ON THE FINITE GENERATION AND PRESENTABILITY OF DIAGONAL ACTS, FINITARY POWER SEMIGROUPS AND SCHÜTZENBERGER PRODUCTS

Peter Gallagher

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Contents

Declaration

I, Peter Gallagher, hereby certify that this thesis, which is approximately 40000 words in length, has been written by me, that it is the record of work carried out by me and that it has not been submitted in any previous application for a higher degree.

Signature **Name** Peter Gallagher **Date** 27/9/2005

I was admitted as a research student in September 2002 and as a candidate for the degree of Ph.D. in September 2003; the higher study for which this is a record was carried out in the University of St Andrews between 2002 and 2005.

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Abstract

The diagonal right act of a semigroup S is the set $S \times S$, with S acting by componentwise multiplication from the right. The diagonal left act of S and the diagonal bi-act of S are defined analogously.

Necessary and sufficient conditions are found for the finite generation of the diagonal bi-acts of completely zero-simple semigroups, completely simple semigroups and Clifford semigroups. It is also proved that various classes of semigroups do not have finitely generated or cyclic diagonal right, left or bi-acts.

The semigroups of full transformations, partial transformations and binary relations on an infinite set each have cyclic diagonal right and left acts. The semigroup of full finite-to-one transformations on an infinite set has a cyclic diagonal right act but its diagonal left act is not finitely generated. The semigroup of partial injections on an infinite set has neither finitely generated diagonal right nor left act, but has a cyclic diagonal bi-act.

The finitary power semigroup of a semigroup S, denoted $\mathcal{P}_f(S)$, is the set of finite subsets of S with respect to the usual set multiplication. We show that $\mathcal{P}_f(S)$ is not finitely generated if S is an infinite semigroup in any of the following classes: commutative; Bruck–Reilly extensions; inverse semigroups that contain an infinite group; completely zero-simple; completely regular.

Finally, the finite generation and presentability of Schützenberger products are investigated. A necessary and sufficient condition is established for finite generation. The Schützenberger product of two groups is finitely presented as an inverse semigroup if and only if the groups are finitely presented, but is not finitely presented as a semigroup unless both groups are finite.

Chapter 1

Introduction

We begin by outlining the chapters of the thesis: for each will be stated the topics, definitions, a brief history of the previous research, and our main results. The most important pieces are to be (or have been) published in [10, 11, 12, 13, 14]. However, this thesis is not simply a compilation of these papers. The results here have been presented in more detail, in many cases have been extended, and a number of supplementary results and examples are also included.

In Chapters 3 and 4 we consider diagonal acts of semigroups. This concept was first mentioned, implicitly, in [2]. It was then formally defined and considered by Robertson, Ruškuc and Thomson in [40] (and those sections which appeared as [35, 36]), initially in relation to wreath products. In [36, 40] the notion of diagonal acts was used to prove necessary and sufficient conditions for a restricted wreath product to be finitely generated or finitely presented. In [35, 40] diagonal acts were studied in their own right and some interesting connections were made with finitary power semigroups. Some interesting results from [40] appear in Chapters 3 and 4 as Propositions 3.1.1, 3.1.4, 3.1.7, 3.3.1, 4.1.1, 4.1.2, 4.2.1, 4.2.2 and Corollaries 3.2.3 and 3.8.2.

In Chapter 3 we investigate which of the 'standard' classes of semigroups can or cannot have finitely generated or cyclic diagonal right, left or bi-acts. Our main findings on this question are summarised in Table 1.1.

Property of semigroup S	**Non-trivial S, cyclic right/left act?**	**Infinite S, f.g. right/left act?**	**Non-trivial S, cyclic bi-act?**	**Infinite S, f.g. bi-act?**
Bruck–Reilly extension	No	No	No	No (Cor 3.6.2)
cancellative	No	No (Thm 3.7.1)	No (Prop 3.5.1)	Yes (Prop 3.3.1)
Clifford	No	No (Thm 3.8.1)	No (Cor 3.5.5)	Yes (Thm 3.4.10)
commutative	No	No	No	No (Thm 3.6.1)
completely regular	No	No (Cor 3.8.5)	No (Cor 3.5.5)	Yes (Prop 3.3.1, Thm 3.4.10)
completely simple	No	No (Cor 3.8.4)	No (Cor 3.5.4)	Yes (Cor 3.3.3)
completely zero-simple	No	No (Thm 3.8.3)	No (Cor 3.5.3)	Yes (Thm 3.3.2)
finite	No	N/A	No (Thm 3.5.2)	N/A
idempotent	No	No	No	No (Cor 3.6.4)
inverse	No	No (Thm 3.8.1)	Yes (Thm 4.3.3)	Yes (Prop 3.3.1, Thms 3.4.10, 4.3.3)
left cancellative	No	No (Thm 3.7.1)	?????	Yes (Prop 3.3.1)
locally finite	No	No (Prop 3.1.10)	?????	?????
right cancellative	No	No (Thm 3.7.7)	?????	Yes (Prop 3.3.1)

Table 1.1: Summary of results on diagonal acts in standard classes of semigroups.

In Chapter 4 we look for examples of infinite semigroups that have finitely generated diagonal acts. There are many families of transformation semigroups for which this question is not immediately answered by the results of Table 1.1, so we search in this area.

Let X be an infinite set. We consider the question of finite generation of the diagonal right, left and bi-acts of the semigroups: $\mathcal{B}_X$ of binary relations; $\mathcal{P}_X$ of partial transformations; $\mathcal{T}_X$ of full transformations; $\mathcal{F}_X$ of full finite-to-one transformations (that is, in which no infinite subset is mapped to a single point); $\mathcal{I}_X$ of partial injective transformations (also called partial bijections); $\mathcal{S}_X$ of bijections; $\mathcal{S}urj_X$ of full surjections; $\mathcal{I}nj_X$ of full injections; $\mathcal{T}_X \setminus \mathcal{S}urj_X$ of full non-surjections; $\mathcal{T}_X \setminus \mathcal{I}nj_X$ of full non-injections; and $\mathcal{PS}urj_X$ of partial surjections on X. Letting X be an infinite totally ordered set, we consider this question for the semigroups $\mathcal{O}_X$ of full monotonic transformations and $\mathcal{Q}_X$ of full strictly monotonic transformations on X. We also consider the diagonal acts of $\mathrm{End}(\mathbf{A})$, the endomorphism semigroup of an infinite independence algebra $\mathbf{A}$. These results are summarised in Table 1.2. We conclude Chapter 4 by exhibiting two families of semigroups, each defined by presentations, which have finitely generated diagonal right acts.

Semigroup	Right act	Left act	Bi-act
$\mathcal{B}_X$	cyclic (Thm 4.1.3)	cyclic (Thm 4.2.3)	cyclic
$\mathcal{P}_X$	cyclic (Cor 4.1.6)	cyclic (Cor 4.2.6)	cyclic
$\mathcal{T}_X$	cyclic (Cor 4.1.7)	cyclic (Thm 4.2.4)	cyclic
$\mathcal{F}_X$	cyclic (Cor 4.1.8)	infinite (Thm 4.2.9)	cyclic
$\mathcal{I}_X$	infinite (Thm 4.1.10)	infinite (Thm 4.2.10)	cyclic (Thm 4.3.3)
$\mathcal{S}_X$	infinite	infinite	infinite (Thm 4.4.1)
$\mathcal{S}urj_X$	infinite	infinite	infinite (Thm 4.4.2)
$\mathcal{I}nj_X$	infinite	infinite	infinite (Thm 4.4.4)
$\mathcal{T}_X \setminus \mathcal{S}urj_X$	infinite	infinite	infinite (Thm 4.4.5)
$\mathcal{T}_X \setminus \mathcal{I}nj_X$	infinite	infinite	infinite (Thm 4.4.6)
$\mathcal{P}\mathcal{S}urj_X$	infinite	infinite	infinite (Thm 4.4.3)
$\mathcal{O}_X$	infinite (Thm 4.5.5)	infinite (Thm 4.5.4)	?????
$\mathcal{Q}_X$	infinite	infinite	?????
End($\mathbf{A}$)	can be cyclic (Thms 4.6.2, 4.6.6)	can be cyclic (Thms 4.6.3, 4.6.6)	can be f.g. (Thm 4.6.6)

Table 1.2: Results on semigroups of transformations.

In Chapter 5 we consider power semigroups and finitary power semigroups. Both of these constructions have received significant attention in the literature over the years. Perhaps most importantly, a number of papers deal with the 'isomorphism problem': if S and T are semigroups and $\mathcal{P}(S)$ is isomorphic to $\mathcal{P}(T)$, is it necessarily true that S is isomorphic to T? Mogiljanskaja [31] showed that this is not the case in general and subsequently a series of papers considered the restrictions of the problem where S and T belong to various distinguished subclasses of semigroups; see for example [5, 17, 28]. Almeida [1] utilizes power semigroups of finite semigroups in the study of pseudovarieties and Putcha [33] studies their maximal subgroups. Easdown and Gould [8] investigate orders in finitary power semigroups.

The general question that we consider here is that of finite generation of the finitary power semigroup $\mathcal{P}_f(S)$. This strand of research originates in [35] (which was also part of [40]). Some interesting results from [35] appear in Chapter 5 as Corollary 5.5.3, Theorem 5.5.6 (in the cyclic case), Proposition 5.2.1 and Example 5.7.1.

An important theme we introduce on this topic is that of properly decomposable finite sets; that is, finite sets which may be written as products of smaller sets. We observe that if $\mathcal{P}_f(S)$ is finitely generated then all finite sets of size larger than some $N \in \mathbb{N}$ are properly decomposable. We show, using this fact, that if S is an inverse semigroup containing an infinite group then $\mathcal{P}_f(S)$ is not finitely generated.

Some connections are demonstrated between the finite generation of finitary power semigroups and that of diagonal acts. Using these, we deduce further results on our general problem. In particular, if S is infinite and $\mathcal{P}_f(S)$ is finitely generated then S is not commutative, completely zero-simple, completely regular or a Bruck–Reilly extension.

Except for some easy constructions on semigroups already known to have this property, no new examples of infinite semigroups with finitely generated finitary power semigroups arise in this work. However, we do consider certain relative ranks of the finitary power semigroups of all those semigroups for which we found positive results on diagonal acts in Chapter 4 (that is, as listed in Table 1.2). Several of these relative ranks have finite values but, most interestingly, this is not the case for the semigroup of partial injective transformations on an infinite set X.

Chapter 6 concerns Schützenberger products of semigroups and groups. This construction arises naturally in language theory (see [24]) and has been studied in several papers; see, for example, [16, 25, 30]. Here we consider Schützenberger products in the context of finite generation and finite presentability. This strand of research originates in [25], in which the authors determine generators and defining relations for $S\Diamond T$, where S and T are monoids. However, unless S and T are both finite, these sets of generators and relations are infinite. There is further work on this topic in [16], in which a more general construction (the Schützenberger product of n groups) is considered and two infinite presentations are exhibited, each of which reflects the structure.

The following are the main results on this topic.

Theorem A *Let S and T be semigroups, at least one of which is infinite. The Schützenberger product $S\Diamond T$ is finitely generated if and only if the following conditions are satisfied:*

(i) S and T are finitely generated;

(ii) S has a unique maximal $\mathcal{R}$-class R and there exists a finite $A \subseteq S$ such that $S = RA$;

(iii) T has a unique maximal $\mathcal{L}$-class L and there exists a finite $B \subseteq T$ such that $T = BL$.

Theorem B *The Schützenberger product $G\Diamond H$ of two groups is finitely presented as an inverse monoid if and only if G and H are finitely presented.*

Theorem C *The Schützenberger product $G\Diamond H$ of two groups is finitely presented as a monoid if and only if both G and H are finite.*

Where S and T are not groups, the question of finite presentability of $S\Diamond T$ remains unsolved.

Chapter 2

Preliminaries

We now give the basic definitions and results from semigroup theory and some related areas which will be used throughout the thesis.

2.1 Basics

A *semigroup* is a non-empty set S together with an associative binary operation, usually called *multiplication* (that is, $(xy)z = x(yz)$ for all $x, y, z \in S$). A *subsemigroup* of S is a subset $T \subseteq S$ which is a semigroup with respect to the same multiplication. We abbreviate 'T is a subsemigroup of S' as $T \leq S$. A *proper subsemigroup* of S is a subsemigroup which is not equal to S.

We say that $e \in S$ is an *identity element* (or identity, usually denoted 1 or 1_S) of S if $es = se = s$ for all $s \in S$. If S contains an identity then it has precisely one identity and is a *monoid*. For any semigroup S we may form a monoid S^1 as follows. If S is not a monoid then, for some symbol $1 \notin S$, we let $S^1 = S \cup \{1\}$ and we extend the multiplication on S by defining $1s = s1 = s$ for all $s \in S^1$. If S is a monoid then $S^1 = S$. For $V \subseteq S$ we write $V^1 = V \cup \{1\} \subseteq S^1$.

If S is a monoid and for every $x \in S$ there is a unique $y \in S$ such that $xy = yx = 1_S$ then we say that S is a *group*. In this case we write $y = x^{-1}$, the (group-theoretic) inverse of x.

We say that $z \in S$ is a *zero element* (or zero, usually denoted 0) of S if $zs = sz = z$ for all $s \in S$. A semigroup can have at most one zero. For any semigroup S, we can obtain a semigroup with a zero, denoted S^0, as follows.

If S does not have a zero then, for some symbol $0 \notin S$, we let $S^0 = S \cup \{0\}$ and we extend the multiplication on S by defining $0s = s0 = 0$ for all $s \in S^0$. If S has a zero then $S^0 = S$. For $V \subseteq S$ we write $V^0 = V \cup \{0\} \subseteq S^0$.

In a number of contexts we will use the notation of multiplying subsets of a semigroup, in which $AB = \{ab : a \in A, b \in B\}$ for $A, B \subseteq S$. In particular, $A^2 = \{a_1a_2 : a_1, a_2 \in A\}$ and $A^n = \{a_1 \dots a_n : a_1, \dots, a_n \in A\}$; we note the distinction between these sets and sets such as A^0 and A^1. In the following section we define the set $A^{(n)}$ which, again, is obviously distinguished from A^n.

2.2 Sets, binary relations and mappings

For sets X and Y we denote the set $\{(x, y) : x \in X, y \in Y\}$ as $X \times Y$. A *binary relation* on X is a subset $R \subseteq X \times X$. If R and Q are two binary relations on X then we may define the composition $R \circ Q$ as

$$\{(x, z) : (\exists y \in X)((x, y) \in R, (y, z) \in Q)\}.$$

In fact, $\circ$ is an associative operation, so the set $\mathcal{B}_X$ of binary relations on X forms a semigroup. The identity relation Δ_X on X, defined as $\{(i, i) : i \in X\}$, is the identity element of $\mathcal{B}_X$.

Let R be a binary relation on X. If xRx, for all $x \in X$, then R is *reflexive.* If xRy implies yRx, for all $x, y \in X$, then R is *symmetric.* If xRy and yRx implies that $x = y$ then R is *anti-symmetric.* If xRy and yRz implies xRz then R is *transitive.* If R is reflexive, anti-symmetric and transitive then it is an *order relation.* If R is reflexive, symmetric and transitive then it is an *equivalence relation.* An equivalence relation R on a semigroup S is a *right congruence* if xRy implies $xzRyz$ for all $x, y, z \in S$. *Left congruences* are defined analogously and a relation is a *congruence* if it is both a left congruence and a right congruence.

A *(full) mapping* f from a set X to a set Y, written as $f : X \to Y$, sends each element of X to precisely one element of Y. Instead of writing 'f sends x to y', we will usually denote this as $(x)f = y$ (the notation $xf = y$ also appears in the literature, but we include the brackets around the x as there appear some instances where this adds clarity). The *image* of f, denoted

$\mathrm{Im}(f)$, is the subset of Y defined as $\{y \in Y : (\exists x \in X)((x)f = y)\}$. We say that f is *surjective* if $\mathrm{Im}(f) = Y$. The *kernel* of f, denoted $\mathrm{Ker}(f)$, is the binary relation on X defined as $\{(x_1, x_2) \in X \times X : (x_1)f = (x_2)f\}$. We say that f is *finite-to-one* if each equivalence class of $\mathrm{Ker}(f)$ is finite. We say that f is *injective* if each equivalence class of $\mathrm{Ker}(f)$ is a singleton. We say that f is *bijective* if it is injective and surjective.

A *partial mapping* $f : X \to Y$ sends each element of X to either one or no element of Y. Instead of writing 'f does not send x to any element of Y', we denote this as $(x)f = -$. The *domain* of a partial mapping $f : X \to Y$, denoted $\mathrm{Dom}(f)$, is the subset of X defined as $\{x \in X : (x)f \neq -\}$. Then f is a full mapping if $\mathrm{Dom}(f) = X$. Images and kernels are defined as for full mappings, noting that $\mathrm{Ker}(f)$ is a binary relation on $\mathrm{Dom}(f)$.

Letting $f : X \to Y$ and $g : Y \to Z$, we define the composition mapping $f \circ g : X \to Z$ by $(i)(f \circ g) = ((i)f)g$. *Transformations*, *partial transformations* and *permutations* are mappings, partial mappings and bijections, respectively, from a set X to itself. It can be shown that $\circ$ is associative, so it follows that the sets of transformations and partial transformations on X each form semigroups with the same multiplication $\circ$. There are many further natural semigroups of transformations, defined by only including those transformations with certain properties; for example, injective.

We may consider a partial transformation $f : X \to X$ as a binary relation on X such that for every $i \in X$ there is at most one $j \in X$ with $(i, j) \in f$ (in this case, $j = (i)f$). Full transformations are binary relations f such that every $i \in X$ has precisely one $j \in X$ with $(i, j) \in f$. Further, the composition of mappings coincides with the composition of binary relations.

We may also consider a binary relation on X as a mapping from X into the power set $\mathcal{P}(X)$ (which is defined in Section 2.6 below). Mimicking the notation that we have stated for mappings, for an element $i \in X$ and a binary relation x on X we may denote the set $\{j \in X : (i, j) \in x\}$ as $(i)x$.

Let I be an index set and for each $i \in I$ let X_i be a set. A *system of distinct representatives* of the family $\{X_i : i \in I\}$ is a collection of elements $\{x_i : i \in I\}$ such that $x_i \in X_i$ for each $i \in I$ and if $i \neq j$ then $x_i \neq x_j$.

We say that a set X is *ordered* if there is an order relation, usually denoted as $\leq$, on its elements. We say that X is *totally ordered* if every pair $x, y \in X$ satisfy either $x \leq y$ or $y \leq x$. If X and Y are ordered then the mapping $f : X \to Y$ is *monotonic* if $i \geq j$ implies $(i)f \geq (j)f$, and it is *strictly monotonic* if $i > j$ implies $(i)f > (j)f$ for all $i, j \in X$.

We make frequent use of some facts and notation from infinite set theory. We denote the cardinality of a set X as $|X|$. For sets X and Y, we have that $|X| = |Y|$ if and only if there is a bijection $\phi : X \to Y$. Also, $|X| \geq |Y|$ if and only if there is a surjection $\phi : X \to Y$ or, equivalently, if and only if there is an injection $\psi : Y \to X$. The set $\mathbb{N}$ of natural numbers has the smallest infinite cardinality, which we call *countable*, or countably infinite.

Let X be an infinite set. Then X may be 'cut' into $|X|$ disjoint pieces, each with the same cardinality as X. More formally, there exists a set Z with the same cardinality as X and a collection of mutually disjoint sets $V_z \subseteq X$ (for $z \in Z$), with each $|V_z| = |X|$, and such that $\bigcup_{z \in Z} V_z = X$. As a consequence of this, there are disjoint subsets $X_1, X_2 \subseteq X$, each with the same cardinality as X and which satisfy $X = X_1 \cup X_2$.

Another well-known fact states that $X \times X$ has the same cardinality as X, so there is a bijection $g : (X \times X) \to X$. Indeed, we may denote $X \times X \times \ldots \times X$ (n times) as $X^{(n)}$, and this also has the same cardinality as X. Finally, the *pigeonhole principle* states that if a set of size $n + 1$ (or larger) is written as the union of n subsets then one of them has size at least 2. There is a similar statement, sometimes termed the *infinite pigeonhole principle*, which states that if an infinite set is written as the union of finitely many subsets, then at least one of them is infinite.

For further information on the general theory of sets and mappings see [27].

2.3 Ideals and Green's relations

A *right ideal* of a semigroup S is a non-empty subset $A \subseteq S$ such that if $a \in A$ and $s \in S$ then $as \in A$. *Left ideals* are defined analogously, and *ideals* are sets which are both right ideals and left ideals. Any of these sets may be called *proper* if it does not equal S.

If I is a proper ideal of S then the *Rees quotient* of S by I is the semigroup with elements $(S \setminus I) \cup \{0\}$ and multiplication $\cdot$ defined as

$$x \cdot y = \begin{cases} xy & \text{if } x, y, xy \in S \setminus I \\ 0 & \text{otherwise.} \end{cases}$$

Let X be a non-empty subset of S. The *right ideal generated by* X is the smallest right ideal of S that contains X. This can be shown to be equal to the set XS^1, which is of course defined as $\{xs : x \in X, s \in S^1\}$. If $|X| = 1$, say $X = \{x\}$, then this set is called a *principal right ideal.* We usually write $\{x\}S^1 = xS^1$ and we say that this is *generated by* x. The definitions and notation for the *generation of left ideals* and *principal left ideals* are analogous. The *ideal generated by* X is the smallest ideal containing X, and equals S^1XS^1. If $|X| = 1$, say $X = \{x\}$, then we write $S^1XS^1 = S^1xS^1$, which we call the *principal ideal* generated by x.

We define the equivalence relation $\mathcal{R}$ on S by the rule that $x\mathcal{R}y$ if and only if x and y generate the same principal right ideal; in other words, if and only if $xS^1 = yS^1$. An equivalent condition is that there exists $a, b \in S^1$ such that $x = ya$ and $y = xb$. Similarly, we define $\mathcal{L}$ by the rule that $x\mathcal{L}y$ if and only if $S^1x = S^1y$. Noting that $\mathcal{L}$ and $\mathcal{R}$ commute (that is, $\mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$), we define $\mathcal{D} = \mathcal{R} \circ \mathcal{L}$ and $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$.

The relation $\mathcal{J}$ is defined by the rule that $x\mathcal{J}y$ if and only if x and y generate the same principal ideal. This is also equivalent to saying that $S^1xS^1 = S^1yS^1$, or that there exist $a, b, c, d \in S^1$ such that $x = ayc$ and $y = cxd$. The relations $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}$ and $\mathcal{J}$ are *Green's relations.* We write R_x for the $\mathcal{R}$-class of S containing x, with similar notation for the other relations.

We may visualise each $\mathcal{D}$-class of S as a rectangular grid in which every row has the same size, every column has the same size and every cell has the same size. We call this an eggbox picture (see Tables 3.1 and 3.2, Section 3.4, for examples). The whole grid represents a $\mathcal{D}$-class, while the rows are $\mathcal{R}$-classes, the columns are $\mathcal{L}$-classes and the cells are $\mathcal{H}$-classes. From [23, 26] we have some useful results in this area. For example, if H is an $\mathcal{H}$-class then H is a group if and only if $H^2 \cap H \neq \emptyset$. If S is finite then $\mathcal{D} = \mathcal{J}$. In the finite case it also holds that if x, y and xy are all $\mathcal{D}$-related then xy is in the $\mathcal{H}$-class $R_x \cap L_y$.

The principal right ideals of S are partially ordered by inclusion. Therefore the set of $\mathcal{R}$-classes of S is also partially ordered in a corresponding manner. That is, there is a natural order $\leq$ on the set of $\mathcal{R}$-classes of S, defined by $R_x \leq R_y$ if and only if $xS^1 \subseteq yS^1$, or if and only if there exists $a \in S^1$ such that $x = ya$. Similarly there are partial orders on the $\mathcal{L}$-classes and the $\mathcal{J}$-classes. For example, $J_x \leq J_y$ if and only if there exist $a, b \in S^1$ such that $x = ayb$.

2.4 Homomorphisms

Let S and T be semigroups. A *homomorphism* from S into T is a mapping $\phi : S \to T$ which satisfies $(s_1 s_2)\phi = (s_1\phi)(s_2\phi)$ for all $s_1, s_2 \in S$. An *epimorphism* is a surjective homomorphism, a *monomorphism* is an injective homomorphism and an *isomorphism* is a bijective homomorphism. We say that T is a *homomorphic image* of S if there is an epimorphism $\phi : S \to T$. We say that S and T are *isomorphic* if there is an isomorphism $\phi : S \to T$.

An *endomorphism* is a homomorphism $S \to S$ and an *automorphism* is an isomorphism $S \to S$. The set of endomorphisms of S, denoted $\mathrm{End}(S)$, is a monoid with respect to composition of mappings. The set of automorphisms of S, denoted $\mathrm{Aut}(S)$, is a group with the same operation.

If $f : S \to T$ is a homomorphism then $\mathrm{Ker}(f)$ is a congruence on S. Conversely, let σ be a congruence on S and denote the σ-class containing $x \in S$ as x/σ. Denote the set of σ-classes of S as S/σ and define a multiplication on S/σ as $(x/\sigma)(y/\sigma) = (xy)/\sigma$. The fact that σ is a congruence means that this multiplication is well-defined and makes S/σ into a semigroup. Then $f : S \to S/\sigma$, defined as $(x)f = x/\sigma$, is an epimorphism and $\mathrm{Ker}(f) = \sigma$. In general, $\mathrm{Im}(f) \cong S/\mathrm{Ker}(f)$.

For example, let T be the Rees quotient of a semigroup S by a proper ideal I of S. Thus T is a homomorphic image of S via the epimorphism $\psi : S \to T$ defined by

$$(x)\psi = \begin{cases} x & \text{if } i \in S \setminus I \\ 0 & \text{if } x \in I. \end{cases}$$

In this case, $\mathrm{Ker}(\psi) = (I \times I) \cup \{(s, s) : s \in S \setminus I\}$.

2.5 Generation and presentation

Let S be a semigroup and let Y be a non-empty subset of S. The *subsemigroup generated by* Y, which we denote as $\langle Y \rangle$, is the smallest subsemigroup of S that contains Y. This also equals the set of all elements which may be written as products of the form $y_1 \dots y_n$ where each $y_i \in Y$. The *rank* of S, denoted $\text{rank}(S)$, is the cardinality (finite or infinite) of a smallest generating set for S. We say that S is *finitely generated* (or *has finite rank*) if there is a finite set $Y \subseteq S$ which generates S as a subsemigroup. One well-known fact states that if T is a subsemigroup of S and the complement $S \setminus T$ is an ideal of S then the rank of T is less than the rank of S. In particular, if S is finitely generated then T is finitely generated. It can also be shown that if a semigroup S is finitely generated then it has finitely many maximal $\mathcal{J}$-classes, with all other $\mathcal{J}$-classes below these. Similar results hold for $\mathcal{R}$- and $\mathcal{L}$-classes.

An *alphabet* X is a non-empty set of symbols. A *word* on X is a finite string of the form $x_1 \dots x_n$ where each $x_i \in X$. Let X^+ denote the set of all words on X. Then there is an associative binary operation which we define on X^+ by

$$(x_1 \dots x_n)(y_1 \dots y_m) = x_1 \dots x_n y_1 \dots y_m.$$

Thus we obtain the *free semigroup on* X. The *free monoid* X^* is $(X^+)^1$.

We call $(u, v) \in X^+ \times X^+$ a *relation* on X and write this as $u = v$. For a set of relations R on X, we let $R^\sharp$ denote the smallest congruence on X^+ that contains R. If $S \cong X^+/R^\sharp$ then we say that $\langle X \mid R \rangle$ *presents* S (*as a semigroup*), and we write $S = \text{Sgp}\langle X \mid R \rangle$. We say that S is *finitely presented* if there is a finite alphabet X and a finite set of relations R on X such that $S = \text{Sgp}\langle X \mid R \rangle$.

There is a similar notion of presentation as a monoid, in which X^* replaces X^+ in the definition. Often we will write $S = \text{Mon}\langle X \mid R \rangle$ for '$\langle X \mid R \rangle$ presents S as a monoid'. A monoid S is finitely presented as a semigroup if and only if it is finitely presented as a monoid.

We say that S is *inverse* if for every $x \in S$ there is a unique $y \in S$ such that $xyx = x$ and $yxy = y$.

For an alphabet X we may define further alphabets $X^{-1} = \{x^{-1} : x \in X\}$

and $X^{\pm 1} = X \cup X^{-1}$. Let $w \in (X^{\pm 1})^+$ be arbitrary. Then we may write $w = x_1^{\delta_1} \dots x_n^{\delta_n}$ with each $x_i \in X$ and $\delta_i \in \{1, -1\}$. We now define the word $w^{-1} = x_n^{-\delta_n} \dots x_1^{-\delta_1}$. The *standard inverse semigroup relations* on X, denoted $\Re_X$, are

$$\{ww^{-1}w = w, ww^{-1}zz^{-1} = zz^{-1}ww^{-1} : w, z \in (X^{\pm 1})^+\}.$$

The *inverse semigroup presentation* $\mathrm{Inv}\langle X \mid R\rangle$, in which X is an alphabet and R is a set of relations on $X^{\pm 1}$, is defined to present the inverse semigroup $\mathrm{Sgp}\langle X^{\pm 1} \mid R, \Re_X\rangle$. We say that S is *finitely presented as an inverse semigroup* if there are finite X and R such that $S = \mathrm{Inv}\langle X \mid R\rangle$. There is a similar notion of presentation as an inverse monoid, in which $(X^{\pm 1})^*$ replaces $(X^{\pm 1})^+$ in the definition. We denote '$\langle X \mid R\rangle$ presents S as an inverse monoid' as $S = \mathrm{InvMon}\langle X \mid R\rangle$.

An inverse monoid is finitely presented as an inverse semigroup if and only if it is finitely presented as an inverse monoid. If a semigroup is inverse and is finitely presented as a semigroup then it is finitely presented as an inverse semigroup. In fact, if S is inverse and $S = \mathrm{Sgp}\langle X \mid R\rangle$ then $S = \mathrm{Inv}\langle X \mid R\rangle$. The converse does not hold; in [39] it is shown that the free inverse semigroup (on any alphabet) is not finitely presented as a semigroup (this result also appeared as Theorem IX.4.7 of [32]). There is an equivalent fact in regard to presentation as a monoid and as an inverse monoid.

The *standard group relations* on X, denoted $\Im_X$, are

$$\{xx^{-1} = x^{-1}x = 1 : x \in X\}.$$

The *group presentation* $\mathrm{Gp}\langle X \mid R\rangle$, in which X is an alphabet and R is a set of relations on $X^{\pm 1}$, is defined to be the monoid $\mathrm{Mon}\langle X^{\pm 1} \mid R, \Im_X\rangle$. The *free group on* X is the group presented by $\mathrm{Gp}\langle X \mid \ \rangle$. (There is another definition for group presentations which uses normal subgroups of the free group on X, but the two are equivalent.) A group is finitely presented as a group if and only if it is finitely presented as a monoid if and only if it is finitely presented as an inverse monoid (and so on).

Let us consider a semigroup presentation $\mathrm{Sgp}\langle X \mid R\rangle$ and let us pick two arbitrary words $w_1, w_2 \in X^+$. We write $w_1 \equiv w_2$ if they are identical words.

We say that $w_1 = w_2$ *holds by one application of a relation in* R if we may write $w_1 \equiv pu_1q$ and $w_2 \equiv pu_2q$ where $p, q \in X^*$ and $(u_1 = u_2) \in R$. We say that $w_1 = w_2$ *may be deduced from the relations in* R if there is a finite sequence of words $v_1, \ldots, v_n \in X^+$ such that $v_1 \equiv w_1$, $v_n \equiv w_2$ and for each $i = 1, \ldots, n-1$ we have that $v_i = v_{i+1}$ holds by one application of a relation from R. Then the words $s, t \in X^+$ are equal in the semigroup $\mathrm{Sgp}\langle X \mid R\rangle$ if and only if this equality can be deduced from the relations R.

The following alterations, which we may perform on a semigroup presentation $\langle A \mid R\rangle$, are called *Tietze transformations.*

(T1) Add a symbol z to the list of generators and add $z = w$, for some $w \in A^+$, to the list of relations.

(T2) If $w_1 = w_2$, for some $w_1, w_2 \in A^+$, may be deduced from the relations R, then add this to the list of relations.

(T3) If there is a symbol $a \in A$ and a relation $a = w$ (for some $w \in (A \setminus \{a\})^+$) in the list of relations R, then remove a from A, remove $a = w$ from R and replace a by w everywhere else that it appears in R.

(T4) If a relation $w_1 = w_2$, which appears in the list of relations R, can be deduced from the other relations, then remove this relation.

Tietze transformations produce equivalent presentations; that is, presentations which define the same semigroup. In fact, two finite presentations define the same semigroup if and only if there is a finite sequence of applications of Tietze transformations which turns one into the other.

We also make use of the following additional facts concerning semigroup presentations; for proofs of these and further information on the theory of semigroup presentations, see [38]. If S is finitely presented and T is a subsemigroup of S whose complement is an ideal of S then T is finitely presented. If S is finitely presented and $S = \mathrm{Sgp}\langle X \mid R\rangle$ where X is finite and R is infinite then there is a finite $Q \subseteq R$ such that $S = \langle X \mid Q\rangle$. If S is generated by X, and there is a set of relations R on X that holds in S, then S is a homomorphic image of $\mathrm{Sgp}\langle X \mid R\rangle$. Moreover, if all equalities of products in S can be deduced (writing both sides as products of elements of X) from the relations in R then $S = \mathrm{Sgp}\langle X \mid R\rangle$.

Finally, there are analogous Tietze transformations which may be performed, and there are analogous statements which hold, in regard to monoid, inverse semigroup and inverse monoid presentations.

2.6 Some classes and constructions

We introduce some standard classes of semigroups, which we will explore in later chapters in a number of contexts.

Elements $x, y \in S$ are said to *commute* if $xy = yx$, and S is *commutative* if all of its elements commute. In a commutative semigroup we see that all of the Green's relations $\mathcal{R}, \mathcal{L}, \mathcal{H}, \mathcal{D}$ and $\mathcal{J}$ coincide.

An element $x \in S$ is (an) *idempotent* if $x^2 = x$, and S is *idempotent* (or a *band*) if all of its elements are idempotent.

A *semilattice* is a commutative band. The relations $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{J}$ in a semilattice coincide and the classes are all singleton sets. Denoting the partial order on its elements (classes) as $\leq$, we observe that $x \leq y$ if and only if $xy = x$. The product of two elements is their (unique) greatest lower bound with respect to this order. If a semilattice is finite and has a unique maximal element (that is, an identity element) then each pair of elements has a unique lowest upper bound, or *supremum*; we denote the supremum of β and γ as $\sup\{\beta, \gamma\}$.

We say that S is *right cancellative* if $xz = yz$ implies $x = y$ for all $x, y, z \in S$. *Left cancellative* is defined analogously, and a semigroup is *cancellative* if it is both left and right cancellative.

We say that S is *locally finite* if every finitely generated subsemigroup of S is finite.

We say that $x \in S$ is a *regular* element if there exists $y \in S$ such that $xyx = x$. We say that $y \in S$ is a (semigroup-theoretic) inverse of $x \in S$ if $xyx = x$ and $yxy = y$. It may be shown that an element is regular if and only if it has an inverse. We say that S is a *regular* semigroup if all of its elements are regular. A semigroup S is *inverse* if each $x \in S$ has a unique inverse, which we denote as x^{-1}. An equivalent condition is that S is regular and its idempotents commute. The obvious examples of inverse semigroups are groups and semilattices, which, in many respects, lie at opposite ends of the spectrum of inverse semigroups.

A *rectangular band* is a semigroup with a set of elements $I \times \Lambda$, for some sets I and Λ, and multiplication given by $(i, \lambda)(j, \mu) = (i, \mu)$. A *left zero semigroup* is a rectangular band of this form with $|\Lambda| = 1$. An equivalent condition for S to be a left zero semigroup is that $xy = x$ for all $x, y \in S$ (so every element is indeed a left zero). *Right zero semigroups* are defined analogously.

We say that S is *simple* if it has no proper ideals. We say that S is *completely simple* if it is simple and has minimal right and left ideals (with respect to inclusion). Rectangular bands are completely simple.

We say that S is *completely regular* if for every $x \in S$ there exists $y \in S$ such that $xyx = x$ and $xy = yx$. An equivalent condition is that S is a union of groups; that is, every element of S lies in a subgroup of S. Completely simple semigroups are completely regular.

If S has a zero (denoted 0) then we say that it is *zero-simple* if the only proper ideal of S is $\{0\}$. We say that S is *completely zero-simple* if it is zero-simple but has minimal non-zero right ideals and minimal non-zero left ideals, and $S^2 \neq \{0\}$. The condition that $S^2 \neq \{0\}$ is necessary to avoid the trivial semigroup and the two-element semigroup $\{x, 0\}$ with $x^2 = x0 = 0x = 0^2 = 0$.

Let G be a group and let I and Λ be non-empty sets. Let P be a matrix, indexed by Λ and I, respectively, with entries from G^0, and such that no row or column of P consists entirely of zeros. We denote $P = (p_{\lambda i})_{\lambda \in \Lambda, i \in I}$. Let S be the semigroup with elements $(I \times G \times \Lambda) \cup \{0\}$ and multiplication defined as

$$(i, g, \lambda)(j, h, \mu) = \begin{cases} (i, gp_{\lambda j}h, \mu) & \text{if } p_{\lambda j} \neq 0 \\ 0 & \text{if } p_{\lambda j} = 0, \end{cases}$$

and $0(i, g, \lambda) = (i, g, \lambda)0 = 0^2 = 0$, for all $i, j \in I, g, h \in G$ and $\lambda, \mu \in \Lambda$. Then S is the *zero Rees matrix semigroup* with respect to G, I, Λ and P. We write $S = \mathcal{M}^0[G; I, \Lambda; P]$. The *Rees–Suschkewitsch theorem* states that a semigroup is completely zero-simple if and only if it is isomorphic to a zero Rees matrix semigroup.

Let Y be a semilattice and for each $\alpha \in Y$, let T_α be a semigroup. A *semilattice of the* T_α is a semigroup whose elements are those in the disjoint union $\dot{\bigcup}_{\alpha \in Y} T_\alpha$ with multiplication defined naturally within each T_α and globally obeying the rule $T_\alpha T_\beta \subseteq T_{\alpha\beta}$.

Strong semilattices of the T_α are nice examples of these. They involve an extra ingredient, which is a family of homomorphisms $\{\phi_{\alpha,\beta} : \alpha, \beta \in Y, \alpha \geq \beta\}$ where each $\phi_{\alpha,\beta}$ maps from T_α to T_β, $\phi_{\alpha,\alpha}$ is the identity mapping on T_α and $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$. The semigroup has elements $\dot{\bigcup}_{\alpha \in Y} T_\alpha$ and multiplication is defined, for $x \in T_\alpha$ and $y \in T_\beta$, as $xy = (x\phi_{\alpha,\alpha\beta})(y\phi_{\beta,\alpha\beta})$. We denote this semigroup as $(Y, T_\alpha, \phi_{\alpha,\beta})$.

A semigroup is completely regular if and only if it is a semilattice of completely simple semigroups. A semigroup is a band if and only if it is a semilattice of rectangular bands.

For a semigroup S the following statements are equivalent: S is inverse and completely regular; S is inverse and every idempotent commutes with every element; S is a semilattice of groups; S is a strong semilattice of groups. A semigroup satisfying one (and hence all) of these conditions is called a *Clifford semigroup*.

Let $\mathbb{N}^0$ be the set of non-negative integers under addition, let T be a monoid and let $\pi : T \to T$ be an endomorphism. The *Bruck–Reilly extension* of T with respect to π, denoted $BR(T, \pi)$, is the set $\mathbb{N}^0 \times T \times \mathbb{N}^0$ with multiplication defined by

$$(m, t_1, n)(p, t_2, q) = (m - n + r, (t_1\pi^{r-n})(t_2\pi^{r-p}), q - p + r),$$

where $r = \max(n, p)$. From [38], if $T = \mathrm{Mon}\langle X \mid R \rangle$ then

$$BR(T, \pi) = \mathrm{Mon}\langle X, b, c \mid R, bc = 1, bx = (x\pi)b, xc = c(x\pi)\ (x \in X)\rangle.$$

The resulting semigroup is infinite and contains a copy of T. Further, if $\mathrm{Im}(\pi)$ is contained in the group of units of T then $BR(T, \pi)$ is simple.

The *power semigroup* of a semigroup S, denoted $\mathcal{P}(S)$, is the set of all non-empty subsets of S under the usual multiplication $AB = \{ab : a \in A, b \in B\}$ for $A, B \subseteq S$. The *finitary power semigroup* $\mathcal{P}_f(S)$ is the subsemigroup of $\mathcal{P}(S)$ consisting of all finite subsets of S.

