^*(-x)$$

$$= \xi(-i)[-\xi'\varphi(-x) - \xi\varphi'(-x)]^*$$

$$= i\xi K_1(d/dx)(\xi\varphi)$$

$$= K_1 T_{10}\varphi .$$

$(\widehat{\xi T_1})_0$ is K-real therefore and hence may be extended to a selfadjoint operator.

For the even case we again show explicitly that $(\widehat{\xi T_2})_0$ is K-real.

(5.2.9) Example

$$\begin{aligned} T_{20} K_2 \varphi(x) &= T_{20} \varphi^*(x) \\ &= \xi(-d^2/dx^2)(\xi \varphi^*) \\ &= \xi(-d^2/dx^2)(\xi\varphi)^* \\ &= \xi[(-d^2/dx^2)(\xi\varphi)]^* \\ &= K_2 T_{20} \quad , \text{ since } \xi \text{ is real.} \end{aligned}$$

Evidently the operator $(\widehat{\xi T_2})_0$ is one of the most important of these operators in physical terms for it corresponds to a localisation of the free Hamiltonian operator (with an appropriate choice of units to make $\hbar^2/2m = 1$). It turns out that we can extend our existence theorems to some other Hamiltonian operators.

(5.2.10) Theorem

Let H^+ be the Hamiltonian differential operator given by

$$H^+ = -d^2/dx^2 + V(x)$$

where V is a real, positive function of x. Then the operator $(\widehat{\xi H^+})_0$ defined by

$$(\widehat{\xi H^+})_0 \varphi = \xi H^+(\xi\varphi) \text{ for all } \varphi \in \mathfrak{D}((\widehat{\xi H^+})_0) \equiv C^\infty(\mathbb{R})$$

has at least one positive selfadjoint extension.

Proof

The domain of $(\widehat{\zeta H^+})_0$ is dense in $\mathcal{H}$ and the symmetry then follows easily from the symmetry of $-d^2/dx^2$ on $C_0^\infty(\mathbb{R})$ and the reality of V. Positivity of $(\widehat{\zeta H^+})_0$ follows from the positivity of H^+ and we use Weidmann Theorem 5.38 again.

(5.2.11) Theorem

Let H be the Hamiltonian differential operator

$$H = -d^2/dx^2 + V(x)$$

where V is real and continuous and let $(\widehat{\zeta H})_0$ be the operator defined by

$$(\widehat{\zeta H})_0\varphi = \zeta H(\zeta\varphi) \text{ for all } \varphi\in\mathcal{D}((\widehat{\zeta H})_0) = C^\infty(\mathbb{R}).$$

Then there exists at least one selfadjoint extension of $(\widehat{\zeta H})_0$.

Proof

We follow again the method of Theorem (5.2.7). Symmetry of $(\widehat{\zeta H})_0$ on $C^\infty(\mathbb{R})$ follows from the symmetry of H on $C_0^\infty(\mathbb{R})$ and the fact that $C^\infty(\mathbb{R})$ is dense in $\mathcal{H}$. We show that $(\widehat{\zeta H})_0$ is K-real with respect to the conjugation $K = *$:

$$\begin{aligned}
\zeta(-d^2/dx^2 + V)(\zeta K\varphi) &= \zeta(-d^2/dx^2 + V)(\zeta\varphi^*) \\
&= \zeta(-d^2/dx^2 + V)(\zeta\varphi)^* \\
&= \zeta(-d^2/dx^2(\zeta^*\varphi^*)) + (\zeta V\zeta\varphi)^* \\
&\qquad \text{(since V, } \zeta \text{ are real)} \\
&= K[\zeta(-d^2/dx^2 + V)\zeta]\varphi .
\end{aligned}$$

These results go some way to establishing a method of localising various rather important unbounded operators. Of course there is a problem in that there may be many selfadjoint extensions and we shall not know which one to choose. This problem is not however always as severe as it seems at first sight. In certain cases it is possible to show that all these extensions have the same spectrum. For example if an operator has a selfadjoint extension with a purely continuous spectrum then all such extensions have the same spectrum [Weidmann 1980 Theorem 8.18].

We mention here that it is also sometimes possible to form selfadjoint local operators by considering differential expressions on certain finite intervals. As an example of this, the momentum operator -id/dx has selfadjoint extensions on closed intervals [a,b]. However,there are an infinity of different such extensions and they each have a different spectrum which renders this method rather unsatisfactory. We refer the reader to Weidmann 1980, Naimark 1968, or Akhiezer and Glasmann 1961 for further details of selfadjoint differential operators of this nature.

5.3 Localisation and Spectral Measures

The spectral measure of any operator T yields spectral projections E(T;b) which are bounded operators for each $b \in \mathcal{B}(\mathbb{R})$ and hence localisable according to (3.3.1) by $(E(T;b))_\Lambda = E(\underline{x};\Lambda)E(T;b)E(\underline{x};\Lambda)$.

Now consider a bounded operator with a discrete, simple spectrum

(5.3.1)

$$B = \sum_i \lambda_i P_i$$

where $P_i = E(B;\{\lambda_i\}) = |\varphi_i><\varphi_i|$ and φ_i are eigenfunctions of B corresponding to eigenvalues λ_i. We can express the localisations $P_{i\Lambda}$ of the individual projectors in terms of truncated eigenfunctions $\varphi_{i\Lambda} = E(\underline{x};\Lambda)\varphi_i$. Then we have $P_{i\Lambda} = E(\underline{x};\Lambda)P_i\,E(\underline{x};\Lambda) = |\varphi_{i\Lambda}><\varphi_{i\Lambda}|$. If we sum these localised projectors over i we can construct an operator $(B,\Lambda) = \sum_i \lambda_i P_{i\Lambda}$ and not surprisingly it is easy to show that in fact $(B,\Lambda) = B_\Lambda$, the localisation of B in Λ. To generalise this to the continuous case we consider the continuous analogue of (5.3.1),

(5.3.2)

$$B = \int \lambda \, dE_\lambda$$

where E_λ is the spectral function $E(B;(-\infty,\lambda))$ of B. E_λ is a projector for each λ and we can localise these

projections by $(E_\lambda)_\Lambda = E(\underline{x};\Lambda)E_\lambda E(\underline{x};\Lambda)$. Notice that, just as $(E(T;b))_\Lambda$ is not necessarily a projector, neither is $(E_\lambda)_\Lambda$ and so it does not (necessarily) define the spectral family of any operator. Nevertheless we can once again define an operator (B,Λ) by:

$$(5.3.3) \qquad \begin{aligned} \langle\varphi|(B,\Lambda)\psi\rangle &= \int \lambda \, d\langle\varphi|(E_\lambda)_\Lambda\psi\rangle \\ &= \int \lambda \, d\langle\varphi_\Lambda | E_\lambda \psi_\Lambda\rangle \\ &= \langle\varphi_\Lambda|B\,\psi_\Lambda\rangle \\ &= \langle\varphi|B_\Lambda\psi\rangle . \end{aligned}$$

So we have $(B,\Lambda) = B_\Lambda$ quite generally so long as B is a bounded operator. If B is not bounded then the interchange of integrals in (5.3.3) is not generally valid so that we cannot construct the desired selfadjoint operator. Hence this method yields no great insight into the localisation of unbounded operators but is of interest only in the construction of bounded local operators from their localised spectral measures.

Of course, in the spirit of the algebraic approach to quantum mechanics we need in theory never consider unbounded observables at all. We might, for instance, postulate that the algebra of observables relevant to the measurement of a system using an apparatus of finite size Λ is the C^*-algebra generated by the set $\mathcal{A}_\Lambda$

$= \{A_\Lambda \in \mathcal{B}(\mathcal{H}); A_\Lambda = E(\underline{x};\Lambda) A_\Lambda E(\underline{x};\Lambda)\}$ [For the algebraic properties of local observables see Mclean 1984. We shall have further recourse to these properties below (chapters 9 and 10)]

In practice however, we require some reinterpretation of the formalism if we are to make sense of localising unbounded observables in this way. To understand this let us recall that a projection $E(T;b)$ corresponds in the conventional interpretation to the measurement of the observable T. The probability that a measurement of T yields a value in the set b is given by $\|E(T;b)\varphi\|^2$ if the state of the system is φ. Now, of course the localised projections $(E(T;b))_\Lambda$ are in $\mathcal{A}_\Lambda$, but these entities, which are not projections themselves unless T is local, have as yet no physical meaning: $E(\underline{x};\Lambda)E(T;b)E(\underline{x};\Lambda)$ does not correspond to a measurement of a localised operator in the usual way but to the localisation of something corresponding to the measurement of an unlocalised operator!

It is beyond the scope of this thesis to propose a coherent formal theory based on localised projections. There are several remarks however that may be considered pertinent to the discussion. For a start we point out that the localised projections have previously appeared in our discussion in the

formulation of the concept of L_Λ-probability. The projections localised there were projections of operators A_Λ which commuted with $E(\underline{x};\Lambda)$ and this enabled us to regard the L_Λ-probabilities as conditional probabilities in a roughly classical sense. The difficulties of generalising the conditional probability concept to quantum mechanics as a whole have already been mentioned. One possibility might be to formulate quite brazenly a theory in which L_Λ-probability does all the work and one allows arbitrary observables, or at least localised projections of arbitrary observables to represent physical magnitudes in finite measurement situations. The entities $E(\underline{x};\Lambda)E(T;b)E(\underline{x};\Lambda)$ would then always be associated with such a measurement. In fact the localised projections $(E(T;b))_\Lambda$ can be shown to be positive operator-valued measures in the sense of Davies [1976 p16] who has proposed a quantum mechanical theory of measurement in which these measures play the dominant role. It would be of interest to investigate to what extent it is possible to apply Davies' work to the suggestions made here.

In addition to their appearance in the L_Λ-probabilities, the localised projections have also been seen in another aspect of the preceding analysis. Recall from Theorem (4.4.4) and Figure (4.4.5) that the localisation $(E(\hat{p};\Delta))_{\Lambda_o}$ of the spectral projections of

momentum give an indication of the agreement between the local momentum observables and the global momentum observables in the measurement situation. In addition $(E(\hat{p};\Delta))_{\Lambda_0}$ provides a measure of the simultaneous confinement of position and momentum in the sense of section (4.5). We can generalise this latter aspect of the localised projections to an arbitrary observable.

In (4.5) we examined the convergence of the fraction $w(\hat{p},\Delta,\varphi,\Lambda_0)$ given by:

$$w(\hat{p},\Delta,\varphi,\Lambda_0) = \int_{\Lambda_0} |E(\hat{p};\Delta)\varphi|^2 dx \Big/ \|E(\hat{p};\Delta)\varphi\|^2,$$

that is the probability that an initially localised wavefunction remains in Λ_0 after a momentum measurement. We can write:

$$w(\hat{p},\Delta,\varphi,\Lambda_0) = \|(E(\hat{p};\Delta))_{\Lambda_0}\varphi\|^2 \Big/ \|E(\hat{p};\Delta)\varphi\|^2.$$

For an arbitrary operator T, let φ_λ be a complete set of (possibly generalised) eigenvectors. Then

$$E(T;b)\varphi = \int_b |\varphi_\lambda\rangle\langle\varphi_\lambda|\varphi\rangle d\lambda$$

and

$$(E(T;b))_{\Lambda_0}\varphi = \int_b |\varphi_{\lambda\Lambda_0}\rangle\langle\varphi_{\lambda\Lambda_0}|\varphi\rangle d\lambda .$$

We have:

(5.3.5)

$$w(T,b,\varphi,\Lambda_o) = \|(E(T;b))_{\Lambda_o}\varphi\|^2 \Big/ \|E(T;b)\varphi\|^2$$

$$= \int_b\int_b K_{\Lambda_o}(\lambda,\lambda')\tilde{\varphi}(\lambda)\tilde{\varphi}^*(\lambda')\,d\lambda\,d\lambda' \Big/ \int|\tilde{\varphi}(\lambda)|^2 d\lambda$$

[cf Proof of Theorem (4.4.4)] where $\tilde{\varphi}(\lambda) = \langle\varphi_\lambda|\varphi\rangle$ and $K_{\Lambda_o}(\lambda,\lambda') = \int_{\Lambda_o}\varphi_\lambda\varphi_{\lambda'}^*\,dx$. If T has discrete spectrum (5.3.5) yields:

$$(5.3.6)\qquad w(T,b,\varphi,\Lambda_o) = \sum_{\lambda_i\in b}\sum_{\lambda_j\in b}K_{\Lambda_o}(i,j)\tilde{\varphi}_i\,\tilde{\varphi}_j \Big/ \sum_{\lambda_i\in b}|\tilde{\varphi}_i|^2$$

where $\tilde{\varphi}_i = \tilde{\varphi}(\lambda_i)$ and $K_{\Lambda_o}(i,j) = K_{\Lambda_o}(\lambda_i,\lambda_j)$.

For the particular case in which b comprises a single discrete eigenvalue $\{\lambda\}$, say, we have:

$$(5.3.7)\qquad w(T,\lambda,\varphi,\Lambda_o) = K_{\Lambda_o}(\lambda,\lambda) = \int_{\Lambda_o}|\varphi_\lambda|^2\,dx,$$

which is independent of the initial wavefunction φ, as we would expect. Let us consider an illustration.

5.4 Example: localisation of the harmonic oscillator

Specifically we shall seek $w(H_{osc},E_n,\varphi,\Lambda_o)$ where H_{osc} is the Hamiltonian for the harmonic oscillator given by

(5.4.1)

$$H_{osc} = (-\hbar^2/2m)d^2/dx^2 + mw^2x^2.$$

The solutions to the eigenvalue equation are given by [Schiff 1955 pp69-71,eg]:

(5.4.2) $$E_n = (n+\tfrac{1}{2})\hbar w$$

$$\varphi_n = (mw/\hbar\pi)^{1/4}(2^n n!)^{-\frac{1}{2}}H_n((mw/\hbar)^{\frac{1}{2}}x)\exp[-mwx^2/\hbar]$$

where H_n are the Hermite polynomials given by

$$H_n(y) = (-1)^n \exp[y^2](\partial^n/\partial y^n)\exp[-y^2]$$

or by the generating function

$$S(y,s) = \exp[-s^2+2sy] = \sum_{n=0}^{\infty} H_n(y)s^n/n!.$$

We wish to find $\int_{\Lambda_0}|\varphi_n|^2dx$. First notice that

(5.4.3)

$$\int_{\Lambda_0}|\varphi_n|^2dx = (mw/\hbar\pi)^{\frac{1}{2}}(2^n n!)^{-1}\int_{-T}^{T}|H_n(y)\exp[-y^2]|^2dx$$

where $y=(mw/h)^{\frac{1}{2}}x$

$$= \pi^{\frac{1}{2}}(2^n n!)^{-1}\int_{-(\frac{mw}{\hbar})^{\frac{1}{2}}T}^{(\frac{mw}{\hbar}T)}|H_n(y)\exp[-y^2]|^2\,dy$$

where we choose $\Lambda_0=[-T,T]$.

Now consider the following integral:

(5.4.4)

$$\int_{\Lambda_0}S(y,s)S(y,t)\exp[-y^2]dy = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{s^n t^m}{n!m!}\int_{\Lambda_0'}H_n(y)H_m(y)\exp[-y^2]dy$$

where $\Lambda_0' = (mw/\hbar)^{\frac{1}{2}}\Lambda_0$. The left hand side of (5.4.4) yields:

$$\int_{\Lambda_0'} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{s^n t^m}{n!m!} H_n(y)H_m(y)e\ dy = \int_{\Lambda_0'} e^{-s^2+2ys} e^{-t^2+2yt} e^{-y^2} dy$$

$$= \int_{\Lambda_0'} e^{-(y-(t+s))^2} e^{2ts} dy$$

$$= \exp[2ts] \int_{-(\frac{mw}{\hbar})^{\frac{1}{2}}T-t-s}^{(\frac{mw}{\hbar})^{\frac{1}{2}}T-t-s} \exp[-y^2]dy.$$

By choosing t and s very small we can ensure that this last expression is as close as we please to the integral:

$$\exp[2ts] \int_{\Lambda_0'} \exp[-y^2]dy$$

which in its turn can be expressed in terms of the error function erf [Dwight 1961 p136,eg] as

(5.4.5)

$$\pi^{\frac{1}{2}}\mathrm{erf}\{(mw/\hbar)^{\frac{1}{2}}T\}\exp[2ts] = \pi^{\frac{1}{2}}\mathrm{erf}\{(mw/\hbar)^{\frac{1}{2}}T\}\sum_{n=0}^{\infty} t^n s^n 2^n/n!\ .$$

By equating equal powers of t and s between (5.4.4) and (5.4.5) we deduce that

(5.4.6)

$$\int_{\Lambda_0} H_n^2(y)\exp[-y^2]dy \simeq \pi^{\frac{1}{2}}\mathrm{erf}\{(mw/\hbar)^{\frac{1}{2}}T\}n!2^n,$$

and it follows that

(5.4.7)

$$\int_{\Lambda_0} |\varphi_n|^2 dx \simeq \mathrm{erf}\{(mw/\hbar)^{\frac{1}{2}}T\}.$$

Notice a rather peculiar feature of the example we have chosen, namely that every eigenfunction has the same concentration in Λ_0. Now, for example erf x is 0.995, ie very close to unity when x is 2. Hence $w(H_{osc}, E_n, \varphi, \Lambda_0) = 0.995$ when $(mw/\hbar)^{\frac{1}{2}}T = 2$, so for each eigenvalue, so long as the size of the localisation 2T is greater than $4(\hbar/mw)^{\frac{1}{2}}$ we have a very high probability of retaining the particle in Λ_0. If m is around the mass of the electron, say 10^{-27}gm and $\hbar$ is 10^{27} erg s we require the size of the box to be greater than $(4/w^{\frac{1}{2}})$cms where w is the classical angular frequency of the harmonic oscillator. For high frequency oscillators in particular therefore a high degree of localisation is achieved.

5.5 Further Remarks

Evidently not every operator will allow the sort of localisation analysis provided by the particular example of the harmonic oscillator. In general the analysis will be complicated by such features as degeneracy of eigenvalues, continuity of the spectrum and substantially less straightforward convergences involving the original wavefunction as well as the (generalised) eigenvectors. However in each case it is

in principle possible to examine the extent of localisation of the measured system by means of the general analysis proposed above and for this analysis the localised projections $(E(T;b))_{\Lambda_0}$ are a measure of the localisation of quantum mechanical systems. Furthermore, let us recall that a particularly desirable feature of the local momentum observable $\hat{\jmath}\hat{p}$ is that within the centre of localisation Λ_0 the local and global measurements provide the same probability distribution. In particular of course we have $(E(\hat{p};\Delta))_{\Lambda_0} = (E(\hat{\jmath}\hat{p};\Delta))_{\Lambda_0}$ and here the localised projections play the part of demonstrating the suitability of local observables in measurement analysis. The analyses of the type pursued in (5.4) then enable us to say how large an apparatus needs to be in order for the boundary region $\Lambda-\Lambda_0$ to be negligible in the measurement results and hence for the local observables to differ negligibly from the global ones. The same role may be fulfilled for any other local observable $\hat{\jmath}\hat{T}$, say provided that the spectral measure of $\jmath T$ satisfies $(E(T;b))_{\Lambda_0} = (E(\hat{\jmath}\hat{T};b))_{\Lambda_0}$. A sufficient condition for this to hold is that the (generalised) eigenfunctions of $\hat{\jmath}\hat{T}$ and T coincide on Λ_0. This is not a straightforward issue however, because there is in general no guarantee that any one-one correspondence exists between global eigenfunctions and local ones.

Finally, we remark in passing, that localisation with respect to an arbitrary spectral measure is a mathematical possibility for bounded observables [McLean 1984]. In the next chapter we shall see how it is possible to localise an operator with respect to its own spectrum and this localisation also has some interesting physical consequences.

CHAPTER 6

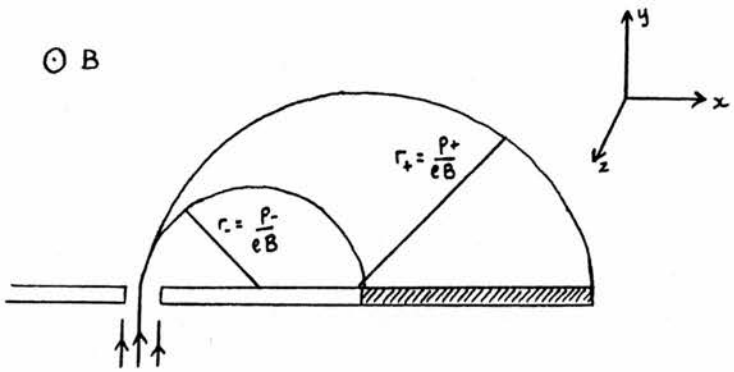
SPECTRUM-LIMITED OBSERVABLES

6.1 Introduction

In chapter 3 we demonstrated how to construct bounded local observables A_Λ satisfying $A_\Lambda = E(\underline{x};\Lambda)A_\Lambda E(\underline{x};\Lambda)$, $\Lambda \in \mathcal{B}_c(\hat{\mathbb{R}})$, and showed that in a certain sense, a family of such observables may be taken to correspond to a single global observable. The physical consequences of such a construction enable us to talk more realistically about the process of measurement using apparatus of finite size. While it is clear that this procedure has a straightforward generalisation to localisation with respect to the spectral measure of an arbitrary observable (eg, we can construct observables A_Δ satisfying $A_\Delta = E(\hat{p};\Delta)A_\Delta E(\hat{p};\Delta)$ where $\hat{p}$ is the usual momentum observable), perhaps a more immediate physical necessity is engendered in the following discussion.

Consider the process of measurement of an arbitrary observable using real physical apparatus. Almost invariably (one might be tempted to say always) such apparatus is limited not only in sensitivity, but also in the range of possible values of measurement attainable using the apparatus. One way in which this can occur is illustrated by the following.

(6.1.1) Example: measurement of momentum by deflection in a magnetic field [Messiah 1961 p146, eg]:



A beam of electrons with momentum in the y-direction lying in some (possibly unbounded) range Δp, say, enters the magnetic field. The particles are deflected according to [Clemmow 1973 p253,eg]

$$r = p/eB, \quad p \in \Delta p,$$

where e is the electronic charge, B is the magnitude of the magnetic field and r is the radius of the resultant deflection.(We neglect here uncertainty introduced by the effect of diffraction due to a finite width of slit. This is of course important but we need not introduce such a complication for the purposes of this discussion. The units are rationalised mks units [cf Clemmow p 4])

Now if p lies in Δp where $\Delta p = (p_1, p_2)$, say, then the radius of curvature r lies in (r_1, r_2) where $r_1 = p_1/eB$ and $r_2 = p_2/eB$. However it is obvious from the figure that the deflected particle will only be detected if r lies in the interval (r_-, r_+) where $2r_-$ is the distance from the slit to the nearest point of the detection screen and $2r_+$ is the distance from the slit to the furthest point of the screen. This means that the range of momentum values which may be detected by this apparatus is (p_-, p_+) where $p_- = eBr$, $p_+ = eBr$. Values of the momentum in $\Delta p \setminus (p_-, p_+)$ remain undetected whatever the range Δp of the initial wavefunction. In particular we see, since p_-, p_+ are necessarily bounded, that the range of possible measurement results $\Delta p \cap (p_-, p_+)$ is also bounded.

Another restriction that we might expect to arise is in the preparation of an initial state for an experiment. In the case of the beam of electrons, the range of possible momentum values is limited by the available energy in the source. In fact, since we are considering nonrelativistic conditions we must have $\max\{|p_1|, |p_2|\} \ll mc$ and relativistic considerations impose an upper bound mc on the possible momentum magnitude.

These considerations, together with the problems associated with unbounded observables (infinite expectation values,domain constraints,difficulties in localisation) should be sufficient to motivate the search for observables that will be at once more physically realistic and mathematically better behaved. Similar motivations prompted the algebraic formulation of quantum mechanics to its use of spectral projectors in place of their corresponding operators. The approach we propose here has the advantage of retaining operators corresponding to physical magnitudes such as momentum,energy,etc as well as the spectral projections of these operators, and hence of preserving a useful concept of expectation values. In section (6.2) we construct these operators. In the following section we present a correspondence, analogous to that of (3.4), between the familiar observables and certain families of bounded observables. The physical aspects of this correspondence will be examined in (6.4) and in (6.5) we consider the particular case of the momentum operator.

6.2 Constructing Spectrum-Limited Observables

We start by defining what we mean by a spectrum-limited observable.

(6.2.1) Definition
An observable A on $\mathcal{H}$ is said to be a spectrum-limited observable in Δ (Δ-limited observable for short) if there exists a $\Delta \in \mathcal{B}_c(\mathbb{R})$ such that $A = E(A;\Delta)AE(A;\Delta)$.

Suppose that T is an arbitrary selfadjoint operator with domain $\mathcal{D}(T)$. We are going to construct our spectrum-limited observables by first formulating them in the spectral representation [Jauch and Misra 1965 p30]. This involves constructing a measure space $L^2(\mathbb{R},\rho^T)$ on which the observable takes the form of a multiplication operator. By exploiting the unitary correspondence between this space and the Hilbert space corresponding to the configuration space of the particle, many results which are easy to formulate in the diagonal representation can be translated in terms of the original operator.

We shall assume here for convenience that T has a simple spectrum [cf Guenin 1966, Jauch and Misra 1965]. Then there is a unitary correspondence $U: \mathcal{H} \to L^2(\mathbb{R}, \rho^T)$ with the property that each $\varphi \in \mathcal{H}$ is represented uniquely by an essentially bounded function $u(\lambda) \in L^2(\mathbb{R}, \rho^T)$ and T is represented by $\lambda u(\lambda)$. The spectral function $E(T;t)$ of T is now given by:

$$(6.2.2) \qquad UE(T;t)U^{-1} = \chi_{(-\infty,t]}(\lambda)$$

where χ is a characteristic function in the usual notation. The domain of T is given by:

$$(6.2.3) \qquad \mathfrak{D}(T) = \{\varphi \in \mathcal{H} : \int \lambda^2 d\|E(T;t)\varphi\|^2 < \infty\}$$

If we denote by $\hat{\lambda}$ the operator of multiplication by λ on $L^2(\mathbb{R}, \rho^T)$, we have

$$(6.2.4) \qquad \hat{\lambda} u(\lambda) = UTU^{-1}u = \lambda u(\lambda) \quad u \in \mathfrak{D}(\hat{\lambda})$$

where $\mathfrak{D}(\hat{\lambda}) = \{u \in L^2(\mathbb{R}, \rho^T) : \int \lambda^2 |u|^2 d\rho^T < \infty\} = U\mathfrak{D}(T)$.

We now formulate the spectrum-limited observables in the spectral representation. Define the operator $\hat{\lambda}_\Delta$ on $L^2(\mathbb{R}, \rho^T)$ by:

$$(6.2.5) \qquad \hat{\lambda}_\Delta u(\lambda) = \lambda \chi_\Delta(\lambda) u(\lambda), \quad u \in \mathfrak{D}(\hat{\lambda}_\Delta)$$

where $\mathcal{D}(\hat{\lambda}_\Delta) = \{u \in L^2(\mathbb{R}, \rho^\tau) : \int_\Delta \lambda^2 |u|^2 d\rho^\tau < \infty\}$.

Concerning the operator $\hat{\lambda}_\Delta$ we now prove the following result.

(6.2.6) Theorem

Let $\Delta \in \mathcal{B}_c(\mathbb{R})$ and suppose that $\hat{\lambda}_\Delta$ is defined according to (6.2.4). Then:

(i) $\hat{\lambda}_\Delta$ is bounded

(ii) $\mathcal{D}(\hat{\lambda}_\Delta) = L^2(\mathbb{R}, \rho^\tau)$

(iii) $\hat{\lambda}_\Delta$ is selfadjoint.

Proof

(i) $\|\hat{\lambda}_\Delta u(\lambda)\| = \|\lambda \chi_\Delta u(\lambda)\|$

$\leq |\sup_{\lambda \in \Delta}\{\lambda\}| \, \|u(\lambda)\|$

$< \infty$, since $\Delta \in \mathcal{B}_c(\mathbb{R})$.

(ii) follows from (i).

(iii) $\langle u_1 | \hat{\lambda}_\Delta u_2 \rangle = \int u_1^* \lambda \chi_\Delta u_2 \, d\rho^\tau$

$= \int_\Delta (u_1 \lambda)^* u_2 \, d\rho^\tau$, since λ is real,

$= \int_\Delta (u_1 \lambda \chi_\Delta)^* u_2 \, d\rho^\tau$

$= \langle \hat{\lambda}_\Delta u_1 | u_2 \rangle$.

Hence $\hat{\lambda}_\Delta$ is a symmetric operator on the entire space and is therefore selfadjoint.

We now use the unitary correspondence $U: \mathcal{H} \to L^2(\mathbb{R}, \varrho^T)$ to define an operator T on $\mathcal{H}$ by:

$$(6.2.7) \quad T_\Delta \varphi = U^{-1}\hat{\lambda}_\Delta U\varphi = U^{-1}\hat{\lambda} UU^{-1}\chi_\Delta U\varphi = TE(T;\Delta)\varphi$$

$$\mathcal{D}(T_\Delta) = U^{-1}\mathcal{D}(\hat{\lambda}_\Delta) = U^{-1}L^2(\mathbb{R}, \varrho^T) = \mathcal{H}.$$

Using the properties of unitary transformations and the Theorem (6.2.6) the following result is obvious:

(6.2.8) Theorem

Let $T_\Delta = E(T;\Delta)TE(T;\Delta)$, $\Delta \in \mathcal{B}_c(\mathbb{R})$, $\mathcal{D}(T_\Delta) = \mathcal{H}$. Then T is a bounded selfadjoint linear operator on $\mathcal{H}$.

Proof

It suffices to remark that $E(T;\Delta)TE(T;\Delta) = TE(T;\Delta)$ and the result follows immediately from the previous analysis.

(6.2.9) Definition

The observable T_Δ defined by (6.2.7) is called the Δ-limitation (or spectrum limitation in Δ) of T.

Obviously we should like to be able to show that the Δ-limitation T_Δ of T is a Δ-limited observable. To do this we need to know the spectral function $E(T_\Delta;t)$ of T_Δ.

(6.2.10) Lemma

The spectral measure $E(T_\Delta;M)$ of $T_\Delta, M \in \mathcal{B}(\mathbb{R})$ is given by:

$E(T_\Delta;M) = E(T;\Delta \cap M) + E(T;\Delta^\perp)E(0;M)$.

Proof

We construct the spectral measure first in the representation space where it has the form of a characteristic function $\chi_M(\lambda\chi_\Delta(\lambda))$. Now we have

$$\chi_M(\lambda\chi_\Delta(\lambda)) = (\chi_M \circ \lambda\chi_\Delta)(\lambda)$$

$$= \chi_{\{t\in\mathbb{R}:\, t\chi_\Delta(t)\in M\}}(\lambda)$$

$$= \begin{cases} 1 \text{ if } \lambda\chi_\Delta(\lambda) \in M \\ 0 \text{ if } \lambda\chi_\Delta(\lambda) \notin M \end{cases}$$

But $\lambda\chi_\Delta(\lambda) \in M$ if and only if either $\lambda\in\Delta$ and $\lambda \in M$ (ie, $\lambda \in \Delta \cap M$) or else $\lambda\notin\Delta$ and $0\in M$, and $\lambda\chi_\Delta(\lambda)\notin M$ if and only if either $\lambda\notin\Delta$ and $0\notin M$ or else $\lambda\in\Delta$ and $\lambda\notin M$. So we have

$$\chi_M(\lambda\chi_\Delta(\lambda)) = \begin{cases} 1 \text{ if } \begin{cases} \text{either } \lambda\in\Delta \text{ and } \lambda\in M \\ \text{or } \lambda\notin\Delta \text{ and } 0\in M \end{cases} \\ 0 \text{ if } \begin{cases} \text{either } \lambda\notin\Delta \text{ and } 0\notin M \\ \text{or } \lambda\in\Delta \text{ and } \lambda\notin M \end{cases} \end{cases}$$

It is straightforward to verify that this function is

represented by:

$$F_{\Delta}(M) = \chi_{\Delta}(\lambda)\chi_{M}(\lambda) + \chi_{M}(0)\chi_{\Delta^{\perp}}(\lambda)$$

and using the unitary transformation we have immediately the result.

(6.2.11) Corollary

The spectral function $E(T_{\Delta};t)$ of T_{Δ} is given by:

$E(T_{\Delta};t) = E(T;t)E(T;\Delta) + E(T;\Delta^{\perp})E(0;t)$.

Proof

Follows immediately from the lemma.

Now it is straightforward to prove the result we wanted.

(6.2.12) Theorem

The Δ-limitation T_{Δ} of an arbitrary observable T is Δ-limited.

Proof

$$\begin{aligned}(T_{\Delta})_{\Delta} &= T_{\Delta}E(T_{\Delta};_{\Delta}) \\ &= TE(T;\Delta)\{E(T;\Delta)E(T;\Delta) + E(T;\Delta^{\perp})E(0;\Delta)\} \\ &= TE(T;\Delta) \\ &= T_{\Delta}.\end{aligned}$$

It is clear that if A is a bounded observable with spectrum $\sigma(A)$ then A is Δ-limited for each $\Delta \supseteq \sigma(A)$ since [Prugovecki p253] $E(A;\Delta) = 1$ for all such Δ. Conversely, what can we say about the spectrum of a Δ-limited operator? The answer to this question is the subject of the following two theorems and corollary.

(6.2.13) Theorem

Let $\Delta \in \mathcal{B}_c(\mathbb{R})$. An observable A is Δ'-limited for each $\Delta' \supseteq \Delta$, $\Delta' \in \mathcal{B}_c(\mathbb{R})$, if and only if $\sigma(A) \subseteq \Delta \cup \{0\}$.

Proof

Suppose that A is Δ-limited. Then $A = A_\Delta$ and it follows that

$E(A;M) = E(A_\Delta;M)$ for each $M \in \mathcal{B}(\mathbb{R})$. Hence

$E(A;M) = E(A;M \cap \Delta) + E(A;\Delta^\perp)E(0;M)$, $M \in \mathcal{B}(\mathbb{R})$.

Take $M = \Delta \cup \{0\}$ and we have

$E(A;\Delta \cup \{0\}) = E(A;\Delta) + E(A;\Delta^\perp) = 1$,

and it follows that $\sigma(A) \subseteq \Delta \cup \{0\}$.

Conversely, suppose that $\sigma(A) \subseteq \Delta \cup \{0\}$. If $0 \in \Delta$, then we have $\sigma(A) \subseteq \Delta$, and $E(A;\Delta') = E(A;\Delta) = 1$ for all $\Delta' \supseteq \Delta$ and so A_Δ is Δ'-limited for all $\Delta' \supseteq \Delta$. Next suppose that $0 \notin \Delta$. Then $\Delta \cap \{0\} = \phi$ and we can write

$1 = E(A;\Delta \cup \{0\}) = E(A;\Delta) + E(A;\{0\})$

and hence

$A = AE(A;\Delta) + AE(A;\{0\})$.

The last term in this expression is zero however since $E(A;\{0\})$ is the projector onto the subspace on which A

$= 0$ so we have $AE(A;\{0\})\varphi = 0$ for every φ in $\mathcal{H}$. It follows that $A = AE(A;\Delta)$

As an immediate consequence of this theorem we observe the following corollary.

(6.2.14) Corollary

Let T be an arbitrary selfadjoint operator and let $\Delta \in \mathcal{B}_c(\mathbb{R})$. Then the spectrum $\sigma(T_\Delta)$ of T_Δ satisfies:

$$\sigma(T_\Delta) \subseteq \Delta \cup \{0\}.$$

Proof

According to Theorem (6.2.12) T_Δ is Δ-limited and the result follows from Theorem (6.2.13).

Actually we are able to prove a stronger result. Namely

(6.2.15) Theorem

Let T be an arbitrary selfadjoint operator on $\mathcal{H}$ with spectrum $\sigma(T)$. The spectrum $\sigma(T_\Delta)$ of T_Δ, $\Delta \in \mathcal{B}_c(\mathbb{R})$, is given by

$$\sigma(T_\Delta) = \begin{cases} (\sigma(T) \cap \Delta) \cup \{0\}, & \text{if } \Delta \not\supseteq \sigma(T) \\ \sigma(T), & \text{if } \Delta \supseteq \sigma(T). \end{cases}$$

Proof

Notice first that $\sigma(T) \cap \Delta$ and $(\sigma(T) \cap \Delta) \cup \{0\}$ are both closed sets since $\sigma(T)$ is closed and $\Delta \in \mathcal{B}_c(\mathbb{R})$. Now if $\Delta \supseteq \sigma(T)$ we have immediately $E(T;\Delta) = E(T;\sigma(T)) = 1$. Suppose that $\sigma(T) \not\subseteq \Delta$. Then it follows that $E(T;\Delta^\perp)$ is strictly greater than zero, and hence that

for each open interval J containing zero

$E(T_\Delta;J) = E(T;\Delta)\,E(T:J) + E(T;\Delta^\perp) \geqslant E(T;\Delta^\perp) > 0.$

and from the definition of the spectrum it follows that $0 \in \sigma(T_\Delta)$. Also for each $t \in \sigma(T) \cap \Delta$ it follows that $t \in \Delta$ and for each open interval I containing t, $E(T;I) > 0$. Let I be such an interval. Since I is an open interval and by hypothesis $t \in \Delta$, there exists another open interval I', say, containing t such that $I' \subseteq I \cap \Delta$, and $E(T;I') > 0$. Therefore we have for every open I

$$\begin{aligned} E(T_\Delta;I) &= E(T;I\cap\Delta) + E(T;\Delta^\perp)E(0;I) \\ &\geqslant E(T;I\cap\Delta) \\ &\geqslant E(T;I') > 0. \end{aligned}$$

Hence $t \in \sigma(T_\Delta)$. We have therefore that $(\sigma(T)\cap\Delta)\cup\{0\} \subseteq \sigma(T_\Delta)$. To prove the opposite inclusion we note first that

$(\sigma(T)\cap\Delta)\cup\{0\} = (\sigma(T)\cup\{0\})\cap(\Delta\cup\{0\})$.

We have already shown in corollary (6.2.13) that $\sigma(T_\Delta) \subseteq \Delta \cup \{0\}$ and it remains only to demonstrate that $\sigma(T_\Delta) \subseteq (\sigma(T)\cup\{0\})$. Let $t \in \sigma(T_\Delta)$. Suppose that $t \notin \sigma(T)\cup\{0\}$; ie $t \neq 0$ and $t \notin \sigma(T)$. Then there exists an open interval I_o, say, such that $t \in I_o$, $0 \notin I_o$ and $E(T;I_o) = 0$. So we have

$$\begin{aligned} E(T_\Delta;I_o) &= E(T;I_o\cap\Delta) + E(T;\Delta^\perp)E(0;I_o) \\ &= E(T;I_o\cap\Delta), \text{ since } 0 \notin I_o, \\ &\leqslant E(T;I_o) = 0. \end{aligned}$$

This implies that $t \notin \sigma(T_\Delta)$ and the result follows by contradiction.

