ON THE PERFORMANCE OF SIMPLE SWITCHING ALGORITHMS FOR SHOCK RESOLUTION

by

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Abstract

In this thesis we introduce some Simple Switching finite-difference schemes for solving the one dimensional linear and non-linear hyperbolic partial differential equation. We also compare the numerical results to those produced by other schemes like the FTBS, Lax-Wendroff and finally the Total Variation Diminishing (TVD) method of Davis [2] which is a particular example of the TVD analysed by Sweby .

Acknowledgment

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Last but not least, I wish to thank my family back home for their constant help and encouragement, particularly my mother.

DECLARATION

I Shahla Suleiman hereby certify that this thesis has been composed by myself, that it is a record of my own work, and that it has not been accepted in partial or complete fulfilment of any other degree or professional qualification.

Signed : Date : 22.11.1990

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CHAPTER ONE

INTRODUCTION AND BASIC THEORY

1.1 The Linear Advection Equation:

Suppose that an elastic string with length L is stretched tightly and fixed at its both ends so that x-axis lies along the string. When the string vibrates, it moves in an infinite number of normal modes in a vertical plane. So let u(x,t) be the vertical displacement experienced by the string at the point (x,t), then a typical mode is shown by the following equation:

$$u_n(x,t) = A_n \sin\left(\frac{n\pi}{L}\right)x \;\; \cos \omega_n t \qquad , \text{ is integer, } t > 0 \quad (1.1)$$

where $\omega_n = \frac{n\pi}{2}\left(\frac{T}{m}\right)$,

(T is the tension of the string and m is the mass per unit length).

We define $\sqrt{\frac{T}{m}} = a$ as the wave speed, equation (1.1) becomes:

$$u_n(x,t) = \frac{1}{2} A_n \sin\left(\frac{n\pi}{L}(x-\sqrt{\frac{T}{m}}\,t)\right) + \frac{1}{2} A_n \sin\left(\frac{n\pi}{L}(x+\sqrt{\frac{T}{m}}\,t)\right) \qquad \text{...... (1.2)}$$

This equation is a description of two traveling waves in opposite directions. Now let us consider the first of the two terms on the right-hand side of the equation (1.2), which is the one moving in the positive direction (figure 1.1) as :

$$u(x,t) = A \sin(\frac{2\pi}{\lambda}(x-at)) \tag{1.3}$$

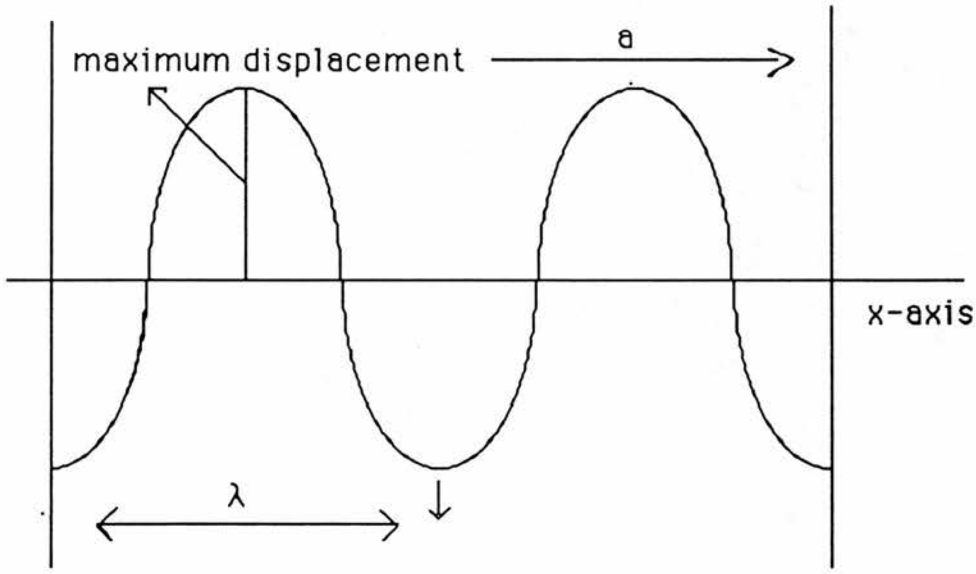
where $\lambda = \frac{2L}{n}$ the wave length. If we take the x and t partial derivatives for u :

$$\frac{\partial u}{\partial x} = \frac{2\pi}{\lambda} A \cos(\frac{2\pi}{\lambda}(x-at))$$

$$\frac{\partial u}{\partial t} = \frac{-2\pi a}{\lambda} A \cos(\frac{2\pi}{\lambda}(x-at))$$

from which we get

$$\frac{\partial u}{\partial t} = -a \frac{\partial u}{\partial x} . \tag{1.4}$$



The wave moves in positive x-axis

Figure 1.1

So equation (1.4) describes the motion of a wave in one direction with no change in the shape of the wave, with constant speed. This equation is called linear first-order wave equation or advection equation in one dimension. In general the independent variable x represents the space coordinate and t represents the time coordinate depending on physical situation, and the dependent variable u has many possible interpretations.

This type of equation is either an initial-value problem or an initial-boundary-value problem, but we will discuss the initial-value problem only here. In order to solve equation (1.4) we find curves in the xt-plane along which this equation reduces to an ordinary differential equation. So let us define the curve Ω by

$$x = x(t) \quad , \text{then } u = u(x(t),t) \quad \text{which is on } \Omega \, .$$

Now

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial t} \tag{1.5}$$

By comparing (1.5) with (1.4) we find that :

If

$$\frac{dx}{dt} = a \quad \text{on } \Omega \tag{1.5)a$$

then

$$\frac{du}{dt} = 0 \quad \text{on } \Omega \, . \tag{1.5)b$$

The solution $x = x(t)$ defines the characteristic curve. Solving the characteristic equation (1.5)a we get

$$x = at + x_0, \quad x_0 \text{ is a constant} \tag{1.6}$$

So equation (1.4) has a single family of characteristics (1.6), which are straight lines in xt-plane, and also equation (1.5)b on Ω implies that $u(x,t)$ is constant along the characteristic lines. When $t = 0$ the characteristic intersects x-axis at x_0 . If a is positive, a point on the characteristic moves a unit in the positive x-direction in unit time, and when a is negative it moves a unit in the negative x-direction (Figure 1.2).

If we now consider equation (1.4) subject to the initial-value condition

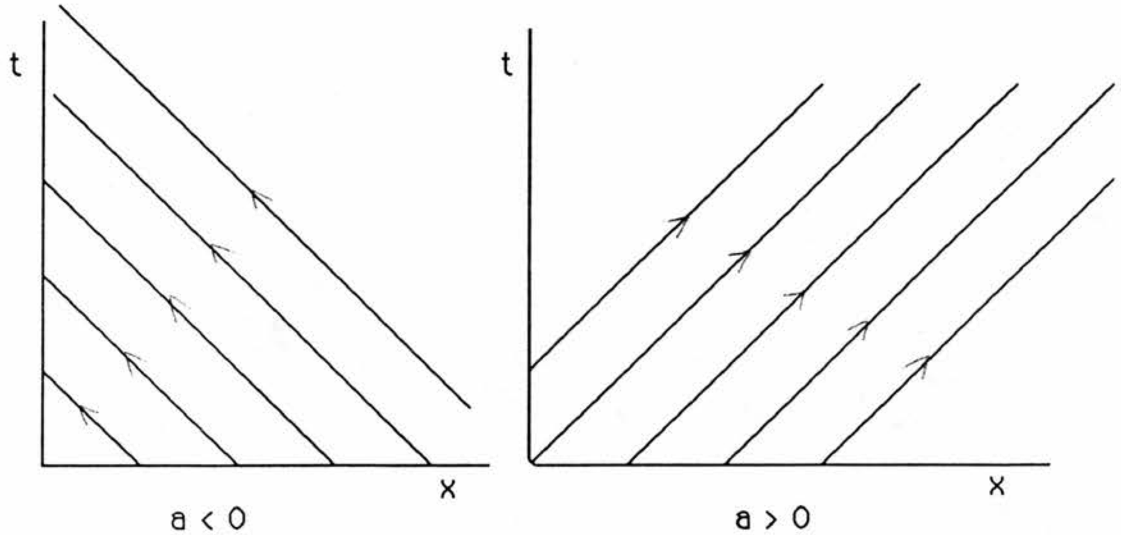
$$u(x,0) = f(x) \tag{1.7}$$

then the exact solution of this IVP is

$$u(x,t) = f(x-at) \tag{1.8}$$

The domain of dependence of the equation (1.4) at (x_0,t_0) is the characteristic line which passes through that point.

The linear advection equation (1.4) is a very simple equation. However, it is ideal for numerical investigations because finite-difference approximations of (1.4) have many of the properties of more complicated problems. A review of the more important properties is presented in sections 1.3 and 1.4 .



The charactaristic lines

Figure 1.2

1.2 Explicit Finite-Difference Equation :

The finite-difference methods for the advection equation have certain advantages, for example they are easy to apply, and they generate numerical approximations to the solution of the equation on a rectangular mesh in the xt-plane.

To show how we obtain some of these methods, consider the simple equation

$$u_t + au_x = 0 \qquad , (a > 0) \qquad (1.9)$$

with the initial condition

$$u(x,0) = f(x) \qquad (1.10)$$

and consider the half plane Ψ as :

$$\Psi = \{(x,t) : -\infty < x < +\infty , t \geq 0\}$$

covered by a grid which has h as the grid spacing in the x-direction and k in the t-direction defined by :

$$x_q = qh \qquad , q = 0, \pm 1, \pm 2, \pm 3, \ldots$$

$$t_p = pk \qquad , p = 0,1, 2, 3, \ldots$$

We define the value of u at the mesh point Q(qh,pk) by

$$u_Q = u(qh,pk) = u_q^p \qquad \text{(see Figure 1.3)}$$

By using a Taylor expansion for u_q^{p+1} :

$$u_q^{p+1} = u + ku_t + \frac{k^2}{2!} u_{tt} + \frac{k^3}{3!} u_{ttt} + \ldots..$$

we obtain the first-order forward difference approximation for $\frac{\partial u}{\partial t}$ at Q as the following :

$$\frac{\partial u}{\partial t} \approx \frac{u_q^{p+1} - u_q^p}{k} + TE$$

and by using the Taylor expansion for u_{q+1}^p , we obtain the forward difference approximation for $\frac{\partial u}{\partial x}$ at Q as :

$$\frac{\partial u}{\partial x} \approx \frac{u_{q+1}^p - u_q^p}{h} + TE$$

and in this way a variety of different approximations can be constructed. Some of the more obvious ones are now considered.

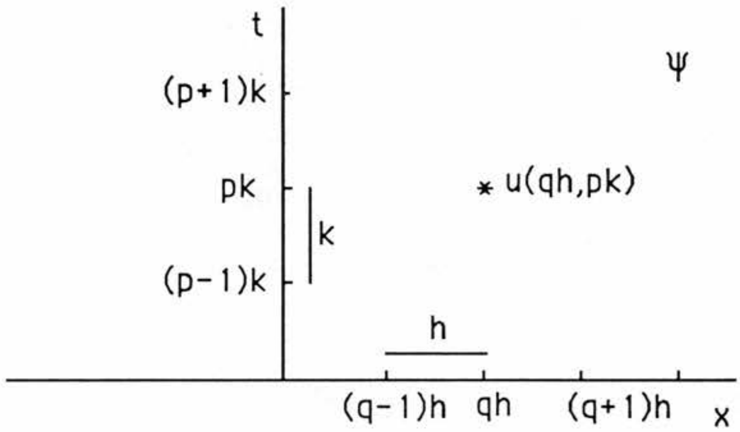


Figure 1.3

Forward in Time-Backward in Space Method (FTBS) :

To obtain the FTBS method we substitute u_x in equation (1.9) by backward difference approximation, and u_t by forward difference approximation.This produces an explicit difference approximation

$$\frac{u_{q+1}^{p} - u_{q}^{p}}{k} + a\frac{u_{q}^{p} - u_{q-1}^{p}}{h} = 0 \tag{1.11}$$

where $\mu = \frac{\delta t}{\delta x} = \frac{k}{h}$. Equation (1.11) can be written as

$$u_q^{p+1} = (1 - a\mu)\, u_q^p + a\mu\, u_{q-1}^p \qquad (1.12)$$

This equation, with first-order accuracy, is called Forward in Time Backward in Space or an Upstream difference method. The truncation error is

$$TE = \frac{-ah}{2}\,(1 - a\mu)\,\frac{\partial^2 u}{\partial x^2} + \frac{ah^2}{6}\,(1 - a^2\mu^2)\,\frac{\partial^3 u}{\partial x^3} + \ldots$$

$$= O(h)$$

Forward in Time-Centered in Space Method (FTCS) :

The FTCS approximation of the equation (1.9) is

$$a\,\frac{u_{q+1}^p - u_{q-1}^p}{2h} + \frac{u_q^{p+1} - u_q^p}{k} = 0$$

or

$$u_q^{p+1} = u_q^p - \frac{a\mu}{2}\,(u_{q+1}^p - u_{q-1}^p) \qquad (1.13)$$

with truncation error :

$$\frac{a^2\mu h}{2}\,\frac{\partial^2 u}{\partial x^2} + O(h^2)$$

or $O(k + h^2)$.

Lax-Friedrichs method :

We use the FTCS approximation, and replace u_q^p in the time derivative by the average of u_{q-1}^p and u_{q+1}^p then we get the equation

$$u_q^{p+1} = \frac{1}{2}\left(u_{q+1}^p + u_{q-1}^p\right) - \frac{1}{2} a\mu\left(u_{q+1}^p - u_{q-1}^p\right) \qquad (1.14)$$

which is called the Lax-Friedrichs approximation with truncation error:

$$\left(\frac{a^2\mu h}{2} - \frac{h}{2\mu}\right)\frac{\partial^2 u}{\partial x^2} + \left(\frac{h^2 a}{6} - \frac{a^3 h^2 \mu^2}{6}\right)\frac{\partial^3 u}{\partial x^3} + \dots$$

or $O(h)$.

Lax-Wendroff method:

By following the same procedure which we used to obtain FTBS, but replacing the x-derivatives by centered-difference approximation.The explicit difference formula of Lax-Wendroff method can be obtained, which may be stated in following form

$$u_q^{p+1} = \frac{1}{2} a\mu(1+a\mu)\, u_{q-1}^p + (1-a^2\mu^2)\, u_q^p - \frac{1}{2} a\mu(1-a\mu)\, u_{q+1}^p \qquad (1.15)$$

with truncation error :

$$\frac{1}{6} ah^2(1- a^2\mu^2)\frac{\partial^3 u}{\partial x^3} + \dots$$

or $O(h^2)$.

This scheme is second order, and is more accurate than the methods we mentioned earlier.

The above schemes (except for FTCS) generate the exact solution for the partial differential equation when $a\mu = 1$.

1.3 <u>(1) The Von Neumann Stability Condition :</u>

To examine the stability of the finite-difference approximation to the partial differential equation :

$$u_t + au_x = 0 \qquad , (a > 0) \tag{1.16}$$

we first use the complex form of the Fourier series of u to express the solution of the finite-difference equation as :

$$u_q^p = \alpha^p \, e^{i\beta q}$$

and substitute this in the finite-difference approximation to obtain α, then the Von Neumann condition of stability for the equation (1.16) is :

$$|\alpha| \le 1 \qquad , \text{for all values of } \beta$$

α is called the Amplification Factor, and it depends on β and h, it is also a complex number. The Von Neumann condition is sufficient as well as necessary for stability.

(2) The Courant-Friedrichs-Lewy (CFL) Condition of Stability :

This condition is necessary for the stability of the explicit finite-difference methods, but not sufficient and it states that the domain of dependence of the finite-difference equation includes the domain of dependence of the partial differential equation at all points (x,t), that is

$$\left|a\mu\right| \leq 1 \quad , \text{ where } \mu = k/h, \text{ a is constant}$$

Let us examine the stability of the following :

(I) FTBS scheme :

The Upstream difference has the form

$$u_q^{p+1} = (1-a\mu)\, u_q^p + a\mu\, u_{q-1}^p$$

By seeking the solution of the form

$$u_q^p = \alpha^p e^{i\beta q}$$

and apply this in the equation (1.17), we find that the amplification factor

$$\alpha = (1-a\mu) + a\mu\,(\cos\beta h - i\sin\beta h)$$

and the modulus of α is

$$\left|\alpha\right| = (1 - 4a\mu(1-a\mu)\,\sin^2(\beta h))^{\frac{1}{2}}$$

It is obvious that $\left|\alpha\right| \leq 1$ if and only if $0 \leq a\mu \leq 1$, which means that FTBS method satisfies the Von Neumann stability condition and it is locally stable only if $0 \leq a\mu \leq 1$,which is identical to the CFL condition of stability.

(**II**) FTCS scheme :

This finite-difference method has the form

$$u_q^{p+1} = u_q^p - \frac{a\mu}{2}\,(u_{q+1}^p - u_{q-1}^p) \qquad (1.18)$$

It is easy to find, by applying the same procedure of the Von Neumann analysis of the stability, that:

$$\alpha = (1 - ia\mu\,\sin(\beta h))$$

and

$$\left|\alpha\right| = (1+(a\mu\,\sin(\beta h))^2)^{\frac{1}{2}}$$

We can see here that $\left|\alpha\right| \geq 1$ for all $\mu > 0$, which means that the numerical solution will eventually grow as μ increases. Even when $\mu \leq 1$ the approximation is still unstable, which means that the FTCS method does not satisfy the Von Neumann condition. This also shows that the CFL condition is not sufficient to prove that the finite-difference method is locally stable. So the FTCS is unconditionally unstable.

(III) Lax-Friedrichs method :

For the Lax-Friedrichs scheme of the form

$$u_q^{p+1} = \frac{1}{2}\left(u_{q+1}^p + u_{q-1}^p\right) - \frac{1}{2} a\mu\left(u_{q+1}^p - u_{q-1}^p\right) \qquad (1.19)$$

the amplification factor is

$$\alpha = \left(\frac{1}{2} - \frac{a\mu}{2}\right) e^{i\beta h} + \left(\frac{1}{2} - \frac{a\mu}{2}\right) e^{-i\beta h}$$

and the modulus of α is

$$\left|\alpha\right| = \left(1 - (1 - a^2\mu^2)\sin^2\beta h\right)^{\frac{1}{2}}$$

It is clear that $\left|\alpha\right| \le 1$ if and only if $(1- a^2\mu^2) \ge 0$, so the Lax-Friedrichs approximation satisfies the Von-Neumann condition, provided that $a\mu \le 1$. The CFL condition of stability is also $a\mu \le 1$.

(IV) Lax-Wendroff scheme :

This scheme is of the form

$$u_q^{p+1} = \frac{1}{2} a\mu(1+a\mu)\, u_{q-1}^p + (1-a^2\mu^2)\, u_q^p - \frac{1}{2} a\mu(1-a\mu)\, u_{q+1}^p \qquad (1.20)$$

has the amplification factor

$$\alpha = 1-a^2\mu^2 + \frac{1}{2} a\mu(1+a\mu)\, e^{-i\beta h} - \frac{1}{2} a\mu\,(1-a\mu)\, e^{i\beta h}$$

and

$$\left|\alpha\right| = (1 - 4a^2\mu^2(1 - a^2\mu^2)\, \sin^4\beta h)^{\frac{1}{2}}$$

From this it follows that $\left|\alpha\right| \leq 1$ if $-1 < a\mu \leq 1$, which means that for $a > 0$ the Lax-Wendroff method is locally stable provided the CFL condition $a\mu \leq 1$ is satisfied.