Let S and T be semigroups and let $\theta : T \to \mathrm{End}(S)$ be a homomorphism. The *semi-direct product* of S and T with respect to θ, denoted $S \rtimes_\theta T$ (writing mappings on the left), has the set of elements $S \times T$ and multiplication

$$(s_1, t_1)(s_2, t_2) = (s_1[\theta(t_1)](s_2), t_1 t_2).$$

If, in addition to this, we have that $\theta(t) \in \mathrm{Aut}(S)$ for all $t \in T$ then this is called the semi-direct product of S and T *in which T acts on S by automorphism.*

The *direct product* of S and T, denoted simply as $S \times T$, is the semi-direct product in which $(t)\theta$ is the identity mapping on S for all $t \in T$. That is, the multiplication is defined by

$$(s_1, t_1)(s_2, t_2) = (s_1 s_2, t_1 t_2).$$

The notation $S \times T$ always means the set $\{(s,t) : s \in S, t \in T\}$, but if the context implies that we are considering a semigroup then we mean the direct product.

Let S and T be presented as semigroups by $\langle A \mid R \rangle$ and $\langle B \mid Q \rangle$ respectively. Then the *free product* of S and T, denoted $S*T$, is defined as $\mathrm{Sgp}\langle A, B \mid R, Q \rangle$. If S and T are monoids, presented by $\mathrm{Mon}\langle A \mid R \rangle$ and $\mathrm{Mon}\langle B \mid Q \rangle$ respectively, then the *monoid free product* of S and T, denoted $\mathrm{Mon}(S * T)$, is defined as $\mathrm{Mon}\langle A, B \mid R, Q \rangle$. Note that, in this construction, the identities of S and T merge to form the identity of $\mathrm{Mon}(S * T)$.

If $S = \mathrm{Inv}\langle A \mid R \rangle$ and $T = \mathrm{Inv}\langle B \mid Q \rangle$ then the *inverse free product* of S and T, denoted $\mathrm{Inv}(S*T)$, is defined as $\mathrm{Inv}\langle A, B \mid R, Q \rangle$. If $S = \mathrm{InvMon}\langle A \mid R \rangle$ and $T = \mathrm{InvMon}\langle B \mid Q \rangle$ then $\mathrm{InvMon}\langle A, B \mid R, Q \rangle$ presents the *inverse monoid free product* of S and T, which we denote as $\mathrm{InvMon}(S * T)$. Each of the free product constructions mentioned here can be shown to be independent of the presentations used for the semigroups S and T.

For more information on these and many further areas of semigroup theory, see [23, 26].

2.7 Acts

We say that a semigroup S acts on a set X from the right, and that X is a *right S-act* via $\alpha : X \times S \to X$ ($\alpha : (x, s) \mapsto xs \in X$) if $(xs_1)s_2 = x(s_1 s_2)$ for all $x \in X, s_1, s_2 \in S$. *Left acts* are defined analogously. We say that X is a *bi-act* if S acts on it both on the left and on the right and $(s_1x)s_2 = s_1(xs_2)$ for all $x \in X, s_1, s_2 \in S$.

If S is a monoid (with identity 1) then for X to be a *right* S-act it is generally also required that $x1 = x$ for all $x \in X$. For all of the acts studied in this thesis, it is easy to see that if S is a monoid then this property holds. Further, many of our results concerning acts are simpler to prove if we assume that S is a monoid, as the definition of generation of the act is made slightly simpler.

If S acts on X via a right action then $A \subseteq X$ *generates* this act if $AS^1 = X$. Generation of left acts and bi-acts is defined analogously. If A can be chosen to be finite then we say that X is *finitely generated*. Further, if A can be chosen to be a singleton, say $A = \{a\}$, then X is *cyclic*. In this case we say that X is generated by a and we write $X = aS^1$. (Of course, if S is a monoid then X is generated by $A \subseteq X$ if $X = AS$.)

Let X be a right S-act, generated by $A \subseteq X$. We say that this act is *free* with respect to generation by A if, for all $a_1, a_2 \in A, s_1, s_2 \in S$, the equality $a_1s_1 = a_2s_2$ implies $a_1 = a_2$ and $s_1 = s_2$. There is a more general notion of a finitely presented act, but we do not consider it in this research. There are analogous definitions for the generation and freeness of left S-acts and bi-S-acts.

We demonstrate some examples. Most obviously, a semigroup S acts on itself by right multiplication. A generating set for this act must contain a set of representative elements of the maximal $\mathcal{R}$-classes of S and, if S has any infinite ascending chains of $\mathcal{R}$-classes without a maximal one (so there is no principal right ideal containing the union of the chain), then we also require elements from arbitrarily high in each such chain. Of course, if S is a monoid then its identity 1_S generates this act. In this case the act is free with respect to generation by 1_S.

Let R be a right ideal of S. Then S acts on R by right multiplication. Noting that R is a union of $\mathcal{R}$-classes of S, a generating set for this act must contain representatives of those $\mathcal{R}$-classes of S that are maximal within R.

If R is a right ideal of S then S acts on $(S \setminus R) \cup \{0\}$ from the right via

$$(x)s = \begin{cases} xs & \text{if } x, xs \in S \setminus R \\ 0 & \text{otherwise.} \end{cases}$$

A generating set for this act must contain representatives of the maximal $\mathcal{R}$-classes of S that are within $S \setminus R$.

Let σ be a right congruence on S (that is, an equivalence relation on S with the additional property that $x \; \sigma \; y$ implies $xz \; \sigma \; yz$ for all $x, y, z \in S$). Then S acts on S/σ, the set of equivalence classes of σ, from the right via $(x/\sigma)s = (xs)/\sigma$. A generating set for this act will consist of the σ-classes of the generators for S as a right S-act.

Let X be a set and let $\mathcal{T}_X$ be the semigroup of full transformations on X. Then $\mathcal{T}_X$ acts on X from the right in the natural manner. An arbitrary element $i \in X$ generates X as a right $\mathcal{T}_X$-act; to show this we observe that for every $j \in X$ there is $f \in \mathcal{T}_X$ such that $(i)f = j$. Of course, unless X is a singleton there will be more than one such f, so this act is not free.

There are some ways to produce more examples of S-acts using simple constructions on an arbitrary right S-act X. Let I be an index set and, for each $i \in I$, let X_i be a copy of X (disjoint from the others) under the bijection $x \leftrightarrow x_i$. Then S acts on $\bigcup_{i \in I} X_i$ from the right via $(x_i)s = (xs)_i$. It is easy to show that if A generates X as a right S-act then $\bigcup_{i \in I} A_i$ (where A_i is the copy of A within X_i) generates $\bigcup_{i \in I} X_i$ as a right S-act. There is an obvious analogue for left S-acts and bi-S-acts.

Again we let X be a right S-act. Then $X \times X$ is also a right S-act, via the natural componentwise action $(x, y)s = (xs, ys)$. There are obvious analogues for left and bi-S-acts.

Extending the actions of S on S to actions of S on $S \times S$ leads to the notion of *diagonal acts*. More formally, the diagonal right act of S is the set $S \times S$, on which S acts via $(x, y)s = (xs, ys)$. The diagonal left act of S is defined analogously, via $s(x, y) = (sx, sy)$. The diagonal bi-act of S is the same set $S \times S$, on which S acts from both left and right. The question of the finite generation of diagonal acts is the main topic in Chapters 3 and 4.

For each of the stated examples in which S acts on one of its own subsets X or on a construction X of itself, the question of the finite generation of

$X \times X$ is not explored formally in this thesis, but we remark that each of these questions is connected to the same question for diagonal acts. Perhaps there is some potential for future research here, to determine a more precise answer to this question.

Considering $X \times X$ as a right $\mathcal{T}_X$-act, we see that any pair (i, j) with $i \neq j$ generates $X \times X$. To show this, we observe that for any pair (k, l) there is $f \in \mathcal{T}_X$ such that $(i)f = k$ and $(j)f = l$, so $(i, j)f = (k, l)$. However, if X contains more than two elements then there will be more than one such mapping f, so this act is not free.

2.8 Independence algebras

We begin this section by defining and briefly discussing universal algebras, although we will be mainly concerned with independence algebras.

Let A be a non-empty set. As usual, $A^{(n)}$ denotes the set of ordered n-tuples (with $n > 0$) of elements of A and we also define $A^{(0)} = \{\emptyset\}$. An *n-ary operation* on A is a mapping $f : A^{(n)} \to A$. We use the word *nullary* instead of 0-ary and we refer to the results of nullary operations as *constants*. We also use the word *unary* for 1-ary and *binary* for 2-ary.

A *language of algebras* is a set $\mathcal{F}$ such that there is a non-negative integer n associated with each $f \in \mathcal{F}$ (again, we describe f as n-ary).

If $\mathcal{F}$ is a language of algebras then an *algebra* $\mathbf{A}$ of type $\mathcal{F}$ is an ordered pair $\langle A, F \rangle$ where A is a non-empty set and F is a family of finitary operations on A indexed by $\mathcal{F}$ such that each n-ary $f \in \mathcal{F}$ is in correspondence with an n-ary operation $f (\in F)$ on A. Each element of F is called a *fundamental operation* of $\mathbf{A}$.

Let X be an alphabet of symbols. We define the set of *terms over* X as follows: this set includes the elements of X and all constants; and if $t_1, \ldots, t_n$ are terms over X and $f \in F$ is an n-ary operation then $f(t_1, \ldots, t_n)$ is another term. The set of all terms over X is the *free algebra* of type $\mathcal{F}$ over X.

In an algebra $\mathbf{A}$ of type $\mathcal{F}$ every term $t(x_1, \ldots, x_n)$ defines an n-ary *term operation* $A^n \to A$ by substitution of variables (that is, replacing the symbols appearing in the term by elements of $\mathbf{A}$).

We say that an algebra $\mathbf{B}$ is a *subalgebra* of $\mathbf{A}$ if: they are of the same type; the element set of $\mathbf{B}$ is a subset of the element set of $\mathbf{A}$; and every fundamental operation of $\mathbf{B}$ is the restriction of a fundamental operation of $\mathbf{A}$. Letting A and B denote the element sets of $\mathbf{A}$ and $\mathbf{B}$ respectively, we say that B is a *subuniverse* of $\mathbf{A}$ and note that B is closed with respect to the fundamental operations of $\mathbf{A}$; that is, if f is a fundamental n-ary operation of $\mathbf{A}$ and $b_1, \ldots, b_n \in B$ then $f(b_1, \ldots, b_n) \in B$.

Given an algebra $\mathbf{A}$ of type $\mathcal{F}$ we define, for every subset X of A,

$$\mathrm{Sg}^{\mathbf{A}}(X) = \bigcap \{B : X \subseteq B \text{ and } B \text{ is a subuniverse of } \mathbf{A}\},$$

which is the smallest subuniverse containing X and which we call *the subuniverse generated by* X. We call $\mathrm{Sg}^{\mathbf{A}}$ the *closure operator* on $\mathbf{A}$.

We say that $X \subseteq \mathbf{A}$ is *independent* if $i \notin \mathrm{Sg}^{\mathbf{A}}(X \setminus \{i\})$ for all $i \in X$. A *basis* of $\mathbf{A}$ is a minimal set (with respect to inclusion) that generates $\mathbf{A}$ as a subuniverse. While there does not necessarily exist a basis of $\mathbf{A}$, we note that if one does exist then it must be independent. The *rank* of $\mathbf{A}$, denoted $\mathrm{rank}(\mathbf{A})$, is the size of a basis (so rank is not defined if there does not exist a basis). Each operation in $\mathbf{A}$ extends to a componentwise operation in $\mathbf{A} \times \mathbf{A}$, endowing this set with the status of a universal algebra.

The transformation $\pi : A \to A$ is an endomorphism of $\mathbf{A}$ if it satisfies

$$[f(a_1, \ldots, a_n)]\pi = f([a_1]\pi, \ldots, [a_n]\pi)$$

for all the n-ary mappings f in $\mathcal{F}$ and sequences $a_1, \ldots, a_n \in A$. In particular, every endomorphism fixes the results of the nullary operations. It follows, for any n-ary term operation t and sequence $a_1, \ldots, a_n \in A$, that

$$[t(a_1, \ldots, a_n)]\pi = t([a_1]\pi, \ldots, [a_n]\pi).$$

Two endomorphisms on $\mathbf{A}$ are equal if and only if they agree on one (and hence every) basis. A bijective endomorphism is called an *automorphism*. The set of endomorphisms of $\mathbf{A}$, denoted $\mathrm{End}(\mathbf{A})$, is a semigroup with respect to the usual composition of mappings. The set $\mathrm{Aut}(\mathbf{A})$ of automorphisms of $\mathbf{A}$ forms a subgroup of $\mathrm{End}(\mathbf{A})$.

We will frequently abuse our notation by writing $\mathbf{A}$ for both an algebra and its underlying set of elements. For more information on universal algebra, see [3].

An *independence algebra* is an algebra $\mathbf{A}$ which satisfies the following conditions:

(i) for all $X \subseteq \mathbf{A}$ and $x, y \in \mathbf{A}$, if $x \in \mathrm{Sg}^{\mathbf{A}}(X, y) \setminus \mathrm{Sg}^{\mathbf{A}}(X)$, then $y \in \mathrm{Sg}^{\mathbf{A}}(X, x)$ (the *exchange property*);

(ii) if X is a basis of $\mathbf{A}$ and $\alpha : X \to \mathbf{A}$ is any mapping then α can be extended to an endomorphism (the *free basis property*).

The exchange property implies that: $\mathbf{A}$ has a (at least one) basis; each basis has the same cardinality; bases are precisely maximal independent sets (with respect to inclusion); $\mathbf{A}$ contains no independent sets larger than this; and every independent set can be extended to a basis. Because of these facts, we often refer to the rank of an independence algebra $\mathbf{A}$ as its dimension, denoted $\dim(\mathbf{A})$.

Two examples of independence algebras are sets (with no operations) and vector spaces. The endomorphism monoid of a set X is the semigroup $\mathcal{T}_X$ of full transformations on X. The endomorphism monoid of a vector space V is the semigroup of linear mappings on V. In the finite-dimensional case (say V has dimension n and base field F) this can be regarded as the semigroup of all $n \times n$ matrices over F with respect to the operation of matrix multiplication.

For further information on independence algebras see [18].

Chapter 3

Finite generation of diagonal acts among some standard classes of semigroups

Throughout this chapter and the next we will study diagonal acts; thus it is worthwhile to restate the definitions at this point. The *diagonal right act* of S is the set $S \times S$ with S acting via $(x, y)s = (xs, ys)$. The *diagonal left act* is defined analogously, via $s(x, y) = (sx, sy)$. The *diagonal bi-act* is the same set $S \times S$ on which S acts from both left and right as above.

We wish to know more about which of the standard classes of infinite semigroups may have finitely generated diagonal acts. Every result which is stated and proved in this chapter and concerns the finite generation of the diagonal right act has an analogous statement for the diagonal left act. There will sometimes be analogous facts regarding the diagonal bi-act, and even concerning cyclic generation.

The majority of this chapter, but not all of it, appeared in [12].

3.1 Some preliminary observations

We begin with the following, which simplifies our basic definitions. This result, and also Propositions 3.1.4 and 3.1.7 below, first appeared in [40].

Proposition 3.1.1 *The diagonal right act of S is finitely generated if and only* *if $S \times S = (U \times U)S^1$ for some finite $U \subseteq S$. The diagonal bi-act is finitely generated if and only if there is a finite $U \subseteq S$ satisfying $S \times S = S^1(U \times U)S^1$. Moreover, if S is infinite then the diagonal right act is finitely generated if and only if there is a finite $U \subseteq S$ such that $S \times S = (U \times U)S$.*

PROOF. We begin with a simple proof for the first statement of the result.

($\Rightarrow$) Assume that the diagonal right act of S is finitely generated, so there is a finite $A \subseteq S \times S$ that satisfies $S \times S = AS^1$. If we define $U \subseteq S$ as $U = \{a : (a, b) \in A \text{ or } (b, a) \in A\}$ then it is clear that U is finite and that $S \times S = (U \times U)S^1$.

($\Leftarrow$) This part is obvious.

The analogous result for the diagonal bi-act is similarly easy to prove. We now consider the final statement of the result.

($\Rightarrow$) Assume that $S \times S = (U \times U)S^1$ for a finite $U \subseteq S$. We let $V = U \cup U^2$ and claim that $S \times S = (V \times V)S$. For arbitrary $x, y \in S$ there are $u_1, u_2 \in U$ and $c \in S^1$ such that

$$(x, y) = (u_1, u_2)c.$$

Of course, if either $x \notin U$ or $y \notin U$ then $c \neq 1$, so $(x, y) \in (V \times V)S$. We now assume that this is not the case, so $x, y \in U$. As S is infinite and V is finite we know that $S \neq V$, so there exists $s \in S \setminus V$. Then there are $u_3, u_4, u_5, u_6 \in U$ and $t_1, t_2 \in S^1$ such that

$$(x, s) = (u_3, u_4)t_1, \quad (y, s) = (u_5, u_6)t_2.$$

As $s \notin V$ it follows that $t_1, t_2 \notin U \cup \{1\}$. There are $u_7, u_8 \in U, t_3 \in S^1$ such that

$$(t_1, t_2) = (u_7, u_8)t_3.$$

As $t_1, t_2 \notin U$ it is clear that $t_3 \neq 1$, so

$$(x, y) = (u_3u_7, u_5u_8)t_3 \in (V \times V)S,$$

completing the proof.

($\Leftarrow$) This part is obvious. ■

We briefly remark on the significance of this result. Firstly, a finite generating set $A \subseteq S \times S$ for a diagonal act can be replaced by one of the form $U \times U$. Further, in the case of the diagonal right act of an infinite semigroup, the identity may be removed from the equality. It is natural to ask whether the identity may be removed from the equality for the diagonal bi-act, but Example 3.2.7 below shows that this is not the case.

Proposition 3.1.2 *If the diagonal right act of S is cyclic and generated by (a, b), for some $a, b \in S$, then $S \times S = (a, b)S$.*

PROOF. From $S \times S = (a, b)S^1$ it follows that there is $u \in S^1$ such that $(b, a) = (a, b)u$. If $u = 1$ then $a = b$, so $|S| = 1$; that is, S is the trivial monoid and our result follows. If $u \neq 1$ then $(a, b) = (a, b)u^2 \in (a, b)S$, so $S \times S = (a, b)S$. ■

From this point we will use Propositions 3.1.1 and 3.1.2 with no further indication.

Proposition 3.1.3 *If the diagonal right act of S is finitely generated and T is a homomorphic image of S then the diagonal right act of T is finitely generated. Analogous statements hold for the diagonal bi-act and cyclic generation.*

PROOF. This follows because if $S \times S = (U \times U)S$, $\phi : S \to T$ is an epimorphism and we let $V = U\phi$, then $T \times T = (V \times V)T$. ■

Proposition 3.1.4 *If the diagonal right act of S is finitely generated, T is a subsemigroup of S and $S \setminus T$ is an ideal of S then the diagonal right act of T is finitely generated. An analogous statement holds for the diagonal bi-act.*

PROOF. This follows because if $S \times S = (U \times U)S$, $T \leq S$, $S \setminus T$ is an ideal of S and we let $V = T \cap U$, then $T \times T = (V \times V)T$. ■

Proposition 3.1.5 *There does not exist a semigroup S whose diagonal right act is cyclic and such that S has a subsemigroup T whose complement is an ideal of S. An analogous statement holds for the diagonal bi-act.*

PROOF. Assume that there are $a, b \in S$ such that $S \times S = (a, b)S$, and that there is $T \leq S$ such that $S \setminus T$ is an ideal of S. It is clear that we must have $a, b \in T$. For $x \in T$ and $y \in S \setminus T$ there is $u \in S$ which satisfies $(x, y) = (au, bu)$. As $x \in T$ it follows that $u \in T$, which in turn implies that $y \in T$, a contradiction. ■

Proposition 3.1.6 *The diagonal right act of S is finitely generated if and only if the diagonal right act of S^0 is finitely generated. An analogous statement holds for the diagonal bi-act.*

PROOF. If S has a zero then $S^0 = S$ and there is nothing more to prove, so we assume that this is not the case. If S is finite then the result is also obvious, so we assume that it is an infinite semigroup.

($\Rightarrow$) Assume that $S \times S = (U \times U)S$ for some finite $U \subseteq S$ and note that $US = S$. Then

$$\begin{aligned}(U^0 \times U^0)S^0 &= (U \times U)S \cup (U \times \{0\})S \cup (\{0\} \times U)S \cup \{(0,0)\} \\ &= (S \times S) \quad \cup (S \times \{0\}) \quad \cup (\{0\} \times S) \quad \cup \{(0,0)\} \\ &= S^0 \times S^0,\end{aligned}$$

so the diagonal right act of S^0 is finitely generated.

($\Leftarrow$) This is a consequence of Proposition 3.1.4. ■

In fact, this proof goes further than we have stated. It shows that adjoining a zero to any semigroup, even if it already has a zero, preserves the finite generation of diagonal acts.

If S does not have a zero then, by Proposition 3.1.5, the diagonal right act, left act and bi-act of S^0 are not cyclic.

The *n-diagonal right act* of S is the direct product of n copies of S (denoted $S^{(n)}$), on which S acts via $(x_1, \ldots, x_n)s = (x_1 s, \ldots, x_n s)$. The n-diagonal left act of S and the n-diagonal bi-act of S are defined analogously. By a proof similar to that of Proposition 3.1.1, the n-diagonal right act of S is finitely generated if and only if there is a finite $U \subseteq S$ such that $S^{(n)} = U^{(n)}S$, while the n-diagonal bi-act of S is finitely generated if and only if there is a finite $U \subseteq S$ such that $S^{(n)} = S^1 U^{(n)} S^1$.

The following appeared in [40, 35].

Proposition 3.1.7 *If the diagonal right act of S is finitely generated then the n-diagonal right act of S is finitely generated for all $n \in \mathbb{N}$.*

PROOF. Assume that $S \times S = (U \times U)S$ for some finite $U \subseteq S$. We will use induction to prove that for any $n \in \mathbb{N}$ with $n \geq 2$ there is a finite $V \subseteq S$ such that $S^{(n)} = V^{(n)}S$.

The base case $n = 2$ is precisely our assumption. For the inductive step we assume, for some $k \geq 2$, that there is a finite $W \subseteq S$ such that $S^{(k)} = W^{(k)}S$. We let $V = W \cup W^2$ and claim that $S^{(k+1)} = V^{(k+1)}S$.

For arbitrary $x_1, \ldots, x_{k+1} \in S$ there are $w_1, \ldots, w_k \in W, p \in S$ such that

$$(x_1, \ldots, x_k) = (w_1, \ldots, w_k)p.$$

Further, there are $w_{k+1}, w_{k+2} \in W$ and $q \in S$ such that

$$(p, x_{k+1}) = (w_{k+1}, w_{k+2})q.$$

Hence

$$(x_1, \ldots, x_k, x_{k+1}) = (w_1 w_{k+1}, \ldots, w_k w_{k+1}, w_{k+2})q \in V^{(k+1)}S,$$

which completes the proof. ■

We observe the following unsolved question. We cannot answer this by adapting the proof of Proposition 3.1.7 in any obvious way because we cannot get the necessary induction to work in this case.

Open Problem 3.1.8 Let S be an infinite semigroup with a finitely generated diagonal bi-act. Is it necessarily true that the n-diagonal bi-act is finitely generated for all $n \in \mathbb{N}$?

Proposition 3.1.9 *Let S be an infinite semigroup with a finite subset U which satisfies $S \times S = (U \times U)S$. Then U generates an infinite subsemigroup.*

PROOF. Suppose that U generates a subsemigroup which is finite of size n. By Proposition 3.1.7 there is a finite subset $W \subseteq S$ which satisfies the equality $S^{(n+1)} = W^{(n+1)}S$. If W is produced as stated in the proof of that result then $W \subseteq \langle U \rangle$, so we must have $|W| \leq n$.

As S is infinite there are $n+1$ distinct elements $s_1, \ldots, s_{n+1} \in S$. There are $w_1, \ldots, w_{n+1} \in W$ and $t \in S$ such that

$$(s_1, \ldots, s_{n+1}) = (w_1, \ldots, w_{n+1})t.$$

By the pigeonhole principle, two of the w_i must be equal, say $w_i = w_j$ with $i \neq j$. Then $s_i = w_i t = w_j t = s_j$, which is a contradiction. ■

From this we have the following consequence.

Corollary 3.1.10 *If S is an infinite locally finite semigroup then the diagonal right act of S is not finitely generated.*

Following these results, we show that there are similar, and even stronger, results in relation to the cyclic generation of the diagonal right act.

Proposition 3.1.11 *Let S be a semigroup with a cyclic diagonal right act, so that $S \times S = (a,b)S$ for some $a, b \in S$. For all $n \in \mathbb{N}$, the n-diagonal right act of S is cyclic. Indeed, if we list the elements of $\{a,b\}^n$ as $w_1, \ldots, w_{2^n}$, then we have that $S^{(2^n)} = (w_1, \ldots, w_{2^n})S$.*

PROOF. If the k-diagonal right act of S is cyclic then, for all $l < k$, the l-diagonal right act of S is also cyclic; the generating l-tuple is the first l co-ordinates of the generating k-tuple for the k-diagonal act. Thus the first statement of the result will follow from the second. We proceed by induction to prove the second statement.

The base case $n = 1$ (that is, a statement concerning $S \times S$) is precisely our assumption. For the inductive step we assume, for some $k \geq 1$, that if we denote $\{a,b\}^k = \{v_1, \ldots, v_{2^k}\}$, then we have $S^{(2^k)} = (v_1, \ldots, v_{2^k})S$. For arbitrary $x_1, \ldots, x_{2^{k+1}} \in S$ there are $s, t \in S$ such that

$$(x_1, \ldots, x_{2^k}) = (v_1, \ldots, v_{2^k})s, \ (x_{2^k+1}, \ldots, x_{2^{k+1}}) = (v_1, \ldots, v_{2^k})t.$$

Further, there is $r \in S$ such that

$$(s, t) = (a, b)r.$$

Then

$$(x_1, \ldots, x_{2^{k+1}}) = (v_1 a, \ldots, v_{2^k} a, v_1 b, \ldots, v_{2^k} b)r,$$

so $S^{(2^{k+1})} = (v_1 a, \ldots, v_{2^k} a, v_1 b, \ldots, v_{2^k} b)S$ and the 2^{k+1}-diagonal right act is finitely generated. We also see that $\{v_1 a, \ldots, v_{2^k} a, v_1 b, \ldots, v_{2^k} b\} = \{a, b\}^{k+1}$, so the result follows by induction. ■

Proposition 3.1.12 *Let S be a non-trivial semigroup with a cyclic diagonal right act, so there are $a, b \in S$ satisfying $S \times S = (a, b)S$. Then the subsemigroup generated by a and b is free of rank 2.*

PROOF. First we claim, for each $n \in \mathbb{N}$, that $\{a, b\}^n$ has size 2^n; that is, all products (words) of a and b with length n (which we write as $w_1, \ldots, w_{2^n}$) are distinct in S. To prove this by contradiction we will assume that it is not the case. This means that there are $n \in \mathbb{N}$ and $w_i, w_j \in \{a, b\}^n$ with $i \neq j$ (say $i < j$) but $w_i = w_j$ in S. As S is non-trivial there are distinct $x, y \in S$. We consider the 2^n-tuple $(x, \ldots, x, y, \ldots, y)$ which has x in the first i coordinates and y in the others. By Proposition 3.1.11 there is $u \in S$ such that

$$(x, \ldots, x, y, \ldots, y) = (w_1, \ldots, w_{2^n})u.$$

Then $x = w_i u$ and $y = w_j u$, so $x = y$, which is a contradiction. From this we conclude, as desired, that $\{a, b\}^n$ has size 2^n for each n.

We now claim that all products of a and b are distinct in S. To prove this by contradiction we suppose that there are two distinct words $v, w \in \{a, b\}^+$ which are equal in S. Let us say that v has length k and that w has length l. We know that $k \neq l$, so without loss of generality we may say that $k < l$. We also say that the $(k+1)$-th letter of w is a (a symmetric argument covers the case where it is b). Then wbv and vbw are words in $\{a, b\}^+$, each with length $k + l + 1$, and are equal in S, but they are distinct products as they differ on the $(k+1)$-th letter. This is a contradiction, and we can conclude, as desired, that all products of a and b are distinct in S. From this there follows the result, that a and b generate a free semigroup of rank 2. ■

An *identity* is a formal equality of two words over an alphabet X. We say that a semigroup S satisfies an identity if every substitution of letters from X by elements of S yields an equality that holds in S. For example, the identity $xy = yx$ is satisfied by every commutative semigroup.

Corollary 3.1.13 *Let S be a non-trivial semigroup that satisfies a non-trivial identity. The diagonal right act of S is not cyclic.*

PROOF. It is clearly not possible for S to contain a free semigroup of rank 2; thus Proposition 3.1.12 gives us this result. ■

The following observation will become particularly useful in Lemma 5.9.4.

Proposition 3.1.14 *If the diagonal right act of S is cyclic and free with respect to a generator (a, b) then the 2^n-diagonal right act of S is cyclic and free with respect to the generator $(w_1, \ldots, w_{2^n})$, where $\{a, b\}^n = \{w_1, \ldots, w_{2^n}\}$.*

PROOF. By Proposition 3.1.11 the 2^n-diagonal right act of S is cyclic, with generating 2^n-tuple $(w_1, \ldots, w_{2^n})$. We will use induction to show that the 2^n-diagonal right act of S is free with respect to this generator. We have assumed that the the diagonal right act of S is cyclic and free, so we already have the base case $n = 1$ (that is, concerning $S \times S$).

We now assume this statement to be true for all $n \leq k$ and we consider the 2^{k+1}-diagonal right act of S. We let $\{u_1, \ldots, u_{2^k}\} = \{a, b\}^k$ and note that $\{u_1 a, \ldots, u_{2^k} a, u_1 b, \ldots, u_{2^k} b\} = \{a, b\}^{k+1}$. We now let $s, t \in S$ be such that

$$(u_1 a, \ldots, u_{2^k} a, u_1 b, \ldots, u_{2^k} b)s = (u_1 a, \ldots, u_{2^k} a, u_1 b, \ldots, u_{2^k} b)t.$$

Breaking this down, we see that

$$(u_1, \ldots, u_{2^k})as = (u_1, \ldots, u_{2^k})at,$$

$$(u_1, \ldots, u_{2^k})bs = (u_1, \ldots, u_{2^k})bt.$$

By our inductive assumption it follows that $as = at$ and $bs = bt$; thus we can write $(a, b)s = (a, b)t$, from which it can be deduced that $s = t$. We have shown that the 2^{k+1}-diagonal right act of S is free with respect to the generator $(u_1a, \ldots, u_{2^k}a, u_1b, \ldots, u_{2^k}b)$ and the result follows by induction. ■

3.2 Structural conditions

We now describe some links between the $\mathcal{J}$-, $\mathcal{R}$- and $\mathcal{L}$-class structure of an infinite semigroup S and the diagonal acts of S.

Proposition 3.2.1 *If S is infinite and has a finitely generated diagonal bi-act then each maximal $\mathcal{J}$-class has the same cardinality as S.*

PROOF. Assume that S is infinite and that $S \times S = S^1(U \times U)S^1$ for some finite $U \subseteq S$, but that J is a maximal $\mathcal{J}$-class whose cardinality is less than that of S.

As the cardinality of J is less than that of S, and as U is finite, it follows that the cardinality of J^1UJ^1 is also less than that of S, so we may select $y \in S \setminus J^1UJ^1$. Letting $x \in J$ be arbitrary, we may find $s, t \in S^1, u_1, u_2 \in U$ such that

$$(x, y) = (su_1t, su_2t).$$

We claim that $s \in J^1$. Indeed, if $s \neq 1$ then it is clear from $x = su_1t$ that $J = J_x \leq J_s$. But J is maximal, so $J = J_s$ and $s \in J$ as desired. Analogously it can be shown that $t \in J^1$. Hence $y = su_2t \in J^1UJ^1$, a contradiction. ■

Theorem 3.2.2 *If S is infinite with a finitely generated diagonal bi-act then S is a principal ideal and the associated (unique maximal) $\mathcal{J}$-class has the same cardinality as S.*

PROOF. Assume that S is infinite and that $S \times S = S^1(U \times U)S^1$ for some finite $U \subseteq S$.

For arbitrary $x, y \in S$ there are $s, t \in S^1, u_1, u_2 \in U$ such that

$$(x, y) = (su_1t, su_2t).$$

Then $S^1xS^1 = S^1su_1tS^1 \subseteq S^1u_1S^1$. That is, every principal ideal is contained in one of the form S^1uS^1 where $u \in U$. As U is finite, this family of principal ideals is also finite. It is clear that some of these are maximal with respect to inclusion amongst themselves, and it quickly follows that these are maximal.

Now suppose that there are distinct maximal $\mathcal{J}$-classes $J_1, J_2 \subseteq S$. By Proposition 3.2.1 both of these classes have the same cardinality as S. Then, for $x \in J_1 \setminus U$ and $y \in J_2 \setminus U$ there are $s, t \in S^1, u_1, u_2 \in U$ such that

$$(x, y) = (su_1t, su_2t).$$

If $s \neq 1$ then $J_1 = J_x \leq J_s$ and hence $J_1 = J_s$. Also, $J_2 = J_y \leq J_s$ and hence $J_2 = J_s$. Therefore $J_1 = J_2$, which is a contradiction. We conclude that $s = 1$. Analogously, it can be shown that $t = 1$. Hence $x, y \in U$, a contradiction. We conclude that there is a unique maximal $\mathcal{J}$-class and that all other $\mathcal{J}$-classes are below this, so S is a principal ideal. ■

If $x \in S$ is indecomposable then $\{x\}$ is a maximal $\mathcal{J}$-class of S. If S has no identity element then $\{1\}$ is a maximal $\mathcal{J}$-class of S^1.

Corollary 3.2.3 *If S is infinite and the diagonal bi-act of S is finitely generated then S contains no indecomposable elements.*

Corollary 3.2.4 *There does not exist an infinite non-monoid S for which the diagonal bi-act of S^1 is finitely generated.*

Proposition 3.2.5 *If S is infinite with a finitely generated diagonal right act then S has a finite number of maximal principal right ideals and every principal right ideal is contained in one of these.*

Proof. Assume that S is an infinite semigroup and that $S \times S = (U \times U)S$ for some finite $U \subseteq S$.

For arbitrary $x, y \in S$ there are $u_1, u_2 \in U, s \in S$ such that

$$(x, y) = (u_1s, u_2s).$$

Thus $xS^1 = u_1sS^1 \subseteq u_1S^1$, so every principal right ideal is contained in one from the finite family $\{uS^1 : u \in U\}$. Again, as U is finite it is clear that this family is finite. Also, some of these must be maximal with respect to inclusion among the family, so it follows that some are maximal. ■

Corollary 3.2.6 *If S is infinite with a finitely generated diagonal right act then at least one of its principal right ideals has the same cardinality as S.*

It is natural to ask whether Proposition 3.2.5 may be strengthened to state that an infinite semigroup with a finitely generated diagonal right act has a unique maximal principal right ideal. In fact the answer is no, as we will show in the following example. This also answers some other interesting questions.

Example 3.2.7 Let $S = \{y, z\} \times \mathcal{T}_X$ where $\{y, z\}$ is a 2-element left zero semigroup and $\mathcal{T}_X$ is the semigroup of full transformations on an infinite set X. Then $\{y\} \times \mathcal{T}_X$ and $\{z\} \times \mathcal{T}_X$ are distinct maximal principal right ideals of S. Theorem 4.1.7 below shows that $\mathcal{T}_X \times \mathcal{T}_X = (a, b)\mathcal{T}_X$ for some $a, b \in \mathcal{T}_X$. Therefore

$$S \times S = \{[(y, a), (y, b)], [(y, a), (z, b)], [(z, a), (y, b)], [(z, a), (z, b)]\}S$$

is a finitely generated diagonal right act. Hence an infinite semigroup with a finitely generated diagonal right act need not have a unique maximal principal right ideal.

We note that for no $\alpha, \beta \in \mathcal{T}_X$ and $U \subseteq S$ is the pair $[(y, \alpha), (z, \beta)]$ in the set $S(U \times U)$, so the diagonal left act of S is not finitely generated.

Similarly, we see that there are no $\alpha, \beta \in \mathcal{T}_X$ and $U \subseteq S$ for which $[(y, \alpha), (z, \beta)]$ is in $S(U \times U)S$. However, we know that the diagonal right act is finitely generated, so the diagonal bi-act of S is also finitely generated. We conclude that in Proposition 3.1.1 the identity elements cannot be removed from the equality in the case of the diagonal bi-act. ■

Proposition 3.2.8 *Let S be an infinite semigroup with maximal principal left ideals and such that the diagonal right act of S is finitely generated. Then S is a principal left ideal of itself and the associated (unique maximal) $\mathcal{L}$-class is a subsemigroup with the same cardinality as S.*

Proof. Let $U \subseteq S$ be a finite set satisfying $S \times S = (U \times U)S$. We suppose that L_1 and L_2 are two distinct maximal $\mathcal{L}$-classes of S. For arbitrary $x \in L_1, y \in L_2$ there are $u_1, u_2 \in U, s \in S$ such that

$$(x, y) = (u_1 s, u_2 s).$$

Then $S^1 x = S^1 u_1 s \subseteq S^1 s$, so $S^1 x = S^1 s$. Analogously it can be shown that $S^1 y = S^1 s$, so $S^1 x = S^1 y$ and hence $L_1 = L_2$, a contradiction. We conclude that there is a unique maximal $\mathcal{L}$-class, which we denote as L.

For arbitrary $x \in L, y \in S$, there are $u_3, u_4 \in U, t \in S$ such that

$$(x, y) = (u_3 t, u_4 t).$$

Then $S^1 x = S^1 u_3 t \subseteq S^1 t$, so $S^1 x = S^1 t$. Further, $S^1 y = S^1 u_4 t \subseteq S^1 t = S^1 x$, so $S^1 x = S$ and S is a principal left ideal of itself. By Corollary 3.2.3 we know that x is not indecomposable, so we may write $x = pq$ for some $p, q \in S$. Then $x \mathcal{L} q$, so there is an element $r \in S^1$ satisfying $q = rx$. Hence $x = prx \in Sx$, so $Sx = S$. We conclude, for all $x \in L$, that $Sx = S$. It now follows that if $s, t \in L$ then $Sst = St = S$, so $st \in L$. Therefore L is a subsemigroup of S.

Now suppose that L has a smaller cardinality than S. As U is finite it follows that UL also has a smaller cardinality than S, so we may select $y \in S \setminus UL$. Letting $x \in L$ be arbitrary, there are $u_5, u_6 \in U$ and $v \in S$ such that

$$(x, y) = (u_5 v, u_6 v).$$

Then $S^1 x = S^1 u_5 v \subseteq S^1 v$, so $S^1 x = S^1 v$ and $v \in L$. We deduce that $y = u_6 v \in UL$, which is a contradiction. Therefore L has the same cardinality as S. ■

This presents an unsolved (and relatively uninteresting) question.

Open Problem 3.2.9 Does there exist a semigroup without maximal principal left ideals but whose diagonal right act is finitely generated?

We also have the following result, which is an interesting statement on semigroups with a particular kind of structure.

Proposition 3.2.10 *Let Y be a semilattice. For each $\alpha \in Y$ let T_α be a semigroup which is either finite or does not have a finitely generated diagonal right act. Let S be an infinite semigroup which is a semilattice of the T_α. Then the diagonal right act of S is not finitely generated. There are analogous statements for the diagonal bi-act and for cyclic generation.*

PROOF. Assume that S is an infinite semigroup which is a semilattice of the T_α and that the diagonal right act of S is finitely generated. By Theorem 3.2.2, S has a unique maximal $\mathcal{J}$-class J, so the underlying semilattice has a unique maximal element μ. Moreover, J has the same cardinality as S. As $J \subseteq T_\mu$ it follows that T_μ is infinite. Further, $T_\mu \leq S$ and $S \setminus T_\mu$ is an ideal of S. By Proposition 3.1.4 the diagonal right act of T_μ is finitely generated, which contradicts our assumptions.

In the cyclic case a contradiction arises because Proposition 3.1.5 implies that $S = T_\mu$. ■

3.3 Classes admitting finitely generated diagonal bi-acts: completely zero-simple semigroups

The next result first appeared in [40] and classifies groups with finitely generated diagonal bi-acts. We use this to deduce more general results, characterising all infinite semigroups in certain classes which have finitely generated diagonal bi-acts.

Proposition 3.3.1 *If G is a group then the diagonal bi-act of G is finitely generated if and only if G has only finitely many conjugacy classes.*

PROOF. ($\Rightarrow$) If the diagonal bi-act of G is finitely generated then there is a finite $U \subseteq G$ such that $G \times G = G(U \times U)G$. For arbitrary $x \in G$ there are $u_1, u_2 \in U, g_1, g_2 \in G$ such that $(x, 1_G) = g_1(u_1, u_2)g_2$. Therefore

$$x = 1_G^{-1}x = g_2^{-1}u_2^{-1}g_1^{-1}g_1u_1g_2 = g_2^{-1}(u_2^{-1}u_1)g_2$$

is conjugate to $u_2^{-1}u_1$, so G has only finitely many conjugacy classes.