6.3 Families of Spectrum-Limited Observables

We shall now proceed to demonstrate a correspondence of a unique and canonical nature between certain families of spectrum limited observables and selfadjoint operators T with (not necessarily limited) spectrum $\sigma(T)$. This correspondence is analogous to that elucidated in Chapter 3 between bounded selfadjoint operators and bounded families of globally related observables. The correspondence provides the mathematical justification to replace arbitrary observables by their corresponding families. We start with a definition.

(6.3.1) Definition

A map $\Upsilon : \mathcal{B}_c(\mathbb{R}) \to \mathcal{B}(\mathcal{H})$ is called a family of spectrum-limited observables if Υ satisfies

$$\Upsilon(\Delta) = (\Upsilon(\Delta))_\Delta ,$$

ie, if $\Upsilon(\Delta)$ is Δ-limited.

The particular families in which we shall be interested are the following.

(6.3.2) Definition

A family Υ of spectrum-limited observables is called a spectrally-related family of spectrum-limited observables (or just a related family of limited

observables, for short) if, for each $\Delta,\Delta' \in \mathcal{B}_c(\mathbb{R})$ such that $\Delta' \supseteq \Delta$ we have

(*) $\qquad \Upsilon(\Delta) = (\Upsilon(\Delta'))_\Delta ,$

ie, $\Upsilon(\Delta)$ is the Δ-limitation of $\Upsilon(\Delta')$.

The following lemma provides the explanation for the terminology of these definitions.

(6.3.3) Lemma

If Υ is a related family of limited observables, then for each $\Delta,\Delta' \in \mathcal{B}_c(\mathbb{R})$ such that $\Delta' \supseteq \Delta$ we have

$\sigma(\Upsilon(\Delta)) = (\sigma(\Upsilon(\Delta'))_\wedge \Delta)_\cup \{0\} \subseteq \sigma(\Upsilon(\Delta'))$.

Proof

Obvious from Theorem (6.2.13) and the definitions.

Now for convenience we introduce the shorthand notation $E_\Delta(t) = E(\Upsilon(\Delta);t)$ and $E_\Delta(M) = E(\Upsilon(\Delta);M)$ for the spectral function and the spectral measures associated with a related family Υ. If Υ is such a family it follows from (6.2.11) that

$$E_\Delta(t) = E_{\Delta'}(t)E_{\Delta'}(\Delta) + E_{\Delta'}(\Delta^\perp)E(0;t)$$

for each $\Delta,\Delta' \in \mathcal{B}_c(\mathbb{R})$ with $\Delta' \supseteq \Delta$, and for all $t \in \mathbb{R}$. Through the following series of lemmas we establish a correspondence between related families of limited observables and a certain observable which appears in the strong limit for increasing Δ.

(6.3.4) Lemma

Let $\{\Delta_i\}$ be an increasing sequence of subsets of $\mathbb{R}$, $\Delta_i \in \mathcal{B}_c(\mathbb{R})$ such that $\Delta_i \to \mathbb{R}$ as $i \to \infty$. Then for each $t \in \mathbb{R}$ and $i \leq j$, and for each $\varphi \in \mathcal{H}$ we have:

$$\langle \varphi | E_{\Delta_i}(t) - E_{\Delta_j}(t) | \varphi \rangle = \langle \varphi | E_{\Delta_j}(\Delta_i^{\perp})(E(0;t) - E_{\Delta_j}(t)) | \varphi \rangle$$

Proof

The left hand side of the equation yields

$$\begin{aligned}
&\langle \varphi | E_{\Delta_i}(t) - E_{\Delta_j}(t) | \varphi \rangle \\
&= \langle \varphi | E_{\Delta_j}(t) E_{\Delta_j}(\Delta_i) + E_{\Delta_j}(\Delta_i^{\perp}) E(0;t) - E_{\Delta_j}(t) | \varphi \rangle \\
&= \langle \varphi | E_{\Delta_j}(t)(E_{\Delta_j}(\Delta_i) - 1) + E_{\Delta_j}(\Delta_i^{\perp}) E(0;t) | \varphi \rangle \\
&= \langle \varphi | E_{\Delta_j}(\Delta_i^{\perp})(E(0;t) - E_{\Delta_j}(t)) | \varphi \rangle .
\end{aligned}$$

(6.3.5) Lemma

(i) For each $t < 0$, $\{E_{\Delta_i}(t)\}$ is a nonincreasing sequence of projections.

(ii) For each $t \geq 0$, $\{E_{\Delta_i}(t)\}$ is a nondecreasing sequence of projections.

Proof

Using the definition [cf(2.1.7)] of $E(0;t)$ and the lemma (6.3.4) we have

(i) if $i \leq j$ and $t < 0$

$$\langle \varphi | E_{\Delta_i}(t) - E_{\Delta_j}(t) | \varphi \rangle = -\langle \varphi | E_{\Delta_j}(\Delta_i^{\perp}) E_{\Delta_j}(t) | \varphi \rangle \leq 0,$$

for all $\varphi \in \mathcal{H}$, since E_{Δ_j} is positive and

(ii) if $i \leq j$ and $t \geq 0$

$$\langle \varphi | E_{\Delta_i}(t) - E_{\Delta_j}(t) | \varphi \rangle = \langle \varphi | E_{\Delta_j}(\Delta_i^{\perp})(1 - E_{\Delta_j}(t)) | \varphi \rangle \geq 0,$$

for all $\varphi \in \mathcal{H}$, since E_{Δ_j} is less than 1 and positive.

(6.3.6) Theorem

A spectral family is defined on $\mathcal{H}$ by

$$E(t) = \underset{i\to\infty}{s\text{-}\lim} E_{\Delta_i}(t), \text{ for each } t \in \mathbb{R} .$$

Proof

Firstly, $E(t)$ is a well-defined projector on $\mathcal{H}$ for each t by virtue of Theorem 3.10 on p209 of Prugovecki's book. Next, let $s \leq t, s, t \in \mathbb{R}$. Then for each i and for all $\varphi \in \mathcal{H}$ we have

$$\| E_{\Delta_i}(s)\varphi \| \leq \| E_{\Delta_i}(t)\varphi \|$$

from which it follows that

$$\lim_{i\to\infty} \| E_{\Delta_i}(s)\varphi \| \leq \lim_{i\to\infty} \| E_{\Delta_i}(t)\varphi \| .$$

and hence

$$\| E(s)\varphi \| \leq \| E(t)\varphi \|, \text{ for all } \varphi \in \mathcal{H}$$

or

$$E(s) \leq E(t).$$

Now let ε be a small positive number. We have

$$\| E(t+\varepsilon)\varphi - E(t)\varphi \|$$

$$= \| (E(t+\varepsilon) - E_{\Delta_i}(t+\varepsilon) + E_{\Delta_i}(t) - E(t) + E_{\Delta_i}(t+\varepsilon) - E_{\Delta_i}(t))\varphi \| ,$$

for all i,

$$\leq \| E(t+\varepsilon)\varphi - E_{\Delta_i}(t+\varepsilon)\varphi \| + \| E_{\Delta_i}(t)\varphi - E(t)\varphi \| + \| E_{\Delta_i}(t+\varepsilon)\varphi - E_{\Delta_i}(t)\varphi \| ,$$

for all i.

Now the first two terms on the right hand side of the inequality can be made arbitrarily small by taking i large enough and hence there is an i_o, say, such that

$$\| E(t+\varepsilon)\varphi - E(t)\varphi \| \leq \| E_{\Delta_i}(t+\varepsilon)\varphi - E_{\Delta_i}(t)\varphi \| + \varepsilon , \text{ for all}$$

$i \geq i_o(\varepsilon)$

which tends to zero as ε tends to zero since E_{Δ_i} is a spectral function.

Finally, let t_Λ be any sequence of real numbers converging monotonically to $-\infty$. Then $E(t_\Lambda)$ is a nondecreasing sequence of projections bounded below by zero and hence converges to a projection $E(-\infty)$ say. Thus $\lim_{\Lambda\to\infty} \lim_{i\to\infty} \| E_{\Delta_i}(t_\Lambda)\varphi\|$ exists for each $\varphi \in \mathcal{H}$. Moreover $\lim_{i\to\infty} \lim_{\Lambda\to\infty} \| E_{\Delta_i}(t_\Lambda)\varphi\|$ exists for each φ and equals zero. Equality of these two limits ensures that $E(t) \to 0$ as $t \to -\infty$. Similarly it may be shown that $E(T) \to 1$ as $t \to +\infty$.

Now $E(t)$ defines a unique spectral measure $E(M), M \in \mathcal{B}(\mathbb{R})$ and of course by the spectral theorem [(2.1.5)] it also defines a unique selfadjoint operator T on $\mathcal{H}$ given by

$$(6.3.7) \qquad T = \int t\,dE(t)$$

$$\mathcal{D}(T) = \{\varphi \in \mathcal{H} : \int t^2\, d\|E(t)\varphi\|^2 < \infty\}$$

Consider the Δ-limitation of this operator T. Let E^Δ denote the spectral measure $E(T_\Delta ; .)$ of T_Δ. That is,

$$E^\Delta(t) = E(t)E(\Delta) + E(\Delta^\perp)E(0;t).$$

The fundamental theorem for our desired correspondence may now be proved.

(6.3.8) Theorem

Let Υ be a related family of limited observables. Then there exists a unique observable T on $\mathcal{H}$ with the property that for each $\Delta \in \mathcal{B}_c(\mathbb{R})$:

(Δ)

$$\Upsilon(\Delta) = T_\Delta$$

where T_Δ is the Δ-limitation of T. Conversely, let T be any observable on $\mathcal{H}$ then there exists a unique related family of limited observables Υ satisfying (Δ).

Proof

We prove the converse first which is easy. Let T be an arbitrary observable. For each $\Delta \in \mathcal{B}_c(\mathbb{R})$ define a map $\Upsilon : \mathcal{B}_c(\mathbb{R}) \to \mathcal{B}(\mathcal{H})$ by (Δ). Let $\Delta' \supseteq \Delta, \Delta, \Delta' \in \mathcal{B}_c(\mathbb{R})$. Then

$$\begin{aligned}(T_{\Delta'})_\Delta &= TE(T;\Delta')E(T_{\Delta'};\Delta) \\ &= TE(T;\Delta')(E(T;\Delta_\wedge\Delta') + E(T;\Delta'^{\perp})E(0;\Delta)) \\ &= TE(T;\Delta) = T_\Delta\end{aligned}$$

and hence Υ is a related family of limited observables. Uniqueness is self-evident.

Conversely, let Υ be a related family of limited observables. Define T by (6.3.7). T is selfadjoint. Moreover for each φ, ψ in $\mathcal{H}$ we have

$$\begin{aligned}&|\langle\varphi|E^\Delta(t) - E_\Delta(t)|\psi\rangle| \\ &= |\langle\varphi|E(t)E(\Delta) + E(\Delta^\perp)E(0;t) - E_\Delta(t)|\psi\rangle| \\ &= |\langle\varphi|E(t)E(\Delta)+E(\Delta^\perp)E(0;t)-E_{\Delta'}(t)E_{\Delta'}(\Delta)-E_{\Delta'}(\Delta^\perp)E(0;t)|\psi\rangle|, \\ &\leq |\langle\varphi|E(t)E(\Delta)-E_{\Delta'}(t)E_{\Delta'}(\Delta)|\psi\rangle| + |\langle\varphi|E(0;t)(E(\Delta^\perp)-E_{\Delta'}(\Delta^\perp)|\psi\rangle|\end{aligned}$$

for each $\Delta' \supseteq \Delta$.

Now the right hand side of this inequality may be made arbitrarily small because the strong convergence of $E_{\Delta'}$

to E implies the weak convergence of $E_{\Delta'}(\Delta^{\perp})$ to $E(\Delta^{\perp})$ and also of $E_{\Delta'}(t)E_{\Delta'}(\Delta)$ to $E(t)E(\Delta)$. It follows that $E_{\Delta} = E^{\Delta}$ and by the uniqueness clause in the spectral theorem we have $\Upsilon(\Delta) = T_{\Delta}$. To complete the proof, suppose that T' is another operator such that $T'_{\Delta} = \Upsilon(\Delta)$ for each $\Delta \in \mathcal{B}_c(\mathbb{R})$. Let E'^{Δ} be the spectral measure of T' and we have $E'^{\Delta} = E_{\Delta}$ for each $\Delta \in \mathcal{B}_c(\mathbb{R})$ exactly as before. In particular if $\Delta_i \to \mathbb{R}$, $E'^{\Delta_i} = E_{\Delta_i}$ for all i. But $E_{\Delta_i} = E^{\Delta_i}$, and so $E'^{\Delta_i} = E^{\Delta_i}$ for all i and it follows that $\text{s-}\lim_{i\to\infty} E'^{\Delta_i} = \text{s-}\lim_{i\to\infty} E^{\Delta_i}$, or $E' = E$. Hence $T' = T$.

As a corollary to this result we have another theorem.

(6.3.9) Theorem

The unique observable T associated with the related family of limited observables Υ is given by:

$$T = \text{s-}\lim_{i\to\infty} \Upsilon(\Delta_i),$$

in the sense that

$$\|T\varphi - \Upsilon(\Delta_i)\varphi\| \to 0 \text{ for each } \varphi \in \mathcal{D}(T)$$

where $\{\Delta_i\}$ is any sequence of sets in $\mathcal{B}_c(\mathbb{R})$ converging to $\mathbb{R}$.

Proof

Notice first that if $\varphi \in \mathcal{D}(T)$ then $T\varphi = \psi$ is a well-defined vector in $\mathcal{H}$. Furthermore, from the properties of the spectral measure we have

$$\|E(\Delta_i)\psi - \psi\| \to 0, \text{ for all } \psi \in \mathcal{H}.$$

In particular, therefore,

$$\| E(\Delta_i)T\varphi - T\varphi \| \to 0, \text{for all } \varphi \in \mathfrak{D}(T).$$

But for each φ in $\mathfrak{D}(T)$ we have

$T_\Delta \varphi = TE(\Delta)\varphi = E(\Delta)T\varphi$

and so $T_{\Delta_i}\varphi \xrightarrow{s} T\varphi$ as $i \to \infty$.

As a physical consequence of this theorem we observe the following result concerning expectation values.

(6.3.10) Corollary

Let φ be any vector in the domain of T. Then for each $\varepsilon > 0$ there exists a $\Delta \in \mathfrak{B}_c(\mathbb{R})$ such that

$$|\langle T_{\Delta'};\varphi\rangle - \langle T;\varphi\rangle| < \varepsilon ,$$

for every $\Delta' \supseteq \Delta$.

Proof

Strong convergence of T_Δ to T entails the convergence of the expectation values $\langle \varphi | T_\Delta \varphi \rangle$.

A further result characterising related families will proof useful in the sequel.

(6.3.11) Theorem

Let Υ be any family of limited observables. Υ is a related family if and only if for every pair $\Delta, \Delta' \in \mathfrak{B}_c(\mathbb{R})$ with $\Delta \cap \Delta' \neq \emptyset$

$$(\Upsilon(\Delta))_{\Delta'} = (\Upsilon(\Delta'))_\Delta .$$

Proof

Evidently the condition (*) of Definition (6.3.2) is a special case of the condition of the theorem.

Conversely suppose that Υ is a related family of limited observables. It therefore satisfies $\Upsilon(\Delta) = (\Upsilon(\Delta'))_\Delta$ for every $\Delta' \supseteq \Delta$. Suppose that Δ_1 and Δ_2 satisfy $\Delta_1 \cap \Delta_2 \neq \phi$, then $\Delta_1 \cap \Delta_2 \subseteq \Delta_1$ and $\Delta_1 \cap \Delta_2 \subseteq \Delta_2$. Hence

$$\begin{aligned}\Upsilon(\Delta_1 \cap \Delta_2) &= (\Upsilon(\Delta_1))_{\Delta_1 \cap \Delta_2} \\ &= \Upsilon(\Delta_1)E(\Upsilon(\Delta_1); \Delta_1 \cap \Delta_2) \\ &= \Upsilon(\Delta_1)E(\Upsilon(\Delta_1); \Delta_1)E(\Upsilon(\Delta_1); \Delta_2) \\ &= \Upsilon(\Delta_1)E(\Upsilon(\Delta_1); \Delta_2) \\ &= (\Upsilon(\Delta_1))_{\Delta_2}.\end{aligned}$$

Similarly $\Upsilon(\Delta_1 \cap \Delta_2) = (\Upsilon(\Delta_2))_{\Delta_1}$, and hence $(\Upsilon(\Delta_1))_{\Delta_2} = (\Upsilon(\Delta_2))_{\Delta_1}$, for all $\Delta_1 \cap \Delta_2 \neq \emptyset$.

This characterisation enables us to introduce a new definition concerning pairs of operators.

(6.3.12) Definition

Suppose that T_Δ is a Δ-limited observable and $T_{\Delta'}$ is a Δ'-limited observable, $\Delta \cap \Delta' \neq \emptyset$. T_Δ and $T_{\Delta'}$ are said to be spectrally related (or just related) if there exists a related family Υ such that $T_\Delta = \Upsilon(\Delta)$ and $T_{\Delta'} = \Upsilon(\Delta')$.

Then we have another result which follows immediately from the definition and the preceding theorem.

(6.3.13) Theorem

T_Δ and $T_{\Delta'}$ are two related observables if and only $(T_\Delta)_{\Delta'} = (T_{\Delta'})_\Delta$.

6.4 The Physical Limitations of Measurement

Physical observables in quantum mechanics are generally regarded as selfadjoint operators on the Hilbert space associated with the system. If the selfadjoint operator corresponding to a particular physical magnitude is unbounded then the set $\{\langle T;\varphi\rangle\}$ of expectation values is also unbounded. This means that for a particular physical apparatus the average values given by the apparatus for the observable in question have no upper bound. This is clearly in contradiction to the general physical tenet advanced in (6.1) namely that in general physical measurements using real apparatus do not yield unbounded sets of measurement results. Let us suppose in fact that a particular physical apparatus is capable of yielding measurement results in the set $\Delta_\upsilon\{0\}$, where $\Delta\epsilon\,\mathfrak{B}_c(\mathbb{R})$. (It makes sense to include the value zero for it is always possible that no measured value is perceived.) This means that the expectation values for the operator T describing the observable to be measured must satisfy

(6.4.1)

$$t_1 \leq \langle T;\varphi\rangle \leq t_2, \quad \varphi\epsilon\mathfrak{D}(T),$$

where $t_1 = \inf\{\Delta_\upsilon\{0\}\}$, $t_2 = \sup\{\Delta_\upsilon\{0\}\}$. Then we can prove the following.

(6.4.2) Theorem

Suppose that Δ is simply connected, $\Delta \in \mathcal{B}_c(\mathbb{R})$. Then

$t_1 \leq \langle T;\varphi\rangle \leq t_2$, for all, $\varphi \in \mathcal{D}(T)$

if and only if T is Δ-limited.

Proof

We can write

$$\langle T;\varphi\rangle = \int t\, d\|E(T;t)\varphi\|^2$$

If T is Δ-limited, then according to (6.2.17) $\sigma(T) \subseteq \Delta \cup \{0\}$ and so

$$\langle T;\varphi\rangle = \int_{\Delta\cup\{0\}} t\, d\|E(T;t)\varphi\|^2.$$

Now $\inf\{\Delta \cup \{0\}\} \int_{\Delta\cup\{0\}} d\|E(T;t)\varphi\|^2 \leq \int_{\Delta\cup\{0\}} t\, d\|E(T;t)\varphi\|^2$

$$\leq \sup\{\Delta \cup \{0\}\} \int_{\Delta\cup\{0\}} d\|E(T;t)\varphi\|^2$$

and since $\int_{\Delta\cup\{0\}} d\|E(T;t)\varphi\|^2 = \int_{\sigma(T)} d\|E(T;t)\varphi\|^2 = \|\varphi\|^2 = 1$

we have $t_1 \leq \langle T;\varphi\rangle \leq t_2$.

Conversely suppose that T is not Δ-limited. Then there exists a $t \in \sigma(T)$ such that $t \notin \Delta \cup \{0\}$. Since $\Delta \cup \{0\}$ is closed there exists an open interval I, say, such that I contains t and $I \cap (\Delta \cup \{0\}) = 0$ and for such I we have

$$E(T;I) \neq 0.$$

Δ is simply connected so we can choose a $t \in \sigma(T)$ such that either $t > \sup \Delta \cup \{0\}$ or $t < \inf \Delta \cup \{0\}$. Let φ be a vector in $\mathcal{H}$ satisfying $E(T;I)\varphi = \varphi$ and suppose there

is a t in the spectrum of T which is greater than $\sup \Delta_{\upsilon}\{0\}$. That is there is an open interval I containing t which is disjoint from $\Delta_{\upsilon}\{0\}$ and $\inf_{I} t > \sup_{\Delta_{\upsilon}\{0\}} t$. Hence

$$\langle T;\varphi\rangle = \int_{I} t\, d\|E(T;t)\varphi\|^2 \geqslant \inf I \int d\|E(T;t)\varphi\|^2 > \sup \Delta_{\upsilon}\{0\}.$$

The result follows by contradiction.

We see therefore that observables measurable in such limited physical situations correspond to the Δ-limited observables of (6.2) and (6.3). Actually, we need to clarify our concepts a little here. The abuse of notation which we have been accustomed to employ which allows us to identify a physical magnitude and the selfadjoint operator representing it under the same term "observable" is no longer valid. For instance, it is no longer immediately obvious how a physical quantity (observable) is to be related to the (Δ-limited) selfadjoint operator (observable) representing it. The theorem above tells us however that a measurement of a physical observable using apparatus sensitive only within a certain range Δ must be represented by a Δ-limited selfadjoint operator.

Generally, then, we have a scheme in which each physical quantity is not represented by an individual selfadjoint operator but by a family of such operators. Now let us suppose that we have a family $\{M_\Delta\}$ of measurement devices such that M_Δ has range $\Delta \cup \{0\}$, $\Delta \in \mathcal{B}_c(\mathbb{R})$ (we assume that each device has the same zero reading). Obviously if each M_Δ purports to measure values of the same physical quantity, the following consistency condition must be satisfied.

(6.4.3) Assumption

The family of measuring devices $\{M_\Delta\}$ measure the same physical quantity if and only if for every pair of apparatus M_Δ and $M_{\Delta'}$, $\Delta, \Delta' \in \mathcal{B}_c(\mathbb{R})$ and for every $I \subseteq \Delta \cap \Delta'$ (ie, for every interval in the range of both the apparatus) the probability for obtaining a value in I is independent of which apparatus is used.

Mathematically this assumption requires that for such M_Δ and $M_{\Delta'}$

$$(6.4.4) \qquad E(T^\Delta; I) = E(T^{\Delta'}; I), \text{ for all } I \text{ in } \Delta \cap \Delta',$$

where T^Δ and $T^{\Delta'}$ are the limited observables corresponding to measurement of the physical quantity in question using the respective devices.

(6.4.5) Lemma

Suppose that two limited observables T^{Δ} and $T^{\Delta'}$ satisfy (6.4.4). Then we have

$$E(T^{\Delta};\Delta') = E(T^{\Delta'};\Delta)$$

Proof

By assumption both Δ and Δ' contain zero so that $\sigma(T^{\Delta}) \subseteq \Delta$ and $\sigma(T^{\Delta'}) \subseteq \Delta'$. Therefore $E(T^{\Delta};\Delta^{\perp}) = E(T^{\Delta'};\Delta'^{\perp}) = 0$ and we have $E(T^{\Delta};\Delta') = E(T^{\Delta};\Delta\cap\Delta')$ and $E(T^{\Delta'};\Delta) = E(T^{\Delta'};\Delta\cap\Delta')$. From (6.4.4) it follows that $E(T^{\Delta'};\Delta) = E(T^{\Delta};\Delta')$.

(6.4.6) Theorem

suppose that T^{Δ} and $T^{\Delta'}$ satisfy (6.4.4). Then

$$(T^{\Delta})_{\Delta'} = (T^{\Delta'})_{\Delta}.$$

Proof

We prove equality of the spectral projections.

Firstly: $E((T^{\Delta})_{\Delta'};I) = E(T^{\Delta};\Delta'\cap I)+E(T^{\Delta};\Delta'^{\perp})E(0;I)$,

for all $I \in \mathcal{B}(\mathbb{R})$,

$$= E(T^{\Delta};\Delta\cap\Delta'\cap I)+E(T^{\Delta};\Delta'^{\perp})E(0;I),$$

for all $I \in \mathcal{B}(\mathbb{R})$. Similarly:

$$E((T^{\Delta'})_{\Delta};I) = E(T^{\Delta'};\Delta\cap\Delta'\cap I)+E(T^{\Delta'};\Delta^{\perp})E(0;I),$$

for all $I \in \mathcal{B}(\mathbb{R})$.

Using (6.4.4) to cancel the first terms on the right hand sides we have for every $I \in \mathcal{B}(\mathbb{R})$:

$$\begin{aligned} E((T^{\Delta})_{\Delta'};I)-E((T^{\Delta'})_{\Delta};I) &= E(0;I)(E(T^{\Delta};\Delta'^{\perp})-E(T^{\Delta'};\Delta^{\perp})) \\ &= E(0;I)(E(T^{\Delta'};\Delta)-E(T^{\Delta};\Delta')) \\ &= 0, \text{ by virtue of (6.4.5).} \end{aligned}$$

Therefore $(T^{\Delta})_{\Delta'} = (T^{\Delta'})_{\Delta}$ as desired.

The theorem (6.3.11) now reveals that the set $\{T^{\Delta}\}$ of limited observables measured by the apparatus $\{M_{\Delta}\}$ is in fact a related family of limited observables. We therefore postulate as follows concerning the physical measurement process.

(6.4.7) Postulate

To each physical quantity in quantum mechanics is associated a related family of limited observables Υ. The observable $\Upsilon(\Delta)$ is the appropriate observable to describe the measurement of that physical quantity when the range of the apparatus is Δ.

In conclusion then, we have argued two main points in the preceding two sections. Firstly we have shown that there is a mathematical correspondence between the selfadjoint operators and the set of related families of limited observables. Secondly we have argued that the description of real measurement situations demands the use of related families of limited operators. It is evident that the first result provides the link between the second result and the conventional formulation and in general terms the discussion is thereby completed. In the next section we shall examine a particular example, namely that of the momentum observable.

6.5 Spectrum-Limited Momentum Observables

In the conventional formulation momentum is represented by the (unbounded) operator defined in (2.1.3). In chapter 4 we discussed the possibility of representing the momentum observable by alternative "local" observables dependent on the size of the measurement apparatus. Here we use slightly different physical constraints to suggest that the momentum observable be represented by a related family of limited observables $\{\hat{p}_\Delta\}$ where

$$(6.5.1) \qquad \hat{p}_\Delta \varphi = (2\pi h)^{-\frac{1}{2}} \int_\Delta p \exp[ipx/\hbar]\tilde{\varphi}\,dp, \quad \varphi \in \mathcal{H}, \ \Delta \in \mathcal{B}(\mathbb{R}).$$

Alternative integral forms for $\hat{p}_\Delta$ are also possible.

(6.5.2) Theorem

$$\hat{p}_\Delta \varphi = (2\pi h)^{-\frac{1}{2}} \int \chi_\Delta(p) p \tilde{\varphi}(p) \exp[ipx/\hbar]dp \quad \text{(i)}$$

$$= (K_\Delta * \hat{p}\varphi)(x) \quad \text{(ii)}$$

$$= \hat{p}(K_\Delta * \varphi)(x) \quad \text{(iii)}$$

where $K_\Delta(x)$ denotes $(2\pi h)^{-\frac{1}{2}} \int_\Delta \exp[ipx/\hbar]dp$ and * denotes convolution [cf Appendix (A.1)].

Proof

(i) follows immediately from (6.5.1).

(ii) Let $\chi_\Delta p\tilde{\varphi} = \tilde{\Phi}_\Delta(p)$. The inverse Fourier transform of $p\tilde{\varphi}$ is $\hat{p}\varphi$ and hence, by the convolution theorem [Papoulis 1962], the inverse Fourier transform of $\tilde{\Phi}_\Delta(p)$ is given by

$$\Phi_\Delta(x) = (K_\Delta * \hat{p}\varphi)(x).$$

Examination of the integral in (i) reveals that it is precisely the inverse Fourier transform of $\tilde{\Phi}_\Delta(p)$ and hence (ii) holds.

(iii) Let $\chi_\Delta\tilde{\varphi} = \tilde{\varphi}_\Delta(p)$. The inverse Fourier transform $\varphi_\Delta(x)$ is given by the convolution theorem as

$$\varphi_\Delta(x) = (K_\Delta * \varphi)(x)$$

and the inverse transform of $p\tilde{\varphi}_\Delta$ is $\hat{p}\varphi_\Delta$ so the integral in (i) yields:

$$\hat{p}_\Delta\varphi = \hat{p}\,\varphi_\Delta(x) = \hat{p}(K_\Delta * \varphi)(x).$$

These integrals may be expressed more explicitly.

(6.5.3) Corollary

$$(i/\hbar)\hat{p}_\Delta\varphi = \int_{\mathbb{R}} K_\Delta(x-x')(d\varphi/dx')dx'$$

$$= \int_{\mathbb{R}} (d/dx)(K_\Delta(x-x'))\varphi(x')dx'$$

Proof

Evaluate (6.5.2) as convolution integrals.

According to (6.2.15), the spectral function of $\hat{p}_\Delta$ is given by

(6.5.4) $E(\hat{p}_\Delta;I) = E(\hat{p};\Delta\cap I) + E(\hat{p};\Delta^\perp)E(0;I)$, $I\in\mathcal{B}(\mathbb{R})$.

This yields, for each vector φ in $\mathcal{H}$:

$$(6.5.5)\quad E(\hat{p}_\Delta;I)\varphi = (2\pi\hbar)^{-\frac{1}{2}}\{\int_{\Delta\cap I}\exp[ipx/\hbar]\tilde{\varphi}(p)dp + \chi_I(0)\int_{I\setminus\Delta}\exp[ipx/\hbar]\tilde{\varphi}(p)dp\}$$

where χ_I is the characteristic function of the set I. The spectrum $\sigma(\hat{p})$ of $\hat{p}$ is the entire real line $\mathbb{R}$, from which we deduce that for each $\Delta\in\mathcal{B}(\mathbb{R})$

$$(6.5.6)\quad \sigma(\hat{p}_\Delta) = \Delta\cup\{0\}$$

using theorem (6.2.17).

The preceding results serve only to give a mathematical form to $\hat{p}_\Delta$. It is of interest to consider in which ways the limited observables $\hat{p}_\Delta$ yield conceptual differences from the usual momentum observable in physical analyses. In particular we shall investigate the uncertainty principle in relation to such limited operators. We start by proving some elementary mathematical results.

(6.5.7) Lemma

Suppose that $\Delta = [p_1, p_2]$, and let φ be any vector in $\mathcal{D}([x,\hat{p}_\Delta])$. Then

$$[x,\hat{p}_\Delta]\varphi = (2\pi\hbar)^{-\frac{1}{2}}\{p_2\exp[ip_2x/\hbar]\tilde{\varphi}(p_2) - p_1\exp[ip_1x/\hbar]\tilde{\varphi}(p_1)\}.$$

Proof

$$[x,\hat{p}_\Delta]\varphi = x\hat{p}E(\hat{p};\Delta)\varphi - \hat{p}E(\hat{p};\Delta)x\varphi$$

$$= (2\pi\hbar)^{-\frac{1}{2}}\{ \int_\Delta x p \exp[ipx/\hbar]\tilde{\varphi}(p)dp$$

$$- \int_\Delta p\exp[ipx/\hbar](\tilde{x\varphi})(p)dp\}$$

Now $(\tilde{x\varphi})(p) = i\hbar(d\tilde{\varphi}/dp)(p)$, so that

$$\int_\Delta p\exp[ipx/\hbar](\tilde{x\varphi})(p)dp$$

$$= \int_\Delta i\hbar p\exp[ipx/\hbar](d\tilde{\varphi}/dp)(p)dp$$

$$= (i\hbar p\exp[ipx/\hbar]\tilde{\varphi}(p))|_\Delta + \int_\Delta \tilde{\varphi}(p)px\exp[ipx/\hbar]dp$$

Substituting in the above we obtain

$$[x,\hat{p}_\Delta]\varphi = (2\pi\hbar)^{-\frac{1}{2}}(p\exp[ipx/\hbar]\tilde{\varphi}(p))$$

$$= (2\pi\hbar)^{-\frac{1}{2}}\{p_2\exp[ip_2x/\hbar]\tilde{\varphi}(p_2) - p_1\exp[ip_1x/\hbar]\tilde{\varphi}(p_1)\}$$

(6.5.8) Corollary

Let Δ,φ be as in (6.5.7). Then

$$\langle[x,\hat{p}_\Delta];\varphi\rangle = -i\hbar\{p_2|\tilde{\varphi}(p_2)|^2 - p_1|\tilde{\varphi}(p_1)|^2\}$$

Proof

$$\langle[x,\hat{p}_\Delta];\varphi\rangle = -i\hbar(2\pi\hbar)^{-\frac{1}{2}}\{p_2\tilde{\varphi}(p_2)\int\varphi^*(x)\exp[ip_2x/\hbar]dx$$

$$+ p_1\tilde{\varphi}(p_1)\int\varphi^*(x)\exp[ip_1x/\hbar]dx\}$$

$$= -i\hbar\{p_2|\tilde{\varphi}(p_2)|^2 - p_1|\tilde{\varphi}(p_1)|^2\}.$$

We now use the general uncertainty analysis (2.5) to deduce "uncertainty relations" for the operators x and

$\hat{P}_\Delta$.

(6.5.9) Theorem

Let Δ, φ be as in (6.5.7). Then

$$\Delta_\varphi x . \Delta_\varphi \hat{P}_\Delta \geq (\hbar/2) \left| \{ p_2 |\tilde{\varphi}(p_2)|^2 - p_1 |\tilde{\varphi}_1(p_1)|^2 \} \right| .$$

Proof

(2.5.1) and Corollary (6.5.8).

What is significant about this result is that the uncertainty product $\Delta_\varphi x . \Delta_\varphi \hat{P}_\Delta$ has no state independent minimum. For certain states therefore the minimum uncertainty can be as small as zero, which allows for the possibility that the product itself may be equal to zero, or at least arbitrarily small. This is a rather surprising consequence of the spectrum-limited approach. It demonstrates in particular the independence of operator relations of the form (6.5.9) and heuristic uncertainty in the sense of some of the early analyses [cf (2.5)], since momentum is no longer represented by an operator which yields a minimum uncertainty product strictly greater than zero.

6.6 Remarks on the Localisation of Limited Observables

It is immediately obvious from the results of (6.2) that an operator T is Δ-limited for some Δ if and only if it is bounded. This obviously has a happy consequence for the programme of localisation of observables. It will be remembered that a satisfactory and mathematically straightforward theory of local observables exists only for bounded observables. Hitherto this has presented difficulties in generalising the theory to many observables of common interest to physics: eg, momentum, energy, which are commonly regarded as unbounded observables. Using the present Δ-limitation of operators it is now possible to set about providing a consistent localisation for such physically essential observables. For example we may localise the momentum by considering the localisation (3.2.1) of the limited momentum operator $\hat{p}_\Delta = \hat{p}E(\hat{p};\Delta)$. The localisation is straightforward. We define an operator $(\hat{p}_\Delta)_\Lambda$ by

$$(6.6.1)\quad (\hat{p}_\Delta)_\Lambda = E(x;\Lambda)\hat{p}_\Delta E(x;\ ) = E(x;\Lambda)\hat{p}E(\hat{p};\Delta)E(x;\Lambda)$$

$$\mathfrak{D}((\hat{p}_\Delta)_\Lambda) = \mathfrak{H}\,.$$

Calculation of the uncertainty product for this operator is straightforward if we use the fact that

$[x,(\hat{p}_\Delta)_\Lambda] = E(x;\Lambda)[x,\hat{p}_\Delta]E(x;\Lambda) = [x_\Lambda, p_\Delta]$

where x_Λ is the Λ-limited position observable. Then it is a matter of straightforward calculation to show that the uncertainty product is exactly as for p_Δ but with replaced by φ_Λ where $\varphi_\Lambda = E(x;\Lambda)\varphi$. Again we see that it is possible for this minimum product to vanish for a particular wavefunction. Another point arises from this. Comparison of the pair $x,(\hat{p}_\Delta)_\Lambda$ and the pair $x_\Lambda,\hat{p}_\Delta$ provides an interesting insight into the processes of localisation and limiting spectra: limiting the spectrum of the position observable corresponds to localising the limited momentum observable here.

Finally we observe that it is also possible to localise simultaneously in position and momentum by considering the spectrum limitation $(\hat{\mathfrak{z}p})_\Delta$ of the local momentum operator of chapter 4 given by

(6.6.2)

$$(\hat{\mathfrak{z}p})_\Delta = E(\hat{\mathfrak{z}p};\Delta)\hat{\mathfrak{z}p}E(\hat{\mathfrak{z}p};\Delta)$$

$$\mathfrak{D}((\hat{\mathfrak{z}p})_\Delta) = \mathfrak{H}.$$

It is an interesting question to consider what physical and mathematical differences exist between these two formulations. In particular for certain wavefunctions

the two operators $(\hat{p}_{\Delta})_{\Lambda}$ and $(\widehat{\xi p})_{\Delta}$ (and their projections) may be expected to differ only in the boundary regions of the apparatus so that the question of which localisation to consider first is resolved.

CHAPTER 7

TIME-DEVELOPMENT OF LOCALISED SYSTEMS

7.1 Introduction

A well-known result on free nonrelativistic wavepackets is the fact that any wavepacket localised in some bounded region of configuration space at time t=0, spreads instantaneously over all space in subsequent time [Amrein 1981 eg]. A similar quite general result is proved even for relativistic quantum systems by Hegerfeldt and Ruijsenaar [1980]. The nonrelativistic result is easily demonstrated.

Suppose $\varphi_0(x)$ is the wavefunction at time t=0 for a free quantum mechanical particle moving in the one-dimensional configuration space $\mathbb{R}$. Suppose further that φ_0 satisfies

(7.1.1) $$\varphi_0(x) = E(x;\Lambda)\varphi_0(x).$$

The time development of such a wavefunction can be given the integral representation [cf (2.1.4)]

(7.1.2)
$$\varphi_t(x) = (2\pi\hbar)^{-\frac{1}{2}}\int_{\mathbb{R}} dp\ dx'\ \varphi_0(x')\exp[ip(x-x')/\hbar]\exp[-ip^2 t/2m\hbar]$$

Completing the square in the argument for the exponential and performing the p-integration one obtains [Feynmann and Hibbs 1965 pp96-8]

(7.1.3)

$$\varphi_t(x) = (m/2\pi i\hbar t)^{\frac{1}{2}} \exp[imx^2/2\hbar t) \int_{\mathbb{R}} \exp[-imxx'/\hbar t + imx'^2/2\hbar t]\varphi_o(x')dx'$$

Put $\psi_t(x) = \exp[imx^2/2\hbar t]\varphi_o(x)$ and it follows from (7.1.1) that $E(x;\Lambda)\psi_t(x) = \psi_t(x)$, while from (7.1.3) we deduce that

$$(7.1.4) \quad \varphi_t(x) = (m/it)^{\frac{1}{2}} \exp[imx^2/2\hbar t]\tilde{\psi}_t(xm/t),$$

where $\tilde{\psi}_t$ denotes the Fourier transform of ψ_t. It is standard result in Fourier analysis [cf (4.1)] that the transform of a function with bounded support is non-vanishing almost everywhere and hence for each $t > 0$, $\varphi_t(x)$ is nonvanishing almost everywhere in x. Now suppose that x_o and x_t represent the time dependent position operators at times 0 and t respectively in the Heisenberg picture [cf (2.1)]. The following theorem expresses the instantaneous spreading of wave packets rather neatly.