1.4 Dissipation and Dispersion :

To examine the dissipation and dispersion of the finite difference we will also use the Von Neumann analysis of stability. Let us consider again equation (1.16) with the initial value condition:

$$u(x,0) = f(x) \qquad , (-\infty \leq x \leq +\infty) \qquad (1.21)$$

then the exact solution of this problem is

$$u(x,t) = f(x-at) \quad .$$

Seek the solution of the form

$$u_q^p = \alpha^p \, e^{i\beta q}$$

and let the amplification factor

$$\alpha = \left|\alpha\right| \, e^{-\theta t}$$

then

$$u_q^{p+1} = \left|\alpha\right|^p \quad e^{i(\beta x - \theta t)}$$

where $\sigma = \theta/\beta$ is the speed of propagation. Apply this in the finite-difference approximation for the equation (1.16) to obtain α . Then the finite-difference method is called nondissipative if $\left|\alpha\right| = 1$ for all β . It is called dissipative if $\left|\alpha\right| < 1$ for at least one β, this means that the different Fourier modes decay with time, or in other words the numerical solution will eventually die away to zero. The difference method is dissipative of order 2b if there is a constant $c > 0$, and integer b such that $\left|\alpha\right| \le 1 - c(\beta h)^{2b}$. For example the FTBS method is dissipative of order 2, while the Lax-Wendroff method is dissipative of order 4.

To discuss the dispersion of the finite-difference methods, let

$$\tan \theta k = \frac{\text{Imaginary part of } \alpha}{\text{Real part of } \alpha} \quad \text{, then} \quad \frac{\theta}{\beta} = \frac{-1}{\beta k} \tan^{-1} \theta k$$

$\frac{\theta}{\beta}$ is called the discrete dispersion relation and it ought to be close to a. The expression $\frac{\theta}{a\beta}$ is called the dispersion ratio. The finite difference method is said to be dispersive if $\frac{\theta}{\beta} \neq a$. That means the different Fourier modes of different wave length or wave number move at different speeds. If $\frac{\theta}{\beta} < a$ then the numerical solution is moving slower than the exact solution, and if $\frac{\theta}{\beta} > a$ then it moves faster than the exact solution. The finite-difference method is called nondispersive if $\frac{\theta}{\beta} = a$ for all β.

Example :

In this example we are going to examine the dispersion and the dissipation of the first-order FTBS method and the second-order Lax-Wendroff method. We will consider the advection equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \qquad , a = 1$$

with the initial condition

$$u(x,0) = \begin{cases} \sin n\pi x & o \le x \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

We choose the interval $0 \le t \le 1$ and $\mu = k/h = 0.5$ so that $h = 0.1$ and $k = 0.05$.

For the Lax-Wendroff method (1.15)

$$\left|\alpha\right| = (1 - \frac{3}{4}\sin^4(\frac{\beta h}{2}))^{\frac{1}{2}}$$

which is less than 1, so the Lax-Wendroff method is dissipative, and

$$\tan \theta k = \frac{a\mu \sin\beta h}{1 - a^2\mu^2(1-\cos\beta h)}$$

$$= \frac{2\sin(\frac{\beta}{10})}{3+ \cos(\frac{\beta}{10})}$$

Now when the initial condition $u(x,0) = \sin n\pi x$ then $\beta = n\pi$. For different values of n we get the following values of the amplification factor and the dispersion ratio

n	$\lvert\alpha\rvert$	θ/β
1	0.9978	0.9878
5	0.9014	0.7487
9	0.5350	0.2072

For the FTBS method (1.12) we have

$$\lvert\alpha\rvert = (1 - 4a\mu(1-a\mu)\ \sin^2(\beta h))^{\frac{1}{2}}$$

and $\tan \theta k = \dfrac{a\mu\sin\beta h}{1-a\mu(1-\cos\beta h)}$

and also for different values of n we get

n	$\lvert\alpha\rvert$	θ/β
1	0.9877	1
5	0.7071	1
9	0.1564	1

Here are some numerical results of the solution

	Lax-Wendroff		FTBS	
x	n=1	n=5	n=1	n=5
0.9	-0.2712	0.0868	-0.2412	-0.0010
1.0	0.0381	-0.0905	0.0000	0.0000
1.1	0.3437	0.0868	0.2412	0.0010
1.2	0.6156	0.0905	0.4588	0.0000
1.3	0.8272	0.0905	0.6315	-0.0010
1.4	0.9579	-0.0905	0.7423	0.0000
1.5	0.9948	-0.8068	0.7805	0.0010

The solution is in the form

$$u_q^p = \left|\alpha\right|^p \sin(\beta x - \theta/\beta t)$$

For instance for the Lax-Wendroff method

$$u_q^p = (1-4a^2\mu^2\sin^4\frac{\pi h}{2}(1-a^2\mu^2))^{\frac{p}{2}}\sin(\pi(\beta x-\theta/\beta t))$$

At p = 20 the solution is

$$c \sin \pi(x-\theta/\beta)$$

If we take the values x = 1.5 and x = 1.0 at $\beta = \pi$, then

$$c \sin \pi(1.5 - \theta/\beta) = 0.9948 = c \cos (\theta/\beta)\pi$$

and

$$c \sin \pi(1 - \theta/\beta) = 0.0381 = c \sin(\theta/\beta)\pi$$

now

$$\tan\left(\frac{\theta}{\beta}\right) = \frac{0.0381}{0.9948} = 0.0383$$

$$\frac{\theta\pi}{\beta} = \pi - 0.0383$$

so $$\frac{\theta}{\beta} = 0.9878$$

which is the same as we computed earlier, and this is right also if we apply the same for the FTBS method with different values of β. From the calculations and the results above we conclude that for the Lax-Wendroff, $|\alpha|$ decreases as the number of waves increases, that is this method becomes more dissipative with large β when $\mu a = 0.5$, and from the values of $\frac{\theta}{\beta}$ above it is clear that the Lax-Wendroff is dispersive and the wave speed becomes slower with large value of β. The FTBS is also dissipative when $\mu a = 0.5$ and becomes more dissipative with bigger β, and we notice that the FTBS method is nondispersive, ie. $\frac{\theta}{\beta} = a$ for all β when $\mu a = 0.5$, which means that it moves at the same speed as the solution of the partial differential equation.

CHAPTER TWO

A Simple Shock Problem

2.1 Numerical Applications

In chapter one we described some finite-difference methods that can be used to approximate the solution of the advection equation. From the form of the truncation error of those methods we could conclude that the Lax-Wendroff scheme should be more accurate than the others.

In this chapter we are going to discuss some numerical applications of some of those schemes, the Lax-Wendroff and the FTBS methods in particular. So let us consider the following initial-value problem :

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \tag{2.1}$$

$$u(x,0) = \begin{cases} 1 & 0 \le x \le 0.5 \\ 0 & 0.5 < x \le 2.5 \end{cases} \tag{2.2}$$

and let $a = 1$. To avoid dealing with boundary conditions the number of time steps will be restricted so that fixed values at the boundary do not alter the nature of the solution. The interval $0 < x < 2.5$ is subdivided into 50 so that the x grid spacing is taken to be $h = 0.05$ and restrict t to the range $0 \leq t \leq 1.25$. The time grid spacing k is chosen so that $\mu = \frac{ak}{h} = 0.5$. The numerical results for the solution of the initial-value problem (2.1) and (2.2) after 20 time steps, at $t = 0.5$, and 40 time steps, at $t = 1.0$, for the FTBS and the Lax-Wendroff are compared with the exact solution and represented in table 2.1, and illustrated in figure 2.1.

It is obvious from the results that the FTBS scheme produces a smooth solution which decays as the calculations progress, this decay reflects the fact that $|\alpha| < 1$ for $0 \leq \beta \leq m\pi$ in a Von Neumann analysis. Although the Lax-Wendroff scheme produces a better solution than the FTBS scheme when they are compared with the exact solution, it produces pronounced oscillation near the discontinuity. The dissipation terms, or diffusion terms, usually introduce damping in the high frequency terms more strongly than in the low frequency terms, and also the high frequency components move more slowly than the low frequency waves. Since we know from the Fourier analysis that the solution is constructed of high frequency components, therefore when there is only a small amount of numerical dissipation, as in the Lax-Wendroff method, the high frequency components which do not decay quickly will move more slowly than the low frequency components. This is shown by the oscillations trailing the shock.

Numerical Results for Different Values of k/h :

In order to see how much the change of the values of μ affects the results in the initial-value problem (2.1) and (2.2), let us increase the value of μ by increasing the value of Δt . For example we will have $\mu = 0.75$ and $\mu = 0.90$ and find the solution after 20 time steps for these values of μ . See figure (2.2). The larger value of μ makes the solution look better, and reduces the oscillations, but those oscillations do not disappear completely. It should be noted that for the simple equation (2.1) both Lax-Wendroff and FTBS schemes will produce the exact solution when μ is chosen to be 1.0 . However, the figures presented here, where $\mu < 1$, typify the performance of these schemes for more complex situations and oscillations are a well-documented feature of the Lax-Wendroff method.

Table 2.1 : The Numerical Solution of Advection Equation

t = 0.5				t = 1.0			
x	Exact	FTBS	L-W	x	Exact	FTBS	L-W
0.2	1.0000	1.0000	1.0000	0.3	1.0000	1.0000	1.0000
0.3	1.0000	1.0000	0.9996	0.4	1.0000	1.0000	0.9999
0.4	1.0000	1.0000	1.0017	0.5	1.0000	1.0000	1.0002
0.5	1.0000	1.0000	0.9942	0.6	1.0000	1.0000	0.9993
0.6	1.0000	1.0000	1.0190	0.7	1.0000	1.0000	1.0017
0.7	1.0000	0.9987	0.9424	0.8	1.0000	1.0000	0.9975
0.8	1.0000	0.9793	1.1009	0.9	1.0000	1.0000	0.9978
0.9	1.0000	0.8684	1.0911	1.0	1.0000	0.9997	1.0249
1.0	1.0000	0.5881	0.4964	1.1	1.0000	0.9968	0.9399
1.1	0.0000	0.2517	0.0964	1.2	1.0000	0.9808	1.0099
1.2	0.0000	0.0577	0.0079	1.3	1.0000	0.9231	1.1977
1.3	0.0000	0.0059	0.0000	1.4	1.0000	0.7852	0.9626
1.4	0.0000	0.0002	0.0000	1.5	1.0000	0.5627	0.4598
1.5	0.0000	0.0000	0.0000	1.6	0.0000	0.3179	0.1324
				1.7	0.0000	0.1342	0.0235
				1.8	0.0000	0.0403	0.0083
				1.9	0.0000	0.0083	0.0002
				2.0	0.0000	0.0011	0.0000

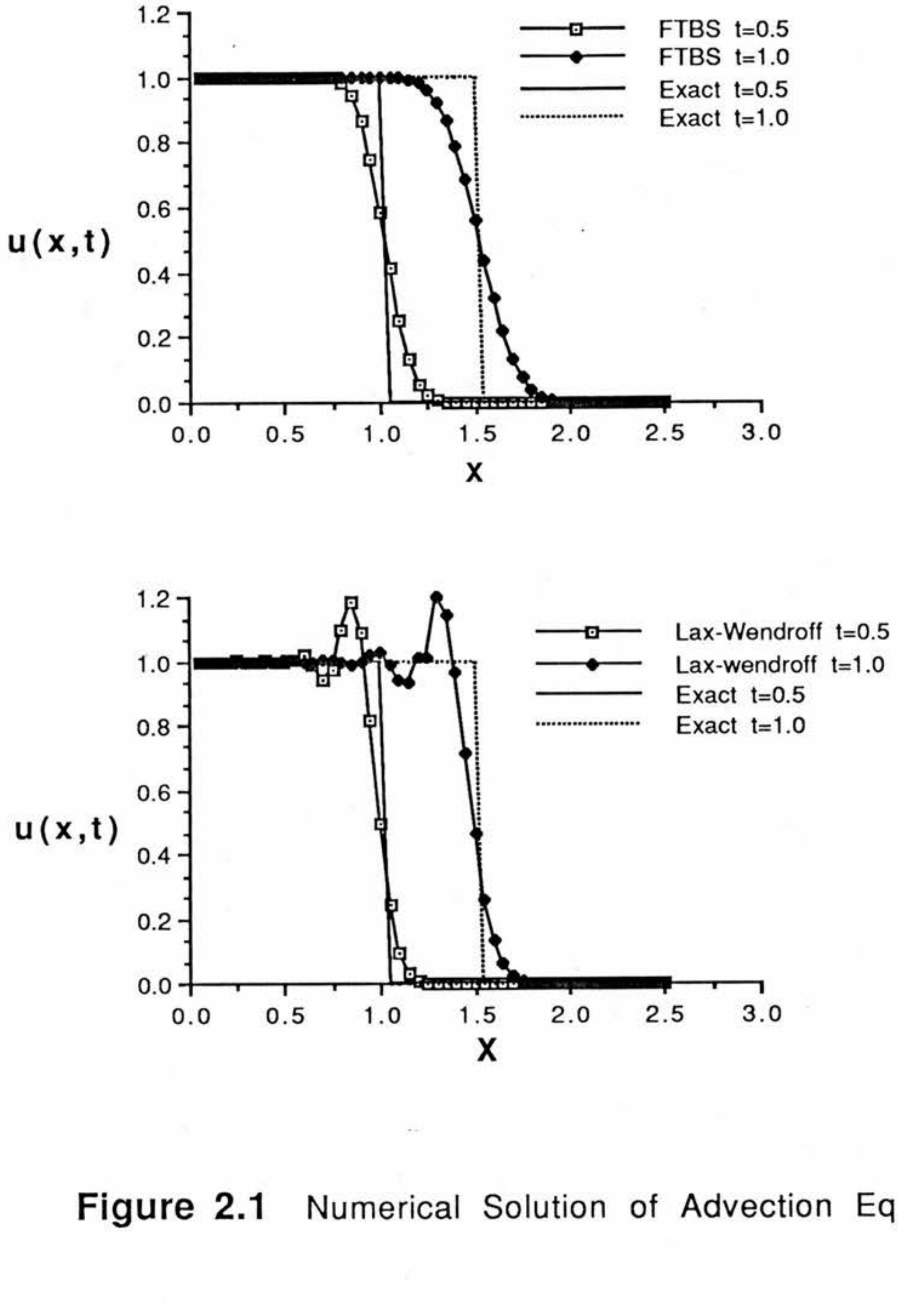
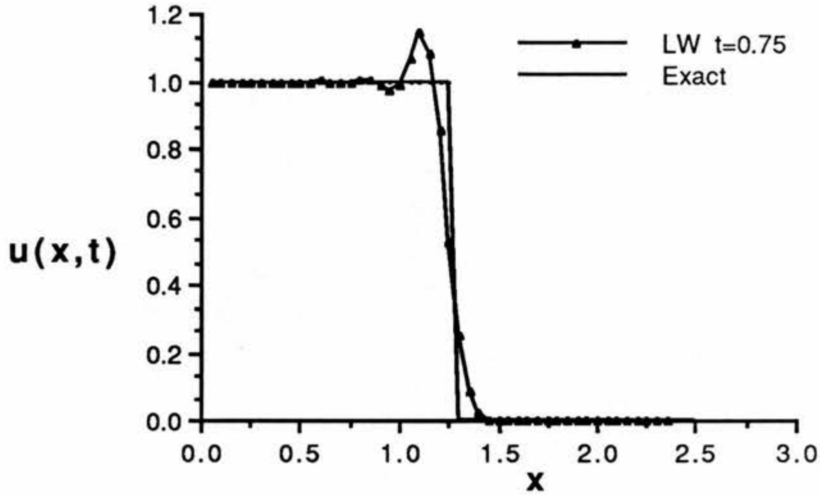


Figure 2.1 Numerical Solution of Advection Equation



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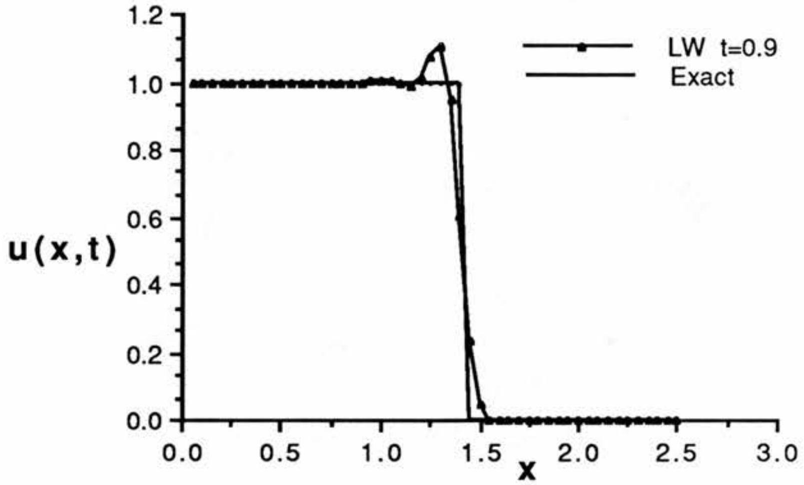


b

a. Lax-Wendroff when k/h = 0.75

b. Lax-Wendroff when k/h = 0.90

Figure2.2

2.2 The Artificial Viscosity

We shall introduce an additional dissipative, or diffusion, term in the finite-difference method and this small amount of dissipation is often called an artificial viscosity, although it is not related to physical viscosity. The affect of adding such a term is to reduce the oscillations by damping the short-wave length (high frequency) component of the solution while allowing the shock to affect only some mesh points and negligible affect in smooth parts as the calculations progress. It can also prevent the non-linear instability in some problems, so if there is an unstable or marginally stable finite-difference method, this can often be stabilized by adding the artificial viscosity term.

Let us consider the advection equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \qquad (2.3)$$

Following Zuchteau and Zuchmann [3], first we add an artificial dissipation term υu_{xx} to the equation (2.3). Indeed each approximation we discussed earlier is consistent with an equation of this form. Consider the modified equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \upsilon u_{xx} \qquad (2.4)$$

where υ is the dissipation coefficient, we shall use the centred difference to approximate x-derivatives and the forward difference for t-derivative. Equation (2.4) can be replaced by the approximation:

$$\frac{u_q^{p+1} - u_q^p}{k} + a\left(\frac{u_{q+1}^p - u_{q-1}^p}{2h}\right) = \upsilon \frac{\left(u_{q+1}^p - 2u_q^p + {}_{q-1}^p\right)}{h^2}$$

$$= \frac{\upsilon}{h^2}\, \delta_x^2\, u_q^p$$

and which is also of the form:

$$u_{q+1}^p = u_q^p - \frac{1}{2} a \mu \,(u_{q+1}^p - u_{q-1}^p) + \frac{k\upsilon}{h^2}\, \delta_x^2\, u_q^p \qquad (2.5)$$

Using the Von Neumann stability analysis we conclude that the formula (2.5) is stable if

$$\frac{(a\mu)^2}{2} \le \frac{k\upsilon}{h^2} \le \frac{1}{2}$$

By choosing a different value of the coefficient υ, equation (2.5) produces different schemes, for instance :

1) When $\frac{k\upsilon}{h^2} = \frac{1}{2}$, equation (2.5) yields the Lax-Friedrichs method (1.14).

2) When $\frac{k\upsilon}{h^2} = \frac{(a\ \mu)^2}{2}$, equation (2.5) yields the Lax-Wendroff method (1.15)

3) When $\frac{k\upsilon}{h^2} = \frac{a\ \mu}{2}$, equation (2.5) yields the FTBS method (1.12).

2.3 Simple Switching Schemes :

In general the higher the order of the approximation, the more accurate the solution. However, the Lax-Wendroff scheme, with its second order accuracy, produces oscillations at some points, while the first order accurate scheme FTBS has a smooth solution. The goal of switching between the two schemes is to have the advantages of the smoothness of the FTBS method and the accuracy of the Lax-Wendroff method at the same time.