($\Leftarrow$) If G has only finitely many conjugacy classes then let $U \subseteq G$ be a set of conjugacy class representatives. We note, as the identity 1_G is only conjugate to itself, that we must have $1_G \in U$. For arbitrary $x, y \in G$ there are $u \in U, g \in G$ such that $xy^{-1} = g^{-1}ug$ and hence

$$(x, y) = g^{-1}(u, 1_G)gy \in G(U \times U)G,$$

so $U \times U$ finitely generates the diagonal bi-act of G. ∎

We now consider diagonal bi-acts of completely zero-simple semigroups.

Theorem 3.3.2 *If S is an infinite completely zero-simple semigroup then the diagonal bi-act of S is finitely generated if and only if S is a group with only finitely many conjugacy classes and a zero adjoined.*

PROOF. ($\Rightarrow$) Assume that $S = \mathcal{M}^0[G; I, \Lambda; P]$ is an infinite completely zero-simple semigroup represented as a zero Rees matrix semigroup and that some finite $U \subseteq S$ satisfies $S \times S = S^1(U \times U)S^1$. We may also assume that U is a set of the form $(I_0 \times G_0 \times \Lambda_0) \cup \{0\}$, where $I_0 \subseteq I, G_0 \subseteq G$ and $\Lambda_0 \subseteq \Lambda$ are finite.

We claim that $|I| = |\Lambda| = 1$. To show this, we begin by supposing that $|I| \geq 2$.

It is obvious that if $(i_0, g_0, \lambda_0) \in U$ then $i_0 \in I_0, g_0 \in G_0, \lambda_0 \in \Lambda_0$. We now consider some arbitrary but fixed element $(i, g, \lambda) \in S \setminus U$ and aim to show that $i \in I_0, g \in G_0G_0^{-1}$ and that $|\Lambda| = 1$. From this it will follow that I, G and Λ are each finite, giving a contradiction. We select arbitrary elements $j \in I \setminus \{i\}, \mu \in \Lambda$ and we consider the pair $[(i, g, \lambda), (j, 1_G, \mu)]$.

There are $(k, d, \nu), (l, e, \pi) \in U$ and $\alpha, \beta \in S^1$ such that

$$[(i, g, \lambda), (j, 1_G, \mu)] = \alpha[(k, d, \nu), (l, e, \pi)]\beta.$$

If $\alpha \neq 1$ and $\beta \neq 1$, say $\alpha = (m, c, \rho)$ and $\beta = (n, f, \sigma)$, then

$$\begin{aligned}[(i, g, \lambda), (j, 1_G, \mu)] &= (m, c, \rho)[(k, d, \nu), (l, e, \pi)](n, f, \sigma) \\ &= [(m, cp_{\rho,k}dp_{\nu,n}f, \sigma), (m, cp_{\rho,l}ep_{\pi,n}f, \sigma)].\end{aligned}$$

Thus $i = j(= m)$, a contradiction. The same contradiction is also obtained if $\alpha \neq 1, \beta = 1$. We conclude that $\alpha = 1$. If $\beta = 1$ then we would have that $(i, g, \lambda) = (k, d, \nu) \in U$, a contradiction. Thus we may let $\beta = (n, f, \sigma)$, which gives

$$[(i, g, \lambda), (j, 1_G, \mu)] = [(k, dp_{\nu,n}f, \sigma), (l, ep_{\pi,n}f, \sigma)].$$

We have shown that the arbitrary element $\mu \in \Lambda$ is equal to λ, so we conclude that $|\Lambda| = 1$. Also, $i(= k) \in I_0$ and hence $I = I_0$.

We also see that $g = dp_{\nu,n}f$ and $1_G = ep_{\pi,n}f$. Then

$$g = g1_G^{-1} = dp_{\nu,n}ff^{-1}p_{\pi,n}^{-1}e^{-1} = dp_{\nu,n}p_{\pi,n}^{-1}e^{-1}.$$

Of course, $|\Lambda| = 1$ implies that $\nu = \pi$, so $p_{\pi,n} = p_{\nu,n}$ and $g = de^{-1} \in G_0G_0^{-1}$. As desired, we are now able to conclude that I, G and Λ are each finite, which is a contradiction.

If we suppose that $|\Lambda| \geq 2$ then we may similarly derive a contradiction. Therefore $|I| = 1$ and $|\Lambda| = 1$, so in fact $S \cong G^0$. By Propositions 3.1.6 and 3.3.1, we further deduce that G has only finitely many conjugacy classes.

($\Leftarrow$) By Propositions 3.1.6 and 3.3.1, if G is a group with only finitely many conjugacy classes and S is G with a zero adjoined then the diagonal bi-act of S is finitely generated. ∎

Corollary 3.3.3 *If S is an infinite completely simple semigroup then the diagonal bi-act of S is finitely generated if and only if S is a group with only finitely many conjugacy classes.*

Proof. ($\Rightarrow$) Let S be an infinite completely simple semigroup with a finitely generated diagonal bi-act. Then S^0 is an infinite completely zero-simple semigroup and, by Proposition 3.1.6, it has a finitely generated diagonal bi-act. By Theorem 3.3.2 it now follows that $S^0 \cong G^0$ where G is a group with finitely many conjugacy classes. Therefore $S \cong G$ and this part of the result is shown.

($\Leftarrow$) This is simply a part of Proposition 3.3.1. ∎

Although these results are positive and cover two important classes, they provide no examples of infinite semigroups with finitely generated diagonal bi-acts beyond those given by Propositions 3.1.6 and 3.3.1. In the next section we find some new examples.

3.4 Classes admitting finitely generated diagonal bi-acts II: Clifford semigroups

The class of Clifford semigroups contains that of groups, so again we will build on Proposition 3.3.1. These are inverse, so Theorem 3.8.1 below implies that an infinite Clifford semigroup does not have a finitely generated diagonal left or right act. We represent Clifford semigroups as strong semilattices of groups, using the notation introduced in Section 2.6. Throughout, we let $S = (Y, G_\alpha, \phi_{\alpha,\beta})$.

Proposition 3.4.1 *If the diagonal bi-act of S is finitely generated then Y is finite and has an identity.*

PROOF. Assume that the diagonal bi-act of S is finitely generated. We define the mapping $\psi : S \to Y$ as $(x)\psi = \alpha$, where $x \in G_\alpha$. This is well-defined because the groups are disjoint and it is surjective because each group is non-empty. For arbitrary $x, y \in S$ let us say that $x \in G_\alpha$ and $y \in G_\beta$, so $(x)\psi = \alpha$ and $(y)\psi = \beta$. Of course, $xy \in G_{\alpha\beta}$, so $(xy)\psi = \alpha\beta = (x\psi)(y\psi)$ and ψ is an epimorphism. Thus Y is a homomorphic image of S; an alternative and shorter proof of this fact could simply use the observation that $\mathcal{H}$ is a congruence on S and that $Y \cong S/\mathcal{H}$.

Now Proposition 3.1.3 implies that the diagonal bi-act of Y is finitely generated. By Theorem 3.6.1 below, it follows that Y is finite (the proof of that result uses nothing from this section, so there is no possibility of a circular argument arising). By Theorem 3.2.2, we know that S must be a principal ideal of itself. Thus Y also has this property, so Y has an identity element. ■

The next main result in our investigation of Clifford semigroups is Proposition 3.4.7 below. To prove this we will need the following small statements. In these we let Z be a subsemilattice of Y and let $T = (Z, G_\alpha, \phi_{\alpha,\beta})$.

Lemma 3.4.2 *The set T is a subsemigroup of S.*

PROOF. Let $x, y \in T$ be arbitrary. Then $x \in G_\gamma$ and $y \in G_\delta$ for some $\gamma, \delta \in Z$. Then $\gamma\delta \in Z$ and $xy \in G_{\gamma\delta} \subseteq T$, so T is a subsemigroup of S. ■

Lemma 3.4.3 *If $Y \setminus Z$ is an ideal of Y then $S \setminus T$ is an ideal of S.*

PROOF. Let $x \in S \setminus T$ and $y \in S$ be arbitrary. Then $x \in G_\epsilon$ (with $\epsilon \in Y \setminus Z$) and that $y \in G_\zeta$ (with $\zeta \in Y$). As $Y \setminus Z$ is an ideal of Y, we must have $\epsilon\zeta \notin Z$. Then $xy, yx \in G_{\epsilon\delta}$, which is disjoint from $(Z, G_\alpha, \phi_{\alpha,\beta})$. Hence $xy, yx \in S \setminus T$, as desired. ■

We are now able to prove the following result, which is effectively the first half of Proposition 3.4.7.

Proposition 3.4.4 *Let Y be a finite semilattice with an identity. For some $\theta \in Y$ let $Z = \{\alpha \in Y : \alpha \geq \theta\}$ and let $T = (Z, G_\alpha, \phi_{\alpha,\beta})$. If the diagonal bi-act of S is finitely generated then the diagonal bi-act of T is finitely generated.*

PROOF. We claim that $Z \leq Y$. To show this we let $\lambda, \mu \in Z$, so that $\theta \leq \lambda$ and $\theta \leq \mu$. Then $\theta\lambda = \theta\mu = \theta$, so $\theta\lambda\mu = \theta$ and hence $\theta \leq \lambda\mu$. Then $\lambda\mu \in Z$, so Z is indeed a subsemigroup of Y. It follows by Lemma 3.4.2 that T is a subsemigroup of S.

We now claim that $Y \setminus Z$ is an ideal of Y. To show this, we let $\lambda \in Y$ and $\mu \in Y \setminus Z$, so that $\theta \nleq \mu$. If we suppose that $\lambda\mu \in Z$, then $\theta \leq \lambda\mu$ and $\theta(\lambda\mu) = \theta$. It follows that $\theta\mu = \theta$, so $\theta \leq \mu$ and $\mu \in Z$, which is a contradiction. We deduce that $\lambda\mu \in Y \setminus Z$, so we have shown our claim that the complement of Z is an ideal of Y. It now follows by Lemma 3.4.3 that $S \setminus T$ is an ideal in S.

If we assume that the diagonal bi-act of S is finitely generated then we conclude, by Proposition 3.1.4, that the diagonal bi-act of T is finitely generated.

■

To prove Proposition 3.4.7 we need further auxiliary results, which we will now prove. Again we let Z be a subsemilattice of Y.

Lemma 3.4.5 *Let $\pi : Y \to Z$ be a homomorphism satisfying $(\alpha)\pi \leq \alpha$ for all $\alpha \in Y$ and $(\alpha)\pi = \alpha$ for all $\alpha \in Z$. Then the mapping $\psi : S \to T$, defined as $(x)\psi = x\phi_{\alpha,(\alpha)\pi}$ (where $x \in G_\alpha$), is a homomorphism which fixes every element of T.*

PROOF. Let $x \in G_\alpha, y \in G_\beta$, so that $xy = (x\phi_{\alpha,\alpha\beta})(y\phi_{\beta,\alpha\beta}) \in G_{\alpha\beta}$. Then

$$\begin{aligned}
(xy)\psi &= [(x\phi_{\alpha,\alpha\beta})(y\phi_{\beta,\alpha\beta})]\phi_{\alpha\beta,(\alpha\beta)\pi} \\
&= (x\phi_{\alpha,\alpha\beta}\phi_{\alpha\beta,(\alpha\beta)\pi})(y\phi_{\beta,\alpha\beta}\phi_{\alpha\beta,(\alpha\beta)\pi}) \\
&= (x\phi_{\alpha,(\alpha\beta)\pi})(y\phi_{\beta,(\alpha\beta)\pi}) \\
&= (x\phi_{\alpha,(\alpha)\pi(\beta)\pi})(y\phi_{\beta,(\alpha)\pi(\beta)\pi}) \\
&= (x\phi_{\alpha,(\alpha)\pi}\phi_{(\alpha)\pi,(\alpha)\pi(\beta)\pi})(y\phi_{\beta,(\beta)\pi}\phi_{(\beta)\pi,(\alpha)\pi(\beta)\pi}) \\
&= (x\phi_{\alpha,(\alpha)\pi})(y\phi_{\beta,(\beta)\pi}) \\
&= (x\psi)(y\psi),
\end{aligned}$$

so ψ is a homomorphism.

As for the claim that ψ fixes each element of T: if $x \in G_\alpha \subseteq T$ then $\alpha \in Z$ and $(\alpha)\pi = \alpha$, so $(x)\psi = (x)\phi_{\alpha,\alpha} = x$. ■

The next result holds in a general semigroup, although we only use it in this section.

Proposition 3.4.6 *Let S be a semigroup and let T_i $(i = 1, \ldots, k)$ be a family of subsemigroups satisfying the following conditions:*

(i) each T_i has a finitely generated diagonal right act;

(ii) for every $x, y \in S$ there is some i such that $x, y \in T_i$.

Then the diagonal right act of S is finitely generated. There is an analogous result for the diagonal bi-act.

PROOF. For each T_i there is a finite $U_i \subseteq T_i$ such that $T_i \times T_i = (U_i \times U_i)T_i^1$. We let $U = \bigcup_{i=1}^{k} U_i$ and claim that $S \times S = (U \times U)S^1$. For all $x, y \in S$ there is some T_i such that $x, y \in T_i$, and $q \in T_i^1, u_1, u_2 \in U_i$ such that $(x, y) = (u_1 q, u_2 q) \in (U \times U)S^1$, completing the proof. ■

Proposition 3.4.7 *Let Y be a finite semilattice with an identity. The diagonal bi-act of S is finitely generated if and only if the diagonal bi-acts of all subsemigroups of S which have the following forms are finitely generated:*

(G) G_β for any $\beta \in Y$;

(T) $G_\beta \cup G_\gamma$ for any comparable $\beta, \gamma \in Y$;

(D) $G_{\beta\gamma} \cup G_\beta \cup G_\gamma \cup G_\delta$, where $\delta = \sup(\beta, \gamma)$, for any incomparable $\beta, \gamma \in Y$.

PROOF. ($\Rightarrow$) Assume that the diagonal bi-act of S is finitely generated. We select and fix arbitrary $\beta, \gamma \in Y$, and we define $Z_1 = \{\alpha \in Y : \beta\gamma \leq \alpha\}$ and the subsemigroup $T_1 = (Z_1, G_\alpha, \phi_{\alpha,\beta})$. By Proposition 3.4.4 it follows that the diagonal bi-act of T_1 is finitely generated.

As Y is a finite semilattice with an identity it follows that Z_1 is also a finite semilattice with an identity. Thus β and γ have a supremum which we will call δ. We let $Z_2 = \{\delta, \beta, \gamma, \beta\gamma\}$ and note that $Z_2 \leq Z_1$. We define $\pi : Z_1 \to Z_2$ as $(\alpha)\pi = \max\{\lambda \in Z_2 : \lambda \leq \alpha\}$. That this mapping is well-defined comes from the fact that there can be no ambiguity regarding the image of each α: the only case which could cause problems is if $\alpha \geq \beta$ and $\alpha \geq \gamma$. However, as δ is the lowest upper bound of β and γ it follows in this case that $\alpha \geq \delta$, so $(\alpha)\pi = \delta$. Further, we have that $(\alpha)\pi \leq \alpha$ for all $\alpha \in Z_1$ and $(\alpha)\pi = \alpha$ for all $\alpha \in Z_2$.

We also claim that π is a homomorphism. To show this, we let $\lambda, \mu \in Z_1$, so that $\lambda, \mu \geq \beta\gamma$ and we consider a number of cases.

First, we suppose that $(\lambda\mu)\pi = \delta$. Then $\lambda\mu \geq \delta$, which can only happen if $\lambda, \mu \geq \delta$. It is now easy to see that

$$(\lambda\mu)\pi = \delta = \delta^2 = (\lambda\pi)(\mu\pi).$$

Next we suppose that $(\lambda\mu)\pi = \beta$. In this case it is clear that $\lambda\mu \geq \beta$ but $\lambda\mu \ngeq \gamma$. This could arise in a number of ways, in each of which both λ and μ are above β, but at least one of them is not above γ. One possibility is that $\lambda \geq \delta$ and $\mu \geq \beta$ but $\mu \ngeq \gamma$. Then $(\lambda)\pi = \delta$ and $(\mu)\pi = \beta$, so

$$(\lambda\pi)(\mu\pi) = \delta\beta = \beta = (\lambda\mu)\pi.$$

A similar manipulation for each of the other cases in which $(\lambda\mu)\pi = \beta$ also gives that $(\lambda\pi)(\mu\pi) = \beta$.

The case in which $(\lambda\mu)\pi = \gamma$ is precisely analogous to the case in which $(\lambda\mu)\pi = \beta$.

Finally, in order to have $(\lambda\mu)\pi = \beta\gamma$, we need $\lambda\mu$ to be above neither β nor γ. For example, we could have $\lambda \geq \beta$ and $\mu \geq \gamma$ but $\lambda \ngeq \gamma$ and $\mu \ngeq \beta$. In this case we have $(\lambda)\pi = \beta$ and $(\mu)\pi = \gamma$, so

$$(\lambda\mu)\pi = \beta\gamma = (\lambda\pi)(\mu\pi).$$

A similar manipulation for each of the other cases in which $(\lambda\mu)\pi = \beta\gamma$ leads to the same equality, so π is a homomorphism.

By Lemma 3.4.5 it follows that if we let $T_2 = (Z_2, G_\alpha, \phi_{\alpha,\beta})$ then there is an epimorphism $\psi : T_1 \to T_2$. By Proposition 3.1.3, the diagonal bi-act of T_2 is finitely generated.

It is clear that T_2 is a subsemigroup of type (G) if $\beta = \gamma$, (T) if β and γ are comparable, and (D) if β and γ are incomparable.

($\Leftarrow$) Let $T_1, \ldots, T_k$ be a list of all those subsemigroups in S of the types listed (of course, this list is finite because Y is finite). If the diagonal bi-acts of each of these is finitely generated then condition (i) of Proposition 3.4.6 (stated for the diagonal bi-act) is satisfied.

We now select arbitrary elements $x, y \in S$, say with $x \in G_\beta$ and $y \in G_\gamma$. If $\beta = \gamma$ then $x, y \in G_\beta$, which is a subsemigroup of type (G) and is therefore one of our T_i. If $\gamma < \beta$ (or $\gamma > \beta$) then $x, y \in G_\beta \cup G_\gamma$, a type (T) subsemigroup. If β and γ are incomparable then x and y are in a type (D) subsemigroup, namely $G_\delta \cup G_\beta \cup G_\gamma \cup G_{\beta\gamma}$ where $\delta = \sup\{\beta, \gamma\}$. That is, the list of subsemigroups $T_1, \ldots, T_k$ satisfies condition (ii) of Proposition 3.4.6, so this part of the result follows. ∎

We have reduced our problem from the case of a general Clifford semigroup to the same problem for three specific and relatively simple types of Clifford semigroup. Representing groups as rectangles, we draw subsemigroups of type (G), (T) and (D) overleaf.

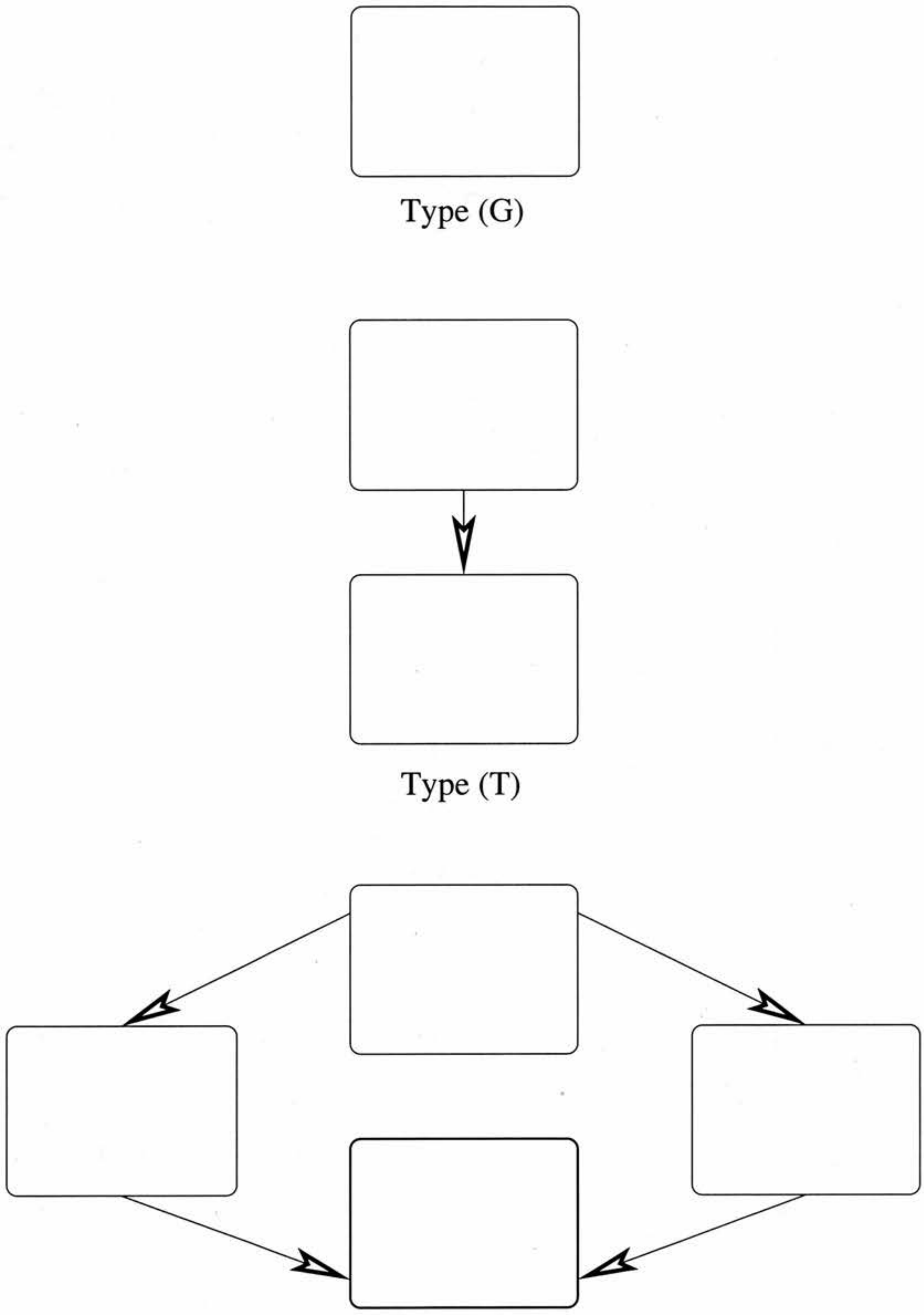

Figure 3.1: Subsemigroup classes (G), (T) and (D)

Proposition 3.3.1 has already answered our question for subsemigroups of type (G). For those of types (T) and (D) we introduce some new notions.

If G is a group and K is a subgroup of G then $x, y \in G$ are K-*conjugate* if some $k \in K$ satisfies $x = k^{-1}yk$. Thus G-conjugacy is the usual form of conjugacy. If H is a subgroup of K and $x, y \in G$ are H-conjugate then they are also K-conjugate. Hence each K-conjugacy class is a union of H-conjugacy classes. In particular, a set of K-conjugacy class representatives contains a set of conjugacy class representatives.

Proposition 3.4.8 *Let $T = (\{\beta, \gamma\}, G_\alpha, \phi_{\alpha,\beta})$ with $\beta > \gamma$ (that is, a type (T) subsemigroup). The diagonal bi-act of T is finitely generated if and only if G_β has only finitely many conjugacy classes and G_γ has only finitely many $\mathrm{Im}(\phi_{\beta,\gamma})$-conjugacy classes.*

PROOF. ($\Rightarrow$) Assume that $T \times T = T(U \times U)T$ for some finite $U \subseteq T$. By Propositions 3.3.1 and 3.4.7 it follows that G_β has only finitely many conjugacy classes.

For arbitrary $x \in G_\gamma$ there are $p, q \in T$ and $u_1, u_2 \in U$ such that

$$(1_{G_\beta}, x) = (pu_1q, pu_2q).$$

From $1_{G_\beta} = pu_1q$ it follows that $p, q, u_1 \in G_\beta$. Now, from $x = pu_2q$ we see that $u_2 \in G_\gamma$. Hence

$$x = 1_{G_\beta}^{-1}x = q^{-1}u_1^{-1}p^{-1}pu_2q = q^{-1}u_1^{-1}u_2q = (q\phi_{\beta,\gamma})^{-1}(u_1^{-1}u_2)(q\phi_{\beta,\gamma})$$

is $\mathrm{Im}(\phi_{\beta,\gamma})$-conjugate to $u_1^{-1}u_2$. Thus $U^{-1}U$ contains a finite set of $\mathrm{Im}(\phi_{\beta,\gamma})$-conjugacy class representatives.

($\Leftarrow$) Assume that G_γ has only finitely many $\mathrm{Im}(\phi_{\beta,\gamma})$-conjugacy classes and that G_β has only finitely many conjugacy classes. Let V and W be sets of representative elements for these, respectively, and let $U = V \cup W$. We will show that $U \times U$ finitely generates the diagonal bi-act of T. From the proof of Proposition 3.3.1 it is clear that $T(U \times U)T$ contains $G_\beta \times G_\beta$ and $G_\gamma \times G_\gamma$. Therefore we only need to demonstrate that it contains $G_\beta \times G_\gamma$ (and $G_\gamma \times G_\beta$, which will follow by a symmetric argument).

For arbitrary $g \in G_\beta$ and $h \in G_\gamma$ it follows that $g^{-1}h \in G_\gamma$, so this is $\mathrm{Im}(\phi_{\alpha,\beta})$-conjugate to some $w \in W$. Thus for some $g_1 \in G_\beta$ we have

$$g^{-1}h = (g_1\phi_{\beta,\gamma})^{-1}w(g_1\phi_{\beta,\gamma}) = g_1^{-1}wg_1.$$

Then

$$(g,h) = gg_1^{-1}(1_{G_\beta}, w)g_1 \in T(U \times U)T,$$

completing the proof. ■

If G is a group and K is a subgroup of G then a *double coset* of K in G is a set of the form $KgK = \{k_1gk_2 : k_1, k_2 \in K\}$ for some $g \in G$. Distinct double cosets of K in G are disjoint and their union equals G; the *double coset index* $[G : K]$ is the number of distinct double cosets of K in G.

We will also use the observation that if $\alpha, \beta, \gamma \in Y$ satisfy $\gamma \leq \beta \leq \alpha$, then $\mathrm{Im}(\phi_{\alpha,\gamma}) \leq \mathrm{Im}(\phi_{\beta,\gamma})$. Therefore if G_γ has only finitely many $\mathrm{Im}(\phi_{\alpha,\gamma})$-conjugacy classes then it has only finitely many $\mathrm{Im}(\phi_{\beta,\gamma})$-conjugacy classes.

Proposition 3.4.9 *Let $T = (\{\delta, \beta, \gamma, \beta\gamma\}, G_\alpha, \phi_{\alpha,\beta})$ where β and γ are incomparable and $\delta = \sup\{\beta, \gamma\}$ (that is, a type (D) subsemigroup) and let $K = \{(g\phi_{\delta,\beta}, g\phi_{\delta,\gamma}) : g \in G_\delta\} \leq G_\beta \times G_\gamma$. The diagonal bi-act of T is finitely generated if and only if the following conditions hold:*

(i) G_δ has finitely many conjugacy classes;

(ii) G_β has finitely many $\mathrm{Im}(\phi_{\delta,\beta})$-conjugacy classes;

(iii) G_γ has finitely many $\mathrm{Im}(\phi_{\delta,\gamma})$-conjugacy classes;

(iv) $G_{\beta\gamma}$ has finitely many $\mathrm{Im}(\phi_{\delta,\beta\gamma})$-conjugacy classes;

(v) $[G_\beta \times G_\gamma : K]$ is finite.

Proof. ($\Rightarrow$) Assume that $T \times T = T(U \times U)T$ for some finite $U \subseteq T$. By Propositions 3.3.1, 3.4.7 and 3.4.8, it follows that conditions (i), (ii), (iii) and (iv) hold.

Let $g \in G_\beta$ and $h \in G_\gamma$ be arbitrary. There are $s, t \in T$ and $u_1, u_2 \in U$ such that

$$(g,h) = (su_1t, su_2t).$$

Then $s, t \in G_\delta$, so

$$(g,h) = (s\phi_{\delta,\beta}, s\phi_{\delta,\gamma})(u_1,u_2)(t\phi_{\delta,\beta}, t\phi_{\delta,\gamma}) \in K(u_1,u_2)K.$$

That is, $U \times U$ contains a finite set of double coset representatives for K in $G_\beta \times G_\gamma$, and condition (v) holds.

($\Leftarrow$) Assume that the stated conditions hold. By (i), (ii), (iii), (iv) and Propositions 3.3.1 and 3.4.8, it follows that all of $T \times T$ can be finitely generated as a diagonal bi-act, except perhaps for $G_\beta \times G_\gamma$ (and $G_\gamma \times G_\beta$, but this will be produced in a symmetric manner).

Let $V \subseteq G_\beta$ and $W \subseteq G_\gamma$ be finite sets such that $V \times W$ contains double coset representatives of K in $G_\beta \times G_\gamma$. For arbitrary $g \in G_\beta$ and $h \in G_\gamma$ there are $v \in V$ and $w \in W$ such that $(v,w) \in K(g,h)K$. Hence, for some $f_1, f_2 \in G_\delta$, we may write

$$\begin{aligned}(g,h) &= (f_1\phi_{\delta,\beta}, f_1\phi_{\delta,\gamma})(v,w)(f_2\phi_{\delta,\beta}, f_2\phi_{\delta,\gamma}) \\ &= (f_1vf_2, f_1wf_2) \subseteq T(V \times W)T,\end{aligned}$$

completing the proof. ■

Propositions 3.3.1, 3.4.7, 3.4.8 and 3.4.9 combine to give the following.

Theorem 3.4.10 *The diagonal bi-act of $(Y, G_\alpha, \phi_{\alpha,\beta})$ is finitely generated if and only if the following conditions are satisfied:*

(i) Y is a finite semilattice with an identity μ;

(ii) for all $\alpha \in Y$ the group G_α has only finitely many $\mathrm{Im}(\phi_{\mu,\alpha})$-conjugacy classes;

(iii) $[G_\beta \times G_\gamma : K]$ is finite for all incomparable $\beta, \gamma \in Y$, where $\delta = \sup(\beta,\gamma)$ and $K = \{(g\phi_{\delta,\beta}, g\phi_{\delta,\gamma}) : g \in G_\delta\}$.

Considering Corollary 3.3.3 and Theorem 3.4.10, one possible next step would be to describe all the completely regular semigroups with finitely generated diagonal bi-acts. Sadly, this is an open problem.

Open Problem 3.4.11 Can we classify all the infinite completely regular semigroups with finitely generated diagonal bi-acts? What about completely regular semigroups for which $\mathcal{H}$ is a congruence (this class properly contains the class of Clifford semigroups)?

Theorem 4.3.3 below gives us another example of an infinite inverse semigroup with a finitely generated diagonal bi-act; from that proof and also from Proposition 3.3.1 and Theorem 3.4.10 it becomes clear that conjugacy in inverse semigroups is a thematic link in these questions. In general, however, the following remains another unsolved question.

Open Problem 3.4.12 Which infinite inverse semigroups have finitely generated diagonal bi-acts?

3.5 Classes which do not admit cyclic diagonal bi-acts

Besides those semigroups with cyclic diagonal right or left acts, Theorem 4.3.3 below shows that the semigroup $\mathcal{I}_X$ of partial injective transformations on an infinite set X has a cyclic diagonal bi-act. We will now prove that certain classes of semigroups contain no such examples.

Proposition 3.5.1 *If S is a cancellative non-trivial semigroup then the diagonal bi-act of S is not cyclic.*

Proof. Assume that S is cancellative, non-trivial and that there are distinct $a, b \in S$ with $S \times S = S^1(a,b)S^1$. For $x \in S$ there are $p, q \in S^1$ with $(x,x) = (paq, pbq)$. From $paq = pbq$ it follows that $a = b$, a contradiction. ■

Theorem 3.5.2 *If S is a finite non-trivial semigroup then the diagonal bi-act of S is not cyclic.*

Proof. Assume that S is finite, non-trivial and that $S \times S = S^1(a,b)S^1$ for some distinct $a, b \in S$. For arbitrary $x, y \in S$ there are $u, v \in S^1$ such that

$$(x,y) = (uav, ubv).$$

			$\bullet q$
	$\bullet b$		$\bullet bq$
$\bullet a$			$\bullet aq$
$\bullet pa$	$\bullet pb$	$\bullet p$	$\bullet paq$ $\bullet pbq$

Table 3.1: The positions of some elements within J.

Hence $J_x \leq J_a$. So J_a is a unique maximal $\mathcal{J}$-class. We rename this class J. Clearly $a, b \in J$, so $|J| \geq 2$. For $s, t \in J \setminus \{a\}$ there are $p, q \in S^1$ such that

$$(s, t) = (paq, pbq).$$

Then $J = J_s \leq J_{pa}$, so $J = J_{pa}$ and hence $pa \in J$. It may analogously be shown that $pb, aq, bq \in J$ and $p, q \in J^1$. We now represent J as the $\mathcal{D}$-class eggbox picture in Table 3.1.

If $p, q \in S$ (that is, if $p \neq 1$ and $q \neq 1$) then $s = paq\mathcal{H}pbq = t$. If $p \in S$ (that is, $p \neq 1$) but $q = 1$ then $s = pa\mathcal{R}pb = t$. If $p = 1$ and $q \in S$ then $s = aq\mathcal{L}bq = t$. If $p = q = 1$ then $s = a$, a contradiction, so this is not the case. In conclusion, any pair of elements from J are either $\mathcal{R}$- or $\mathcal{L}$-related, so J is either a single $\mathcal{R}$-class or a single $\mathcal{L}$-class. Let us assume that it is a single $\mathcal{L}$-class.

There are $g, h \in S^1$ such that

$$(b, a) = (gah, gbh).$$

By the earlier argument, $ga, gb, ah, bh \in J$ and $g, h \in J^1$. We now draw J as the eggbox picture in Table 3.2.

If $g, h \in S$ (that is, $g \neq 1$ and $h \neq 1$) then $a = gbh\mathcal{H}gah = b$. Similarly, if $g \in S$ (that is, $g \neq 1$) but $h = 1$ then $a = gb\mathcal{H}ga = b$. If $g = 1$ and $h \in S$ then $a = bh$ and $ah = b$, so $a\mathcal{R}b$ and hence $a\mathcal{H}b$. From $a\mathcal{H}b$ we conclude that $s\mathcal{H}t$ for all $s, t \in J$. That is, J is a single $\mathcal{H}$-class. Analogously this conclusion may be deduced from the assumption that J is a single $\mathcal{R}$-class.

$\bullet h$
$\bullet g \quad \bullet ga \quad \bullet gb$ $\bullet gah \quad \bullet gbh$
$\bullet a$ $\bullet ah$
$\bullet b$ $\bullet bh$

Table 3.2: The positions of some more elements within J.

Clearly $J^2 \cap J \neq \emptyset$, so J is a group. As $J \leq S$ and $S \setminus J$ is either empty or is an ideal of S, it follows by Proposition 3.1.5 that it must be empty, so $S = J$. By Proposition 3.5.1 it follows that J is trivial. ■

The proof of this result seems unnaturally long and intricate. Perhaps there is a more concise argument which remains undiscovered.

Corollary 3.5.3 *If S is a completely zero-simple semigroup then the diagonal bi-act of S is not cyclic.*

PROOF. Assume that S is a completely zero-simple semigroup (so it is not trivial) and that the diagonal bi-act of S is cyclic. Theorem 3.5.2 tells us that S is infinite. Then, by Theorem 3.3.2, we see that $S \cong G^0$ where G is an infinite group with only finitely many conjugacy classes. By Proposition 3.1.5 there is a contradiction, as G is a subsemigroup with an ideal complement in S. ■

Corollary 3.5.4 *If S is a non-trivial completely simple semigroup then the diagonal bi-act of S is not cyclic.*

PROOF. Assume that S is a non-trivial completely simple semigroup and that the diagonal bi-act of S is finitely generated. Theorem 3.5.2 tells us that S is infinite. Then, by Corollary 3.3.3, we see that S is a group with only finitely many conjugacy classes. This is a contradiction to Proposition 3.5.1. ■

In [26] it is shown that a semigroup is completely regular if and only if it is a semilattice of completely simple semigroups. Using this fact, the next result follows from Corollary 3.5.4 and Proposition 3.2.10.

Corollary 3.5.5 *If S is a non-trivial completely regular semigroup then the diagonal bi-act of S is not cyclic.*

3.6 Classes which do not admit finitely generated diagonal bi-acts

If the diagonal right, left or bi-act of S is cyclic or finitely generated, then the diagonal bi-act of S is finitely generated. However, examples of infinite semigroups with finitely generated diagonal bi-acts are still hard to find among the standard classes. In this section we show that several classes of semigroups contain no such examples.

Theorem 3.6.1 *If S is an infinite commutative semigroup then the diagonal bi-act of S is not finitely generated.*

PROOF. Assume that S is an infinite commutative semigroup and that there is a finite $A \subseteq S \times S$ such that $S \times S = S^1 A S^1$.

By Theorem 3.2.2 there are infinitely many elements $y \in S$ such that $yS = S$. We fix one such $y \in S$ and define the partial mapping $\psi : A \to S$ as $(a, b)\psi = yz$ where there exist $p, q \in S^1$ such that

$$(y, z) = (paq, pbq).$$

This is clearly surjective and we will show that it is well-defined. Suppose that

$$(y, z_1) = (sat, sbt),\ (y, z_2) = (uav, ubv),$$

with $(a, b) \in A$, $z_1, z_2 \in S$ and $s, t, u, v \in S^1$. Then $(a, b)\psi = yz_1,\ (a, b)\psi = yz_2$ and

$$yz_1 = uavsbt = satubv = yz_2.$$

So ψ is well-defined and S is finite, a contradiction. ■

Corollary 3.6.2 *If S is a Bruck–Reilly extension then the diagonal bi-act of S is not finitely generated.*

PROOF. Let S be the Bruck–Reilly extension of the monoid $T = \mathrm{Mon}\langle X \mid R\rangle$ with respect to the endomorphism $\pi : T \to T$ and assume that the diagonal bi-act of S is finitely generated. Then S is the monoid presented by

$$\mathrm{Mon}\langle X, b, c \mid R, bc = 1, bx = (x\pi)b, xc = c(x\pi)\ (x \in X)\rangle.$$

The mappings $(b)\phi = z^{-1}, (c)\phi = z$ and $(x)\phi = 1_G$ (for all $x \in X$) can be extended to a full epimorphism $\phi : S \to G$ where G is the infinite cyclic group generated by z. By Proposition 3.1.3, the diagonal bi-act of G is finitely generated, contradicting Theorem 3.6.1. ■

Rectangular bands are completely simple, so the following is a consequence of Corollary 3.3.3.

Corollary 3.6.3 *If S is an infinite rectangular band then the diagonal bi-act of S is not finitely generated.*

In [26] it is shown that a semigroup is a band if and only if it is a semilattice of rectangular bands. Using this fact, the next result is a consequence of Corollary 3.6.3 using Proposition 3.2.10.

Corollary 3.6.4 *If S is an infinite band then the diagonal bi-act of S is not finitely generated.*

3.7 Classes which do not admit finitely generated diagonal right acts: left cancellative and right cancellative

Examples of infinite semigroups with finitely generated diagonal right acts appear in [35] and in Section 4.1 below. In this and the next section we list some classes which contain no such examples.

Theorem 3.7.1 *If S is an infinite left cancellative semigroup then the diagonal right act of S is not finitely generated.*

PROOF. Assume that S is an infinite left cancellative semigroup and that there is a finite $A \subseteq S \times S$ such that $S \times S = AS$.

We arbitrarily select and fix $y \in S$. We now define the partial mapping $\psi : A \to S$ as $(a,b)\psi = z$ where there exists $q \in S$ such that

$$(y,z) = (aq, bq).$$

This may be a partial mapping, but we see that it is surjective because A generates the diagonal right act. Of course, it is not yet clear that ψ is well-defined. To show this, we let

$$(y, z_1) = (au, bu),\ (y, z_2) = (av, bv)$$

with $(a,b) \in A$ and $z_1, z_2, u, v \in S$, then $(a,b)\psi = z_1$ and $(a,b)\psi = z_2$. However, $y = au = av$ implies $u = v$, so $z_1 = bu = bv = z_2$. Thus ψ is well-defined and we conclude that S is finite, a contradiction. ■

We will shortly consider the finite generation of the diagonal right acts of right cancellative semigroups. Before we do so, we include the following four propositions, which describe some useful (for our purposes) connections between right cancellative semigroups and the semigroup $\mathcal{I}nj_X$ of full injective transformations on a set X.