(7.1.5) Theorem

For each Λ, Δ in $\mathcal{B}_c(\mathbb{R})$ we have

$$E(x_o;\Lambda)_\wedge \; E(x_t;\Delta) = 0.$$

Proof

Suppose the theorem does not hold. Then there exists a nonzero vector ψ in $L^2(\mathbb{R})$ such that the simultaneous equations

$$(i) \qquad E(x_o;\Lambda)\psi = \psi$$

$$(ii) \quad E(x_t;\Delta)\psi = \psi$$

are satisfied. The second of these equations implies that

$$U_t^{-1} E(x_o;\Delta) U_t \psi = \psi$$

which in turn implies that

$$E(x_o;\Delta)\psi_t = \psi_t .$$

But it follows from the above reasoning [(7.1.2)-(7.1.4)] that this equation cannot be simultaneously satisfied with (i) and the result follows by contradiction.

This result is a neat analogy to (and indeed arises from) the complementarity theorem (2.5.6). Another analogy can be drawn between time development and the canonically conjugate pair. Namely, there exists a sort of uncertainty relation linking the position at time 0 and the position at time t.

(7.1.6) Theorem

Let φ be a member of the appropriate domain [cf (2.5)]. Then

$$\Delta_\varphi x_o . \Delta_\varphi x_t \geqslant \hbar t/2m.$$

We start by evaluating formally the commutator bracket $[x_o, U_t]$.

$$\begin{aligned}
[x_o, U_t] &= x_o \exp[-i\hat{p}^2 t/2m\hbar] - \exp[-i\hat{p}^2 t/2m\hbar] x_o \\
&= \sum_{n=0}^{\infty} (-it/2m\hbar)^n [x_o, \hat{p}^{2n}]/n! \\
&= \sum_{n=0}^{\infty} (-it/2mh)^n \, 2ni\hbar\hat{p}^{2n-1}/n! \quad \text{[Prugovecki 1971 p333]} \\
&= \sum_{n=1}^{\infty} (-it/2m\hbar)^{n-1} (t/m)\hat{p}^{2n-1}/(n-1)!
\end{aligned}$$

$$= (\hat{p}t/m)\sum_{n=0}^{\infty}(-it/2m\hbar)^n \hat{p}^{2n}/n!$$

$$= (\hat{p}t/m)\exp[-i\hat{p}^2 t/2m\hbar]$$

$$= (\hat{p}t/m)U_t .$$

Now we evaluate the commutator bracket $[x_o, x_t]$ using the above result.

$$[x_o, x_t] = [x_o, U_t^{-1} x_o U_t]$$

$$= x_o U_t^{-1} x_o U_t - U_t^{-1} x_o U_t x_o$$

$$= x_o + x_o U^{-1}[x_o, U_t] - x_o - U_t^{-1}[x_o, U_t]x_o$$

$$= (t/m)[x_o, \hat{p}]$$

$$= i\hbar t/m$$

The result now follows by substituting this commutation bracket in the usual mathematical analysis:

$$\Delta_\varphi x_o . \Delta_\varphi x_t \geq |\langle [x_o, x_t]; \varphi \rangle| \ , \ [\text{cf}(2.5.4)].$$

It may be noted that this inequality embodies the phenomenon known as the spreading of the wavepacket in the sense that as t tends to infinity, $\Delta_\varphi x_o . \Delta_\varphi x_t$ tends to infinity. No matter how small the original uncertainty, for large times the uncertainty in position becomes increasingly greater. A similar result associated with the spreading of wavepackets is the following theorem.

(7.1.7) Theorem

Let $\varphi \in L^2(\mathbb{R})$, V_t an evolution group such that $V_t \in L^\infty(p)$. Then for every bounded $\Lambda \in \mathcal{B}(\mathbb{R})$ we have

$$\lim_{t \to \infty} \| E(x; \Lambda) V_t \varphi \| = 0.$$

In particular of course the result holds for $V_t = U_t =$

$\exp[-i\hat{p}^2 t/2m\hbar]$.

Proof

Amrein 1981 p132

The physical interpretation of this result is that the probability of finding a quantum mechanical particle in any finite region Λ at time t later approaches zero as t tends to infinity (or negative infinity).

All of this does not bode well for our programme of localising quantum systems. The measurement situations we have been at pains to respect seem doomed by instantaneous spreading of systems as soon as we allow time to pass. In this chapter we shall investigate possible solutions to this dilemma. We propose, for instance, a time evolution "in a box" based on the time evolution operator $U_{\mathfrak{Z}t} = \exp[-i\hat{\mathfrak{Z}}p^2 t/2m\hbar]$. This time evolution will at least allow for systems to remain localised within an apparatus. Comparison of this operator with the usual time evolution is made for initially localised systems. We also describe some results of the WM-theory [Wan and McLean 1983] which indicate how a localisation of sorts is possible asymptotically. This localisation is within unbounded regions and we discuss whether it is possible to localise in bounded regions at infinity. Finally we shall discuss the separation

of particles in two-particle systems.

7.2 Time Evolution in a Box

We consider the time evolution operator $U_{\xi t}$ given by

(7.2.1)

$$U_{\xi t} = \exp[-i\hat{\xi p}^2 t/2m\hbar]$$

where ξ is a C^∞-function of compact support in $\Lambda = (a,b)$ which takes the value 1 on $\Lambda_0 = [a_0, b_0] \subsetneq (a,b)$. $\hat{\xi p}$ is therefore a local momentum observable as defined in (4.2.1):

$$\hat{\xi p}\varphi = -i\hbar(\xi(d\varphi/dx) + (1/2)\varphi(d\xi/dx))$$

$$\mathcal{D}(\hat{\xi p}) = \{\varphi \in L^2(\mathbb{R}) : \varphi \in AC(X,\mathbb{R}), \hat{\xi p}\varphi \in L^2(\mathbb{R})\}$$

with localised eigenfunctions [(4.4.1)]

$$F(\lambda,x) = (2\pi\xi\hbar)^{-\frac{1}{2}} \exp[(i\lambda/\hbar)\int_{x_0}^{x} \xi(x')^{-1}dx)] \quad x_0, x \in (a,b),$$

and spectral measure

$$E(\hat{\xi p};\Delta)\varphi = \int_\Delta F(\lambda,x)\{\int_\Lambda F^*(\lambda,x')\varphi(x')dx'\}d\lambda.$$

The completeness of the eigenfunctions in (a,b) [MacFarlane 1980] ensures that any wavefunction $\varphi(x)$ with support in (a,b) can be expressed in local

momentum representation space by

$$\langle F|\varphi\rangle = \int_{\Lambda} F(\lambda,x)^{*}\varphi(x)dx$$

In this space (7.2.1) becomes

$$U_{\mathfrak{s}t} = \exp[-i\lambda^{2}t/2m\hbar]$$

and hence we can express $U_{\mathfrak{s}t}\varphi(x)$ as follows

$$(7.2.2)\quad U_{\mathfrak{s}t}\varphi(x) = \int_{\mathbb{R}} F(\lambda,x)\langle F|\varphi\rangle\exp[-i\lambda^{2}t/2m\hbar]d\lambda.$$

Now for all x in Λ_{o} we have [(4.4.4)]

$$F(\lambda,x) = \exp[-i\lambda x_{o}/\hbar]f(\lambda,x),$$

where $f(\lambda,x)$ are the generalised momentum eigenfunctions. Furthermore if we impose the restriction that φ vanishes outside the centre of localisation Λ_{o} then it can be demonstrated [cf(4.4.4)] that

$$\langle F|\varphi\rangle = \exp[i\lambda x_{o}/\hbar]\langle f|\varphi\rangle$$

where $\langle f|\varphi\rangle$ is the representation of φ in the momentum space given by [(2.1.3)]

$$\langle f|\varphi\rangle = (2\pi h)^{-\frac{1}{2}}\int\varphi(x)\exp[-i\lambda x/\hbar].$$

From all this it is straightforward to deduce the following result.

(7.2.3) Theorem

Let $\varphi_{o}\in L^{2}(\mathbb{R})$, $\varphi_{o}(x) = 0$, $x\notin\Lambda_{o}$, then

$$U_{\mathfrak{z}t}\varphi_o(x) = U_t\varphi_o(x), \text{ for all } x \in \Lambda_o .$$

In other words the time development in the centre of localisation of a wavefunction φ_o initially lying entirely within the centre of localisation may be described equally by the usual time evolution operator U_t or by the local time evolution operator $U_{\mathfrak{z}t}$ for all subsequent time. Physically speaking, as far as the interior of the apparatus is concerned the local operator is exactly the same as the global one in describing the time evolution of the system. The difference between the two is that, in contrast to Theorem (7.1.5) above there are many solutions to the simultaneous equations

$$(7.2.4) \qquad \begin{aligned} E(x;\Lambda_o)\varphi_o &= \varphi_o \\ E(x;\Lambda)U_{\mathfrak{z}t}\varphi_o &= U_{\mathfrak{z}t}\varphi_o ; \end{aligned}$$

it is possible to retain a localised system within the apparatus for subsequent time. Equally it is obvious that

$$(7.2.5) \qquad \lim_{t\to\infty} \| E(x;\Lambda)U_{\mathfrak{z}t}\varphi_o \| = 1,$$

which is in sharp contrast to Theorem (7.1.6).

Now suppose that we denote by $\hat{\hat{x}}_t$ the time development of the position operator using the local evolution operator. In other words $\hat{\hat{x}}_t$ is given by

$$(7.2.6) \qquad \hat{\hat{x}}_t = \exp[i\hat{\xi}p^2 t/2m\hbar]x_o \exp[-i\hat{\xi}p^2 t/2m\hbar].$$

It is worthwhile to consider whether we may draw some kind of analogy with the "uncertainty" theorem (7.1.6) by considering the uncertainty product $\Delta_\varphi x_o . \Delta_\varphi \hat{\hat{x}}_t$ for the local time evolution. In fact we can prove the following, remarkably similar result for the local case.

(7.2.7) Theorem

Let $\varphi_o \in L^2(\mathbb{R})$ satisfy $E(x_o;\Lambda_o)\varphi_o = \varphi_o$. Then for all $t > 0$

$$\Delta_\varphi x_o . \Delta_\varphi \hat{\hat{x}}_t \geqslant \hbar t/2m.$$

Proof

We start by computing the commutation relations for x_o and $\hat{\xi}p^{2\Lambda}$. Proceeding formally, and using the equivalent representation $\hat{\xi}p = \xi^{\frac{1}{2}}\hat{p}\xi^{\frac{1}{2}}$, we obtain first

$$[x_o, \hat{\xi}p^{2\Lambda}] = \xi^{\frac{1}{2}}[x_o\hat{p}\xi \ldots \xi\hat{p} - \hat{p}\xi \ldots \xi\hat{p}x_o]\xi^{\frac{1}{2}}.$$

Using the operator relation $[f,\hat{p}] = -\hat{p}(f)$ (cf $[x,\hat{p}] = i\hbar$), and assuming a compatible domain, the commutator relation becomes (after some computation)

$$[x_o, \xi\hat{p}^{2\Lambda}] = i\hbar\xi^{\frac{1}{2}}\{2n\hat{p}\xi \ldots \hat{p}\xi + \ldots$$
$$\ldots + (-1)^{r-1}(2n!/(2n-r)!r!)\overbrace{\hat{p}\xi \ldots \hat{p}}^{2\Lambda-r}\xi^{\frac{1}{2}}\xi^{\frac{1}{2}}\hat{p}\xi \ldots \xi\hat{p}(\xi) + \ldots$$
$$\ldots + (-1)^{2\Lambda-1}\hat{p}\xi \ldots \hat{p}\xi\}\xi^{\frac{1}{2}}$$

$$= i\hbar\xi^{1/2}\{\sum_{r=1}^{2n}(-1)^{r-1}(2n!/(2n-r)!r!)\overbrace{\hat{p}\xi..\xi\hat{p}}^{2n-r}\xi^{\frac{1}{2}}\xi^{\frac{1}{2}}\overbrace{\hat{p}\xi..\xi\hat{p}}^{r}\}\xi^{1/2}$$

This rather complicated looking expression is substantially simplified when we remember that for $r \geqslant 2$ the summands contain the term $d\xi/dx$ which vanishes on Λ_o. It follows that

$$[x_o, \hat{\xi p}^{2n}] = 2ni\hbar\hat{\xi p}^{2n-1}.$$

Hence the commutator bracket is exactly the same as for the global momentum case [cf proof of (7.1.6)] but with $\hat{p}$ replaced by $\hat{\xi p}$. The analysis of (7.1.6) goes through exactly as before then and we obtain

$$[x_o, U_{\xi t}] = (\hat{\xi p}t/m)U_{\xi t}$$

and

$$\begin{aligned}[x_o, \hat{\hat{x}}_t]\varphi &= (t/m)[x_o, \hat{\xi p}]\varphi \\ &= -(i\hbar t/m)(x\hat{\xi p}\varphi - \hat{\xi p}(x\varphi)) \\ &= -(i\hbar t/m)(x\hat{\xi p}\varphi - x\hat{\xi p}\varphi - \varphi\hat{\xi p}(x)) \\ &= (i\hbar\xi t/m)\varphi.\end{aligned}$$

Since φ_o vanishes outside Λ_o we deduce that

$$\langle[x_o, \hat{\hat{x}}_t]; \rangle = \hbar t/m, \quad t \geqslant 0,$$

and the result follows from the usual mathematical relation concerning the product of the variances [cf (2.5.4)].

7.3 Asymptotic Localisation and Separation

The result of theorem (7.1.6) indicates that as time tends to infinity a quantum mechanical system governed by the conventional time evolution cannot be localised in any finite region in space. There is however a sense in which we may consider a system to be asymptotically localised for large times even when the global time evolution is used.

There is a much discussed experimental procedure in quantum mechanics known as the "time of flight" measurement of momentum [de Broglie 1930 p156, Feynmann and Hibbs 1965 pp96-8, Gottfried 1966 p12, Jayaram 1966, Kemble 1937 p58 et seq, Park and Margenau 1968, Raith 1976]. The idea of this experiment is to determine the momentum of a quantum mechanical particle by measuring the time taken for the particle to travel from one specified point in the configuration space to another. In classical mechanics this is a straightforward enough thing to do. Suppose that initially the (classical) particle is at position x and after time t it arrives at x_t. Assuming that the particle travels freely, ie no external forces act on the particle, then the velocity of the particle between the specified points is $(x_t - x_o)/t$ and hence the momentum is given by $m(x_t - x_o)/t$. In quantum mechanics,

as we know, no such straightforward description is possible due to the probabilistic nature of the theory, the spreading of the wavepacket, the incompatibility of position and momentum measurements etc. It can be shown instead [Schiff 1955 p29, cf also Farina 1984] that the average (or expectation) values for the quantum mechanical position and momentum measurements satisfy $\langle x;\varphi\rangle = \langle \hat{p};\varphi\rangle t/m$.

There is another way of looking at the problem. Consider a classical ensemble of particles all of whose momenta lie in some finite range [p,p'] in $\mathbb{R}$. Suppose that at time t=0 all the particles are located at x=0. After time t we can say that the ensemble may be found within the region [pt/m,p't/m] in the configuration space. A quantum mechanical counterpart to this result is proved by Park and Margenau (eg) in their analysis of the time of flight experiment for a restricted class of wavefunctions. A more general result has been proved by Wan and McLean.

(7.3.1) Theorem [Wan and McLean 1983(a) Theorem 1] Let $\varphi \in L^2(\mathbb{R}^n)$, and suppose that $[\underline{v},\underline{v}']$ is any proper closed interval in $\mathbb{R}^n$. Then

$$\lim_{t\to\infty} \| E(\underline{x};[\underline{v}t,\underline{v}'t])U_t\varphi \| = \| E(\underline{\hat{p}};[m\underline{v},m\underline{v}'])\varphi \| .$$

Physically speaking, this result mimics asymptotically the property of the classical ensemble. Notice that (7.3.1) does not conflict with (7.1.6) since the "localisation" takes place only in an unbounded interval. Now we provide a formal definition for this "localisation".

(7.3.2) Definition [Wan and McLean 1983(a)]
A quantum mechanical particle described by the state vector $\varphi \in L^2(\mathbb{R}^n)$ is said to be asymptotically localisable if there exists a proper closed interval $[\underline{v},\underline{v}']$ in $\mathcal{B}(\mathbb{R}^n)$ such that $\|E(\underline{x};[\underline{v}t,\underline{v}'t])U_t\varphi\| = 1$. We shall also say that the particle is asymptotically localised in $[\underline{v}t,\underline{v}'t]$.

We have immediately the following result.

(7.3.3) Corollary [op.cit]
A quantum mechanical particle described by the state vector φ is asymptotically localisable if and only if there exists a proper, closed interval $[\underline{p},\underline{p}']$ such that $\|E(\hat{\underline{p}};[\underline{p},\underline{p}'])\varphi\| = 1$.
Proof
Evidently we have only to take $[\underline{p},\underline{p}'] = [m\underline{v},m\underline{v}']$ where m is the mass of the particle and the result follows from theorem (7.3.1).

We introduce a further definition.

(7.3.4) Definition [op.cit]
Two states φ,ψ of a one particle system are said to be asymptotically separating if there exist disjoint proper closed intervals $[\underline{u},\underline{u}']$ and $[\underline{v},\underline{v}']$ in $\mathbb{R}^n$ such that φ is asymptotically localised in $[\underline{u}t,\underline{u}'t]$ and ψ is asymptotically localised in $[\underline{v}t,\underline{v}'t]$.

The effect of the definition is that the asymptotically separating states are those which are in disjoint spatial regions for large times.

(7.3.5) Corollary [op.cit]
Two vector states φ and ψ are asymptotically separating if and only if there exist disjoint proper closed intervals $[\underline{p},\underline{p}']$ and $[\underline{r},\underline{r}']$ such that
$\|E(\hat{\underline{p}};[\underline{p},\underline{p}'])\varphi\| = 1 = \|E(\hat{\underline{p}};[\underline{r},\underline{r}'])\psi\|$.

7.4 Asymptotic Separation for Two Particle Systems

Now let us consider the case of a two-particle system. Obviously we can extend the notion of asymptotic localisation of the states. We state first the two-particle equivalent of theorem (7.3.1).

(7.4.1) Theorem [Wan and McLean 1984(c) Theorem 2]
Let $\Phi \in L^2(\mathbb{R}^{2n})$. Let m_1 be the mass of particle 1 and m_2 be the mass of particle 2. Then for every proper closed interval $[\underline{v},\underline{v}']$ in $\mathbb{R}^n$ we have

$$\lim_{t\to\infty} \|(E(\underline{x}_1;[vt,v't])\otimes 1)U_t\Phi\| = \|(E(\hat{\underline{p}}_1;[m_1\underline{v},m_1\underline{v}'])\otimes 1)\Phi\|$$

and

$$\lim_{t\to\infty} \|(1\otimes E(\underline{x}_2;[\underline{v}t,\underline{v}'t]))U_t\Phi\| = \|(1\otimes E(\hat{\underline{p}}_2;[m_2\underline{v},m_2\underline{v}']))\Phi\| .$$

One can introduce definitions of the asymptotic localisation and separation of states quite analogously to the one-particle case [op.cit Definition 2]. We remark here that in the present analysis we confine ourselves to the term asymptotically separating to describe such states. In the previous WM analysis, such states are sometimes called asymptotically separable. However it does not in general hold that states which are separating are separable as we shall see later on [Chapter 9]. For the purposes of the theory to be presented in this thesis a more useful concept for two-particle systems is the notion of asymptotically separating particles.

(7.4.2) Definition [McLean 1984]
The two particles in the vector state $\Phi \in L^2(\mathbb{R}^{2n})$ are said to be asymptotically separating if there exist disjoint proper closed intervals $[\underline{v}_1,\underline{v}_1']$ and $[\underline{v}_2,\underline{v}_2']$ in $\mathbb{R}^n$ such that

$$1 = \lim_{t\to\infty} \|(E(\underline{x}_1;[\underline{v}_1 t,\underline{v}_1' t])\otimes 1)U_t\Phi\|$$
$$= \lim_{t\to\infty} \|(1\otimes E(\underline{x}_2;[\underline{v}_2 t,\underline{v}_2' t]))U_t\Phi\| .$$

For a simple tensor state $\Phi = \varphi \otimes \psi$ the effect of this is easy to visualise physically. The two particles are separating if their wavefunctions end up in disjoint spatial regions "at infinity". There is an obvious extension of Corollary (7.3.5).

(7.4.3) Corollary [McLean 1984]

The two particles in the state Φ are asymptotically separating if and only if there exist disjoint intervals $[\underline{p}_1, \underline{p}_1']$ and $[\underline{p}_2, \underline{p}_2']$ such that

$$\|(E(\hat{\underline{p}}_1; [\underline{p}_1, \underline{p}_1']) \otimes 1)\Phi\| = 1 = \|(1 \otimes E(\hat{\underline{p}}_2; [\underline{p}_2, \underline{p}_2']))\Phi\| .$$

When we come to consider two particle systems with spin a further degree of generalisation is required.

(7.4.4) Theorem

Let Φ^σ be a vector state of a system comprising two spin-half particles with masses m_1 and m_2. Let $[\underline{y}_1, \underline{y}_1']$ and $[\underline{y}_2, \underline{y}_2']$ be proper closed intervals in $\mathbb{R}^n$. Then

$$\lim_{t \to \infty} \|((E(\underline{x}_1; [\underline{y}_1 t, \underline{y}_1' t]) \otimes 1) \dot{\otimes} 1) U_t^\sigma \Phi^\sigma\|$$
$$= \|((E(\hat{\underline{p}}_1; [m_1 \underline{y}_1, m_1 \underline{y}_1']) \otimes 1) \dot{\otimes} 1)\Phi^\sigma\| \quad \text{and}$$
$$\lim_{t \to \infty} \|((1 \otimes E(\underline{x}_2; [\underline{y}_2 t, \underline{y}_2' t])) \dot{\otimes} 1) U_t^\sigma \Phi^\sigma\|$$
$$= \|((1 \otimes E(\hat{\underline{p}}_2; [m_2 \underline{y}_2, m_2 \underline{y}_2'])) \dot{\otimes} 1)\Phi^\sigma\| .$$

where U_t^σ is the time-evolution operator for the two-particle spin system, namely: $U_t^\sigma = U_t \dot{\otimes} 1$.

Proof

The degree of generalisation in taking the tensor product of

the L^2 space with the four-dimensional spin space is in fact less then the generalisation already incurred in taking the tensor product of the two L^2 spaces. The form of Wan and McLean 's proof for (7.4.1) may therefore be employed a fortiori for the present case.

Of course we can then generalise the definition (7.4.2) and the corollary (7.4.3) to incorporate the spin, simply by including the tensor product with the identity operator on the spin algebra. Finally we point out that generalisations of these results exist in two further senses. Firstly it is possible to prove the following.

(7.4.5) Theorem

Let w^σ be any normal state on $\mathcal{B}(\mathcal{H}_c^\sigma)$ the algebra of bounded operators on a system of two spin-half particles, and let $[\underline{y}_1, \underline{y}_1']$, $[\underline{y}_2, \underline{y}_2']$ be proper closed intervals in $\mathbb{R}^\wedge$. Then

$$\lim_{t\to\infty} w_t^\sigma((E(\underline{x}_1; [\underline{y}_1 t, \underline{y}_1' t]) \otimes 1) \dot{\otimes} 1)$$
$$= w^{\sigma\infty}((E(\hat{\underline{p}}_1; [m_1 \underline{y}_1, m_1 \underline{y}_1']) \otimes 1) \dot{\otimes} 1), \text{ and}$$

$$\lim_{t\to\infty} w_t^\sigma((1 \otimes E(\underline{x}_2; [\underline{y}_2 t, \underline{y}_2 t])) \dot{\otimes} 1)$$
$$= w^{\sigma\infty}((1 \otimes E(\hat{\underline{p}}_2; [m_2 \underline{y}_2, m_2 \underline{y}_2'])) \dot{\otimes} 1),$$

where $w_t^\sigma(A) = w^\sigma(A_t)$ for every $A \in \mathcal{B}(\mathcal{H}_c^\sigma)$, A_t is the time developed operator in the Heisenberg picture [cf (2.1)], and $w^{\sigma\infty}$ is the state at infinity generated by w^σ.

Proof

Notice firstly that $w^{\sigma\infty}((E(\hat{\underline{p}}_1; [m_1 \underline{y}_1, m_1 \underline{y}_1']) \otimes 1) \dot{\otimes} 1)$ is equal to $w^\sigma((E(\hat{\underline{p}}_1; [m_1 \underline{y}_1, m_1 \underline{y}_1']) \otimes 1) \dot{\otimes} 1)$ since the operator argument is in $L^\infty(\underline{p}_1, \underline{p}_2) \dot{\otimes} \mathcal{S}_c$ and that if w^σ is given by a pure

state Φ^σ, then the theorem reduces to (7.4.4). For the general case we remark that a normal state w^σ is given by a convex linear sum of pure states

$$w^\sigma = \sum_i \lambda_i \rho^\sigma_{\Phi_i} \quad , \quad \lambda_i > 0$$

where $\sum_i \lambda_i = 1$. Hence we have

$$\lim_{t\to\infty} w^\sigma_t((E(\underline{x}_1;[\underline{y}_1\, t, \underline{y}_1' t]) \otimes 1) \dot{\otimes} 1)$$
$$= \lim_{t\to\infty} \| \sum_i \lambda_i ((E(\underline{x}_1;[\underline{y}_1\, t, \underline{y}_1' t]) \otimes 1) \dot{\otimes} 1) U^\sigma_t \Phi^\sigma_i \|.$$

Now the terms of the sum are uniformly bounded by λ_i since $\|((E(x_1;[\underline{y}_1\, t, \underline{y}_1' t]) \otimes 1) \dot{\otimes} 1) U^\sigma_t \Phi^\sigma_i\|$ is bounded uniformly by 1 and so, by the Weierstrass M-test [Apostol 1957 p396, eg] the sum converges uniformly. Hence we can interchange the limit and the sum to obtain

$$\lim_{t\to\infty} w^\sigma_t((E(\underline{x}_1;[\underline{y}_1\, t, \underline{y}_1' t]) \otimes 1) \dot{\otimes} 1)$$
$$= \sum_i \lambda_i \lim_{t\to\infty} \| ((E(\underline{x}_1;[\underline{y}_1\, t, \underline{y}_1' t]) \otimes 1) \dot{\otimes} 1) U^\sigma_t \Phi^\sigma_i \|$$
$$= \sum_i \lambda_i \| ((E(\underline{\hat{p}}_1;[m, \underline{y}_1, m, \underline{y}_1']) \otimes 1) \dot{\otimes} 1) \Phi^\sigma_i \|$$
$$= w^{\sigma\infty}((E(\underline{\hat{p}}_1;[m, \underline{y}_1, m, \underline{y}_1']) \otimes 1) \dot{\otimes} 1).$$

Similarly for the second particle.

A further generalisation is possible. In each of (7.3.1), (7.4.1),(7.4.4) and (7.4.5) it is possible to replace the projections of the position operator by any essentially bounded complex-valued function g(mx/t). A theorem in Amrein [1981,p123] ensures that the limit yields the function g(p). For details of this and an application of the more general result to states at infinity we refer the reader to Wan and Jackson [1985].

7.5 Spatial Separation for Finite Time

Finally we shall define a notion of spatial separation which is not asymptotic but exists for finite time t.

(7.6.1) Definition

Two states $\varphi, \psi \in \mathcal{H}$ are said to be spatially separate at time t if there exist $\Lambda, \Lambda' \in \mathcal{B}(\mathbb{R}^n)$ such that

$$\varphi_t = E(\underline{x}; \Lambda)\varphi_t$$

$$\psi_t = E(\underline{x}; \Lambda')\psi_t .$$

Two systems and described by the two-particle state vector Φ at time t are said to be spatially separate at time t if there exist $\Lambda, \Lambda' \in \mathcal{B}(\mathbb{R}^n)$ such that $\Lambda \cap \Lambda' = \emptyset$ and

$$\Phi = E(\underline{x}_1 \otimes \underline{x}_2 ; \Lambda \times \mathbb{R}^n)\Phi$$

$$= E(\underline{x}_1 \otimes \underline{x}_2 ; \mathbb{R}^n \times \Lambda')\Phi .$$

Evidently the instantaneous spreading of the wavepacket [(7.1)] means that even if such a localisation of subsystems in disjoint regions occurs for a particular time t there is going to be a subsequent overlapping of the systems for later time. However this does not prevent us considering the possibility that at a particular time t we have separation of the two systems (or states) into disjoint spatial regions. It will emerge later [cf (9.8)] that consideration of such spatial separation for finite time yields some insight into the problems of correlated subsystems in quantum mechanics. There is an obvious

extension of the concept of spatial separation for finite times to systems with spin.

CHAPTER 8

REDUCED STATISTICAL OPERATORS AND CORRELATIONS BETWEEN SUBSYSTEMS

8.1 Introduction

At the heart of the so-called paradoxes in quantum mechanics concerning two-particle systems (eg the EPR experiment [Einstein,Podolski and Rosen 1935, Bohm 1951]) lies the relationship between the state of the overall system I+II and the states of the individual component subsystems, namely the reduced statistical operators. In particular therefore we must elucidate the following two questions.

(8.1.1) Given a state w of I+II what are the states w_1 and w_2 of the individual component subsystems ?

(8.1.2) Given states w_1 and w_2 of systems I and II, what is the state w of the combined system ?

The next section in this chapter discusses these questions in the context of the conventional theory. Following this we examine the problem for the WM-algebra [cf(2.4)]. Defining, in section (8.4), several precise (and distinct) formulations used in the literature under the blanket term "correlations", we then show in what sense, correlations do and do not exist in the WM-theory.

We shall assume for most of what follows that we are dealing with a system of two distinguishable particles of masses m_1 and m_2, and to start with we are going to consider systems without spin. The time evolution will be taken as $U_t = U_{1t} \otimes U_{2t}$. This amounts to an assumption that the systems are non-interactive. Since we are concerned with free particles here, U_{1t} and U_{2t} will be the respective free particle evolution operators [cf (2.1.2)] for I and II.

8.2 Reduced Statistical Operators: Conventional Theory

Consider a state w of a two particle system given by the density operator ϱ on $\mathcal{H}_c$. Now one defines the one particle observables in a two particle system as the observables of the form $A_1 \otimes 1, 1 \otimes A_2$, with $A_1 \in \mathcal{B}(\mathcal{H}_1)$, $A_2 \in \mathcal{B}(\mathcal{H}_2)$. We shall follow the usual, very reasonable assumption that the statistics of the individual subsystems of the system, reflected in the reduced particle states w_1 and w_2, are consistent with the statistics of the one particle observables in the state w of the combined system. That is we have the following requirement [cf Beltrametti and Cassinelli 1981 p65]

(8.2.1) Consistency Requirement

The probability of obtaining a value in the set b on measurement of the observable A_1 of system I in state w_1 is equal to the probability of obtaining a value in the set $b \times \mathbb{R}$ on measurement of the observable $A_1 \otimes 1$ of the composite system in state w. Similarly for system II.

Mathematically requirement (8.2.1) yields the following.

$$
\begin{aligned}
&w_1(E(A_1;b)) = w(E(A_1 \otimes 1; b \times \mathbb{R})), \text{ for all } A_1 \in \mathcal{B}(\mathcal{H}_1) \\
&w_2(E(A_2;b)) = w(E(1 \otimes A_2; \mathbb{R} \times b), \text{ for all } A_2 \in \mathcal{B}(\mathcal{H}_2)
\end{aligned} \tag{8.2.2}
$$

for every $b \in \mathcal{B}(\mathbb{R})$.

Since we are considering the conventional formulation in which every bounded operator on $\mathcal{H}$ belongs to $\mathcal{A}$, (8.2.3) may be expressed in terms of the projections P_1 on $\mathcal{H}_1$, P_2 on $\mathcal{H}_2$ and in terms of the traces of the density operators ρ_1 and ρ_2 corresponding to w_1 and w_2. So we have

$$
\begin{aligned}
&\mathrm{Tr}(\rho_1 P_1) = \mathrm{Tr}(\rho(P_1 \otimes 1)), \text{ for all } P_1 \in \mathcal{L}(\mathcal{H}_1), \\
&\text{and} \\
&\mathrm{Tr}(\rho_2 P_2) = \mathrm{Tr}(\rho(1 \otimes P_2)), \text{ for all } P_2 \in \mathcal{L}(\mathcal{H}_2).
\end{aligned} \tag{8.2.3}
$$

We point out that this consistency requirement could as easily have been formulated in terms of expectation values, since these are interderivable with probabilities [cf(2.6)]. Thus we have, equally

$$\begin{aligned} &\mathrm{Tr}(\rho_1 A_1) = \mathrm{Tr}(\rho(A_1 \otimes 1)), \text{ for all } A_1 \in \mathfrak{B}(\mathcal{H}_1) \\ &\text{and} \\ &\mathrm{Tr}(\rho_2 A_2) = \mathrm{Tr}(\rho(1 \otimes A_2)), \text{ for all } A_2 \in \mathfrak{B}(\mathcal{H}_2). \end{aligned} \tag{8.2.4}$$

It turns out that these consistency requirements enable us to provide fairly concise answers to the questions (8.1.1) and (8.1.2). We shall state these results without proof and refer the reader to Beltrametti and Cassinelli [1981, Chapter 7] and von Neumann [1955] for further details.

(8.2.5) Lemma

Let ρ be the state of the composite system. Suppose ρ and ρ' both satisfy (8.2.2)-(8.2.4). Then the respective reduced statistical operators ρ_1 and ρ_1' for system I satisfy $\rho_1 = \rho_1'$. Similarly for system II.

We can derive explicit expressions for the unique reduced statistical operators generated by a specific state ρ of the composite system.

(8.2.6) Theorem

(i) Let $\{\varphi_i \otimes \psi_j\}$, $\varphi_i \in \mathcal{H}_1$, $\psi_j \in \mathcal{H}_2$, be an orthonormal basis for $\mathcal{H}_c$. Suppose ρ is a pure state on $\mathcal{B}(\mathcal{H}_c)$ given by the state vector Φ whose decomposition in the orthonormal basis is

$$\Phi = \sum_{i,j} \lambda_{ij} \varphi_i \otimes \psi_j \quad , \quad \lambda_{ij} \in \mathbb{C} \, , \, \sum_{i,j} |\lambda_{ij}|^2 = 1$$

Then ρ_1 and ρ_2 are given by

$$\rho_1 \varphi = \sum_{i,j} c_{ij} |\varphi_j\rangle\langle\varphi_i|\varphi\rangle$$

$$\rho_2 \psi = \sum_{i,j} d_{ij} |\psi_j\rangle\langle\psi_i|\psi\rangle$$

where $c_{ij} = \sum_k \lambda^*_{ik} \lambda_{jk}$, and $d_{ij} = \sum_k \lambda^*_{ki} \lambda_{kj}$.

(ii) If the state of the composite system is given by the mixture

$$\rho = \mu \rho' + (1-\mu)\rho'' \, , \, 0 < \mu < 1$$

then the reduced statistical operators are given by

$$\rho_1 = \mu \rho_1' + (1-\mu)\rho_1''$$

$$\rho_2 = \mu \rho_2' + (1-\mu)\rho_2''$$

where ρ_1' and ρ_2' are the reduced statistical operators corresponding to ρ' and ρ_1'', ρ_2'' are those corresponding to ρ''.

This theorem ensures that once the state of the composite system is known the reduced statistical operators are also determined. Moreover everything can be reduced (according to (ii)) to the case in which is pure and for which ρ yields precise expressions for the operators ρ_1 and ρ_2. The following corollaries detail certain specific cases.

(8.2.7) Corollary

(i) Suppose that $\Phi = \sum_i \lambda_i (\varphi_i \otimes \psi_i)$. Then

$$\rho_1 = \sum_i |\lambda_i|^2 P_{\varphi_i}$$

$$\rho_2 = \sum_i |\lambda_i|^2 P_{\psi_i}.$$

(ii) If ρ is a pure state represented by the simple tensor $\varphi \otimes \psi$, then ρ_1, ρ_2 are also pure states and are represented by the vectors φ and ψ respectively.

We now tackle the converse problem (8.1.2).

(8.2.8) Theorem

Let ρ_1 and ρ_2 be the density operators associated with two subsystems I and II of a composite system of particles I+II. Then

(i) The state $\rho = \rho_1 \otimes \rho_2$ is always a possible state for the system.

(ii) If ρ_1 and ρ_2 are pure states then the state of the composite system is uniquely determined as $\rho = \rho_1 \otimes \rho_2$.

(iii) If ρ_1 and ρ_2 are not pure then there may be a number of possible states for the composite system.

It is of interest to note that the proof of (iii) in (8.2.8) illustrates a feature peculiar to quantum mechanics, namely that the composite state may be pure while the reduced statistical states are mixed. We shall come across a particular example of this in the isotropic spin-zero vector of the EPR paradox.

8.3 Reduced Statistical States; The WM-Theory

Let us now turn our attention to the WM theory, in which the algebra of observables is given by the asymptotic algebra $\mathcal{A}_{WMc}$ [cf (2.4)] with states which include the normal NPLF's on $\mathcal{A}_{WMc}$ and the normal NPLF's at infinity on $\mathcal{A}_{WMc}$ [cf (2.4.3)]. As in the conventional case we shall adopt the physical requirement (8.2.1). Now however we must generalise the mathematical requirements to include all the states on the system. For every Borel set b we require:

$$\begin{aligned} &w_1(E(A_1;b)) = w(E(A_1;b)\otimes 1), \text{ for all } A_1 \in \mathcal{A}_{WM1} \\ &\text{and} \\ &w_2(E(A_2;b)) = w(1\otimes E(A_2;b)), \text{ for all } A_2 \in \mathcal{A}_{WM2} \end{aligned} \tag{8.3.1}$$

We shall also make use of the following equalities

corresponding to (8.2.4):

$$(8.3.2)\quad \begin{aligned} &w_1(A_1) = w(A_1 \otimes 1) \text{ for all } A_1 \in \mathcal{A}_{WM1}\ , \\ &\text{and} \\ &w_2(A_2) = w(1 \otimes A_2) \text{ for all } A_2 \in \mathcal{A}_{WM2}\ . \end{aligned}$$

If A is a selfadjoint element of the algebra $\mathcal{A}_{WM}$ where $\mathcal{A}_{WM}$ is a WM-algebra then [Wan and McLean 1984(a) Theorem 8] every projection of A is in $\mathcal{A}_{WM}$. It follows that (8.3.2) implies (8.3.1). Conversely the spectral theorem ensures that (8.3.1) implies (8.3.2).So these two conditions are again equivalent. For the case of the WM-theory we recall that states are of two distinct kinds. There are normal states given as usual by the density operators on $\mathcal{A}_{WM}$, and the states at infinity, generated from the normal states as the limits in time of the time-developed normal states [(2.4.3)]. We shall use the notation w_ϱ to specify normal states and w_τ^∞ to specify states at infinity. In both cases the subscripted greek letter denotes the density operator generating the state. A general state w is of the form [Wan and McLean 1984(b) Postulate 3] $w = \mu w_\varrho + (1-\mu) w_\tau^\infty$, $0 \leq \mu \leq 1$. Let us suppose first that the state w of the composite system has been given.