This idea is to use the Lax-Wendroff method at all mesh points until the solution starts jumping or until there is a trouble spot then switch to the FTBS scheme at that point and then back immediately to the Lax-Wendroff. In order to do so we must have some kind of discriminator or indicator to catch the trouble points, and we shall use the artificial viscosity term to switch between the schemes by alternating the viscosity coefficient, as we will see in the next three schemes.

(I) The Basic Scheme :

We use the sign of the second difference as an indicator to decide the trouble spots. Let us consider again the initial-value problem

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \qquad , a = 1 \qquad (2.10)$$

$$u(x,0) = \begin{cases} 1 & 0 \le x \le 0.5 \\ 0 & 0.5 < x \le 2.5 \end{cases} \tag{2.11}$$

which we discussed in section 2.1 . Let $\mu = k/h = 0.5$ where $h = 0.5$.

By adding the artificial viscosity υu_{xx} to the equation (2.10) we get

$$u_q^{p+1} = u_q^p - \frac{1}{2} a \mu \left(u_{q+1}^p - u_{q-1}^p\right) + \frac{k\upsilon}{h^2} \delta_x^2 u_q^p \tag{2.12}$$

When the artificial coefficient $\upsilon = 0.0125$ then equation (2.12) is the Lax-Wendroff method (see section 2.1), and when $\upsilon = 0.025$ equation (2.12) reduces to the FTBS scheme. The second difference is

$$\delta_x^2 u_q^p = u_{q+1}^p - 2u_q^p + u_{q-1}^p$$

Now if $\delta_x^2 u_q^p$ and $\delta_x^2 u_{q-1}^p$ have the same sign, then we put υ to be 0.0125 and keep using the Lax-Wendroff method until their signs are different. When a change of sign occurs put υ to be 0.025 and repeat the calculation at the previous point q-1 using the FTBS method then apply FTBS at the point q . Continue to use FTBS until the signs are same again then go back to the Lax-Wendroff method. The numerical results of the Basic scheme, FTBS , Lax-Wendroff, and exact solution after 40 time steps ($t = 1.0$), are shown in table 2.2 and illustrated in figure 2.3. Using the second difference as an indicator means that the switch operates after moving through a discontinuity. Hence the need to retrace at least one step. The basic scheme always returns one step when switching to FTBS method.

There is no doubt that this scheme produces a better solution than the FTBS and quite successful although there are still some oscillations at some points which suggests that Lax-Wendroff scheme is being used at inappropriate points, but these oscillations are not as bad as those produced by Lax-Wendroff.

(II) The Modified Scheme :

We use here the same procedure as in the Basic scheme, but in addition to the sign change, the difference in size of consecutive second differences is also used as an indicator in the following way: At each point we examine if $\delta_x^2 u_q^p$ and $\delta_x^2 u_{q-1}^p$ have the same signs, also if $\left| \delta_x^2 u_q^p - \delta_x^2 u_{q-1}^p \right|$ is less than or equal to a small amount (let this amount in our problem (2.10) and (2.11) be $6h^2$), then use the Lax-Wendroff scheme by setting $\upsilon = 0.0125$ in the equation (2.12). If either of these conditions is not satisfied then there is either an oscillation or a rapid change, so we put $\upsilon = 0.025$ and use the FTBS scheme.

The numerical solution of the Modified scheme after 40 time steps (t = 1.0), compared with the FTBS, Lax-Wendroff schemes and exact solution are shown is in table 2.2 and illustrated in figure 2.4.

By applying this method the discontinuity will be identified earlier. Although the solution is smooth with no oscillations, it is little improvement on the FTBS results. Indeed the switch to FTBS is being triggered too frequently and at too many points. Thus the Modified scheme is not very successful.

(III) The Refined scheme :

In preparing for the solution of non-linear problem, let the scheme be written as a two steps process:

$$u_q^{(1)} = u_q^p - a\,\mu\,(u_q^p - u_{q-1}^p) \qquad (1.13a)$$

$$u_q^{(2)} = \frac{1}{2}\,(u_q^p + u_q^{(1)} - a\,\mu\,(u_{q+1}^{(1)} - u_q^{(1)})) \qquad (1.13b)$$

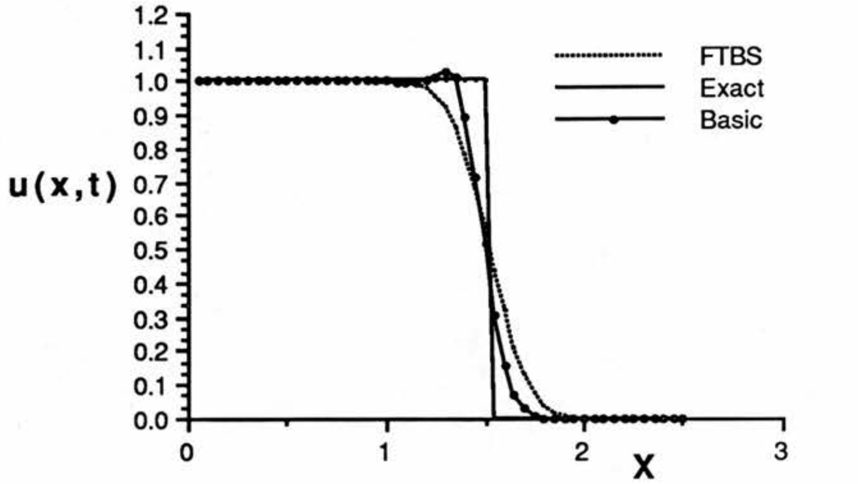
This scheme is the MacCormack form of the Lax-Wendroff method for the linear advection equation [8]. The first step is the FTBS scheme, when the first step is substituted into the second one the Lax-Wendroff method is obtained. So instead of switching between values of υ as before, the decision to be taken is whether to apply the second stage or not. We shall use the sign of the second difference and its size to identify rapid changes in the solution.

In this method the values of $u_q^{(1)}$ are evaluated then used to evaluate $u_q^{(2)}$ as follows :

At each grid point, first calculate the value of $u_q^{(1)}$ in the first step and the value of second difference $\delta_x^2 u_{q-1}^p$. If the value of $\left| \delta_x^2 u_{q-1}^p \right|$

is very small (say less than 0.001 in our problem (2.10) and (2.11)), then let $\delta_x^2 u_{q-1}^p = 0.0$. Now if the signs of $\delta_x^2 u_{q-1}^p$ and $\delta_x^2 u_q^p$ are the same and the absolute value of the difference between them is less than a small amount (say $6h^2$), then we go to the second step to calculate the value of $u_q^{(2)}$ using $u_q^{(1)}$ that was calculated in the first step. If any of above conditions are not satisfied then stage 2 is not carried out and the scheme moves on to the next grid point. The values here are computed in order of decreasing q .

The numerical results of the solution using this scheme are shown in table 2.2 and illustrated in figure 2.5 after 40 time steps (t = 1.0) compared with the Lax-Wendroff, the FTBS and the exact solution. As the difference in sizes of the second difference picks up the discontinuity earlier there is no need to retrace a step. Also the use of a tolerance on the second difference allows some minor oscillations to go unnoticed and prevents too frequent use of the FTBS method. From the results, we note that this scheme strikes a compromise between the Basic and the Modified schemes. It eliminates all visual oscillations on graphical output without over-damping the solution. Compared with the Modified scheme the discontinuity is steeper, diffused over fewer mesh points. However, resolution at a discontinuity is still quite poor, but how does this compare with more sophisticated schemes?. To judge the performance of the Refined scheme it is necessary to compare these results with those produced by more sophisticated methods and this is undertaken in chapter four.



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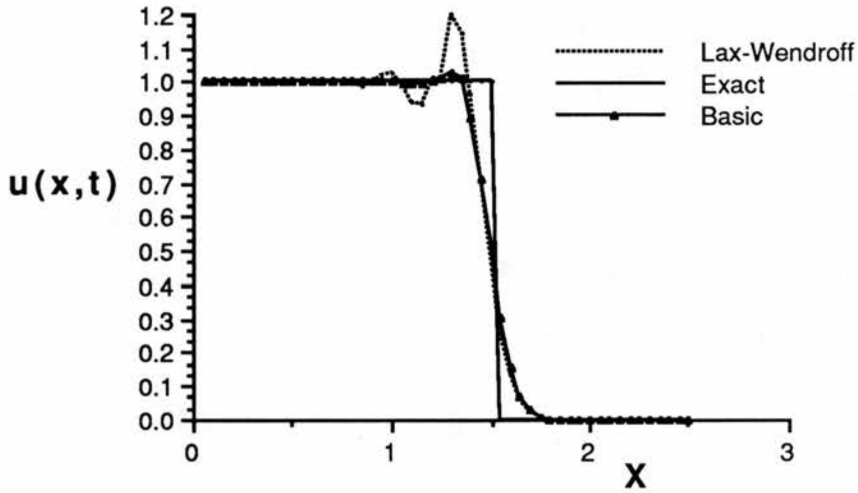

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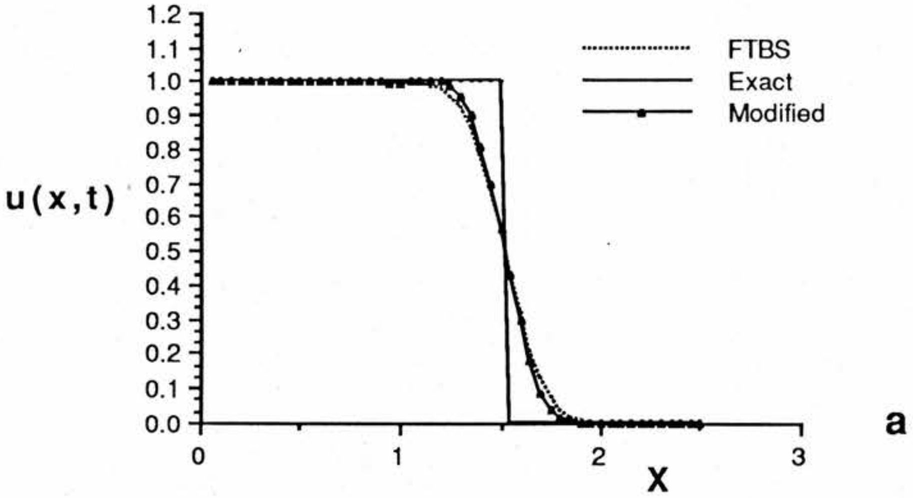
Figure 2.3



1.2
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u(x,t)
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2
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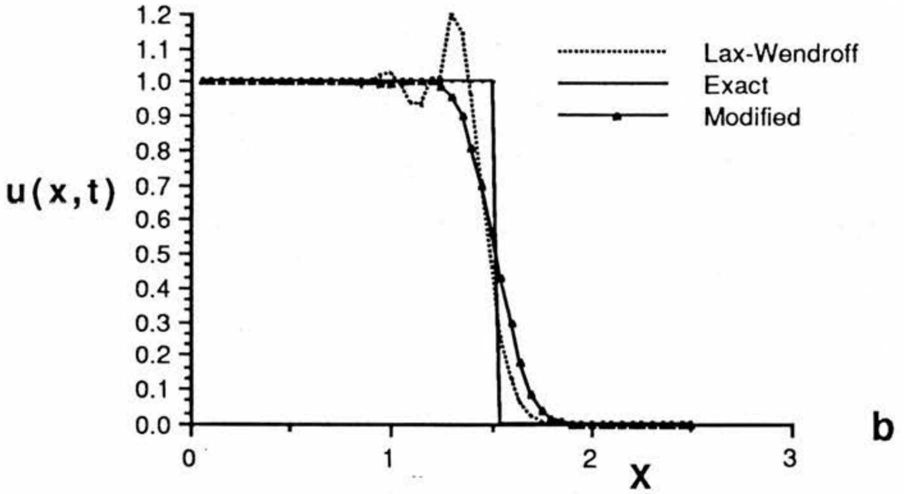
X
FTBS
Exact
Modified
a



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u(x,t)
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2
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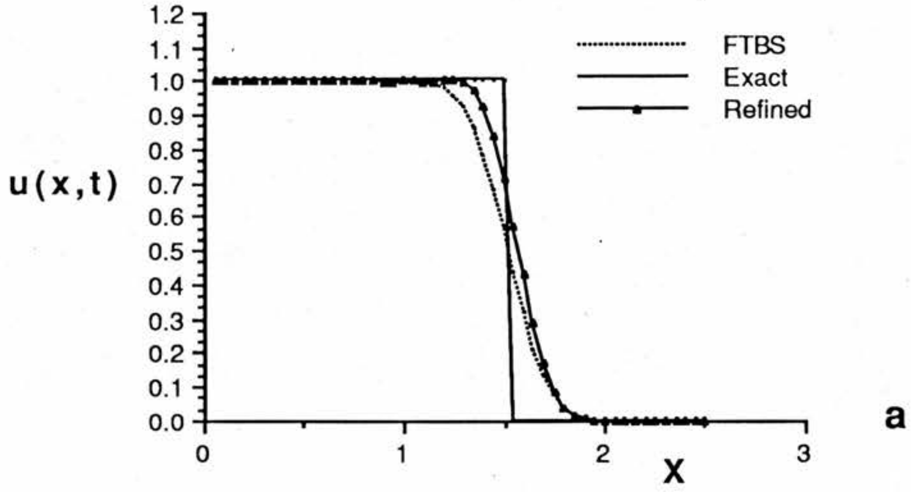
X
Lax-Wendroff
Exact
Modified
b

Figure 2.4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FTBS
Exact
Refined
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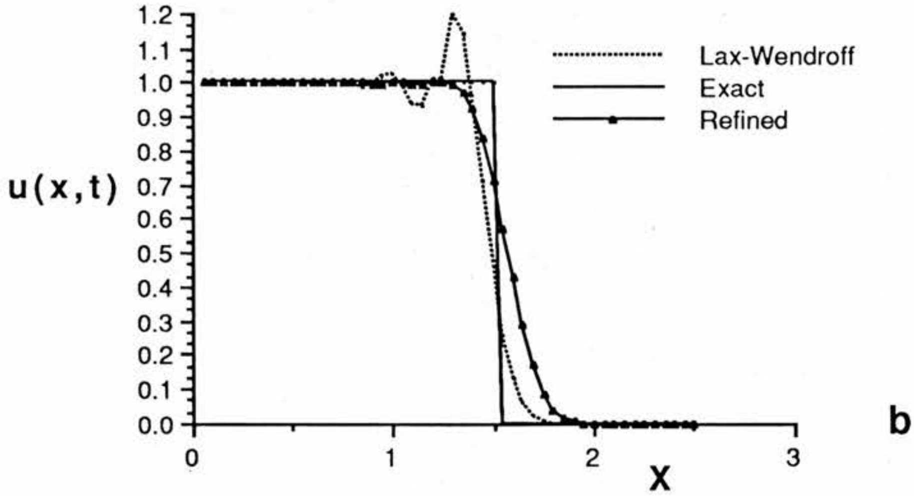
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0.0
u(x,t)
0
1
2
3
X
Lax-Wendroff
Exact
Refined
b

Figure 2.5

Table 2.2 : The Numerical Solution of the Advection Equation by Simple Switching Schemes , at t = 1.0

				Simple Switching Schemes		
x	Exact	FTBS	L-W	Basic	Modified	Refined
0.5	1.0000	1.0000	1.0002	1.0000	1.0000	1.0000
0.6	1.0000	1.0000	0.9993	1.0000	1.0000	1.0000
0.7	1.0000	1.0000	1.0017	1.0000	1.0000	1.0000
0.8	1.0000	1.0000	0.9975	1.0000	1.0000	1.0000
0.9	1.0000	1.0000	0.9978	1.0000	1.0000	0.9999
1.0	1.0000	0.9997	1.0249	1.0001	0.9998	1.0001
1.1	1.0000	0.9968	0.9399	0.9991	1.0016	0.9992
1.2	1.0000	0.9808	1.0099	1.0031	1.0031	1.0040
1.3	1.0000	0.9231	1.1977	1.0273	0.9548	0.9710
1.4	1.0000	0.7852	0.9626	0.8959	0.8094	0.8375
1.5	1.0000	0.5627	0.4598	0.5153	0.5637	0.5753
1.6	0.0000	0.3179	0.1324	0.1562	0.2962	0.2869
1.7	0.0000	0.1341	0.0235	0.0283	0.0898	0.0890
1.8	0.0000	0.0403	0.0083	0.0002	0.0148	0.0140
1.9	0.0000	0.0083	0.0002	0.0000	0.0014	0.0013
2.0	0.0000	0.0011	0.0000	0.0000	0.0001	0.0003
2.1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001

CHAPTER THREE

The Non-Linear Advection Problem

3.1 Conservation Laws

A conservation law states that the change in the total amount of a material within some fixed region, is equal to the flux of that material across the boundary of the region. Let $u = u(x,t)$ represent the linear density of the material at a particular position x and time t , then the total amount of the material in the closed interval $[a,b]$ is

$$\int_a^b u(x,t)\ dx \tag{3.1}$$

Let F denote the one-dimensional flux of the quantity, then the flux across any boundary is

$$-\ F(u(x,t))$$

If we assume that $F(u)$ is continuously differentiable and that u is continuously differentiable, then the conservation law may be written as

$$\frac{\partial}{\partial t}\int_a^b u(x,t)\ dx = -\ F(u(x,t))\Big]_a^b$$

$$= F(u(a,t)) - F(u(b,t)) \qquad (3.2)$$

This integral form of conservation law states that the change in the amount of material within the region [a,b] at the time t is equal to the material flow into [a,b] across x = a minus the flow out of [a,b] across x = b . Equation (3.2) can be written in the form

$$\int_a^b \frac{\partial}{\partial t} u(x,t)\ dx = - F(u(x,t))\Big]_a^b \qquad (3.3)$$

but as $F(u(x,t))\Big]_a^b = \int_a^b \partial_x F(u)\ dx$, equation (3.3) becomes

$$\int_a^b \left(\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} F(u)\right) dx = 0 \qquad (3.4)$$

Since the interval [a,b] is arbitrary, and since u and F are continuously differentiable, then

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} F(u) = 0 \qquad (3.5)$$

The partial differential equation (3.5) is an example of a one-dimensional first order conservation law. F depends on u, and as $u = u(x,t)$, F depends on x and t . In equation (3.4) if there is no material flow into or out of [a,b] then

$$\int_a^b \frac{\partial u}{\partial t}\ dx = 0 \qquad , \text{or}$$

$$\int_a^b u\,dx = \text{constant} \tag{3.6}$$

that is the material is conserved.

3.2 The Conservation Law Difference Methods

The conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} F(u) = 0 \tag{3.7}$$

can be written as

$$\frac{\partial u}{\partial t} + \frac{\partial F}{\partial u}\,\frac{\partial u}{\partial x} = 0$$

By replacing $\frac{\partial u}{\partial x}$ by $a(u)$, where $a(u)$ is non-constant, the conservation law (3.7) can be rewritten as

$$\frac{\partial u}{\partial x} + a(u)\,\frac{\partial u}{\partial x} = 0 \tag{3.8}$$

The hyperbolic equation (3.8) is called the non-linear advection equation. If $\frac{\partial F}{\partial u} = a$, a is constant, then equation (3.8) reduces to the linear advection equation which we discussed earlier.

The characteristic curves of the equation (3.8) are

$$\frac{dx}{dt} = a(u)$$

and as $u(x,t)$ is constant along the characteristics, it follows that $a(u)$ is constant along the characteristics, so the characteristics are straight lines.

Let us consider the conservation law (3.8) with the initial-value condition

$$u(x,0) = f(x) \tag{3.9}$$

then the exact solution for the initial-value problem (3.8) and (3.9) is

$$u = f(x - a(u)t) \quad .$$

Let us consider again the non-linear conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} F(u) = 0 \quad , \quad \frac{\partial F}{\partial u} = a(u) \tag{3.10}$$

The usual finite-difference schemes that are used in the solution of the linear advection equation can be used for the non-linear one.