Proposition 3.7.2 *Let S be a subsemigroup of the semigroup $\mathcal{I}nj_X$ of full injective transformations on a set X. Then S is right cancellative.*

PROOF. Let $x, y, z \in S$ be arbitrary such that $xz = yz$. Then, for all $i \in X$, we have $(i)xz = (i)yz$, or $[(i)x]z = [(i)y]z$. As $z \in S$ it must be injective, so $(i)x = (i)y$ for all $i \in X$ and hence $x = y$, completing the result. ■

Proposition 3.7.3 *For each $z \in S$ let $\rho_z : S \to S$ be defined as $(x)\rho_z = xz$. Then S is right cancellative if and only if every ρ_z is injective.*

PROOF. ($\Rightarrow$) Assume that S is right cancellative and consider an arbitrary ρ_z. Let $x, y \in S$ be arbitrary and suppose that $(x)\rho_z = (y)\rho_z$, so $xz = yz$. By right cancellativity we have that $x = y$, so ρ_z is injective.

($\Leftarrow$) Assume that every ρ_z is injective. Let $x, y, z \in S$ be arbitrary such that $xz = yz$. We may rewrite this as $(x)\rho_z = (y)\rho_z$ and then use the fact that ρ_z is injective to conclude that $x = y$. Thus S is right cancellative. ■

The 'Cayley theorem for semigroups' states that every semigroup is isomorphic to a subsemigroup of a semigroup of full transformations. This is proven in [26]. We apply this to right cancellative semigroups.

Proposition 3.7.4 *Let S be a right cancellative semigroup and for each $z \in S$ let $\rho_z : S \to S$ be defined as $(x)\rho_z = xz$. Then $\phi : S \to \mathcal{I}nj_S$, defined as $(z)\phi = \rho_z$, is a well-defined homomorphism. Furthermore, if S is a monoid then ϕ is injective and S is isomorphic to a subsemigroup of $\mathcal{I}nj_S$.*

PROOF. That $\phi : S \to \mathcal{I}nj_S$ is well-defined comes from Proposition 3.7.3.

To show that ϕ is a homomorphism, we let $y, z \in S$ be arbitrary and consider $(yz)\phi$, which obviously equals ρ_{yz}. For an arbitrary $x \in S$ we have that

$$(x)\rho_{yz} = xyz = (xy)\rho_z = (x)\rho_y\rho_z,$$

so $\rho_{yz} = \rho_y\rho_z$. That is, $(yz)\phi = (y\phi)(z\phi)$, so ϕ is a homomorphism.

Now assume that S is a monoid with identity 1_S. To show that ϕ is injective, we let $y, z \in S$ be arbitrary such that $(y)\phi = (z)\phi$. Then $\rho_y = \rho_z$ and in particular we must have that $y = (1_S)\rho_y = (1_S)\rho_z = z$, so ϕ is indeed injective. It quickly follows that S is isomorphic to $\mathrm{Im}(\phi)$, which is a subsemigroup of $\mathcal{I}nj_S$. ■

We ask whether all right cancellative semigroups are subsemigroups of $\mathcal{I}nj_X$ for some set X. The presence of the identity element in S is essential in Proposition 3.7.4 to show that ϕ is injective. If we define $\rho_z : S^1 \to S$ then ϕ would certainly be injective. However, if we do this then it may turn out that ρ_z is not injective as we could have $(1)\rho_z = (x)\rho_z$ for some $x \in S$. We now show this to be the case for left zero semigroups (which are, of course, right cancellative).

Proposition 3.7.5 *Let S be a non-trivial left zero semigroup. There is no set X such that S is isomorphic to a subsemigroup of the semigroup $\mathcal{I}nj_X$ of full injective transformations on a set X.*

PROOF. To obtain a proof by contradiction, let us suppose that $S \leq \mathcal{I}nj_X$. We select an arbitrary $x \in S$ and observe that $x^2 = x$. As S is non-trivial we may suppose that x is not the identity transformation. Then there are distinct $i, j \in X$ such that $(i)x = j$. But then $(i)x = (i)x^2 = [(i)x]x = (j)x$, so x is not injective, a contradiction. We conclude that x is the identity transformation, which contradicts our assumptions. ■

Considering these connections, we begin to tackle our question on the generation of diagonal right acts of right cancellative semigroups by proving the following theorem. As well as answering our question for a sub-class of right cancellative semigroups, we will soon utilise this to answer the general question.

Theorem 3.7.6 *Let S be an infinite subsemigroup of the semigroup $\mathcal{I}nj_X$ of full injective transformations on an infinite set X. The diagonal right act of S is not finitely generated.*

PROOF. Assume that there is a finite $A \subseteq S \times S$ such that $S \times S = AS$.

For $x, y \in S$ there are $(a, b) \in A, u \in S$ such that $(x, y) = (au, bu)$. For every $l \in (\mathrm{Im}(y))x^{-1}$ there is $m \in X$ such that $(l)x = (m)y$, so $(l)au = (m)bu$. Thus $(l)a = (m)b$ and $l \in (\mathrm{Im}(b))a^{-1}$. We see that

$$(\mathrm{Im}(y))x^{-1} \subseteq (\mathrm{Im}(b))a^{-1}.$$

The reverse inclusion may be shown similarly, giving

$$(\mathrm{Im}(b))a^{-1} = (\mathrm{Im}(y))x^{-1}.$$

We define a finite family of sets F as $\{(\mathrm{Im}(b))a^{-1} : (a, b) \in A\}$. Then $(\mathrm{Im}(y))x^{-1} \in F$ for all $x, y \in S$.

If S only contained bijections then it would be left cancellative and contradict Theorem 3.7.1, so there is a non-bijection $x \in S$. For all $n \in \mathbb{N}$ we see that

$$(\mathrm{Im}(x^{n+1}))x^{-1} = \mathrm{Im}(x^n) \in F.$$

As x is injective but not surjective, it follows that $\mathrm{Im}(x^{n_1}) \neq \mathrm{Im}(x^{n_2})$ where $n_1 \neq n_2$. Hence F is infinite, which is a contradiction. ■

At this point we note, by Proposition 3.7.4, that we have already answered our question for right cancellative monoids. However, by Proposition 3.7.5, we know that we have not answered it for all right cancellative semigroups. The following theorem completes this task.

Theorem 3.7.7 *If S is an infinite right cancellative semigroup then the diagonal right act of S is not finitely generated.*

PROOF. Assume that S is an infinite right cancellative semigroup and that there is a finite $A \subseteq S \times S$ such that $S \times S = AS$.

We define $\rho_z : S \to S$ (for $z \in S$) and $\phi : S \to \mathcal{I}nj_S$ as in Proposition 3.7.4. Then ϕ is a homomorphism, so Proposition 3.1.3 states that the diagonal right act of $\mathrm{Im}(\phi)$ is finitely generated. But $\mathrm{Im}(\phi)$ is a subsemigroup of $\mathcal{I}nj_S$, so Theorem 3.7.6 states that $\mathrm{Im}(\phi)$ is finite, say with size r. We select representatives $z_1, \ldots, z_r \in S$ such that $\mathrm{Im}(\phi) = \{\rho_{z_1}, \ldots, \rho_{z_r}\}$. Then, for any $x \in S$, we see that

$$\begin{aligned} xS^1 &= \{xz : z \in S^1\} \\ &= \{(x)\rho_z : z \in S\} \cup \{x\} \\ &= \{(x)\rho_z : z = z_1, \ldots, z_r\} \cup \{x\} \\ &= \{x, xz_1, \ldots, xz_r\} \end{aligned}$$

is finite, contradicting Corollary 3.2.6. ■

As stated at the start of the chapter, each of our results concerning the diagonal right act have dual statements for the diagonal left act. Note, however, that the dual of Theorem 3.7.1 concerns the diagonal left acts of right cancellative semigroups, while the dual of Theorem 3.7.7 concerns the diagonal right acts of left cancellative semigroups.

3.8 Classes which do not admit finitely generated diagonal right acts II: inverse, completely zero-simple and completely regular

Theorem 3.8.1 *If S is an infinite inverse semigroup then the diagonal right act of S is not finitely generated.*

Proof. Assume that S is an infinite inverse semigroup and that there is a finite $A \subseteq S \times S$ such that $S \times S = AS$.

We arbitrarily select and fix an idempotent $e \in S$. We define the partial mapping $\psi : A \to Se$ as $(a,b)\psi = ze$ where there exists $q \in S$ such that

$$(e,z) = (aq, bq).$$

This is clearly surjective and we will show that it is well-defined. If

$$(e, z_1) = (au, bu),\ (e, z_2) = (av, bv)$$

with $(a,b) \in A$ and $z_1, z_2, u, v \in S$, then $(a,b)\psi = z_1 e$ and $(a,b)\psi = z_2 e$. However,

$$(ea)(ue)(ea) = e(au)e^2 a = e^4 a = ea,$$

$$(ue)(ea)(ue) = ue^2(au)e = ue^4 = ue.$$

It follows that $ue = (ea)^{-1}$ and it can analogously be shown that $ve = (ea)^{-1}$. Therefore $ue = ve$ and $z_1 e = bue = bve = z_2 e$. So ψ is well-defined and Se is finite.

The transformation $\phi : S \to S$, defined by $(x)\phi = x^{-1}$, is a bijection and $(Se)\phi = eS$, so eS is finite. That is, the principal right ideal generated by an arbitrary idempotent is finite. Further, for all $x \in S$ we see that $xx^{-1}S^1 \subseteq xS^1$ and $xS^1 = xx^{-1}xS^1 \subseteq xx^{-1}S^1$, so $xS^1 = xx^{-1}S^1$. Noting that xx^{-1} is idempotent, this means that every principal right ideal is equal to one generated by an idempotent. Thus every principal right ideal of S is finite, contradicting

Corollary 3.2.6. ■

From any of Theorems 3.7.1, 3.7.7 or 3.8.1 there follows the next result, which first appeared in [40].

Corollary 3.8.2 *If S is an infinite group then the diagonal right act of S is not finitely generated.*

Corollary 3.8.3 *If S is an infinite completely zero-simple semigroup then the diagonal right act of S is not finitely generated.*

PROOF. If S is an infinite completely zero-simple semigroup and has a finitely generated diagonal right act then, by Theorem 3.3.2, it is an infinite group with a zero adjoined. By Corollary 3.8.2 and Proposition 3.1.6 there is a contradiction. ■

Corollary 3.8.4 *If S is an infinite completely simple semigroup then the diagonal right act of S is not finitely generated.*

PROOF. Let S be an infinite completely simple semigroup with a finitely generated diagonal right act. By Proposition 3.1.6, S^0 is an infinite completely zero-simple semigroup with a finitely generated diagonal right act. This contradicts Corollary 3.8.3. ■

Recalling that a semigroup is completely regular if and only if it is a semilattice of completely simple semigroups, Corollary 3.8.4 and Proposition 3.2.10 imply the following result.

Corollary 3.8.5 *If S is an infinite completely regular semigroup then the diagonal right act of S is not finitely generated.*

To conclude the chapter, we include the table overleaf, which summarises the main results. So far most of these have been negative, so it seems that very few infinite semigroups have finitely generated diagonal acts. However, in the next chapter we prove some positive results.

Property of semigroup S	Non-trivial S, cyclic right/left act?	Infinite S, f.g. right/left act?	Non-trivial S, cyclic bi-act?	Infinite S, f.g. bi-act?
Bruck–Reilly extension	No	No	No	No (Cor 3.6.2)
cancellative	No	No (Thm 3.7.1)	No (Prop 3.5.1)	Yes (Prop 3.3.1)
Clifford	No	No (Thm 3.8.1)	No (Cor 3.5.5)	Yes (Thm 3.4.10)
commutative	No	No	No	No (Thm 3.6.1)
completely regular	No	No (Cor 3.8.5)	No (Cor 3.5.5)	Yes (Prop 3.3.1, Thm 3.4.10)
completely simple	No	No (Cor 3.8.4)	No (Cor 3.5.4)	Yes (Cor 3.3.3)
completely zero-simple	No	No (Thm 3.8.3)	No (Cor 3.5.3)	Yes (Thm 3.3.2)
finite	No	N/A	No (Thm 3.5.2)	N/A
idempotent	No	No	No	No (Cor 3.6.4)
inverse	No	No (Thm 3.8.1)	Yes (Thm 4.3.3)	Yes (Prop 3.3.1, Thms 3.4.10, 4.3.3)
left cancellative	No	No (Thm 3.7.1)	?????	Yes (Prop 3.3.1)
locally finite	No	No (Prop 3.1.10)	?????	?????
right cancellative	No	No (Thm 3.7.7)	?????	Yes (Prop 3.3.1)

Table 3.3: Results concerning some standard classes of semigroups.

Chapter 4

Examples of infinite semigroups with finitely generated diagonal acts

Having established some classes of infinite semigroups which do not admit finitely generated diagonal right, left or bi-acts and only a restricted class of Clifford semigroups that do, we now search for further specific examples that have these properties.

The majority of this chapter, but not all of it, appeared in [11].

4.1 Diagonal right acts

We begin with two results from [35, 40] (the first of which also appeared in [2]).

Proposition 4.1.1 *The diagonal right act of $T_{\mathbb{N}}$, the semigroup of full transformations on the natural numbers, is cyclic.*

The following refers to the monoid $R_{\mathbb{N}}$ of partial recursive functions of one variable. In [35] it is shown that $R_{\mathbb{N}}$ is finitely generated but not finitely presented; for further information see [4].

Proposition 4.1.2 *The diagonal right act of $R_{\mathbb{N}}$, the monoid of partial recursive functions of one variable, is cyclic.*

We now consider the semigroup $\mathcal{B}_X$ of binary relations on a set X in the same context.

Theorem 4.1.3 *The diagonal right act of $\mathcal{B}_X$, the semigroup of binary relations on an infinite set X, is cyclic.*

PROOF. Let X_1 and X_2 be disjoint subsets of X, each with the same cardinality as X. We fix bijections $a : X \to X_1$ and $b : X \to X_2$, consider them as binary relations

$$a = \{(i, (i)a) : i \in X\}, \quad b = \{(i, (i)b) : i \in X\},$$

and claim that $\mathcal{B}_X \times \mathcal{B}_X = (a, b)\mathcal{B}_X$.

Select arbitrary $x, y \in \mathcal{B}_X$ and construct $u \in \mathcal{B}_X$ as

$$u = \{((i)a, j) : (i, j) \in x\} \cup \{((i)b, j) : (i, j) \in y\}.$$

Then $(x, y) = (au, bu)$ and hence the diagonal right act of $\mathcal{B}_X$ is cyclic. ■

We can also characterise the pairs which generate this act, as we now show. We use the notation $(i)x = \{j \in X : (i, j) \in x\}$ for $i \in X, x \in \mathcal{B}_X$.

Theorem 4.1.4 *The diagonal right act of $\mathcal{B}_X$, the semigroup of binary relations on an infinite set X, is generated by (a, b) if and only if every set in the family $\{(i)a : i \in X\} \cup \{(i)b : i \in X\}$ contains an element that is not in any other set of this family.*

PROOF. ($\Rightarrow$) We assume that $a, b \in \mathcal{B}_X$ satisfy $\mathcal{B}_X \times \mathcal{B}_X = (a, b)\mathcal{B}_X$. We let $x, y \in \mathcal{B}_X$ be an arbitrary pair of full injective transformations with disjoint image sets, and we find $u \in \mathcal{B}_X$ such that

$$(x, y) = (au, bu).$$

In order to derive a proof by contradiction, we suppose that that there is some set in the family $\{(i)a : i \in X\} \cup \{(i)b : i \in X\}$ which does not contain any element that is outside all other sets of this family. Let us say that this set is $(i)a$, for some particular $i \in X$. We let $k \in (i)a$ be arbitrary and note that there is some $l \in X$ with either $k \in (l)a$ (and $i \neq l$) or $k \in (l)b$. We claim that $(k)u$ is empty, where $u \in \mathcal{B}_X$ is as above.

Let us consider the first possibility for k, in which we have $k \in (l)a$ for some $l \neq i$. To show our claim, we suppose that $(k)u$ is not empty, so there is some $m \in (k)u$. Then $(i,k),(l,k) \in a$ and $(k,m) \in u$, so $(i,m),(l,m) \in au = x$. But x is an injective transformation, so this means that $m = (i)x = (l)x$, which is a contradiction.

Now we consider the second possibility, in which $k \in (l)b$ for some $l \in X$. If there is some $m \in (k)u$ then $(i,k) \in a, (k,m) \in u$ and hence $(i,m) \in x$, so $(i)x = m$. Similarly we may derive $(l)y = m$, which contradicts the assumption that x and y have disjoint image sets.

So $(k)u$ is empty. Indeed, we have shown this for an arbitrary $k \in (i)a$, so it follows that $(i)x$ is empty, a contradiction. We conclude that the stated condition is necessary.

($\Leftarrow$) Returning to the proof of Theorem 4.1.3 we can see that $(x,y) = (au, bu)$ even if we 'enlarge' a and b by adding pairs (i,j) for any $i \in X$ and $j \in X \setminus (X_1 \cup X_2)$. This is because u is not defined outside of $X_1 \cup X_2$, so the resulting relation is the same in each case. In this way, a and b may become any pair of binary relations for which our stated condition holds. ∎

We now ask for which generators the diagonal right act of $\mathcal{B}_X$ is free.

Theorem 4.1.5 *The diagonal right act of $\mathcal{B}_X$ is free with respect to the generator (a,b) if and only if a and b are full injective transformations with disjoint image sets and* $\mathrm{Im}(a) \cup \mathrm{Im}(b) = X$.

PROOF. ($\Rightarrow$) Suppose that (a,b) generates the diagonal right act of $\mathcal{B}_X$ and that this act is free with respect to (a,b). By Theorem 4.1.4 we know that every set in the family $\{(i)a : i \in X\} \cup \{(i)b : i \in X\}$ contains an element that is not in any other set of this family. First, we will claim that every set of this family contains a unique such element.

For each set of this family, say $(i)a$ or $(i)b$, we select and fix a representative element $r_{(i)a}$ or $r_{(i)b}$, which is outside all other sets of this family. Then, as in the proof of Theorem 4.1.3 (although with slightly different notation as we are letting a and b be more general binary relations), we let $x, y \in \mathcal{B}_X$ be arbitrary, define $u \in \mathcal{B}_X$ as

$$u \ = \ \{(r_{(i)a}, j) : (i,j) \in x\} \cup \{(r_{(i)b}, j) : (i,j) \in y\} \tag{4.1}$$

and note that $(x, y) = (au, bu)$. If we now suppose that there are two distinct possible choices for one of the $r_{(i)a}$ (or $r_{(i)b}$), then by (4.1) for every $x, y \in \mathcal{B}_X$ we may construct two distinct binary relations u that satisfy $(x, y) = (au, bu)$. This contradicts the assumption that the act is free.

Next, we claim that $\{r_{(i)a} : i \in X\} \cup \{r_{(i)b} : i \in X\} = X$. Again, we suppose that this is not the case, so there is an element j outside of this union, and we aim to obtain a contradiction. We let $x \in \mathcal{B}_X$ be a full constant transformation, say with $(i)x = k$ for all $i \in X$ and some fixed $k \in X$. From (4.1) we know that

$$u = \{(r_{(i)a}, k) : i \in X\} \cup \{(r_{(i)b}, k) : i \in X\}$$

satisfies $(x, x) = (au, bu)$. However, if we let $v = u \cup \{(j, k)\}$ then we have $(x, x) = (av, bv)$ while $u \neq v$, which is a contradiction.

Combining these observations, the only possibility is that a and b are full injective transformations with disjoint image sets and that $\mathrm{Im}(a) \cup \mathrm{Im}(b) = X$.

($\Leftarrow$) Assume that $a, b \in \mathcal{B}_X$ are full injective transformations on X with disjoint image sets and that $\mathrm{Im}(a) \cup \mathrm{Im}(b) = X$, so they clearly satisfy $\mathcal{B}_X \times \mathcal{B}_X = (a, b)\mathcal{B}_X$. Now assume that this act is not free with respect to the generator (a, b), so there are distinct $u, v \in \mathcal{B}_X$ that satisfy $(au, bu) = (av, bv)$. As u and v are distinct, there is some $i \in X$ such that $(i)u \neq (i)v$. We also see that there is a unique $j \in X$ and exactly one of either a or b (let us say a) such that $(j)a = i$. It now follows that $(j)au \neq (j)av$, so $au \neq av$, a contradiction.

■

We now turn to some other semigroups of transformations. First we note that, in the proof of Theorem 4.1.3, the relations a and b are full injective transformations. Moreover, if x and y are both partial transformations then our construction yields u which is also a partial transformation. Similarly, if x and y are both full or full finite-to-one transformations then our construction yields u which may be extended to a full or full finite-to-one transformation, respectively. Note that to extend u to a full finite-to-one transformation may require a little care to avoid it mapping an infinite set to a single point, but this can always be done. Hence we have the following results.

Corollary 4.1.6 *The diagonal right act of $\mathcal{P}_X$, the semigroup of partial transformations on an infinite set X, is cyclic.*

Corollary 4.1.7 *The diagonal right act of $\mathcal{T}_X$, the semigroup of full transformations on an infinite set X, is cyclic.*

(This is an extension of Proposition 4.1.1.)

Corollary 4.1.8 *The diagonal right act of $\mathcal{F}_X$, the semigroup of full finite-to-one transformations on an infinite set X, is cyclic.*

Indeed, we may even derive Proposition 4.1.2 in this manner, as is shown in [35]. In a similar manner to Theorem 4.1.4, we have the following result.

Corollary 4.1.9 *The diagonal right act of $\mathcal{P}_X$, the semigroup of partial transformations on an infinite set X, is generated by (a, b) if and only if a and b are full, injective and their image sets are disjoint. Further, the diagonal right act of $\mathcal{P}_X$ is free with respect to the generator (a, b) if and only if $\mathrm{Im}(a) \cup \mathrm{Im}(b) = X$. Analogous statements hold for $\mathcal{T}_X$, $\mathcal{F}_X$ and $R_{\mathbb{N}}$.*

It is not possible to extend Theorem 4.1.3 to partial (or full) injective mappings. Indeed, Theorem 3.8.1 states that no infinite inverse semigroup has a finitely generated diagonal right act. Therefore we have the following result.

Theorem 4.1.10 *The diagonal right act of $\mathcal{I}_X$, the semigroup of partial injective transformations on an infinite set X, is not finitely generated.*

4.2 Diagonal left acts

From [35] we have the following two propositions.

Proposition 4.2.1 *The diagonal left act of $T_{\mathbb{N}}$, the semigroup of full transformations on the natural numbers, is cyclic.*

Proposition 4.2.2 *The diagonal left act of $R_{\mathbb{N}}$, the monoid of partial recursive functions of one variable, is cyclic.*

Again, we consider the semigroup $\mathcal{B}_X$ of binary relations on a set X.

For $x \in \mathcal{B}_X$ we define $x^R \in \mathcal{B}_X$ (the 'reverse' of x) as $\{(j,i) : (i,j) \in x\}$. Then the mapping $x \mapsto x^R$ is a permutation on $\mathcal{B}_X$ and is its own inverse. It is important to note, for all $x, y \in \mathcal{B}_X$, that $(xy)^R = y^R x^R$. In the following, we are showing and using the well-known fact that $\mathcal{B}_X$ is 'left-right dual'.

Theorem 4.2.3 *The diagonal left act of $\mathcal{B}_X$, the semigroup of binary relations on an infinite set X, is cyclic.*

PROOF. We let $a, b \in \mathcal{B}_X$ be defined as in Theorem 4.1.3, and claim that $\mathcal{B}_X \times \mathcal{B}_X = \mathcal{B}_X(a^R, b^R)$. For arbitrary $x, y \in \mathcal{B}_X$ there is $u \in \mathcal{B}_X$ such that $(x^R, y^R) = (au, bu)$. Then

$$\begin{aligned}(x,y) &= ((x^R)^R, (y^R)^R) \\ &= ((au)^R, (bu)^R) \\ &= (u^R a^R, u^R b^R) \in \mathcal{B}_X(a^R, b^R),\end{aligned}$$

so the diagonal left act is cyclic, as desired. ∎

We may also characterise all possible generating pairs for the diagonal left act of $\mathcal{B}_X$, and those with respect to which the act is free, as the reverses of those stated in Theorems 4.1.4 and 4.1.5.

However, we cannot use the proof of Theorem 4.2.3 to show equivalent results for $\mathcal{P}_X$, $\mathcal{T}_X$ or $\mathcal{F}_X$, as the reverse of a transformation is not, in general, a transformation. We turn to $\mathcal{T}_X$ and use a different method of proof. We note that this is an extension of Proposition 4.2.1.

Theorem 4.2.4 *The diagonal left act of $\mathcal{T}_X$, the semigroup of full transformations on an infinite set X, is cyclic.*

PROOF. We fix a surjection $g : X \to X \times X$ and let $p_1, p_2 : X \times X \to X$ denote projections onto the first and second co-ordinates, respectively. We let

$a, b \in \mathcal{T}_X$ be defined as gp_1 and gp_2 respectively and we claim that the equality $\mathcal{T}_X \times \mathcal{T}_X = \mathcal{T}_X(a, b)$ holds.

We let $f : X \times X \to X$ be a left inverse of g, so that fg is the identity mapping on $X \times X$ (note that in general there is more than one mapping f with this property). For $x, y \in \mathcal{T}_X$ we define $u \in \mathcal{T}_X$ as $(i)u = ((i)x, (i)y)f$ for all $i \in X$. For all $i \in X$ we see that

$$\begin{aligned} (i)ua &= ((i)x, (i)y)fa \\ &= ((i)x, (i)y)fgp_1 \\ &= ((i)x, (i)y)p_1 \\ &= (i)x, \end{aligned}$$

so $x = ua$. Similarly we can deduce that $y = ub$, so $(x, y) = (ua, ub)$ and hence the diagonal left act of $\mathcal{T}_X$ is cyclic. ■

We may also characterise all pairs of transformations which generate the diagonal left act of $\mathcal{T}_X$.

Corollary 4.2.5 *The diagonal left act of $\mathcal{T}_X$, the semigroup of full transformations on an infinite set X, is generated by (a, b) if and only if the mapping $(a, b) : X \to X \times X$, defined as $(i)(a, b) = ((i)a, (i)b)$, is a surjection. Furthermore, the diagonal left act of $\mathcal{T}_X$ is free with respect to the generator (a, b) if and only if the mapping (a, b) is a bijection.*

PROOF. ($\Rightarrow$) Assume that $\mathcal{T}_X \times \mathcal{T}_X = \mathcal{T}_X(a, b)$. Let $j, k \in X$ be arbitrary and let $i \in X$ and $x, y \in \mathcal{T}_X$ be arbitrary such that $(i)(x, y) = (j, k)$. There is $u \in \mathcal{T}_X$ such that

$$(x, y) = (ua, ub).$$

Hence $(j, k) = (i)(x, y) = (i)(ua, ub) = ((i)u)(a, b)$, so (a, b) is a surjection.

To prove the second statement of the result we suppose that the diagonal left act of $\mathcal{T}_X$ is free with respect to the generator (a, b), but that the mapping $(a, b) : X \to X \times X$ is not a bijection. As it is certainly a surjection it must not be an injection, so there are distinct $p, q \in X$ with $(p)(a, b) = (q)(a, b)$. Let us define $u \in \mathcal{T}_X$ as $(p)u = q$, $(q)u = p$ and $(i)u = i$ for all $i \in X \setminus \{p, q\}$.

Now we note that $(a, b) = (ua, ub)$ but that u is not equal to the identity transformation on X, a contradiction.

($\Leftarrow$) The mapping (a, b) corresponds precisely to what we called g in the proof of Theorem 4.2.4, where it was shown that if g is a surjection then (a, b) generates the diagonal left act of $\mathcal{T}_X$.

To show the statement regarding freeness, let us assume that the mapping (a, b) is a bijection but that the diagonal right act of $\mathcal{T}_X$ is not free with respect to the generator (a, b). Then there are two distinct elements $u, v \in \mathcal{T}_X$ satisfying

$$(ua, ub) = (va, vb).$$

As u and v are distinct there exists $i \in X$ such that $(i)u \neq (i)v$. As (a, b) is a bijection it follows that $[(i)u](a, b) \neq [(i)v](a, b)$, so (ua, ub) and (va, vb) disagree on i. This is a contradiction and the result is shown. ■

We now turn to $\mathcal{P}_X$, which we may consider as the subsemigroup of $\mathcal{T}_{X\cup\{-\}}$ consisting of all the transformations x with $(-)x = -$. If we apply the proof of Theorem 4.2.4 to $\mathcal{T}_{X\cup\{-\}}$ ensuring that $(-)g = (-, -)$, then it is clear that $a, b \in \mathcal{P}_X$, and that if $x, y \in \mathcal{P}_X$ then $u \in \mathcal{P}_X$. Thus we have the following result.

Corollary 4.2.6 *The diagonal left act of $\mathcal{P}_X$, the semigroup of partial transformations on an infinite set X, is cyclic.*

We can also apply the same logic as Theorem 4.2.5 to show the following result.

Corollary 4.2.7 *The diagonal left act of $\mathcal{P}_X$, the semigroup of partial transformations on an infinite set X, is generated by (a, b) if and only if the mapping*

$$(a, b) : (X \cup \{-\}) \to (X \cup \{-\}) \times (X \cup \{-\}),$$

which we define as $(i)(a, b) = ((i)a, (i)b)$, is surjective. Further, the diagonal left act of $\mathcal{P}_X$ is free with respect to the generator (a, b) if and only if (a, b) is a bijection.

From [35], we know that the proof of Proposition 4.2.2 uses the same logic as the proof of Corollary 4.2.4. Similarly, the same proof as Corollary 4.2.5 may be used to show the next result.

Proposition 4.2.8 *The diagonal left act of $R_{\mathbb{N}}$, the monoid of partial recursive functions of one variable, is generated by (a, b) if and only if the mapping $(a, b) : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, defined as $(i)(a, b) = ((i)a, (i)b)$ is a surjection. Moreover, this act is free with respect to (a, b) if and only if (a, b) is a bijection.*

By way of contrast with Theorem 4.2.4 and Corollary 4.2.6, we have the following result.

Theorem 4.2.9 *The diagonal left act of $\mathcal{F}_X$, the semigroup of full finite-to-one transformations on an infinite set X, is not finitely generated.*

PROOF. Assume that the diagonal left act of $\mathcal{F}_X$ is finitely generated, so there is a finite $A \subseteq \mathcal{F}_X \times \mathcal{F}_X$ such that $\mathcal{F}_X \times \mathcal{F}_X = \mathcal{F}_X A$. For an arbitrary pair $i, j \in X$ there are $k \in X$ are $x, y \in \mathcal{F}_X$ such that $(k)x = i$ and $(k)y = j$. There are also $(a, b) \in A$ and $u \in \mathcal{F}_X$ such that $(x, y) = (ua, ub)$. Then $(i, j) = ((k)u)(a, b)$, so

$$\{((k)a, (k)b) : k \in X, (a, b) \in A\} = X \times X.$$

Fixing $l \in X$, this implies that

$$\{l\} \times X \subseteq \{((k)a, (k)b) : k \in X, (a, b) \in A\}.$$

As A is finite and X is infinite, we can use the pigeonhole principle to see that there is a particular $(a, b) \in A$ and infinite $Y \subseteq X$ such that

$$\{l\} \times Y \subseteq \{((k)a, (k)b) : k \in X\}.$$

From this we can see that $l \in X$ has infinitely many pre-images under a, and hence $a \notin \mathcal{F}_X$, a contradiction. ■

The dual of Theorem 3.8.1 states that no infinite inverse semigroup has a finitely generated diagonal left act. Therefore we have the following result.

Theorem 4.2.10 *The diagonal left act of $\mathcal{I}_X$, the semigroup of partial injective transformations on an infinite set X, is not finitely generated.*

4.3 Diagonal bi-acts

As the diagonal right acts of $\mathcal{B}_X$, $\mathcal{P}_X$, $\mathcal{T}_X$ and $\mathcal{F}_X$ (X infinite) are cyclic, it follows that the diagonal bi-acts of these semigroups are also cyclic. We now consider $\mathcal{I}_X$, the semigroup of partial injective transformations on an infinite set X, and show that, in contrast to its diagonal right and left acts, its diagonal bi-act is cyclic.

This semigroup is important in inverse semigroup theory; just as every group is a subgroup of the symmetric group $\mathcal{S}_X$ of bijections on some set X, every inverse semigroup is a subsemigroup of some $\mathcal{I}_X$. In [29], the notion of cycle representations of permutations is extended to that of path representations of partial injective transformations. We will use this notion, in an infinite context, in the proof of our result.

For $x \in \mathcal{I}_X$ we let Γ_x be a digraph with vertex set X and edges specified by

$$i \to j \Leftrightarrow (i)x = j.$$

Let Ω_X be the set of all digraphs on X in which every vertex has in-degree either 0 or 1 and out-degree either 0 or 1. Then there is a natural bijection $\phi : \mathcal{I}_X \to \Omega_X$, defined as $(x)\phi = \Gamma_x$.

For graphs Λ_1 and Λ_2, a *graph isomorphism* $\psi : \Lambda_1 \to \Lambda_2$ is a bijection between the vertex sets such that

$$i \to j \text{ in } \Lambda_1 \Leftrightarrow (i)\psi \to (j)\psi \text{ in } \Lambda_2.$$

If there exists a graph isomorphism from Λ_1 to Λ_2 then they are *isomorphic*, which is denoted $\Lambda_1 \cong \Lambda_2$.

We say that $p, q \in \mathcal{I}_X$ are *conjugate* if there exists $s \in \mathcal{S}_X$ such that $q = sps^{-1}$. In [29] it is shown, for a finite set X, that $p, q \in \mathcal{I}_X$ are conjugate if and only if they have the same path structure. We state and prove this in terms of digraphs, and where X may be infinite or finite.

Lemma 4.3.1 *Elements $p, q \in \mathcal{I}_X$ are conjugate if and only if Γ_p and Γ_q are isomorphic.*

PROOF. ($\Rightarrow$) Let us assume that $p, q \in \mathcal{I}_X$ are conjugate. That is, there is $s \in \mathcal{S}_X$ such that $q = sps^{-1}$, so

$$\begin{aligned} i \to j \text{ in } \Gamma_p &\Leftrightarrow (i)p = j \\ &\Leftrightarrow [(i)s^{-1}]sps^{-1} = (j)s^{-1} \\ &\Leftrightarrow [(i)s^{-1}]q = (j)s^{-1} \\ &\Leftrightarrow (i)s^{-1} \to (j)s^{-1} \text{ in } \Gamma_q, \end{aligned}$$

and $s^{-1} : \Gamma_p \to \Gamma_q$ is a graph isomorphism.

($\Leftarrow$) Let us assume that Λ_q and Λ_p are graph isomorphic, so there is a graph isomorphism $\psi : \Gamma_p \to \Gamma_q$. This is a bijection on X, so we may consider it as some $s \in \mathcal{S}_X$. Furthermore,

$$\begin{aligned} (i)p = j &\Leftrightarrow i \to j \text{ in } \Gamma_p \\ &\Leftrightarrow (i)\psi \to (j)\psi \text{ in } \Gamma_q \\ &\Leftrightarrow [(i)\psi]q = (j)\psi \\ &\Leftrightarrow (i)sq = (j)s \\ &\Leftrightarrow (i)sqs^{-1} = j \end{aligned}$$

so $p = sqs^{-1}$ and p is conjugate to q. ■

The *components* of a digraph are the connected components of its underlying undirected graph. Up to isomorphism, the only components which may appear in $\Lambda \in \Omega_X$ are:

- finite paths of size r (for all $r \in \mathbb{N}$) $i_1 \to i_2 \to \cdots \to i_r$;
- finite cycles of size r (for all $r \in \mathbb{N}$) $i_1 \to i_2 \to \cdots \to i_r \to i_1$;
- left infinite paths $\cdots \to i_{-2} \to i_{-1} \to i_0$;
- right infinite paths $i_1 \to i_2 \to i_3 \to \cdots$;
- bi-infinite paths $\cdots \to i_{-1} \to i_0 \to i_1 \to \cdots$.

In particular, there are only countably many isomorphism classes for components, so these can be indexed by the set $\mathbb{N}$ of natural numbers. To be more precise, we fix a family $\{R_n : n \in \mathbb{N}\}$ where R_n is a component in some digraph $\Lambda_n \in \Omega_X$ and such that for any component C in any $\Lambda \in \Omega_X$ there is a unique $n \in \mathbb{N}$ such that $C \cong R_n$. In this case we refer to n as the isomorphism class of C.

The restriction of $p \in \mathcal{I}_X$ to $Y \subseteq X$ is denoted $p\restriction_Y$ and is the mapping defined by $(i)p\restriction_Y = (i)p$ if $i, (i)p \in Y$ and $(i)p\restriction_Y = -$ otherwise. It is clear that $p\restriction_Y \in \mathcal{I}_X$.

Lemma 4.3.2 *There exists $c \in \mathcal{I}_X$ such that for all $p \in \mathcal{I}_X$ there exists $s \in \mathcal{I}_X$ such that $p = scs^{-1}$.*

PROOF. We let Z be an index set with the same cardinality as X and we partition X as the disjoint union

$$X = \dot{\bigcup_{z \in Z, n \in \mathbb{N}}} V_{z,n}$$

where each $|V_{z,n}| = |R_n|$. We let $\Upsilon \in \Omega_X$ be arbitrary such that each $V_{z,n}$ is the vertex set of a component of isomorphism class n and let $c \in \mathcal{I}_X$ be such that $\Gamma_c = \Upsilon$.

Let $p \in \mathcal{I}_X$ be arbitrary and let F_q $(q \in Q)$ be the components of Γ_p. As every graph in Ω_X has at most $|X|$ components of each isomorphism class, we may assume without loss of generality that $Q \subseteq Z$. For each F_q we consider $V_{q,n}$, where n is the isomorphism class of F_q. Clearly $F_q \cong V_{q,n}$, so the restrictions $p\restriction_{F_q}$ and $c\restriction_{V_{q,n}}$ are conjugate, by Lemma 4.3.1. That is, there exists $s_q \in \mathcal{S}_X$ such that $p\restriction_{F_q} = s_q c\restriction_{V_{q,n}} s_q^{-1}$. We can see that $s_q\restriction_{F_q}$ has domain F_q, has the range $V_{q,n}$, and that it is injective. Now we define $s : X \to X$ as $(i)s = (i)s_q$ when $i \in F_q$. As the F_q are disjoint it follows that s is well-defined; as the $V_{q,n}$ are disjoint it follows that s is injective. We can also see that s is a full transformation, but it is not necessarily surjective. Finally, for any $i \in X$ (say with $i \in F_q$) we have

$$\begin{aligned}
(i)scs^{-1} &= (i)s_q c\restriction_{V_{q,n}} s_q^{-1} \\
&= (i)p\restriction_{F_q} \\
&= (i)p,
\end{aligned}$$

so $p = scs^{-1}$, as desired. ■

We are now ready to prove the main result of this section.

Theorem 4.3.3 *The diagonal bi-act of $\mathcal{I}_X$, the semigroup of partial injective transformations on an infinite set X, is cyclic.*

PROOF. We claim that $\mathcal{I}_X \times \mathcal{I}_X = \mathcal{I}_X(c, 1)\mathcal{I}_X$ where c is as in Lemma 4.3.2. Let $x, y \in \mathcal{I}_X$ be arbitrary. By Lemma 4.3.2 there exists $s \in \mathcal{I}_X$ such that $xy^{-1} = scs^{-1}$, and we note that

$$(i)x = (j)y \Leftrightarrow (i)xy^{-1} = j \Leftrightarrow (i)scs^{-1} = j \Leftrightarrow (i)sc = (j)s. \qquad (4.2)$$

Let $t : X \to X$ be defined on $\mathrm{Im}(sc) \cup \mathrm{Im}(s)$ such that $[(i)sc]t = (i)x$ and $[(j)s]t = (j)y$.

Suppose that a contradiction is implicit here, in that t is required to map one point to two different image points. This could only be the case if $(i)sc = (j)s$ but $(i)x \neq (j)y$ for some $i, j \in X$, which by (4.2) cannot happen.

Suppose that t is not injective. This is only the case if $(i)x = (j)y$ but $(i)sc \neq (j)s$ for some $i, j \in X$. Again, by (4.2), this does not happen.

So $t \in \mathcal{I}_X$, $(x, y) = (sct, st) = s(c, 1)t$, and the theorem is proved. ■

4.4 Semigroups of transformations without any finitely generated diagonal acts

During the search for the positive results that we have demonstrated so far in this chapter, we also found a number of natural semigroups of transformations without finitely generated diagonal right, left or bi-acts. We first consider $\mathcal{S}_X$, the symmetric group on an infinite set X. The following result is an immediate consequence of Proposition 3.3.1.