(8.3.3) Lemma

Let w be the state of I+II. If w_1 and w_1' are states of I which both satisfy (8.3.2) then $w_1 = w_1'$. If w_2 and

w_2 are states of II which both satisfy (8.3.2) then $w_2 = w_2'$.

Proof

The states w_1, w_1' are by definition elements of the dual space $\mathcal{A}^*_{wM1}$ of $\mathcal{A}_{wM1}$ [Bratteli and Robinson 1979 p48]. $w_1 - w_1'$ is a well-defined, though not necessarily positive, element of this space. Since $w_1(A_1) = w_1'(A_1)$ for every $A \in \mathcal{A}_{wM}$, we have $w_1 - w_1' = 0$. Hence the result.

We have shown here that the reduced statistical states (as we shall call them in a natural extension of the term reduced statistical operator) are uniquely determined by the state w of the composite system and we shall therefore refer to (8.3.3) as "the uniqueness lemma". We use this lemma to generalise Theorem (8.2.6) above to our present case.

(8.3.4) Theorem

(i) Let w be the state of the composite system I+II and suppose that w is pure. Then w can be represented by some vector Φ in $\mathcal{H}_c$ with

$$\Phi = \sum_{ij} \lambda_{ij} \varphi_i \otimes \psi_j$$

where $\{\varphi_i \otimes \psi_j\}$ is an orthonormal basis in $\mathcal{H}_c$. Moreover we have

$$w_1 = w_{\rho_1} \; ; \; w_2 = w_{\rho_2}$$

where ρ_1 and ρ_2 are the reduced statistical operators

determined by the vector Φ according to Theorem (8.2.6).

(ii) If w is a mixture of the form

$$w = \mu w' + (1-\mu) w''$$

where $0 < \mu < 1$, then the states w_1 and w_2 satisfy

$$w_1 = \mu w_1' + (1-\mu) w_1''$$

$$w_2 = \mu w_2' + (1-\mu) w_2''$$

where w_1', w_2' are the reduced statistical states of w' and w_1'', w_2'' are those of w''.

Proof

(i) Theorem 5 in Wan and McLean 1984(b) tells us that Φ has the given form in the orthonormal basis. Theorem (8.2.6) ensures that the states w_1 and w_2 satisfy (8.3.2) which has the trace form (8.2.4) for this particular case. Their uniqueness as solutions to the problem follows from the uniqueness lemma.

(ii) This follows easily from the uniqueness lemma and the linearity of the states.

(8.3.5) Corollary

(i) If w is a pure state $\Phi = \sum_i \lambda_i \varphi_i \otimes \psi_i$, then $w_1 = w_{\rho_1}$, $w_2 = w_{\rho_2}$, where

$$\rho_1 = \sum_i |\lambda_i|^2 P_{\varphi_i}$$

$$\rho_2 = \sum_i |\lambda_i|^2 P_{\psi_i}$$

(ii) If $\Phi = \varphi \otimes \psi$, then w_1 and w_2 are also pure states represented by the vectors φ and ψ respectively.

Proof

Both (i) and (ii) follow easily from (8.3.4).

There is another useful consequence of (8.3.3) and (8.3.4).

(8.3.6) Corollary

If w is a normal state, then so are w_1 and w_2.

Proof

Follows immediately from (8.3.4)(ii).

In the case where w is normal therefore, all the states, including reduced particle states are given by density operators and the results that hold for the conventional formalism concerning the form of the reduced statistical states and their uniqueness are reproduced for the asymptotic theory (WM). We now investigate the problem of the states at infinity, which of course do not exist in the conventional theory.

(8.3.7) Theorem

Let w be a state at infinity given by $w = w^{\infty}_{\tau}$. Then the reduced statistical states for the subsystems I and II are given by $w^{\infty}_{\tau_1}$ and $w^{\infty}_{\tau_2}$ respectively, where τ_1 and τ_2 are the reduced statistical operators determined uniquely by τ.

Proof

For each A_1 in $\mathcal{A}_{WM1}$ we have

$$w^{\infty}_{\tau_1}(A_1) = \lim_{t\to\infty} \mathrm{Tr}(\tau_1 A_{1t}) = \lim_{t\to\infty} \mathrm{Tr}(\tau(A_{1t}\otimes 1)) = w^{\infty}_{\tau}(A_1\otimes 1),$$

and

$$w^{\infty}_{\tau_2}(A_2) = \lim_{t\to\infty} \mathrm{Tr}(\tau_2 A_{2t}) = \lim_{t\to\infty} \mathrm{Tr}(\tau(1\otimes A_{2t})) = w^{\infty}_{\tau}(1\otimes A_2).$$

The uniqueness lemma ensures that the states $w^{\infty}_{\tau_1}$ and $w^{\infty}_{\tau_2}$ are the appropriate reduced statistical states.

Of course the results of Theorem (8.2.6) suffice to elucidate the form of the density operators τ_1 and τ_2 which generate $w^{\infty}_{\tau_1}$ and $w^{\infty}_{\tau_2}$. In addition, when τ is a mixed state we have a further result

(8.3.8) Corollary

Let $w = w^{\infty}_{\tau}$ where τ is a mixed state $\tau = \mu\tau' + (1-\mu)\tau''$. Then $w_1 = \mu w_1' + (1-\mu)w_1''$, where $w_1' = w^{\infty}_{\tau_1'}$, $w_1'' = w^{\infty}_{\tau_2'}$ and similarly for w_2.

Proof

For every A_1 in $\mathcal{A}_{WM1}$ we have

$$\begin{aligned} w_1(A_1) = w^{\infty}_{\tau_1}(A_1) &= \lim_{t\to\infty} \mathrm{Tr}(\tau_1 A_{1t}) \\ &= \lim_{t\to\infty} \mathrm{Tr}((\mu\tau_1' + (1-\mu)\tau_1'')A_{1t}) \\ &= \mu \lim \mathrm{Tr}(\tau_1' A_{1t}) + (1-\mu)\lim \mathrm{Tr}(\tau_1'' A_{1t}) \\ &\quad \text{(by linearity of Tr and the limit)} \\ &= \mu w_1'(A_1) + (1-\mu)w_1''(A_1) \end{aligned}$$

Similarly $w_2(A_2) = \mu w_2'(A_2) + (1-\mu)w_2''(A_2)$.

It was noted above that a general state of the WM-theory is given as a linear sum of a normal state and a state at infinity. By the following lemma we can transfer this mixture to the subsystems.

(8.3.9) Lemma

Let $w = \mu w_\rho + (1-\mu) w_\tau^\infty$, $0 < \mu < 1$. Then

$$w_1 = \mu w_{\rho_1} + (1-\mu) w_{\tau_1}^\infty$$

$$w_2 = \mu w_{\rho_2} + (1-\mu) w_{\tau_2}^\infty$$

where $\rho_1, \rho_2, \tau_1, \tau_2$ are determined according to Theorems (8.3.4) and (8.3.8).

Proof

Using the uniqueness lemma we have only to show that w_1 and w_2 satisfy (8.3.2). This is straightforward since w_1 (eg) is a linear sum of states w_{ρ_1} and $w_{\tau_1}^\infty$ which have already been shown to satisfy the consistency requirement.

We now consider the reverse problem. Given the states w_1 and w_2 of the subsystems of a composite system, when and how can we determine the state of the composite system ? Firstly, let us prove a preli minary lemma .

(8.3.10) Lemma

(i) If w_1 and w_2 are both normal states then w is normal.

(ii) If w_1 and w_2 are both states at infinity, then w is a state at infinity.

Proof

(i) Suppose that w is not normal; ie suppose that

$$w = \mu w_\rho + (1-\mu) w_\tau^\infty$$

where $0 \leq \mu < 1$. Then from lemma (8.3.9)

$$w_1 = \mu w_{\ell_1} + (1-\mu) w^{\infty}_{r_1},$$

so that w_1 is not normal. The result follows by contradiction.

(ii) The proof is similar to that of (i).

Now, suppose that w_1 and w_2 are any two states of I and II. By definition, w_1 and w_2 are members of $\mathcal{A}^*_{WM1}$ and $\mathcal{A}^*_{WM2}$ respectively. We can form the tensor product $\mathcal{A}^*_{WM1} \otimes \mathcal{A}^*_{WM2}$ of these dual spaces, consisting of linear combinations of simple tensor elements $w_1 \otimes w_2$ which act on $\mathcal{A}_{WMC}$ in the following way: let $\sum_{i,j} \lambda_{ij} A_i \otimes A_j$ be an arbitrary element of $\mathcal{A}_{WMC}$; then

$$(w_1 \otimes w_2)\left(\sum_{i,j} \lambda_{ij} A_i \otimes A_j\right) = \sum_{i,j} \lambda_{ij} w_1(A_i) w_2(A_j).$$

We can now generalise Theorem (8.2.8) to the present case.

(8.3.11) Theorem

Let w_1 and w_2 be the states of the systems I and II. Then:

(i) A possible state of the combined system is always

$$w = w_1 \otimes w_2$$

(ii) If w_1 and w_2 are pure states with state vectors φ and ψ respectively, then w is pure and has state vector $\varphi \otimes \psi$.

(iii) If w_1 and w_2 are not pure, then the solution w to (8.3.2) for the state of the composite system is not

necessarily unique.

Proof

(i) $\mathcal{A}^*_{WM1} \otimes \mathcal{A}^*_{WM2}$ is a space of continuous linear functionals on $\mathcal{A}_{WMC}$. Hence the functional defined by $w = w_1 \otimes w_2$ is continuous (ie bounded) and hence [cf(2.2.2)] positive. Moreover $w(1) = w_1(1)w_2(1) = 1$, so w defines a state on $\mathcal{A}_{WMC}$. If $A_1 \in \mathcal{A}_{WM1}$ we have

$$(w_1 \otimes w_2)(A_1 \otimes 1) = w_1(A_1)w_2(1) = w_1(A_1).$$

Similarly for system II.

(ii) Since w_1 and w_2 are pure they are normal [Theorem 5 Wan and McLean 1984(b)]. According to Lemma (8.3.10) w is also normal and the problem is reduced to the density matrix form of the conventional case so that the result follows from Theorem (8.2.8).

(iii) The example of (8.2.8) suffices here.

8.4 Correlations Between Subsystems

We have made an extensive comparison of the relationships between composite and reduced statistical states in the conventional theory to the relationships between such states in the asymptotic (WM) theory. It transpires that many of the results for the conventional theory in which states correspond to the density operators in $\mathcal{B}(\mathcal{H})$ are echoed in the asymptotic theory. In both cases, the state of the

composite system I+II uniquely determines the reduced statistical states for the component subsystems. In both cases the state of the composite system is uniquely determined by the states of the subsystems if and only if the states of the subsystems are both pure. In addition the form of the reduced statistical states for a given pure state of the composite system is identical in both theories. One important difference emerges however, which we shall now elucidate by examining the concept of correlations between subsystems. Unfortunately, the quantum mechanical literature is littered freely with the term correlation (or sometimes correlation function) and it is possible to isolate three distinct usages.

Firstly, the term correlation or correlation function is often used to refer to the joint expectation function $\langle A \otimes B; \varrho \rangle$ or the joint probability function $\langle A \otimes B; \Lambda_1 \times \Lambda_2 ; \varrho \rangle$. In particular many authors employ this term when considering such joint probabilities in Bell-type analyses of two-particle systems [cf many of the references in Appendix A.3]. To avoid confusion, which is particularly likely in the precise situation in which we are talking about "correlations" in more than one way, we shall always refer to these "correlation functions" as a joint expectation functions or joint probability functions.

In another sense, the term correlation is sometimes used to refer to the effects of coherence which characterise the difference between a pure state and a mixed one. Given some expectation value $\langle A;(\varphi+\psi)/\sqrt{2}\rangle$, say, the so-called correlation terms are those given by $\langle\varphi|A\psi\rangle$ for example and yield the statistical difference between $\langle A;(\varphi+\psi)/\sqrt{2}\rangle$ and the mixed state $\langle A;(1/2)(P_{\varphi}+P_{\psi})\rangle$. As an example of this meaning of the term correlations the WM-algebra is an algebra of "asymptotically vanishing correlations" in precisely this sense [Wan and McLean 1984(a)]. When describing a system in which two states φ and ψ are asymptotically separating [cf (7.3.4)] the Wan and McLean algebra ensures that the coherence terms vanish in the asymptotic limit. This means that pure states evolve to mixtures "at infinity" and these proper [cf Cufaro Petroni 1977, Jauch 1968,eg] mixtures provide an essentially classical statistics for large times. The concept of correlations in this sense, which from now on we shall refer to as coherence, arises in the extension of asymptotic separation of states to the two-particle case and is cited by Wan and McLean [1984(c)] in their resolution of the momentum formulation of the EPR paradox. The resolution consists in demonstrating that at infinity a system comprising two particles whose states are a superposition of asymptotically separating vectors is

described by an incoherent mixture. We shall see later that this analysis has no straightforward extension to the spin case. We shall attempt to provide such an extension by employing a stronger notion of correlations.

Classically, [Cramer 1946, eg] correlations are supposed to exist in a combined statistical system if the joint expectation function E for the overall system and the expectation functions E_1 and E_2 for the individual systems fail to satisfy the following "no correlation" condition.

(8.4.1)

$$E(A_1 A_2) = E_1(A_1)E_2(A_2)$$

for all pairs of random variables A_1 and A_2 (ie, physical magnitudes) on the probability spaces Ω_1 and Ω_2.

In quantum mechanics a simplistic extension of this condition for the conventional formulation is given by [Beltrametti and Cassinelli 1981 p68]

(8.4.2)

$$Tr(\varrho(A_1 \otimes A_2)) = Tr(\varrho_1 A_1)Tr(\varrho_2 A_2)$$

where ϱ is the state of the composite system and ϱ_1, ϱ_2 are the reduced statistical operators for the individual subsystems, and A_1, A_2 are bounded

observables on $\mathcal{H}_1$ and $\mathcal{H}_2$. For a general quantum system, therefore we might require as a no-correlation condition

(8.4.3)

$$w(A_1 \otimes A_2) = w_1(A_1)w_2(A_2),$$

where w is the state of the composite system and w_1, w_2 are the reduced statistical states; A_1 and A_2 are elements of the C^*-algebra associated with the systems I and II. This is in fact the no-correlation condition given by Beltrametti and Cassinelli [1981] for example. It corresponds, moreover to the separability requirement usually made in the analysis of Bell-type systems [Aspect 1976, eg]. The classical correlations themselves are defined by

(8.4.4)

$$C(A_1, A_2; \mu) = \frac{\langle A_1 A_2; \mu \rangle - \langle A_1; \mu_1 \rangle \langle A_2; \mu_2 \rangle}{[V(A_1; \mu_1)]^{1/2} [V(A_2; \mu_2)]^{1/2}}$$

where $V(A_1; \mu_1), V(A_2; \mu_2)$ are the variances of the random variables with respect to the probability measures μ_1 and μ_2 respectively. A quantum mechanical counterpart of this classical correlation is defined in Beltrametti and Cassinelli [1981 p68].

We shall adopt the usage of the term correlation in this third sense, since it obviously corresponds most exactly with the classical usage. We suggest one generalisation , however, designed to take into account the possibility of considering sums of simple tensor observables. Namely, we propose as the no-correlation condition

(8.4.5)

$$w(A) = w_1 \otimes w_2(A)$$

for every observable A in the composite algebra $\mathfrak{A}$ for the combined system. We shall also call this a separability requirement for a two-particle system, and say the the two particles are separable if they are correlation-free in the sense of (8.4.5).

Now for a classical system if the state is pure then the system is always correlation-free. For a quantum mechanical system this is also true when the pure state vector Φ is of the simple tensor form $\Phi = \varphi \otimes \psi$, as it is easy to verify from (8.4.5). Such a system is always separable therefore. However, in quantum mechanics there exist pure states which are not correlation-free.

(8.4.6) Example [cf Beltrametti and Cassinelli 1981 §7.4, eg]

Consider a system of two particles I+II described by the spin half algebra $\mathcal{S}_c = \mathcal{S} \otimes \mathcal{S}$ of bounded operators on the Hilbert space $\mathbb{C}^4$. Suppose that the state of the system is $\gamma_0 = (1/\sqrt{2})(\alpha_1 \otimes \beta_2 - \beta_1 \otimes \alpha_2)$ where α_i, β_j are [cf (2.3)] the eigenvectors of spin z in the up and down directions (respectively) for the (respective) systems. In the matrix notation the spin observable in the z-direction is given by

$$J_{1z} = J_{2z} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2}\sigma_z$$

(where we have chosen units with $\hbar = 1$). Hence the observable $J_z = J_{1z} \otimes J_{2z}$ has the matrix representation

$$J_z = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then we have

$$\langle \gamma_0 | J_{1z} \otimes 1 | \gamma_0 \rangle = \frac{1}{4}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\-1\\-1\\0\end{pmatrix} = 0 = \langle \gamma_0 | 1 \otimes J_{2z} | \gamma_0 \rangle$$

While

$$\langle \gamma_0 | J_z | \gamma_0 \rangle = \frac{1}{8}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}0\\-1\\-1\\0\end{pmatrix}$$

$$= \frac{1}{8}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}0\\-1\\1\\0\end{pmatrix} = -1/4$$

It follows that $w(J_{1z} \otimes J_{2z}) \neq w_1(J_{1z}) w_2(J_{2z})$ and therefore this is an example of a correlated system.

We have here an example of a system in a pure state which is not correlation-free therefore. This is an undeniably nonclassical aspect to quantum theory responsible for such effects as the EPR paradox. If we examine the algebraic theory proposed by Wan and McLean we find that we still have correlations in the sense of (8.4.5) even when the states are asymptotically separating.

(8.4.7) Example

Let $w = w_\varrho$ where ϱ is the projection operator corresponding to $\Phi = (1/\sqrt{2})(\varphi_1 \otimes \psi_2 - \psi_1' \otimes \varphi_2' >$ and φ_1, ψ_1' and φ_2', ψ_2 are pairs of asymptotically separating vectors. Let $A = A_1 \otimes A_2$ be any observable in $\mathcal{A}_{\text{tot}}$. Then we have $s\text{-}\lim_{t\to\infty} A_t = f_1 \otimes f_2$ for some $f_1 \in L^\infty(\underline{p}_1)$, $f_2 \in L^\infty(\underline{p}_2)$, and hence

$$w_\varrho^\infty(A) = \lim_{t\to\infty} \mathrm{Tr}(\varrho(A_1 \otimes A_2)_t)$$

$$= \mathrm{Tr}(\varrho(f_1 \otimes f_2))$$

$$= (1/2)(<\varphi_1 | f_1 \varphi_1 ><\psi_2 | f_2 \psi_2 > + <\psi_1' | f_1 \psi_1' ><\varphi_2' | f_2 \varphi_2' >$$

while

$$w_{\varrho_1}^\infty(A_1) = (1/2)(<\varphi_1 | f_1 \varphi_1 > + <\psi_1' | f_1 \psi_1' >)$$

and

$$w_{\varrho_2}^\infty(A_2) = (1/2)(<\varphi_2' | f_2 \varphi_2' > + <\psi_2 | f_2 \psi_2 >).$$

Easy computation shows that $w_\varrho^\infty(A) \neq w_{\varrho_1}^\infty(A_1) w_{\varrho_2}^\infty(A_2)$ and hence the state is not correlation free.

The point of the WM analysis is that the coherence terms, that is the correlations in the second sense mentioned above, vanish and hence the correlations that remain are essentially those of a classical mixture. At infinity, therefore, the pure state has evolved to a mixture and it is quite all right, even classically, for a mixed state to exhibit correlations in the sense that (8.4.5) is not satisfied.

8.5 Remarks

It is illuminating to compare the WM treatment of the EPR paradox (in the momentum formulation) to the discussions concerning "sensitive" and "indifferent" observables in quantum mechanics [Capasso,Fortunato and Selleri 1973, eg]. In a sense, the observables in the WM-algebra are indifferent to the difference between a pure state "of the second type" [op.cit] and a mixture of state vectors "of the first type", at infinity, when the states are asymptotically separating. Finally we mention that, while it is not possible to determine the state of a composite system from a knowledge of the reduced statistical states [Theorems (8.2.8) and (8.3.11)], one can introduce a physical operator, called the correlation operator and it turns out that a knowledge of this operator together with a knowledge of just one of the reduced statistical states provides all the information concerning the composite system [Herbut and Vujicic 1975 and 1984].

CHAPTER 9

LOCAL OBSERVABLES
FOR
TWO-PARTICLE SYSTEMS AND SYSTEMS WITH SPIN

9.1 Introduction

In Chapter 3 we discussed the concept of local observables and the localisation of bounded one-particle observables according to a well-defined scheme. We now wish to extend this discussion to the two-particle case and to the case of systems with spin. Sections (9.2) and (9.3) are concerned with defining these local observables. The next few sections elucidate the algebraic properties of sets of these local operators and the correlation properties of states on these algebras. We shall apply the analysis of Chapter 8 to some pertinent examples and thereby draw some conclusions as to the relevance of these local observables to a certain naive resolution of the EPR-type paradoxes. This resolution is based on the idea [cf Chapter 1] that when the wavefunctions of the subsystems become disjoint, there are no correlations between the subsystems. Finally we include a few brief examples to illustrate how local observables apply to the case of identical particle systems.

9.2 Local Observables for Two-Particle Systems

Following the motivation behind the formulation of L_Λ-observables [Chapter 3] we postulate for the present case as follows.

(9.2.1) Measurement Postulate
A measuring device or apparatus of finite size Λ cannot detect either particle of a two-particle system if that particle lies outside Λ.

Now in the description of two-particle systems in quantum mechanics we retain the option of only considering one of the particles by allowing for the existence of one-particle observables [cf(2.4)] of the form $A_1 \otimes 1$ (or $1 \otimes A_2$). The mathematical requirement for one-particle observables which arises from (9.2.1) is

(9.2.2) One-particle observables $A_1 \otimes 1$, $1 \otimes A_2$ measurable with finite apparatus of size Λ must satisfy
(i) $\langle \Phi | A_1 \otimes 1 \Phi \rangle = 0$, for all $\Phi = (E(\underline{x}_1 ; \Lambda^\perp) \otimes 1)\Phi$
(ii) $\langle \Phi | 1 \otimes A_2 \Phi \rangle = 0$, for all $\Phi = (1 \otimes E(\underline{x}_2 ; \Lambda^\perp))\Phi$.

When we consider two-particle observables however we have

(9.2.3) Two-particle observables A measurable with finite apparatus of size Λ must satisfy

$$\langle \Phi | A \Phi \rangle = 0$$

for every $\Phi = (E(\underline{x}_1 ; \Lambda^{\perp}) \otimes 1)\Phi$ and for every $\Phi = (1 \otimes E(\underline{x}_2 ; \Lambda^{\perp})\Phi$.

We now define our local observables for these systems. Although we restrict our attention to bounded observables here, it should be noted that the definition itself can be generalised to arbitrary observables in much the same way as (3.2.1).

(9.2.4) Definitions

A one-particle observable (OP-observable) $A_1 \otimes 1$ $(1 \otimes A_2)$, $A_1 \in \mathcal{B}(\mathcal{H}_1)$ $(A_2 \in \mathcal{B}(\mathcal{H}_2))$, is a local one-particle observable in Λ for system I (II), or $L_{\Lambda 1}$-observable ($L_{\Lambda 2}$-observable) for short, if A_1 (A_2) is an L_Λ-observable on $\mathcal{H}_1$ $(\mathcal{H}_2)$. An arbitrary observable A on $\mathcal{H}_c$ is a local two-particle observable in Λ, or LTP_Λ-observable for short, if A satisfies

$$E(\underline{x}_1 \otimes \underline{x}_2 ; \Lambda \times \Lambda) A E(\underline{x}_1 \otimes \underline{x}_2 ; \Lambda \times \Lambda) = A.$$

It turns out, as in the one particle case that these definitions provide exactly the appropriate observables to satisfy the measurement requirement (9.2.1).

(9.2.5) Theorem

(i) The selfadjoint operator $A \in \mathcal{B}(\mathcal{H}_c)$ is an LTP_Λ-observable if and only if it satisfies (9.2.3).

(ii) The selfadjoint OP-operator $A_1 \otimes 1$, $A_1 \in \mathcal{B}(\mathcal{H}_1)$ is an $L_{\Lambda 1}$-observable if and only if it satisfies (9.2.2). Similarly for $L_{\Lambda 2}$-observables.

Proof

(i) It is easy to show that every LTP_Λ-observable satisfies (9.2.3). Conversely, suppose that (9.2.3) holds for some observable A in $\mathcal{B}(\mathcal{H}_c)$. Then we have

$$\langle \Phi | A \Phi \rangle = 0 \text{ for all } \Phi = E^\perp(\underline{x}_1 \otimes \underline{x}_2 ; \Lambda \times \Lambda)$$

This implies that $AE^\perp(\underline{x}_1 \otimes \underline{x}_2 ; \Lambda \times \Lambda)\Psi = 0$ for all Ψ in $\mathcal{H}_c$, and hence we have

$$0 = E(\underline{x}_1 \otimes \underline{x}_2 ; \Lambda \times \Lambda) A E^\perp(\underline{x}_1 \otimes \underline{x}_2 ; \Lambda \times \Lambda)$$

$$= E^\perp(\underline{x}_1 \otimes \underline{x}_2 ; \Lambda \times \Lambda)\, A E^\perp(\underline{x}_1 \otimes \underline{x}_2 ; \Lambda \times \Lambda),$$

and since A is selfadjoint we also have

$$0 = E^\perp(\underline{x}_1 \otimes \underline{x}_2 ; \Lambda \times \Lambda) A E(\underline{x}_1 \otimes \underline{x}_2 ; \Lambda \times \Lambda) \text{ [cf Theorem (3.2.2)]}.$$

Therefore $A = E(\underline{x}_1 \otimes \underline{x}_2 ; \Lambda \times \Lambda) A E(\underline{x}_1 \otimes \underline{x}_2 ; \Lambda \times \Lambda)$.

(ii) Straightforward using the Theorem (3.2.1).

9.3 Local Spin Observables

We shall follow a procedure similar to the above when considering systems with spin. We start with a one-particle spin system. The usual measurement requirement [cf (3.1.1) and (9.2.1)] yields the mathematical requirement

(9.3.1) All observables A^σ on a one particle spin system must satisfy

$$\langle \varphi^\sigma | A^\sigma \varphi^\sigma \rangle = 0$$

for all $\varphi^\sigma \in \mathcal{H}^\sigma$ such that $\varphi^\sigma = (E(\underline{x};\Lambda) \dot{\otimes} 1)\varphi^\sigma$.

(9.3.2) Definition

A bounded observable A^σ on $\mathcal{H}^\sigma$ is a local observable in Λ, or L^σ_Λ-observable for short, if A^σ satisfies

$$(E(\underline{x};\Lambda) \dot{\otimes} 1)A(E(\underline{x};\Lambda) \dot{\otimes} 1) = A \ .$$

Once again we can show that the L^σ_Λ-observables are precisely the observables which satisfy the measurement requirement.

(9.3.3) Theorem

Let $A^\sigma \in \mathcal{B}(\mathcal{H}^\sigma)$ and suppose that A^σ is selfadjoint. Then A^σ is an L^σ_Λ-observable if and only if A^σ satisfies (9.3.1).

Proof

It is evident that each local observable satisfies

(9.3.1). Conversely, (9.3.1) implies that $A^\sigma\varphi^\sigma = 0$ for all $\varphi^\sigma \in \mathcal{H}^\sigma(\Lambda)^\perp$, and the proof follows exactly as for (3.2.2).

For two-particle systems with spin we again use the measurement requirement (9.2.1) to deduce a mathematical requirement.

(9.3.4) OP-observables measurable with apparatus of finite size must satisfy

$$\langle \Phi^\sigma | A_1^\sigma \otimes 1 \, \Phi^\sigma \rangle = 0, \quad \text{for all} \quad \Phi^\sigma = (E(\underline{x}_1 \otimes \underline{x}_2 ; \Lambda \times \hat{\mathbb{R}}) \dot{\otimes} 1)\Phi^\sigma$$

$$\langle \Phi^\sigma | 1 \otimes A_2^\sigma \Phi^\sigma \rangle = 0, \quad \text{for all} \quad \Phi^\sigma = (E(\underline{x}_1 \otimes \underline{x}_2 ; \hat{\mathbb{R}} \times \Lambda) \dot{\otimes} 1)\Phi^\sigma.$$

Two-particle observables must satisfy

$$\langle \Phi^\sigma | A^\sigma \Phi^\sigma \rangle = 0,$$

for all $\Phi^\sigma = E^\perp(\underline{x}_1 \otimes \underline{x}_2 ; \Lambda \times \Lambda) \dot{\otimes} 1)\Phi^\sigma$.

We define local observables in the expected way.

(9.3.5) Definitions

A bounded two-particle observable A^σ in $\mathcal{B}(\mathcal{H}_c^\sigma)$ is a local two-particle observable in Λ, or LTP_Λ^σ-observable for short, if A^σ satisfies

$$(E(\underline{x}_1 \otimes \underline{x}_2 ; \Lambda \times \Lambda) \dot{\otimes} 1) A^\sigma (E(\underline{x}_1 \otimes \underline{x}_2 ; \Lambda \times \Lambda) \dot{\otimes} 1) = A^\sigma$$

A one-particle observable $A_1^\sigma \otimes 1$ ($1 \otimes A_2^\sigma$) is a local one particle observable in Λ for system 1 (2), $L_{\Lambda 1}^\sigma$-observable ($L_{\Lambda 2}^\sigma$-observable) for short, if A_1 (A_2) is an L_Λ^σ-observable on $\mathcal{H}_1$ ($\mathcal{H}_2$).

(9.3.6) Theorem

An observable A^σ in $\mathcal{B}(\mathcal{H}^\sigma_c)$ is an LTP^σ_Λ-observable if and only if A^σ satisfies (9.3.4). An OP-observable $A^\sigma_1 \otimes 1$ in $\mathcal{B}(\mathcal{H}^\sigma_1) \otimes 1$ is an $L^\sigma_{\Lambda 1}$-observable if and only if $A^\sigma_1 \otimes 1$ satisfies the appropriate equality in (9.3.4). Similarly for $L^\sigma_{\Lambda 2}$-observables

The proof of this theorem is a straightforward extension of what has gone before. As a consequence of the above analysis we see that the L^σ_Λ-observables are tensor products of local observables with spin observables. Thus for instance, the simplest form of L^σ_Λ-observable is $E(\underline{x};\Lambda) \dot{\otimes} S$, for some S in $\mathcal{S}$, the algebra of spin operators. This is a significant feature of the formulation, for it embodies the insistence that we consider the locality of the particle and the confines of the measurement situation, even when it is only the spin parts of the observables that appear to be of interest. It will be seen below that this fact has important consequences in the physical analysis of spin measurements on two-particle systems.

9.4 Localisation of Bounded Observables

Exactly as for the localisation of bounded one-particle observables for systems without spin, it is possible to introduce a procedure for the localisation of bounded two-particle observables and observables for systems with spin.

(9.4.1) Definitions

For each $A^\sigma \in \mathcal{B}(\mathcal{H}^\sigma)$ where $\mathcal{H}^\sigma$ is the Hilbert space corresponding to a one-particle system the localisation A^σ_Λ of A^σ in Λ is defined by

$$A^\sigma_\Lambda = (E(\underline{x};\Lambda) \dot{\otimes} 1) A^\sigma (E(\underline{x};\Lambda) \dot{\otimes} 1),$$

for each $\Lambda \in \mathcal{B}_c(\mathbb{R}^n)$.

For each A^σ in $\mathcal{B}(\mathcal{H}^\sigma_c)$, the localisation A^σ_Λ of A^σ in Λ is defined by

$$A^\sigma_\Lambda = (E(\underline{x}_1 \otimes \underline{x}_2 ; \Lambda \times \Lambda) \dot{\otimes} 1) A^\sigma (E(\underline{x}_1 \otimes \underline{x}_2 ; \Lambda \times \Lambda) \dot{\otimes} 1),$$

for all $\Lambda \in \mathcal{B}_c(\mathbb{R}^n)$.

Obviously the localisation in Λ of $A^\sigma \in \mathcal{B}(\mathcal{H}^\sigma)$ is an L^σ_Λ-observable, while the localisation in Λ of $A^\sigma \in \mathcal{B}(\mathcal{H}^\sigma_c)$ is an LTP^σ_Λ-observable. Similar localisation procedures can be defined for one-particle observables in two-particle systems. Without much difficulty it is possible to prove several convergence results corresponding to those in (3.3). Essentially, these amount to the fact that in the limit of large

apparatus size we regain the usual global observables. In addition it is of some interest to explore formulation of bounded globally related families of L_Λ^σ-observables and LTP_Λ^σ-observables. Since we have carried all this through in some detail for the one-particle case however, we shall not detail the extension to the present cases. Rather, we shall concern ourselves here with an investigation into the algebraic properties of the local observables. In particular we shall investigate the correlation properties [cf (8.4)] of certain states with respect to algebras of local operators.

9.5 Algebras of Local Operators

We start by considering sets of L_Λ-operators on one-particle systems without spin. Throughout Λ will denote an element of $\mathcal{B}_c(\mathbb{R}^n)$.

(9.5.1) Lemma

Let $\mathcal{A}_\Lambda = \{A \in \mathcal{B}(\mathcal{H}) : A \text{ is an } L_\Lambda\text{-operator}\}$

$= \{A \in \mathcal{B}(\mathcal{H}) : A = E(\underline{x};\Lambda)AE(\underline{x};\Lambda)\}$.

Then $\mathcal{A}_\Lambda$ is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$.

Proof

Lemma (8.2) in McLean 1984.

(9.5.2) Lemma

Let $\mathcal{A}_{\Lambda 1} = \{A \in \mathcal{B}(\mathcal{H}_c): A \text{ is an } L_{\Lambda 1}\text{-operator}\}$

$= \{A \in \mathcal{B}(\mathcal{H}_c): A = A_1 \otimes 1, A_1 \in \mathcal{A}_\Lambda\}$, and

let $\mathcal{A}_{\Lambda 2} = \{A \in \mathcal{B}(\mathcal{H}_c): A \text{ is an } L_{\Lambda 2}\text{-operator}\}$

$= \{A \in \mathcal{B}(\mathcal{H}_c): A = 1 \otimes A_2, A_2 \in \mathcal{A}_\Lambda\}$.

Then $\mathcal{A}_{\Lambda 1}$ and $\mathcal{A}_{\Lambda 2}$ are C^*-subalgebras of $\mathcal{B}(\mathcal{H}_c)$.

Proof

$\mathcal{A}_{\Lambda 1}$ is the tensor product algebra of $\mathcal{A}_\Lambda$ with the trivial C^*-algebra $\{1\}$ and by the properties of tensor products of C^*-algebras [cf (2.4)] is itself a C^*-algebra. Similarly for $\mathcal{A}_{\Lambda 2}$.

(9.5.3) Theorem

Let $\mathcal{A}_L = \{A \in \mathcal{B}(\mathcal{H}): A \in \mathcal{A}_\Lambda \text{ for some } \Lambda \in \mathcal{B}_c(\mathbb{R}^n)\}$

$= \bigcup_\Lambda \mathcal{A}_\Lambda$,

and let $\bar{\mathcal{A}}_L$ denote the closure in the operator norm of $\mathcal{A}_L$. Then

(i) $\mathcal{A}_L$ is a proper *-subalgebra of $\mathcal{B}(\mathcal{H})$.

(ii) $\bar{\mathcal{A}}_L$ is a proper C^*-subalgebra of $\mathcal{B}(\mathcal{H})$.

Proof

(i) Theorem (8.5) in McLean 1984.

(ii) Evidently $\bar{\mathcal{A}}_L$ is a C^*-algebra. Moreover $1 \notin \mathcal{A}_L$, nor is 1 the uniform limit of members of $\mathcal{A}_L$ since $\|E(\underline{x};\Lambda) - 1\| = \|E(\underline{x};\Lambda^\perp)\| = 1$ for all Λ. Hence $1 \notin \bar{\mathcal{A}}_L$ and $\bar{\mathcal{A}}_L$ is a proper C^*-subalgebra of $\mathcal{B}(\mathcal{H})$.

(9.5.4) Theorem

Let $\mathcal{A}_1 = \{A \in \mathcal{B}(\mathcal{H}_c): A \in \mathcal{A}_{\Lambda 1} \text{ for some } \Lambda \in \mathcal{B}_c(\mathbb{R}^n)\}$

$\mathcal{A}_2 = \{A \in \mathcal{B}(\mathcal{H}_c) : A \in \mathcal{A}_{\Lambda 2} \text{ for some } \Lambda \in \mathcal{B}_c(\mathbb{R}^n)\}$,

and let $\bar{\mathcal{A}}_1$, $\bar{\mathcal{A}}_2$ denote the norm closures of these sets. Then we have

(i) $\mathcal{A}_1$ and $\mathcal{A}_2$ are proper *-subalgebras of $\mathcal{B}(\mathcal{H}_c)$.

(ii) $\bar{\mathcal{A}}_1$ and $\bar{\mathcal{A}}_2$ are proper C*-subalgebras of $\mathcal{B}(\mathcal{H}_c)$.

Proof

(i) It is sufficient to observe that $\mathcal{A}_1$ and $\mathcal{A}_2$ are tensor product algebras of proper*-subalgebras.

(ii) As for the proof of (i).

(9.5.5) Corollary

Let $\mathcal{A}_{LC} = \mathcal{A}_1 \vee \mathcal{A}_2$. $\mathcal{A}_{LC}$ is a *-subalgebra of $\mathcal{B}(\mathcal{H}_c)$. $\bar{\mathcal{A}}_{LC}$ is a C*-subalgebra of $\mathcal{B}(\mathcal{H}_c)$.

Proof

Follows immediately from (9.5.4) and definition.

(9.5.6) Theorem

Let $\mathcal{A}_{LTP_\Lambda} = \{A \in \mathcal{B}(\mathcal{H}_c) : A \text{ is an } LTP_\Lambda\text{-operator}\}$

and $\mathcal{A}_{LTP} = \{A \in \mathcal{B}(\mathcal{H}_c) : A \in \mathcal{A}_{LTP_\Lambda}, \text{ for some } \Lambda \in \mathcal{B}_c(\mathbb{R}^n)\}$.

Then

(i) $\mathcal{A}_{LTP_\Lambda}$ is a proper C*-subalgebra of $\mathcal{B}(\mathcal{H}_c)$.