The FTBS, or the upstream approximation, for the equation (3.10) is

$$u_q^{p+1} = u_q^p - \mu\ (F_q^p - F_{q-1}^p) \qquad , (a(u) > 0) \tag{3.11}$$

where $\mu = k/h$, h and k denote the x and t grid spacing, and $F_q^p = F(u(qh,pk))$. The FTBS is a first order accurate method.

The FTCS method has the form

$$u_q^{p+1} = u_q^p - \frac{1}{2}\mu\,(F_{q+1}^p - F_{q-1}^p) \qquad (3.12)$$

when $a(u) = a$ it reduces to the unconditionally unstable method, which we mentioned in chapter one. So the FTCS method (3.12) is not useful since it is unstable.

To obtain the Lax-Wendroff method of the conservation law (3.10), we use the Taylor expansion for $u(qh,pk)$:

$$u(qh,(p+1)k) = u(qh,pk) + ku_t\,(qh,pk) + \frac{k^2}{2}\,u_{tt}\,(qh,pk) + O(k^3)$$

.... (3.13)

From equation (3.10) we have

$$\frac{\partial u}{\partial t} = \frac{-\partial F(u)}{\partial x}$$

and $$\frac{\partial^2 u}{\partial t^2} = -\,(F'(u)\;u_t)_x$$

$$= \frac{\partial}{\partial x}\,(a(u)\,\frac{\partial}{\partial x}\;(F(u)))$$

Now $\frac{\partial u}{\partial t}$ in (3.13) can be approximated as

$$\frac{\partial u}{\partial t} = \frac{-(F_{q+1}^p - F_{q-1}^p)}{2h} + O(h^2) \qquad (3.14)$$

The difference operator δ_x is defined by

$$\delta_x \, u_q^p = u^p_{q+\frac{1}{2}} - u^p_{q-\frac{1}{2}}$$

Thus

$$\frac{\partial}{\partial x} F(u) = \frac{1}{h} \, \delta_x F_q^p + O(h^2)$$

which is the central difference for F(u) with $\frac{1}{2}$ h step size. Then we have

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{h^2} \delta_x \, (a(u_q^p) \, \delta_x \, F(u_q^p)) + O(h^2)$$

$$= \frac{1}{h^2} [(a^p_{q+\frac{1}{2}}) \; \{ F^p_{q+1} - F_q^p \} - (a^p_{q-\frac{1}{2}})\{ F_q^p - F^p_{q-1} \}] + O(h^2)$$

.... (3.15)

where

$$a^p_{q+\frac{1}{2}} = \frac{a(u_q^p + u^p_{q+1})}{2} \qquad \text{and}$$

$$a^p_{q-\frac{1}{2}} = \frac{a(u_q^p + u^p_{q-1})}{2}$$

Now by replacing u_t and u_{tt} in equation (3.13) by formulas (3.14) and (3.15), the conservation law Lax-Wendroff is written as the following

$$u_q^{p+1} = u_q^p - \frac{1}{2}\mu (F_{q+1}^p - F_{q-1}^p) + \frac{1}{2}\mu^2 \, [(a_{q+\frac{1}{2}}^p) \{F_{q+1}^p - F_q^p \} -$$

$$(a_{q-\frac{1}{2}}^p)\{F_q^p - F_{q-1}^p \}] \qquad \text{........(3.16)}$$

which is a second-order accurate method. When F(u) = au , (a is constant), the formula (3.16) reduces to the Lax-Wendroff method (1.15) for the linear advection equation which is stable by Von Neumann stability analysis,(see section 1.3), provided that $|\mu a| \leq 1$. Although the Von Neumann analysis of stability can not be applied when F(u) is non-linear, a guide to the stability of formula (3.16) can be obtained by linearizing the equation. A linearized stability suggests that Lax-Wendroff scheme (3.16) is stable provided

$$\left|\mu a \, (\, u_q^p \,)\right| \leq 1 \qquad \text{for all q, p}$$

The Lax-Wendroff scheme (3.16) is not efficient because it requires the evaluation of both $a_{q-\frac{1}{2}}^p$ and $a_{q+\frac{1}{2}}^p$ once for each grid point. In order to avoid the complication of computing a(u) between grid points as well as F(u) at grid points, two-step modifications of the Lax-Wendroff method have been constructed. An example of one of these modifications is the MacCormack method, which is one of the most efficient and widely used schemes.

The MacCormack method for the conservation law (3.10) has the following form [2] :

$$\tilde{u}_q^{p+1} = u_q^p - \mu [F_q^p - F_{q-1}^p] \qquad (3.17a)$$

$$u_q^{p+1} = \frac{1}{2} [u_q^p + \tilde{u}_q^{p+1} - \mu (\tilde{F}_{q+1}^{p+1} - \tilde{F}_q^{p+1})] \qquad (3.17b)$$

where

$$\tilde{F}_q^p = F(\tilde{u}_q^p)$$

The first step (3.17a) is the first order method FTBS and this step calculates the values of $\tilde{u}_q^{p+1}$. The second step (3. 17b), with second order accuracy, is the correction step and it corrects those values of u_q^{p+1} which are computed in the first step approximation. By reversing the direction of the difference we obtain another MacCormack formula

$$\tilde{u}_q^{p+1} = u_q^p - \mu [F_{q+1}^p - F_q^p] \qquad (3.18a)$$

$$u_q^{p+1} = \frac{1}{2} [u_q^p + \tilde{u}_q^{p+1} - \mu (\tilde{F}_q^{p+1} - \tilde{F}_{q-1}^{p+1})] \qquad (3.18b)$$

which is equivalent to the formula (3.17a) and (3.17b). When $F(u) = au$ the two-step MacCormack method is the same as the one-step Lax-Wendroff method for the linear advection equation with constant a (equation(1.15)).

3.3 Difference Approximations in Conservation Law Form

Consider the non-linear advection equation

$$u_t + F(u)_x = 0 \tag{3.19}$$

where $\frac{\partial F}{\partial u} = a(u)$. We say that a finite difference method is in conservation law form if it can be expressed in the form

$$\frac{Q^p_{q+\frac{1}{2}} - Q^p_{q-\frac{1}{2}}}{h} + \frac{u^{p+1}_q - u^p_q}{k} = 0 \tag{3.20}$$

where $Q^p_{q+\frac{1}{2}} = Q(u^p_q , u^p_{q+1})$ and $Q^p_{q-\frac{1}{2}} = Q(u^p_{q-1} , u^p_q)$ represent the numerical approximations to the fluxes $F(u^p_{q+\frac{1}{2}})$ and $F(u^p_{q-\frac{1}{2}})$ respectively. Q varies from scheme to scheme, and it has to be consistent with the conservation flux F, which means

$$Q(u,u) = F(u) \tag{3.21}$$

Equation (3.20) can be written as

$$u^{p+1}_q = u^p_q + \mu (Q^p_{q-\frac{1}{2}} - Q^p_{q+\frac{1}{2}}) \tag{3.22}$$

Now if we apply the equation (3.22) at the interval $q = 1,2,...,m-1$ then we have

$$\sum_{q=1}^{m-1} u_q^{p+1} = \sum_{q=1}^{m-1} u_q^p - \mu \sum_{q=1}^{m-1} (Q_{q+\frac{1}{2}}^p - Q_{q-\frac{1}{2}}^p) \qquad (3.23)$$

which can be rearranged as

$$\sum_{q=1}^{m-1} (u_q^{p+1} - u_q^p) = -\mu (Q_{m+\frac{1}{2}}^p - Q_{m-\frac{1}{2}}^p) \qquad (3.24)$$

Formula (3.24) is the conservation law difference equation, and it states that the change in amount of the material within some region is equal to the difference between the flow of the material in and out across the boundary of that region during a period of time, (from time t_p to t_{p+1}). If the flux on the boundary is zero then

$$\sum_{q=1}^{m-1} u_q^{p+1} = \sum_{q=1}^{m-1} u_q^p \qquad (3.25)$$

and u_q^p is conserved,(see equation (3.6)).

A conservation law difference formula has the following property, (which is expressed by a theorem due to the Lax and Wendroff [11] :

Theorem 3.1 : If the numerical solution u_q^p of a conservation- law difference formula converges to a bounded function $u(x,t)$, then $u(x,t)$ is a generalized solution of the partial differential equation (3.19).

The FTBS scheme (3.11) is in conservation law form with numerical fluxes defined by

$$Q^{p}_{q-\frac{1}{2}} = F^{p}_{q-1} = F(u^{p}_{q-1}) \tag{3.26a}$$

and

$$Q^{p}_{q+\frac{1}{2}} = F^{p}_{q} = F(u^{p}_{q}) \tag{3.26b}$$

The Lax-Wendroff method (3.16) is also seen to be in conservation law form with Q defined by

$$Q^{p}_{q+\frac{1}{2}} = F^{p}_{q+1} + \mu\, a_{q+\frac{1}{2}}(F^{p}_{q} - F^{p}_{q+1}) \tag{3.27a}$$

and

$$Q^{p}_{q-\frac{1}{2}} = F^{p}_{q-1} + \mu\, a_{q-\frac{1}{2}}(F^{p}_{q} - F^{p}_{q-1}) \tag{3.27b}$$

where $a_{q+\frac{1}{2}} = a(u^{p}_{q}\, , u^{p}_{q+1})$

$$= \frac{a(u^{p}_{q} + u^{p+1}_{q})}{2}$$ and

$$a_{q-\frac{1}{2}} = a(u^{p}_{q-1}\, , u^{p}_{q})$$

$$= \frac{a(u^{p}_{q-1} + u^{p}_{q})}{2}$$

and Q in (3.27a) and (3.27b) is consistent with the flux F .

3.4 A Test Example

In this section we will consider a test example of a conservation law and discuss its exact solution and then the numerical solution using the two-step MacCormack method (3.17a) and (3.17b). Also a comparison of the results obtained using some of the other schemes discussed earlier will be presented. So let us first consider the following scalar conservation law (Burgers equation)

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = 0 \tag{3.30}$$

with the continuous initial condition

$$u(x,0) = \begin{cases} 4x-1 & 0.25 < x \leq 0.5 \\ 3-4x & 0.5 < x \leq 0.75 \\ 0 & \text{elsewhere} \end{cases} \tag{3.31}$$

We will choose $0 \leq x \leq 2.5$, where $h = 0.05$ and choose $0 \leq t \leq 1.25$.

I. Generalized solution:

Since $a(u) = u$ the equation (3.30) has a solution of the form

$$u = f(x - ut) \tag{3.32}$$

The x and t derivatives of u are

$$u_t = (-u - u_t\, t)\, f'$$

$$u_x = (1 - u_x\, t)\, f'$$

then $\quad u_t + u\, u_x = - (u_t + u\, u_x) f' t$

which can be rearranged as

$$(1 + f' t)(u_t + u\, u_x) = 0$$

and provided that $1 + f' t \neq 0$ then $u_t + u\, u_x = 0$. Therefore

$$x - ut = \text{constant}$$

is a characteristic line of the partial differential equation (3.30). For the above initial condition (3.31) the characteristic lines are :

$$x = \text{constant} \qquad \text{when } u = 0$$

$$x - t = \frac{1}{2} \qquad \text{when } u = 1$$

$$2x - t = \frac{3}{4} \text{ or } \frac{5}{4} \qquad \text{when } u = \frac{1}{2}$$

and these are illustrated in figure 3.1 .

For $t < \frac{1}{4}$, the solution is

$$u(x,t) = \begin{cases} \dfrac{4(x - \frac{1}{4})}{1 + 4t} & \frac{1}{4} \leq x \leq \frac{1}{2} + t \\ \dfrac{4(\frac{3}{4} - x)}{1 - 4t} & \frac{1}{2} + t \leq x \leq \frac{3}{4} \\ 0 & \text{elsewhere} \end{cases}$$

Note that the characteristic lines $x = \frac{3}{4}$ and $x - t = \frac{1}{2}$ (which correspond to $u = 0$ and $u = 1$) intersect at the point $(\frac{3}{4}, \frac{1}{4})$, see figure 3.1). As a result of this intersection we have a double-valued solution which leads to a contradiction at the point $(\frac{3}{4}, \frac{1}{4})$. The above formal solution is no longer valid everywhere so we introduce the idea of a generalized solution which contains a discontinuity, called a chock. Let $H : x = H(t)$ be a curve in xt-plane across which a solution of (3.30) and (3.31) is discontinuous (figure 3.2). Assume that the conservation law (3.30) has the following exact solution

$$u(x,t) = \begin{cases} u_1 & x < H(t) \\ u_2 & H(t) < x \end{cases} \tag{3.33}$$

(u_1 is a solution behind the shock and u_2 is a solution in front of the shock), where $u_1 \neq u_2$, and assume that in the neighbourhood of H, u_1 and u_2 are continuously differentiable, H is called the shock provided the following conditions hold along H :

$$H_1 : \frac{\partial H}{\partial t} = \frac{F(u_1) - F(u_2)}{u_1 - u_2} \tag{3.34a}$$

$$H_2 : a(u_1) > \frac{\partial H}{\partial t} > a(u_2) \tag{3.34b}$$

The first condition ensures the conservation of the material across the shock and the second one is called the <u>entropy condition</u> [4]. Let $x_1 < H(t) < x_2$, and u is a linear density then the amount A of material in the interval (x_1, x_2) at time t is

$$A(t) = \int_{x_1}^{x_2} u(x,t)\,dx = \int_{x_1}^{H(t)} u_1(x,t)\,dt + \int_{H(t)}^{x_2} u_2(x,t)\,dt \tag{3.35}$$

then

$$\frac{\partial A}{\partial t} = u_1(H(t),t)\,\frac{\partial H}{\partial t} - u_2(H(t),t)\,\frac{\partial H}{\partial t} \tag{3.36}$$

For the material to be conserved we have

$$\frac{\partial A}{\partial t} = F(u_1(x_1,t)) - F(u_2(x_2,t)) \tag{3.37}$$

As $x_1 \to \bar{H}(t)$ and $x_2 \to \overset{+}{H}(t)$, (see [3]), equations (3.36) and (3.37) imply :

$$\frac{\partial H}{\partial t} = \frac{(\frac{1}{2}u^2)_{\bar{H}} - (\frac{1}{2}u^2)_{\overset{+}{H}}}{u_{\bar{H}} - u_{\overset{+}{H}}} \tag{3.38}$$

Since in our test example $u_{\overset{+}{H}} = 0$

$$\frac{\partial H}{\partial t} = \frac{1}{2} u_{\bar{H}}$$

$$= \frac{\frac{1}{2}(4(H - \frac{1}{4}))}{1 + 4t}$$ which can be integrated to give

$$H - \frac{1}{4} = c\sqrt{1+4t}$$, c is constant.

When $H = \frac{3}{4}$, $t = \frac{1}{4}$ so $c = \frac{1}{2\sqrt{2}}$ and hence

$$H(t) = \frac{1}{4} + \frac{1}{2\sqrt{2}}(1 + 4t)^{\frac{1}{2}}$$

This is the location of the jump (or shock).

So for $t > \frac{1}{4}$

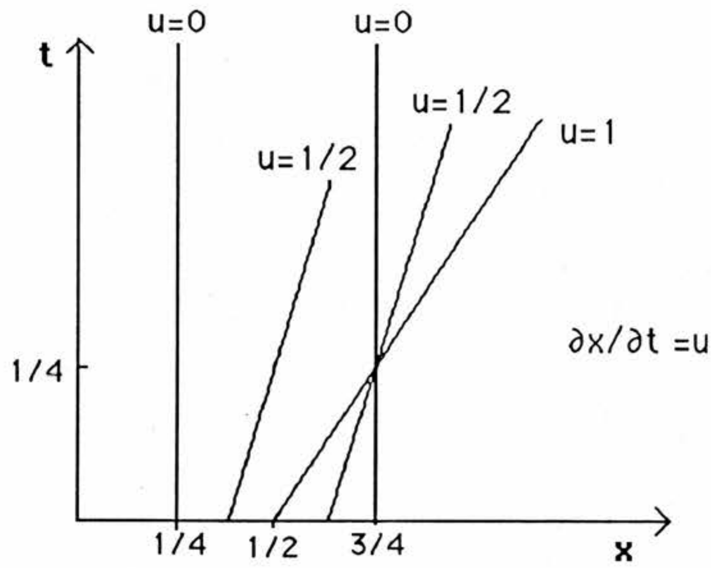
$$u(x,t) = \begin{cases} \dfrac{4(x - \frac{1}{4})}{1+4t} & \frac{1}{4} \leq x < H(t) \\ 0 & \text{elsewhere} \end{cases}$$

u_H is called the height of the jump, where

$$u_H = u(H(t)) = \frac{4}{2\sqrt{2}\ \sqrt{1+4t}}$$

$$= \frac{\sqrt{2}}{\sqrt{1+4t}}$$

For instance when $t = \frac{1}{2}$, $H(t) = 0.8624$ and $u_H = 0.8165$. When $t = 1.0$, $H(t) = 1.0406$ and $u_H = 0.6325$.



The Characteristic Lines for equation (3.31)

Figure 3.1

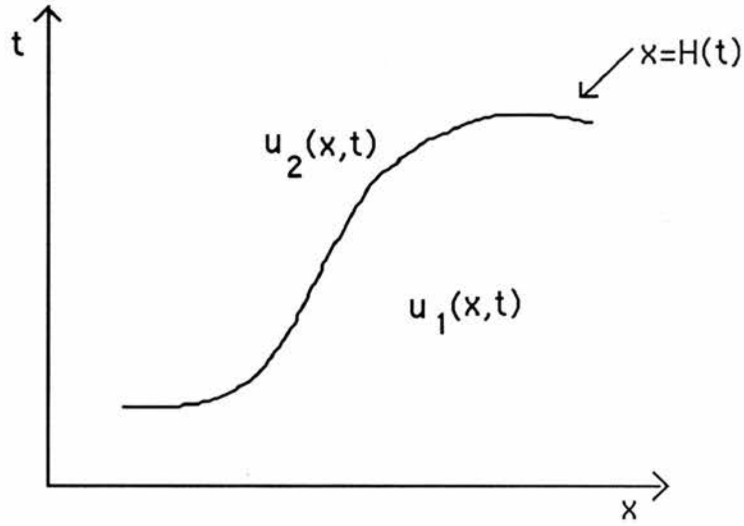


Figure 3.2

II. Numerical Solution :

Here we are going to use the same ideas as used earlier in chapter two, and were referred to as the Modified and the Refined schemes. In the case of the Modified scheme the two-step MacCormack method is now used and the indicators determine whether the second step should be applied at each point. For convenience, the indicators are summarised below.

The Modified Scheme : At each mesh point we calculate the value of u_q^{p+1} using the first step (3.17a), and calculate the second difference $\delta_x^2 u_{q-1}^p$. Now if

$$(\delta_x^2 u_q^p)(\delta_x^2 u_{q-1}^p) \geq 0 \quad \text{and} \quad \left| (\delta_x^2 u_q^p) - (\delta_x^2 u_{q-1}^p) \right| < 6h^2$$

then we use the second step (3.17b) to correct the values of u_q^{p+1} that we computed in the first step. If either of these above conditions are not satisfied, then only the first step is used to calculate u_q^{p+1}. We do the calculations of the values here in the order of decreasing q .

The Refined Scheme : We follow the same procedure as in the Modified scheme but with the added condition that just after computing the first step and the second difference a tolerance is introduced, as described in chapter two. This is that if

$$\left| \delta_x^2 u_{q-1}^p \right| < \text{some small amount (say 0.001)}$$

then set $\delta_x^2 u_{q-1}^p$ to be zero and continue with the rest of the procedure. The numerical results of the above two methods compared with the exact solution ,the two-step MacCormack and the FTBS scheme when $\mu = 0.5$ and $\mu = 0.25$,for different values of t are shown in tables 3.1 and 3.2 and also illustrated in figures 3.3 and 3.4.