Corollary 4.4.1 *The diagonal bi-act of $\mathcal{S}_X$, the symmetric group on an infinite set X, is not finitely generated.*

We now examine $\mathcal{S}urj_X$, the semigroup of full surjective transformations on an infinite set X.

Theorem 4.4.2 *The diagonal bi-act of $\mathcal{S}urj_X$, the semigroup of full surjective transformations on an infinite set X, is not finitely generated.*

PROOF. We first note that $\mathcal{S}_X$ is a subsemigroup of $\mathcal{S}urj_X$, and we claim that $\mathcal{S}urj_X \setminus \mathcal{S}_X$ is an ideal of $\mathcal{S}urj_X$. To prove this, we let $\alpha \in \mathcal{S}urj_X \setminus \mathcal{S}_X$ and $\beta \in \mathcal{S}urj_X$ be arbitrary and we examine $\alpha\beta$ and $\beta\alpha$. As α is not an injection, it is easy to see that $\alpha\beta$ is also not an injection, so $\alpha\beta \in \mathcal{S}urj_X \setminus \mathcal{S}_X$. Let us now suppose that $\beta\alpha \in \mathcal{S}_X$; that is, it is an injection. Again it is easy to see that β must be an injection and hence is a bijection. As $\alpha = \beta^{-1}(\beta\alpha)$ it follows that $\alpha \in \mathcal{S}_X$, a contradiction. We have shown our claim that $\mathcal{S}urj_X \setminus \mathcal{S}_X$ is an ideal.

If we assume that the diagonal bi-act of $\mathcal{S}urj_X$ is finitely generated then we conclude, by Proposition 3.1.4, that the diagonal bi-act of $\mathcal{S}_X$ is finitely generated. This contradicts Corollary 4.4.1. ■

Let $\mathcal{PS}urj_X$ be the semigroup of partial surjective transformations on an infinite set X; that is,

$$\mathcal{PS}urj_X = \{x \in \mathcal{P}_X : \mathrm{Im}(x) = X\}.$$

Theorem 4.4.3 *The diagonal bi-act of $\mathcal{PS}urj_X$, the semigroup of partial surjective transformations on an infinite set X, is not finitely generated.*

PROOF. We note that $\mathcal{S}urj_X \leq \mathcal{PS}urj_X$ and we claim that the complement $\mathcal{PS}urj_X \setminus \mathcal{S}urj_X$ is an ideal of $\mathcal{PS}urj_X$. To show this, we select arbitrary elements $\alpha \in \mathcal{PS}urj_X \setminus \mathcal{S}urj_X$ and $\beta \in \mathcal{PS}urj_X$ and we examine $\alpha\beta$ and $\beta\alpha$. We fix an element $i \notin \mathrm{Dom}(\alpha)$, note that $i \notin \mathrm{Dom}(\alpha\beta)$, and conclude that $\alpha\beta \in \mathcal{PS}urj_X \setminus \mathcal{S}urj_X$. As β is surjective there exists $j \in X$ such that $(j)\beta = i$. Then $j \notin \mathrm{Dom}(\beta\alpha)$, so $\beta\alpha \in \mathcal{PS}urj_X \setminus \mathcal{S}urj_X$. We have shown our claim that $\mathcal{PS}urj_X \setminus \mathcal{S}urj_X$ is an ideal of $\mathcal{PS}urj_X$.

If we assume that the diagonal bi-act of $\mathcal{PS}urj_X$ is finitely generated, then by Proposition 3.1.4 we conclude that the diagonal bi-act of $\mathcal{S}urj_X$ is finitely generated. This contradicts Theorem 4.4.2. ■

Let $\mathcal{I}nj_X$ be the semigroup of full injective transformations on an infinite set X.

Theorem 4.4.4 *The diagonal bi-act of $\mathcal{I}nj_X$, the semigroup of full injective transformations on an infinite set X, is not finitely generated.*

PROOF. We note that $\mathcal{S}_X \leq \mathcal{I}nj_X$. In a similar style to the previous proofs, we claim that $\mathcal{I}nj_X \setminus \mathcal{S}_X$ is an ideal of $\mathcal{I}nj_X$. To show this, we let $\alpha \in \mathcal{I}nj_X \setminus \mathcal{S}_X$ and $\beta \in \mathcal{I}nj_X$ be arbitrary and we examine $\alpha\beta$ and $\beta\alpha$. As $\alpha \notin \mathcal{S}_X$ we know that it is not surjective. So $\beta\alpha$ is not surjective either; that is, $\beta\alpha \in \mathcal{I}nj_X \setminus \mathcal{S}_X$. Now we suppose that $\alpha\beta \in \mathcal{S}_X$. Again it follows that β must be surjective, so β is a bijection. Hence $\alpha = (\alpha\beta)\beta^{-1} \in \mathcal{S}_X$, a contradiction. We have shown our claim that $\mathcal{I}nj_X \setminus \mathcal{S}_X$ is an ideal of $\mathcal{I}nj_X$.

If we assume that the diagonal bi-act of $\mathcal{I}nj_X$ is finitely generated then, by Proposition 3.1.4, we conclude that the diagonal bi-act of $\mathcal{S}_X$ is finitely generated. This contradicts Corollary 4.4.1. ■

Theorem 4.4.5 *The diagonal bi-act of $\mathcal{T}_X \setminus \mathcal{S}urj_X$, the semigroup of full non-surjective transformations on an infinite set X, is not finitely generated.*

PROOF. For brevity we will write $S = \mathcal{T}_X \setminus \mathcal{S}urj_X$. We begin by assuming that there is a finite $A \subseteq S \times S$ which satisfies $S \times S = S^1AS^1$.

Let $j, k, l, m \in X$ be distinct and let $x, y \in S$ be defined as follows. For all $i \neq j$ we let $(i)x = i$; we also let $(j)x = k$. For all $i \neq l$ we let $(i)y = i$; we also let $(l)x = m$. Note that x and y are not surjective but that $\text{Im}(x) \cup \text{Im}(y) = X$.

As usual, there are $u, v \in S^1$ and $(a, b) \in A$ such that

$$(x, y) = (uav, ubv).$$

Then $\text{Im}(x) \subseteq \text{Im}(v)$ and $\text{Im}(y) \subseteq \text{Im}(v)$, so $X = \text{Im}(v)$. In other words, v is surjective, so it cannot be an element of S and we conclude that $v = 1$. We may now write

$$(x, y) = (ua, ub).$$

We now see that $\mathrm{Im}(x) \subseteq \mathrm{Im}(a)$ and as a is not surjective it follows that $\mathrm{Im}(a) = X \setminus \{j\}$. That is, for all $j \in X$ there exists $(a, b) \in A$ with $\mathrm{Im}(a) = X \setminus \{j\}$. Therefore A is infinite, which is a contradiction. ∎

Theorem 4.4.6 *The diagonal bi-act of $\mathcal{T}_X \setminus \mathcal{I}nj_X$, the semigroup of full non-injective transformations on an infinite set X, is not finitely generated.*

Proof. For brevity we write $S = \mathcal{T}_X \setminus \mathcal{I}nj_X$. We begin by assuming that there is a finite $A \subseteq S \times S$ which satisfies $S \times S = S^1 A S^1$.

We consider the mappings x and y from the proof of Theorem 4.4.5, but this time we utilise the fact that x and y are not injective but $\mathrm{Ker}(x) \cap \mathrm{Ker}(y) = \Delta_X$. As before, there are $u, v \in S^1$ and $(a, b) \in A$ such that

$$(x, y) = (uav, ubv).$$

It is clear that $\mathrm{Ker}(x) \supseteq \mathrm{Ker}(u)$ and that $\mathrm{Ker}(y) \supseteq \mathrm{Ker}(u)$. We conclude that $\mathrm{Ker}(u) = \Delta_X$, so u is injective and cannot be an element of S. The only remaining possibility is $u = 1$, which now allows us to write

$$(x, y) = (av, bv).$$

We see that $\mathrm{Ker}(x) \supseteq \mathrm{Ker}(a)$ and we know that a is not injective, so we are able to deduce that $\mathrm{Ker}(a) = \Delta_X \cup \{(j, k), (k, j)\}$. We have shown that for all distinct $j, k \in X$ there is $(a, b) \in A$ in which a has this kernel. We conclude that A is infinite, which is a contradiction. ∎

4.5 Semigroups of monotonic transformations

We now consider the question of the finite generation of the diagonal acts of the semigroups $\mathcal{O}_X$ and $\mathcal{Q}_X$.

Let X be an infinite totally ordered set. Let $\mathcal{O}_X$ denote the semigroup of full monotonic transformations on X; that is,

$$\mathcal{O}_X = \{x \in \mathcal{T}_X : i \geq j \Rightarrow (i)x \geq (j)x \ (i, j \in X)\}.$$

Let $\mathcal{Q}_X$ denote the semigroup of full strictly monotonic transformations on X; that is,

$$\mathcal{Q}_X = \{x \in \mathcal{T}_X : i > j \Rightarrow (i)x > (j)x \ (i, j \in X)\}.$$

Unlike those semigroups of transformations that we have previously considered, $\mathcal{O}_X$ and $\mathcal{Q}_X$ do not only depend on the cardinality of X but also on its structure. For example, although the set $\mathbb{Q}$ of rational numbers has the same cardinality as the set $\mathbb{N}$ of natural numbers, the semigroups $\mathcal{O}_\mathbb{N}$ and $\mathcal{O}_\mathbb{Q}$ behave differently. To illustrate this point, we note that the inverse of a monotonic bijection is also a monotonic bijection, so the group of units of $\mathcal{O}_X$ or $\mathcal{Q}_X$ is the set of monotonic bijections on X. Hence the group of units of $\mathcal{O}_\mathbb{Q}$ is infinite, whereas the group of units of $\mathcal{O}_\mathbb{N}$ is trivial.

As $\mathcal{O}_X$ and $\mathcal{Q}_X$ rely on the structure of X, we include the following proposition and discussion on this subject. We begin by introducing a new notion. If X and Y are totally ordered sets and there exists a monotonic bijection $\phi : X \to Y$ then we say that X and Y are *order isomorphic*. Although $\mathbb{Q}$ and $\mathbb{N}$ are totally ordered sets with the same cardinality, it is easy to see that there is no monotonic bijection between them.

Proposition 4.5.1 *Let X and Y be totally ordered sets. If X and Y are order isomorphic then $\mathcal{O}_X$ and $\mathcal{O}_Y$ are isomorphic.*

Proof. Assume that there is a monotonic bijection $\phi : X \to Y$. We define $\psi : \mathcal{O}_X \to \mathcal{O}_Y$ as $(x)\psi = \phi^{-1}x\phi$. As $x : X \to X$ it quickly follows that $\phi^{-1}x\phi : Y \to Y$. Noting that ϕ^{-1} is monotonic, we also see that the composition $\phi^{-1}x\phi$ is monotonic. That is, ψ is well-defined.

We claim that ψ is a bijection. To show this, we suppose that there are $x, y \in \mathcal{O}_X$ with $(x)\psi = (y)\psi$. Then $\phi^{-1}x\phi = \phi^{-1}y\phi$. As ϕ is a bijection it follows that $x = y$, so ψ is injective. Also, for an arbitrary $y \in \mathcal{O}_Y$ we may write $y = \phi^{-1}\phi y\phi^{-1}\phi = (\phi y\phi^{-1})\psi$, so ψ is surjective.

To show that ψ is a homomorphism, we select arbitrary $x, y \in \mathcal{O}_X$ and observe that

$$(x\psi)(y\psi) = (\phi^{-1}x\phi)(\phi^{-1}y\phi) = \phi^{-1}xy\phi = (xy)\psi,$$

which completes the proof. ■

To show that the converse of Proposition 4.5.1 does not hold, we define some new notions. If X and Y are totally ordered sets then a mapping $\phi : X \to Y$ is *anti-monotonic* if $i \geq j$ implies $(i)\phi \leq (j)\phi$. If there is an anti-monotonic bijection $\phi : X \to Y$ then we say that X and Y are *order anti-isomorphic*. Observing that the composition of two anti-monotonic mappings is monotonic, the proof of Proposition 4.5.1 may be used to show that if X and Y are order anti-isomorphic then $\mathcal{O}_X \cong \mathcal{O}_Y$. Thus $\mathcal{O}_{\mathbb{N}} \cong \mathcal{O}_{-\mathbb{N}}$ although the sets $\mathbb{N}$ and $-\mathbb{N}$ are not order isomorphic.

On the topic of diagonal acts, the following question remains unsolved.

Open Problem 4.5.2 Does there exist an infinite totally ordered set X for which the diagonal bi-act of $\mathcal{O}_X$ or $\mathcal{Q}_X$ is finitely generated, or even cyclic? If so, can we characterise all such X?

However, we will show several partial results relating to this problem. In particular, the diagonal right and left acts of $\mathcal{O}_X$ and $\mathcal{Q}_X$ are never finitely generated. The diagonal bi-act of $\mathcal{O}_X$ is not finitely generated if X has the property of being discrete, which we will define and characterise. We finish by showing one possible approach to the questions of the finite generation of the diagonal bi-acts of $\mathcal{O}_X$ and $\mathcal{Q}_X$, where X is a general totally ordered infinite set.

We start by pointing out an important property of $\mathcal{Q}_X$.

Proposition 4.5.3 *The semigroup $\mathcal{Q}_X$ of full strictly monotonic transformations on an infinite totally ordered set X is a subsemigroup of the semigroup $\mathcal{I}nj_X$ of full injective transformations on X.*

PROOF. We only need to show that an arbitrary $x \in \mathcal{Q}_X$ is injective. Let $i, j \in X$ be arbitrary with $(i)x = (j)x$. Of course, if $i > j$ then $(i)x > (j)x$ and if $i < j$ then $(i)x < (j)x$. We can immediately conclude that $i = j$, so x is injective. ■

By Proposition 3.7.3 we know that $\mathcal{Q}_X$ is right cancellative. Then Theorem 3.7.7 and the dual of Theorem 3.7.1 imply that $\mathcal{Q}_X$ has neither a finitely generated diagonal right nor left act.

Theorem 4.5.4 *The diagonal left act of $\mathcal{O}_X$, the semigroup of full monotonic transformations on an infinite totally ordered set X, is not finitely generated.*

PROOF. Assume that $\mathcal{O}_X \times \mathcal{O}_X = \mathcal{O}_X A$ for some finite $A \subseteq \mathcal{O}_X \times \mathcal{O}_X$.

For $x, y \in \mathcal{O}_X$ we define the mapping $(x, y) : X \to X \times X$ as

$$(i)(x, y) = ((i)x, (i)y).$$

For any $i, j \in X$ there are $x, y \in \mathcal{O}_X$ and $k \in X$ such that $(k)(x, y) = (i, j)$. By writing $(x, y) = (ua, ub)$, with $u \in \mathcal{O}_X$ and $(a, b) \in A$, it quickly follows that $(i, j) = (k)(x, y) = [(k)u](a, b) \in \mathrm{Im}(a, b)$. Therefore we conclude that

$$\bigcup_{(a,b)\in A} \mathrm{Im}(a, b) = X \times X,$$

which will be very important in this proof.

Let $n = |A| + 1$ and let $j_1, \ldots, j_n \in X$ satisfy $j_i < j_{i+1}$ for $i = 1, \ldots, n-1$. For $i = 1, \ldots, n$ the ordered pair (j_i, j_{n+1-i}) is contained in some $\mathrm{Im}(a, b)$ with $(a, b) \in A$. The pigeonhole principle implies that two such pairs are in the same $\mathrm{Im}(a, b)$. That is, there are $k, l \in \mathbb{N}$ (say with $1 \leq k < l \leq n$) such that $(j_k, j_{n+1-k}), (j_l, j_{n+1-l}) \in \mathrm{Im}(a, b)$. Hence there are $p, q \in X$ with

$$((p)a, (p)b) = (j_k, j_{n+1-k}) \text{ and } ((q)a, (q)b) = (j_l, j_{n+1-l}).$$

But $(p)a = j_k < j_l = (q)a$, so $p < q$. Similarly, $(p)b = j_{n+1-k} > j_{n+1-l} = (q)b$, so $p > q$, a contradiction. ∎

Theorem 4.5.5 *The diagonal right act of $\mathcal{O}_X$, the semigroup of full monotonic transformations on an infinite totally ordered set X, is not finitely generated.*

PROOF. Assume that $\mathcal{O}_X \times \mathcal{O}_X = A\mathcal{O}_X$ for some finite $A \subseteq \mathcal{O}_X \times \mathcal{O}_X$.

For $x, y \in \mathcal{O}_X$ we define $C_{x,y} \subseteq X$ as $\{k : (k)x \neq (k)y\}$; that is, the set of points on which x and y disagree. It is clear that if x and y disagree on an infinite set of points and we write $(x, y) = (au, bu)$ then a and b disagree on an infinite set of points. Therefore, there is $(a, b) \in A$ for which the set $C_{a,b}$ is infinite. Let F be a system of distinct representatives of the family of sets $\{C_{a,b} : (a, b) \in A, C_{a,b} \text{ is infinite}\}$. Let us say that F (which is obviously finite) has size n and let us write $F = \{f_1, \dots, f_n\}$ with $f_i < f_{i+1}$ for $i = 1, \dots, n-1$. As each $C_{a,b}$ in this family is infinite, there are infinitely many choices for each element of F; thus, for each i, we may ensure that there is at least one element of X between each f_i and f_{i+1} and also that f_n is not the maximal element of X (if there is one). That is, for $i = 1, \dots, n-1$ there exists $g_i \in X$ with $f_i < g_i < f_{i+1}$ and there also exists $g_n \in X$ with $f_n < g_n$.

We define mappings $x, y : \{f_1, \dots, f_n\} \to \{f_1, g_1, \dots, f_n, g_n\}$ as follows. Each $i = 1, \dots, n$ has some $(a, b) \in A$ associated with it, in that f_i was chosen as a representative of the set $C_{a,b}$. We consider each f_i with its associated (a, b). If $(f_i)a > (f_i)b$ then we let $(f_i)x = f_i$ and $(f_i)y = g_i$. On the other hand, if $(f_i)a < (f_i)b$ then we let $(f_i)x = g_i$ and $(f_i)y = f_i$. At this stage x and y are clearly monotonic where they have been defined.

We define x, for all $j \in X$, as

$$(j)x = \begin{cases} \min\{(f_i)x : j \leq (f_i)x, i = 1, \dots, n\} & \text{if } j \leq (f_n)x \\ (f_n)x & \text{if } j > (f_n)x \end{cases}$$

Then x is a full monotonic transformation on X. We may extend y in an analogous manner, so that it also becomes a full monotonic transformation on X. We observe that $\mathrm{Im}(x)$ and $\mathrm{Im}(y)$ are disjoint, so x and y disagree everywhere.

There are $u \in \mathcal{O}_X$ and $(a, b) \in A$ such that

$$(x, y) = (au, bu).$$

As x and y disagree on infinitely many points, it follows that a and b also disagree on infinitely many points, so $C_{a,b}$ is infinite. Hence there is $f_i \in F \cap C_{a,b}$, so $(f_i)a \neq (f_i)b$. Suppose that $(f_i)a > (f_i)b$. Then, by our method of construction, we have that $(f_i)x = f_i$ and $(f_i)y = g_i$, so $(f_i)x < (f_i)y$. Thus $(f_i)au < (f_i)bu$ and we conclude that $(f_i)a < (f_i)b$, a contradiction. Supposing that $(f_i)a < (f_i)b$ similarly leads to a contradiction. ■

We now consider the finite generation of the diagonal bi-act of $\mathcal{O}_X$, where X is an infinite totally ordered discrete set. Briefly, we define and characterise the notion of a discrete set.

A totally ordered set is *discrete* if every pair of points has only finitely many points between them. If we define $I_{j,k}$, the interval set between $j, k \in X$ with $j \leq k$, as $\{i \in X : j \leq i \leq k\}$, then X is discrete if every $I_{j,k}$ is finite. it is well-known that sets with these properties can only be of three types, up to order isomorphism. We state this in the following proposition and include a proof for completeness.

Proposition 4.5.6 *Let X be an infinite totally ordered discrete set. Then X is order isomorphic to one of: the set $\mathbb{N}$ of positive integers; the set $-\mathbb{N}$ of negative integers; or the set $\mathbb{Z}$ of all integers.*

PROOF. In the first case, suppose that X has a minimal element z. We define $\phi : X \to \mathbb{N}$ as $(i)\phi = |I_{z,i}|$ and we claim that ϕ is a bijection. Suppose that ϕ is not injective, so there are distinct $i, j \in X$ such that $|I_{z,i}| = |I_{z,j}|$. As X is totally ordered we know that i and j are comparable, so let us say that $i < j$. Then $I_{z,i} \subset I_{z,j}$ and hence $|I_{z,i}| < |I_{z,j}|$, a contradiction.

Now suppose that ϕ is not surjective, so there is some $n \in \mathbb{N} \setminus \mathrm{Im}(\phi)$. We know that ϕ is injective, so $\mathrm{Im}(\phi)$ is infinite and therefore contains no largest value. So there is some $m \in \mathrm{Im}(\phi)$ with $m > n$. This means that there is some $i \in X$ such that $|I_{z,i}| = m$. Denoting

$$I_{z,i} = \{p_1, \ldots, p_m\}$$

with $p_1 = z$, $p_m = i$ and $p_l < p_{l+1}$ for $l = 1, \ldots, m-1$, it is easily seen that $(p_n)\phi = |I_{z,p_n}| = n$, a contradiction. Thus our claim is shown and ϕ is a bijection.

We now claim that ϕ is monotonic. To show this, we select $i, j \in X$ with $i < j$. By the same argument that we used to show the injectiveness of ϕ, we see that $|I_{z,i}| < |I_{z,j}|$, or in other words $(j)\phi < (i)\phi$. Our claim is shown, so X is order isomorphic to $\mathbb{N}$.

In the second case, we suppose that X has a maximal element. By a symmetric argument it follows that X is order isomorphic to $-\mathbb{N}$.

In the third case we suppose that X has neither a minimal element nor a maximal element. We select an arbitrary element $z \in X$ and define $\phi : X \to \mathbb{Z}$ as follows. If $z \leq i$ then $(z)\phi = |I_{z,i}|$ If $i < z$ then $(i)\phi = -|I_{i,z}|$. An analogous argument to the first case shows that ϕ is a monotonic bijection, so X is order isomorphic to $\mathbb{Z}$.

As an infinite totally ordered discrete set cannot have both a maximal element and a minimal element, there are no further cases to consider and the proof is complete. ■

We now state and prove the main result concerning discrete sets within this topic.

Theorem 4.5.7 *Let X be an infinite totally ordered discrete set. The diagonal bi-act of $\mathcal{O}_X$, the semigroup of full monotonic transformations on X, is not finitely generated.*

PROOF. Assume that $\mathcal{O}_X \times \mathcal{O}_X = \mathcal{O}_X A \mathcal{O}_X$ for some finite $A \subseteq \mathcal{O}_X \times \mathcal{O}_X$. If there are any pairs $(a, b) \in A$ for which $\mathrm{Im}(a)$ is a finite set then let

$$m = 1 + \max\{|\mathrm{Im}(a)| : (a, b) \in A, |\mathrm{Im}(a)| < \infty\}.$$

If there are no such $(a, b) \in A$ then let $m = 1$. By Proposition 4.5.6 we know that X contains arbitrarily large interval sets; in particular, there are $p, q \in X$ with $p < q$ and $|I_{p,q}| = m$. We fix and consider these points as follows.

Let $\mathrm{id} : X \to X$ be the identity transformation on X and let $x : X \to X$ be defined as

$$(i)x = \begin{cases} p & \text{if } i \leq p \\ i & \text{if } p < i < q \\ q & \text{if } i \geq q. \end{cases}$$

As $\mathrm{id}, x \in \mathcal{O}_X$ there are $(a, b) \in A$ and $s, t \in \mathcal{O}_X$ such that

$$(x, \mathrm{id}) = (sat, sbt).$$

As $\mathrm{id} = sbt$ is not eventually constant in either direction, it follows that neither are s, b or t‘. As $x = sat$ it follows that a must be eventually constant in both directions; if this was not the case then we would have a contradiction, as it is impossible for the product of three mappings, none of which are eventually constant, to be eventually constant. As X is discrete this means that $\mathrm{Im}(a)$ is a finite set, but of course we must have that

$$|\mathrm{Im}(a)| \geq |\mathrm{Im}(x)| = |I_{p,q}| = m,$$

which is a contradiction to our definition for m.

(If X is order isomorphic to $\mathbb{N}$ or $-\mathbb{N}$ then the phrase 'eventually constant in both directions' makes little sense. However, our proof still works; we focus on whether these transformations are eventually constant in the positive direction for $\mathbb{N}$, and in the negative direction for $-\mathbb{N}$.) ■

The proof of Theorem 4.5.7 cannot be applied to $\mathcal{Q}_X$, as no full strictly monotonic transformation on an infinite set X is eventually constant.

We do not know the answer to the general case of Open Problem 4.5.2 in which X may be any infinite totally ordered set. However, we show one possible approach, from which we have made some progress. We begin by introducing some new notions.

A *pattern* on a set $Y \subseteq X$ is a mapping

$$f : (Y \times Y) \to \{<,=,>\} \times \{<,=,>\} \times \{<,=,>\}.$$

For elements $x, y \in \mathcal{O}_X$ (or $\mathcal{Q}_X$) and $Y \subseteq X$ we define the pattern $p_{x,y,Y}$ on Y by $(i,j)p_{x,y,Y} = (\sigma_1, \sigma_2, \sigma_3)$ (with $\sigma_1, \sigma_2, \sigma_3 \in \{<,=,>\}$) where $(i)x \; \sigma_1 \; (j)y$, $(i)x \; \sigma_2 \; (j)x$ and $(i)y \; \sigma_3 \; (j)y$. For brevity we denote $p_{x,y,X}$ as $p_{x,y,Y}$. Intuitively, the pattern $p_{x,y}$ records how the images of elements of Y under the mappings x and y compare to each other.

Let Y and Z be totally ordered sets, let f be a pattern on Y and let g be a pattern on Z. If there exists a monotonic bijection $h : Y \to Z$ which satisfies $(i,j)f = ((i)h, (j)h)g$ for all $i, j \in Y$, then we say that f is *pattern isomorphic* to g and we write $f \cong g$.

We state and prove the following result for $\mathcal{Q}_X$, as it is neater than its analogue for $\mathcal{O}_X$.

Proposition 4.5.8 *Let X be an infinite totally ordered set, let $\mathcal{Q}_X$ be the semigroup of strictly monotonic transformations on X and let $(x,y) = (sat, sbt)$ with $x, y, s, t, a, b \in \mathcal{Q}_X$. Then $p_{a,b,\mathrm{Im}(s)} \cong p_{x,y}$.*

PROOF. Assume that $(x, y) = (sat, sbt)$. At this point we note that (with $i, j \in X$) if $(i, j)p_{x,y} = (\sigma_1, \sigma_2, \sigma_3)$ then $(i)x\ \sigma_2\ (j)x$ and $(i)y\ \sigma_3\ (j)y$; as x and y are strictly monotonic it follows that $i\ \sigma_2\ j$, $i\ \sigma_3\ j$, and $\sigma_2 = \sigma_3$.

Now, for all $i, j \in X$ we see that

$$\begin{aligned}
(i,j)p_{x,y} = (\sigma_1, \sigma_2, \sigma_3) &\Leftrightarrow (i)x\ \sigma_1\ (j)y \quad \text{and} \quad i\ \sigma_2\ j \\
&\Leftrightarrow (i)sat\ \sigma_1\ (j)sbt \quad \text{and} \quad i\ \sigma_2\ j \\
&\Leftrightarrow (i)sa\ \sigma_1\ (j)sb \quad \text{and} \quad i\ \sigma_2\ j \\
&\Leftrightarrow ((i)s, (j)s)p_{a,b,\mathrm{Im}(s)} = (\sigma_1, \sigma_2, \sigma_3).
\end{aligned}$$

The proof is already complete as $s : X \to \mathrm{Im}(s)$ is a monotonic bijection and

$$(i,j)p_{x,y} = ((i)s, (j)s)p_{a,b,\mathrm{Im}(s)}$$

for all $i, j \in X$. ■

We state and prove the next result for $\mathcal{O}_X$, as it is neater than the analogue for $\mathcal{Q}_X$.

Proposition 4.5.9 *Let X be an infinite totally ordered set, let $\mathcal{O}_X$ be the semigroup of full monotonic transformations on X and let us assume that if Y is a subset of X and $f : Y \to X$ is monotonic then we may extend f to a full monotonic transformation $f : X \to X$. If $x, y, a, b \in \mathcal{O}_X$ satisfy $p_{x,y} \cong p_{a,b,Y}$ then there are $s, t \in \mathcal{O}_X$ such that*

$$(x, y) = (sat, sbt).$$

PROOF. Assume that $p_{x,y} \cong p_{a,b,Y}$. So there is a monotonic bijection $s : X \to Y$ which satisfies $(i,j)p_{x,y} = ((i)s, (j)s)p_{a,b,Y}$ for all $i, j \in X$. Then

$$\begin{aligned}
(i)x\ \sigma_1\ (j)y &\Leftrightarrow (i,j)p_{x,y} = (\sigma_1, \sigma_2, \sigma_3) \ \text{ for some } \ \sigma_2, \sigma_3 \\
&\Leftrightarrow ((i)s, (j)s)p_{a,b,Y} = (\sigma_1, \sigma_2, \sigma_3) \ \text{ for some } \ \sigma_2, \sigma_3 \qquad (4.3) \\
&\Leftrightarrow (i)sa\ \sigma_1\ (j)sb.
\end{aligned}$$

Similarly, we may show that $(i)x\ \sigma\ (j)x$ if and only if $(i)sa\ \sigma\ (j)sa$. Further, $(i)y\ \sigma\ (j)y$ if and only if $(i)sb\ \sigma\ (j)sb$.

We now define $t : \mathrm{Im}(sa) \cup \mathrm{Im}(sb) \to X$ as $[(i)sa]t = (i)x$ and $[(i)sb]t = (i)y$ and we extend this to a full monotonic transformation $t \in \mathcal{O}_X$ (which we have assumed to be possible).

Suppose that this method of construction requires t to map some $i \in X$ to two distinct points $j, k \in X$. This would occur if $i = (l)sa = (m)sb$, for some $l, m \in X$, but that $j = (l)x \neq (m)y = k$. By (4.3), this case does not happen. This problem would also arise if we had $i = (l)sa = (m)sa$ for some $l, m \in X$, while $j = (l)x \neq (m)x = k$. However, by the remark following (4.3)we see that this problem cannot occur.

At this point we make the straightforward observation that $x = sat$ and $y = sbt$.

It only remains to show that that t is monotonic. Suppose, in order to derive a proof by contradiction, that it is not. Then there are $i, j \in \mathrm{Im}(sa) \cup \mathrm{Im}(sb)$ with $i > j$ but $(i)t < (j)t$.

As a first case to consider, let us say that $i, j \in \mathrm{Im}(sa)$, so that $i = (k)sa$ and $j = (l)sa$ for some $k, l \in X$. As $i = (k)sa > (l)sa = j$ it follows that $k > l$. Then $(k)x \geq (l)x$, so $(k)sat \geq (l)sat$ and hence $(i)t \geq (j)t$, a contradiction. The case $i, j \in \mathrm{Im}(sb)$ analogously leads to contradiction.

Finally, we suppose that $i \in \mathrm{Im}(sa)$ and $j \in \mathrm{Im}(sb)$, so that $i = (k)sa$ and $j = (l)sb$ for some $k, l \in X$. Then $(k)sa > (l)sb$, and by (4.3) it follows that $(k)x > (l)y$. Then $(k)sat > (l)sbt$, so $(i)t > (j)t$, which is a contradiction. The case $i \in \mathrm{Im}(sb), j \in \mathrm{Im}(sa)$ analogously leads to contradiction and our proof is complete. ■

This result raises a number of questions. Firstly, the condition referring to the extension of partial monotonic transformations to full ones is mysterious. We now show that, in fact, this condition does not always hold.

Example 4.5.10 Let X be the set $\mathbb{R} \setminus \{0\}$ of real numbers except 0, which is clearly an infinite totally ordered set. Let Y be the set $\mathbb{R} \setminus \{0, 1\}$ of real numbers except 0 and 1, which is clearly a subset of X. We define $f : Y \to X$ as $(i)f = i - 1$. Then f is monotonic on Y but cannot be extended to a full monotonic transformation on X as there is no possible value to assign to $(1)f$

which would preserve its monotonic nature. ■

In Proposition 4.5.8 we consider $\mathcal{Q}_X$ while in Proposition 4.5.9 we consider $\mathcal{O}_X$; the latter also includes the condition about extending partial monotonic transformations to full ones. Excepting for these discrepancies, these results imply a connection between Open Problem 4.5.2 and the following unsolved question.

Open Problem 4.5.11 Does there exist a finite $A \subseteq \mathcal{O}_X \times \mathcal{O}_X$ such that for all $x, y \in \mathcal{O}_X$ there are $(a, b) \in A$ and $Y \subseteq X$ such that $p_{x,y} \cong p_{a,b,Y}$? How about for $\mathcal{Q}_X$?

Sadly, we have no more knowledge on this topic.

As an aside, we note that we may regard a totally ordered infinite set X as a semilattice with the same order. We then have that $\mathcal{O}_X \cong \mathrm{End}(X)$. On a similar note, the semigroup $\mathcal{P}_X$ of partial transformations on some set X is the endomorphism monoid of the semilattice which consists of that set X (considered as an antichain) with a zero. Corollaries 4.1.6 and 4.2.6 state that $\mathcal{P}_X$ has some interesting diagonal act properties, and the results of this section indicate that the diagonal bi-act of $\mathcal{O}_X$ may well have interesting properties, so there arises the following question.

Open Problem 4.5.12 For which infinite semilattices are the diagonal right, left and bi-acts of the endomorphism monoid finitely generated or cyclic?

We may even ask this question for diagonal acts of the endomorphism monoids of general infinite semigroups and other algebraic structures. This provides the topic for the next section.

4.6 Endomorphism monoids of independence algebras

Corollary 4.1.7 and Theorem 4.2.4 have shown that the monoid $\mathcal{T}_X$ of full transformations on an infinite set X has very interesting diagonal act properties. Of course, a set X with no operations is an independence algebra and

$\mathcal{T}_X$ is its endomorphism monoid. It is also known that $\mathcal{T}_X$ behaves very similarly in many aspects to the endomorphism monoid $\mathrm{End}(\mathbf{A})$ of an arbitrary independence algebra $\mathbf{A}$ (see [18]), so it is particularly natural to investigate $\mathrm{End}(\mathbf{A})$ in the context of the generation of its diagonal acts.

For our purposes, we want the endomorphism monoid $\mathrm{End}(\mathbf{A})$ to be infinite.

Proposition 4.6.1 *Let $\boldsymbol{A}$ be an independence algebra. The endomorphism monoid* $\mathrm{End}(\boldsymbol{A})$ *of $\boldsymbol{A}$ is infinite if and only if $\boldsymbol{A}$ is infinite and its dimension is not zero.*

PROOF. ($\Rightarrow$) Assume that $\mathrm{End}(\mathbf{A})$ is infinite. It is clear that if $\mathbf{A}$ is finite then $\mathrm{End}(\mathbf{A})$ is finite; thus we see that $\mathbf{A}$ must be infinite.

Now suppose that $\mathbf{A}$ has zero dimension. Then $\mathrm{Sg}^{\mathbf{A}}(\emptyset) = \mathbf{A}$. It is clear, in general, that every endomorphism fixes every element of $\mathrm{Sg}^{\mathbf{A}}(\emptyset)$, so in this case the only endomorphism on $\mathbf{A}$ is the identity transformation. Thus the endomorphism monoid is trivial and we have a contradiction.

($\Leftarrow$) Assume that $\mathbf{A}$ is infinite and has non-zero dimension. Then there exists an independent set of size 1, say $\{a\}$, in $\mathbf{A}$. For all $b \in \mathbf{A}$ we may extend the map $x : a \mapsto b$ to an endomorphism on $\mathbf{A}$. This gives infinitely many distinct endomorphisms, so $\mathrm{End}(\mathbf{A})$ is infinite. ∎

In our study of the diagonal acts of $\mathrm{End}(\mathbf{A})$ we begin with the following extension of Corollary 4.1.7.

Theorem 4.6.2 *Let $\boldsymbol{A}$ be an infinite-dimensional independence algebra and let* $\mathrm{End}(\boldsymbol{A})$ *be the monoid of endomorphisms of $\boldsymbol{A}$. The diagonal right act of* $\mathrm{End}(\boldsymbol{A})$ *is cyclic.*

PROOF. Let B be an (infinite) basis of $\mathbf{A}$ and let B_1 and B_2 be disjoint subsets of B, each with the same cardinality as B and satisfying $B = B_1 \cup B_2$. We fix bijections $f : B \to B_1$, $g : B \to B_2$ and let $a : \mathbf{A} \to \mathbf{A}$, $b : \mathbf{A} \to \mathbf{A}$ be the extensions of f and g, respectively, to endomorphisms. We claim that $\mathrm{End}(\mathbf{A}) \times \mathrm{End}(\mathbf{A}) = (a, b)\mathrm{End}(\mathbf{A})$.

For arbitrary $x, y \in \mathrm{End}(\mathbf{A})$ let $v : B \to \mathbf{A}$ be defined as

$$(i)v = \begin{cases} (i)f^{-1}x & \text{if } i \in B_1 \\ (i)g^{-1}y & \text{if } i \in B_2, \end{cases}$$

and let $u : \mathbf{A} \to \mathbf{A}$ be the extension of v to a homomorphism. Then x and au agree on every point of B, so $x = au$. Similarly it follows that $y = bu$, so $(x, y) = (au, bu)$. ■

The following result, which concerns the diagonal left act of $\mathrm{End}(\mathbf{A})$, is an extension of Theorem 4.2.4 but also provides further examples of semigroups whose diagonal right and left acts behave differently (see Corollary 4.6.5 below).

We refer to the direct square $\mathbf{A} \times \mathbf{A}$ of an independence algebra $\mathbf{A}$; we do not know whether this is, in general, also an independence algebra (see Open Problem 4.6.11 below), so we refer to its rank and not its dimension.

Theorem 4.6.3 *Let $\boldsymbol{A}$ be an infinite-dimensional independence algebra and let $\mathrm{End}(\boldsymbol{A})$ be the monoid of endomorphisms of $\boldsymbol{A}$. The diagonal left act of $\mathrm{End}(\boldsymbol{A})$ is finitely generated if and only if there is a generating set of cardinality $\dim(\boldsymbol{A})$ for $\boldsymbol{A} \times \boldsymbol{A}$ (so $\dim(\boldsymbol{A}) = \mathrm{rank}(\boldsymbol{A} \times \boldsymbol{A})$ if the rank of $\boldsymbol{A} \times \boldsymbol{A}$) is well defined). In addition, in this case the diagonal left act of $\mathrm{End}(\boldsymbol{A})$ is cyclic.*

Proof. ($\Rightarrow$) Assume that there is a finite $F \subseteq \mathrm{End}(\mathbf{A}) \times \mathrm{End}(\mathbf{A})$ satisfying

$$\mathrm{End}(\mathbf{A}) \times \mathrm{End}(\mathbf{A}) = \mathrm{End}(\mathbf{A})F.$$

Let $k, l \in \mathbf{A}$ be arbitrary. As $\mathbf{A}$ has non-zero dimension and satisfies the free basis property, there is an element $j \in \mathbf{A}$ and endomorphisms $x, y \in \mathrm{End}(\mathbf{A})$ such that $(j)x = k$ and $(j)y = l$. Further, there are $u \in \mathrm{End}(\mathbf{A})$ and $(a, b) \in F$ such that

$$(x, y) = (ua, ub).$$

We regard (x, y) as the homomorphism $(x, y) : \mathbf{A} \to \mathbf{A} \times \mathbf{A}$, defined as $(i)(x, y) = ((i)x, (i)y)$. Then $(k, l) = (j)(ua, ub) = [(j)u](a, b) \in \mathrm{Im}(a, b)$, so $(k, l) \in \mathrm{Im}(a, b)$. This is true for arbitrary $k, l \in \mathbf{A}$, so we see that

$$\mathbf{A} \times \mathbf{A} = \bigcup_{(a,b) \in F} \mathrm{Im}(a, b).$$

Let B be an (infinite) basis of $\mathbf{A}$. Then the set $(B)(a,b)$ (which we define as $\{(i)(a,b) : i \in B\}$) generates the subalgebra $\operatorname{Im}(a,b)$ of $\mathbf{A} \times \mathbf{A}$, so $\bigcup_{(a,b)\in F}(B)(a,b)$ generates $\mathbf{A} \times \mathbf{A}$. This is a finite union of sets $(B)(a,b)$, each of which have cardinality less than or equal to that of B; thus the whole generating set has a cardinality less than or equal to that of B. That is, there is a generating set of cardinality $\dim(\mathbf{A})$ for $\mathbf{A} \times \mathbf{A}$. Of course, this is the smallest possible cardinality for such a generating set because $\mathbf{A}$ is a homomorphic image of $\mathbf{A} \times \mathbf{A}$.