(ii) $\bar{\mathcal{A}}_{LTP}$ is a proper C*-subalgebra of $\mathcal{B}(\mathcal{H}_c)$.

(iii) $\bar{\mathcal{A}}_{LTP} \subset \bar{\mathcal{A}}_{LC} \subset \mathcal{A}_0^S \subset \mathcal{B}(\mathcal{H}_c)$.

Proof

(i) $\mathcal{A}_{LTP_\Lambda} = \mathcal{A}_{\Lambda 1} \otimes \mathcal{A}_{\Lambda 2}$.

(ii) Let $A, B \in \mathcal{A}_{LTP}$. Then there exist Λ, Λ' such that A is an LTP_Λ-operator, B is an $LTP_{\Lambda'}$-operator. It is a straightforward matter to show that A+B is an

$LTP_{\Lambda\cup\Lambda'}$-operator, while AB is an $LTP_{\Lambda\cap\Lambda'}$-operator. Closure under the adjoint operation is straightforward and it follows that $\mathcal{A}_{LTP}$ is a *-subalgebra of $\mathcal{B}(\mathcal{H}_c)$. 1 is not in $\mathcal{A}_{LTP}$ [cf proof of (9.5.3)] and hence $\mathcal{A}_{LTP}$ is proper. $\bar{\mathcal{A}}_{LTP}$ is therefore a proper C^*-subalgebra of $\mathcal{B}(\mathcal{H}_c)$.

(iii) Every element of $\mathcal{A}_{LTP}$ is a sum of elements $A_{1\Lambda}\otimes A_{2\Lambda}$ for some Λ. But $A_{1\Lambda}\otimes A_{2\Lambda}$ is a product of $A_{1\Lambda}\otimes 1$ and $1\otimes A_{2\Lambda}$ which are both in $\mathcal{A}_{LC}$. Now $A_{1\Lambda}\otimes 1$ is in $\mathcal{A}_{LC}$ but not in $\mathcal{A}_{LTP}$ and we have $\mathcal{A}_{LTP}\subset\mathcal{A}_{LC}$. For each $L_{\Lambda 1}$-operator A, say, we have

$$s\text{-}\lim_{t\to\infty}((A_{1\Lambda}\otimes 1)_t) = s\text{-}\lim_{t\to\infty}(A_{1\Lambda t}) = 0$$

by Theorem (9.20) in McLean 1984. Similarly $s^*\text{-}\lim_{t\to\infty}((1\otimes A_{2\Lambda})_t) = 0$. Hence each $L_{\Lambda 1}$-operator or $L_{\Lambda 2}$-operator is in $\mathcal{A}_0^s$. Since $\mathcal{A}_0^s$ is a C^*-algebra [Wan and McLean 1984(b)] it follows that arbitrary sums and products of such operators are in $\mathcal{A}_0^s$. This exhausts all the operators in $\mathcal{A}_{LC}$; ie $\mathcal{A}_{LC}\subset\mathcal{A}_0^s$. The final inclusion is obvious. The result follows by closure.

When we introduce spin quantities into the discussion we shall use, as before, the superscript σ. C^*-algebras are generated by taking the tensor product of the local algebras $\mathcal{A}_{1\Lambda}$, $\mathcal{A}_{LC_\Lambda}$, etc with the relevant spin algebras. The algebras inherit the nomenclature of the operators. Thus, for example, $\mathcal{A}^\sigma_{\Lambda 1}$ is the $L^\sigma_{\Lambda 1}$-algebra comprising all $L^\sigma_{\Lambda 1}$-operators, $\mathcal{A}^\sigma_{LC_\Lambda}$ is the LC^σ_Λ-algebra comprising all LC^σ_Λ-operators, $\mathcal{A}^\sigma_{LTP}$ is the

LTP^{σ}-algebra of all LTP^{σ}-operators, and so on.

9.6 Correlation Properties for Local Spin Algebras

In Chapter 8 it was shown in the proof of Theorem (8.4.) that the spin state $\gamma_0 = (1/\sqrt{2})(\alpha_1 \otimes \beta_2 - \beta_1 \otimes \alpha_2)$ is not separable (ie not correlation free) on the algebra of observables $\mathcal{S}_c$. That analysis was carried out using purely spin quantities. We shall examine this case (among others) anew in the light of our present understanding that we must take into account the spatial location of the various states and subsystems, due to the finite limitations of the measuring device.

Before proceeding we need to clarify the concepts "pure" and "mixed" as applied to states. In particular, when we have a separable product $w = w_{space} \otimes w_{spin}$ of a spin state with a space state, for example it is possible for the overall state to be a mixture in the conventional sense, ie

$$w = \mu w' + (1-\mu) w'',$$

and yet only one part of the state, say the space part w_{space}, is mixed. Since the pure part of the state, the spin part for example, may be a superposition of correlated vectors, it will not necessarily be sufficient for our purposes (in examining correlations

between subsystems) to show simply that the overall state is mixed.

(9.6.1) Definition

We call a normal state w on a spin algebra, spin-separable if we can write

$$w = w_{space} \dot{\otimes} w_{spin}$$

with the obvious nomenclature.

The states $\Phi \dot{\otimes} \chi$, $\Phi \in \mathcal{H}_c$, $\chi \in \mathcal{H}_c$, are all spin-separable, for example. Notice that a spin-separable state is not necessarily separable in the sense of being correlation-free. When a distinction is necessary between these two concepts we shall refer to the latter as particle-separability as opposed to spin-separability.

(9.6.2) Definitions

We call a spin-separable state $w_{space} \dot{\otimes} w_{spin}$ spin-mixed if w_{spin} is a mixed state, and we call it space mixed if w_{space} is a mixed state.

(9.6.3) Lemma

A spin-separable state w is mixed if and only if it is either spin-mixed or space-mixed.

The proof of the lemma is obvious. Notice however that w being mixed does not imply that w is spin-mixed (eg).

(9.6.4) Example

Consider the state $\Phi \dot{\otimes} \gamma_0$ where γ_0 is the isotropic spin-zero vector, and Φ is the example of (8.4.7) given by

$$\Phi = (1/\sqrt{2})(\varphi_1 \otimes \psi_2 + \psi_1' \otimes \varphi_2'),$$

We recall from (8.4.7) that the state Φ is mixed on the algebra $L^\infty(\underline{p}_1, \underline{p}_2)$ and hence $\Phi \dot{\otimes} \gamma_0$ is space-mixed on the algebra $L^\infty(\underline{p}_1, \underline{p}_2) \dot{\otimes} \mathcal{S}_c$. But γ_0 is not mixed on $\mathcal{S}_c$. So this state is not spin-mixed on $L^\infty(\underline{p}_1, \underline{p}_2) \dot{\otimes} \mathcal{S}_c$.

It is evident that to a certain extent at least, just as the notion of mixed or pure is dependent on the algebra on which a state acts, so is the notion of separability: both particle separability and spin-separability. As an illustration of this we observe that, while a state given by a pure vector $\varphi \otimes \psi$ is separable on any algebra, any state w is separable on a one-particle algebra, since by definition $w(A_1 \otimes 1) = w_1(A_1) = w_1 \otimes w_2(A_1 \otimes 1)$. Equally a vector state $\Phi^\sigma = \Phi \dot{\otimes} \gamma$ is spin-separable on any algebra, while any state is spin-separable on an algebra $\mathcal{A} \dot{\otimes} 1$ or $1 \dot{\otimes} \mathcal{S}$.

We shall formulate some results concerning spin-separable states. First we must introduce one or two further definitions.

(9.6.5) Definitions

Let w be a spin-separable state $w = w_{space} \dot{\otimes} w_{spin}$. We shall say that w is spin-simple if w_{spin} is a pure state of the simple tensor form. We shall say that w is space-simple if w_{space} is a pure state of the simple tensor form. Finally we shall say that w is simple if w is both spin-simple and space-simple: ie w is given by a pure state $\Phi^{\sigma} = (\varphi \otimes \psi) \dot{\otimes} (\gamma_1 \otimes \gamma_2)$.

The notions of spin and space simple are not algebra dependent. However it does follow that a simple state is both particle separable and spin-separable on any algebra. Some immediate results are presented in the following.

(9.6.6) Theorem

Let $\mathcal{A}$ be any one-particle algebra. Then

(i) Every state on $\mathcal{A}$ is particle-separable (ie correlation free).

Let $\mathcal{A}^{\sigma}$ be an OP^{σ}-algebra. Then

(ii) Every spin-separable state is either space-simple or space-mixed.

(iii) Every spin-separable state is either spin-simple

or spin-mixed.

(iv) Every spin-separable state is either simple (and hence correlation free) or mixed.

Proof

(i) Evident from the preceding remarks.

(ii) Let $w = w_{space} \otimes w_{spin}$. Let $A \otimes 1 \in \mathcal{A}^{\sigma}$, and suppose that $A^{\sigma} = \sum_{i,j} B_i \otimes S_j$, $B_i \in \mathcal{A}$, $S \in \mathcal{S}'$. Then we have

$$w(A^{\sigma} \otimes 1) = \sum_{i,j} w_{space}(B_i \otimes 1) \cdot w_{spin}(S_j \otimes 1)$$
$$= \sum_{i,j} w^{(1)}_{space}(B) \cdot w^{(1)}_{spin}(S),$$

Where $w^{(1)}_{space}$ and $w^{(1)}_{spin}$ are reduced particle states. Now it follows from Theorem (8.2.8) that $w^{(1)}_{space}$ is mixed unless w_{space} is given by a simple tensor, and hence w is either space-simple or space-mixed.

(iii) The proof is identical to (ii).

(iv) Follows from (ii) and (iii).

The physical importance of this result may best be seen as follows. Recall [(8.4.1)] that in classical probability theory every pure state is correlation free. What we have shown in the above is that for a certain class of states (namely spin-separable ones) on a certain class of algebras (one-particle algebras) the classical respect for pure states is echoed. Every state which is not mixed is correlation free. As a corollary we examine a particular example.

(9.6.7) Corollary

Let $\Phi = (1/\sqrt{2})(\Phi_{1\Lambda} \otimes \Psi_{2N} + \Psi'_{1\Lambda'} \otimes \Phi'_{2\Lambda})$, where the subscripts

Λ, Λ' denote the support of the wavefunctions. Let $w = P_{\Phi} \dot{\otimes} P_{\gamma_0}$. Then w is a mixed, space-mixed and spin-mixed, correlation-free state on each of

(a) $\mathcal{A}^{\sigma}_{\Lambda 1}$ (b) $\mathcal{A}^{\sigma}_{\Lambda 2}$ (c) $\bar{\mathcal{A}}^{\sigma}_{L1}$ (d) $\bar{\mathcal{A}}^{\sigma}_{L2}$.

Proof

w is spin separable but is neither space-simple or spin-simple and the result follows from Theorem (9.6.6) therefore.

(9.6.8) Theorem

Let w be the state defined in (9.6.7) and let $\mathcal{A}^{\sigma}_{LC_{\Lambda''}}$ be any $LC^{\sigma}_{\Lambda''}$-algebra with $\Lambda'' \cap \Lambda' = \emptyset$. Then w is either space-mixed, spin-mixed and correlation free or else zero on all elements of this algebra.

Proof

For all one-particle observables w is spin-mixed, space-mixed and correlation-free. The other observables in $\mathcal{A}^{\sigma}_{LC_{\Lambda''}}$ are generated by the $LTP^{\sigma}_{\Lambda''}$-observables and it is straightforward to show that w is zero on these observables since every term in the wavefunction Φ contains a vector in $\mathcal{H}^{\perp}_{\Lambda''}$.

(9.6.9) Theorem

Let w be as in (9.6.7). Consider the LTP^{σ}-algebra $\bar{\mathcal{A}}^{\sigma}_{LTP}$. There are observables in $\bar{\mathcal{A}}^{\sigma}_{LTP}$ for which w is neither zero, nor mixed, nor correlation free.

Proof

Take $A^{\sigma} \in \bar{\mathcal{A}}^{\sigma}_{LTP}$, $A^{\sigma} = E(\underline{x}_1 \otimes \underline{x}_2 ; \Lambda'' \times \Lambda'') \otimes J_{\mathbb{Z}}$, where $J_{\mathbb{Z}}$ is as

in example (8.4.6), and $\Lambda''_{\Lambda}\Lambda \neq 0$ and $\Lambda''_{\Lambda}\Lambda' \neq 0$. The space part of the state is nonzero and the spin part is not correlation free. Neither part is mixed.

(9.6.10) Corollary
Let w be as above and consider the LC^{σ}-algebra $\bar{\mathcal{A}}^{\sigma}_{LC}$. There are observables in $\bar{\mathcal{A}}^{\sigma}_{LC}$ for which w is neither zero, nor mixed, nor correlation free.
Proof
$\bar{\mathcal{A}}^{\sigma}_{LTP} \subseteq \bar{\mathcal{A}}^{\sigma}_{LC}$ by (9.5.8).

We remark that the results of (9.6.7)-(9.6.10) can be proved a fortiori for the state given by vector $(\varphi_{1\Lambda} \otimes \psi_{2\Lambda'}) \dot{\otimes} \gamma_0$.

9.7 The EPR-Paradox

Recall from (2.7) that the spin vector relevant to the EPR paradox is the isotropic spin vector γ_0. Almost invariably no attention is paid to the space part of the wavefunction in consideration of this problem in the literature. We have learned in the present chapter, however that this neglect is not only unjustified but represents a major oversight in the analysis of the issue. We must therefore reconsider the problem, trying to take into account the space-part

of the wavefunction in order to be able to specify the locality of the particles in the system at the time of the measurement. Remember that the two particles are supposed to fly off in opposite directions after time t = 0 at which they are in the spin-zero state, so that when a measurement is made on them, the two particles are (arbitrarily) separated. This can come about in two ways. Either the particles are asymptotically separating, so that for large times they occupy disjoint spatial regions. Or else, for some finite time t, say, the particles are spatially separate in the sense of (7.6.1). In either case the form of the overall wavefunction for the EPR system must be:

$$(9.7.1) \qquad \Phi_o^\sigma = (1/\sqrt{2})(\varphi \otimes \psi)(\alpha_1 \otimes \beta_2 - \beta_1 \otimes \alpha_2)$$

where φ and ψ are either asymptotically separating or spatially separate for some finite time t. In this chapter we shall consider the case for finite time and suppose that at that time (we suppress reference to the exact time itself) the two particles are spatially separate. Thus, (9.7.1) becomes a special case of the wavefunction Φ of (9.6.7)-(9.6.10). In particular Theorem (9.6.8) holds for this case.

Our resolution of the EPR paradox as formulated in (2.7) consists in the following line of reasoning;

(9.7.2)

(i) The appropriate wavefunction to describe the system is Φ_o^σ.

(ii) A measurement of one of the particles in the system can only satisfy the nondisturbance criterion if the measurement apparatus does not interfere with the other particle. In particular this means that the finite domain Λ'' of the measurement apparatus and the support of the wavefunction of the second particle are disjoint.

(iii) The C^*-algebra of observables measurable with finite apparatus of size Λ'' is $\mathcal{A}^\sigma_{LC_{\Lambda''}}$ [This follows from (9.2.2)].

(iv) The state given by Φ_o^σ and all possible algebras of observables measurable in such a way as to satisfy the EPR non-disturbance criterion satisfy the conditions of Theorem (9.6.8).

(v) The state w is mixed, spin-mixed and space-mixed, and correlation-free (ie separable) for all non-trivial observables in all possible non-disturbing algebras.

We consider (v) to constitute a resolution to the paradox in this limited finite time case: w is not only mixed, but demonstrably separable with respect to the physical algebras relevant to the situation. The limitations of the resolution are obviously the ones already mentioned concerning the instantaneous spreading of the wavepacket. This resolution exists only instantaneously. The next instant, the wavefunctions will have spread irrevocably, no further spatial separation is possible and the two systems become inseparable according to the standard laws of time-evolution for quantum mechanics [cf(7.1)]. We cannot alleviate this problem by considering a sequence of local algebras because we would then involve ourselves in considering the algebra $\bar{\mathcal{A}}^{\sigma}_{\mathcal{L}}$, and we have seen [(9.6.10)] that $\bar{\mathcal{A}}^{\sigma}_{\mathcal{L}}$ is neither mixed nor separable on this algebra. The only way out of this dilemma would be to claim that the time-evolution in quantum mechanics is faulty ,at least globally. (We have already suggested an alternative time-evolution in a box [(7.2)].) But such drastic measures must await elucidation elsewhere. Another solution is possible in the asymptotic limit where a "localisation" in disjoint spatial regions occurs [(7.4.2)]. We shall investigate this situation in the next chapter.

9.8 Identical Particle Systems

Although it is not our purpose here to discuss identical particle systems in great depth we shall briefly mention one or two examples in which the use of local observables illuminates the breaking of particle symmetry suggested by Pauli [cf Chapter 1].

(9.8.1) Example

Consider the state w given by the vector

$$\Phi = (1/\sqrt{2})(\varphi \otimes \psi + \psi \otimes \varphi)$$

where $\varphi(x) = 0$, $x \notin \Lambda$, and $\psi(x) = 0$, $x \notin \Lambda'$, $\Lambda \cap \Lambda' = \emptyset$, and let A'_Λ be the symmetrised observable given by

$$A'_\Lambda = A_\Lambda \otimes 1 + 1 \otimes A_\Lambda$$

where A_Λ is an L_Λ-observable. Notice that the observable A'_Λ is a sum of local one-particle observables and hence belongs to $\mathcal{A}_{\mathcal{L}_\Lambda}$. The symmetry of both state and observable suggest this description as the appropriate one for two identical symmetric particles, that is bosons, and we have

$$\begin{aligned} w(A'_\Lambda) &= (1/2)(\langle \varphi | A_\Lambda \varphi \rangle + \langle \varphi | A_\Lambda \varphi \rangle) \\ &= \langle \varphi | A_\Lambda \varphi \rangle \\ &= (1/2)\mathrm{Tr}(P_{\varphi\otimes\psi} A') + (1/2)\mathrm{Tr}(P_{\psi\otimes\varphi} A') \end{aligned}$$

It is evident from the above that (a) w is a mixed state with respect to observables of the given form and (b) A'_Λ is in some sense a one-particle observable. The sense in which A'_Λ may be regarded as a one-particle

observable is as follows. With respect to observables measurable using apparatus of finite size Λ and appropriately symmetrised w is no longer a pure superposition of state $\varphi\otimes\psi$ in which I is on the left, say and II on the right with state $\psi\otimes\varphi$ in which I is on the right, II on the left; it is now a mixture of these states. Since it is a mixture we are entitled to conceptualise the problem in the following manner: one (and only one) of the particles is on the left (ie, in Λ), one (and only one) is on the right (in Λ') but we do not know which is which. This is the classical ignorance interpretation of a mixture. The use of local observables therefore enables us to break the identical particle symmetry, when the particles become disjoint [cf Pauli 1973 p168].

(9.8.2) Example

Consider the non-spin-separable identical particle (boson) state w given by the vector

$$\Phi^{\sigma} = (1/\sqrt{2})(\varphi\alpha\otimes\psi\beta + \psi\beta\otimes\varphi\alpha),$$

where we have suppressed the spin tensor product for ease of writing. Consider an observable $A'_{\Lambda} = A^{\sigma}_{\Lambda}\otimes 1 + 1\otimes A^{\sigma}_{\Lambda}$, where this time A^{σ}_{Λ} is an L^{σ}_{Λ}-observable. Let us take a simple case: suppose A^{σ}_{Λ} is the local spin observable in the z-direction $J_{z\Lambda} = E(\underline{x};\Lambda)J_z$ and write $J'_{z\Lambda} = J_{z\Lambda}\otimes 1 + 1\otimes J_{z\Lambda}$. Then $w(J')$ is easily calculated and we find $w(J') = 1/2$. In other words, the expected value for a measurement of spin in the z-direction in

region Λ is 1/2. This makes physical sense since it is seen by inspection of the wavefunction that the spin up state is always allied to the wavefunction whose support is in Λ. What is surprising is that if we tried to proceed in the conventional way by considering a spin measurement corresponding to the spin observable $J'_z = J_z \otimes 1 + 1 \otimes J_z$ we discover that

$$w(J'_z) = \langle \Phi^\sigma | 1 \dot{\otimes} (J_z \otimes 1 + 1 \otimes J_z) | \Phi^\sigma \rangle$$

$$= 1/2(\langle \alpha\otimes\beta | (J_z \otimes 1 + 1 \otimes J_z)\alpha\otimes\beta \rangle + \langle \beta\otimes\alpha | (J_z \otimes 1 + 1 \otimes J_z)\beta\otimes\alpha \rangle$$

$$= (1/2)\{((1/2) - (1/2)) + ((1/2 - (1/2))\}$$

$$= 0.$$

The identical particle symmetry seems to prevent us making a non-trivial measurement on a single particle. But once again we see that the use of a local spin-observable does allow such a measurement once the particle symmetry is broken by the fact that the wave functions no longer overlap. It turns out in addition that Φ^σ is an eigenvector of the local operator $J'_{z\Lambda}$ with eigenvalue 1/2, and not surprisingly it is also an eigenvector of $J'_{z\Lambda'}$ with eigenvalue -1/2.

CHAPTER 10

ASYMPTOTICALLY SEPARABLE QUANTUM MECHANICS AND THE EPR EXAMPLE

10.1 Introduction

The Einstein Podolski and Rosen experiment in the spin formulation [Bohm 1951] hinges on the separability of systems described by wavevectors of the form

$$(10.1.1) \qquad \Phi_0^{\sigma} = (1/\sqrt{2})(\varphi \otimes \psi) \dot{\otimes} (\alpha_1 \otimes \beta_2 - \beta_1 \otimes \alpha_2)$$

[cf (2.7),(9.7)]. In chapter 9 we showed that if φ and ψ were considered to be spatially separate states [(7.5)] localised in disjoint spatial regions Λ and Λ' respectively at time t, say, then the state w given by Φ_0^{σ} could indeed be regarded as separable with respect to any algebra $\mathcal{A}_{\Lambda''}$ of observables relevant to the non-disturbative measurement of the system in Λ', say. This is one analysis of the EPR problem. Quite a different analysis has recently been proposed by Wan and McLean [1984(c)] for the momentum formulation of the paradox. They consider the wavefunction

$$(10.1.2) \qquad \Phi = (1/\sqrt{2})(\varphi_1 \otimes \psi_2 + \psi_1' \otimes \varphi_2'),$$

where φ_1 and ψ_1' are a pair of asymptotically separating states for particle I and φ_2' and ψ_2 are a pair of asymptotically separating states for particle II. Their solution consists in demonstrating that the state Φ has vanishing coherence terms (sometimes called

correlation terms [cf(8.4)]) in the asymptotic limit of large times with respect to the algebra $\mathcal{A}_{WMc}$ [cf (2.4)]. Our aim now is to generalise these results, or rather to approach the problem in a slightly different fashion which will allow us to present an algebraic solution to the EPR problem by providing asymptotic separability of the relevant state whenever the systems are asymptotically separating. This generalisation will be applicable to both the spin and the momentum formulation of the problem.

Let us suppose first that we can extend the WM-theory to the spin case by simply considering the algebra

(10.1.3) $$\mathcal{A} = \mathcal{A}_{WMc} \dot{\otimes} \mathcal{S}_c,$$

and let us investigate to what extent separability may be achieved for the wavefunction Φ_o^σ, say of (10.1.1). Although the space part of the spin separable wavefunction is obviously correlation free (and also mixed at infinity) the spin part χ_o yields expectation values of the form $\langle \chi_o | S \chi_o \rangle, S \in \mathcal{S}_c$, where χ_o is the isotropic spin-zero vector, and it has already been demonstrated [(8.4.6) eg] that this state exhibits correlations for certain observables. Neither is w a spin-mixed state with respect to $\mathcal{A}$.

The problem we have encountered here stems from the fact that the usual quantum mechanical description of two independent systems proceeds via the specification of the tensor product of the individual algebras and Hilbert spaces. Using the usual state specification on the tensor product algebra, one arrives at what is supposedly a joint probability measure for results of experiments carried out with respect to observables on the two-particle system. Unfortunately, however, the quantum mechanical superposition of wavefunctions yields coherence effects in the probability measure which mean that two systems described in this fashion can never be treated as independent or isolated. That this is an intrinsic property of the logic (and hence probability) of quantum mechanics has been pointed out recently by Aerts [1984]. It is evident then that the mathematical formulation of the conventional formulation is actually inadequate to the description of a situation where we demand that two systems be isolated from each other and independent of each other. This remains the case no matter how far apart the systems might be. Of course, this fact (and the philosophy behind it) is exactly what constitutes the huge sledgehammer that allows followers of the Copenhagen interpretation of quantum mechanics to knock the EPR argument on the head: EPR's reasoning is faulty, they will say, because no matter

how far apart the particles it is not possible to measure one system "without in any way disturbing" the other.

Our aim in this thesis has largely been to mitigate the extremity of this position. For the finite case , a sort of uneasy truce was negotiated by restricting the observables relevant to the physical situation. Unfortunately the truce is short-lived. In the long-term, namely the asymptotic limit, as we have seen before [cf 7.4)] it is possible to consider a localisation of particles in disjoint regions at infinity. In this chapter we shall pursue a theory which provides separability in the asymptotic limit of large times, when the particles in a two-particle system are asymptotically separating. To this end we shall consider an approach in which we make use of the freedom of the algebraic theory of quantum mechanics to regard the C^*-algebra of observables as the fundamental building block and consider as states the NPLF's on the algebra. Our asymptotically separable theory will be based on a restriction of the NPLF's on the WM -algebra $\mathcal{A}$.

10.2 Asymptotic Separation

Before constructing the asymptotically separable theory, we return to the asymptotic separation of particles once again [cf(7.4.2)] and introduce some concepts and notation that will be of use in the subsequent analysis.

Firstly we shall use the abbreviation $E(\underset{\sim}{p}_1 \otimes \underset{\sim}{p}_2 ; m_1\Delta \times m_2\Delta') = E(m_1\Delta \times m_2\Delta')$ throughout this chapter for the spectral measure of the momentum operator. Let us consider a vector state Φ^σ of a two-particle spin system. For each such vector, we define the following sets:

(10.2.1)

$$\Delta_1(\Phi^\sigma) = \bigcap\{\Delta \in \mathcal{B}(\mathbb{R}^n) : \|(E(m_1\Delta \times m_2\mathbb{R}^n) \dot{\otimes} 1)\Phi^\sigma\| = 1\}$$

$$\Delta_2(\Phi^\sigma) = \bigcap\{\Delta \in \mathcal{B}(\mathbb{R}^n) : \|(E(m_1\mathbb{R}^n \times m_2\Delta) \dot{\otimes} 1)\Phi^\sigma\| = 1\}$$

$$\Delta_c \equiv \Delta_c(\Phi^\sigma) = \Delta_1(\Phi^\sigma) \cap \Delta_2(\Phi^\sigma).$$

(10.2.2) Definition

Let $\Delta_s = \mathbb{R}^n \setminus \Delta_c$. We call Δ_s the region of asymptotic separation of the (vector) state Φ^σ. We define a parameter of separation $\lambda(\Phi^\sigma)$ by

$$\lambda(\Phi^\sigma) = \|(E(m_1\Delta_s \times m_2\Delta_s) \dot{\otimes} 1)\Phi^\sigma\|$$

and we say that a state has no (asymptotic) separation if the parameter of separation is zero.

(10.2.3) Theorem

The following statements are equivalent:

(i) Φ^σ is an asymptotically separating state.

(ii) The region of asymptotic separation of Φ^σ is $\mathbb{R}^n$.

(iii) The parameter of separation for Φ^σ is 1.

Proof

(i) $\Rightarrow$ (ii): If Φ^σ is asymptotically separating, then there exist disjoint Borel sets Δ', Δ such that $\|(E(m_1\Delta \times m_2\mathbb{R}^n) \dot{\otimes} 1)\Phi^\sigma\| = 1 = \|(E(m_1\mathbb{R}^n \times m_2\Delta')\otimes 1)\Phi^\sigma\|$, and it follows that $\Delta_c(\Phi^\sigma) = \Delta_1 \wedge \Delta_2 \leq \Delta \wedge \Delta' = \emptyset$. Hence $\Delta_s = \mathbb{R}^n$.

(ii) $\Rightarrow$ (iii): Obvious since $E(m_1\mathbb{R}^n \times m_2\mathbb{R}^n) = 1$.

(iii) $\Rightarrow$ (i): $\|(E(m_1\Delta_s \times m_2\Delta_s) \dot{\otimes} 1)\Phi^\sigma\| = 1$ implies that $m_1\Delta_s \times m_2\Delta_s \supseteq m_1\Delta_1 \times m_2\Delta_2$ and hence $\Delta_s \supseteq \Delta_1 \cup \Delta_2$. It follows that $\Delta_c = 0$ and hence Φ^σ is asymptotically separating.

Now we define two vectors Φ_s^σ and Φ_c^σ by:

(10.2.4)

$$\Phi_s^\sigma = (1/\lambda(\Phi^\sigma))(E(m_1\Delta_s \times m_2\Delta_s) \dot{\otimes} 1)\Phi^\sigma$$

$$\Phi_c^\sigma = (1/(1-\lambda^2))^{\frac{1}{2}}(E^\perp(m_1\Delta_s \times m_2\Delta_s) \dot{\otimes} 1)\Phi^\sigma.$$

It is straightforward to show that these vectors are normalised and that

(10.2.5)

$$\Phi^\sigma = \lambda(\Phi^\sigma)\Phi_s^\sigma + (1-\lambda^2(\Phi^\sigma))^{1/2}\Phi_c^\sigma .$$

(10.2.6) Theorem

Let Φ_s^σ and Φ_c^σ be as defined in (10.2.4). Then we have $\lambda(\Phi_s^\sigma) = 1$ and $\lambda(\Phi_c^\sigma) = 0$. Furthermore the decomposition (10.2.5) of Φ^σ into two orthogonal vectors, one of which is separating, and the other which has no separation, is unique.

Proof

Let $\Delta_1' = \Delta_1 \setminus \Delta_c$, $\Delta_2' = \Delta_2 \setminus \Delta_c$. Then

$$(E(m_1\Delta_1' \times m_2\mathcal{R}^\cap) \dot{\otimes} 1)\Phi_s^\sigma = \Phi_s^\sigma \text{ and}$$

$$(E(m_1\mathcal{R}^\cap \times m_2\Delta_2') \dot{\otimes} 1)\Phi_s^\sigma = \Phi_s^\sigma .$$

But $\Delta_1' \cap \Delta_2' = 0$ and it follows that Φ_s^σ is asymptotically separating, and (by (10.2.3)) $\lambda(\Phi_s^\sigma) = 1$.

Next, we have $(E(m_1\Delta_1 \times m_2\mathcal{R}^\cap) \dot{\otimes} 1)\Phi_c^\sigma = \Phi_c^\sigma$, since the projections commute and so $\Delta_1(\Phi_c^\sigma) \subseteq \Delta_1 \equiv \Delta_1(\Phi^\sigma)$. Now suppose that $\Delta_1(\Phi_c^\sigma) \subsetneq \Delta_1$. Then

$$(1-\lambda^2)^{1/2}\|E(m_1\Delta_1(\Phi_c^\sigma) \times m_2\mathcal{R}^\cap) \dot{\otimes} 1)\Phi_c^\sigma\| =$$

$$\|(E(m_1\Delta_1(\Phi_c^\sigma) \times m_2\mathcal{R}^\cap) \dot{\otimes} 1)(E^\perp(m_1\Delta_s \times m_2\Delta_s) \dot{\otimes} 1)\Phi^\sigma\| =$$

$$\|(E(m_1\Delta_1(\Phi_c^\sigma) \times m_2\Delta_c) + E(m_1(\Delta_1(\Phi_c^\sigma) \cap \Delta_c) \times m_2\Delta_s)) \dot{\otimes} 1\Phi^\sigma\| =$$

$$\|(E(m_1\Delta_1(\Phi_c^\sigma) \times m_2\Delta_c) \dot{\otimes} 1)\Phi^\sigma\| + \|(E(m_1(\Delta_1(\Phi_c^\sigma) \cap \Delta_c) \times m_2\Delta_s) \dot{\otimes} 1)\Phi^\sigma\| <$$

$$\|(E(m_1\mathcal{R}^\cap \times m_2\Delta_c) \dot{\otimes} 1)\Phi^\sigma\| + \|(E(m_1\Delta_c \times m_2\Delta_s) \dot{\otimes} 1)\Phi^\sigma\| =$$

$$\|(E^\perp(m_1\Delta_s \times m_2\Delta_s) \dot{\otimes} 1)\Phi^\sigma\| = (1-\lambda^2)^{1/2}\|\Phi_c^\sigma\|.$$

It follows by contradiction that $\Delta_1(\Phi_c^\sigma) = \Delta_1$. Similarly we have $\Delta_2(\Phi_c^\sigma) = \Delta_2$ and hence $\Delta_c(\Phi_c^\sigma) = \Delta_c$ and $\Delta_s(\Phi_c^\sigma) = \Delta_s$. Therefore we have

$$\begin{aligned}\lambda(\Phi_c^\sigma) &= \|(E(m_1\Delta_s(\Phi_c^\sigma) \times m_2\Delta_s(\Phi_c^\sigma) \dot{\otimes} 1)\Phi_c^\sigma\| \\ &= \|(E(m_1\Delta_s \times m_2\Delta_s) \dot{\otimes} 1)\Phi_c^\sigma\| \\ &= 0.\end{aligned}$$

For the uniqueness, suppose that

$$\Phi^{\sigma} = \mu \Phi_s' + (1-\mu^2)^{1/2} \Phi_c'$$

we observe that if Φ_s' is asymptotically separating then there exist Δ_{1s}', Δ_{2s}' with $\Delta_{1s}' \cap \Delta_{2s}' = \emptyset$ and

$$\|(E(m_1\Delta_{1s}' \times m_2\Delta_{2s}') \dot{\otimes} 1)\Phi_s'\| = 1.$$

Since Φ_s' and Φ_c' are orthogonal we must have $(E(m_1\Delta_{1s}' \times m_2\Delta_{2s}') \dot{\otimes} 1)\Phi_c' = 0$, and hence

(i) $(E(m_1\Delta_{1s}' \times m_2\Delta_{2s}') \dot{\otimes} 1)\Phi^{\sigma} = \mu\Phi_s'$, and

(ii) $\|(E(m_1\Delta_{1s}' \times m_2\Delta_{2s}') \dot{\otimes} 1)\Phi^{\sigma}\| = \mu$.

Now

$$1 = \|(E(m_1\Delta_1 \times m_2\Delta_2) \dot{\otimes} 1)\Phi^{\sigma}\|^2$$

$$= \mu^2 \|(E(m_1\Delta_1 \times m_2\Delta_2) \dot{\otimes} 1)\Phi_s'\|^2 + (1-\mu^2)\|(E(m_1\Delta_1 \times m_2\Delta_2) \dot{\otimes} 1)\Phi_c'\|^2$$

Each of the norms on the right of the equality is less than or equal to one and actually we must have $(E(m_1\Delta_1 \times m_2\Delta_2) \dot{\otimes} 1)\Phi_s' = \Phi_s'$ and since Φ_s' is separating the stronger result, $(E(m_1\Delta_1' \times m_2\Delta_2') \dot{\otimes} 1)\Phi_s' = \Phi_s'$, must also hold. Then we have $m_1\Delta_{1s}' \times m_2\Delta_{2s}' \subseteq m_1\Delta_1' \times m_2\Delta_2'$ or $\Delta_1' \supseteq \Delta_{1s}'$, $\Delta_2' \supseteq \Delta_{2s}'$, and so from (i)

(iii) $$\Phi_s' = (1/\mu)(E(m_1\Delta_{1s}' \times m_2\Delta_{2s}') \dot{\otimes} 1)\Phi^{\sigma}$$

$$= (\lambda/\mu)(E(m_1\Delta_{1s}' \times m_2\Delta_{2s}') \dot{\otimes} 1)\Phi_s^{\sigma}$$

and from (ii) we have $\lambda \geq \mu$. Suppose that $\Delta_{1s}' = \Delta_1'$ and $\Delta_{2s}' = \Delta_2'$. Then $\lambda = \mu$ and

$$\Phi_s' = (\lambda/\mu)(E(m_1\Delta_1' \times m_2\Delta_2') \dot{\otimes} 1)\Phi_s^{\sigma} = \Phi_s^{\sigma}.$$

Suppose that $\Delta_{1s}' \subsetneq \Delta_1'$ and $\Delta_{2s}' \subsetneq \Delta_2'$. Then from (ii) $\mu < \lambda$ and from (iii), we have

$$1 = \|\Phi_s'\| = (\lambda/\mu)\|(E(m_1\Delta_{1s}' \times m_2\Delta_{2s}') \dot{\otimes} 1)\Phi_s^{\sigma}\| < 1.$$

The result follows by contradiction.

The generalisation of the preceding formulation to arbitrary normal states is relatively straightforward. Let w_ϱ be a normal state given by the density operator ϱ on $\mathcal{H}_c^\sigma$. Then the definitions (10.2.1) become

(10.2.7)

$$\Delta_1(w_\varrho) = \bigcap\{\Delta \in \mathcal{B}(\mathbb{R}^n): w_\varrho(E(m_1\Delta \times m_2\mathbb{R}^n)\dot{\otimes}1) = 1\}$$

$$\Delta_2(w_\varrho) = \bigcap\{\Delta \in \mathcal{B}(\mathbb{R}^n): w_\varrho(E(m_1\mathbb{R}^n \times m_2\Delta)\dot{\otimes}1) = 1\}$$

$$\Delta_c(w_\varrho) = \Delta_1(w_\varrho) \cap \Delta_2(w_\varrho).$$

Evidently this definition reduces to the previous case whenever ϱ is a projection operator corresponding to a vector in $\mathcal{H}_c^\sigma$. The region of asymptotic separation is defined exactly as in (10.2.2), and we can define a parameter of asymptotic separation by: $\lambda(w_\varrho) = w_\varrho(E(m_1\Delta_S \times m_2\Delta_S)\dot{\otimes}1)$, and prove results analogous to (10.2.3), without any great difficulty. If however, we look for a decomposition of an arbitrary state w_ϱ in terms of an asymptotically separating state and a state with no separation, we shall not be successful. In fact it is easy enough to see that even for the vector case with $\varrho = P_{\Phi^\sigma}$, say, it is not in general true that $w_\varrho = \lambda^2 w_S + (1-\lambda^2)w_c$, since w_ϱ is generally a pure state and not a mixed one. Nevertheless, it is still possible to define an asymptotically separating state $(w_\varrho)_S$ and a state $(w_\varrho)_c$ with no separation by

(10.2.8)

$$(w_\varrho)_s = w_{\varrho_s}$$

where

$$\varrho_s = \frac{(E(m_1\Delta_s \times m_2\Delta_s)\dot{\otimes}\,1)\,\varrho\,(E(m_1\Delta_s\times m_2\Delta_s)\dot{\otimes}\,1)}{\mathrm{Tr}(\varrho\,(E(m_1\Delta_s\times m_2\Delta_s)\dot{\otimes}\,1)}$$

and

$$(w_\varrho)_c = w_{\varrho_c}$$

where

$$\varrho_c = \frac{(E^\perp(m_1\Delta_s \times m_2\Delta_s)\dot{\otimes}\,1)\,\varrho\,(E^\perp(m_1\Delta_s\times m_2\Delta_s)\dot{\otimes}\,1)}{\mathrm{Tr}(\varrho\,(E^\perp(m_1\Delta_s\times m_2\Delta_s)\dot{\otimes}\,1)}$$

and actually ϱ is decomposable in terms of these states on the algebra at infinity:

(10.2.9) Theorem

Let $F = F(\underline{p}_1, \underline{p}_2, \sigma)$ be any operator in the algebra at infinity $\mathcal{A}^{\sigma\infty} = L^\infty(\underline{p}_1, \underline{p}_2)\otimes \mathcal{S}_c$. Then for every normal state w_ϱ we have

$$w_\varrho(F) = \lambda^2(w_\varrho)_s(F) + (1-\lambda^2)(w_\varrho)_c(F),$$

where $\lambda = \lambda(w_\varrho)$ is the parameter of separation for w_ϱ.