Table 3.1: Numerical Solution of the IVP (3.30) and (3.31) when $\frac{k}{h} = 0.5$:

(3.1a) After 8 time steps (t= 0.2)

x	Exact	Modified	Refined	MacCormack	FTBS
0.20	0.0000	0.0000	0.0000	-0.0002	0.0000
0.25	0.0000	0.0000	0.0000	-0.0215	0.0000
0.30	0.1111	0.1464	0.1411	0.1167	0.1411
0.35	0.2222	0.2589	0.2475	0.2162	0.2492
0.40	0.3333	0.3677	0.3496	0.3555	0.3565
0.45	0.4444	0.4792	0.4522	0.4380	0.4638
0.50	0.5556	0.5961	0.5638	0.5400	0.5710
0.55	0.6667	0.7085	0.6710	0.6749	0.6742
0.60	0.7778	0.7985	0.7641	0.8228	0.7603
0.65	0.8889	0.8081	0.8035	0.9594	0.7903
0.70	1.0000	0.6230	0.6827	0.7066	0.6757
0.75	0.0000	0.2111	0.2848	0.1363	0.2974
0.80	0.0000	0.0090	0.0180	0.0019	0.0212
0.85	0.0000	0.0000	0.0000	0.0000	0.0000

(3.1b) After 20 time steps (t=0.5)

x	Exact	Modified	Refined	MacCormack	FTBS
0.20	0.0000	0.0000	0.0000	-0.0015	0.0000
0.25	0.0000	0.0000	0.0000	-0.0346	0.0000
0.30	0.0667	0.0982	0.0982	0.0800	0.0982
0.35	0.1333	0.1645	0.1645	0.1209	0.1649
0.40	0.2000	0.2253	0.2230	0.2052	0.2297
0.45	0.2667	0.2869	0.2808	0.2805	0.2941
0.50	0.3333	0.3492	0.3450	0.3512	0.3584
0.55	0.4000	0.4135	0.4104	0.3681	0.4225
0.60	0.4667	0.4792	0.4759	0.4951	0.4861
0.65	0.5333	0.5463	0.5438	0.5591	0.5486
0.70	0.6000	0.6113	0.6081	0.4317	0.6076
0.75	0.6667	0.6588	0.6576	0.7405	0.6554
0.80	0.7333	0.6563	0.6525	0.9444	0.6495
0.85	0.8000	0.4310	0.4221	0.2591	0.4205
0.90	0.0000	0.0623	0.0637	0.0040	0.0641
0.95	0.0000	0.0003	0.0003	0.0000	0.0004

(3.1c) After 40 time steps (t=1.0)

x	Exact	Modified	Refined	MacCormack	FTBS
0.20	0.0000	0.0000	0.0000	-0.0058	0.0000
0.25	0.0000	0.0000	0.0000	-0.0465	0.0000
0.30	0.0400	0.0654	0.0654	0.0627	0.0654
0.35	0.0800	0.1073	0.1073	0.0731	0.1074
0.40	0.1200	0.1449	0.1426	0.1230	0.1474
0.45	0.1600	0.1823	0.1788	0.1486	0.1867
0.50	0.2000	0.2212	0.2142	0.1972	0.2257
0.55	0.2400	0.2589	0.2512	0.2738	0.2646
0.60	0.2800	0.2963	0.2897	0.3027	0.3033
0.65	0.3200	0.3342	0.3287	0.2673	0.3420
0.70	0.3600	0.3717	0.3670	0.3141	0.3807
0.75	0.4000	0.4098	0.4055	0.5011	0.4191
0.80	0.4400	0.4509	0.4469	0.4828	0.4572
0.85	0.4800	0.4933	0.4906	0.2471	0.4941
0.90	0.5200	0.5277	0.5257	0.3825	0.5263
0.95	0.5600	0.5369	0.5343	0.7641	0.5334
1.00	0.6000	0.4281	0.4253	0.4613	0.4201
1.05	0.0000	0.1298	0.1240	0.0366	0.1234
1.10	0.0000	0.0037	0.0033	0.0001	0.0033

Table 3.2 : **Numerical Solution of the IVP (3.30) and (3.31) when $\frac{k}{h} = 0.25$:**

(3.2a) After 16 time steps (t=0.2)

x	Exact	Modified	Refined	MacCormack	FTBS
0.20	0.0000	0.0000	0.0000	-0.0002	0.0000
0.25	0.0000	0.0000	0.0000	-0.0219	0.0000
0.30	0.1111	0.1420	0.1420	0.1155	0.1420
0.35	0.2222	0.2509	0.2498	0.2103	0.2520
0.40	0.3333	0.3548	0.3510	0.3694	0.3614
0.45	0.4444	0.4573	0.4517	0.4157	0.4706
0.50	0.5556	0.5712	0.5674	0.5291	0.5799
0.55	0.6667	0.6765	0.6737	0.6918	0.6809
0.60	0.7778	0.7545	0.7526	0.8629	0.7550
0.65	0.8889	0.7699	0.7686	0.9597	0.7667
0.70	1.0000	0.6543	0.6500	0.6833	0.6501
0.75	0.0000	0.3084	0.2999	0.1445	0.3106
0.80	0.0000	0.0274	0.0274	0.0028	0.0308
0.85	0.0000	0.0000	0.0001	0.0000	0.0001
0.90	0.0000	0.0000	0.0000	0.0000	0.0000

(3.2b) After 40 time steps (t=0.5)

x	Exact	Modified	Refined	MacCormack	FTBS
0.20	0.0000	0.0000	0.0000	-0.0016	0.0000
0.25	0.0000	0.0000	0.0000	-0.0356	0.0000
0.30	0.0667	0.0991	0.0991	0.0844	0.0991
0.35	0.1333	0.1666	0.1663	0.1007	0.1669
0.40	0.2000	0.2278	0.2247	0.2160	0.2330
0.45	0.2667	0.2894	0.2820	0.2936	0.2987
0.50	0.3333	0.3555	0.3499	0.3387	0.3642
0.55	0.4000	0.4206	0.4149	0.3549	0.4294
0.60	0.4667	0.4857	0.4809	0.5200	0.4936
0.65	0.5333	0.5521	0.5490	0.4868	0.5553
0.70	0.6000	0.6121	0.6098	0.3929	0.6109
0.75	0.6667	0.6496	0.6474	0.8586	0.6494
0.80	0.7333	0.6246	0.6211	0.9675	0.6245
0.85	0.8000	0.3999	0.3932	0.2724	0.4006
0.90	0.0000	0.0727	0.0671	0.0067	0.0733
0.95	0.0000	0.0008	0.0007	0.0000	0.0009
1.00	0.0000	0.0000	0.0000	0.0000	0.0000

(3.2c) after 80 time steps (t=1.0)

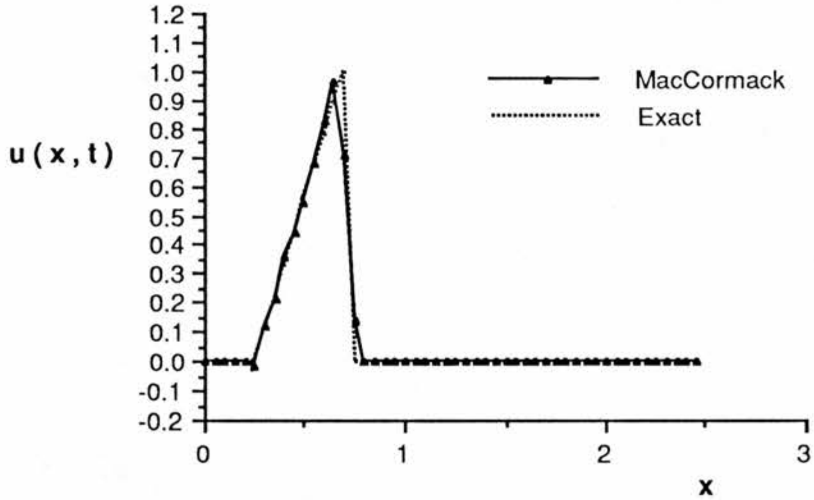
x	Exact	Modified	Refined	MacCormack	FTBS
0.20	0.0000	0.0000	0.0000	-0.0064	0.0000
0.25	0.0000	0.0000	0.0000	-0.0509	0.0000
0.30	0.0400	0.0661	0.0661	0.0756	0.0661
0.35	0.0800	0.1085	0.1084	0.0564	0.1086
0.40	0.1200	0.1458	0.1442	0.1170	0.1492
0.45	0.1600	0.1833	0.1789	0.1354	0.1891
0.50	0.2000	0.2230	0.2188	0.2254	0.2288
0.55	0.2400	0.2618	0.2560	0.2980	0.2683
0.60	0.2800	0.3007	0.2939	0.2489	0.3077
0.65	0.3200	0.3393	0.3316	0.2231	0.3470
0.70	0.3600	0.3777	0.3686	0.4085	0.3860
0.75	0.4000	0.4159	0.4078	0.5407	0.4248
0.80	0.4400	0.4566	0.4520	0.3191	0.4626
0.85	0.4800	0.4980	0.4959	0.1236	0.4984
0.90	0.5200	0.5274	0.5255	0.3377	0.5264
0.95	0.5600	0.5232	0.5196	0.8365	0.5216
1.00	0.6000	0.3968	0.3876	0.6610	0.3944
1.05	0.0000	0.1175	0.1086	0.1052	0.1172
1.10	0.0000	0.0039	0.0032	0.0009	0.0039
1.15	0.0000	0.0000	0.0000	0.0000	0.0000

Figure 3.3a [(1),(2),(3)]

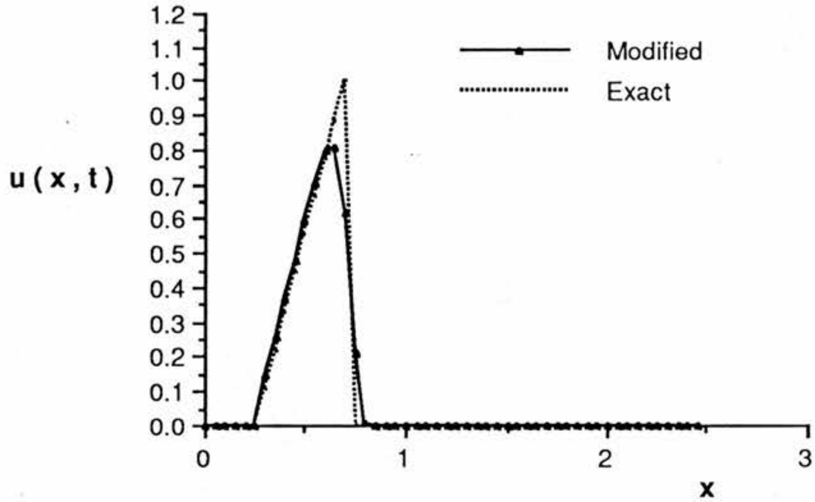
Numerical Solution of the IVP (3.30) and (3.31) When $\frac{k}{h} = 0.5$

After 8 Time Steps (t=0.2)

(1)



(2)



(3)

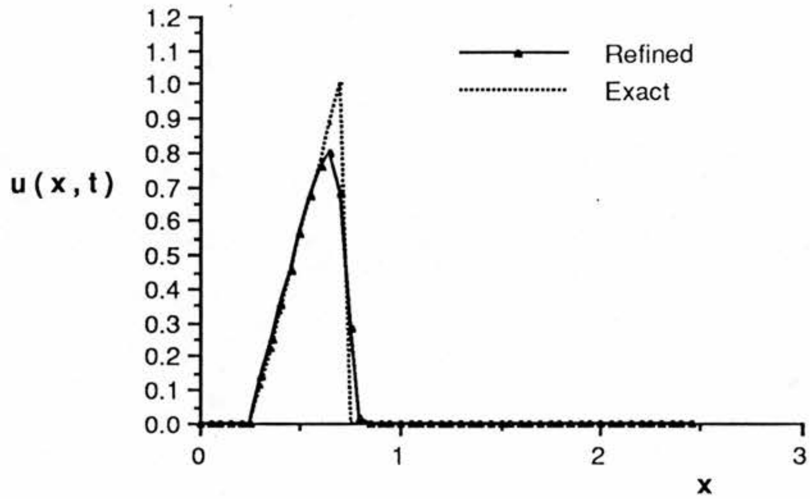
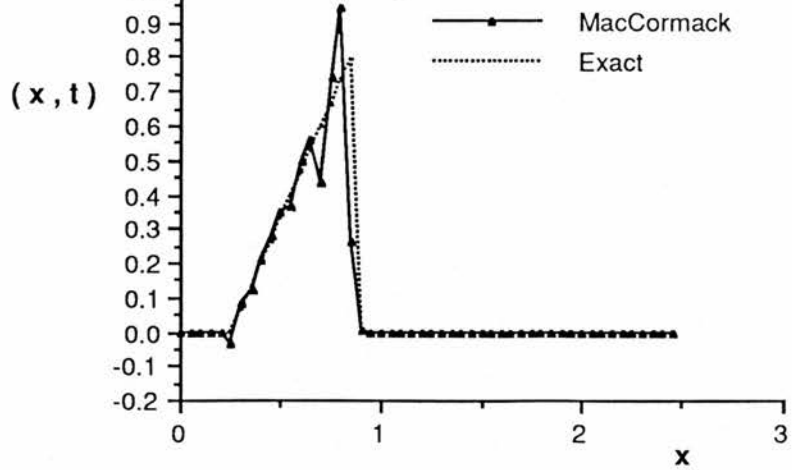


Figure 3.3b [(4),(5),(6)]

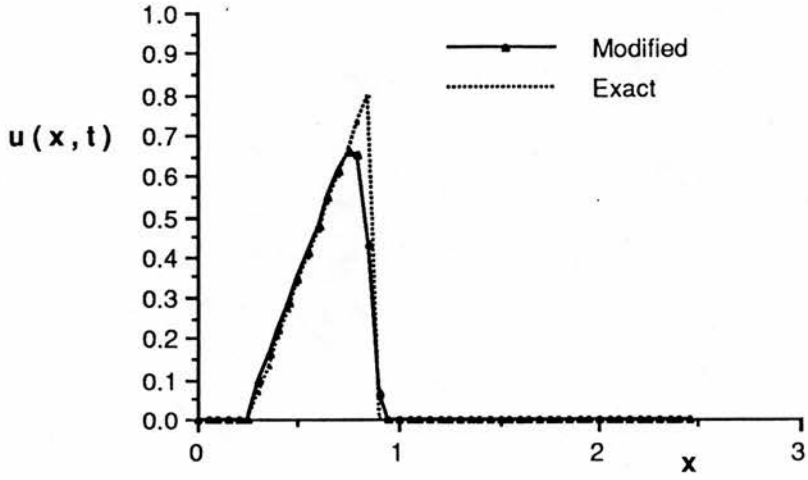
Numerical Solution of the IVP (3.30) and (3.31) When $\frac{k}{h} = 0.5$

After 20 Time Steps (t=0.5)

(4)



(5)


1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
u (x , t)
Modified
Exact
0
1
2
3
x

(6)

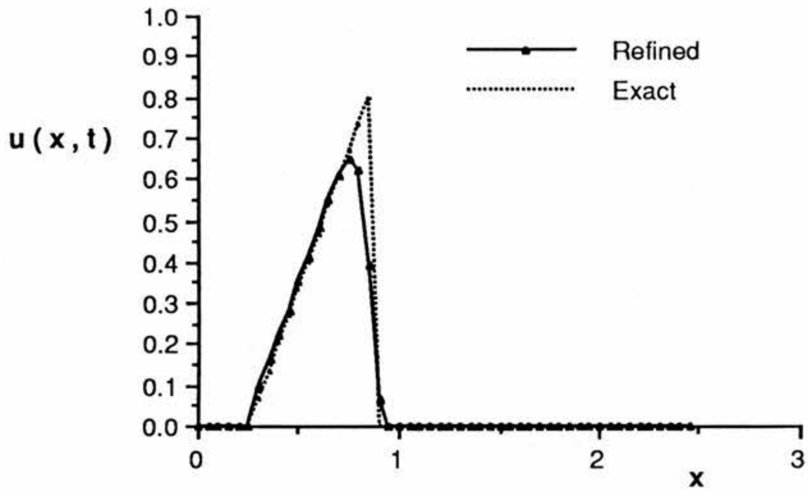
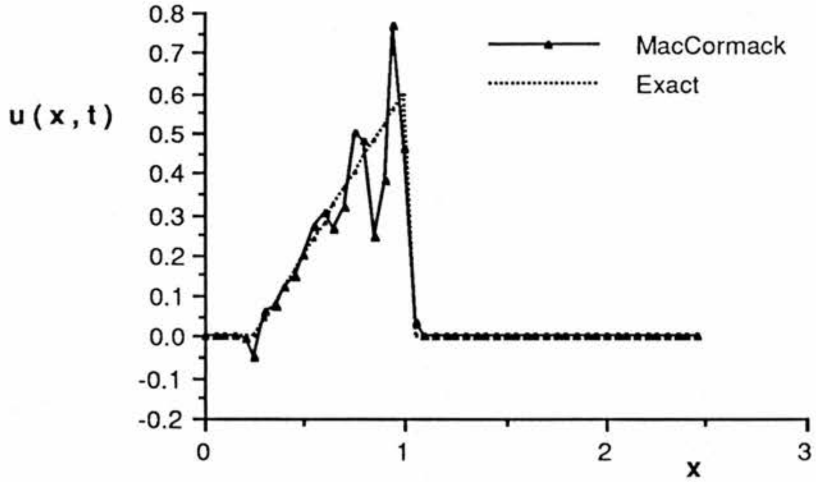

1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
u (x , t)
Refined
Exact
0
1
2
3
x

Figure 3.3c [(7),(8),(9)]

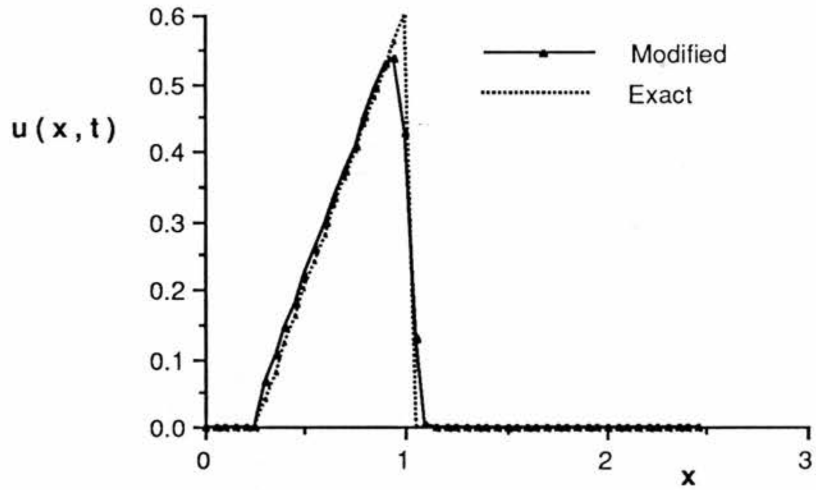
Numerical Solution of the IVP (3.30) and (3.31) When $\frac{k}{h} = 0.5$

After 40 Time Steps (t=1.0)

(7)



(8)



(9)