($\Leftarrow$) We assume that there is a generating set of cardinality $\dim(\mathbf{A})$ for $\mathbf{A}\times\mathbf{A}$, and we let B be an (infinite) basis of $\mathbf{A}$. By our assumption there exists a set $C(\subseteq \mathbf{A} \times \mathbf{A})$, which has the same cardinality as B and which generates $\mathbf{A} \times \mathbf{A}$. We fix mutually inverse bijections $g : B \to C$ and $f : C \to B$ and we let $p_1, p_2 : \mathbf{A} \times \mathbf{A} \to \mathbf{A}$ be the natural projections onto the first and second co-ordinates respectively. We let a and b be the extensions of gp_1 and gp_2, respectively, to endomorphisms on $\mathbf{A}$.

We now claim that $\operatorname{End}(\mathbf{A}) \times \operatorname{End}(\mathbf{A}) = \operatorname{End}(\mathbf{A})(a,b)$. To show this we let $x, y \in \operatorname{End}(\mathbf{A})$ be arbitrary and we construct an appropriate $u \in \operatorname{End}(\mathbf{A})$ as follows. For each $i \in B$ we write

$$(i)(x,y) = t((c_1,d_1),\ldots,(c_n,d_n))$$

(where t is a n-ary term operation and each $(c_j,d_j) \in C$), we define $v : B \to \mathbf{A}$ as

$$(i)v = t((c_1,d_1)f,\ldots,(c_n,d_n)f)$$

and we let u be the extension of v to an endomorphism on $\mathbf{A}$. Then, for each

$i \in B$, we observe that

$$\begin{aligned}
(i)ua &= (i)va \\
&= [t((c_1, d_1)f, \ldots, (c_n, d_n)f)]a \\
&= t([(c_1, d_1)f]a, \ldots, [(c_n, d_n)f]a) \\
&= t([(c_1, d_1)f]gp_1, \ldots, [(c_n, d_n)f]gp_1) \\
&= t((c_1, d_1)[fg]p_1, \ldots, (c_n, d_n)[fg]p_1) \\
&= t((c_1, d_1)p_1, \ldots, (c_n, d_n)p_1) \\
&= t(c_1, \ldots, c_n) \\
&= (i)x.
\end{aligned}$$

We see that x agrees with ua on the basis B, so $x = ua$. Analogously we may show that $y = ub$, so $(x, y) = (ua, ub)$, completing the proof. ■

The condition stated in Theorem 4.6.3 is a little mysterious in that we do not know which infinite-dimensional independence algebras $\mathbf{A}$ satisfy the equality $\dim(\mathbf{A}) = \operatorname{rank}(\mathbf{A} \times \mathbf{A})$. This is clearly the case if $\mathbf{A}$ is an infinite set or an infinite-dimensional vector space. For some more examples, let $\mathbf{A}$ be any infinite-dimensional independence algebra with only finitely many operations. Then $\dim(\mathbf{A}) = |\mathbf{A}| = |\mathbf{A} \times \mathbf{A}|$. Of course, we must also have the inequality $\dim(\mathbf{A}) \leq \operatorname{rank}(\mathbf{A} \times \mathbf{A}) \leq |\mathbf{A} \times \mathbf{A}|$, so it follows that $\operatorname{rank}(\mathbf{A} \times \mathbf{A}) = \dim(\mathbf{A})$.

The following is an example of an independence algebra that does not have this property.

Example 4.6.4 Let $\mathbf{A}$ be the set $\mathbb{R}$ of real numbers. For every $x \in \mathbb{R} \setminus \mathbb{N}$ we define a nullary operation ψ_x, whose result is x. We define no other operations.

We observe that $\mathrm{Sg}^{\mathbf{A}}(X) = X \cup (\mathbb{R} \setminus \mathbb{N})$ for any $X \subseteq \mathbf{A}$. To show that the exchange property holds, we let $x, y \in \mathbf{A}$ and $X \subseteq \mathbf{A}$ be arbitrary such that $x \in \mathrm{Sg}^{\mathbf{A}}(X, y) \setminus \mathrm{Sg}^{\mathbf{A}}(X)$. But $\mathrm{Sg}^{\mathbf{A}}(X) = X \cup (\mathbb{R} \setminus \mathbb{N})$ and $\mathrm{Sg}^{\mathbf{A}}(X, y) = X \cup \{y\} \cup (\mathbb{R} \setminus \mathbb{N})$, so it quickly follows that $x = y$ and hence $y \in \mathrm{Sg}^{\mathbf{A}}(X, x)$.

The unique basis for $\mathbf{A}$ is $\mathbb{N}$, and the endomorphisms of $\mathbf{A}$ are precisely those transformations which fix every element of $\mathbb{R} \setminus \mathbb{N}$. It is now clear that any mapping from the basis into the algebra can be extended to an endomorphism, so the free basis property holds and $\mathbf{A}$ is an independence algebra.

To compare the cardinalities of the generating sets, we already know that $\dim(\mathbf{A})$ is countably infinite. Of course, the only operations in the direct square $\mathbf{A} \times \mathbf{A}$ are the nullaries ψ_x, for each $x \in \mathbb{R} \setminus \mathbb{N}$, whose result is (x, x). Thus the unique minimum generating set for $\mathbf{A} \times \mathbf{A}$ is $(\mathbb{R} \times \mathbb{R}) \setminus \{(x, x) : x \in \mathbb{R} \setminus \mathbb{N}\}$, so $\operatorname{rank}(\mathbf{A} \times \mathbf{A})$ is uncountably infinite. ■

We can also ask which finite-dimensional independence algebras $\mathbf{A}$ satisfy the equality $\dim(\mathbf{A}) = \operatorname{rank}(\mathbf{A} \times \mathbf{A})$. This is answered in Proposition 4.6.10 below.

Corollary 4.6.5 *Let $\boldsymbol{A}$ be the independence algebra in Example 4.6.4. The diagonal right act of* $\operatorname{End}(\boldsymbol{A})$ *is cyclic but the diagonal left act of* $\operatorname{End}(\boldsymbol{A})$ *is not finitely generated.*

We now tackle our main question for endomorphism monoids of finite-dimensional independence algebras.

Theorem 4.6.6 *Let $\boldsymbol{A}$ be an infinite independence algebra with finite non-zero dimension, let* $\operatorname{End}(\boldsymbol{A})$ *be the monoid of endomorphisms of $\boldsymbol{A}$ and let* $\operatorname{Aut}(\boldsymbol{A})$ *be the group of automorphisms of $\boldsymbol{A}$. The diagonal right and left acts of* $\operatorname{End}(\boldsymbol{A})$ *are not finitely generated and its diagonal bi-act is not cyclic. Furthermore, if the diagonal bi-act of* $\operatorname{End}(\boldsymbol{A})$ *is finitely generated then* $\operatorname{Aut}(\boldsymbol{A})$ *is an infinite group with only finitely many conjugacy classes.*

Proof. As $\mathbf{A}$ is infinite with non-zero dimension, it follows from Proposition 4.6.1 that $\operatorname{End}(\mathbf{A})$ is infinite.

From [18] we have that $x, y \in \operatorname{End}(\mathbf{A})$ satisfy $x \mathcal{J} y$ if and only if the dimensions of the subalgebras $\operatorname{Im}(x)$ and $\operatorname{Im}(y)$ are equal. Further, $J_x \leq J_y$ if and only if the dimension of $\operatorname{Im}(x)$ is less than or equal to the dimension of $\operatorname{Im}(y)$. From this we deduce that $\operatorname{End}(\mathbf{A})$ has a unique maximal $\mathcal{J}$-class, namely the set of endomorphisms x for which $\dim(\operatorname{Im}(x)) = \dim(\mathbf{A})$. As $\mathbf{A}$ has finite dimension, the only subalgebra with the same dimension as $\mathbf{A}$ is $\mathbf{A}$ itself. So this $\mathcal{J}$-class is precisely the set of surjective endomorphisms on $\mathbf{A}$.

Of course, the image of a basis under an endomorphism generates the image of the endomorphism. From this it follows that a surjective endomorphism maps a basis to a basis. Let us say that $x \in \mathrm{End}(\mathbf{A})$ is surjective, that B is a basis of $\mathbf{A}$ and that $\{(i)x : i \in B\} = C$. Then C is also a basis and we may define a mapping $y : C \to B$ as $y : (i)x \mapsto i$. We extend y to an endomorphism and observe that xy and yx act as the identity on B and C, respectively. Hence $xy = yx = \mathrm{id}$, so y is the inverse of x, which is therefore a bijection. We conclude that the unique maximal $\mathcal{J}$-class of $\mathrm{End}(\mathbf{A})$ is $\mathrm{Aut}(\mathbf{A})$.

By Theorem 3.2.2 it follows that $\mathrm{Aut}(\mathbf{A})$ is infinite. It is clear that $\mathrm{Aut}(\mathbf{A})$ is a subsemigroup of $\mathrm{End}(\mathbf{A})$ and that its complement is either empty or is an ideal. Hence, if the diagonal right or left act of $\mathrm{End}(\mathbf{A})$ is finitely generated then it follows by Proposition 3.1.4 that $\mathrm{Aut}(\mathbf{A})$ has the same property. This contradicts Corollary 3.8.2. If the diagonal bi-act of $\mathrm{End}(\mathbf{A})$ is cyclic then Proposition 3.1.5 states that we must have $\mathrm{End}(\mathbf{A}) = \mathrm{Aut}(\mathbf{A})$, and hence this is an infinite group with a cyclic diagonal bi-act. This contradicts Proposition 3.5.1.

If the diagonal bi-act of $\mathrm{End}(\mathbf{A})$ is finitely generated then, by Proposition 3.1.4, the diagonal bi-act of $\mathrm{Aut}(\mathbf{A})$ is finitely generated. Proposition 3.3.1 implies that that $\mathrm{Aut}(\mathbf{A})$ has only finitely many conjugacy classes. ■

We show that there exist infinite finite-dimensional independence algebras whose endomorphism monoids have finitely generated diagonal bi-acts.

Example 4.6.7 Let G be an infinite group with only finitely many conjugacy classes. Let $\mathbf{A}$ have elements G and for each $g \in G$ let there be a unary operation which pre-multiplies its argument by g; that is, it is defined as $h^g = gh$. This is an example of a free G-act over a group, which is well-known to be an independence algebra; we provide a short proof of this fact.

To show that $\mathbf{A}$ satisfies the exchange property, we let $x, y \in \mathbf{A}$ and $X \subseteq \mathbf{A}$ be such that $x \in \mathrm{Sg}^{\mathbf{A}}(X, y) \setminus \mathrm{Sg}^{\mathbf{A}}(X)$. As the only operations present are unary it follows that $x \in \mathrm{Sg}^{\mathbf{A}}(y)$, so $x = y^g$ for some $g \in G$. Then $x = gy$, so $y = g^{-1}x = x^{g^{-1}} \in \mathrm{Sg}^{\mathbf{A}}(x) \subseteq \mathrm{Sg}^{\mathbf{A}}(X, x)$, as desired.

The endomorphisms of $\mathbf{A}$ are precisely those transformations $\phi_g : \mathbf{A} \to \mathbf{A}$ defined by $(h)\phi_g = hg$, for each $g \in G$. A basis of $\mathbf{A}$ is any single element set $\{g\}$. Then any mapping α on this basis, say $(g)\alpha = k$, can be extended to an endomorphism, namely $\phi_{g^{-1}k}$. Thus **A** satisfies the free basis property and is an independence algebra.

Composition of endomorphisms follows the rule $\phi_g \phi_k = \phi_{gk}$, so the endomorphism monoid is isomorphic to the group G. By Proposition 3.3.1 this has a finitely generated diagonal bi-act. ∎

We consider this question for vector spaces.

Proposition 4.6.8 *Let V be an infinite finite-dimensional vector space. The diagonal bi-act of the endomorphism monoid of V is not finitely generated.*

PROOF. Let $n = \dim(V)$ and denote the base field of V as F. It is clear that F is infinite, that $\mathrm{End}(V)$ is the semigroup of all $n \times n$ matrices over F and that $\mathrm{Aut}(V)$ is the general linear group of all invertible $n \times n$ matrices over F.

The result follows from Theorem 4.6.6, as the general linear group over an infinite field has infinitely many conjugacy classes (with representative elements the Jordan normal forms of the matrices).

Alternatively, denote the multiplicative semigroup of F as F^* and define a mapping $\phi : \mathrm{End}(V) \to F^*$ by $(M)\phi = \det(M)$. Then ϕ is an epimorphism, so if we assume that the diagonal bi-act of $\mathrm{End}(V)$ is finitely generated then we conclude, by Proposition 3.1.3, that the diagonal bi-act of F^* is finitely generated. This contradicts Theorem 3.6.1. ∎

The second argument also leads to the following more general question.

Open Problem 4.6.9 Let R be an infinite ring (or even semiring), let R^* be the multiplicative semigroup of R and let S be the semigroup of all $n \times n$ matrices over R, with the operation of matrix multiplication. Can the diagonal right, left or bi-act of S, or indeed R^*, be cyclic or finitely generated?

We include the table overleaf as a summary of our results on the diagonal acts of transformation semigroups.

Semigroup	Right act	Left act	Bi-act
$\mathcal{B}_X$	cyclic (Thm 4.1.3)	cyclic (Thm 4.2.3)	cyclic
$\mathcal{P}_X$	cyclic (Cor 4.1.6)	cyclic (Cor 4.2.6)	cyclic
$\mathcal{T}_X$	cyclic (Cor 4.1.7)	cyclic (Thm 4.2.4)	cyclic
$\mathcal{F}_X$	cyclic (Cor 4.1.8)	infinite (Thm 4.2.9)	cyclic
$\mathcal{I}_X$	infinite (Thm 4.1.10)	infinite (Thm 4.2.10)	cyclic (Thm 4.3.3)
$\mathcal{S}_X$	infinite	infinite	infinite (Thm 4.4.1)
$\mathcal{S}urj_X$	infinite	infinite	infinite (Thm 4.4.2)
$\mathcal{I}nj_X$	infinite	infinite	infinite (Thm 4.4.4)
$\mathcal{T}_X \setminus \mathcal{S}urj_X$	infinite	infinite	infinite (Thm 4.4.5)
$\mathcal{T}_X \setminus \mathcal{I}nj_X$	infinite	infinite	infinite (Thm 4.4.6)
$\mathcal{PS}urj_X$	infinite	infinite	infinite (Thm 4.4.3)
$\mathcal{O}_X$	infinite (Thm 4.5.5)	infinite (Thm 4.5.4)	?????
$\mathcal{Q}_X$	infinite	infinite	?????
End($\mathbf{A}$)	can be cyclic (Thms 4.6.2, 4.6.6)	can be cyclic (Thms 4.6.3, 4.6.6)	can be f.g. (Thm 4.6.6)

Table 4.1: Results on semigroups of transformations.

Using our results we can make the following observation, although this proof seems unnecessarily indirect. Perhaps there is a direct proof which remains undiscovered.

Proposition 4.6.10 *There does not exist a non-trivial finite-dimensional independence algebra for which the rank of the direct square equals the dimension of the algebra.*

PROOF. Suppose that $\mathbf{A}$ is a non-trivial finite-dimensional independence algebra and that $\text{rank}(\mathbf{A} \times \mathbf{A}) = \dim(\mathbf{A})$. Note that the "$\Leftarrow$" part of the proof of Theorem 4.6.3 only uses the assumption that $\dim(\mathbf{A}) = \text{rank}(\mathbf{A} \times \mathbf{A})$, and therefore may be applied to $\mathbf{A}$. By this we may conclude that the diagonal left act of $\text{End}(\mathbf{A})$ is cyclic. Then Theorem 4.6.6 implies that $\text{End}(\mathbf{A})$ is finite, so Theorem 3.5.2 implies that $\text{End}(\mathbf{A})$ is trivial.

We now claim that $\mathbf{A}$ is either a single element or has zero dimension. To prove this by contradiction, we suppose that neither of these are the case; that is, $|\mathbf{A}| \geq 2$ and $\dim(\mathbf{A}) \geq 1$. So there are distinct elements $a_1, a_2 \in \mathbf{A}$ and an independent set of size 1 in $\mathbf{A}$, say $\{b\}$. Then the mappings $\alpha : b \mapsto a_1$ and $\beta : b \mapsto a_2$ can be extended to distinct endomorphisms. This is a contradiction to our earlier conclusion that $\text{End}(\mathbf{A})$ is trivial, so our claim is shown.

We now claim that $\mathbf{A}$ has a single element. To prove this by contradiction we suppose that it is not the case; that is, there are distinct $a_1, a_2 \in \mathbf{A}$. By our earlier conclusion we know that $\dim(\mathbf{A}) = 0$. Then $\text{rank}(\mathbf{A} \times \mathbf{A}) = 0$, so $\mathbf{A} \times \mathbf{A}$ is generated by the results of its nullary operations. But these must be of the form (a, a), for some $a \in \mathbf{A}$, and they can only generate other pairs of the same form. Thus they cannot generate the pair (a_1, a_2), a contradiction. We conclude that $\mathbf{A}$ must be a single element. ■

Continuing on the theme of the direct square $\mathbf{A} \times \mathbf{A}$, we ask the following question; although it is at a tangent to the main topic of this section, this is very interesting as it seems to be a simple question but it remains unanswered.

Open Problem 4.6.11 If $\mathbf{A}$ is an independence algebra, does it necessarily follow that the direct square $\mathbf{A} \times \mathbf{A}$ is an independence algebra?

Considering examples gives us little intuition here, as it is very simple to show that the direct squares of sets and vector spaces are independence algebras. From [19] we have the following partial result.

Theorem 4.6.12 *If $\boldsymbol{A}$ is an infinite-dimensional independence algebra then the direct square $\boldsymbol{A} \times \boldsymbol{A}$ satisfies the exchange property.*

PROOF. Let $(x, y), (p, q) \in \mathbf{A} \times \mathbf{A}$ and $X \subseteq \mathbf{A} \times \mathbf{A}$ be such that

$$(x, y) \in \mathrm{Sg}^{\mathbf{A}\times\mathbf{A}}(X, (p, q)) \setminus \mathrm{Sg}^{\mathbf{A}\times\mathbf{A}}(X).$$

To prove the result directly, we aim to show that $(p, q) \in \mathrm{Sg}^{\mathbf{A}\times\mathbf{A}}(X, (x, y))$. There is a collection of elements $(x_1, y_1), \ldots, (x_n, y_n) \in X$ and some $(n+1)$-ary term operation t such that

$$\begin{aligned}(x, y) &= t((x_1, y_1), \ldots, (x_n, y_n), (p, q)) \\ &= (t(x_1, \ldots, x_n, p), t(y_1, \ldots, y_n, q)).\end{aligned}$$

Now suppose further that there is an independent set $\{z_1, \ldots, z_n, z\} \subseteq \mathbf{A}$ and a term operation s such that

$$t(z_1, \ldots, z_n, z) = s(z_1, \ldots, z_n).$$

We define $\alpha : z_i \mapsto x_i$ and $\alpha : z \mapsto p$. As $\{z_1, \ldots, z_n, z\}$ is independent the mapping α can be extended to an endomorphism of $\mathbf{A}$, so it follows that

$$\begin{aligned}x &= t(x_1, \ldots, x_n, p) \\ &= t(z_1\alpha, \ldots, z_n\alpha, z\alpha) \\ &= [t(z_1, \ldots, z_n, z)]\alpha \\ &= [s(z_1, \ldots, z_n)]\alpha \\ &= s(z_1\alpha, \ldots, z_n\alpha) \\ &= s(x_1, \ldots, x_n).\end{aligned}$$

Similarly we define $\alpha : z_i \mapsto y_i$ and $\alpha : z \mapsto q$, extend this to an endomorphism and deduce that $y = s(y_1, \ldots, y_n)$. Therefore

$$(x, y) = s((x_1, y_1), \ldots, (x_n, y_n)) \in \mathrm{Sg}^{\mathbf{A}\times\mathbf{A}}(X),$$

which is a contradiction.

We conclude that if $\{z_1, \ldots, z_n, z\} \subseteq \mathbf{A}$ is independent then

$$t(z_1, \ldots, z_n, z) \notin \mathrm{Sg}^{\mathbf{A}}(z_1, \ldots, z_n).$$

Of course, as $\dim(\mathbf{A})$ is infinite, there does exist an independent set $Z \subseteq \mathbf{A}$ of size $n+1$. Writing $Z = \{z_1, \ldots, z_n, z\}$ we see that

$$t(z_1, \ldots, z_n, z) \in \mathrm{Sg}^{\mathbf{A}}(z_1, \ldots, z_n, z) \setminus \mathrm{Sg}^{\mathbf{A}}(z_1, \ldots, z_n).$$

Then the exchange property implies that

$$z \in \mathrm{Sg}^{\mathbf{A}}(z_1, \ldots, z_n, t(z_1, \ldots, z_n, z)),$$

so there is a term operation u such that

$$z = u(z_1, \ldots, z_n, t(z_1, \ldots, z_n, z)).$$

We now define $\alpha : z_i \mapsto x_i$, $\alpha : z \mapsto p$, $\beta : z_i \mapsto y_i$ and $\beta : z \mapsto q$. Extending these to endomorphisms, we conclude that

$$\begin{aligned} p &= u(x_1, \ldots, x_n, t(x_1, \ldots, x_n, p)) \\ &= u(x_1, \ldots, x_n, x), \\ q &= u(y_1, \ldots, y_n, t(y_1, \ldots, y_n, q)) \\ &= u(y_1, \ldots, y_n, y). \end{aligned}$$

Then

$$(p, q) = u((x_1, y_1), \ldots, (x_n, y_n), (x, y)) \in \mathrm{Sg}^{\mathbf{A} \times \mathbf{A}}(X, (x, y)),$$

which completes the proof. ■

4.7 Some finitely presented examples

We now construct two semigroups, which we denote as $\mathcal{S}(X, A, C, D, E, F)$ and $\mathcal{M}(X, A, C, D, (a', b'), r)$. These are interesting for our purposes because they are based on presentations (which can be finite) and they each have finitely generated diagonal right acts. To begin, we describe a list of ingredients:

(X) We fix an alphabet X, which may be finite or infinite. Letting 1 denote an identity (or the empty word from X^*) it will be useful for our notation to write $Z = (X \times \{1\}) \cup (\{1\} \times X)$.

(A) We fix a finite set $A \subseteq X^+ \times X^+$.

(C) For each $(a,b) \in A$ and $(y,z) \in Z$ we fix a word $w_{(y,z),(a,b)} \in X^+$ and we denote the set $\{w_{(y,z),(a,b)} : (y,z) \in Z, (a,b) \in A\}$ as C.

(D) For each $(y,z) \in Z$ we fix $\alpha_{y,z} \in \mathcal{T}_A$, the semigroup of full transformations on A. We denote the set $\{\alpha_{y,z} : (y,z) \in Z\}$ as D.

(E) For all $(v,x) \in X \times X$ we fix $(a_{v,x}, b_{v,x}) \in A$. We denote the set $\{(a_{v,x}, b_{v,x}) : v, x \in X\}$ as E.

(F) For all $(v,x) \in X \times X$ we fix $r_{v,x} \in X^+$. We define a new set F as $\{r_{v,x} : v, x \in X\}$.

We also define $p_1, p_2 : X^+ \times X^+ \to X^+$ as the natural projections onto the first and second co-ordinates respectively. Then $\mathcal{S}(X, A, C, D, E, F)$ is defined as

$$\begin{aligned}\mathrm{Sgp}\langle X \mid\ & a_{v,x}r_{v,x} = v,\ b_{v,x}r_{v,x} = x, ya = [(a,b)\alpha_{y,z}p_1]w_{(y,z),(a,b)}, \\ & zb = [(a,b)\alpha_{y,z}p_2]w_{(y,z),(a,b)}\ ((a,b) \in A, (v,x) \in X \times X, (y,z) \in Z)\rangle.\end{aligned} \tag{4.4}$$

Theorem 4.7.1 *Let S be an infinite semigroup. The diagonal right act of S is finitely generated if and only if S is a homomorphic image of a semigroup of the form $\mathcal{S}(X, A, C, D, E, F)$, for some sets X, A, C, D, E and F.*

Proof. ($\Rightarrow$) Assume that S is an infinite semigroup with a finitely generated diagonal right act, so there is a finite set $B \subseteq S \times S$ which satisfies $S \times S = BS$.

We let S be generated by a (finite or infinite) set $Q (\subseteq S)$ and we define an alphabet $X = \{x_q : q \in Q\}$ to be in correspondence $x_q \leftrightarrow q$ with Q. For an element $s \in S$, say with $s = q_1 \dots q_r$ (each $q_i \in Q$) we denote by w_s the word $x_{q_1} \dots x_{q_r} \in X^+$. We let $A \subseteq X^+ \times X^+$ be defined as the set $\{(w_a, w_b) : (a,b) \in B\}$.

We consider arbitrary $v, x \in X$ and let these correspond to $q_1, q_2 \in Q$ (so we should really write $v = x_{q_1}, x = x_{q_2}$). There are $(a, b) \in B$ and $s \in S$ which satisfy

$$(q_1, q_2) = (a, b)s.$$

In this case we let $r_{v,x} \in X^+$ be defined as the word w_s and we also define the pair $(a_{v,x}, b_{v,x}) = (w_a, w_b) \in A$. Letting $E = \{(a_{v,x}, b_{v,x}) : v, x \in X\}$ and $F = \{r_{v,x} : v, x \in X\}$, we observe that the equalities corresponding to the relations $v = a_{v,x}r_{v,x}$ and $x = b_{v,x}r_{v,x}$ are satisfied in S, for all $v, x \in X$.

We now let $I = (Q \times \{1\}) \cup (\{1\} \times Q)$, let $Z = (X \times \{1\}) \cup (\{1\} \times X)$, and extend the correspondence between Q and X to a correspondence between I and Z.

We consider arbitrary $(y, z) \in Z$ and $(a, b) \in A$, and we let these correspond to $(p, q) \in I$ and $(a', b') \in B$. There is $s \in S$ and $(a'', b'') \in B$ such that

$$(pa', qb') = (a'', b'')s.$$

In this case we denote $w_s = w_{(y,z),(a,b)}$ and we let $(w_{a''}, w_{b''}) \in A$ be the result of mapping (a, b) by some transformation $\alpha_{y,z} : A \to A$ (which clearly depends on (y, z)). We now label the sets $\{w_{(y,z),(a,b)} : (y, z) \in Z, (a, b) \in A\}$ and $\{\alpha_{y,z} : (y, z) \in Z\}$ as C and D respectively. It is now clear that all of the equalities corresponding to the relations in (4.4) hold.

We are not claiming that these relations are sufficient to define the semigroup, but simply that they hold in it; from this it follows that S is a homomorphic image of a semigroup with a presentation of this form, so this part of the result is shown.

($\Leftarrow$) We let S be presented by (4.4), for some choice of the necessary ingredients. We claim that $S \times S = AS$. Once this is shown the result will follow by Proposition 3.1.3. For any $s, t \in S$ there are $(y_1, z_1), \ldots, (y_{n-1}, z_{n-1}) \in Z$ and $(y_n, z_n) \in X \times X$ such that

$$(s, t) = (y_1, z_1) \ldots (y_n, z_n).$$

While this is not necessarily unique, we refer to the smallest possible value of n as the *length* of (s, t). We will use induction on n to show that if (s, t) has length k then $(s,t) \in AS$. To consider the base case: if $n = 1$ then $s, t \in X$ and hence

$$(s,t) = (a_{s,t}, b_{s,t})r_{s,t} \in AS.$$

For the inductive step we let $n \geq 1$ be arbitrary and assume that if a pair (s,t) has length l (with $l \leq n$) then $(s,t) \in AS$. We now fix and consider a particular pair (s,t) which has length $n+1$. By the definition of length this means that we can write $(s,t) = (y,z)(y_1,z_1)\dots(y_n,z_n)$ where $(y,z),(y_1,z_1),\dots,(y_{n-1},z_{n-1}) \in Z$ and $(y_n,z_n) \in X \times X$. Clearly the pair $(y_1\dots y_n, z_1\dots z_n)$ has length n, so by our inductive assumption there are $(a,b) \in A$ and $u \in S$ such that

$$(y_1 \dots y_n, z_1 \dots z_n) = (a,b)u.$$

Then

$$\begin{aligned}(s,t) &= (yy_1\dots y_n, zz_1\dots z_n)\\ &= (ya, zb)u\\ &= [(a,b)\alpha_{y,z}]w_{(y,z),(a,b)}u \in AS,\end{aligned}$$

completing the proof. ■

We now describe a related monoid construction. The sets X, Z, A, C and D and the mappings p_1 and p_2 are the same as before, except that we replace X^+ by X^* in the definitions. The final two ingredients are single elements: we distinguish a fixed pair $(a',b') \in A$ and a fixed word $r \in X^*$. Then $\mathcal{M}(X, A, C, D, (a',b'), r)$ is defined as

$$\begin{aligned}\mathrm{Mon}\langle X \ \mid\ & a'r = 1, b'r = 1, ya = [(a,b)\alpha_{y,z}p_1]w_{(y,z),(a,b)}, \\ & zb = [(a,b)\alpha_{y,z}p_2]w_{(y,z),(a,b))]}\ ((a,b) \in A, (y,z) \in Z)\rangle.\end{aligned} \tag{4.5}$$

Theorem 4.7.2 *Let M be a monoid. The diagonal right act of M is finitely generated if and only if it is a homomorphic image of a monoid of the form $\mathcal{M}(X, A, C, D, (a',b'), r)$, for some sets X, A, C, D, and elements (a',b') and r.*

PROOF. This proof is very similar to that of Theorem 4.7.1. We will outline the main differences and omit some parts that are identical.

($\Rightarrow$) Let us assume that M is a monoid with a finitely generated diagonal right act, so there is a finite $B \subseteq M \times M$ such that $M \times M = BM$.

We let M be generated by a (finite or infinite) set $Q(\subseteq M)$ and we define an alphabet $X = \{x_q : q \in Q\}$ to be in correspondence $x_q \leftrightarrow q$ with Q. For an element $s \in M$, say with $s = q_1 \dots q_r$ (each $q_i \in Q$) we denote by w_s the word $x_{q_1} \dots x_{q_r} \in X^+$. We let $A \subseteq X^* \times X^*$ be defined as the set $\{(w_a, w_b) : (a, b) \in B\}$.

It is clear that there are $(a_1, b_1) \in B$ and $s \in M$ such that

$$(1, 1) = (a_1, b_1)s.$$

We define the words $r = w_s$, $a' = w_{a_1}$ and $b' = w_{b_1}$, so $(a', b') \in A$ and the equalities corresponding to the relations $a'r = 1$ and $b'r = 1$ hold in M.

The mappings $\alpha_{y,z} : A \to A$, the words $w_{(y,z),(a,b)} \in X^*$ (for each $(y, z) \in Z$ and $(a, b) \in A$) and the sets C and D are defined precisely as in the proof of Theorem 4.7.1, and it similarly follows that all of the relations in (4.5) hold. This part of the result is complete.

($\Leftarrow$) Let M be presented by (4.5) for some choice of the ingredients involved. We claim that $M \times M = AM$; once this is shown the result will follow by Proposition 3.1.3.

For $s, t \in M$ there are $(y_1, z_1), \dots, (y_n, z_n) \in Z$ such that

$$(s, t) = (y_1, z_1) \dots (y_n, z_n).$$

Again it is not necessarily a unique decomposition but we refer to the smallest possible value of n as the *length* of (s, t). Also, the length of $(1, 1)$ is 0. (Note that this is slightly different to the notion of length that we used in the proof of Theorem 4.7.1.) We will use induction on n to show that if (s, t) has length n then $(s, t) \in AM$. For the base case $k = 0$ we have $s = t = 1$ and we may write

$$(s, t) = (1, 1) = (a', b')r \in AM.$$

For the inductive step we let $n \geq 0$ be arbitrary and assume that if (s, t) has length less than or equal to n then $(s, t) \in AM$. We now fix and consider a particular pair (s, t) with length $n + 1$. This means that we may write

$(s,t) = (y,z)(y_1,z_1)\dots(y_n,z_n)$ with $(y,z),(y_1,z_1),\dots,(y_n,z_n) \in Z$. The inductive step of this proof is now identical to that of Theorem 4.7.1. ■

We note that there are obvious duals of these presentations, which present semigroups with finitely generated diagonal left acts. We may also present a semigroup by a presentation including the relations of (4.4) (or (4.5)) with those of its dual. The resulting semigroup has both finitely generated diagonal right and left acts.

However, there are no obvious constructions which give analogous results for diagonal bi-acts. This is because the inductive part of the proofs of Theorems 4.7.1 and 4.7.2 cannot be applied to a semigroup defined by relations which are the obvious two-sided analogues of those given here.

If X is finite then $\mathcal{S}(X,A,C,D,E,F)$ and $\mathcal{M}(X,A,C,D,(a',b'),r)$ are finitely presented. Given that the examples of infinite semigroups with finitely generated diagonal acts that we have exhibited so far have been uncountably infinite, a finitely presented one would be interesting. Sadly, each particular selection of the necessary ingredients produces a semigroup that stands an unknown chance of being finite (and therefore unexciting).

Open Problem 4.7.3 For which choices of the ingredients are the semigroups $\mathcal{S}(X,A,C,D,E,F)$ and $\mathcal{M}(X,A,C,D,(a',b'),r)$ infinite?

It is easier to consider $\mathcal{M}(X,A,C,D,(a',b'),r)$, as it has fewer defining relations. If X is a singleton then the semigroup defined is commutative and hence is finite, by Theorem 3.6.1. If $X = \{x,y\}$ and $A = \{(a,b)\}$ then C contains four words from $\{x,y\}^*$, the transformations of D are identities on the single element of the set A, and $(a',b') = (a,b)$. Further, by Theorem 3.5.2, the semigroup defined is either infinite or trivial. If we relabel the words of C from $w_{(1,x),(a,b)}, w_{(1,y),(a,b)}, w_{(x,1),(a,b)}$ and $w_{(y,1),(a,b)}$ to w_1, w_2, w_3 and w_4, respectively, then $\mathcal{M}(X,A,C,D,(a',b'),r)$ is presented by

$$\begin{aligned}\mathrm{Mon}\langle x,y \mid\ & ar=1,\ br=1,\ xa=aw_1,\ b=bw_1,\ ya=aw_2\\ & b=bw_2,\ a=aw_3,\ xb=bw_3,\ a=aw_4,\ yb=bw_4\rangle. \qquad (4.6)\end{aligned}$$

Unfortunately, the answer to Open Problem 4.7.3 is not known even in this restricted case. We can deduce some conditions which are necessary for the semigroup defined by this presentation to be infinite, but we do not know any sufficient conditions. For example, by Theorems 3.7.1 and 3.7.7 we know that our semigroup is neither left cancellative nor right cancellative, so we cannot have both x and y being left invertible, nor both of them being right invertible. Thus a and b must start with the same letter. We can similarly argue that neither x nor y can be invertible, so r must end with the other letter.

By the proof of Theorem 4.7.2 it follows that if there exists a two-generated monoid with a cyclic diagonal right act then it is a homomorphic image of a monoid defined by a presentation of form (4.6). In [40] it is shown that $R_{\mathbb{N}}$, the monoid of partial recursive functions in one variable, has precisely these properties, and it follows that there exist choices of words for which (4.6) presents an infinite monoid.

Chapter 5

Finite generation of finitary power semigroups

In this chapter we shall be concerned with finitary power semigroups. Although these were defined in Section 2.6, we remind the reader of the central notions.

The *power semigroup* of a semigroup S, denoted $\mathcal{P}(S)$, is the set of all non-empty subsets of S under the usual multiplication $AB = \{ab : a \in A, b \in B\}$ for $A, B \subseteq S$. The *finitary power semigroup* $\mathcal{P}_f(S)$ is the subsemigroup of $\mathcal{P}(S)$ consisting of all finite subsets of S. (Note that we do not define $\mathcal{P}(S)$ or $\mathcal{P}_f(S)$ to include the empty set as this would simply be a removable zero.)

The general question that we consider here is that of finite generation of the finitary power semigroup $\mathcal{P}_f(S)$. The same problem for $\mathcal{P}(S)$ is trivial. If S is finite then $\mathcal{P}(S)$ is finite and hence finitely generated, while if S is infinite then $\mathcal{P}(S)$ is uncountable and hence not finitely generated.

The majority of this chapter, but not all of it, appeared in [10, 13].

5.1 Some basic properties

Before we begin our study of the finite generation of finitary power semigroups, we consider when the finitary power semigroup is in some of the 'standard' classes. For example, it is clear that $\mathcal{P}_f(S)$ is a monoid if and only if S is a monoid and that $\mathcal{P}_f(S)$ is commutative if and only if S is commutative.

To classify when $\mathcal{P}_f(S)$ is a band, we use the notion of royal semigroups. A semigroup S is *royal* if for every $x, y \in S$ we have either $xy = x$ or $xy = y$.

Equivalently, S is royal if every subset of S is a subsemigroup, or if the only generating set for S is S itself. For further information on royal semigroups see citegirh.

Proposition 5.1.1 *The finitary power semigroup $\mathcal{P}_f(S)$ is a band if and only if S is royal.*

PROOF. ($\Rightarrow$) Assume that $\mathcal{P}_f(S)$ is a band. Let $x, y \in S$ be arbitrary and observe that $\{x, y\}^2 = \{x^2, xy, yx, y^2\} = \{x, y\}$. Therefore either $xy = x$ or $xy = y$, so S is royal.

($\Leftarrow$) Assume that S is royal. Let $Q \in \mathcal{P}_f(S)$ be arbitrary and note that $Q^2 = \{xy : x, y \in Q\}$. We know that every xy either equals x or y, so we see that $Q^2 \subseteq Q$. Of course, every $x \in Q$ satisfies $x^2 = x$, so $Q \subseteq Q^2$ and hence $Q = Q^2$. That is, S is a band. ■

A more interesting question would be when $\mathcal{P}_f(S)$ is inverse, as this could potentially lead to a study of the finite presentability of $\mathcal{P}_f(S)$ as an inverse semigroup. It is easy to see that a semigroup is commutative and royal if and only if it is a totally ordered semilattice. Thus the following result is implied by Proposition 5.1.1 and provides non-trivial examples where $\mathcal{P}_f(S)$ is inverse.

Corollary 5.1.2 *The finitary power semigroup $\mathcal{P}_f(S)$ is a semilattice if and only if S is a totally ordered semilattice.*

It is also easy to see that $\mathcal{P}_f(C_2)$ is inverse, where C_2 is the cyclic group of order 2. However, the following general question remains unanswered.

Open Problem 5.1.3 Can we classify all those semigroups S for which the finitary power semigroup $\mathcal{P}_f(S)$ is inverse? What about regular?

There are, of course, many other classes of semigroups, like cancellative and completely simple, but the author feels that this construction is too 'wild' for a non-trivial finitary power semigroup to have such properties. All of the observations in this section can equally be applied to the power semigroup.

5.2 Comparing the ranks

We begin our study of the generation of finitary power semigroups with the following important observation, which first appeared in [35].

Proposition 5.2.1 *For any semigroup S,* $\mathrm{rank}(\mathcal{P}_f(S)) \geq \mathrm{rank}(S)$. *In particular, if $\mathcal{P}_f(S)$ is finitely generated then S is finitely generated.*

PROOF. Let us say that $\mathrm{rank}(\mathcal{P}_f(S)) = r$, so there are $A_1, \ldots, A_r \in \mathcal{P}_f(S)$ such that $\mathcal{P}_f(S) = \langle A_1, \ldots, A_r \rangle$. We select and fix $a_i \in A_i$ for $i = 1, \ldots, r$. For all $s \in S$ we can write $\{s\} = A_{i_1} \ldots A_{i_k}$ with each $i_j \in \{1, \ldots, r\}$. Then $s = a_{i_1} \ldots a_{i_k}$, so $S = \langle a_1, \ldots, a_r \rangle$ and hence $\mathrm{rank}(S) \leq r$, completing the proof. ∎

For the rest of this section we will be concerned with the question of whether Proposition 5.2.1 can be strengthened; that is, apart from the obvious example where S is the trivial semigroup and $\mathrm{rank}(S) = \mathrm{rank}(\mathcal{P}_f(S)) = 1$, are there any other examples in which the stated bound is actually an equality? Of course, for any countable non-finitely generated semigroup S, the rank of S and the rank of $\mathcal{P}_f(S)$ are both countably infinite, but we discount such examples as they are relatively uninteresting. In other words, we ask the following question.

Open Problem 5.2.2 Does there exist a non-trivial semigroup S for which the rank of $\mathcal{P}_f(S)$ and the rank of S are equal and finite?

We conjecture that the answer is no, but cannot provide a conclusive argument. However, we give the next six propositions by way of a partial proof.