Proof

Set $E(m_1\Delta_s\times m_2\Delta_s)\dot{\otimes}\,1 = E$, $E^\perp(m_1\Delta_s\times m_2\Delta_s)\dot{\otimes}\,1 = E^\perp$. Then E and $E^\perp$ commute with F and

$$\begin{aligned} w_\varrho(F) &= \mathrm{Tr}(\varrho\,F) \\ &= \mathrm{Tr}(\varrho\,(E+E^\perp)F(E+E^\perp)) \\ &= \mathrm{Tr}(\varrho\,EFE) + \mathrm{Tr}(\varrho\,E^\perp FE^\perp) \end{aligned}$$

$$= \mathrm{Tr}(E\varrho\, EF) + \mathrm{Tr}(E^{\perp}\varrho\, E^{\perp}F)$$
$$= \mathrm{Tr}(E\varrho\,)(w_{\varrho})_s(F) + \mathrm{Tr}(E^{\perp}\varrho\,)(w_{\varrho})_c(F)$$
$$= \lambda^2(w_{\varrho})_s(F) + (1-\lambda^2)(w_{\varrho})_c(F).$$

For the special case of the algebra at infinity therefore w_{ϱ} has the decomposition into an asymptotically separating state and a state with no separation. We shall say that a state at infinity is separating if the normal state from which it is generated is asymptotically separating, and that it has no separation if the normal state from which it is generated has parameter of separation zero.

(10.2.10) Corollary

Every state at infinity is decomposable as a sum of a separating state and a state with no separation.

Proof

We are required to prove that

$$w^{\infty} = \lambda^2(w_s^{\infty}) + (1-\lambda^2)(w_c)$$

for some separating state w_s^{∞} and state w_c^{∞} with no separation. It is obvious that the states $(w_{\varrho})_c$ and $(w_{\varrho})_s$ will suffice and the result follows immediately from Theorem (10.2.8).

10.3 Asymptotically Separable States on the Algebra .

We define an asymptotically separable state to be any state w for which the state at infinity w^∞ generated from w by $w^\infty = \lim_{t\to\infty} w(A_t)$ exists and is separable. We attempt to formulate a theory for which the following physical requirement is satisfied.

(10.3.1) Every asymptotically separating two-particle state is asymptotically separable.

(10.3.2) Lemma
The WM-theory comprising the algebra $\mathcal{A}$ and the normal states and normal states at infinity on $\mathcal{A}$ does not satisfy (10.3.1).
Proof
Example (8.4.7) suffices to demonstrate the lemma.

We overcome this difficulty by dropping the restriction to normal states and by defining a new class of state $\tilde{w}_\varrho$ determined by density operators but satisfying (10.3.1). For each density operator ϱ on $\mathcal{H}_c^\sigma$ we denote, as before, the asymptotically separating part of w_ϱ by $(w_\varrho)_s$ and the state with no separation by $(w_\varrho)_c$. These states are defined as in (10.2.8). We denote further the reduced statistical operators corresponding to ϱ_s and ϱ_c by ϱ_{s1}, ϱ_{s2} and ϱ_{c1}, ϱ_{c2}

respectively.

(10.3.3) Lemma

For each density operator ϱ on $\mathcal{H}_c^\sigma$ let $\tilde{w}_\varrho$ be the functional defined on $\mathcal{A}$ by

$$\tilde{w}_\varrho(A_o + F) = w_\varrho(A) + \lambda^2 w_{\varrho_{s_1}} \otimes w_{\varrho_{s_2}}(F) + (1-\lambda^2) w_{\varrho_c}(F),$$

where λ is $\lambda(\varrho)$ the parameter of separation associated with the density operator ϱ and $A + F$ is an arbitrary element of $\mathcal{A}$. Then $\tilde{w}_\varrho$ is a well-defined linear functional on $\mathcal{A}$.

Proof

For each fixed ϱ, $\lambda(\varrho)$ is a constant and $w_{\varrho_{s_1}} \otimes w_{\varrho_{s_2}} = w_{\varrho_{s_1} \otimes \varrho_{s_2}}$ is a well-defined linear functional on $\mathcal{A}$. It follows that $\tilde{w}_\varrho$ is a linear sum of well-defined linear functionals on $\mathcal{A}$ and is therefore a linear functional on $\mathcal{A}$.

(10.3.4) Lemma

$\tilde{w}_\varrho$ is an NPLF on $\mathcal{A}$.

Proof

Firstly we have $\tilde{w}_\varrho(1) = \lambda^2 w_{\varrho_{s_1}} \otimes w_{\varrho_{s_2}}(1) + (1-\lambda^2) w_{\varrho_c}(1)$

$$= \lambda^2 + (1-\lambda^2) = 1.$$

For positivity it is sufficient [Arveson 1976, p33] to show boundedness on $\mathcal{A}$. Now $\mathcal{A}$ is a C^*-subalgebra of $\mathcal{B}(\mathcal{H}_c^\sigma)$ and so $w_\varrho(A) \leq \|A\| < \infty$ and $w_{\varrho_c}(F) \leq \|F\| < \infty$ for all $A+F$ in $\mathcal{A}$. Similarly, $w_{\varrho_{s_1}} \otimes w_{\varrho_{s_1}}(F) \leq \|F\| < \infty$ and $\lambda^2, (1-\lambda^2)$ are both less than 1 so that

$$\tilde{w}_\varrho(A+F) = w_\varrho(A) + \lambda^2 w_{\varrho_{s_1}} \otimes w_{\varrho_{s_2}}(F) + (1-\lambda^2) w_{\varrho_c}(F)$$

$$\leq \|A\| + \lambda^2\|F\| + (1-\lambda^2)\|F\|$$

$$< \infty .$$

(10.3.5) Lemma

Let $\tilde{w}_\varrho^\infty$ be defined by

$$\tilde{w}_\varrho^\infty(A) = \lim_{t\to\infty} \tilde{w}_\varrho(A_t), \text{ for each } A \in \mathcal{A} .$$

Then $\tilde{w}_\varrho^\infty$ is a well-defined NPLF on $\mathcal{A}$.

Proof

Let $A = A + F$. Then

$$\begin{aligned}\tilde{w}_\varrho(A) &= \lim_{t\to\infty} \{w_\varrho(A_t) + \lambda^2 w_{\varrho_{s_1}} \otimes w_{\varrho_{s_2}}(F_t) + (1-\lambda^2) w_{\varrho_c}(F_t)\} \\ &= \lim_{t\to\infty} \lambda^2 w_{\varrho_{s_1}} \otimes w_{\varrho_{s_2}}(F_t) + \lim_{t\to\infty} (1-\lambda^2) w_{\varrho_c}(F_t) \\ &= \lambda^2 w_{\varrho_{s_1}} \otimes w_{\varrho_{s_2}}(F) + (1-\lambda^2) w_{\varrho_c}(F).\end{aligned}$$

It follows that each of the limits is well-defined and that $\tilde{w}_\varrho^\infty$ is an NPLF on $\mathcal{A}$.

Now we can define reduced statistical states corresponding to the states $\tilde{w}_\varrho$. Obviously we require these states to satisfy the usual physical requirement (8.2.1). Indeed we define the reduced statistical states by

(10.3.6)

$$\tilde{w}_{\varrho_1}(A_1) = \tilde{w}_\varrho(A_1 \otimes 1), \text{ for each } A_1 \text{ in } \mathcal{A}_1 ,$$

$$\tilde{w}_{\varrho_2}(A_2) = \tilde{w}_\varrho(1 \otimes A_2), \text{ for each } A_2 \text{ in } \mathcal{A}_2 .$$

(10.3.7) Lemma

The reduced statistical states $\tilde{w}_{\varrho_1}$ and $\tilde{w}_{\varrho_2}$ satisfy

$$\tilde{w}_{\varrho_1} = w_{\varrho_1} \quad ; \quad \tilde{w}_{\varrho_2} = w_{\varrho_2}$$

where w_{ϱ_1} and w_{ϱ_2} are the reduced statistical states

for the normal state w_ϱ.

Proof

$$\begin{aligned}\tilde{w}_{\varrho_1}(A_1+F_1) &= \tilde{w}_\varrho(A_1\otimes 1+F_1\otimes 1)\\ &= w_\varrho(A_1\otimes 1) + \lambda^2 w_{\varrho_{s1}}\otimes w_{\varrho_{s2}}(F_1\otimes 1) + (1-\lambda^2)w_{\varrho_c}(F_1\otimes 1)\\ &= w_\varrho(A_1\otimes 1) + \lambda^2 w_{\varrho_s}(F_1\otimes 1) + (1-\lambda^2)w_{\varrho_c}(F_1\otimes 1)\\ &= w_\varrho(A_1\otimes 1 + F_1\otimes 1), \text{ by (10.2.8)},\\ &= w_{\varrho_1}(A_1+F_1).\end{aligned}$$

Similarly we show that $\tilde{w}_{\varrho_2} = w_{\varrho_2}$.

It is also possible to show that $\tilde{w}_{\varrho_1}^\infty = w_{\varrho_1}^\infty$ and $\tilde{w}_{\varrho_2}^\infty = w_{\varrho_2}^\infty$. These results are straightforward consequences of (10.3.7). Now we shall say that a state $\tilde{w}_\varrho$ is asymptotically separating if there exist disjoint sets Δ and Δ' such that

$$\tilde{w}_\varrho(E(\hat{\underline{p}}_1;m_1\Delta)\dot{\otimes}\,1) = 1 = \tilde{w}_\varrho(E(\hat{\underline{p}}_2;m_2\Delta')\dot{\otimes}\,1).$$

This is obviously a straightforward extension of (7.4.2) and indeed we see from (10.3.7) that $\tilde{w}_\varrho$ is asymptotically separating if and only if w_ϱ is asymptotically separating. We can now show that the states $\tilde{w}_\varrho$ satisfy the physical requirement (10.3.1).

(10.3.8) Theorem

Let $\tilde{w}_\varrho$ be an asymptotically separating state. Then

(i) $\tilde{w}_\varrho$ is asymptotically separable

(ii) $\tilde{w}_\varrho$ is separable if and only if w_ϱ is separable.

Proof

$\tilde{w}_\varrho$ is asymptotically separating implies that ϱ is asymptotically separating and hence that $\lambda(\varrho) = 1$ and

we have

(i) $\tilde{w}^{\infty}(A+F) = \tilde{w}_{\varrho}(F) = w_{\varrho_{s1}} \otimes w_{\varrho_{s2}}(F) = \tilde{w}^{\infty}_{\varrho_1} \otimes \tilde{w}^{\infty}_{\varrho_2}(A+F)$.

(ii) Suppose $\tilde{w}_{\varrho}(A+F) = \tilde{w}_{\varrho_1} \otimes \tilde{w}_{\varrho_2}(A+F)$ for all $A+F$ in $\mathcal{A}$. In particular therefore $\tilde{w}_{\varrho_1}(A) = \tilde{w}_{\varrho_1} \otimes \tilde{w}_{\varrho_2}(A)$ for all A in $\mathcal{A}^{s\sigma}_{oc}$. Hence $w_{\varrho}(A) = w_{\varrho_1} \otimes w_{\varrho_2}(A)$ for every A in $\mathcal{A}^{s\sigma}_{oc}$ and it follows that $\varrho = \varrho_1 \otimes \varrho_2$ and therefore w_{ϱ} is separable on $\mathcal{A}$.

Conversely, if $w_{\varrho} = w_{\varrho_1} \otimes w_{\varrho_2}$ we have $\varrho = \varrho_1 \otimes \varrho_2$ and so

$$\begin{aligned}\tilde{w}_{\varrho}(A+F) &= w_{\varrho_1} \otimes w_{\varrho_2}(A) + \lambda^2 w_{\varrho_{s1}} \otimes w_{\varrho_{s2}}(F) + (1-\lambda^2) w_{\varrho_{c1}} \otimes w_{\varrho_{c2}}(F) \\ &= w_{\varrho_1} \otimes w_{\varrho_2}(A+F) \\ &= \tilde{w}_{\varrho_1} \otimes \tilde{w}_{\varrho_2}(A+F).\end{aligned}$$

We shall call the states $\tilde{w}$ AS-states to indicate that they satisfy the requirement of asymptotic separability whenever they are asymptotically separating as is evident from the above theorem. It follows, moreover, that these AS-states are not unnecessarily restrictive when $\tilde{w}$ is not separating.

As we remarked above we have obtained the required separability by first allowing states on $\mathcal{A}$ which are non-normal and then restricting this set to the state of the form (10.3.3). We shall see nevertheless that the states $\tilde{w}_{\varrho}$ are "almost" normal states in certain senses.

(10.3.9) Theorem

(i) The restriction of $\tilde{w}_{\varrho}$ to the algebra $\mathcal{A}^{s\sigma}_{oc}$ is a

normal state on $\mathcal{A}_{oc}^{s\sigma}$.

(ii) The restriction of $\tilde{w}_\varrho$ to the algebra at infinity $\mathcal{A}^{\infty\sigma}$ is a normal state on $\mathcal{A}^{\infty\sigma} \equiv L^\infty(\underline{\rho}_1, \underline{\rho}_2) \dot{\otimes} \mathcal{S}_c$.

(iii) Every state at infinity $\tilde{w}_\varrho^\infty$ is a normal state at infinity.

(iv) A state $\tilde{w}_\varrho$ is a normal state on $\mathcal{A}$ if and only if $\varrho = \lambda^2 \varrho_{s1} \otimes \varrho_{s2} + (1-\lambda^2)\varrho_c$.

Proof

(i) $\tilde{w}_\varrho(A_o) = w_\varrho(A_o)$, for all A_o in $\mathcal{A}_{oc}^{s\sigma}$.

(ii)
$$\begin{aligned}\tilde{w}_\varrho(F) &= \lambda^2 w_{\varrho_{s1}} \otimes w_{\varrho_{s2}}(F) + (1-\lambda^2) w_{\varrho_c}(F) \\ &= \lambda^2 \mathrm{Tr}(\varrho_{s1} \otimes \varrho_{s2} F) + (1-\lambda^2)\mathrm{Tr}(\varrho_c F) \\ &= \mathrm{Tr}(\lambda^2 \varrho_{s1} \otimes \varrho_{s2} F + (1-\lambda^2)\varrho_c F) \\ &= \mathrm{Tr}(\tilde{\varrho} F) \\ &= w_{\tilde{\varrho}}(F),\end{aligned}$$

where $\tilde{\varrho} = \lambda^2 \varrho_{s1} \otimes \varrho_{s2} + (1-\lambda^2)\varrho_c$ is a density operator on $\mathcal{H}_c^\sigma$.

(iii) Follows immediately from (ii).

(iv) Suppose that $\tilde{w}_\varrho$ is a normal state. Then there exists a density operator $\tilde{\varrho}$ on $\mathcal{H}_c^\sigma$ such that $\tilde{w}_\varrho = w_{\tilde{\varrho}}$, ie

$$\mathrm{Tr}(\tilde{\varrho}(A_o+F)) = \mathrm{Tr}(\varrho A_o) + \lambda^2 \mathrm{Tr}(\varrho_{s1} \otimes \varrho_{s2} F) + (1-\lambda^2)\mathrm{Tr}(\varrho_c F)$$

for every A_o+F in $\mathcal{A}$. In particular therefore we have

$\mathrm{Tr}(\tilde{\varrho} A_o) = \mathrm{Tr}(\varrho A_o)$ for each A_o in $\mathcal{A}_{oc}^{s\sigma}$;

$\mathrm{Tr}(\tilde{\varrho} F) = \mathrm{Tr}((\lambda^2 \varrho_{s1} \otimes \varrho_{s2} + (1-\lambda^2)\varrho_c)F)$,

for each F in $\mathcal{A}^{\infty\sigma} = L^\infty(\underline{\rho}_1, \underline{\rho}_2) \dot{\otimes} \mathcal{S}_c$. It follows that

$$\tilde{\varrho} = \varrho = \lambda^2 \varrho_{s1} \otimes \varrho_2 + (1-\lambda^2)\varrho_c \quad (*)$$

Conversely if $\varrho = \lambda^2 \varrho_{s1} \otimes \varrho_{s2} + (1-\lambda^2)\varrho_c$, it is easy to show that $\tilde{w}_\varrho = w_\varrho$.

(10.3.10) Corollary

(i) If ϱ has no separation then $\tilde{w}_\varrho$ is normal.

(ii) If ϱ is separable then $\tilde{w}_\varrho$ is normal.

Proof

Using (iv) of Theorem (10.3.9) we see that (i) if $\varrho = \varrho_c$ then (*) is satisfied trivially and (ii) if ϱ is separable so is ϱ_s and therefore $\varrho_s = \varrho_{s1} \otimes \varrho_{s2}$ and hence (*) holds again.

(10.3.11) Corollary

Not every AS-state is a normal state.

Proof

It is evident from (10.3.9)(iv) that we can construct a state $\tilde{w}_\varrho$ for which (*) is not satisfied.

10.4 Asymptotically Separable Quantum Mechanics

We shall formulate a quantum mechanical theory which is asymptotically separable in the sense that it satisfies requirement (10.3.1).

(10.4.1) Postulate I

A composite system of two non-identical free spin-half quantum particles moving in configuration space $\mathbb{R}^n$ has associated with it the C^*-algebra $\mathcal{A}$. Selfadjoint

elements of $\mathcal{A}$ correspond to physical observables of the system.

The time-development of the system is the usual one.

(10.4.2) Postulate II

The time evolution of the system is described by a one-parameter group $\{\alpha_t : t \in \mathbb{R}\}$ of automorphisms of $\mathcal{A}$ defined by

$$\alpha_t(A) = U_t A U_t ,$$

where U_t is the time-evolution operator given in (2.3).

Notice that U_t is in $\mathcal{A}$ for each t, so that the theory is self-contained in the sense that the time-evolution operator is an operator in the algebra of the theory. We now specify the states of the system.

(10.4.3)Postulate III

Physical states of the system are given by the AS-states $\tilde{w}_\varrho$ where ϱ is a density operator on the Hilbert space $\mathcal{H}_c^\sigma$, and by the states at infinity $\tilde{w}_\varrho^\infty$ generated by $\tilde{w}_\varrho$ according to (10.3.5). $\tilde{w}(A), A \in \mathcal{A}$, represents the expectation value of the observable A upon measurement.

10.5 The EPR Example

In the conventional theory, the state vector for the spin formulation of the EPR example is given by a vector of the form (10.1.1). We now make the assumption that the state in the new theory is $\tilde{w}_{P_o^\sigma}$ where P_o^σ is the projector onto the subspace generated by Φ_o^σ. This assumption is certainly reasonable, for $\tilde{w}_{P_o^\sigma}$ yields the same statistics as the state vector Φ_o^σ in the conventional formulation for every observable in $\mathcal{A}_{oc}^{s\sigma}$ and particularly therefore for every observable measurable with finite apparatus. In addition $\tilde{w}_{P_o^\sigma}$ is asymptotically separable and at infinity the systems are localised in the same regions as for the conventional formulation. Now however a significant difference between the usual theory and the asymptotically separable theory emerges. In the usual theory the state Φ_o^σ is not separable even at infinity. For the AS-theory however we have

$$\tilde{w}_{P_o^\sigma}(A+F) = w_{\ell_1} \otimes w_{\ell_2}(F) = \tilde{w}_{\ell_1}^{\infty} \otimes \tilde{w}_{\ell_2}^{\infty}(A+F)$$

where ℓ_1 and ℓ_2 are the reduced statistical operators determined by Φ_o^σ. This is evidently a separable state. Moreover it turns out that the state at infinity $\tilde{w}_{P_o^\sigma}^{\infty}$ is both spin mixed and space mixed.

We have demonstrated then how a system which for finite times and overlapping wavefunctions possesses correlated subsystems may in the asymptotic limit separate spatially and be moreover separable. No correlations exist between the subsystems when this has occurred and the state is a statistical mixture. This provides an asymptotic resolution to the EPR paradox. We remark that this algebraic theory includes the local theory proposed in Chapter 9 in so far as the local spin algebras $\mathcal{A}^{\sigma}_{LC_\Lambda}$ are all contained in $\mathcal{A}^{s\sigma}_{oc}$.

10.6 Another Theory

In a recent paper [Wan and Jackson 1985] we have presented a slightly different theory to describe a system of two spin-half particles. Essentially this latter theory is based on the assumption that at infinity it is not possible to measure two-particle observables. This assumption arises from the concept of "chronological disordering" [Wan and Timson 1985]. The algebra of observables at infinity then turns out to be a direct sum algebra. Details of this theory may be found in Appendix A.3 or Wan and Jackson [1985]. What we have attempted in the present chapter is to extend the slightly simplified direct sum theory so as

to include the possibility that two systems which do not separate at all may still be correlated at infinity. In addition, the present formulation provides immediately the joint expectation functions which we shall need for our discussion of hidden variable issues in the next chapter.

CHAPTER 11

HIDDEN VARIABLES AND BELL'S THEOREM

11.1 Introduction

In classical mechanics every pure state is dispersion free. That is to say that the value each idempotent random variable takes on the pure states is either one or zero. We can easily see that this is the case. Idempotent random variables in classical mechanics are the characteristic functions χ_Δ on the phase space.

(11.1.1) $$\chi_\Delta(\underline{x},\underline{p}) = \begin{cases} 1 \text{ if } (\underline{x},\underline{p}) \in \Delta \\ 0 \text{ if } (\underline{x},\underline{p}) \notin \Delta \end{cases} .$$

The pure states in classical mechanics are simply the points $(\underline{x},\underline{p})$ in phase space. The value taken by an observable f in state $(\underline{x},\underline{p})$ is $f(\underline{x},\underline{p})$ and it follows immediately from (11.1.1) that every idempotent random variable takes either one or zero on the pure states.

In quantum mechanics it is no longer the case that the idempotent random variables take only the values one or zero on all pure states. For now the pure states are vectors φ in an appropriate Hilbert space $\mathcal{H}$. The idempotent random variables are projectors E in the lattice of projections $\mathcal{L}(\mathcal{H})$ on $\mathcal{H}$. In general only expectation values are determined [cf(2.6)] and

generally we have

(11.1.2) $0 \leq \langle E;\varphi \rangle \leq 1$.

In fact $\langle E;\varphi \rangle$ can only take the value 1 if φ is a vector in the range of E and the value 0 if φ is a vector in the range of $E^{\perp}$.

We have a situation in quantum mechanics therefore in which the extent of our knowledge of any system in a pure state is still apparently incomplete. We can determine only expectation values for observables and not actual values. In other words the pure states are not dispersion-free. This apparent incompleteness in quantum mechanics led people to wonder whether it is perhaps possible to introduce some extra parameters or hidden variables λ, say, such that (φ, λ) together constitutes a dispersion free state on some state space.

The literature on this subject, which has inspired, perplexed, and occasionally enlightened quantum physicists for fifty years or more, is vast. It was Einstein who became the arch proponent of the view that quantum mechanics was incomplete. His paper [1935] with Podolski and Rosen caused unrest and controversy that has remained unabated. Various authors [von Neumann 1932/55, Jauch and Piron 1963, Gleason 1957 inter alia] have proven theorems which allege the impossibility of hidden variables in quantum

mechanics. But these "no-go" theorems have themselves been criticised [cf Bell 1966 and refs therein] and hidden variables theories of one kind and another have actually been proposed [Bohm 1951, Bohm and Bub 1966]. Part of the problem has been an insufficiently exact definition of a hidden variables theory, or rather the inability of the antagonists in the dispute to agree on one. Moreover the "no-go" theorems themselves have been shown to rest on questionable premises. A thorough historical and theoretical background to all this may be found in Belinfante [1973] or Jammer [1974].

Whatever the case for hidden variables generally, the situation as regards the EPR gedankenexperiment seemed to receive a significant clarification when Bell published his paper [1964] on the impossibility of a hidden variables theory which satisfied not only the quantum mechanical statistics but also a certain locality condition, demanded,so it seemed, by Einstein locality [Einstein 1948; cf also Aspect 1976, Selleri 1978, Jammer 1974]. Bell's theorem as it has come to be known consisted of deducing a certain inequality necessary for the so-called local hidden variable (LHV) theories, but violated by quantum mechanical systems. Since the publication of this work, Bell's inequality and various other forms of inequality which collectively are referred to as Bell inequalities, have

been the subject of extensive discussion and supposedly crucial experimentation to determine th validity of quantum mechanics against the LHV theories. We shall later have occasion to discuss both the relevance and the effectiveness of such theories it perform this determination [cf Chapter 12]. As for the vast literature which has sprung up concerning Bell's theorem we content ourselves here with mentioning that it is largely the "locality condition" which has caused controversy. We shall examine this in more detail later [(11.6)] and mention here only our belief that a locality condition which makes no reference to spatial location is almost bound to cause confusion. A survey of some of the literature is included as an Appendix to this thesis [Appendix A.4]. The immediate task in hand is to clarify the issue of what exactly we take a hidden variables theory to be. We do this in the next section. The following three sections are devoted to several, variously successful, hidden variables theories. We then turn our attention to Bell's theorem and the locality condition , and examine these issues in the light of the asymptotically separable quantum mechanics proposed in the previous chapter.

11.2 Hidden Variables and Hidden Variables at Infinity

A fairly concise definition of a hidden variables theory is provided by Jauch [1968]

(11.2.1) Definition

A physical system is said to admit hidden variables if there exists a measure space Γ together with a finite measure μ (normalised so that $\mu(\Gamma) = 1$) on Γ such that every state W of the system can be represented as a mixture

$$W(E) = \int_{\Gamma} d\mu(\lambda) W_{\lambda}(E)$$

of the dispersion free states W_{λ}.

For several reasons this definition is not really satisfactory for our purposes, however. Firstly it will be more useful for us to have a definition expressed more generally in terms of expectation values rather than in terms of Jauch's more limited definition of a state on the lattice of projections. More importantly, Definition (11.2.1) is more stringent than it needs to be and in fact seems to be too stringent to be fulfilled by any theory. In particular there seems to be no reason why we cannot allow the particular finite measure μ to be dependent on the state of the

system. This is generally the case in a classical theory. We shall therefore use a definition closer to the ones given by Bub [1969] or Jammer [1974].

(11.2.2) Definition

A quantum mechanical theory is said to admit hidden variables if there exists a measure space Γ of hidden parameters λ (λ may itself be a set), a set of measures μ_φ on Γ and a set of real-valued functions f_A, such that the following conditions hold:

(i) Each (statistical) state vector φ of the quantum mechanical system is associated with a measure μ_φ on Γ normalised so that $\mu_\varphi(\Gamma) = 1$ and subject to the interpretation that for each measurable subset Λ of Γ we have

$\mu_\varphi(\Lambda)$ = probability that the hidden state λ of the system lies in Λ.

(ii) Each observable (selfadjoint operator) A of the physical system is associated with a function $f_A : \Gamma \to \mathbb{R}$. f_A is single-valued and has the interpretation that $f_A(\lambda)$ is the result of measuring A when the state is characterised by the parameter λ. (Thus the λ may be seen as dispersion-free states in the same way that the points in phase space are for classical mechanics.)

(iii) The expectation value of the observable A when the (statistical) state is φ, seen as a mixture with distribution μ_φ over the values $f_A(\lambda)$ of states λ, agrees with the usual quantum mechanical expectation

value. That is:

$$\langle A; \mu_\varphi \rangle = \int_\Gamma f_A(\lambda)\, d\mu_\varphi(\lambda).$$

We remark that this definition is less stringent in its conditions than some others. It is not uncommon, for example, to find the following additional condition [cf Bub 1969 p103]:

(iv) If $f_A(\lambda) = a$, then $f_{g(A)}(\lambda) = g(a)$.

This condition is, however, close to the sort of assumptions which have been objected to in impossibility proofs. We will bear it in mind for future comparison, but not include it in our definition.

We now turn our attention to asymptotic theories.

(11.2.3) Definition

We shall say that a quantum mechanical theory admits hidden variables at infinity if:

(0) The algebra of observables of the system is an asymptotic algebra in the sense of Wan and McLean [1984(a)], ie

$$\lim_{t\to\infty} \langle A_t; \varphi \rangle \text{ exists for each } \varphi \text{ in } \mathcal{H},\ A \in \mathcal{A},$$

and there exists a measure space Γ of parameters λ, called (hidden) states at infinity, a set of measures μ_φ on Γ and a set of functions f_A such that

(i) Each statistical state vector φ of the quantum mechanical system is associated with a measure μ_φ on Γ

normalised so that $\mu_\varphi(\Gamma) = 1$ and subject to the interpretation that for each measurable subset Λ of Γ we have:

$\mu_\varphi(\Gamma)$ = probability that the (hidden) state at infinity λ lies in Λ.

(ii) Each observable of the system is associated with a function $f_A : \Gamma \to \mathbb{R}$ such that f_A is single-valued and with the interpretation that $f_A(\lambda)$ is the result of measuring A when the state at infinity is λ.

(iii) The expectation value of A over the space of hidden states at infinity when the statistical state vector is φ agrees with the quantum mechanical expectation value of A in the state at infinity w_φ^∞ generated by φ:

$$w_\varphi^\infty(A) = \lim_{t\to\infty} \langle A_t ; \varphi \rangle = \int f_A(\lambda) d\mu_\varphi(\lambda).$$

The proximity of this definition to (11.2.2) is obvious. It raises the possibility however, that it makes some sense to describe a certain aspect of a particular system by means of hidden variables. We shall discuss this further later on in our analysis. We shall now examine whether it is possible to construct hidden variable theories at infinity for certain simple quantum mechanical systems. First let us conclude this section with another definition.

(11.2.4) Definition

If a system admits hidden variables at infinity, we

shall call the triplet $(\Gamma, \{f_A\}, \{\mu_\varphi\})$ comprising state space, measure set and set of functions, a hidden variables theory at infinity.

11.3 The Single Particle without Spin

We prove the main result of this section in the form of the following theorem.

(11.3.1) Theorem

Let $\mathcal{S}$ be a quantum mechanical system described by the WM-algebra $\mathcal{A}_{WM} = \mathcal{A}_0^S + L^\infty(\underline{p})$. Then the system admits a hidden variables theory $(\Gamma, \{f_A\}, \{\mu_\varphi\})$ at infinity with $\Gamma = \mathbb{R}^n$, $\{f_A\} = L^\infty(\underline{p})$, where $d\mu_\varphi/d\underline{p}$ the probability density function is given by

$$d\mu_\varphi/d\underline{p} = |\langle \underline{p}|\varphi\rangle|^2 \qquad (*)$$

where $\langle p|\varphi\rangle$ is the momentum representation of φ [cf (2.1.3)].

Proof

(0) Firstly it makes sense to talk of a hidden variables theory at infinity because $\mathcal{A}_{WM}$ is by construction an asymptotic algebra.

(i) Suppose that μ_φ is as given by (*). Then we have:

$$\mu_\varphi(\Lambda) = \int_\Lambda |\langle \underline{p}|\varphi\rangle|^2 \, d\underline{p}$$

which can be interpreted as the requisite probability that the parameter $\underline{p}$ is in the measurable subset Λ of $\mathbb{R}^n$ since

$$\mu_\varphi(\Gamma) = \int_{\mathbb{R}^n} |\langle \underline{p}|\varphi\rangle|^2 \, d\underline{p}$$

$$= \int_{\mathbb{R}^n} |\varphi(x)|^2 dx = 1.$$

Notice that this is nothing more than the usual quantum mechanical interpretation of the probability measure defined by the momentum observable.

(ii) Each observable A in $\mathcal{A}_{WH}$ may be written

$$A = A_0 + F$$

where $F \in L^\infty(p)$. We prescribe the functions f_A corresponding to the observables A by setting $f_A = F$ for each A of the above form in the algebra. The interpretation that $f_A(\underline{p})$ is the result of measuring A when the state at infinity is $\underline{p}$ is reminiscent of the phase space interpretation of classical mechanics and makes sense here since f_A is single valued and real-valued.

(iii) We need to show that the expectation values determined by the statistical distribution φ over the hidden parameters $\underline{p}$ are equal to the quantum mechanical expectation values for the state at infinity w_φ^∞. This is straightforward since:

$$\langle A;\mu_\varphi\rangle_{HV} \equiv \int_\Gamma f_A(\underline{p})\,d\mu(\underline{p})$$

$$= \int_{\mathbb{R}^n} |\langle \underline{p}|\varphi\rangle|^2 F(\underline{p})\,d\underline{p}$$

$$= \langle \varphi \mid F(\underline{p})\varphi\rangle$$

$$= w_\varphi^\infty(A)$$

For any system described by the WM-algebra then, a hidden variables theory exists to describe the system at infinity. In terms of dispersion-free states, we have a theory for which such states exist at infinity, namely they are given by the hidden parameters $\underline{p}$. Of course $\underline{p}$ are simply the momentum values and hence it is rather misleading to call them hidden. Nevertheless the essential form of a hidden variables theory has been achieved.

Now in quantum mechanics the states are usually thought of as NPLF's on the algebra of observables. In the hidden variables theory, the states appear as points in a "phase space". These two descriptions can be rendered compatible for the hidden variables theory at infinity proposed here in the following way: for each hidden state $\underline{p}$ at infinity define an NPLF by:

$$\Omega_{\underline{p}}(A) = F(\underline{p}),\quad A = A_0 + F \in \mathcal{A}_{WM}.$$

This is a pure state on $\mathcal{A}_{WM}$ [cf Wan and McLean 1984(b)], and it is straightforward to verify that these states are dispersion free on the WM algebra.

As a point of interest it is perhaps worth noting that if g is any polynomial function, then $g(A) = g(A_0+F) = g(A') + g(F)$ where $A' \in \mathcal{A}_0^S$, and hence we have

$$f_{g(A)}(\underline{p}) = g(F)(\underline{p}) = g(f_A(p)),$$

so that condition (iv) of (11.2) is in fact satisfied in the hidden variables theory at infinity for a single particle without spin.

11.4 The Single Spin Half Particle

In Bell's original paper [1964] and a later paper [1966], he brought attention to the fact that it is possible to construct a hidden variables theory for the spin of a spin-half particle. In Appendix A.2 we have elaborated a version of such a theory based on Bell's suggestions. Essentially the theory consists of the triplet $(\Gamma, \{\mu_{\underline{p}}\}, \{f_s\})$ where Γ the state space is given by:

$$(11.4.1) \qquad \Gamma = \mathcal{S}^{(2)} = \{\underline{\lambda} \in \mathbb{R}^3 : |\underline{\lambda}| = 1\}.$$

The measures $\mu_{\underline{p}}$ are determined for each polarisation

vector $\underline{P}$ (or for each spin vector γ) by

$$(11.4.2) \qquad \mu_{\underline{P}}(\Lambda) = (1/2\pi)\iint_{\Lambda \cap H^{+}_{\underline{P}}} \sin\theta \, d\theta \, d\varphi \, ,$$

where $H^{+}_{\underline{P}}$ is the hemisphere given by $\underline{\lambda}.\underline{P} > 0$, and the functions f_S are given for each spin observable $S = \alpha 1 + \underline{\sigma}.\underline{a}$ by:

$$(11.4.3) \qquad f_S(\underline{\lambda}) = \alpha + a \, \mathrm{sign}\,\underline{\lambda}.\underline{a}'$$

where $\underline{a}'$ is obtained by rotating $\underline{a}$ until

$$(11.4.4) \qquad \underline{a}.\underline{P} = 1 - (2/\pi)\cos^{-1}(\underline{a}'.\underline{P})$$

[cf Appendix A.2 for details and a proof that this theory yields the usual quantum mechanical results under the appropriate statistical averaging.]

A rather important point now arises. Evidently Γ and $\mu_{\underline{P}}$ satisfy the conditions in (i) of definition (11.2.2) and, from the results of Appendix A2 , the condition (iii) is also satisfied, namely the quantum mechanical expectations coincide with the hidden variables expectations. If we look carefully at condition (ii) however, and compare it with the prescription (11.4.3) and (11.4.4) for the function f_S we discover a crucial difference. While f_A in condition (iii) of Definition (11.2.2) is a function

from the hidden state space Γ to the reals, it may be seen that the prescription (11.4.3)-(11.4.4) is essentially for a function $f_S : \Gamma \times \mathcal{H} \to \mathbb{R}$, where $\mathcal{H}$ is the Hilbert space of quantum states ψ. In other words, the particular value $f_S(\underline{\lambda})$ of the observable S is not in fact determined solely by λ, but by ψ (or $\underline{P}$) as well. This seems to indicate that after all Definition (11.2.2) is too stringent to include a hidden variable theory of the type proposed here. In order to include such theories we would need to amend the definition as follows.

(11.4.5) Definition

A quantum mechanical system is said to admit hidden variables if there exists a triplet $(\Gamma, \{\mu_\varphi\}, \{f_A\})$ as before such that

(i) As in Definition (11.2.2)

(ii) Each observable A of the system is associated with a map $f_A : \mathcal{H} \to \mathcal{F}$ where $\mathcal{H}$ is the space of state vectors and $\mathcal{F}$ is the set of real-valued functions on the hidden state space Γ. We denote by f_A^φ the image $f_A(\varphi)$ of φ under the map f_A. We require further that the functions f_A^φ are single-valued and we interpret $f_A^\varphi(\lambda)$ as the value of the observable A when the quantum state is φ and the hidden state is λ.

(iii) The expectation value of A as a statistical average over the values $f_A^\varphi(\lambda)$ under the distribution φ must coincide with the usual quantum mechanical

expectation values:

$$\langle A; \varphi \rangle = \int_{\Gamma} f_A^{\varphi}(\lambda) d\mu_{\varphi}(\lambda).$$

In other words, we have amended our definition in such a way that the quantum mechanical state vector φ now not only plays the role of determining the statistical distribution function but also partly determines the the values of the observables. Given that the role of the hidden variables was originally to "complete" the quantum mechanical states this does not seem unreasonable . Evidently it is this form of hidden variables theory that Bell had in mind when he wrote [1966 p398]:

> "The question at issue is whether the quantum mechanical states can be regarded as ensembles of states specified by additional variables, such that given values of these variables together with the state vector determine precisely the results of individual measurements."