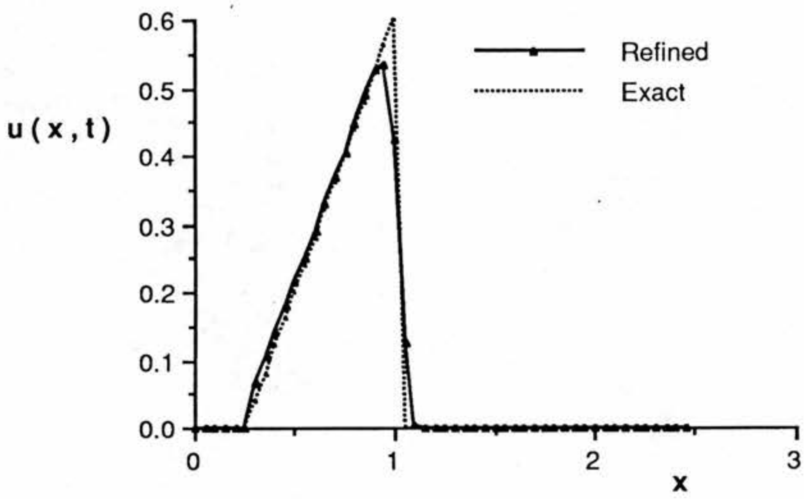
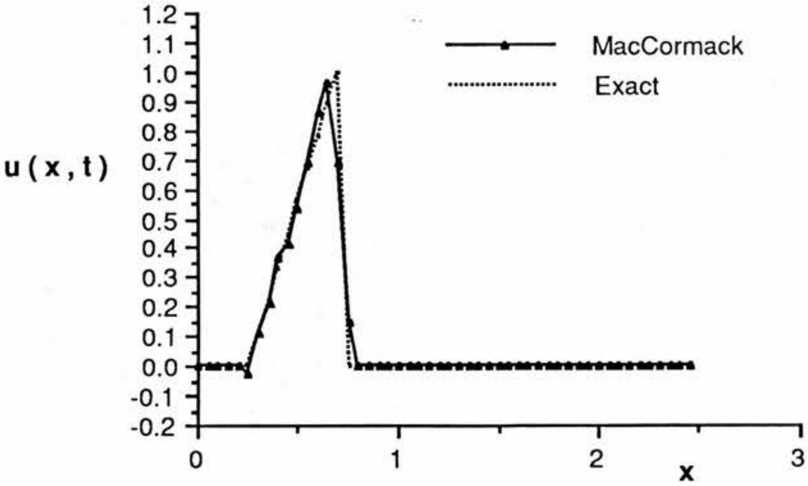


Figure 3.4a [(10),(11),(12)]

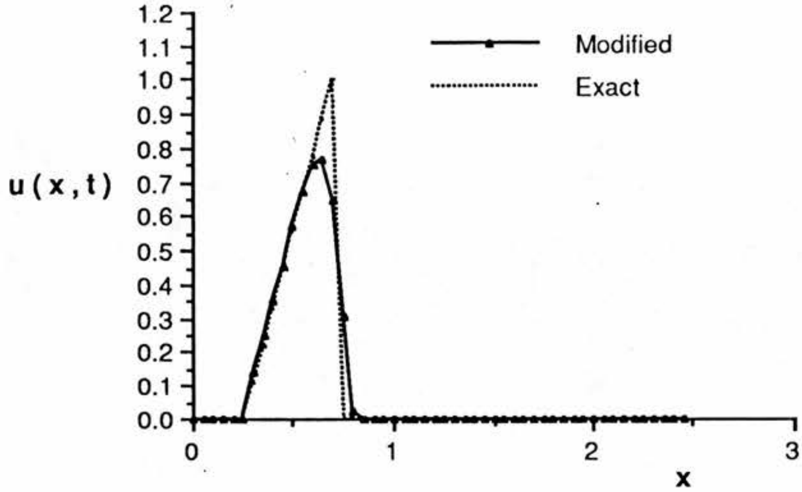
Numerical Solution of the IVP (3.30) and (3.31) When $\frac{k}{h} = 0.25$

After 16 Time Steps (t=0.2)

(10)



(11)


Modified
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u (x , t)
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1.1
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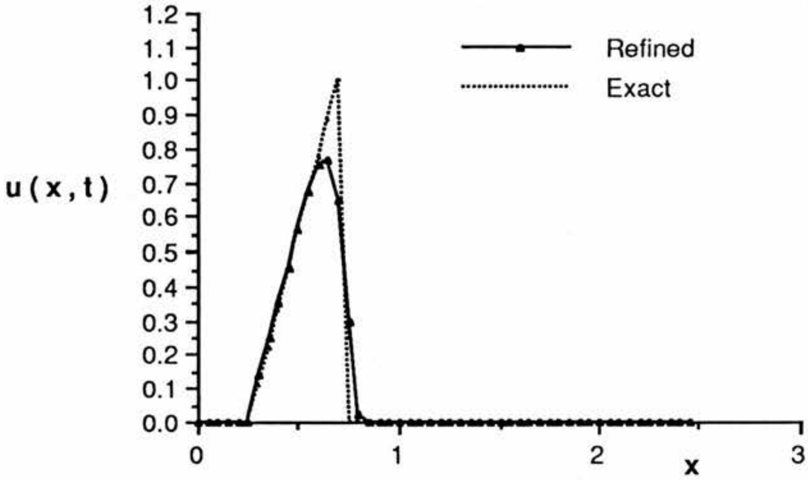
x

(12)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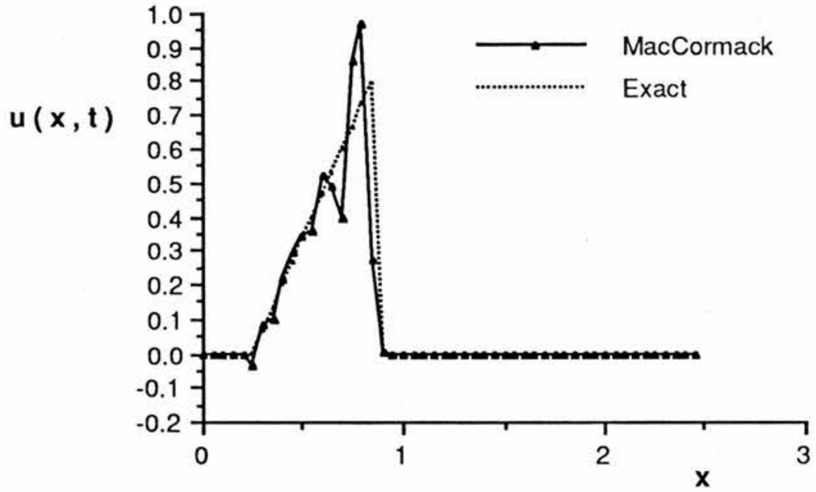
0.2
0.1
0.0
0
1
2
3
x

Figure 3.4b [(13),(14),(15)]

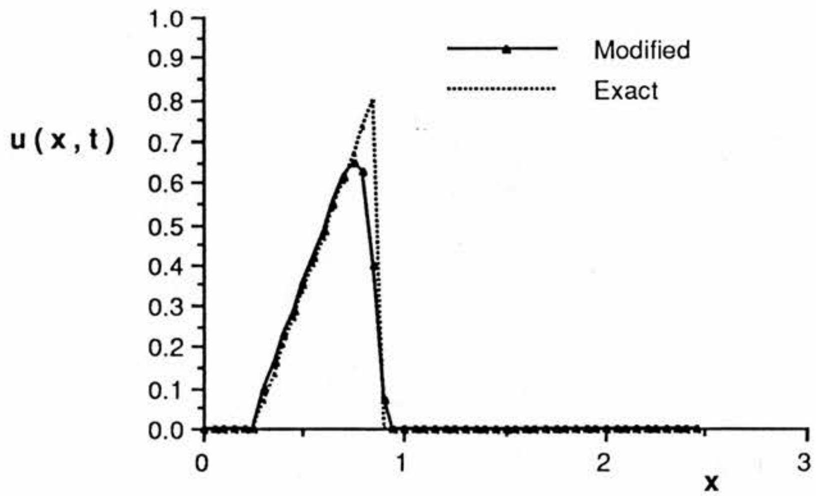
Numerical Solution of the IVP (3.30) and (3.31) When $\frac{k}{h} = 0.25$

After 40 Time Steps (t=0.5)

(13)



(14)



(15)

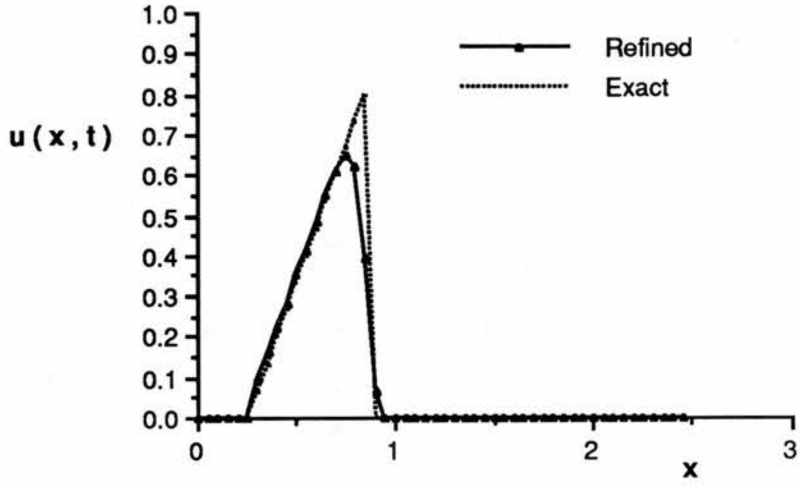
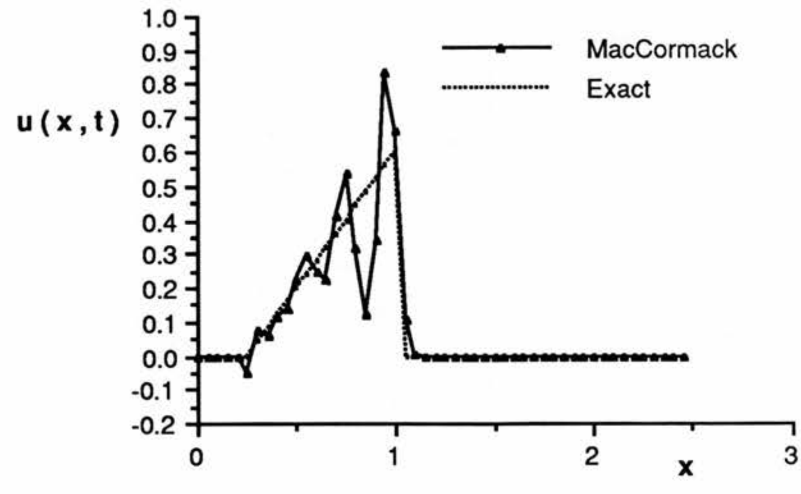


Figure 3.4c [(16),(17),(18)]

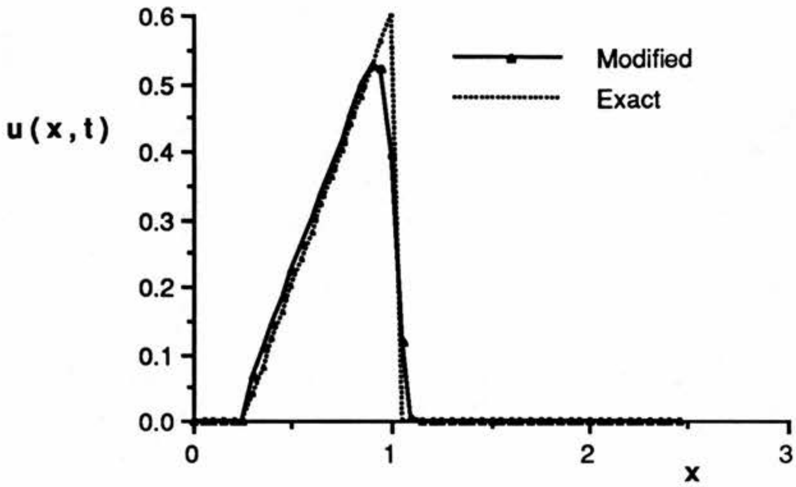
Numerical Solution of the IVP (3.30) and (3.31) When $\frac{k}{h} = 0.25$

After 80 Time Steps (t=1.0)

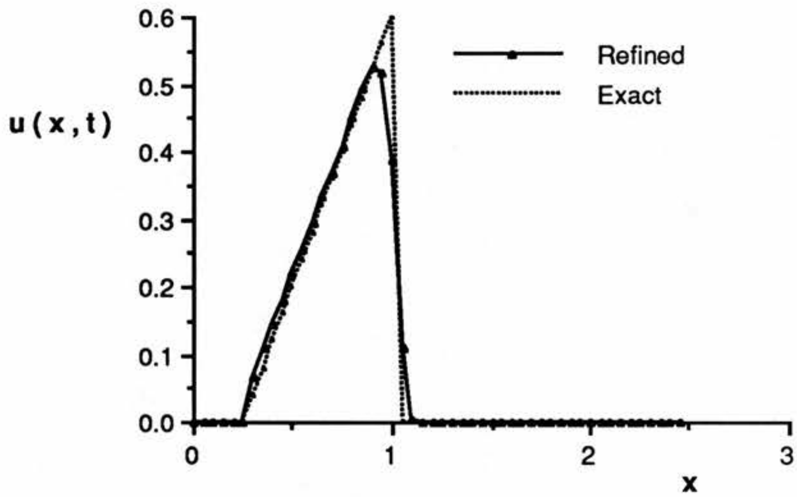
(16)



(17)


0.6
0.5
0.4
0.3
0.2
0.1
0.0
u(x,t)
Modified
Exact
0
1
2
3
x

(18)


0.6
0.5
0.4
0.3
0.2
0.1
0.0
u(x,t)
Refined
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x

CHAPTER FOUR

Total Variation Diminishing (TVD) Finite-Difference Schemes

4.1 Introduction :

The finite-difference methods with second order accuracy, such as the Lax-Wendroff scheme, usually work well and are thus preferable when approximating a smooth solution of either the linear or non-linear advection equation. However, they often produce oscillations when the solution has a discontinuity. Those oscillations in the numerical solution, which are responsible for the poor representations of the discontinuity, are in many cases just a minor annoyance, but in some other situations they can generate non-linear instability in the numerical process, and it may result in convergence to a weak solution that does not satisfy the entropy condition (3.34b). Thus, it is desirable to try to remove those oscillations, and to obtain suitable finite-difference methods that do not produce spurious oscillations when the solution is discontinuous. That is why we are interested in properties like monotonicity and monotonicity preserving which we are going to identify and discuss in the next section.

4.2 Monotonicity Preserving

Definition 1 : The finite-difference equation of the form

$$u_q^{p+1} = G\,(\,u_{q-m}^p\,,\,u_{q-m+1}^p,\,\ldots\,,\,u_{q+m}^p\,) \qquad (4.1)$$

is monotone provided

$$\frac{\partial G}{\partial u_j^p} \geq 0 \qquad \text{for } j = q\text{-}m\,,\,\ldots\,,\,q\text{+}m$$

where G is some finite-difference operator.

Definition 2 : The numerical solution u_q^p is called monotone increasing if $n < q$ implies $u_n^p \leq u_q^p$, and the numerical solution is monotone decreasing if $n < q$ implies $u_n^p \geq u_q^p$. The solution is said to be monotone if it is either monotone increasing or monotone decreasing.

Definition 3 : The finite-difference operator G in equation (4.1) is monotonicity preserving provided all solutions of (4.1) have the following property that if u_q^p is monotone then u_q^{p+1} is monotone of the same type.

Theorem 4.1 : A monotone finite-difference method of the form (4.1) is monotonicity preserving.

Proof [3] : As in equation (4.1) we define

$$u_q^{p+1} = G (u_{q-1}^p , u_q^p , u_{q+1}^p) \qquad (4.2)$$

where u is any monotone mesh function, and

$$u_{q+1}^{p+1} = G (u_q^p , u_{q+1}^p , u_{q+2}^p)$$

By the Mean-Value theorem, there exists a number λ where $0 < \lambda < 1$ such that

$$\begin{aligned} u_{q+1}^{p+1} - u_q^{p+1} &= G (u_q^p , u_{q+1}^p , u_{q+2}^p) - G (u_{q-1}^p , u_q^p , u_{q+1}^p) \\ &= \frac{\partial G}{\partial u_{q-1}^p} (\overline{u}_{q-1}^p , \overline{u}_q^p , \overline{u}_{q+1}^p) [u_q^p - u_{q-1}^p] + \\ &\quad \frac{\partial G}{\partial u_q^p} (\overline{u}_{q-1}^p , \overline{u}_q^p , \overline{u}_{q+1}^p) [u_{q+1}^p - u_q^p] + \\ &\quad \frac{\partial G}{\partial u_{q+1}^p} (\overline{u}_{q-1}^p , \overline{u}_q^p , \overline{u}_{q+1}^p) [u_{q+2}^p - u_{q+1}^p] \end{aligned}$$

where

$$(\overline{u}_{q-1}^p , \overline{u}_q^p , \overline{u}_{q+1}^p) = (\lambda u_q^p + (1- \lambda) u_{q-1}^p , \lambda u_{q+1}^p + (1- \lambda) u_q^p ,$$

$$\lambda u_{q+2}^p + (1- \lambda) u_{q+1}^p)$$

Now since G is monotone, all of the partial derivatives are non-negative. Also $[u_q^p - u_{q-1}^p]$, $[u_{q+1}^p - u_q^p]$ and $[u_{q+2}^p - u_{q+1}^p]$ are all positive (or negative), so $[u_{q+1}^{p+1} - u_q^{p+1}]$ and $[u_{q+1}^p - u_q^p]$ have the same sign. Thus G preserves the monotonicity of u , and the proof is completed.

In the linear case with constant coefficient, monotone is equivalent to monotonicity preserving, i.e. the converse of the theorem 4.1 is true. While in the non-linear case the converse of this theorem is not true, this means that the class of monotonicity preserving finite-difference methods is larger than the class of monotone finite-difference methods. Theorem 4.1 shows that the finite-difference method that preserves monotonicity will not have oscillations provided that the initial-condition is monotonic. Since the methods of order of accuracy greater than one often produce oscillations, a monotone finite-difference method is necessarily first order accurate. For example the Lax-Friedrichs and the FTBS are monotone schemes for the linear and non-linear cases, and they are also monotonicity preserving provided the CFL condition is satisfied. The Lax-Wendroff method is neither monotone nor monotonicity preserving.

Theorem 4.2 : Consider equation (4.1) to be a monotone finite-difference method in conservation law form. If the solution of this finite-difference method u_q^p converges boundedly almost everywhere to some function $u(x,t)$, then $u(x,t)$ is a weak solution and the entropy condition (3.34b) is satisfied at all discontinuities of u .

As we mentioned earlier, if the solution of the advection equation, (either linear or non-linear), is smooth then the higher order schemes (such as Lax-Wendroff) are suitable. But if the solution contains one or more shocks, the second order accurate methods exhibit spurious oscillations, and the monotone difference methods should be considered. Monotonicity preserving methods such as FTBS and Lax-Friedrichs , which are of first order accuracy, always provide poor representation of discontinuities as they suffer from numerical diffusion. So monotonicity preserving alone does not guarantee the accuracy. However, it is desirable to attempt to construct a second order finite-difference method that preserves monotonicity in the locality of discontinuity. In recent years there has been a great deal of effort put into constructing this kind of numerical method. For example the TVD scheme which we are going to discuss in the next section.

4.3 The TVD Scheme

One of the main advantages of the second order accurate Total Variation Diminishing methods (TVD) is that they do not produce spurious oscillations across the discontinuities. To understand the idea of the TVD schemes let us first consider the following initial-value problem :

$$\frac{\partial u}{\partial t} + \frac{\partial F(u)}{\partial x} = 0 \qquad , \frac{\partial F}{\partial u} = a(u), \ t > 0 \tag{4.3}$$

$$u(x,0) = f(x,t) \qquad -\infty < x < \infty \tag{4.4}$$

where f(x) is assumed to have bounded total variation. A weak solution of this scalar initial-value problem has the following monotonicity property :

i) No new extrema in u(x) may be created.

ii) The value of a local minimum is nondecreasing and the value of a local maximum is nonincreasing.