Proposition 5.2.3 *If S is a non-trivial semigroup then $\mathcal{P}_f(S)$ is not monogenic.*

PROOF. Suppose that S is non-trivial and that $\mathcal{P}_f(S) = \langle A \rangle$ for some $A \in \mathcal{P}_f(S)$. We can see that $|A| \geq 2$, so there are certainly distinct elements $a_1, a_2 \in A$. By the proof of Proposition 5.2.1, $S = \langle a_1 \rangle = \langle a_2 \rangle$, so S is monogenic. If S is not a group then the generator of S is unique, so $a_1 = a_2$, a contradiction. Hence S is a group. But then all powers of A contain at least two distinct elements, contradicting that singleton sets lie in $\langle A \rangle$. ■

Proposition 5.2.4 *Let S be a non-trivial semigroup with rank r and assume that $\mathcal{P}_f(S)$ is generated by the collection of r finite sets $A_1, \ldots, A_r \subseteq S$. The sets A_i are mutually disjoint.*

PROOF. From the proof of Proposition 5.2.1, we see that $S = \langle a_1, \ldots, a_r \rangle$ where $a_i \in A_i$ for $i = 1, \ldots, r$ (an arbitrary cross-section of the A_i). As r is the rank of S, it must be the case that $\{a_1, \ldots, a_r\}$ is a minimum generating set for S. If two of the A_i had a non-empty intersection then there would exist a smaller cross-section of the sets A_i and hence S would have a generating set of size less than r, which would be a contradiction. Hence they are mutually disjoint. ■

Proposition 5.2.5 *Let S be a non-trivial semigroup with rank r and assume that $\mathcal{P}_f(S)$ is generated by the collection of r finite sets $A_1, \ldots, A_r \subseteq S$. For $i = 1, \ldots, r$ the set A_i is contained within one $\mathcal{J}$-class.*

PROOF. We consider a particular A_i. If $|A_i| = 1$ then it is clearly contained within its $\mathcal{J}$-class and there is nothing to prove.

Now suppose that $|A_i| \geq 2$, so we can select an arbitrary pair of distinct elements $b, c \in A_i$. We now fix an arbitrary cross-section $a_1, \ldots, a_r$ of the A_i, so that $a_i \in A_i$ for each i. As shown in the proof of Proposition 5.2.1, both $\{a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_r\}$ and $\{a_1, \ldots, a_{i-1}, c, a_{i+1}, \ldots, a_r\}$ are minimum generating sets for S. If we write c in terms of the generating set $\{a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_r\}$ then this product must contain at least one occurrence of b, or else $\{a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_r\}$ would also be a generating set; thus we may write $c = sbt$ (with $s, t \in S^1$). By a similar argument, there are $p, q \in S^1$ such that $b = pcq$. We conclude that $b \mathcal{J} c$. As we have shown this for an arbitrary pair within an arbitrary A_i it follows that each A_i is contained within a $\mathcal{J}$-class and the proof is complete. ■

Proposition 5.2.6 *Let S be a non-trivial semigroup such that the rank of S and the rank of $\mathcal{P}_f(S)$ are equal and finite. Then S has a unique maximal $\mathcal{J}$-class.*

PROOF. Let us say that $\text{rank}(S) = r$, so $\mathcal{P}_f(S)$ is generated by a collection of r finite sets $A_1, \ldots, A_r \subseteq S$.

As the rank of S is finite (that is, S is finitely generated) we know that S has finitely many maximal $\mathcal{J}$-classes, and that all other $\mathcal{J}$-classes are below these. To prove the proposition, we suppose that J_1 and J_2 are distinct maximal $\mathcal{J}$-classes and we aim to obtain a contradiction.

We select elements $j_1 \in J_1, j_2 \in J_2$ and we write $\{j_1, j_2\} = A_{i_1} \ldots A_{i_n}$, with each $i_l \in \{1, \ldots, r\}$. By Proposition 5.2.5 we know that A_{i_1} is contained within one $\mathcal{J}$-class, so let us denote this class as J. Then there are $x_1, x_2 \in A_{i_1} (\subseteq J)$ and $y_1, y_2 \in A_{i_2} \ldots A_{i_n}$ such that $j_1 = x_1 y_1$ and $j_2 = x_2 y_2$. It is now clear that $J_1 = J_{j_1} \leq J_{x_1} = J$ and that $J_2 = J_{j_2} \leq J_{x_2} = J$. As J_1 and J_2 are maximal we see that $J_1 = J$ and that $J_2 = J$, so $J_1 = J_2$. This is a contradiction and completes the proof. ■

Proposition 5.2.7 *Let S be a non-trivial semigroup for which the rank of S and the rank of $\mathcal{P}_f(S)$ are equal and finite. Either S is simple or there exists T, a zero-simple Rees quotient of S with a non-removable zero (that is, $T \setminus \{0\} \not\leq T$), such that the rank of T and the rank of $\mathcal{P}_f(T)$ are equal and finite.*

PROOF. It is possible that S is simple, in which case there is nothing to prove. Therefore we assume that S is not simple. Let us say that the rank of S (and the rank of $\mathcal{P}_f(S)$) is r, so that $\mathcal{P}_f(S)$ is generated by a finite collection of finite sets $A_1, \ldots, A_r \subseteq S$.

By Proposition 5.2.6 there is a unique maximal $\mathcal{J}$-class J. As S is not simple we know that $S \neq J$, so $S \setminus J$ is an ideal of S. As it is the complement of a maximal $\mathcal{J}$-class, this is a maximal ideal, in the sense that there is no ideal I with $(S \setminus J) \subset I \subset S$.

Suppose that J is a proper subsemigroup of S. Selecting $x \in J$ and $y \in S \setminus J$, we write

$$\{x, y\} = A_{i_1} \dots A_{i_l}$$

as usual. As $x \in A_{i_1} \dots A_{i_l} \cap J$ it follows that each of these sets A_{i_j} contains elements from J, so by Proposition 5.2.6 they are each entirely contained in J. As J is a subsemigroup we deduce that $y \in J$, which is a contradiction. Therefore we conclude that J is not a subsemigroup of S.

Now let T be the Rees quotient of S by $S \setminus J$. As $S \setminus J$ is a maximal ideal it follows that T is zero-simple. As J is not a subsemigroup it follows that the zero element of T is not removable, in that there exist $t_1, t_2 \in T \setminus \{0\}$ with $t_1 t_2 = 0$. As T is a homomorphic image of S it follows that the rank of T is finite. We will complete the proof by showing that the rank of T is equal to the rank of $\mathcal{P}_f(T)$.

We choose $a_i \in A_i$ for $i = 1, \dots, r$ and note, as in the proof of Proposition 5.2.1, that $\{a_1, \dots, a_r\}$ generates S. Let $B = \{a_i : a_i \in J\}$ and let $D = \{A_i : A_i \subseteq J\}$. We note that B and D have the same size, which we call n.

We claim that the rank of T is at least n. To show this, we suppose that T has a generating set C of size less than n, and we aim to obtain a contradiction. Clearly the non-zero elements of C generate all of $T \setminus \{0\}$, which then generates 0, so we may assume that C does not contain 0. We may then consider B and C as subsets of J (in S). Noting that C generates B, it follows that $\{a_i : a_i \notin J\} \cup C$ generates S. But this generating set has size less than r, which is a contradiction. We conclude, as desired, that $n \leq \text{rank}(T)$.

Now we claim that the rank of $\mathcal{P}_f(T)$ is at most n. To show this, we will demonstate that D generates $\mathcal{P}_f(T)$. We consider an arbitrary $Q \in \mathcal{P}_f(T)$.

Let us first suppose that $0 \notin Q$. Then we may consider Q as an element of $\mathcal{P}_f(S)$ and write

$$Q = A_{i_1} \dots A_{i_l}$$

with each $i_j \in \{1, \dots, r\}$. As $Q \subseteq J$ it follows that each of these A_{i_j} is contained in J; that is, each of these A_{i_j} is an element of D, so $Q \in \langle D \rangle$.

Second, let us suppose that the set Q contains 0 but that Q is not $\{0\}$. We let $x \in S \setminus J$ be arbitrary and consider the set $(Q \setminus \{0\}) \cup \{x\}$. This is an element of $\mathcal{P}_f(S)$, so we may write

$$(Q \setminus \{0\}) \cup \{x\} = A_{i_1} \dots A_{i_l}$$

with each $i_j \in \{1, \dots, r\}$. As the set $(Q \setminus \{0\}) \cup \{x\}$ contains elements from J, we must have that each of these sets A_{i_j} contains elements from J, and hence by Proposition 5.2.5 they are all contained in J. That is, they are all elements of D. Interpreting this equality in $\mathcal{P}_f(T)$ (that is, mapping under the natural epimorphism $\phi : S \to T$), the element x is mapped to 0 (that is, $(x)\phi = 0$) but everything else is unchanged and we see that

$$Q = A_{i_1} \dots A_{i_l},$$

so $Q \in \langle D \rangle$, as desired.

The final possibility is $Q = \{0\}$. As this zero is not removable, there are $t_1, t_2 \in T \setminus \{0\}$ with $t_1 t_2 = 0$. We have already shown that $\{t_1\}, \{t_2\} \in \langle D \rangle$, so $Q = \{0\} = \{t_1\}\{t_2\} \in \langle D \rangle$ as well. We have shown our claim, that D generates $\mathcal{P}_f(T)$. Thus $\text{rank}(\mathcal{P}_f(T)) \leq n$.

Using Proposition 5.2.1, we now see that

$$n \leq \text{rank}(T) \leq \text{rank}(\mathcal{P}_f(T)) \leq n.$$

Hence $\text{rank}(T) = \text{rank}(\mathcal{P}_f(T))$, so T satisfies all of the stated conditions and the proof is complete. ■

From [20] we have a final partial result on this question. We omit the proof, but mention that it utilises Proposition 5.2.7 and the characterisation of finite zero-simple semigroups as finite zero Rees matrix semigroups.

Proposition 5.2.8 *Let S be a non-trivial finite semigroup. The rank of $\mathcal{P}_f(S)$ is strictly larger than the rank of S.*

Corollary 5.8.1 (below) states that if S is infinite with a finitely generated finitary power semigroup then it has a unique maximal $\mathcal{J}$-class, which is infinite. Thus if the semigroup S in Proposition 5.2.7 is infinite then the given Rees quotient T is also infinite. In other words, there is an example of a non-trivial finitely generated semigroup S for which $\text{rank}(S) = \text{rank}(\mathcal{P}_f(S))$ if and only if there is an infinite simple semigroup or infinite zero-simple semigroup which satisfies the same equality.

Sadly, we gain little further intuition on this question by applying it to the 'standard' examples of infinite simple semigroups and infinite zero-simple semigroups, as Corollaries 5.8.5, 5.8.6 and 5.8.7 (below) show that Bruck–Reilly extensions, infinite completely simple semigroups and infinite completely zero-simple semigroups never have finitely generated finitary power semigroups.

Despite the length of the partial proof exhibited here, and the fact that this research is not conclusive, Open Problem 5.2.2 does not seem as important as some of the other questions in this area. It also seems likely that there is a more concise and direct proof.

5.3 Proper decompositions and a necessary and sufficient condition for $\mathcal{P}_f(S)$ to be finitely generated

From this point we will only be concerned with the question of the finite generation of the finitary power semigroups of infinite semigroups. Thus we introduce the following proposition, which eases notation but loses some information about minimum generating sets, and hence the rank, of $\mathcal{P}_f(S)$.

For $A \subseteq S$ we let $\mathcal{P}(A)$ denote the set of all non-empty subsets of A. The following result gives an equivalent definition of the finite generation of $\mathcal{P}_f(S)$.

Proposition 5.3.1 *The finitary power semigroup of S is finitely generated if and only if it is generated by $\mathcal{P}(A)$ for some finite $A \subseteq S$.*

PROOF. ($\Rightarrow$) Assume that $\mathcal{P}_f(S) = \langle A_1, \ldots, A_l \rangle$. Letting $A = \bigcup_{i=1}^{l} A_i$, it is clear that $A_1, \ldots, A_l \in \mathcal{P}(A)$ and hence that $\mathcal{P}_f(S) = \langle \mathcal{P}(A) \rangle$.

($\Leftarrow$) This part is obvious. ■

We will often use Proposition 5.3.1 without indication.

Let S be a semigroup and let $A_1, \ldots, A_r \in \mathcal{P}_f(S)$. We say that $a \in A_i$ is a *surplus element* of the product $A_1 \dots A_r$ if

$$A_1 \dots A_r = A_1 \dots A_{i-1}(A_i \setminus \{a\})A_{i+1} \dots A_r.$$

Proposition 5.3.2 *Let S be a semigroup and let $Q \in \mathcal{P}_f(S)$. If $Q = A_1 \dots A_r$ then there are $C_1, \dots, C_r \in \mathcal{P}_f(S)$ with the following properties:*

(1) each $C_i \subseteq A_i$;

(2) each $|C_i| \leq |Q|$;

(3) $Q = C_1 \dots C_r$;

(4) the product $C_1 \dots C_r$ has no surplus elements.

Moreover, if some $|C_j| = |Q|$ then all other $|C_i| = 1$.

PROOF. We write $Q = \{q_1, \dots, q_n\}$ be a set of size n and we let $A_1 \dots A_r = Q$. For $m = 1, \dots, n$ there are $a_{mi} \in A_i$ such that $q_m = a_{m1} \dots a_{mr}$. We let $C_i = \{a_{1i}, \dots, a_{ni}\} \subseteq A_i$ and observe that each $|C_i| \leq n$ and that $Q = C_1 \dots C_r$. We may also assume that no C_i contains any surplus elements, as if $c \in C_i$ is surplus then $C_1 \dots C_{i-1}(C_i \setminus \{c\})C_{i+1} \dots C_r$ is another product with the given properties.

To prove the final statement of the result, we suppose that some $|C_j| = n$. Then all of the a_{mj} are distinct and none are surplus. For $k = 1, \dots, n$ we let $X_k = C_1 \dots C_{j-1}\{a_{kj}\}C_{j+1} \dots C_r$ and note that $Q = \bigcup_{k=1}^{n} X_k$. We now claim that $|X_k| = 1$ for each k. We aim to show this by contradiction, so we suppose that $|X_k| \geq 2$ for some k. Then we may remove a set, say X_l, from this union without reducing it. Therefore

$$Q = C_1 \dots C_{j-1}(C_j \setminus \{a_{lj}\})C_{j+1} \dots C_r,$$

so $a_{lj} \in C_j$ is surplus in this product, a contradiction. Hence each $|X_k| = 1$, so now, since no element of C_i is surplus, $|C_i| = 1$ for all $i \neq j$. ■

Let Q be a finite subset of a semigroup S. A *proper decomposition* of Q is a product $A_1 \dots A_r$ (with $A_1, \dots, A_r \subseteq S$) which equals Q and such that each $|A_i| < |Q|$. We say that Q is *properly decomposable* if it has a proper decomposition; in fact this is equivalent to having a proper decomposition which is a product of only two sets, which we will show in Proposition 5.3.5 below. The following result shows the importance of this notion.

Proposition 5.3.3 *If $\mathcal{P}_f(S)$ is finitely generated then there exists $N \in \mathbb{N}$ such that all n-sets in S with $n > N$ are properly decomposable.*

PROOF. Suppose that $\mathcal{P}_f(S) = \langle \mathcal{P}(A) \rangle$ where $A \subseteq S$ is finite. Let $N = |A|$ and let $Q \in \mathcal{P}_f(S)$ be an n-set with $n > N$. We may write $Q = A_1 \dots A_r$ with $A_1, \dots, A_r \subseteq A$. Clearly each $|A_i| < n$, so Q is properly decomposable. ∎

For $n \in \mathbb{N}, A \subseteq S$ we let $B_n(A) = \{Q \subseteq A : 1 \leq |Q| \leq n\}$ and let $D_n(A)$ be the subfamily of $B_n(A)$ consisting of the properly decomposable sets. It is clear that $S \cong B_1(S) \leq \mathcal{P}_f(S)$.

Proposition 5.3.4 *If $\mathcal{P}_f(S)$ is finitely generated and N is the size of a largest finite subset of S that is not properly decomposable, then there exists a finite $A \subseteq S$ such that, for all integers $n \leq N$, we have*

$$D_n(S) \cup B_1(S^1) B_n(A) B_1(S^1) = B_n(S).$$

PROOF. Assume that $\mathcal{P}_f(S)$ is generated by $\mathcal{P}(A)$ for some finite $A \subseteq S$, so that $N \leq |A|$ by the proof of Proposition 5.3.3.

We let Q be an n-set that is not properly decomposable. In other words, we have that $Q \in B_n(S) \setminus D_n(S)$ and we must also have $n \leq N$. By our assumption there are $A_1, \dots, A_r \subseteq A$ such that $Q = A_1 \dots A_r$. By Proposition 5.3.2 there are $C_i \subseteq A_i$ with each $|C_i| \leq n$ and $Q = C_1 \dots C_r$, with no surplus elements. But Q is not properly decomposable, so some $|C_j| = n$. We conclude, by Proposition 5.3.2, that $|C_i| = 1$ for all $i \neq j$. Writing $C_j = \{a_1, \dots, a_n\}$ and $C_i = \{c_i\}$ for all $i \neq j$, we have

$$Q = \{c_1 \dots c_{j-1}\}\{a_1, \dots, a_n\}\{c_{j+1} \dots c_r\} \in B_1(S^1) B_n(A) B_1(S^1),$$

which completes the proof. ∎

The next result gives an equivalent definition for proper decomposition.

Proposition 5.3.5 *A finite subset of a semigroup S is properly decomposable if and only if it has a proper decomposition that is a product of only two sets.*

PROOF. ($\Rightarrow$) Assume that P is a properly decomposable finite subset of S. Let $|P| = n$ and observe that $n \geq 3$. Then P equals a product $A_1 \dots A_r$ of sets $A_1, \dots, A_r \subseteq S$ with each $|A_i| < n$. We may also assume that each $|A_i| > 1$, for if some $|A_i| = 1$ then another proper decomposition of P is $A_1 \dots A_{i-1}(A_i A_{i+1})A_{i+2} \dots A_r$, or $A_1 \dots A_{r-2}(A_{r-1}A_r)$ if $i = r$. Using Proposition 5.3.2 we may add the condition that the product $A_1 \dots A_r$ has no surplus elements. It is also clear that $r \geq 2$. If $r = 2$ then there is nothing to prove. Now suppose that $r > 2$.

Let us also suppose that P does not have a proper decomposition which is a product of only two sets. That is, there do not exist $Q, R \in B_{n-1}(S)$ such that $P = QR$. Of course, $P = A_1(A_2 \dots A_r)$, so Proposition 5.3.2 states that there are $Q' \subseteq A_1$ and $R' \subseteq A_2 \dots A_r$ such that $P = Q'R'$, $|Q'| \leq n$ and $|R'| \leq n$. In fact, as A_1 has no surplus elements it follows that $Q' = A_1$. We know that $|A_1| < n$, so our assumption implies that $|R'| = n$. It now follows, again by Proposition 5.3.2, that $|A_1| = 1$, a contradiction.

($\Leftarrow$) This part is obvious. ■

Proposition 5.3.6 *Let $Q, R \in \mathcal{P}_f(S)$ be such that $|QR| > \max\{|Q|, |R|\}$. There exist $q_1, q_2 \in Q, r_1, r_2 \in R$ such that $|\{q_1, q_2\}\{r_1, r_2\}| \geq 3$.*

PROOF. Without loss of generality we may assume that the product QR has no surplus elements. To prove this proposition, we suppose that the inequality $|\{q_1, q_2\}\{r_1, r_2\}| \leq 2$ holds for all $q_1, q_2 \in Q, r_1, r_2 \in R$, and we aim to obtain a contradiction.

We will use induction on $|W|$ to show that if $W \subseteq Q$ and $|W| \geq 2$ then $|WR| \leq |W|$. For the base case we let $W = \{q_1, q_2\} \subseteq Q$ be arbitrary of size 2. As $q_1 (\in Q)$ is not a surplus element in the product QR, there exists $r_1 \in R$ such that $q_1 r_1 \notin (Q \setminus \{q_1\})R$, so $q_1 r_1 \neq q_2 r_1$. But, for every $r_2 \in R$, we have $|W\{r_1, r_2\}| = 2$ and hence $W\{r_2\} \subseteq W\{r_1\}$. Thus $WR \subseteq W\{r_1\}$, so $|WR| = 2$.

For the inductive step we let $k \geq 2$ be arbitrary and assume that all $W \subseteq Q$ with $2 \leq |W| \leq k$ satisfy $|WR| \leq |W|$. We let $W = \{q_1, \ldots, q_{k+1}\} \subseteq Q$ be arbitrary of size $k+1$, and let $W_1 = \{q_1, \ldots, q_k\}$ and $W_2 = \{q_k, q_{k+1}\}$. Then $|W_1R| \leq k$, $|W_2R| = 2$, $|W_1R \cap W_2R| \geq 1$, and hence

$$|WR| = |W_1R| + |W_2R| - |W_1R \cap W_2R| \leq k + 2 - 1 = |W|.$$

It follows by induction that $|QR| \leq |Q|$, a contradiction. ∎

The following result provides a characterisation of semigroups having finitely generated finitary power semigroups. Although it does not constitute a proper classification, it breaks our general problem into smaller questions on decomposing subsets in particular ways, which will be used in the subsequent sections.

Theorem 5.3.7 *The finitary power semigroup of S is finitely generated if and only if the following conditions hold:*

(1) S is finitely generated; and

(2) there exists $N \in \mathbb{N}$ and a finite $Y \subseteq S$ such that:

(i) for all $n = 2, \ldots, N$, we have

$$D_n(S) \cup B_1(S^1)B_n(Y)B_1(S^1) = B_n(S);$$

(ii) for all $n > N$, every n-set is properly decomposable.

Proof. ($\Rightarrow$) The necessity of conditions (1), (2.i) and (2.ii) have been shown by Propositions 5.2.1, 5.3.4 and 5.3.3, respectively.

($\Leftarrow$) Assume that conditions (1), (2.i) and (2.ii) hold. We let X finitely generate S and let $A = X \cup Y$. We will use induction on $|P|$ to prove that all $P \in \mathcal{P}_f(S)$ have $P \in \langle \mathcal{P}(A) \rangle$. From this we will conclude that $\mathcal{P}(A)$ generates $\mathcal{P}_f(S)$.

For the base case we let $P = \{p\}$ be arbitrary of size 1. There are $x_1, \ldots, x_r \in X$ such that $p = x_1 \ldots x_r$, so

$$P = \{x_1\} \ldots \{x_r\} \in \langle \mathcal{P}(A) \rangle.$$

For the inductive step we let $k \geq 1$ be arbitrary and assume that all subsets of S with size less than or equal to k (that is, $B_k(S)$) are generated by $\mathcal{P}(A)$. We consider an arbitrary $(k+1)$-set P, and first suppose that P is not properly decomposable. Then $k+1 \leq N$ by condition (2.ii). By condition (2.i) there are $\{s\}, \{t\} \in B_1(S^1)$ and $\{y_1, \ldots, y_{k+1}\} \subseteq A$ such that

$$P = \{s\}\{y_1, \ldots, y_{k+1}\}\{t\} \in \langle P(A) \rangle.$$

Finally, suppose that P is properly decomposable. By Proposition 5.3.5 there are $Q, R \subseteq S$ with $|Q|, |R| \leq k$ (so $Q, R \in \langle \mathcal{P}(A) \rangle$) such that $P = QR$. Again, it follows that $P \in \langle \mathcal{P}(A) \rangle$, completing the proof. ■

5.4 Inverse semigroups

In this section we generalise the results of [10], which dealt only with infinite groups. We are now able to answer our question for a wider class of inverse semigroups. We use considerations of proper decomposition, as introduced above, to prove Theorem 5.4.3, which states that if S is an inverse semigroup which contains an infinite group then $\mathcal{P}_f(S)$ is not finitely generated.

We now introduce a new notion, which we call dependence. We note the obvious difference between this and the more usual notion of dependence but, as we only use this definition within this section, it creates no problems in the thesis. The following result, Lemma 5.4.1, shows why this is relevant to this topic.

A subset P of a group G is said to be *dependent* if it satisfies at least one of the following two conditions:

(D1) there exist distinct $x, y, z, t \in P$ such that $x = yz^{-1}t$;

(D2) there exist distinct $x, y, z \in P$ such that $x = yz^{-1}y$.

Otherwise we say that P is *independent*. We denote the set of all independent sets as $\mathcal{I}$.

A product of elements from a set $\{x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}\}$ in an inverse semigroup is called a *Dyck word* if, when interpreted as an element of the free group on $\{x_1, \ldots, x_n\}$, it equals the identity. It can be shown that if an element may be represented as a Dyck word then it is idempotent.

Lemma 5.4.1 *Let S be an inverse semigroup and let P be a finite properly decomposable subset of S, that is contained within one subgroup G of S. Then P is dependent.*

PROOF. By Proposition 5.3.5 there are $Q, R \subseteq S$ with $P = QR$ and $|Q|, |R| < |P|$. By Proposition 5.3.6 there are $q_1, q_2 \in Q, r_1, r_2 \in R$ with $|\{q_1, q_2\}\{r_1, r_2\}| \geq 3$.

Let $x = q_1r_1, y = q_1r_2, z = q_2r_2, t = q_2r_1$, so that $x, y, z, t \in P \subseteq G$. We also have $\{x, y, z, t\} = \{q_1, q_2\}\{r_1, r_2\}$, but we do not know where q_1, q_2, r_1 and r_2 lie. However,

$$xt^{-1}zy^{-1} = q_1r_1r_1^{-1}q_2^{-1}q_2r_2r_2^{-1}q_1^{-1}$$

is a Dyck word on the set $\{q_1, q_2, r_1, r_2\}$, so it is idempotent in S. But $xy^{-1}zt^{-1} \in G$, so we must have $xt^{-1}zy^{-1} = 1_G$; that is, $x = yz^{-1}t$.

If $|\{q_1, q_2\}\{r_1, r_2\}| = 4$ then x, y, z, t are distinct, so (D1) holds. Suppose then that $|\{q_1, q_2\}\{r_1, r_2\}| = 3$. If $x = y$ then $z = t$, so $|\{q_1, q_2\}\{r_1, r_2\}| \leq 2$, a contradiction. Similarly a contradiction is reached if $y = z, z = t$ or $t = x$. If $y = t$ then x, y, z are distinct and $x = yz^{-1}y$, so (D2) holds. If $x = z$, the final possibility, then x, y, t are distinct and $t = xy^{-1}x$, so (D2) holds. ■

Lemma 5.4.2 *If G is an infinite group and $n \in \mathbb{N}$ then G contains an independent set of size n.*

PROOF. For every independent set P, let $E(P)$ be the set of elements that can be added to P without destroying independence; that is,

$$E(P) = \{g \in G \setminus P : P \cup \{g\} \in \mathcal{I}\}.$$

Since every 2-element set is independent, $E(\{g\})$ is infinite for every $g \in G$. If we assume that the lemma is not true and that n is the size of a largest independent subset of G, then $E(P) = \emptyset$ for all $P \in \mathcal{I}$ of size n. Let $Q \in \mathcal{I}$ be an independent set of maximal size such that $E(Q)$ is infinite, say $|Q| = m(< n)$. Let us pick $m + 1$ distinct elements $r_1, \ldots, r_{m+1} \in E(Q)$. For $i = 1, \ldots, m + 1$ the set $Q \cup \{r_i\}$ is independent but $E(Q \cup \{r_i\})$ is finite.

Note that the set

$$E(Q) \setminus (\bigcup_{i=1}^{m+1} E(Q \cup \{r_i\}) \cup \{yz^{-1}t : y, z, t \in Q \cup \{r_1, \ldots, r_{m+1}\}\})$$

is infinite and pick an arbitrary element h in it. Since $h \in E(Q)$, the set $Q \cup \{h\}$ is independent. For $i = 1, \ldots, m+1$ the set $Q \cup \{r_i\}$ is independent, but as $h \notin E(Q \cup \{r_i\})$ the set $Q \cup \{r_i, h\}$ is dependent. That is, some distinct elements x, y, z and possibly t from $Q \cup \{r_i, h\}$ satisfy $x = yz^{-1}t$ or $x = yz^{-1}y$. Two of these elements must be h and r_i and the remaining one or two elements must come from Q. The relation these elements satisfy cannot be $x = yz^{-1}t$, because this could be rewritten as $h = uv^{-1}w$ with $u, v, w \in Q \cup \{r_i\}$, contradicting the choice of h. Similarly, we see that in the equality $x = yz^{-1}y$, we must have $y = h$. So we conclude that $r_i = hq_i^{-1}h$ for some $q_i \in Q$.

Since the $m+1$ elements $q_1, \ldots, q_{m+1}$ come from the m-element set Q, there must exist distinct $i_1, i_2 \in \{1, \ldots, m+1\}$ such that $q_{i_1} = q_{i_2}$. But then we have $r_{i_1} = hq_{i_1}^{-1}h = hq_{i_2}^{-1}h = r_{i_2}$, a contradiction.

Thus our assumption that G contains a largest independent subset is false, and as all subsets of independent sets are independent, the lemma is proved true. ∎

Theorem 5.4.3 *If S is an inverse semigroup containing an infinite group then $\mathcal{P}_f(S)$ is not finitely generated.*

PROOF. We assume that S is inverse, that G is an infinite subgroup of S and that $\mathcal{P}_f(S)$ is finitely generated. By Theorem 5.3.7 there is $N \in \mathbb{N}$ such that all subsets of S with size greater than N are properly decomposable. By Lemma 5.4.2, G contains an independent set of size $N+1$. This can be properly decomposed so, by Lemma 5.4.1, it is dependent, a contradiction. ∎

There now follows the next result, which was the main theorem of [10].

Corollary 5.4.4 *If G is an infinite group then $\mathcal{P}_f(G)$ is not finitely generated.*

The proof of Theorem 5.4.3 relies on the existence of an infinite subgroup, so there remains an unsolved question.

Open Problem 5.4.5 Does there exist an infinite inverse semigroup S such that $\mathcal{P}_f(S)$ is finitely generated?

5.5 Some connections with acts

We now describe some connections between the finitary power semigroup of a semigroup S and its diagonal acts. We begin by defining some new acts, which we will show to have obvious links with diagonal acts.

The *n-oblique right act* of S is the set $B_n(S)$ considered as a right $B_1(S)$-act. That is, those sets of size less than or equal to n, with singletons acting via multiplication from the right. The *n-oblique left act* of S and *n-oblique bi-act* of S are defined analogously.

Several of the results in this chapter which concern acts are stated and proved only for the right act; all of these have analogous results for the left act. However, only those specifically indicated have analogous results for the bi-act.

Lemma 5.5.1 *The n-oblique right act of S is finitely generated if and only if there is a finite $A \subseteq S$ such that $B_n(S) = B_n(A)B_1(S^1)$. There is an analogous result for the oblique bi-act.*

PROOF. ($\Rightarrow$) Assume that there are sets $A_1, \ldots, A_l \in B_n(S)$ which generate $B_n(S)$ as a right act. That is, $B_n(S) = \{A_1, \ldots, A_l\}B_1(S^1)$. If we define the set A by $A = \bigcup_{i=1}^{l} A_i$, then it is clear that $B_n(S) = B_n(A)B_1(S^1)$.

($\Leftarrow$) This part is obvious. ■

Lemma 5.5.2 *The n-diagonal right act is finitely generated if and only if the n-oblique right act is finitely generated. There is an analogous result in terms of the bi-acts.*

PROOF. ($\Rightarrow$) This follows because if $S^{(n)} = U^{(n)}S$ (with U a finite subset of S), then for all $\{p_1, \ldots, p_m\} \in B_n(S)$ there are $(u_1, \ldots, u_m, \ldots, u_m) \in U^{(n)}$ and $s \in S$ satisfying

$$(p_1, \ldots, p_m, \ldots, p_m) = (u_1, \ldots, u_m, \ldots, u_m)s.$$

Therefore $\{p_1, \ldots, p_m\} = \{u_1, \ldots, u_m\}\{s\}$ and hence $B_n(S) = B_n(U)B_1(S)$.

($\Leftarrow$) Similarly, this follows because if $B_n(S) = B_n(A)B_1(S^1)$ and $A \subseteq S$ is finite, then for all $(p_1, \ldots, p_n) \in S^{(n)}$ there are $s \in S^1, a_1, \ldots, a_n \in A$ satisfying

$$\{p_1, \ldots, p_n\} = \{a_1, \ldots, a_n\}\{s\}.$$

Then $(p_1, \ldots, p_n) = (a_{1\pi}, \ldots, a_{n\pi})s$ for some permutation π of $\{1, \ldots, n\}$ and hence $S^{(n)} = A^{(n)}S^1$. ■

Using this proof and also Lemma 3.1.1, it follows that the n-oblique right act of S is finitely generated if and only if there is a finite $A \subseteq S$ such that $B_n(S) = B_n(A)B_1(S)$.

The conditions of Theorem 5.3.7 are quite difficult to check for many examples but, using Lemma 5.5.2, we give some corollaries to that theorem, which may be applied more easily. The following result appeared in [35].

Corollary 5.5.3 *If $\mathcal{P}_f(S)$ is finitely generated then the diagonal bi-act of S is finitely generated.*

PROOF. No 2-set may be properly decomposed, so $D_2(S)$ is empty. If $\mathcal{P}_f(S)$ is finitely generated then, by condition (2.i) of Theorem 5.3.7, there is a finite $Y \subseteq S$ such that

$$B_1(S^1)B_2(Y)B_1(S^1) = B_2(S).$$

The 2-oblique bi-act of S is finitely generated, so the result follows by Lemma 5.5.2. ■

We now give two corollaries of Theorem 5.3.7 which state sufficient conditions for the finite generation of the finitary power semigroup.

Corollary 5.5.4 *If S is finitely generated, the diagonal right act of S is finitely generated and there exists $N \in \mathbb{N}$ such that all n-sets with $n > N$ can be properly decomposed, then $\mathcal{P}_f(S)$ is finitely generated.*

PROOF. Our assumptions include conditions (1) and (2.ii) of Theorem 5.3.7. The diagonal right act of S is finitely generated, so by Proposition 3.1.7

the n-diagonal right act of S is finitely generated for all $n \in \mathbb{N}$. By Lemma 5.5.2, the n-oblique right act of S is finitely generated. That is, there is a finite $Y \subseteq S$ such that $B_n(Y)B_1(S) = B_n(S)$. Condition (2.i) of Theorem 5.3.7 is satisfied, so $\mathcal{P}_f(S)$ is finitely generated. ∎

To prove our next important result, Corollary 5.5.6, we use the following lemma, which connects oblique acts to proper decomposition.

Lemma 5.5.5 *Let $a_1, b_1, a_2, b_2 \in S$ satisfy $B_2(S) = \{\{a_1, b_1\}, \{a_2, b_2\}\}B_1(S)$ (that is, the 2-oblique right act is generated by a set containing at most two sets). Then all n-sets with $n > 2$ can be properly decomposed.*

PROOF. We let $n > 2$ and let $Q = \{q_1, \ldots, q_n\} \subseteq S$ be arbitrary of size n. For $l = 1, 2$ we define $U_l \subseteq B_2(Q)$ as

$$U_l = \{\{q_i, q_j\} : i \neq j \text{ and } \{q_i, q_j\} = \{a_l, b_l\}\{s\} \text{ for some } s \in S\}.$$

We let

$$V_l = \bigcup_{\{q_i,q_j\} \in U_l} \{q_i, q_j\}$$

and observe that $Q = V_1 \cup V_2$. We now claim that either $Q = V_1$ or $Q = V_2$; that is, either for every $q_i \in Q$ there is a 2-set $\{q_i, q_j\}$ that can be written as $\{a_1, b_1\}\{s\}$ for some $s \in S$, or that an equivalent statement regarding $\{a_2, b_2\}$ holds. To show this, we suppose that $Q \neq V_1$. Then there is $q_j \in Q \setminus V_1$, so no 2-sets containing q_j are in U_1. Then every 2-set containing q_j is in U_2, so $Q = V_2$. Our claim is shown, that either $Q = V_1$ or $Q = V_2$.

Say $Q = V_l$. There is $W \subseteq U_l$ such that $|W| < n$ and

$$Q = \bigcup_{\{q_i,q_j\} \in W} \{q_i, q_j\}.$$

For each $\{q_i, q_j\} \in W$ we can write $\{q_i, q_j\} = \{a_l, b_l\}\{s_r\}$ with $s_r \in S$ and $r = 1, \ldots, |W|$. Thus $Q = \{a_l, b_l\}\{s_1, \ldots, s_{|W|}\}$ can be properly decomposed. ∎

Theorem 5.5.6 *If S is finitely generated and there exists $A \subseteq S \times S$ with $S \times S = AS$ and $|A| \leq 2$ (that is, the diagonal right act of S is generated by a set containing at most two pairs) then $\mathcal{P}_f(S)$ is finitely generated.*

PROOF. Let S be finitely generated and let the diagonal right act of S be (at most) two-generated. By the proof of Lemma 5.5.2, the 2-oblique right act is generated by (at most) two sets. By Lemma 5.5.5, the conditions of Corollary 5.5.4, with $N = 2$, are satisfied, completing the proof. ■

This result, restricted to cyclic generation of the diagonal right act, appeared in [35]. We have clearly extended that result, but it produces few new examples, save perhaps Example 5.7.2 below.

We also provide the following alternative proof, which explicitly states a generating set for $\mathcal{P}_f(S)$ in this case.

Proposition 5.5.7 *Let S be a semigroup generated by a finite set X and let there be a set $A = \{(a_1, b_1), (a_2, b_2)\} \subseteq S \times S$ with $S \times S = AS$ (that is, the diagonal right act of S is generated by a set containing at most two pairs). Then $\mathcal{P}_f(S)$ is generated by the set of sets $\{\{x\} : x \in X\} \cup \{\{a_1, b_1\}, \{a_2, b_2\}\}$.*

PROOF. We will prove this result by induction on $|Q|$, showing that every $Q \in \mathcal{P}_f(S)$ is generated by the given set of sets.

For the base case $|Q| = 1$ we let $Q = \{q\}$ and write $q = x_1 \dots x_r$ with each $x_i \in X$. Then $Q = \{x_1\} \dots \{x_r\}$, which is clearly generated by the given set.

For the inductive step we assume that every set of size less than n can be generated by the given collection of sets. We now consider an arbitrary set $Q = \{q_1, \dots, q_n\}$ of size n. Exactly as in the proof of Lemma 5.5.5, we deduce that $Q = \{a_l, b_l\}R$ where either $l = 1$ or $l = 2$ and $|R| < n$. Therefore Q is also generated by this collection of sets, and the result follows by induction. ■

5.6 Some further connections with acts

Considering Corollaries 5.5.4 and 5.5.6, it is natural to ask the following question.

Open Problem 5.6.1 Is it true that if S is finitely generated and the diagonal right act of S is finitely generated then $\mathcal{P}_f(S)$ is finitely generated?

We conjecture that the answer is yes and we will piece together several small results to show that we have made some progress in this area. We begin with the following proposition (which, admittedly, has little obvious relevance).

Proposition 5.6.2 *If the diagonal right act of $\mathcal{P}_f(S)$ is finitely generated then the diagonal right act of S is finitely generated. There are analogous statements for the diagonal bi-act and for cyclic generation.*

PROOF. This result clearly holds if S is finite, so we let S be infinite. We assume that the diagonal right act of $\mathcal{P}_f(S)$ is finitely generated, so there is a finite collection of finite sets $F \subseteq \mathcal{P}_f(S)$ satisfying

$$\mathcal{P}_f(S) \times \mathcal{P}_f(S) = (F \times F)\mathcal{P}_f(S).$$

We now let $U = \bigcup_{A \in F} A$ and claim that $S \times S = (U \times U)S$. For $x, y \in S$ there are $A_1, A_2 \in F$ and $Q \in \mathcal{P}_f(S)$ such that

$$(\{x\}, \{y\}) = (A_1Q, A_2Q).$$

If we let $a_1 \in A_1, a_2 \in A_2$ and $q \in Q$, then we have

$$(x, y) = (a_1q, a_2q) \in (U \times U)S,$$

and we are finished. ■

The question of whether the converse of Proposition 5.6.2 holds also remains unsolved.

Open Problem 5.6.3 Is it true that if the diagonal right act of S is finitely generated then the diagonal right act of $\mathcal{P}_f(S)$ is finitely generated?

The remainder of this section will focus on showing a connection between Open Problems 5.6.1 and 5.6.3. This will mainly be due to the following proposition.

Proposition 5.6.4 *Let S be an infinite semigroup with the property that the diagonal right act of $\mathcal{P}_f(S)$ is finitely generated. There is $N \in \mathbb{N}$ such that every finite subset of S with size larger than N is properly decomposable.*

PROOF. Let us assume that the equality $\mathcal{P}_f(S) \times \mathcal{P}_f(S) = (F \times F)\mathcal{P}_f(S)$ holds for some finite $F \subseteq \mathcal{P}_f(S)$.