In this respect some of the literature on the subject is manifestly unclear. Jammer's definition [1974 p262] of hidden variables contains the clause:

> "(1) Each individual quantum system described by the usual state function φ is characterised by <u>additional</u> hidden states labelled by a parameter λ; the totality of all hidden states is the "phase space" Γ of hidden states; <u>φ and λ</u> determine the result of measuring any observable on the system."

While the underlined parts of the clause (our underlining) indicate a sympathy with Bell's view of a hidden variable theory and our definition (11.4.5), the phrase in quotation marks (our quotation marks) indicates a sympathy with the previous definition. Indeed when Jammer explicitly characterises the functions f_A he does so in the stricter sense of Definition (11.2.2). There is obviously some confusion here. For the sake of the present analysis we shall adopt the view that the quantum mechanical state vector may in part determine the results of measurement of an observable, these results being completely determined only by additional variables λ. Notice that the sentence in parentheses in Definition (11.2.2)(ii) is now no longer strictly true. We cannot regard the λ as dispersion-free states or as points in a phase space in the same way as we can with classical mechanics. We must now say that a dispersion-free state is determined by λ and φ and write (φ, λ), say, for such an object.

Presumably we would then want to regard (φ, λ) as points in some phase space in the classical sense. Certainly, in the theory we are considering here this can be done: each vector γ is determined by a polarisation vector $\underline{P}$ which in its turn is specified by two angles θ and φ ; we can regard the phase space therefore as the space $I_o^{2\pi} \times I_o^{\pi} \times \Gamma$ of all points $(\varphi, \theta, \underline{\lambda})$ where $I_o^{2\pi} = [0, 2\pi]$, $I_o^{\pi} = [0, \pi]$ and Γ is as given by (11.4.1).

The question now arises whether we can formulate a hidden variables theory at infinity which incorporates the spin and the spatial part of the system. Since our spin hidden variables theory is specified in terms of the broader definition (11.4.5) we shall use this as our criterion to judge whether or not such a theory exists. Or, more precisely, we should use the obvious extension of (11.4.5) to the case at infinity.

We consider a quantum mechanical system $\mathcal{S}^{\sigma}$ comprising one particle with spin half described by the WM-algebra $\mathcal{A}_{WM}^{\sigma}$. Now the problem we have to face here is how we may combine the two hidden variable measures for the tensor product states φ^{σ} of the combined spin and space system. We prove the following theorem.

(11.4.6) Theorem

Let $\mu_{\varphi^{\sigma}}$ be a measure on $\Gamma \times \mathbb{R}^3$ given by

$$\mu_{\varphi^\sigma}(\Lambda \times \Lambda') = (1/2\pi)\int_{\Lambda'}\iint_{\Lambda \times H_{\underline{p}}^+} |\langle \underline{p}|\varphi\rangle|^2 \sin\theta\, d\theta\, d\varphi\, dp,$$

where $\varphi^\sigma = \varphi \dot{\otimes} \gamma$ is a spin separable [cf(9.6.1)] vector in the product space $\mathcal{H}^\sigma$. Then the expectation value for the hidden variable theory at infinity of this spin system is the same as the quantum mechanical expectation value.

Proof

Firstly of course, we must clarify the hidden variables theory itself by specifying the value $f_{A^\sigma}^{\varphi^\sigma}(\underline{p},\underline{\lambda})$ of each observable A^σ in $\mathcal{A}_{WM}^\sigma$. We define for each observable $A \otimes S$ in $\mathcal{A}_{WM}^\sigma$ the function $f_{A\otimes S}^{\varphi^\sigma}(\underline{p},\underline{\lambda})$ by:

$$f_{A\otimes S}^{\varphi^\sigma}(\underline{p},\underline{\lambda}) = f_A(\underline{p}) f_S(\underline{\lambda}).$$

It then follows that for each such $A \otimes S$ we have:

$$\langle A \otimes S; \mu_{\varphi^\sigma}\rangle_{HV} = (1/2\pi)\int_{\mathbb{R}^3}\int_\Gamma |\langle \underline{p}|\varphi\rangle|^2 f_A(\underline{p}) f_S(\underline{\lambda})\, d\underline{\lambda}\, d\underline{p}$$

$$= \int_{\mathbb{R}^3} |\langle \underline{p}|\varphi\rangle|^2 f_A(\underline{p})\, d\underline{p} \int_\Gamma f_S(\underline{\lambda})\, d\underline{\lambda}$$

$$= \langle A; \mu_\varphi\rangle_{HV} \langle S; \mu_{\underline{p}}\rangle_{HV}$$

$$= \langle A; w_\varphi^\infty\rangle . \langle S; \gamma\rangle$$

$$= \langle A \otimes S; w_{\varphi^\sigma}^\infty\rangle .$$

For a sum of observables $A = \sum_{ij} A_i \otimes S_j$ we put

$$f_A^{\varphi^\sigma}(\underline{p},\underline{\lambda}) = \sum_{i,j} f_{A_i\otimes S_j}^{\varphi^\sigma}(\underline{p},\underline{\lambda}).$$

It follows that $\langle \sum A_i\otimes S_j ; \mu_{\varphi^\sigma}\rangle_{HV} = \sum_{ij} \langle A_i\otimes S_j;\mu_{\varphi^\sigma}\rangle = \sum_{ij} \langle A_i\otimes S_j; w_{\varphi^\sigma}^{\infty}\rangle = \langle \sum_{ij} A_i\otimes S_j; w_{\varphi^\sigma}^{\infty}\rangle$. The swopping of sum and limit being justified by the uniform convergence of the partial sums [cf (7.4.5)].

The case for an arbitrary state vector φ^σ presents a considerable complication in the question of which is the correct measure in the hidden variable theory. Even for the simplest case where the observable is a simple tensor we do not have an obvious way of combining the measures that will yield the quantum mechanical expectations. Suppose that we consider a vector $\varphi^\sigma = \sum_{ij} c_{ij}\varphi_i\otimes\gamma_j$ in $\mathcal{H}^\sigma$. Then the expectation values $\langle A\otimes S;\varphi^\sigma\rangle$ for a simple tensor observable are given by

$$\langle A\otimes S;\varphi^\sigma\rangle = \sum_{ijkl} c_{ij}^* c_{kl} \langle \varphi_i\otimes\gamma_j | A\otimes S | \varphi_k\otimes\gamma_l\rangle.$$

Note that these expectation values contain crossterms arising from coherence effects in the quantum mechanical measure and hence the hidden variable measure will also have to contain crossterms. Let us attempt to construct the appropriate measure. We observe first that the spin measure $\mu_{\underline{p}}$ satisfies

$$\mu_{\underline{\rho}}(\Lambda) = (1/2\pi)\int_{\Lambda} \chi_{H^+_{\underline{\rho}}} d\underline{\lambda},$$

where $\chi_{H^+_{\underline{\rho}}}$ is the characteristic function of the hemisphere $H^+_{\underline{\rho}}$, so that

$$d\mu_{\underline{\rho}}/d\underline{\lambda} = \chi_{H^+_{\underline{\rho}}}/2\pi$$

and we can think of $\chi_{H^+_{\underline{\rho}}}/2\pi$ as an appropriately normalised probability density function $d\mu_{\underline{\rho}}/d\underline{\lambda}$. Likewise of course $|\langle \underline{p}|\varphi\rangle|^2$ is the probability density function for the spatial part and we can think of $|\langle \underline{p}|\varphi\rangle|^2 \chi_{H^+_{\underline{\rho}}}/2\pi$ as the probability density function $d\mu_{\varphi^\sigma}/d(\underline{p},\underline{\lambda})$ for the combined state, when we are considering a simple tensor $\varphi^\sigma = \varphi \dot{\otimes} \chi$. Now suppose that we have a state vector of the second type, that is an arbitrary vector of the form:

$$\varphi^\sigma = \sum_{ij} c_{ij} \varphi_i \otimes \chi_j$$

where the $\{\varphi_i\}$ and $\{\chi_j\}$ constitute orthonormal systems in the respective spaces. For simplicity we shall consider only two terms:

$$\varphi^\sigma = c\,\varphi_1 \dot{\otimes} \chi_1 + d\,\varphi_2 \otimes \chi_2, \quad c^2 + d^2 = 1.$$

How are we to construct the measure on the hidden variable space $\mathbb{R}^3 \times \mathcal{S}^{(2)}$? Suppose that we try to include crossterms in the measure. Then the probability density functions must have terms $\chi_{H^+_{\underline{\rho}_1}} \chi_{H^+_{\underline{\rho}_2}}$ in them corresponding to the crossterms arising from different spin vectors. Since these spin vectors are orthogonal however, the polarisation vectors $\underline{P}_1$ and $\underline{P}_2$

satisfy $\underline{P}_1 = -\underline{P}_2$ and it follows that the two hemispheres $\underline{\lambda}.\underline{P}_1 > 0$ and $\underline{\lambda}.\underline{P}_2 > 0$ are disjoint. Hence the probability density function vanishes on the crossterms and we are left with

$$d\mu_{\varphi^\sigma}/d\underline{p}d\underline{\lambda} = c^2\, d\mu_{\varphi_1^\sigma}/d\underline{p}d\underline{\lambda} + d^2\, d\mu_{\varphi_\sigma^2}/d\underline{p}d\underline{\lambda}$$

$$= c^2 |\langle \underline{p}|\varphi_1\rangle|^2 \frac{\chi_{H^+_{\underline{P}_1}}}{2\pi} + d^2 |\langle \underline{p}|\varphi_2\rangle|^2 \frac{\chi_{H^+_{\underline{P}_2}}}{2\pi}$$

where $\varphi_i^\sigma = \varphi_i \dot{\otimes} \gamma_i$. Now we can get agreement with quantum mechanics generally only for (mixtures of) spin-separable states of the first kind or alternatively if we consider only spin observables $1\dot{\otimes} S$ or only space observables $A\dot{\otimes} 1$. In the case of spin observables we have

$$\langle 1\dot{\otimes} S;\mu_{\varphi^\sigma}\rangle_{HV} = \int_{\mathbb{R}^3}\iint_{H^+_{\underline{P}_1}} c^2 |\langle \underline{p}|\varphi_1\rangle|^2 f_S(\underline{\lambda})\, d\underline{\lambda}\, d\underline{p} + \int_{\mathbb{R}^3}\iint_{H^+_{\underline{P}_2}} d^2 |\langle \underline{p}|\varphi_2\rangle|^2 f_S(\underline{\lambda})\, d\underline{\lambda}\, d\underline{p}$$

$$= c^2 \iint_{S^{(2)}} \chi_{H^+_{\underline{P}_1}} f_S(\underline{\lambda})\, d\underline{\lambda} + d^2 \iint_{S^{(2)}} \chi_{H^+_{\underline{P}_2}} f_S(\underline{\lambda})\, d\underline{\lambda}$$

$$= c^2 \langle S;\mu_{\underline{P}_1}\rangle_{HV} + d^2 \langle S;\mu_{\underline{P}_2}\rangle_{HV}$$

$$= c^2 \langle S;\gamma_1\rangle + d^2 \langle S;\gamma_2\rangle$$

$$= \langle 1\otimes S;\varphi^\sigma\rangle .$$

We are left then in the position of being unable to provide a hidden variables theory which agrees with quantum mechanics for an arbitrary state vector even at infinity, unless we restrict the observables even further. Moreover, the existence or lack of existence of such a theory remains eminently undecidable. The usual "no-go" theorems do not apply in this instance, either because the underlying logic is not the usual one or because the kind of hidden variables theory we are attempting to construct lacks the necessary algebraic premises - we tried to make it as broad as possible.

In a sense we could say that we have here a contextual hidden variable theory at infinity in that, for either spin observables or space observables the theory in question reproduces quantum mechanical results. This is actually broader than the usual definition of a contextual hidden variable theory [Beltrametti and Cassinelli 1981, eg] since it is usual to take a contextual hidden variable theory as meaning a theory for a set of compatible observables whereas here we have a theory which embraces, contextually, a set of incompatible observables, namely the spin observables.

What is of particular interest in this case is that the obvious combination of two hidden variable theories has not provided us with a hidden variables theory (that satisfies quantum mechanics). This of course should not surprise us,

since it is precisely the difficulty encountered in the attempt to provide a hidden variables theory for two spin half particles. It is obviously of extreme interest to know whether in the case we have examined here it is possible to provide a noncontextual hidden variable theory at infinity which completely satisfies condition (iii) of (11.4.5). But this will have to remain an open question for the present. Supposing that the answer to such a question were negative, what could we still glean from all this ? Firstly, of course, we might be tempted to drop the requirement that the theory should agree with quantum mechanics, argue that quantum mechanics is wrong, and construct an experiment designed to test the hidden variable predictions against those of quantum mechanics. We do not know of any experiment which has been motivated in this way for the case of spin and space vectors, although of course this is precisely what is being tested for the case of two spin-particles by the many experimental tests [cf Aspect 1976, Aspect et al 1981,1982, Lamehi-Rachti and Mittig 1976] involving photon cascades and the like. It seems then that there should be interesting lessons to be learned concerning the latter case by considering carefully what prevented the straightforward formulation of a hidden variable theory in the former. Specifically we remark that what seemed to stand in the way of agreement was the coherence in the quantum mechanical measure which had no counterpart in the hidden variable measure space and it is not at all clear that this has anything to do with whether or not the

measurement results in the asymptotic theory are separable.

There is another option open to us in the face of the failure of such a hidden variables theory to conform to the results of quantum mechanics. It could be that our definition of a hidden variable theory is still too stringent. Perhaps a further broadening could allow hidden variables which do reproduce quantum mechanical expectations. In fact the quantum potential theories of Bohm, Hiley et al are very much along these lines. In this thesis however we shall restrict our attention to the sort of theories outlined here. This is primarily because our main interest is in the locality questions which (11.6 et seq) are best investigated in the present form.

11.5 Two Particles without Spin at Infinity

We are going to consider first the theory described by the WM-algebra $\mathcal{A}_{WMC}$.

(11.5.1) Theorem

Let I+II be a two particle system described by the algebra $\mathcal{A}_{WMC}$ with states given by the normal states w_ρ on $\mathcal{A}_{WMC}$, the normal states at infinity w_ρ^∞, and the singular states at infinity of the form:

$$\Omega^\infty_{\underline{p}_1\underline{p}_2}(A_o + F) = F(\underline{p}_1, \underline{p}_2)$$

with the obvious notation. Then I+II admits a hidden variable theory at infinity in which the hidden variables $(\underline{p}_1, \underline{p}_2)$ are in one-one correspondence with the dispersion-free singular states $\Omega^\infty_{\underline{p}_1\underline{p}_2}$.

Proof

(0) $\mathcal{A}_{WMC}$ is an asymptotic algebra [Wan and McLean 1984(c).

(i) Define for each Φ in $\mathcal{H}_c$ the measure μ_Φ by

$$\mu_\Phi(\Lambda) = \iint_\Lambda |\tilde{\Phi}(\underline{p}_1, \underline{p}_2)|^2 \, d\underline{p}_1 \, d\underline{p}_2,$$

where Λ is a measurable subset in the phase space $\Gamma = \mathbb{R}^{2n}$ and $\tilde{\Phi}$ denotes the Fourier transformed

wavefunction. Evidently we have

$$\mu_{\underline{\Phi}}(\Gamma) = \iint_{\mathbb{R}^3}\int_{\mathbb{R}^3} |\tilde{\Phi}(\underline{p}_1, \underline{p}_2)|^2 \, d\underline{p}_1 \, d\underline{p}_2 = 1.$$

(ii) We can define a single valued function $f_A(\underline{p}_1, \underline{p}_2)$ for each $A = A_0 + F$ in $\mathcal{A}_{WHC}$ by $f_A(\underline{p}_1, \underline{p}_2) = F(\underline{p}_1, \underline{p}_2) = \Omega^{\infty}_{\underline{p}_1\underline{p}_2}(A)$. The one-one correspondence of singular states to phase space points $(\underline{p}_1, \underline{p}_2)$ is then clear and the dispersion-free nature of the singular states can be demonstrated exactly as for the one particle case.

(iii) Finally we must show that $\langle A; \mu_{\underline{\Phi}} \rangle_{HV} = \langle A; w^{\infty}_{\underline{\Phi}} \rangle$. We have

$$\langle A; \mu_{\underline{\Phi}} \rangle_{HV} = \iint_{\Gamma} |\tilde{\Phi}(\underline{p}_1, \underline{p}_2)|^2 \, f_A(\underline{p}_1, \underline{p}_2) \, d\underline{p}_1 \, d\underline{p}_2$$

$$= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \tilde{\Phi}^*(\underline{p}_1, \underline{p}_2) F(\underline{p}_1, \underline{p}_2) \tilde{\Phi}(\underline{p}_1, \underline{p}_2) \, d\underline{p}_1 \, d\underline{p}_2$$

$$= \langle F; \underline{\Phi} \rangle$$

$$= \langle A_0 + F; w^{\infty}_{\underline{\Phi}} \rangle$$

$$= \langle A; w^{\infty}_{\underline{\Phi}} \rangle$$

So we have a very straightforward extension of the previous hidden variables theory at infinity (11.3) to the two particle case. This theory also satisfies the condition (iv) of (11.2).

Next we consider the theory of chapter 10 for the spinless case. Once again the algebra is $\mathcal{A}_{WMC}$. But now the states are a different set of NPLF's.

(11.5.2) Theorem

Let I+II be a system of two-particles without spin described by the algebra $\mathcal{A}_{WMC}$. Let the physical states on $\mathcal{A}_{WMC}$ comprise the states $\tilde{w}_\varrho$ defined in (10.2) and the states at infinity $\tilde{w}^\infty_\varrho$ generated by these states. Then I+II admits a hidden variables theory at infinity.

Proof

We must show that an appropriate measure $\tilde{\mu}$ exists which will fulfill the normalisation condition and yield the quantum mechanical expectation values. Firstly, we remark that the measure μ_Φ defined for a state vector Φ in (11.5.1) can be extended to any normal state by taking

$$d\mu_\varrho / d\underline{p}_1 d\underline{p}_2 = \langle \underline{p}_1, \underline{p}_2 | \varrho | \underline{p}_1, \underline{p}_2 \rangle$$

where $|\underline{p}_1, \underline{p}_2\rangle$ are the generalised momentum eigenfunctions. This reduces to the previous case when $\varrho = P_\Phi$ for some vector Φ since [cf A Bohm 1979 p48 et seq.] $\langle \underline{p}_1, \underline{p}_2 | \Phi \rangle = \tilde{\Phi}(\underline{p}_1, \underline{p}_2)$. In the more general case it is easy enough to show that the measure μ_ϱ yields the quantum mechanical expectation values $\langle F; \varrho \rangle$ in the state ϱ for all observables in $L^\infty(\underline{p}_1, \underline{p}_2)$. Now for each vector Φ in $\mathcal{H}_c$ we have $\Phi = \lambda \Phi_S + (1-\lambda^2)^{\frac{1}{2}} \Phi_c$ [cf (10.2.5)]. We define our measure for the present theory by

$$d\tilde{\mu}_\Phi / d\underline{p}_1 d\underline{p}_2 = \lambda^2 \langle \underline{p}_1, \underline{p}_2 | \varrho_{S1} \otimes \varrho_{S2} | \underline{p}_1, \underline{p}_2 \rangle + (1-\lambda^2) |\tilde{\Phi}_c(\underline{p}_1, \underline{p}_2)|^2$$

where ϱ_{s1} and ϱ_{s2} are the reduced statistical operators determined by Φ_s. Then we have

$$\tilde{\mu}_{\Phi}(\Gamma) = \lambda^2 \int_{\Gamma} \langle \underline{p}_1, \underline{p}_2 | \varrho_{s1} \otimes \varrho_{s2} | \underline{p}_1, \underline{p}_2 \rangle d\underline{p}_1 \, d\underline{p}_2$$

$$+ (1-\lambda^2) \int_{\Gamma} |\tilde{\Phi}_c(\underline{p}_1, \underline{p}_2)|^2 d\underline{p}_1 \, d\underline{p}_2$$

$$= 1$$

since Φ_c and $\varrho_{s1} \otimes \varrho_{s2}$ are both normalised states. Moreover for each observable $F(\underline{p}_1, \underline{p}_2)$ in $L^\infty(\underline{p}_1, \underline{p}_2)$ we have

$$\langle F; \tilde{\mu}_{\Phi} \rangle_{HV} = \lambda^2 \langle F; \varrho_{s1} \otimes \varrho_{s2} \rangle + (1-\lambda^2) \langle F; \varrho_c \rangle$$

$$= \langle F; \tilde{W}^{\infty}_{\Phi} \rangle .$$

11.6 Local Hidden Variables and Bell's Theorem

Since Bell's original paper [1964] in which he proved an "impossibility theorem" for local hidden variable theories in quantum mechanics by providing an inequality which he claimed should be satisfied by every LHV theory but which was violated by quantum mechanics, a number of generalised inequalities have been proposed which go under the collective name of Bell inequalities. [cf refs in Appendix A.4].

In this chapter we shall use the following Bell inequality.

(11.6.1)

$$|P(a,b) - P(a,b')| + |P(a',b) + P(a',b')| \leqslant 2$$

This inequality is Selleri's [1972] amended version of a similar inequality deduced by Clauser, Horne, Shimony, and Holt (CHSH) [1969]. The striking feature of this amendment is not so much that it strengthens CHSH's inequality, but that it does away with the physical requirement that for certain parameter values b,b' say, a definite (or even slightly indefinite) joint expectation P(b,b') exists. It is now possible to prove (11.6.1) using only the fact that the two dichotomic observables satisfy a certain "locality requirement" which we shall specify below.

Let us begin at the beginning; that is to say, with a system I+II of two particles on which we are to carry out measurements of certain physical quantities. Let us suppose that we wish to measure the values A and B of two physical quantities A and B. Suppose also that the two systems are separated and that there are measurement parameters a,b determining the measurements at system I and II respectively. Then the systems are said to fulfill the condition of Bell locality if the following holds.

(11.6.2) Bell Locality

The measurement results A and B for the measurement of the quantities A and B on I and II respectively satisfy

$A = A(a,\lambda)$; $B = B(b,\lambda)$

The important point here is that A depends only on the state λ and the parameter at I and B depends only on λ and the parameter at II and neither measurement result is influenced by the parameter at the distant system. This condition has also been called Einstein locality, or Einstein separability [cf Aspect, eg].

Bell's theorem then consists of the following line of reasoning. A local hidden variable theory must satisfy (11.6.1). Quantum mechanical expectation values do not always satisfy (11.6.1). Therefore no local hidden variable theory is possible for quantum mechanics.

This result was proved by Bell for the specific case of spin observables. Notice, however, that we can apply the reasoning to any observables. In particular, for any observable T_1 on system I the observable given by

$$A_1 = 2E(T_1;a) - 1$$

represents a dichotomic physical quantity. Similarly

$$B_2 = 2E(T_2;b) - 1$$

represents such a quantity for system II. Now if we denote by $P_\Phi(a,b)$ the joint expectation values for these two observables in state Φ we can proceed to examine whether Bell's inequality is satisfied. If there exist a,b,a',b'

for which the inequality (11.6.1) is violated then it can be said that no local hidden variable theory of the Bell type is possible for the observables T_1 and T_2. (Notice that we are asking the question of hidden variables contextually here; ie in terms of specific observables.) The following result is extremely important for our subsequent analysis

(11.6.3) Theorem

Let T_1, T_2 be observables on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, and let A_1 and B_2 be as above. Suppose that the joint expectation value in state ϱ is $P_\varrho(a,b)$ where ϱ is a mixture of state vectors of the first type (ie, simple tensors). Then Bell's inequality is always satisfied for any a,b,a',b'.

Proof

Capasso,Fortunato and Selleri 1972.

11.7 LHV-Theories and Bell's Theorem at Infinity

Suppose that f_1 and f_2 are any dichotomic observables in $L^\infty(\underline{p}_1)$ and $L^\infty(\underline{p}_2)$ respectively. Then $E_1 = (1/2)(f_1 + 1)$ defines a projection in $L^\infty(\underline{p}_1)$ for which $f_1 = 2E_1 - 1$. Similarly for f_2. It follows that every dichotomic observable in the algebra at infinity $\mathcal{A}^\infty$ in the WM-theory is of the form $2E_1 - 1$. The parameters of measurement are then subsets of $\mathbb{R}^n$.

(11.7.1) Theorem

Suppose that the two-particle system I+II without spin is described by the WM-algebra $\mathcal{A}_{WMC}$ with states given by the normal NPLF's on $\mathcal{A}_{WMC}$ and the normal states at infinity generated by normal NPLF's. For every vector state Φ of the form

$$\Phi = (1/\sqrt{2})(\varphi_1 \otimes \psi_2 + \psi_1' \otimes \varphi_2')$$

where φ_1, ψ_1' are asymptotically separating states and ψ_2, φ_2' are asymptotically separating, the joint expectation values P_Φ^∞ at infinity satisfy the Bell inequality for every pair of dichotomic observables.

Proof

The dichotomic observables with nonzero expectation values at infinity are all derived from projectors in $L^\infty(\underline{p}_1)$ and $L^\infty(\underline{p}_2)$. For each $F(\underline{p}_1, \underline{p}_2)$ in $L^\infty(\underline{p}_1, \underline{p}_2)$ we have

$$w_\Phi(F) = \langle \Phi \mid F \Phi \rangle$$

$$= (1/2)\langle \varphi_1 \otimes \psi_2 \mid F \varphi_1 \otimes \psi_2 \rangle + (1/2)\langle \psi_1 \otimes \varphi_2' \mid F \mid \psi_1' \otimes \varphi_2' \rangle$$

since φ_1, ψ_1' and φ_2', ψ_2 are asymptotically separating pairs of states. Hence w_Φ^∞ corresponds to a mixture of state vectors of the first type and therefore by Theorem (11.6.3) satisfies the Bell inequality for every pair of dichotomic observables.

Now let us consider the asymptotically separable algebra of Chapter 10.

(11.7.2) Theorem

Let I+II be a two particle spinless system described by the

algebra $\mathcal{A}_{WMc}$ and the AS-states and AS-states at infinity. Suppose that the two particles are asymptotically separating. ie, the AS-state is asymptotically separating. Then the joint expectation values at infinity for dichotomic observables all satisfy the Bell inequality.

Proof

If $\tilde{w}$ is asymptotically separating then $\tilde{w}^{\infty}$ is separable. That is $\tilde{w}^{\infty} = w_1^{\infty} \otimes w_2^{\infty}$. This state is a mixture of states of the first type and hence [(11.6.3)] satisfies the Bell inequality.

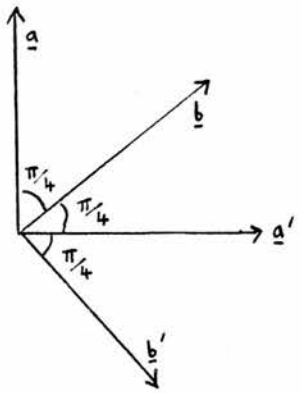
What we have demonstrated then is that for both the asymptotic theory of Wan and McLean and the AS-theory of chapter 10, the quantum mechanical expectations satisfy the Bell inequality when we take appropriate account of the asymptotic localisation of the particles. Specifically, the inequality is satisfied when the particles are separating. Hence we have shown that by making inherent reference to the localisation of the two particles in the formalism we can recover separability in the form of the Bell inequality.

11.8 Bell's Theorem for Two Particles with Spin

We are going to consider spin observables as in the usual Bell analysis, but in the present analysis the (asymptotic) localisation of the particles plays an important role, so we must consider wavefunctions which involve both spin and space parts.

Let us consider first the WM-theory in which the algebra of observables is $\mathcal{A}$ and the states are normal NPLF's on $\mathcal{A}$ and the normal states at infinity. We consider joint expectation values $P_\Phi(a,b)$ for the observable $1 \otimes S_1 \otimes S_2$ where S_1 and S_2 are dichotomic observables on H_1 and H_2 respectively. For such an observable we deduce by a similar analysis to tht used for the one particle algebras [cf (9.6.6)] that every state is spin separable, so that the expectation values reduce to the ones encountered in the usual Bell analysis no matter whether the state is asymptotically separating or not. It follows that there are states and measurement parameters a,b,a',b' for which the Bell inequality is violated. For example if the state is given by a simple tensor product of any space state with the isotropic spin zero vector and the parameters a,b,a',b' are as shown in the diagram

(11.8.1)



then [cf Beltrametti and Cassinelli 1981 Chapter 7,eg] the quantum mechanical expectation values violate (11.6.1). The WM-theory does not satisfy the Bell inequality therefore. Indeed the discussion in (10.1) concerning the nonseparability of WM even asymptotically should already have led us to expect such violation.

Next we examine whether the AS-theory of chapter 10 violates the inequality.

(11.8.2) Theorem

Let $\tilde{w}_\varrho$ be any asymptotically separable AS-state. Let S_1 and S_2 be dichotomic spin observables on H_1 and H_2 respectively, and let $\tilde{P}^\infty(a,b)$ denote the joint expectation function for the observable $1\dot{\otimes} S_1 \otimes S_2$ in the AS-state at infinity generated by $\tilde{w}_\varrho$. Then the expectation values $\tilde{P}^\infty(a,b)$ always satisfy the Bell inequality (11.6.1).

Proof

Firstly, the observable $1\dot{\otimes} S_1 \otimes S_2$ lies in the algebra at infinity $\mathcal{L}^\infty(\underline{p}_1, \underline{p}_2)\dot{\otimes}\mathcal{S}_c$ and $\tilde{w}_\varrho$ is asymptotically separating, so the expectation values $\tilde{P}^\infty(a,b)$ are given by

$$\tilde{P}^{\infty}(a,b) = \tilde{w}_1 \otimes \tilde{w}_2 \ (1 \dot{\otimes} S_1 \otimes S_2)$$
$$= \tilde{w}_{\varrho_1}(1 \otimes S_1)\tilde{w}_{\varrho_2}(1 \otimes S)$$
$$= w_{\varrho_{1\sigma}}(S_1) w_{\varrho_{2\sigma}}(S_2),$$

where $\varrho_{1\sigma}$, $\varrho_{2\sigma}$ are the unique reduced statistical operators on H_1 , H_2 determined by ϱ. It follows that on observables of the form $1 \dot{\otimes} S_1 \otimes S_2$, $\tilde{P}^{\infty}$ is either equivalent to a state corresponding to a vector of the first type on H_c or a mixture of such vectors and hence [cf Theorem(11.6.3)] satisfies the Bell inequality.

This result means that Bell's inequality is satisfied asymptotically by every asymptotically separating state in the AS-theory. Does every AS-state satisfy the Bell inequality asymptotically ? The answer to this question must be negative. Consider the spin-separable state vector $\Phi_c^{\sigma} = \Phi_c \dot{\otimes} \gamma_0$ where γ_0 is the isotropic spin zero vector and Φ_c is any space state with parameter of separation zero. Then the relevant expectation values $\tilde{P}^{\infty}_{\Phi_c^{\sigma}}(a,b)$ are given by

$$\tilde{P}^{\infty}_{\Phi_c^{\sigma}}(a,b) = \langle \Phi_c^{\sigma} | 1 \dot{\otimes} (S_1 \otimes S_2) \Phi_c^{\sigma} \rangle$$
$$= \langle \gamma_0 | S_1 \otimes S_2 | \gamma_1 \rangle,$$

and it is known [(11.8.1)] that there are spin directions for which these expectation values violate the Bell inequality.

11.9 A Hidden Variable Theory for Two Particles with Spin

Bell's theorem is an impossibility theorem. Violation of the inequality implies the impossibility of a local hidden variables description. Satisfaction of the inequality does not imply the existence of such a description. As far as the AS-theory is concerned therefore, what we have shown is that no hidden variable theory exists which will describe every AS-state at infinity. However it remains a possibility (though not a certainty) that a hidden variable description at infinity exists which will describe the asymptotically separating states. Happily it turns out that we can demonstrate just such a theory.

(11.9.1) Theorem

Let $S=S_1 \otimes S_2$ where S_1, S_2 are any two spin observables on H_1, H_2 respectively. Then for each asymptotically separating AS-state $\tilde{w}_\rho$ there exists a set Γ of (hidden) parameters $\underline{\mu}$, a measure $\tilde{\mu}_{\rho\sigma}$ on Γ and a set of functions $f(a,b,\underline{\mu})$ such that

(i) $\tilde{\mu}_{\rho\sigma}(\Gamma) = 1$

(ii) $\int_\Gamma d\tilde{\mu}_{\rho\sigma} f(a,b,\underline{\mu}) = \tilde{P}^{\infty}_{\rho}(a,b)$

Proof

Define the set Γ of hidden states by $\Gamma = S^{(2)} \times S^{(2)}$ where $S^{(2)}$ is the unit sphere [cf (11.4) and Appendix (A.2)]. Now denote by ρ_σ the unique reduced statistical

operator defined on H_c by ϱ and suppose first that ϱ_σ is a pure state on H_c. Each pure state can be written as a projection P_γ where γ is some vector in H_c and each such vector can be written as

$$\gamma = \sum_{ij} \lambda_{ij} \alpha_i \otimes \beta_j , \quad \sum |\lambda_{ij}|^2 = 1,$$

where $\{\alpha_i\}$ is any pair of orthonormal base vectors in H_1 and $\{\beta_j\}$ is any pair of orthonormal base vectors in H_2. For each such γ we define the measure $\tilde{\mu}_\gamma$ on Γ by

$$\tilde{\mu}_\gamma(\Lambda \times \Lambda') = \sum_{ij} |\lambda_{ij}|^2 \mu_{\alpha_i}(\Lambda) . \mu_{\beta_j}(\Lambda')$$

where μ_{α_i} is the hidden variable measure defined on $S^{(2)}$ for α_i by (11.4.2) and μ_{β_j} is the corresponding measure for β_j. Immediately we have $\tilde{\mu}_\gamma(\Gamma) = \sum |\lambda_{ij}|^2 \mu_{\alpha_i}(S^{(2)}) . \mu_{\beta_j}(S^{(2)})$ $= \sum |\lambda_{ij}|^2 = 1$. Next define the functions $f(a,b,\underline{\mu})$ by

$$f(a,b,\underline{\mu}) = f_{S_1}(a,\underline{\lambda}) . f_{S_2}(b,\underline{\lambda}'),$$

where the single valued functions $f_{S_1}(a,\underline{\lambda})$ and $f_{S_2}(b,\underline{\lambda}')$ are as given in (11.4.3). It follows that

$$\int d\tilde{\mu}_\gamma f(a,b,\underline{\mu}) = \sum_{ij} |\lambda_{ij}|^2 \int_\Gamma d\mu_{\alpha_i}(\underline{\lambda}) f_{S_1}(a,\underline{\lambda}) . \int_\Gamma d\mu_{\beta_j}(\underline{\lambda}') f_{S_2}(b,\underline{\lambda}')$$

$$= \sum_{ij} |\lambda_{ij}|^2 \langle S_1 ; \alpha_i \rangle . \langle S_2 ; \beta_j \rangle$$

$$= \sum_{ij} |\lambda_{ij}|^2 \langle S_1 \otimes S_2 ; \alpha_i \otimes \beta_j \rangle$$

$$= \langle S_1 \otimes S_2 ; \sum_{ij} |\lambda_{ij}|^2 P_{\alpha_i \otimes \beta_j} \rangle$$

$$= \langle 1 \dot{\otimes} S_1 \otimes S_2 ; \tilde{\omega}_\varrho^\infty \rangle .$$

For a mixture ϱ_σ on H_c, let us suppose that

$$\varrho_\sigma = \sum_i \mu_i P_{\gamma_i}, \quad \Sigma \mu_i = 1$$

where γ_i are pure states on H_c. Then we define the measure by

$$\tilde{\mu}_{\varrho_\sigma}(\Lambda \times \Lambda') = \sum_i \mu_i \mu_{\gamma_i}(\Lambda \times \Lambda'),$$

and it is straightforward to show that this measure yields the quantum mechanical expectation values so long as the interchange of the integral and the arbitrary sum of pure states is valid. We justify this by considerations of uniform convergence of the partial sums.

The extension of the theorem to arbitrary spin operators is a straightforward matter of the interchangeability of integral and arbitrary sum. The extension to arbitrary observables in the algebra $\mathcal{A}$ is not so straightforward however. The reason for this is the difficulty already encountered in providing a hidden variable measure to echo the tensor product between spin and space systems. In the particular case in which the state is spin separable, we recall from (11.4.6) that this is unproblematic. We state the following theorem without formal proof.

(11.8.4) Theorem

Let $A \dot{\otimes} S_1 \otimes S_2$ be an observable in $\mathcal{A}$. Then for each spin-separable, asymptotically separating AS-state $\tilde{w}_{\Phi^\sigma}$

there exists a measure $\tilde{\mu}$ on the space $\Gamma = \mathbb{R}^{2\Lambda} \times S^{(2)} \times S^{(2)}$ of hidden states at infinity and a set of functions $f(\underline{p}_1, \underline{p}_2, a, b, \underline{\mu})$, given by

$$f(\underline{p}_1, \underline{p}_2, a, b, \underline{\mu}) = F(\underline{p}_1, \underline{p}_2).f_{s_1}(\underline{\lambda}).f_{s_2}(\underline{\lambda}')$$

such that

$$\langle A \dot{\otimes} S \otimes S \; ; \tilde{\mu} \rangle_{HV} = \langle A \dot{\otimes} S_1 \otimes S_2 \; ; \tilde{w}^{\infty}_{\underline{\mu}\sigma} \rangle .$$

Sketch of Proof

The measure is defined by

$$d\tilde{\mu} / d\underline{\mu} \, d\underline{p}_1 \, d\underline{p}_2 = (d\mu_{\underline{\mu}} / d\underline{\mu}) . \langle \underline{p}_1, \underline{p}_2 | \hat{P}_{\underline{\mu}} | \underline{p}_1, \underline{p}_2 \rangle ,$$

and the integral becomes separable into the space hidden variable integral for two particle systems as in (11.4) and the spin integral in (11.8.3).

CHAPTER 12

CONCLUSIONS AND PROSPECTS

In 1936, shortly after the appearance of the EPR article, Furry [1936(a)] published a paper in which he investigated the statistical differences between coherent wavefunctions of the EPR type and mixtures of factorisable state vectors. He demonstrated explicitly that the statistical inferences which may be obtained from a wavefunction

(12.1)

$$\psi_0 = \frac{1}{\sqrt{2}}(\alpha_1 \otimes \beta_2 - \beta_1 \otimes \alpha_2)$$

(eg), are not generally the same as those obtained by assuming that the state is a mixture

(12.2)

$$W_0 = (1/2)P_{\alpha_1 \otimes \beta_2} + (1/2)P_{\beta_1 \otimes \alpha_2}$$

of factorisable wavefunctions, even though such inferences may agree in certain instances. For reasons which are now obscured by the tide of history, the thesis that the state of the EPR example somehow evolves into a mixture (12.2) of factorisable state vectors (and hence provides a resolution to the

paradox) has become known as Furry's hypothesis. It is clear from that paper and a subsequent note [1936(b)] that Furry did not in fact espouse this view, which was actually advanced first by Schrodinger [1935]. Nevertheless we shall refer to the hypothesis in question as Furry's hypothesis. (Interestingly, Furry himself seems [1936(b)] to have attributed the interference terms in the coherent state to the overlap of wavefunctions and also refers to Pauli's discussion in support of this.)