The concept of the total variation in x of the numerical solution u(x,t) first introduced by Harten [6], is denoted by TV(u(t)) and defined as :

$$TV(u) = \sum_{q=0}^{m-1} \left| u_{q+1}^{p} - u_{q}^{p} \right| \tag{4.5}$$

From the above monotonicity property it follows that the total variation in x does not increase with t , that is

$$TV(u(t_1)) \geq TV(u(t_2)) \quad \text{for all} \quad t_2 \geq t_1 \tag{4.6}$$

Now we consider an explicit finite-difference scheme in conservation law form which approximate (4.3) and (4.4) denoted by

$$u^{p+1} = L \, . \, u^{p} \tag{4.7}$$

A numerical scheme is called <u>Total Variation Diminishing (TVD)</u> if :

$$TV(\, u^{p+1} \,) = TV(\, L \, . \, u^{p} \,) \leq TV(\, u^{p}) \tag{4.8}$$

Furthermore, a finite-difference scheme is called monotonicity preserving if the finite-difference operator L is monotonicity preserving, that is if u_q^p is a monotone grid function, then so is L . u^p . Equation (4.8) also provides a bound on the total variation of the solution.

The following two theorems are due to Harten [6]

Theorem (4.3) : A monotone scheme is TVD .

Theorem (4.4) : A TVD scheme is monotonicity preserving .

The above results say that the TVD methods will not produce oscillations.

Now the general scheme (4.7) can be rewritten in the form

$$u_q^{p+1} = u_q^p - C_q \nabla u_q^p + D_q \Delta u_q^p \tag{4.9}$$

and define

$$C_q = C(u_q^p) \text{ and } D_q = D(u_q^p)$$

where C_q and D_q are some bounded functions of u^p . The choice of those functions is not unique. A method of the form (4.9) can be proved to be TVD if the conditions on coefficients C_q and D_q satisfy the following lemma :

Lemma 4.1 (Harten): If the coefficients C and D in (4.9) satisfy the inqualities

$$C_q \geq 0 \text{ , } D_q \geq 0 \quad \text{for all q}$$

and $$0 \leq C_q + D_{q-1} \leq 1 \quad \text{for all q} \tag{4.10}$$

then the scheme (4.9) is Total Variation Diminishing.

The FTBS and the Lax-Friedrichs schemes are examples of TVD methods. While The Lax-Wendroff method is not. So the TVD property of the scheme does not ensure the accuracy of that scheme, nor good resolution of discontinuities. In recent research, the idea of constructing a class of high resolution second order TVD schemes has been developed. In the following section we are going to use Lemma (4.1) to derive such a scheme.

4.4 <u>The Linear Case</u>

Consider first the scalar linear advection equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \qquad a > 0 \text{ (a is constant)} \tag{4.11}$$

and we shall consider that the scheme for this equation can be written in the form of (4.9). The second-order Lax-Wendroff scheme may be written as:

$$u_q^{p+1} = u_q^p - a\mu\nabla u_q^p - \nabla\{\frac{1}{2}(1 - a\mu)\, a\mu\,\Delta u_q^p\} \tag{4.12}$$

where $\mu = \frac{\Delta t}{\Delta x} = \frac{k}{h}$. The formula (4.12) is seen as a first-order scheme (FTBS) :

$$u_q^{p+1} = u_q^p - a\mu\nabla u_q^p \tag{4.13}$$

with additional term (correction term)

$$- \nabla \{ \frac{1}{2} (1 - a \mu) a \mu \Delta u_q^p \} \qquad (4.14)$$

added. Since the Lax-Wendroff scheme is not TVD, some form of limiter Φ_q was introduced by Sweby [21], into the correction term. Therefore equation (4.12) takes the form

$$u_q^{p+1} = u_q^p - a \mu \nabla u_q^p - \nabla \{ \frac{1}{2} \ \Phi_q (1 - a \mu) a \mu \Delta u_q^p \} \qquad (4.15)$$

which is equation (4.9) with

$$C_q = a \mu \ \{1 + \frac{1}{2} \left(\frac{1 - a \mu}{\nabla u_q^p} \right) \nabla [\Phi_q \nabla u_q^p] \}$$

and $\quad D_q = 0$

For this linear case we assume that Φ_q is a function of

$$r_q^p = \frac{\nabla u_q^p}{\Delta u_q^p} \ ,$$

then

$$C_q = a \mu \left\{ \{1 + \frac{1}{2} \ (1 - a \mu) \left[\frac{\Phi(r_q^p)}{r_q^p} - \Phi(r_{q-1}^p) \right] \right\}$$

So for C_q and D_q to satisfy the TVD inqualities (4.10) they reduce to

$$0 \leq C_q \leq 1 \qquad \text{for all } q$$

then

$$\frac{-2}{1 - a\mu} \leq \left[\frac{\Phi(r_q^p)}{r_q^p} - \Phi\left(r^p_{q-1}\right) \right] \leq \frac{2}{a\mu} \qquad \text{for all } q \text{ ,}$$

and this is satisfied when

$$\left| \frac{\Phi(r_q^p)}{r_q^p} - \Phi\left(r^p_{q-1}\right) \right| \leq 2 \tag{4.16}$$

where $\mu \leq 1$. Sweby also suggested that if in addition to requiring $\Phi(r)$ to be nonnegative he also insisted on

$$\Phi(r) = 0 \quad \text{for } r_q^p \leq 0$$

then equation (4.16) is satisfied when either

$$0 \leq \frac{\Phi(r)}{r} \leq 2 \quad \text{and} \quad 0 \leq \Phi(r) \leq 2 \tag{4.17}$$

which are sufficient conditions on Φ for the scheme to be TVD. Now if $\Phi(r) = 0$ formula (4.15) reduces to the FTBS scheme (4.13) . If $\Phi(r) = 1$ it reduces to the Lax-Wendroff method (4.12). If we choose $\Phi = r$, the formula (4.15) becomes

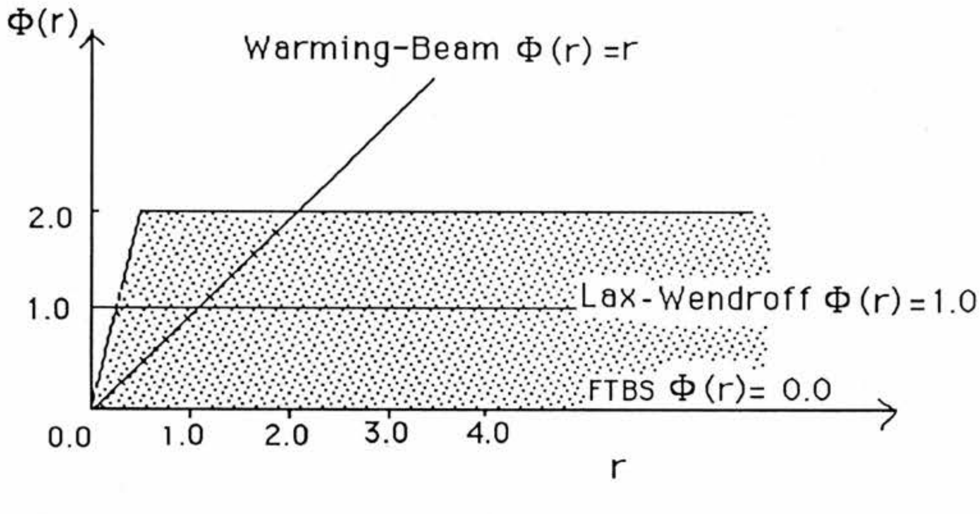
$$u_q^{p+1} = u_q^p - a\,\mu\,\nabla\, u_q^p - \nabla\,\{\,\frac{1}{2}\,(1 - a\mu)\, a\mu\,\nabla\, u_q^p\}$$

or

$$u_q^{p+1} = u_q^p - a\,\mu\,\nabla u_q^p - \frac{1}{2}\{(1 - a\mu)\,a\mu\,\nabla^2 u_q^p\}$$

which is the second-order Warming-Beam scheme [22] , and it based on the 5 grid points (q-2,p), (q-1,p), (q,p), (q+1,p), and (q,p+1) .

The region defined by equation (4.17) is shown in figure 4.1 along with $\Phi(r)$, corresponding to the FTBS, Lax-Wendroff and the Warming-Beam schemes, where the dotted area is guaranteed to be TVD region.



TVD region

Figure 4.1

The Warming-Beam scheme will not necessarily be TVD as $\Phi = r$ does not lie entirely within the dotted region. We note from figure

4.1 that both second-order methods pass through the point $\Phi(1) = 1$, which is a general requirement for the accuracy of any second-order method, and also note that any second-order method using at most those above 5 grid points must be a weighted average of Lax-Wendroff scheme and Warming-Beam scheme [13]. The arithmetic average of the two schemes (Fromm's scheme [21]) is

$$\Phi(r) = 1 - \theta(r)\ \Phi^{LW}(r) + \theta(r)\ \Phi^{WB}(r) \tag{4.19}$$

where the smoothness monitor $\theta_q = \dfrac{\Delta u_q}{\nabla u_q}$, and $0 \le \theta(r) \le 1$ is the internal average. Since $\Phi^{LW}(r) = 1$ and $\Phi^{WB}(r) = r$, equation (4.19) reduces to

$$\Phi(r) = 1 - \theta(r) + r\,\theta(r)$$

$$= 1 + \theta(r)\ (r - 1)$$

and now $\Phi(r)$ is confined to lie in the dotted area in figure 4.2 . This means that a sufficient condition for a second order TVD scheme is for $\Phi(r)$ to lie in that dotted region.

Sweby has shown that the average scheme that was proposed by Van Leer [13] corresponds to the limiter

$$\Phi^{VL}(r) = \frac{|r| + r}{1 + |r|}$$

Note that

$$\Phi^{VL}(r) = \begin{cases} 0 & r \le 0 \\ \dfrac{2r}{1 + r} & r > 0 \end{cases}$$

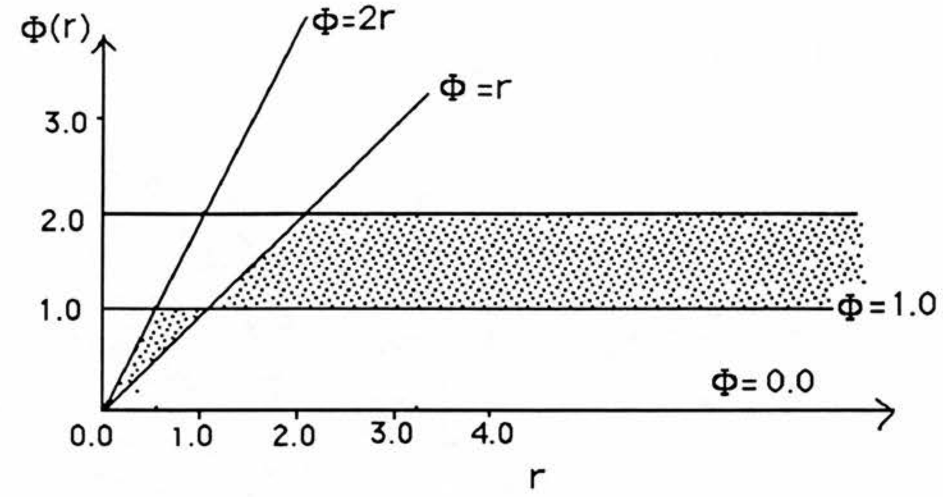
that is for $r > 0$ the scheme is smooth curve lying in the dotted area in figure 4.2 , and for $r \leq 0$ the scheme reduces to the first order FTBS scheme (1.12). Now assuming that when $\nabla u_q^p = \Delta u_q^p = 0$, r is defined to be 1 then we may have the following cases for monotonic data :

i) $r_q^p > 0$ and $r_{q-1}^p > 0$ the formula is second order and monotonicity preserving.

ii) $r_q^p = 0$ and $r_{q-1}^p > 0$ the formula is first order.

iii) $r_q^p > 0$ and $r_{q-1}^p = 0$ the formula is first order.

Note that $r_q^p \geq 0$ for monotonic data and consecutive values of r can not be zero by definition, so the only type of grid points at which a first-order formula is applied are shown in figure 4.3, and when a distribution changes from a constant the lower order formula is applied at the 2 points where the change takes place. The Refined scheme described in chapter two essentially trying to do the same thing by monitoring the shape of the solution and operating a simple switch.



Second Order TVD Region

Figure 4.2

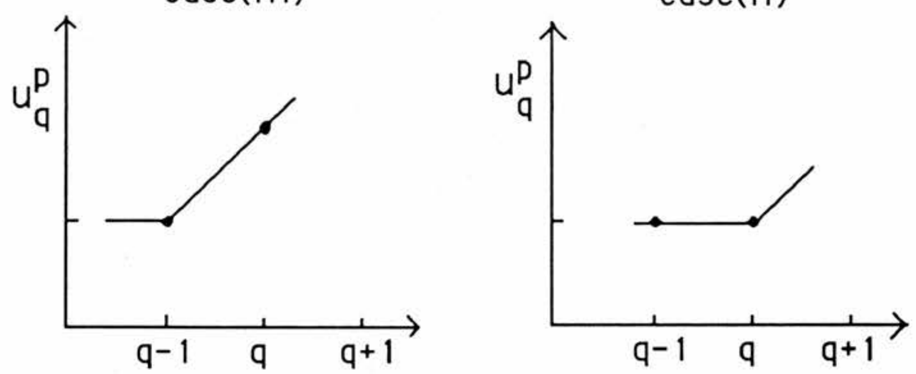


Figure 4.3

4.5 Davis' Algorithm of Sweby Scheme [2]

Since many computed codes are based on the Lax-Wendroff method or its variants, and as this method does not need a determination of an upwind direction, Davis suggested the possibility of adding a correction term to those existing codes to produce a TVD scheme.

For the linear advection equation (4.11) Davis proposed the scheme

$$u_q^{p+1} = u_q^p - a\,\mu\,\nabla u_q^p - \nabla \{ \frac{1}{2}(1 - a\mu)\, a\mu\, \nabla u_q^p \} + K_q(r_q)\,\Delta u_q^p - K_q(r_q)\nabla u_q^p \qquad \text{......(4.20)}$$

which reduces to Sweby's scheme (4.15) by choosing

$$K_q(r) = \frac{a\mu}{2}(1 - a\mu)[1 - \Phi(r)]$$

Now for the non-linear advection equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(F(u)) = 0 \qquad , \text{ where } \frac{\partial F}{\partial u} = a(u) \qquad (4.21)$$

we add a correction term of the form

$$K_q^+(r_q)\,\Delta u_q^p - K_q^-(r_{q-1})\,\nabla u_q^p \qquad (4.22)$$

to the existing MacCormack method codes, with

$$K_q^{\mp}(r) = \frac{a^{\mp}}{2}\mu\,(1- a^{\mp}\mu\,)[\,1- \Phi(r)\,]$$

where

$$a^{+} = \frac{\Delta F_q^p}{\Delta u_q^p} \qquad , \Delta u_q^p \neq 0$$

$$a^{-} = \frac{\nabla F_q^p}{\nabla u_q^p} \qquad , \nabla u_q^p \neq 0$$

and $a^{\mp} = a(u_q^p)$ when $\Delta u_q^p = 0$ or $\nabla u_q^p = 0$. The choice of $\Phi(r)$ here is

$$\Phi(r) = \mathrm{Max}(\ \mathrm{Min}\ (r,1),0\ ) \qquad (4.23)$$

so that:

For $r \leq 0$ the approximation becomes the FTBS scheme.

For $r \geq 1$ the approximation becomes the Lax-Wendroff scheme.

For $0 < r < 1$ the approximation becomes the Warming-Beam scheme.

Here again since the solution is monotonic, r is nonnegative, and if the scheme is TVD, r will never become negative. So the summary of the computation process is :

Stage 2 : Calculate Φ_q^p (equation (4.23)).

Stage 3 : Calculate the second step of MacCormack (equation(3.18b)).

Stage 4 : Add correction term (equation (4.22)).

The four stages require just two sweep of grid. The first with decreasing q and the second with increasing q . Stages 1 and 2 are completed in the first sweep. Stages 3 and 4 are performed in the second sweep. The algorithm for determining Φ is as described above. The process in terms of a flow chart is shown in figure 4.4.

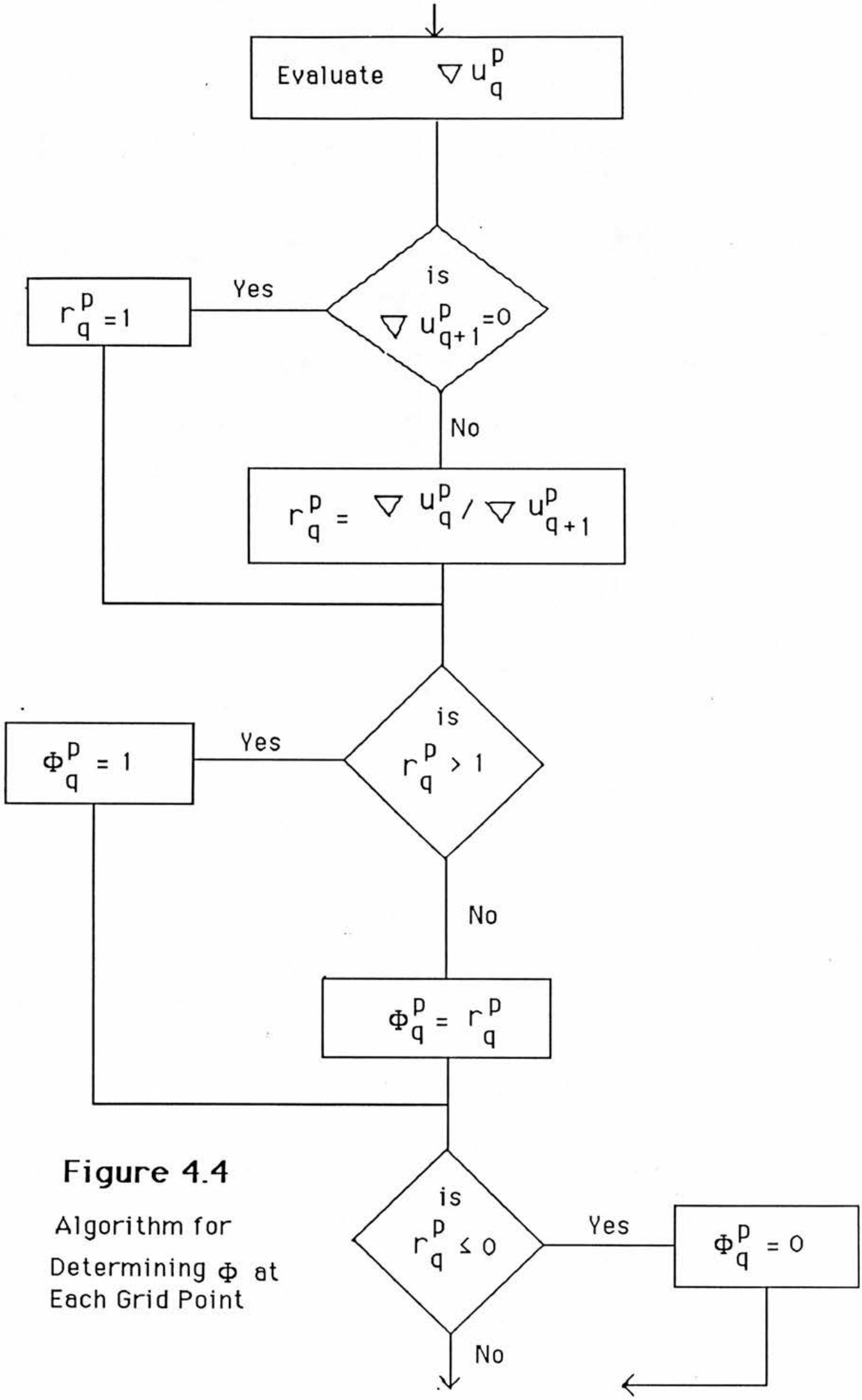


Figure 4.4

Algorithm for Determining Φ at Each Grid Point

4.6 Numerical Examples

Linear Case :

Consider the linear advection equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \qquad , a > 0 , a \text{ is constant} \tag{4.23}$$

Subject to the initial condition

$$u(x,0) = \begin{cases} 1 & 0 \le x \le 0.5 \\ 0 & 0.5 < x \le 2.5 \end{cases} \tag{4.24}$$

we shall choose $\mu = \frac{k}{h} = 0.5$ where $h = 0.05$ and $a = 1.0$. The numerical solution by the TVD Sweby scheme (Davis' algorithm) for this initial-value problem compared with the exact solution and the Refined scheme, (see 2 (III)), and exact solution are shown in table 4.1 and illustrated in figure 4.5 after 20 time steps ($t = 0.5$) and after 40 time steps ($t = 1.0$).

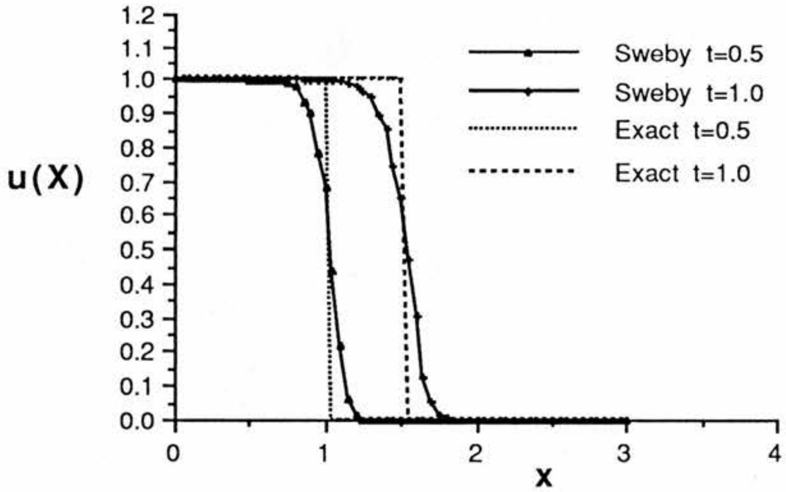
Table 4.1 : Numerical Solution of the IVP (4.23) and (4.24) by TVD.

t = 0.5				**t = 1.0**			
x	TVD	Refined	Exact	x	TVD	Refined	Exact
0.45	1.0000	1.0000	1.0000	0.90	0.9998	0.9999	1.0000
0.50	0.9999	0.9999	1.0000	0.95	0.9994	1.0003	1.0000
0.55	0.9997	1.0002	1.0000	1.00	0.9991	1.0001	1.0000
0.60	0.9994	1.0001	1.0000	1.05	0.9973	0.9994	1.0000
0.65	0.9975	0.9994	1.0000	1.10	0.9958	0.9992	1.0000
0.70	0.9958	1.0003	1.0000	1.15	0.9896	1.0012	1.0000
0.75	0.9859	1.0019	1.0000	1.20	0.9841	1.0040	1.0000
0.80	0.9771	0.9956	1.0000	1.25	0.9645	0.9975	1.0000
0.85	0.9370	0.9658	1.0000	1.30	0.9477	0.9710	1.0000
0.90	0.9021	0.9009	1.0000	1.35	0.8965	0.9230	1.0000
0.95	0.7824	0.7745	1.0000	1.40	0.8529	0.8375	1.0000
1.00	0.6832	0.5971	0.0000	1.45	0.7437	0.7152	1.0000
1.05	0.4388	0.4050	0.0000	1.50	0.6516	0.5753	0.0000
1.10	0.2163	0.2322	0.0000	1.55	0.4693	0.4278	0.0000
1.15	0.0641	0.1024	0.0000	1.60	0.3033	0.2869	0.0000
1.20	0.0166	0.0427	0.0000	1.65	0.1289	0.1723	0.0000
1.25	0.0034	0.0121	0.0000	1.70	0.0512	0.0890	0.0000
1.30	0.0005	0.0023	0.0000	1.75	0.0179	0.0378	0.0000
1.35	0.0001	0.0003	0.0000	1.80	0.0056	0.0140	0.0000
				1.85	0.0015	0.0046	0.0000
				1.90	0.0004	0.0013	0.0000

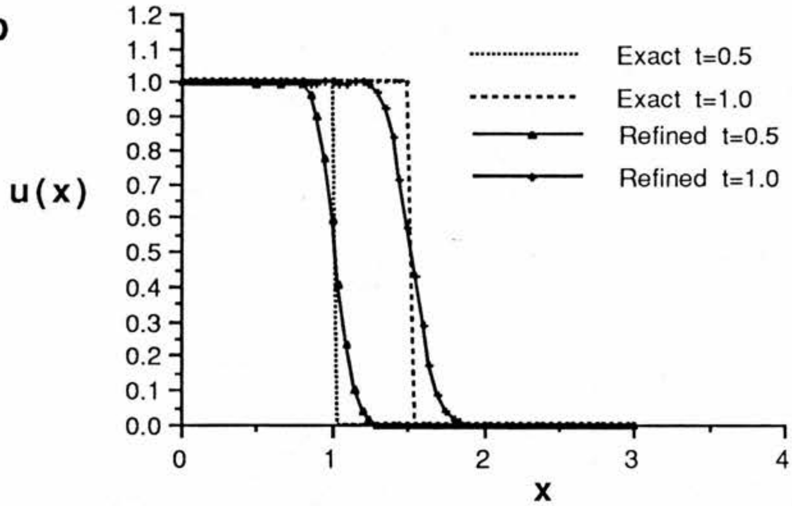
Figure 4.5 (a, b, c, d)

Numerical Solution of the IVP (4.23) and (4.24) by TVD.

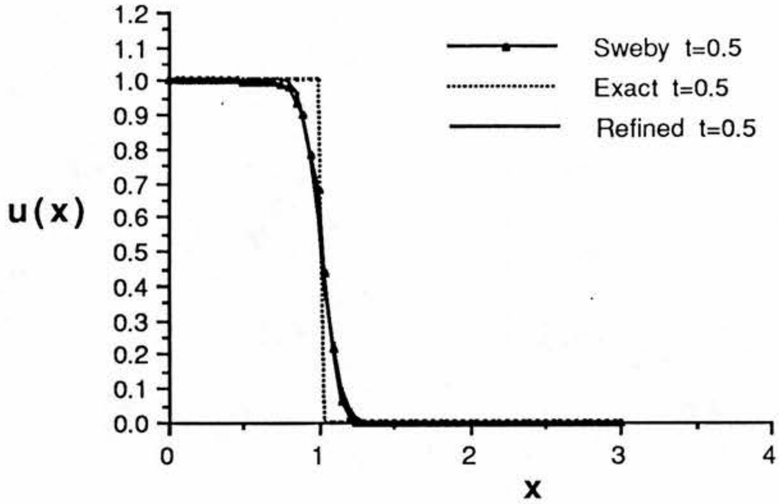
a



b



c


1.2
1.1
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
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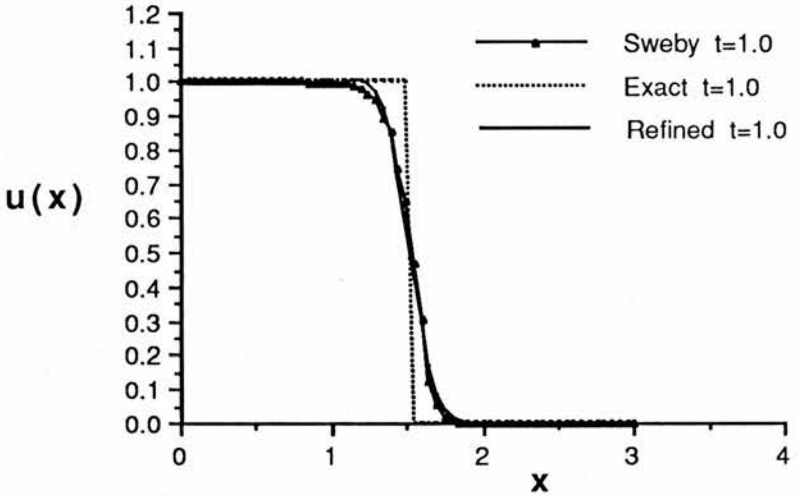
0.0
u(x)
0
1
2
3
4
x
Sweby t=0.5
Exact t=0.5
Refined t=0.5

d


1.2
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1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
u(x)
0
1
2
3
4
x
Sweby t=1.0
Exact t=1.0
Refined t=1.0

Non-Linear Case :

Consider the conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2} u^2\right) = 0 \qquad , \text{where } \frac{\partial F}{\partial u} = a(u) \qquad (2.25)$$

with the initial condition

$$u(x,0) = \begin{cases} 4x - 1 & 0.25 \le x \le 0.5 \\ 3 - 4x & 0.5 < x \le 0.75 \end{cases} \qquad (4.24)$$

and we choose $\mu = 0.5$ with $h = 0.5$. Note that $a(u) = u$. The numerical solution for the above problem for TVD Sweby scheme (Davis' algorithm) after 20 time steps ($t = 0.5$), and 40 time steps ($t = 1.0$) compared with the Refined scheme, (see 2 (III)), and the exact solution are shown in table 4.2 and illustrated in figure 4.6.

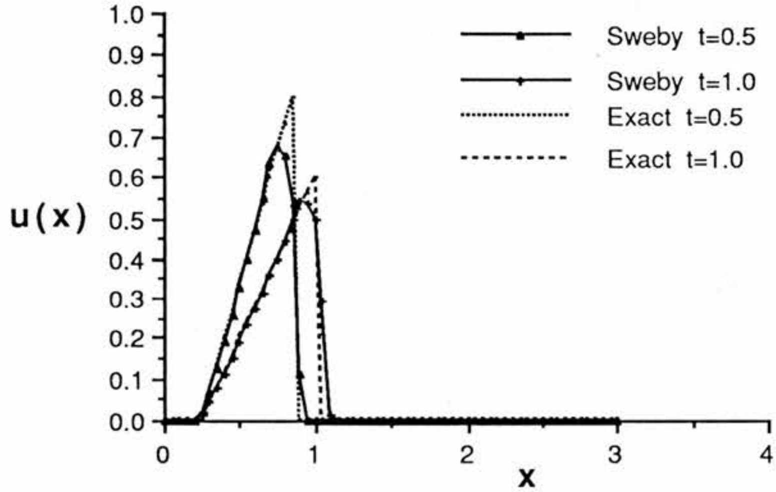
Table 4.2 : Numerical Solution of the IVP (4.25) and (4.26) by TVD.

t = 0.5				t = 1.0			
x	TVD	Refined	Exact	x	TVD	Refined	Exact
0.20	0.0007	0.0000	0.0000	0.20	0.0014	0.0000	0.0000
0.25	0.0195	0.0000	0.0000	0.25	0.0170	0.0000	0.0000
0.30	0.0638	0.0982	0.0667	0.30	0.0431	0.0654	0.0400
0.35	0.1222	0.1645	0.1333	0.35	0.0758	0.1073	0.8000
0.40	0.1877	0.2230	0.2000	0.40	0.1124	0.1426	0.1200
0.45	0.2566	0.2808	0.2667	0.45	0.1513	0.1788	0.1600
0.50	0.3273	0.3450	0.3333	0.50	0.1916	0.2142	0.2000
0.55	0.3990	0.4104	0.4000	0.55	0.2326	0.1512	0.2400
0.60	0.4721	0.4751	0.4667	0.60	0.2741	0.2897	0.2800
0.65	0.5484	0.5438	0.5333	0.65	0.3160	0.3287	0.3200
0.70	0.6365	0.6081	0.6000	0.70	0.3584	0.3670	0.3600
0.75	0.6708	0.6576	0.6667	0.75	0.4017	0.4055	0.4000
0.80	0.6514	0.6525	0.7333	0.80	0.4472	0.4469	0.4400
0.85	0.3542	0.4221	0.8000	0.85	0.4987	0.4906	0.4800
0.90	0.1096	0.0637	0.0000	0.90	0.5400	0.5257	0.5200
0.95	0.0003	0.0003	0.0000	0.95	0.5385	0.5343	0.5600
				1.00	0.4936	0.4253	0.6000
				1.05	0.2934	0.1240	0.0000
				1.10	0.0130	0.0033	0.0000

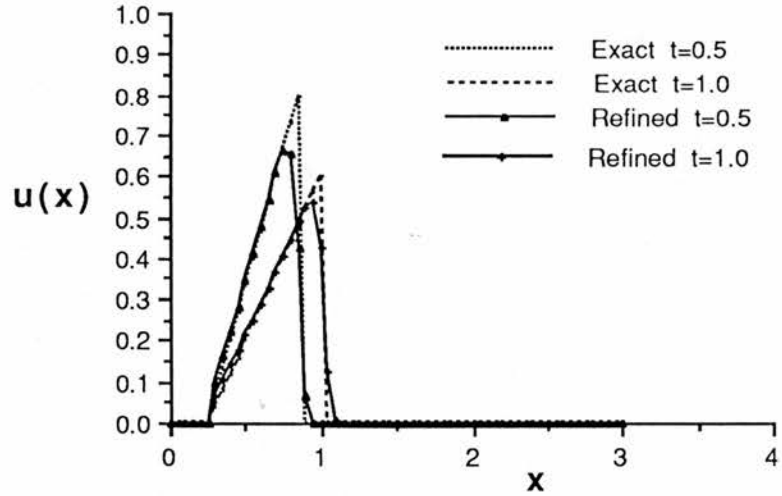
Figure 4.6 (e, f, g, h)

Numerical Solution of the IVP (4.25) and (4.26) by TVD.

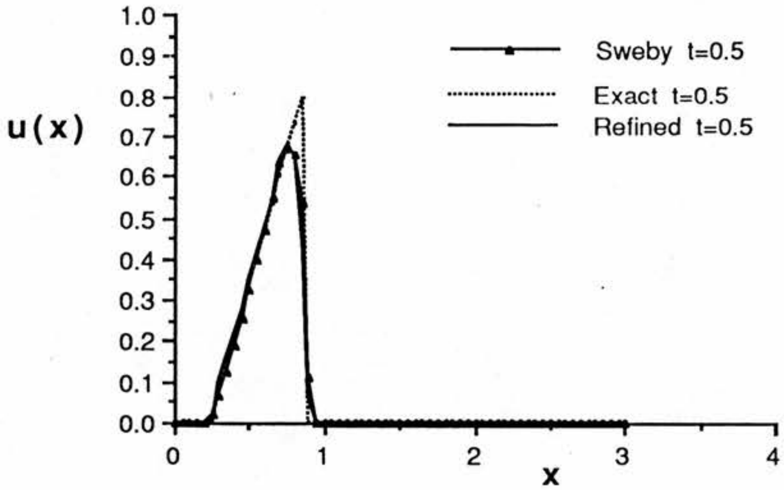
e



f

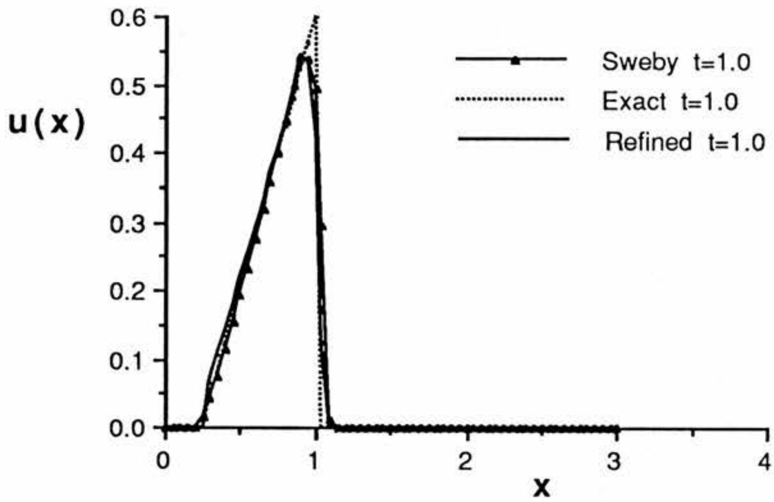


g



1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
u(x)
0
1
2
3
4
x
Sweby t=0.5
Exact t=0.5
Refined t=0.5

h



0.6
0.5
0.4
0.3
0.2
0.1
0.0
u(x)
0
1
2
3
4
x
Sweby t=1.0
Exact t=1.0
Refined t=1.0

CHAPTER FIVE

Discussion and Conclusion

5.1 Discussion of the Results :

In this thesis we have discussed some simple switching schemes for solving the linear and non-linear advection equation and compared them with a total variation diminishing finite-difference scheme (TVD).

The results obtained for the linear advection equation in chapter two suggested that the Refined scheme is a simple and effective way of removing the oscillations from the Lax-Wendroff results. Although the lower order formula is only used at a few points there is smearing out of the shock in a similar way to the FTBS method.

For the non-linear problem in chapter three the Refined scheme has a similar effect to that discussed in chapter two. The oscillations are removed but the sharp front is diffused. For both $\mu = \frac{1}{4}$ and $\mu = \frac{1}{2}$ at $t = 0.2$ (tables 3.1a , 3.2a , figures 3.3a(1) , 3.4a(10)), which corresponds to a stage immediately preceeding the shock formulation, MacCormack produces a reasonable solution with a peak value almost correct although there are early signs of an oscillatory behaviour. For later times $t = 0.5$ and $t = 1.0$ the MacCormack scheme is highly oscillatory, while the Refined scheme produces smooth results with a modest improvement upon the FTBS method.

From table 4.1 and figure 4.5 for the linear problem we could see the following :

i) The TVD results confirm the monotonicity preserving property.

ii) Slight oscillations on the Refined results -due to use of tolerances in the switch.

iii) On comparing the two solutions, Davis' (TVD) solution is not noticeably better than the Refined scheme and if anything the Refined results produce a slightly steeper shock front, although with a more significant leading disturbance.

Davis' scheme is not just more complicated but also calculates a correction at each point, even at those points where no correction is required. So it is less computationally efficient and little improvement (if any) on the simple switching scheme described in this thesis. Therefore there is a little evidence to support the claim that the TVD scheme is superior to the more simple minded approach.

From table 4.2 and figure 4.6 for the non-linear problem, the initial data is not monotonic and therefore the TVD properties may not be reproduced. However, both the TVD and the Refined schemes produce oscillation free solutions and the graphical results are virtually indistinguishable (see figures (4.6)g and (4.6)h).

5.2 Concluding Remarks :

Despite the sophistication of the TVD schemes of Sweby the results produced are little better than those of the more simple minded schemes developed here. The TVD approach automates the selection of a formula at a particular point by monitoring local nature of the solution but it would seem that this in itself is not enough. Removing oscillations seems to severely diffuse the distribution and smear a discontinuity. The Refined scheme suggested here allows the presence of some small oscillations in an attempt not to insert too much diffusion but the results still leave a lot to be desired.

What has been shown is that simple switching schemes can be just as effective (not substantially worse) than the more sophisticated approach. Only one choice of Φ was used here and perhaps other choices might prove better. However the TVD theory suggests that perhaps a simple switch based on r, the ratio of successive gradients, rather than the second difference of u might be more successful. There is clearly plenty of scope for further investigation. Indeed, the use of Lax-Wendroff whenever $r > \frac{1}{2}$ and only switch to some other scheme when this is not the case has its attractions. In a sense the Davis' method is doing precisely this although the process of determining a zero correction is then followed through. There are still many aspects of shock modelling to be investigated and no doubt the subject will remain an active area of research for years to come.

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