Evidently the theory we have proposed in Chapter 10 as an asymptotic solution to the EPR paradox is closely related to the Furry hypothesis. We have, in effect proposed an explicit evolution of the quantum mechanical state into a mixture of factorisable state vectors. Recall from (10.5), that the AS-state relevant to the EPR example is given by $\tilde{w}_{\Phi_o^\sigma}$ where Φ_o^σ is the vector given by

(12.3)

$$\Phi_o^\sigma = (1/\sqrt{2})(\varphi \otimes \psi) \dot{\otimes} \chi_o ,$$

where φ and ψ are asymptotically separating state vectors. Hence, in the asymptotic limit the spin part of the state is given by

(12.4)

$$\begin{aligned} \tilde{w}_o &= (1/2)(P_{\alpha_1} + P_{\beta_1}) \otimes (1/2)(P_{\alpha_2} + P_{\beta_2}) \\ &= (1/4)(P_{\alpha_1 \otimes \alpha_2} + P_{\alpha_1 \otimes \beta_2} + P_{\beta_2 \otimes \alpha_2} + P_{\beta_1 \otimes \beta_2}), \end{aligned}$$

which is again a mixture of factorisable state vectors, although not the same mixture (12.2) that generally appears in the Furry hypothesis.

Now let us consider this solution in the light of certain remarks and objections that have been raised in the literature concerning the separability of quantum systems and the Furry hypothesis in particular.

Firstly, it must be observed that the state χ_0 given in (12.1) is isotropic, ie spherically symmetric in configuration space, so that under any change of coordinate basis in the Hilbert space H_c, χ_0 retains the form of (12.1) [cf Beltrametti and Cassinelli 1981 p70, eg]. The state given by (12.2), on the other hand is non-isotropic. This has led to some rather complicated averaging procedures [cf Bohm and Aharanov 1957, Baracca et al 1975 eg] in order to regain spherical symmetry for the separable state. In the case of the AS-theory however we observe that the spin state (12.4) is already spherically symmetric. It is straightforward to verify this using the direct matrix product notation. The projectors $P_{\alpha_1\otimes\alpha_2}$, $P_{\alpha_1\otimes\beta_2}$, $P_{\beta_1\otimes\alpha_2}$, and $P_{\beta_1\otimes\beta_2}$ are given by the matrices

$$P_{\alpha_1\otimes\alpha_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad P_{\alpha_1\otimes\beta_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$P_{\beta_1\otimes\alpha_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad P_{\beta_1\otimes\beta_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Hence the state (12.4) is simply (1/4)1 where 1 is the identity operator in the space. Since the identity operator is invariant under any change of coordinate bases the state $\tilde{w}_0$ is spherically symmetric as desired. In our theory, therefore, no complicated averaging is required; the symmetry falls out quite naturally from the limiting process.

The reason why our state at infinity (12.4) retains spherical symmetry while (12.2) does not is the inclusion of the terms $P_{\alpha_1\otimes\alpha_2}$ and $P_{\beta_1\otimes\beta_2}$ which yield the unpolarised mixture. Ironically this feature seems to produce a new complication in the physical analysis of the solution. In previous analyses, the only factorisable state vectors considered as permissible in the mixed state for the solution of the paradox have been states $\alpha_1\otimes\beta_2$ and $\beta_1\otimes\alpha_2$ corresponding to systems

whose z-component of spin (eg) are correlated so that the total spin in that direction is zero. Allowing states $\alpha_1 \otimes \alpha_2$ and $\beta_1 \otimes \beta_2$ seems to deny the conservation law that preserves this spin zero component. In an ensemble of identically prepared systems, therefore, the mixture (12.2) indicates that a measurement of the z-component of spin on (say) the leftgoing particle would yield about half of the systems having spin up and half having spin down. Similarly for the rightgoing particle. The state (12.4) in the AS-theory yields the same proportions for each particle. The difference between the former analysis and ours is that (12.2) predicts that for each particle pair on which we conduct a simultaneous measurement of spin in the z-direction one will have spin-up and the other will have spin down. (12.4) on the other hand predicts that only a half of the particle pairs measured will be correlated in this way. Of the rest of the systems measured half will yield both particles having spin up and half will yield spin down for both particles. We still have

$$\langle J_{1z} \otimes 1 + 1 \otimes J_{2z}; \tilde{W}_0 \rangle = \mathrm{Tr}((J_{1z} \otimes 1 + 1 \otimes J_{2z})(1/4)1)$$

$$= (1/4)\mathrm{Tr}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$= (1/4)(1 - 1) = 0$$

so that, on the average, the z-component (and indeed every component) of spin of the system is conserved as zero. Now however, there exist pairs of particles for which the conservation law seems to be violated.

It has recently been shown by Wan and Timson [1985] that this apparent violation of conservation is precisely what we would expect to be observed physically for separating particles. The spread of the wavepacket and the finite size of the measurement device means that a "chronological disordering" of particle pairs is introduced and we are no longer able to correlate particular pairs of particles after they have left the source. It is this chronological disordering which allows the existence of states which seem to violate the conservation laws.

We now turn our attention to some rather different sorts of objections to separability in quantum mechanics. Since the time of Bell's original analysis of the EPR experiment and before, people have proposed a number of variants on experimental situations designed to provide conclusive proof of the essential nonlocality of quantum mechanics. With each such experiment performed the experimenters concerned have claimed the right of experimentum crucis in the

determination: which of Einstein locality or quantum mechanics is right. The most recent experiments concerning this issue are those of Aspect et al [1981,1982]. These experiments have been criticised from the point of view of the statistical significance of the data [cf Marshall 1983,1984], but have nevertheless been widely accepted as providing firm evidence for the nonlocality of quantum mechanics and hence for the denial of Einstein locality.

Aspect's experiments were carried out using photon cascades. For this reason no direct comparison between his experimental results and our theory is possible. We have only tackled the case of massive particles for which a well-defined position operator enables us to describe the locality of the particles in question and hence to undergo the analysis of the AS-theory for separating particles. Photons are massless particles. They do not fall under the sphere of description of our theory. Nevertheless, there are one or two remarks that perhaps are worth making concerning the Aspect results. In particular, we reiterate the belief that in order to discuss the question of nonlocality it is of paramount importance that we be able to represent the spatial location of the particles. For photons this is not entirely straightforward, since the definition of a position operator for massless particles is not without problems. Until such a

definition is possible, however, and until a firm theoretical basis in which to discuss localisation is available, it seems rather dangerous to draw the sort of conclusions concerning locality that are being drawn from the experimental results of the photon cascade experiments.

So far as experiments using massive particles are concerned, there are unfortunately few of these and the ones that have been carried out are not able to help us in the determination of the truth of our hypothesis concerning the separability of systems when the particles are separating. Lamehi-Rachti and Mittig, who performed such an experiment using protons [1976] remark

> "Our device does not fulfil the conditions of spacelike separation.. We assume that this does not affect the result of the measurement."

Obviously, from our point of view, it precisely this lack of spacelike separation which does affect the results of the experiment. Equally obviously, if we wish to test our theory we must perform experiments for which such separation is achieved (at least to a large extent).

Finally, we mention that apart from the extension of the present analysis to massless particles, there are various other avenues of exploration which would seem to lead directly from the present study. For instance, it has been shown by Aerts [1984] that (conventional) quantum mechanics is logically incapable of describing separated systems. The logical axioms which are responsible for this failure are, according to Aerts, the weak modularity and the covering law on the lattice of propositions. It would be worth investigating the logical structure of the AS-theory in respect of these points. In particular we might examine the hypothesis that separation involves a sort of superselection principle in the logic which overcomes the inability of the conventional theory to describe separated systems. In addition we remark that our theory which is asymptotic in nature and hence provides a complete resolution only in the (unattainable) limit of infinite time might be extended by considering the following variation. We define the sets Λ_1 and Λ_2 for the wavefunction Φ of a twoparticle system at time t by

(12.6)

$$\Lambda_1 = \bigcap \{ \Lambda \in \mathcal{B}(\mathbb{R}^n) : \| E(\underline{x}_1 \otimes \underline{x}_2 ; \Lambda_t \otimes \mathbb{R}^n) U_t \Phi \| = 1 \}$$

$$\Lambda_2 = \bigcap \{ \Lambda \in \mathcal{B}(\mathbb{R}^n) : \| E(\underline{x}_1 \otimes \underline{x}_2 ; \mathbb{R}^n \times \Lambda_t) U_t \Phi \| = 1 \}$$

and a "region of separation" for the two particles by

$\Lambda_s = \mathbb{R}^n \setminus \Lambda_c$, where $\Lambda_c = \Lambda_1 \cap \Lambda_2$. The analogy with (10.2.1), and (10.2.2) is obvious. Now, however, we are describing the localisation of the two particles in disjoint regions for finite times. If it were possible to construct some algebra, and set of states in analogy to chapter 10 for finite times, we believe that this theory would be an even better description of the way in which the spatial separation of particles determines the degree of separability of the systems.

In summary, we have made an extensive study of local observables in quantum mechanics paying close attention to the physical limitations of measurement. We have used the concepts involved in localisation of observables to provide a sort of resolution to the EPR paradox for finite times and in addition, we have attempted to provide a quantum mechanical theory which accounts for the separation of systems and allows for the separability of such systems in the event that they separate completely. It seems that a consequence such a theory is that we may return a measure of reality, we might say local reality, to the quantum mechanical wavefunction.

APPENDICES

A.1 Maximisation of $w(\hat{p}, \Delta, \varphi, \Lambda_0)$

Let $\varphi \in H, \varphi(x)=0, x \notin \Lambda_0$. The state $E(\hat{p};\Delta)\varphi$ is given by (Byron and Fuller 1969):

$$(E(\hat{p};\Delta)\varphi)(x) = (2\pi\hbar)^{-\frac{1}{2}} \int_{\Delta} \tilde{\varphi}(\lambda) \exp[i\lambda x/\hbar] d\lambda,$$

where $\tilde{\varphi}(\lambda)=\langle f|\varphi\rangle$ is the Fourier transform of φ. It follows that:

$$\int_{\Lambda_0} |(E(\hat{p};\Delta)\varphi)(x)|^2 \, dx = \iint_{\Delta\,\Delta} K_{\Lambda_0}(\lambda-\lambda')\tilde{\varphi}(\lambda)\tilde{\varphi}^*(\lambda') d\lambda \, d\lambda',$$

where

$$K_{\Lambda_0}(\lambda-\lambda') = (2\pi h)^{-1} \int_{\Lambda_0} \exp[i(\lambda-\lambda')x/\hbar] dx.$$

And since (as in the proof of Theorem (4.4.4))

$$\|E(\hat{p};\Delta)\varphi\|^2 = \int_{\Delta} |\tilde{\varphi}(\lambda)|^2 d\lambda,$$

we deduce that the maximum value w_{max} of w is given by the maximum value of

$$\iint_{\Delta\,\Delta} K_{\Lambda_0}(\lambda-\lambda')\tilde{\varphi}(\lambda)\tilde{\varphi}^*(\lambda') d\lambda \, d\lambda' \Big/ \int_{\Delta} |\tilde{\varphi}(\lambda)|^2 d\lambda.$$

The maximum value of this fraction is given [Landau and Pollak 1961 & 1962] by the maximum eigenvalue μ_0, say, of the integral equation

(A.1.1)

$$\mu \tilde{\eta}(\lambda) = \int_{\Delta} K_{\Lambda_0}(\lambda-\lambda')\tilde{\eta}(\lambda')d\lambda', \quad \lambda \in \Delta .$$

The corresponding eigenfunction $\tilde{\eta}_0(\lambda)$, $\lambda \in \Delta$, provides a restriction on the initial wave function φ_{max}. Explicitly φ_{max} must satisfy $\tilde{\Phi}_{max}(\lambda) = \tilde{\eta}_0(\lambda)$ for $\lambda \in \Delta$. When this is satisfied we have $w = w_{max} = \mu_0$.

In order to specify φ_{max} completely, we recall first that φ_{max} must vanish outside Λ_0 and, therefore, that $\tilde{\Phi}_{max}(\lambda)$ must be nonzero almost everywhere. We can in fact use (A.1.1) to define $\tilde{\eta}(\lambda)$ everywhere by

(A.1.2)

$$\tilde{\eta}(\lambda) = 1/\mu \int_{\Delta} K_{\Lambda_0}(\lambda-\lambda')\tilde{\eta}(\lambda')d\lambda', \quad \lambda \in \mathbb{R} .$$

In particular we can so extend $\tilde{\eta}_0(\lambda)$ and we claim that $\tilde{\eta}_0(\lambda)$, $\lambda \in \mathbb{R}$, provides the appropriate initial wave function $\tilde{\Phi}_{max}(\lambda)$ to maximise w. To show this, we need only demonstrate that any solution $\tilde{\eta}(\lambda)$, $\lambda \in \mathbb{R}$, of (A.1.2) (and hence $\tilde{\eta}_0(\lambda)$ in particular) is position limited in the range Λ_0.

Let $\eta(x)$ be the inverse Fourier transform of $\tilde{\eta}(\lambda)$. Denote by χ_A the characteristic function of the set A and let $\tilde{\psi}(\lambda) = \chi_\Delta(\lambda)\tilde{\eta}(\lambda)$. Let $\psi(x)$ be the inverse Fourier transform of $\tilde{\psi}(\lambda)$. Now (A.1.2) is equivalent to

$$\mu\,\eta(\lambda) = \tilde{\psi}(\lambda) * K_{\Lambda_0}(\lambda),$$

where * denotes convolution. Then by the convolution Theorem [Papoulis,eg] we have

(A.1.3)

$$\mu\,\eta(x) = \chi_{\Lambda_0}(x)\psi(x).$$

Thus we have shown that solutions of (A.1.2) are position limited in Λ_0. In particular, for η_0 we have $\eta_0(x)=0$, $x \notin \Lambda_0$, and hence we can take as maximising initial wave function $\varphi_{max}(x) = \eta_0(x)$.

We want to find out now what the function $\eta_0(x)$ is like and the dependence of the eigenvalue μ_0 on Δ and Λ_0. We cast the problem into an already investigated form (ops.cit) by looking at the function $\psi_0(x)$ from which $\eta_0(x)$ is obtained by truncation in Λ_0 and, observing that, since $\tilde{\psi}(\lambda) = \chi_\Delta(\lambda)\tilde{\eta}(\lambda)$, we have, by the convolution theorem again,

$$\psi(x) = K_\Delta(x) * \eta(x) = \int_{\mathbb{R}} K_\Delta(x-x')\eta(x')dx'.$$

Now we use (A.1.3) to deduce that

(A.1.4)

$$\mu\psi(x) = \int_{\Lambda_0} K_\Delta(x-x')\psi(x')dx',$$

which is essentially the same equation as (A.1.2) with the same eigenvalues. In particular, the maximum eigenvalue is μ_0. Now

$K_\Delta(x-x') = \exp[i\Omega_0(x-x')/\hbar]\frac{\sin((x-x')\Omega/\hbar)}{\pi(x-x')}$ so that (A.1.4) is equivalent to

(A.1.5)

$$\mu\Psi(x) = \int_{\Lambda_0} \frac{\sin((x-x')\Omega/\hbar)}{\pi(x-x')}\Psi(x')dx',$$

where $\Psi(x) = \exp[-i\Omega_0 x/\hbar]\psi(x)$. The solutions to (A.1.5) are known to be the prolate spheroidal wave functions. The maximum eigenvalue μ_0 and the corresponding eigenfunction $\Psi_0(x)$ are both dependent on the product $\Omega T/\hbar$ (ops.cit).

Since $\psi_0(x)$ satisfies (A.1.4) with $\mu=\mu_0$, we have

$$\psi_0(x) = \exp[i\Omega_0 x/\hbar]\Psi_0(x),$$

where Ψ_0 is the appropriate prolate spheroidal wave function. And it is now clear from (A.1.3) that the initial wave function required to attain $w = w_{max} = \mu_0$ is given by

$$\Phi_{max} = \eta_0(x) = \mu^{-1}E(x;\Lambda_0)\exp[i\Omega_0 x/\hbar]\Psi_0(x).$$

A.2 A Hidden Variable Theory for a Spin-half Particle

This model is an elaboration on the one proposed by Bell in his famous paper [1964] on local hidden variables and the EPR experiment. We observe first that the observables of a spin-half system are generated by the four 2×2 matrices

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

A general observable S for the system may be written in the form

$$S = \alpha 1 + \underline{\sigma} . \underline{a}$$

where $\alpha \in \mathbb{R}$ and $\underline{a}$ is a real vector ($\underline{\sigma}.\underline{a}$ represents the spin observable in the direction $\underline{a}$). Pure states of the system may be represented [cf Fano 1983, eg] either by a two-component spinor

$$\gamma(\theta,\varphi) = \begin{pmatrix} e^{-i\varphi/2} \cos\theta/2 \\ e^{i\varphi/2} \sin\theta/2 \end{pmatrix}$$

or alternatively by a unit vector $\underline{P}$ of direction (θ,φ) known as the polarisation vector. The expectation value of any observable S in state γ is given by

(A.2.1) $$\langle S;\gamma \rangle = \langle \gamma | S \gamma \rangle$$
$$= \langle \gamma | \alpha 1 + \underline{\sigma} . \underline{a} | \gamma \rangle$$

$$= \alpha + \underline{a}.\underline{P}$$

Now we can construct a hidden variable theory by taking the state space Γ to be the set of all unit vectors $\underline{\lambda}$ in $\mathbb{R}^3$ with $\underline{\lambda}.\underline{P} > 0$ and specifying the result of measurement of an observable S of the form (A.2.1) to be

(A.2.2)

$$S(\underline{\lambda}) = \alpha + a \operatorname{sign}[\underline{\lambda}.\hat{\underline{a}}']$$

where a is the magnitude of $\underline{a}$, ie $\underline{a} = a\hat{\underline{a}}$, and where $\hat{\underline{a}}'$ is a unit vector obtained by rotating $\hat{\underline{a}}$ towards $\underline{P}$ until the angle η' between $\hat{\underline{a}}'$ and $\underline{P}$ and the angle η between $\underline{a}$ and $\underline{P}$ satisfy

(A.2.3)

$$1 - 2\eta'/\pi = \cos\eta .$$

This procedure amounts to the specification of a function $f_S : \Gamma \to \mathbb{R}$ [cf (11.2.2)] by

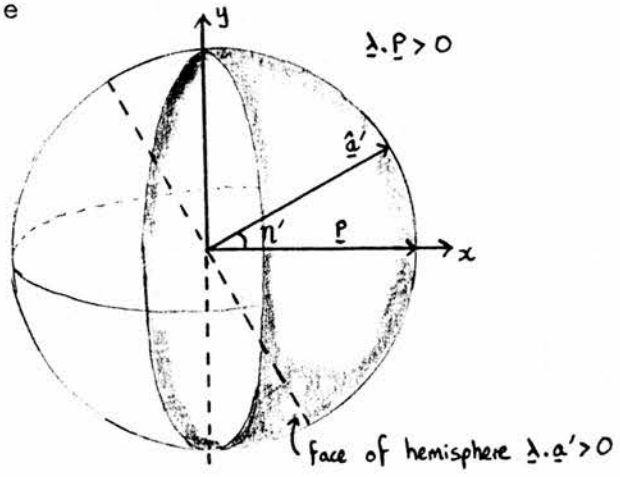
(A.2.4)

$$f_S(\underline{\lambda}) = \alpha + a \operatorname{sign} \underline{\lambda}.\hat{\underline{a}}' .$$

Evidently f_S is single-valued and therefore fulfils the condition (ii) of Definition (11.2.2). For the probability measure μ_γ or $\mu_{\underline{P}}$ on Γ we take a uniform distribution over all vectors $\underline{\lambda}$ in the hemisphere $\underline{\lambda}.\underline{P} > 0$. We average uniformly over all these hidden states to get the hidden variable expectation values $\langle S; \mu_{\underline{P}} \rangle_{HV}$ corresponding to a statistical state γ or $\underline{P}$. If we choose our axes so that the x-axis coincides with $\underline{P}$ we see [Fig (A.2.5)] that the

hemisphere $\underline{\lambda}.\underline{P} > 0$ is the right hand side of the sphere depicted in the diagram.

(A.2.5) Figure



Now let θ' be the angular variable in the zy-plane, and let φ' be the angular variable in the xy-plane. We shall choose our y-axis as in the figure (A.2.5) so that $\hat{\underline{a}}'$ is in the xy-plane. Then η' becomes the φ'-coordinate of $\hat{\underline{a}}'$. Also, the hemisphere $\underline{\lambda}.\underline{P}$ is then characterised by the ranges $0 \leq \theta \leq \pi$, $-\pi/2 \leq \varphi \leq \pi/2$, so that the uniform distribution $\mu_{\underline{P}}$ over the given hemisphere $\underline{\lambda}.\underline{P} > 0$ is given by the constant probability density k where

$$1 = \int_0^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} k \sin\theta' \, d\varphi' \, d\theta'$$

$$= 2\pi k .$$

Hence if we take $k = 1/2\pi$ the measure is appropriately normalised and for each measurable subset $\Lambda(\theta,\varphi)$ in the hemisphere we have:

$$\mu_{\underline{P}}(\Lambda) = (1/2\pi)\iint_{\Lambda} \sin\theta \, d\theta \, d\varphi .$$

Thus $\mu_{\underline{\rho}}$ provides the normalised measure required by (i) of Definition (11.2.2). It remains to show that the expectation value $\langle S;\mu_{\underline{\rho}}\rangle$ given by the hidden variable theory is equal to the expectation value $\langle S;\gamma\rangle$ of the quantum mechanical theory. To this end we observe that, since $\hat{\underline{a}}'$ is in the same plane as φ', we have $\underline{\lambda}.\hat{\underline{a}}' > 0$ for $\varphi' \in (-(\pi/2)+\eta', \pi/2]$ while $\underline{\lambda}.\underline{a}' < 0$ for $\varphi' \in [-\pi/2, -(\pi/2)+\eta)$. It follows that

(A.2.6)

$$\langle S;\mu_{\underline{\rho}}\rangle = (1/2\pi)\int_0^{\pi}\int_{-\pi/2}^{\pi/2} S(\underline{\lambda})\,\sin\theta'\,d\varphi'\,d\theta'$$

$$= (1/2\pi)\int_0^{\pi}\int_{-\pi/2}^{\pi/2}(\alpha+\text{asign}\,\underline{\lambda}.\hat{\underline{a}}')\sin\theta'\,d\theta'\,d\varphi$$

$$= 1/2\int_0^{\pi}\sin\theta'\,d\theta'\,\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\text{asign}\,\underline{\lambda}.\hat{\underline{a}}'\,d\varphi' + \alpha$$

$$= 1/\pi\int_{-\frac{\pi}{2}+\eta'}^{\pi/2} a\,d\varphi' - 1/\pi\int_{-\pi/2}^{-\frac{\pi}{2}+\eta'} a\,d\varphi' + \alpha$$

$$= \alpha + (1-2\eta'/\pi)a$$

$$= \alpha + a\cos\eta$$

$$= \alpha + \underline{a}.\underline{P} = \langle S;\gamma\rangle.$$

A.3 Another Asymptotically Separable Theory.

In a recent paper [Wan and Jackson 1985] we proposed an asymptotically separable theory for two spin half particles based on the construction of a direct sum algebra for the description of two isolated systems. Recall from (2.2) that the direct sum of two C^*-algebras is a well-defined C^*-algebra. Our description of two isolated systems formally as a composite system, to be called a composite description of isolated systems, consists of (i) the association of the C^*-algebra $\mathcal{A}_1 \oplus \mathcal{A}_2$ with the composite system, where $\mathcal{A}_1$ and $\mathcal{A}_2$ are the respective algebras for systems I and II; (ii) the identification of the set $\{w_1 \oplus w_2, 0 \oplus w_2, w_1 \oplus 0 : w_1, w_2$ are normal NPLF's on $\mathcal{A}_1, \mathcal{A}_2$ respectively$\}$ of positive linear functionals on $\mathcal{A}_1 \oplus \mathcal{A}_2$ as the set of states of the composite system. Clearly the composite description is simply as a pair of totally independent one-particle descriptions which automatically excludes all observables capable of correlating the two systems. When dealing with the first particle we can ignore the second particle entirely by considering observables $A_1 \oplus 0$ and states $w_1 \oplus 0$. Such states are normalised to one while the two-particle state $w_1 \oplus w_2$ is normalised to two. The two particle states therefore find a natural interpretation as a sum of probability measures on the

lattice of projections of the respective algebras, rather than as a probability measure on the direct sum lattice associated with the two-particle algebra. For the particular case of two spin half particles let us spell out the postulates on observables and states.

(A.3.1) Postulate 1

(i) A composite system of two free, nonidentical spin-half particles in configuration space $\mathbb{R}^{\wedge}$ has associated with it the C^*-algebra

$$\mathcal{A} = \mathcal{A}_{oc}^{s\sigma} \oplus \mathcal{A}_1^{\infty\sigma} \oplus \mathcal{A}_2^{\infty\sigma}$$

where $\mathcal{A}_{oc}^{s\sigma}$ is the finite spin algebra [cf (9.5.8)], $\mathcal{A}_1^{\infty\sigma}$ is $L^{\infty}(\underline{p}_1)\dot{\otimes}\mathcal{S} \otimes 1$ and $\mathcal{A}_2^{\infty\sigma}$ is $1 \otimes L^{\infty}(\underline{p}_2)\dot{\otimes}\mathcal{S}$.

(ii) The time evolution of the system is described by a one-parameter group $\{\alpha_t : t \in \mathbb{R}\}$ of automorphisms of $\mathcal{A}_1 \oplus \mathcal{A}_2$ defined by

$$\alpha_t(A_o \oplus F_1 \oplus F_2) = U_t^{-1} A_o U_t \oplus F_1 \oplus F_2 .$$

(iii) Selfadjoint members of $\mathcal{A}_{oc}^{s\sigma}$ correspond to bounded observables of the system at finite times and selfadjoint members of $\mathcal{A}_1^{\infty\sigma} \oplus \mathcal{A}_2^{\infty\sigma}$ correspond to bounded observables pertaining to the system at infinity.

(A.3.2) Postulate 2

Any state of the composite system is represented by

$w_0 \oplus 0 \oplus 0$ or $0 \oplus w_1^\infty \oplus w_2^\infty$ or $w_0 \oplus w_1^\infty \oplus w_2^\infty$, where w_0 are NPLF's on $\mathcal{A}_\alpha^{s\sigma}$, w_1^∞ are NPLF's on $\mathcal{A}_1^{\infty\sigma}$ and w_2^∞ are NPLF's on $\mathcal{A}_2^{\infty\sigma}$.

Acting on observables, $w_0 \oplus 0 \oplus 0$ has a nonvanishing expectation only with respect observables in the finite spin algebra $\mathcal{A}_{0c}^{s\sigma}$, while $w^\infty = w_1^\infty \oplus w_2^\infty$ has a nonzero expectation value only with respect to an observable at infinity. At infinity there are basically only one-particle observables, ie those in $\mathcal{A}_1^{\infty\sigma}$, $\mathcal{A}_2^{\infty\sigma}$ and their direct sums for which no correlations between the systems exist. Further details of all of this and some convergence results concerning states at infinity in the theory may be found in Wan and Jackson [1985].

A.4 Bibliography of References on Hidden Variables and EPR

This bibliography gives a brief resume of a number of important papers concerning the hidden variables question and the EPR experiment. It is in chronological order and the exact reference for each item is to be found in the alphabetically ordered reference section below.

1964

J S Bell : "On the EPR Paradox"

> Bell's original paper in which he derives the inequality. Proves that with "the vital assumption..that the result B for particle 2 does not depend on the setting a of the magnet for particle 1, nor A on b", Bell's inequality is satisfied. He then shows that quantum mechanics violates this inequality. Also of interest in the paper is an illustration of a hidden variable theory for one-particle

1966

J S Bell : "On the Problem of Hidden Variables in Quantum Mechanics"

Actually written prior to Bell [1964] this paper looks at the impossibility proofs of von Neumann, Jauch and Piron, and Gleason for hidden variable theories and argues that the validity of these is limited by unreasonable assumptions concerning the additivity of expectation values. An explicit hidden variable model is proposed for a one-particle system. The model does not retain the additivity assumption.

1967

S Kochen and E P Specker :

Impossibility proof for hidden variables based on showing that one cannot represent the statistics of quantum mechanics by measures on a classical probability space if the random variables representing the magnitudes are required to preserve the algebraic structure of the magnitudes. This includes the controversial assumption concerning additivity of the magnitudes challenged by Bell [1966].

1969

J F Clauser, M A Horne, A Shimony, and R A Holt : "Proposed Experiment to Test Local Hidden Variable Theories"

Proposes an experimental test for the Bell inequality
$|P(a,b) - P(a,b')| + |P(a',b) + P(a'b')| \leq 2$,
using the polarisation correlation of a pair of optical photons.

J Bub : "What is a Hidden Variable Theory of Quantum Phenomena?"

Resolves the conflict between the impossibility proofs of von Neumann [1955], Kochen and Specher [1967] and Jauch and Piron [1963] and the hidden variables theory of Bohm [1952(b)] by giving an explicit definition of a hidden variable theory. It is shown that the impossibility proofs have additional assumptions.

1971

H P Stapp : "S-Matrix Interpretation of Quantum Mechanics"

Of interest to the present discussion is the thesis that the Bell argument is essentially between locality and a lawful description of the world.

1972

S J Freedman and J F Clauser : "Experimental Test of Local Hidden Variable Theories"

Experimental tests of CHSH's generalisation of Bell's inequality using photon correlation give results in agreement with quantum mechanics and in violation of the inequality.

F Selleri : "A Stronger Form of the Bell Inequality"

A proof of the inequality (11.6.1).

1973

V Capasso, D Fortunato, and F Selleri : " Sensitive Observables of Quantum Mechanics"

Defines state vectors of the first and second type, mixtures if the first and second type, and sensitive and indifferent observables. Proves that for a mixture of the first type

Bell's inequality is always satisfied, but for a mixture of the second type we can find parameters such that Bell's inequality is violated.

1974

A Baracca, S Bergia, R Bigoni and A Cecchini: "Statistics of Observations for "Proper" and "Improper" Mixtures in Quantum Mechanics"

"Roughly speaking it turns out that, with reference to certain physical situations QM implies the existence of "improper mixtures" which are .. incompatible with a description in terms of hidden variables." An investigation of von Neumann's results and Furry's analysis. Specifically it is proven that for every Φ there are observables which are correlated by Φ.

J Clauser and M Horne: "Experimental Consequences of Objective Local Theories"

Extends the discussion of CHSH to probabilistic local theories.

1975

A Baracca, D Bohm, B Hiley, and A Stuart : "On some New Notions Concerning Locality and Nonlocality in the Quantum Theory"

Proposes a new form for the dynamical laws which might induce the Furry hypothesis, namely that the wavefunction of a many body system factorises into a product of localised states at large distances. Also gives an explicit Bell separable average over the spin directions.

1976

A Aspect: "Proposed Experiment to Test the Nonseparability of Quantum Mechanics"

Proposes an experiment based on Einsteinian separability: the setting of a measuring device at a certain time (event A) does not influence the result obtained with another device (event B) if B is not in the forward light cone of A. Bell locality implies Einstein separability but not vice versa. The experiment proposed uses photon cascades.

D Fortunato and F Selleri: "Sensitive Observables on Infinite Dimensional Hilbert Spaces"

> Extends the previous analysis to infinite dimensional spaces.

F Herbut and M Vujicic: "Distant Measurement"

> Expounds a " strictly quantum mechanical theory of distant correlations which completely replaces [the two particle state vector] by [the reduced statistical operators] and a third operator U which expresses precisely the correlations inherent in [the two particle state vector]." In fact either of the reduced statistical states together with U is sufficient to completely determine the two particle state.

M Lamehi-Rachti and W Mittig: "Quantum Mechanics and Hidden Variables: A Test of Bell's Inequalities by the Measurement of Spin Correlations in Low Energy Proton-Proton Scattering."

One of the few correlation experiments carried out using massive particles and therefore of considerable importance to our analysis. The author's motivation for the use of protons as opposed to photons includes the remark that the coherence length of the photons used in cascade experiments were of the order of the dimensions of the apparatus and hence separation could not be guaranteed. However even this experiment does not fulfil the conditions for spacelike separation. The authors assume that "This does not affect the result of the measurement."

1977

N Cufaro Petroni: "On the Observable Difference Between Proper and Improper Mixtures."

Constructs a particular sensitive observable for each state vector Ψ of the second type. The observable is $\sum_n \lambda_n |\psi_n\rangle\langle\psi_n|$ where $\Psi = \psi_n$.

D Fortunato: "Observable Consequences from Second Type State Vectors of Quantum Mechanics."

Using the singlet state vector, the paper presents an observable for which the expectation is 3 according to quantum mechanics while it has a maximum value of 1 if only state vectors of the first type are considered.

A Garuccio, G Scalera and F Selleri: "On Local Causality and the Quantum Mechanical State Vector"

Shows that it is not possible to drop state vectors of the second type and that state vectors of the second type and local causality are quite different things by considering two inequalities satisfied (resp.)(i) by all locally causal theories as well as all vectors of the first type and (ii) by all state vectors of the first type but not all locally causal theories.

1978

J Clauser and A Shimony: "Bell's Theorem: Experimental Tests and Implications"

Extensive review article on the subject, from Bell's early work to the experimental tests and much historical background. Interpretation biassed in favour of the success of experiment in refuting objective local theories. NB Appendix on Furry's hypothesis.

A Garuccio: "Generalised Inequalities Following From Einstein Locality"

Deduces a set of inequalities of which Bell's is the strongest. Einstein locality is taken to be "the assumption that some variables exist which together with the variable instrumental parameter a determine.. the results of measurement of some observable A."

A Garuccio and F Selleri: "On the Equivalence of Deterministic and Probabilistic Local Theories"

The deterministic approach has $P(a,b) = \int \mu(\lambda) A(a,\lambda) B(b,\lambda) d\lambda$ while the the probabilistic one has $P(a,b) = \int \mu(\lambda) p(a,\lambda) p(b,\lambda) d\lambda$. The paper shows that the class of DLT's is equivalent to the class of PLT's as far as inequalities for linear

combinations of correlation functions goes.

F Selleri: "On the Consequences of Einstein Locality"

More inequalities following from Einstein locality, namely $A=A(a,\lambda)$ or "Measurements on distant correlated systems by apparatus with variable parameters cannot influence the result of the measurement of A." Both DLT's and PLT's are considered. Consequences at small angles are investigated.

1980

A Garuccio and F Selleri: "Systematic Derivation of all the Inequalities of Einstein Locality"

Deduces inequalities of the type $\sum_{ij} c_{ij} P(a_i, b_j) \leq M$ from Einstein locality for probabilistic models.

F Selleri and G Tarozzi: "Is Clauser and Horne's Factorability a Necessary Requirement for a Probabilistic Local Theory ?"

An explicit example is constructed of a PLT satisfying Bell's inequality for which the factorability hypothesis is not valid. Therefore "all considerations developed from

the factorability hypothesis are generally valid only for DLT's." Rather surprising in view of Garuccio and Selleri [1978]. The following statement is of interest to our present analysis of hidden variables: "If one found that local hidden variable models of a probabilistic nature ..exist which do not satisfy Bell's inequality, the latter would lose the general empirical and epistemological significance which is commonly attributed to it and the door for a reconciliation between quantum mechanics and local causality would be open."

H Stapp: "Locality and Reality"

More on the view that Bell's theorem is concerned only with locality. "No process that selects observations that conform to the contingent predictions of quantum theory can be local." Stapp proposes a theory without hidden variables (and also without objective reality) for which locality and quantum mechanics are still incompatible.

1981

A Aspect, P Grangier, and G Roger: "Experimental Tests of Realistic Local Theories via Bell's Theorem."

F Selleri and G Tarozzi: "Quantum Mechanics, Reality and Separability"

Another comprehensive review article with more of the theoretical background and philosophical and epistemological implications than Clauser and Shimony [1978]. De Broglie's paradox and the EPR paradox are discussed.

1982

D Aerts: "Description of Many Separated Physical Entities without the Paradoxes Encountered in Quantum Mechanics"

The paper shows how from a logical point of view quantum mechanics cannot describe separated systems. Five axioms are given for a quantum logical system. The axioms of weak modularity and the covering law are found to be responsible for the inability to describe separate systems. Of interest with reference to our description of separated systems.

A Aspect, P Grangier and G Roger: "Experimental Realisation of the EPR Gedankenexperiment: A New Violation of Bell's Inequalities"
A Aspect, J Dalibard and G Roger: "Experimental Tests of Bell's Inequalities Using Time-Varying Analysers"

Further experiments claiming to support quantum mechanics over Bell inequalities. All three Aspect experiments use photon cascades.

F Selleri: "Generalised EPR Paradox"

Expounds on the difficulties of existing theories concerning deterministic criterion or probabilistic factorability and presents a generalised theory starting from a generalised reality criterion. Of particular interest to us is the observation that the factorability criterion is not in general satisfied by local probabilistic models and the example is given where λ is a set of variables λ' and λ''.

1983

T Angelidis: "Bell's Theorem: Does the CH-inequality hold for all Local Theories ?"

Argues that the universality claimed for the CH inequality does not hold and hence quantum mechanics is not necessarily nonlocal. The argument involves showing that the universality claim is incompatible with the conservation of angular momentum.

D Dieks: "Stochastic Locality and Conservation Laws"

A proof that stochastic hidden variable theories which obey Bell's inequalities do not admit the usual conservation laws. Hence physically interesting stochastic theories must violate Bell inequalities.

D Liddy: "On Locality, Correlation and Hidden Variables"

Since the factorability criterion of CH is not necessary for PLT's, what is ? Liddy concludes that the theory is already local in underlying structure and produces a hidden variables theory which is also therefore

local.

T Marshall: "The Distance Separating Quantum Theory from Reality"
T Marshall, E Santos and F Selleri: "Local Realism has not been Refuted by Atomic Cascade Experiments"

Arguing that the data from the Aspect experiments is inconclusive.

M Vujicic and F Herbut: "A Quantum Mechanical Theory of Distant Correlations"

further exploration of EPR type correlations in terms of the reduced statistical operators and the correlation operator [cf Herbut and Vujicic 1976].

1984
A Barut and P Meystre: "A Classical Model of the EPR experiment with Quantum Mechanical Correlations and Bell Inequalities"

A simple model of a classical breakup has correlation of spin components identical to the q.m. one. It is local but the normalisation procedure for correlation functions is different. Discretisation

reproduces q.m. fully.

D Home and S Sengupta: "Bell's Inequality and Non-contextual Dispersion-free States"

Argues that Bell's inequality is derivable from general consequences of noncontextual hidden variables and that gedankenexperiments can be formulated for which locality is not an issue but Bell's inequality is violated.

D Liddy: "An Objective, Local Hidden-Variables Theory of the Clauser-Horne Experiment"

Extends ideas in Liddy [1983] by proposing a modle for the photon experiments.

T Marshall: "Testing for Reality with Atomic Cascades"

More arguments on the inconclusiveness of the Aspect experiments.

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