We let $N = \max\{2, |A| : A \in F\}$ and claim that every finite subset of S with size larger than N is properly decomposable. To show this we let $Q = \{q_1, \ldots, q_n\}$ be an arbitrary n-set with $n > N$. We suppose that Q is not properly decomposable and we aim to obtain a contradiction.

There are $A, B \in F$ and $U \in \mathcal{P}_f(S)$ such that

$$(Q, \{q_1\}) = (AU, BU).$$

By Proposition 5.3.2 there are $A' \subseteq A$ and $U' \subseteq U$ such that $|A'|, |U'| \leq n$, $Q = A'U'$ and this product has no surplus elements. Of course, we know that $N < |Q|$, $|A| \leq N$ and $A' \subseteq A$, so we must have $|A'| < n$. As Q is not properly decomposable it now follows that $|U'| = n$ (which is at least 3). Applying Proposition 5.3.2 once again, we see that $|A'| = 1$. That is, there is some $a \in A$ such that $Q = \{a\}U'$. For convenience we will write $U' = \{u_1, \ldots, u_n\}$ such that $au_i = q_i$ for each $i = 1, \ldots, n$. Thus $Q \setminus \{q_1\} = \{a\}\{u_2, \ldots, u_n\}$. We also note that for any $b \in B$ we have $\{q_1\} = \{b\}\{u_2, \ldots, u_n\}$. Thus $Q = \{a, b\}\{u_2, \ldots, u_n\}$, which is clearly a proper decomposition. This contradicts our assumption. ∎

Corollary 5.6.5 *Let S be a finitely generated semigroup with the property that the diagonal right act of $\mathcal{P}_f(S)$ is finitely generated. Then $\mathcal{P}_f(S)$ is finitely generated.*

PROOF. By Propositions 5.6.2 and 5.6.4 respectively, the diagonal right act of S is finitely generated and every finite subset of S with size larger than a certain $N \in \mathbb{N}$ can be properly decomposed. The result follows by Corollary 5.5.4. ∎

By this corollary, we see that if Open Problem 5.6.3 has a positive answer then so has Open Problem 5.6.1. We conjecture that it has, but we can only show this in the following specific case.

Proposition 5.6.6 *If the diagonal right act of S is cyclic then the diagonal right act of $\mathcal{P}_f(S)$ is cyclic.*

PROOF. Let us assume that there are $a, b \in S$ satisfying $S \times S = (a, b)S$. We claim that $\mathcal{P}_f(S) \times \mathcal{P}_f(S) = (\{a\}, \{b\})\mathcal{P}_f(S)$.

To show this, we select arbitrary $P, Q \in \mathcal{P}_f(S)$ and write $P = \{p_1, \ldots, p_k\}$ and $Q = \{q_1, \ldots, q_l\}$, with $k \geq l$ (say). There are $u_1, \ldots, u_l \in S$ such that $(p_i, q_i) = (a, b)u_i$ for $i = 1, \ldots, l$. Further, there are $u_{l+1}, \ldots, u_k \in S$ such that $(p_j, q_l) = (a, b)u_j$ for $j = l + 1, \ldots, k$. Then

$$(P, Q) = (\{a\}, \{b\})\{u_1, \ldots, u_k\} \in (\{a\}, \{b\})\mathcal{P}_f(S),$$

completing the proof. ■

Unfortunately it is difficult to apply a similar argument to the more general case, where the diagonal right act of S is finitely generated but not cyclic.

Using Corollary 5.6.5, Proposition 5.6.6 implies a positive answer to Open Problem 5.6.1 in the case where the diagonal right act of S is cyclic. Of course, we have already shown this in Theorem 5.5.6, but these results can also be combined to give the following pleasing consequence.

Corollary 5.6.7 *Let S be a finitely generated semigroup with a cyclic diagonal right act and let us define a sequence of semigroups $\{S_i : i \in \mathbb{N}\}$ by $S_1 = S$ and $S_{i+1} = \mathcal{P}_f(S_i)$ for all i. Every S_i is finitely generated.*

We have now reached the limit of our knowledge on this question.

5.7 Examples

We ask whether certain infinite semigroups, which are known to have finitely generated diagonal acts, also have finitely generated finitary power semigroups. The first comes from [35].

Example 5.7.1 The monoid $R_{\mathbb{N}}$ of partial recursive functions of one variable is two-generated and its diagonal right act is cyclic. By Corollary 5.5.6, $\mathcal{P}_f(R_{\mathbb{N}})$ is finitely generated. ■

Example 5.7.2 If X is finite and $|A| \leq 2$ then both $\mathcal{S}(X, A, C, D, E, F)$ and $\mathcal{M}(X, A, C, D, (a', b'), r)$ have finitely generated finitary power semigroups, again by Corollary 5.5.6. Of course, it remains unknown when these are infinite. ■

We let X be an infinite set. From earlier results we know some interesting facts regarding the generation of the diagonal acts of the semigroups $\mathcal{B}_X$ of binary relations, $\mathcal{P}_X$ of partial transformations, $\mathcal{T}_X$ of full transformations, $\mathcal{F}_X$ of full finite-to-one transformations and $\mathcal{I}_X$ of partial injective transformations on X. None of these are finitely generated so, by Proposition 5.2.1, their finitary power semigroups are not finitely generated. We also know about the diagonal acts of the endomorphism monoid $\text{End}(\mathbf{A})$ of an independence algebra $\mathbf{A}$. However, there are some interesting facts regarding relative ranks for these examples.

For a semigroup S and a subset $Y \subseteq S$ the *relative rank* of S to Y, denoted by $\mathrm{r}(S : Y)$, is the size of a smallest set $Z \subseteq S$ such that $\langle Y \cup Z \rangle = S$. For further information on relative ranks see [22].

Returning to finitary power semigroups, if S is not finitely generated then we ask if there exists $n \in \mathbb{N}$ such that the relative rank $\mathrm{r}(\mathcal{P}_f(S) : B_n(S))$ is finite. For this, we will state and use the proposition below. However, we omit the proof as it is simply a repetition of the proof of Theorem 5.5.6 (with a similar level of intricacy in being built up through a number of smaller results), except that it is stated in terms of relative ranks. Also, we will shortly use Proposition 5.7.5 below (which follows from the same reasoning as Theorem 5.4.3) but we omit the proof of that for a similar reason.

Proposition 5.7.3 *If the diagonal right act of S is generated by a set containing at most two pairs then the relative rank $\mathrm{r}(\mathcal{P}_f(S) : B_1(S))$ is finite.*

Example 5.7.4 The relative rank $\mathrm{r}(\mathcal{P}_f(S) : B_1(S))$ is finite if S is any of the semigroups: $\mathcal{B}_X$ of binary relations; $\mathcal{P}_X$ of partial transformations; $\mathcal{T}_X$ of full transformations; or $\mathcal{F}_X$ of full finite-to-one transformations on an infinite set X (by Proposition 5.7.3 and Theorem 4.1.3, Corollaries 4.1.6, 4.1.7 and 4.1.8

respectively). This is also the case for the monoid $\mathrm{End}(\mathbf{A})$ of endomorphisms of an infinite-dimensional independence algebra $\mathbf{A}$, by Theorem 4.6.2. Furthermore, each of these relative ranks equals one, as $\mathcal{P}_f(S)$ is generated by $B_1(S)$ and the 2-set $\{a,b\}$ where $a,b \in S$ satisfy $S \times S = (a,b)S$. ■

We have not classified when $\mathrm{End}(\mathbf{A})$ is finitely generated, so it seems possible that $\mathcal{P}_f(\mathrm{End}(\mathbf{A}))$ could be finitely generated for some infinite independence algebra (with non-zero dimension) $\mathbf{A}$. We show that this is not the case in Theorem 5.7.12 below.

We turn to $\mathcal{I}_X$ and use the next result.

Proposition 5.7.5 *If S is an inverse semigroup containing an infinite group then the relative rank* $\mathrm{r}(\mathcal{P}_f(S) : B_n(S))$ *is not finite for any* $n \in \mathbb{N}$.

Example 5.7.6 Let $\mathcal{I}_X$ be the semigroup of partial injective transformations on an infinite set X. The relative rank $\mathrm{r}(\mathcal{P}_f(\mathcal{I}_X) : B_n(\mathcal{I}_X))$ is not finite for any $n \in \mathbb{N}$. ■

As examples are in short supply, we ask which simple constructions preserve the finite generation of finitary power semigroups, and hence could create new examples. We first consider a homomorphic image.

Proposition 5.7.7 *If $\mathcal{P}_f(S)$ is finitely generated and T is a homomorphic image of S then $\mathcal{P}_f(T)$ is finitely generated.*

Proof. If T is a homomorphic image of S then $\mathcal{P}_f(T)$ is a homomorphic image of $\mathcal{P}_f(S)$. Thus if $\mathcal{P}_f(S)$ is finitely generated then so is $\mathcal{P}_f(T)$. ■

Example 5.7.8 Let J be the maximal $\mathcal{J}$-class of $R_{\mathbb{N}}$ and let S be the Rees quotient of $R_{\mathbb{N}}$ by $R_{\mathbb{N}} \setminus J$. Then $\mathcal{P}_f(S)$ is finitely generated by Proposition 5.7.7. ■

It is also clear from Proposition 5.7.7 that if $\mathcal{P}_f(S \times T)$ is finitely generated then so are $\mathcal{P}_f(S)$ and $\mathcal{P}_f(T)$. However, the following example shows that the converse of this fact does not hold. This comes from [35], in which we also find the definitions of the 'constant extension' and the 'opposite' of a semigroup.

Example 5.7.9 If S is the constant extension of $R_{\mathbb{N}}$ and T is the opposite of S then $\mathcal{P}_f(S)$ and $\mathcal{P}_f(T)$ are finitely generated but $\mathcal{P}_f(S \times T)$ is not. ■

Proposition 5.7.10 *If $\mathcal{P}_f(S)$ is finitely generated, $T \leq S$ and $S \setminus T$ is an ideal of S then $\mathcal{P}_f(T)$ is finitely generated.*

PROOF. If $T \leq S$ and $S \setminus T$ is an ideal of S then $\mathcal{P}_f(T) \leq \mathcal{P}_f(S)$ and $\mathcal{P}_f(S) \setminus \mathcal{P}_f(T)$ is an ideal of $\mathcal{P}_f(S)$. Thus if $\mathcal{P}_f(S)$ is finitely generated then so is $\mathcal{P}_f(T)$. ■

Proposition 5.7.11 *The finitary power semigroup of S is finitely generated if and only if the finitary power semigroup of S^0 is finitely generated.*

PROOF. If S has a zero then $S^0 = S$ so there is nothing more to prove, so we assume that this is not the case.

($\Rightarrow$) Suppose that $\mathcal{P}_f(S) = \langle \mathcal{P}(A) \rangle$ where $A \subseteq S$ is finite. For $Q \in \mathcal{P}_f(S^0)$ we write $Q \setminus \{0\} = A_1 \ldots A_r$ with $A_1, \ldots, A_r \subseteq A$. If $0 \notin Q$ then $Q = A_1 \ldots A_r$, while if $0 \in Q$ then $Q = (A_1 \cup \{0\})A_2 \ldots A_r$. Thus $\mathcal{P}(A^0)$ generates $\mathcal{P}_f(S)$.

($\Leftarrow$) This follows from Proposition 5.7.10. ■

We note that, in fact, this proof shows more than we have stated. It shows that adjoining a zero to any semigroup, even if it already has one, preserves the finite generation of finitary power semigroups.

Continuing in the theme of connections between finitary power semigroups and diagonal acts, we note that Propositions 5.7.7, 5.7.10 and 5.7.11 mirror Propositions 3.1.3, 3.1.4 and 3.1.6, which made equivalent statements for diagonal acts.

We are now able to use our results to answer our question for the monoid $\mathrm{End}(\mathbf{A})$ of endomorphisms of an independence algebra $\mathbf{A}$.

Theorem 5.7.12 *Let $\boldsymbol{A}$ be an infinite independence algebra with non-zero dimension and let* $\mathrm{End}(\boldsymbol{A})$ *be the endomorphism monoid of $\boldsymbol{A}$. The finitary power semigroup of* $\mathrm{End}(\boldsymbol{A})$ *is not finitely generated.*

PROOF. By Proposition 4.6.1 we know that $\mathrm{End}(\mathbf{A})$ is infinite. Suppose that $\mathcal{P}_f(\mathrm{End}(\mathbf{A}))$ is finitely generated.

Let us further suppose that $\mathbf{A}$ has an infinite basis B. Every mapping $f : B \to \mathbf{A}$ can be extended to a distinct endomorphism, so we can see that $\mathrm{End}(\mathbf{A})$ is uncountably infinite and hence is not finitely generated. By Proposition 5.2.1 we have a contradiction.

We have concluded that $\dim(\mathbf{A})$ is finite, and Corollary 5.5.3 implies that the diagonal bi-act of $\mathrm{End}(\mathbf{A})$ is finitely generated. We may therefore apply Theorem 4.6.6 (and its proof) to conclude that $\mathrm{Aut}(\mathbf{A})$, the automorphism group of $\mathbf{A}$, is an infinite subgroup of $\mathrm{End}(\mathbf{A})$, that it has only finitely many conjugacy classes, and that it has an ideal complement in $\mathrm{End}(\mathbf{A})$. By Proposition 5.7.10 we conclude that $\mathcal{P}_f(\mathrm{Aut}(\mathbf{A}))$ is finitely generated. This is a contradiction to Corollary 5.4.4. ■

5.8 Further restrictions

We now give several consequences of Corollary 5.5.3, using our knowledge of the finite generation of the diagonal bi-act. The following is an immediate consequence of Theorem 3.2.2 and Corollary 5.5.3.

Corollary 5.8.1 *If S is infinite and $\mathcal{P}_f(S)$ is finitely generated then S is a principal ideal and the associated (unique maximal) $\mathcal{J}$-class is infinite.*

If $x \in S$ is indecomposable then $\{x\}$ is a maximal $\mathcal{J}$-class of S. If S has no identity then $\{1\}$ is the unique maximal $\mathcal{J}$-class of S^1.

Corollary 5.8.2 *If S is infinite and has at least one indecomposable element then $\mathcal{P}_f(S)$ is not finitely generated.*

Corollary 5.8.3 *If S is infinite and has no identity then $\mathcal{P}_f(S^1)$ is not finitely generated.*

We now have a consequence of Theorem 3.6.1 and Corollary 5.5.3.

Corollary 5.8.4 *If S is an infinite commutative semigroup then $\mathcal{P}_f(S)$ is not finitely generated.*

From Theorem 3.6.2 and Corollary 5.5.3 we have the next result.

Corollary 5.8.5 *If S is a Bruck–Reilly extension then $\mathcal{P}_f(S)$ is not finitely generated.*

Corollary 5.8.6 *If S is an infinite completely zero-simple semigroup then $\mathcal{P}_f(S)$ is not finitely generated.*

PROOF. If $\mathcal{P}_f(S)$ is finitely generated then, by Corollary 5.5.3, the diagonal bi-act of S is finitely generated. If S is an infinite completely zero-simple semigroup then, by Theorem 3.3.2, it follows that $S \cong G^0$ where G is an infinite group. By Proposition 5.7.11 we deduce that $\mathcal{P}_f(G)$ is finitely generated, contradicting Theorem 5.4.4. ■

Corollary 5.8.7 *If S is an infinite completely simple semigroup then $\mathcal{P}_f(S)$ is not finitely generated.*

PROOF. If S is an infinite completely simple semigroup then S^0 is an infinite completely zero-simple semigroup. If $\mathcal{P}_f(S)$ is finitely generated then, by Proposition 5.7.11, it follows that $\mathcal{P}_f(S^0)$ is finitely generated, contradicting Corollary 5.8.6. ■

The following result mirrors Proposition 3.2.10, which is an equivalent statement for diagonal acts.

Proposition 5.8.8 *Let Y be a semilattice. For each $\alpha \in Y$ let T_α be a semigroup which is either finite or $\mathcal{P}_f(T_\alpha)$ is not finitely generated. If S is an infinite semigroup that is a semilattice of the T_α then $\mathcal{P}_f(S)$ is not finitely generated.*

Proof. We let S be a semilattice of the T_α and assume that $\mathcal{P}_f(S)$ is finitely generated. By Corollary 5.8.1, S has a unique maximal $\mathcal{J}$-class J, which has infinite cardinality. It follows that the underlying semilattice has a unique maximal element μ. Clearly $J \subseteq T_\mu$, so T_μ is infinite. Further, $S \setminus T_\mu$ is an ideal of S so $\mathcal{P}_f(T_\mu)$ is finitely generated by Lemma 5.7.10. This contradicts our assumptions. ■

In [26] it is shown that a semigroup is completely regular if and only if it is a semilattice of completely simple semigroups. The next result is a consequence of Corollary 5.8.7 and Theorem 5.8.8.

Corollary 5.8.9 *If S is an infinite completely regular semigroup then $\mathcal{P}_f(S)$ is not finitely generated.*

Unlike the question of the finite generation of the diagonal bi-acts of infinite completely regular semigroups, we have completely answered the question of the finite generation of finitary power semigroups of infinite completely regular semigroups. Of course, this result for the diagonal bi-act is clearly more complicated, as shown by the (partial) result Theorem 3.4.10. This is due to the difference between Proposition 3.3.1 and Corollary 5.4.4. Of course, there are many classes which we have not even considered in this context (for example, left cancellative and right cancellative). For each of these our questions remain open.

5.9 Finite presentability

Another natural question in the study of the combinatorial properties of finitary power semigroups is that of finite presentability. We begin this section by admitting defeat.

Open Problem 5.9.1 Does there exist an infinite semigroup S for which $\mathcal{P}_f(S)$ is finitely presented?

There are, however, some partial results. We begin with a necessary condition, which may also be regarded as an analogue of Proposition 5.2.1.

Proposition 5.9.2 *If $\mathcal{P}_f(S)$ is finitely presented then S is finitely presented.*

PROOF. Let us assume that $\mathcal{P}_f(S)$ is finitely presented. By Proposition 5.3.1 there must be a finite $A \subseteq S$ such that $\mathcal{P}_f(S) = \langle \mathcal{P}(A) \rangle$. Hence there is a finite presentation $\langle X \mid R \rangle$ which presents $\mathcal{P}_f(S)$ and in which the set of generators X corresponds to the set $\mathcal{P}(A)$. That is, there is a finite set of relations R on the alphabet $X = \{x_B : B \subseteq A\}$ such that every equality in the set $\{B_1 \dots B_l = C_1 \dots C_m : (x_{B_1} \dots x_{B_l} = x_{C_1} \dots x_{C_m}) \in R\}$ holds in $\mathcal{P}_f(S)$ and, moreover, every equality between products of the generators $\mathcal{P}(A)$ that holds in $\mathcal{P}_f(S)$ can be deduced from equalities in this set.

By the proof of Proposition 5.2.1 we know that S is generated by the set A. We define a new alphabet $Y = \{y_a : a \in A\}$ and aim to construct a presentation $\langle Y \mid Q \rangle$ for S, in which $y_a \in Y$ represents $a \in A$. For each relation $u = v$ in R we construct a set of relations $Q_{u=v}$ on Y as follows. We write $u = x_{B_1} \dots x_{B_l}$ and $v = x_{C_1} \dots x_{C_m}$, with each $B_i, C_j \subseteq A$. We include the relation $y_{b_1} \dots y_{b_l} = y_{c_1} \dots y_{c_m}$ in $Q_{u=v}$ if and only if each $b_i \in B_i, c_j \in C_j$ and the equality $b_1 \dots b_l = c_1 \dots c_m$ actually holds in S. We let $Q = \bigcup_{(u=v) \in R} Q_{u=v}$ and we claim that $S = \mathrm{Sgp}\langle Y \mid Q \rangle$.

Let us now fix arbitrary elements $b_1, \dots, b_l, c_1, \dots, c_m \in A$ which satisfy the equality $b_1 \dots b_l = c_1 \dots c_m$. We will complete the proof by demonstrating that the equality $y_{b_1} \dots y_{b_l} = y_{c_1} \dots y_{c_m}$ can be deduced from the relations Q.

We see that each $\{b_i\}, \{c_j\} \in \mathcal{P}(A)$ and that $\{b_1\} \dots \{b_l\} = \{c_1\} \dots \{c_m\}$ holds in $\mathcal{P}_f(S)$. Therefore the equality $x_{\{b_1\}} \dots x_{\{b_l\}} = x_{\{c_1\}} \dots x_{\{c_m\}}$ can be deduced from the relations R. So there is a chain of words $w_1, \dots, w_n \in X^+$ such that $w_1 = x_{\{b_1\}} \dots x_{\{b_l\}}$, $w_n = x_{\{c_1\}} \dots x_{\{c_m\}}$, and for $i = 1, \dots, n-1$ the equality $w_i = w_{i+1}$ holds by one application of a relation from R.

We will use induction on i (with $1 \leq i \leq n$) to show that there exist words $v_1, \dots, v_i \in X^+$ such that for every $q (\leq i)$: if $w_q = x_{D_1} \dots x_{D_r}$ then $v_i = y_{d_1} \dots y_{d_r}$ where $d_j \in D_j$ for each j; and (with $q \neq i$) the equality $v_q = v_{q+1}$ holds by one application of a relation from Q.

For the base step $i = 1$, we define $v_1 = y_{b_1} \dots y_{b_l}$ and the claim is obviously true.

For the inductive step, we suppose that we have defined the words $v_1, \ldots, v_i$, for some particular i with $i < n$, and that these words satisfy the stated properties. Let us say that $v_i = y_{d_1} \ldots y_{d_r}$, and that $w_i = x_{D_1} \ldots x_{D_r}$ with each $d_j \in D_j$. Let us also say that the application of a relation in R which transforms w_i into w_{i+1} does so by changing the subword $x_{D_j} \ldots x_{D_k}$ (with $1 \leq j \leq k \leq r$) into $x_{E_1} \ldots x_{E_p}$. That is,

$$w_{i+1} = x_{D_1} \ldots x_{D_{j-1}} x_{E_1} \ldots x_{E_p} x_{D_{k+1}} \ldots x_{D_r}$$

and $x_{D_j} \ldots x_{D_k} = x_{E_1} \ldots x_{E_p}$ is a relation in R. As this is a relation it follows that $D_j \ldots D_k = E_1 \ldots E_p$ holds in $\mathcal{P}_f(S)$. Therefore we can find elements $e_1 \in E_1, \ldots, e_p \in E_p$ such that the equality $d_j \ldots d_k = e_1 \ldots e_p$ holds in S. We fix these elements and note that $y_{d_j} \ldots y_{d_k} = y_{e_1} \ldots y_{e_p}$ is a relation in Q.

We apply this relation $y_{d_j} \ldots y_{d_k} = y_{e_1} \ldots y_{e_p}$ to the subword $y_{d_j} \ldots y_{d_k}$ of $y_{d_1} \ldots y_{d_r}$, and replace the left hand side of the relation by the right. This produces the word

$$y_{d_1} \ldots y_{d_{j-1}} y_{e_1} \ldots y_{e_p} y_{d_{k+1}} \ldots y_{d_r},$$

which we define to be v_{i+1}. Thus we have produced a sequence of words $v_1, \ldots, v_{i+1} \in Y^+$ with all the desired properties.

We continue this inductive proof to $i = n$. As $w_n = x_{\{c_1\}} \ldots x_{\{c_m\}}$ it follows that $v_n = y_{c_1} \ldots y_{c_m}$. We have shown that there are words $v_1, \ldots, v_n \in Y^+$ with $v_1 = y_{b_1} \ldots y_{b_l}$, $v_n = y_{c_1} \ldots y_{c_m}$, and such that for $i = 1, \ldots, n-1$ the equality $v_i = v_{i+1}$ follows by one application of one relation from Q. Therefore $y_{b_1} \ldots y_{b_l} = y_{c_1} \ldots y_{c_m}$ is a consequence of the relations in Q, and the proof is complete. ∎

We list and label some properties which seem likely to lead to an example of an infinite semigroup S for which $\mathcal{P}_f(S)$ is finitely presented.

(F1) S is a monoid with identity 1.

(F2) S is finitely presented by $\mathrm{Mon}\langle X \mid R \rangle$.

(F3) The diagonal right act of S is cyclic; that is, there are $a, b \in S$ such that $S \times S = (a, b)S$.

(F4) The diagonal right act of S is free with respect to the generator (a, b); that is, $(a, b)s = (a, b)t$ implies $s = t$.

From Proposition 4.1.2, Theorem 4.1.4 and Corollary 4.1.9 we know that there exist infinite semigroups satisfying (F1), (F3) and (F4); the semigroup $\mathcal{B}_X$ of binary relations on an infinite set X and the monoid $R_{\mathbb{N}}$ of partial recursive functions of one variable are two such examples, but neither of these satisfy (F2). The example $\mathcal{M}(X, A, C, D, (a', b'), r)$ with X finite and $|A| = 1$ (from Section 4.7) satisfies (F1), (F2) and (F3) but we do not know whether it satisfies (F4), nor do we even know when it is infinite. In fact, we know of no semigroups that satisfy all of these properties, but we choose to ignore this problem.

In what follows we construct presentations for $\mathcal{P}_f(S)$, based on (and assuming) the properties (F1), (F2), (F3) and (F4). Each of these have the alphabet $X \cup \{z\}$. The letter z represents $\{a, b\} \in \mathcal{P}_f(S)$ while, for each $x \in X$, the letter x represents $\{x\} \in \mathcal{P}_f(S)$. It follows from the proof of Theorem 5.5.6 that these generate $\mathcal{P}_f(S)$.

We will utilise some structural properties of S that are not connected to the presentation $\mathrm{Mon}\langle X \mid R\rangle$. In this way there may arise some ambiguity between an element of S and a word that represents it. To tackle this problem, we adopt the convention of denoting by $w_s \in X^*$ a word representing an element $s \in S$. If there is more than one word representing an element s (as is likely) then we break ties arbitrarily. In a slight deviation from this rule, we let S contain X, so, for example, x and w_x are the same.

We let $r \in S$ be the unique element which satisfies $(1, 1) = (a, b)r$. For each $x \in X$ we let $u(x) \in S$ be the unique element which satisfies the equality $(xa, xb) = (a, b)u(x)$. We define the sets of relations Q_1 and Q_2 on $X \cup \{z\}$ by:

$$Q_1 = \{R,\ zw_r = 1,\ xz = zw_{u(x)} : x \in X\};$$
$$Q_2 = \{z^n w_s = z^n w_t : n \in \mathbb{N}, s, t \in S \text{ and } \{a, b\}^n\{s\} = \{a, b\}^n\{t\}\}.$$

In this section we will frequently use this notation, without any reminder of the definitions.

Lemma 5.9.3 *Let S satisfy (F1), (F2), (F3) and (F4). The finitary power semigroup of S is presented as a monoid by* $\mathrm{Mon}\langle X, z \mid Q_1, Q_2\rangle$.

PROOF. It is clear that the relations Q_1 and Q_2 hold in $\mathcal{P}_f(S)$. Let $A_1 \ldots A_r$ and $B_1 \ldots B_l$ (where each of the A_i and B_j either equals $\{a, b\}$ or $\{x\}$ for some $x \in X$) be equal in $\mathcal{P}_f(S)$. We represent these products in the natural manner as words $q_1, q_2 \in (X \cup \{z\})^*$ respectively. We will complete the proof by showing that $q_1 = q_2$ is a consequence of the relations Q_1 and Q_2.

By applying relations of the form $xz = zv_{u(x)}$ (from Q_1) we may transform q_1 and q_2 into $z^m q_1'$ and $z^n q_2'$, respectively, where $m, n \geq 0$ and $q_1', q_2' \in X^*$. Let us say, without loss of generality, that $m \geq n$. By applying the relation $zw_r = 1$ we may further transform $z^n q_2'$ into $z^m q_2''$, where $q_2'' \in X^*$. As $A_1 \ldots A_r = B_1 \ldots B_l$ holds in $\mathcal{P}_f(S)$, the relation $z^m q_1' = z^m q_2''$ is in Q_2. Therefore $q_1 = q_2$ holds as a consequence of Q_1 and Q_2. ∎

While Q_1 is a finite set of relations, Q_2 is infinite, so we ask whether Q_2 can be reduced. To answer this, we want to know when $\{a, b\}^n\{s\} = \{a, b\}^n\{t\}$ holds in $\mathcal{P}_f(S)$.

We denote $\{w_1, \ldots, w_{2^n}\} = \{a, b\}^n$. We also say that $p \in S$ *permutes* $\{a, b\}^n$ if $\{a, b\}^n\{p\} = \{a, b\}^n$ or, alternatively, if

$$(w_{(1)\pi}, \ldots, w_{(2^n)\pi}) = (w_1, \ldots, w_{2^n})p$$

for some $\pi \in \mathcal{S}_{2^n}$.

Lemma 5.9.4 *Let S be a semigroup satisfying (F1), (F2), (F3) and (F4) and let $s, t \in S, n \in \mathbb{N}$ be such that $\{a, b\}^n\{s\} = \{a, b\}^n\{t\}$. There is an element $p \in S$ which permutes $\{a, b\}^n$ and which satisfies $s = pt$.*

PROOF. Assuming that $\{w_1, \ldots, w_{2^n}\}\{s\} = \{w_1, \ldots, w_{2^n}\}\{t\}$, there is clearly a permutation π of $\{1, \ldots, 2^n\}$ for which

$$(w_1, \ldots, w_{2^n})s = (w_{(1)\pi}, \ldots, w_{(2^n)\pi})t.$$

By Proposition 3.1.11 the 2^n-diagonal right act of S is cyclic, generated by $(w_1, \ldots, w_{2^n})$. Hence there exists $p \in S$ such that

$$(w_{(1)\pi}, \ldots, w_{(2^n)\pi}) = (w_1, \ldots, w_{2^n})p.$$

We note that this element p permutes $\{a,b\}^n$ and that

$$(w_1,\ldots,w_{2^n})s=(w_1,\ldots,w_{2^n})pt.$$

By Proposition 3.1.14, the 2^n-diagonal right act of S is free with respect to the generator $(w_1,\ldots,w_{2^n})$, so we conclude that $s=pt$, as desired. ■

We define a set of relations Q_3 on $X\cup\{z\}$ by

$$Q_3=\{z^n=z^nw_p : n\in\mathbb{N}, p\in S \text{ such that } p \text{ permutes } \{a,b\}^n\}$$

and we note that $Q_3\subseteq Q_2$.

Lemma 5.9.5 *Let S satisfy (F1), (F2), (F3) and (F4). The finitary power semigroup of S is presented as a monoid by* $\mathrm{Mon}\langle X,z \mid Q_1,Q_3\rangle$.

Proof. Let $z^nw_s=z^nw_t$ be a relation in Q_2 (and hence is in the presentation in Lemma 5.9.3), so $\{a,b\}^n\{s\}=\{a,b\}^n\{t\}$. By Lemma 5.9.4 there is $p\in S$ which permutes $\{a,b\}^n$ and satisfies $s=pt$. Hence $w_s=w_pw_t$ (perhaps they are not identical as words, but they are certainly equal in the presentation $\mathrm{Mon}\langle X\mid R\rangle$). We may now rewrite the relation as $z^nw_pw_t=z^nw_t$. It is clear that this relation can be deduced from $z^nw_p=z^n$, which is a relation in Q_2 and is also in the subset Q_3. Therefore, using the Tietze transformation (T4) we may remove the set of relations $Q_2\setminus Q_3$ from the Lemma 5.9.3 presentation to give the desired result. ■

We note that while we have reduced and simplified our set of relations, it is still infinite. At this point we are unable to reduce the presentation any further, so the task of demonstrating a finite presentation for $\mathcal{P}_f(S)$ or showing that one does not exist remains open. The following corollary sums up our progress, but we first require notation for another set of relations.

For each $n\in\mathbb{N}$ we define the set of relations L_n on $X\cup\{z\}$ by

$$L_n=\{z^n=z^nw_p : p\in S \text{ where } p \text{ permutes } \{a,b\}^n\}.$$

Then each L_n is finite, while $Q_3=\bigcup_{n\in\mathbb{N}}L_n$ is infinite. For $N\in\mathbb{N}$ we define a further set of relations $Q_{4,N}$ on $X\cup\{z\}$ by $Q_{4,N}=\bigcup_{n=1}^{N}L_n$.

Corollary 5.9.6 *Let S be a semigroup satisfying (F1), (F2), (F3) and (F4). The finitary power semigroup of S is finitely presented if and only if there is $N \in \mathbb{N}$ such that*

$$\mathcal{P}_f(S) = \mathrm{Mon}\langle X, z \mid Q_1, Q_{4,N}\rangle.$$

PROOF. ($\Rightarrow$) Suppose that $\mathcal{P}_f(S)$ is finitely presented. We use the well-known fact that if T is a finitely presented monoid, A is a finite alphabet, P is an infinite set of relations and $T = \mathrm{Mon}\langle A \mid P\rangle$ then there exists a finite $P_0 \subseteq P$ such that $T = \mathrm{Mon}\langle A \mid P_0\rangle$. Applying this to the Lemma 5.9.5 presentation, we see that there is a finite $Q' \subseteq Q_1 \cup Q_3$ such that $\mathcal{P}_f(S) = \mathrm{Mon}\langle X, z \mid Q'\rangle$. Clearly there exists $N \in \mathbb{N}$ such that $Q_1 \cup Q_{4,N}$ (which is obviously a finite set of relations) contains Q', so the result follows.

($\Leftarrow$) This part is obvious. ■

We now see that if the following question, which remains unsolved, has a positive answer then $\mathcal{P}_f(S)$ is finitely presented.

Open Problem 5.9.7 Does there exist $N \in \mathbb{N}$ such that we may deduce all of the relations Q_3 from only Q_1 and $Q_{4,N}$?

The rest of this section will show a partial proof that the answer to Open Problem 5.9.7 is no. Unfortunately, if this is the case then it means very little, as it is possible that there exists an infinite semigroup S that does not satisfy (F1), (F2), (F3) and (F4) but where $\mathcal{P}_f(S)$ is finitely presented. It seems (at least to the author) that this in unlikely, and that if there is no infinite semigroup S with (F1), (F2), (F3), (F4) and $\mathcal{P}_f(S)$ finitely presented then there should not be any infinite S with a finitely presented $\mathcal{P}_f(S)$.

We begin by describing the set of elements of S which permute $\{a, b\}^n$; then we use some obvious links between this set and the set of relations $Q_{4,N}$ to show that we cannot deduce all of the relations of L_{N+1} using Q_1 and $Q_{4,N}$ in the 'most obvious' ways.

Lemma 5.9.8 *Let $M_n = \{p \in S : p \text{ permutes } \{a, b\}^n\}$. Then M_n is a subgroup of S and is isomorphic to the symmetric group $\mathcal{S}_{2^n}$ of bijections on $\{1, \ldots, 2^n\}$.*

PROOF. We define $\phi : M_n \to \mathcal{S}_{2^n}$ as $(p)\phi = \pi$ where $\pi \in \mathcal{S}_{2^n}$ is the unique permutation which satisfies $(w_1, \ldots, w_{2^n})p = (w_{(1)\pi}, \ldots, w_{(2^n)\pi})$. As $(w_1, \ldots, w_{2^n})$ generates the diagonal right act of S, and because the act is free with respect to this generator, it follows that ϕ is a well-defined bijection.

To show that ϕ is a homomorphism, we let $p_1, p_2 \in M_n$ and write $(p_1)\phi = \pi_1$ and $(p_2)\phi = \pi_2$. Then

$$(w_1, \ldots, w_{2^n})p_1 = (w_{(1)\pi_1}, \ldots, w_{(2^n)\pi_1}),$$

$$(w_1, \ldots, w_{2^n})p_2 = (w_{(1)\pi_2}, \ldots, w_{(2^n)\pi_2}),$$

and hence

$$(w_1, \ldots, w_{2^n})p_1p_2 = (w_{(1)\pi_1\pi_2}, \ldots, w_{(2^n)\pi_1\pi_2}).$$

We see that $(p_1p_2)\phi = \pi_1\pi_2 = (p_1\phi)(p_2\phi)$, so ϕ is an isomorphism. ■

This result gives us some intuition for the set of relations L_n, as there is an obvious correspondence between L_n and M_n. The notion of generation in M_n also has some relevance in L_n, in that the relations $z^n w_p = z^n$ and $z^n w_q = z^n$ in L_n (perhaps also using the relations R) may be used to deduce the relation $z^n w_{pq} = z^n$, and so on for all other products of p and q. In particular, if p and q are generators for the finite symmetric group M_n then we can deduce all of L_n from the relations $z^n w_p = z^n$ and $z^n w_q = z^n$.

Of course, the question remains open on whether we can deduce such 'generating' relations for L_{N+1} using only Q_1 and $Q_{4,N}$. We cannot answer this in general, but we will show in Corollary 5.9.11 below that it cannot be done in what we might think of as the 'most obvious' ways.

We will require the following technical result on finite symmetric groups.

Lemma 5.9.9 *Let X be a finite set, let X_1 and X_2 be disjoint copies of X under the bijections $i \leftrightarrow i_1 \leftrightarrow i_2$ and let $\pi \in \mathcal{S}_X$ be arbitrary. If the transformation $\pi' : X_1 \cup X_2 \to X_1 \cup X_2$ satisfies $(i_1)\pi' = (i\pi)_1$ and $(i_2)\pi' = (i\pi)_2$ for all $i \in X$ then π' is an even permutation of $X_1 \cup X_2$.*

PROOF. Clearly the restriction $\pi' \restriction_{X_1}$ is a permutation on X_1 and the restriction $\pi' \restriction_{X_2}$ is a permutation on X_2. We also see that these permutations have the same cycle shape as π. Let us say that they are each a product of r transpositions. Further, π' is the product of these restrictions; that is, $\pi' = \pi' \restriction_{X_1} \pi' \restriction_{X_2}$, so it can be written as a product of $2r$ transpositions and is, therefore, even. ■

For an element $s \in S$ we may write $s = x_1 \dots x_m$ where each $x_i \in X$. We now define $u(s) = u(x_1) \dots u(x_m)$. A simple inductive argument shows that $u(s)$ (is the unique element which) satisfies $(sa, sb) = (a, b)u(s)$. Similarly, $w_s z = z w_{u(s)}$ holds in the presentation $\mathrm{Mon}\langle X, z \mid Q_1, Q_{4,N}\rangle$ (which comes from Corollary 5.9.6).

Lemma 5.9.10 *Let S satisfy (F1), (F2), (F3) and (F4) and let $p \in S$ permute $\{a,b\}^n$. Then p and $u(p)$ are even permutations of $\{a,b\}^{n+1}$.*

PROOF. As p permutes $\{a,b\}^n$, we know that $\{a,b\}^n\{p\} = \{a,b\}^n$. Then

$$\{a,b\}^{n+1}\{p\} = \{a,b\}^{n+1},$$

$$\{a,b\}^{n+1}\{u(p)\} = \{a,b\}^n\{p\}\{a,b\} = \{a,b\}^{n+1},$$

and so we see that p and $u(p)$ each permute $\{a,b\}^{n+1}$.

First we consider p as a permutation of $\{a,b\}^{n+1}$. We write $X_1 = \{a\}\{a,b\}^n$ and $X_2 = \{b\}\{a,b\}^n$. Then X_1 and X_2 are disjoint copies of the set $\{a,b\}^n$ under the bijections $w \leftrightarrow aw \leftrightarrow bw$, and $X_1 \cup X_2 = \{a,b\}^{n+1}$. Let $w \in \{a,b\}^n$ be arbitrary and note that $wp \in \{a,b\}^n$. Then

$$(aw)p = a(wp) \in X_1,$$

$$(bw)p = b(wp) \in X_2,$$

so p, when considered as a permutation of $\{a,b\}^{n+1}$, satisfies the condition stated in Lemma 5.9.9 and hence is even.

Next, we consider $u(p)$ as a permutation of $\{a,b\}^{n+1}$. Let us now define the sets $Y_1 = \{a,b\}^n\{a\}$ and $Y_2 = \{a,b\}^n\{b\}$. Then Y_1 and Y_2 are disjoint copies of the set $\{a,b\}^n$ under the bijection $w \leftrightarrow wa \leftrightarrow wb$, and $Y_1 \cup Y_2 = \{a,b\}^{n+1}$. Again let $w \in \{a,b\}^n$ be arbitrary and note that $wp \in \{a,b\}^n$. Thus

$$(wa)u(p) = (wp)a \in Y_1,$$

$$(wb)u(p) = (wp)b \in Y_2,$$

so $u(p)$ satisfies the condition stated in Lemma 5.9.9, and hence is an even permutation of $\{a, b\}^{n+1}$. ∎

Corollary 5.9.11 *Let S be a semigroup satisfying (F1), (F2), (F3) and (F4). The relations in L_{N+1} which may be deduced from Q_1 and $Q_{4,N}$ by pre- or post-multiplying a relation in L_N by z are of the form $z^{N+1}w_p = z^{N+1}$, where $p \in S$ is an even permutation of $\{a, b\}^{N+1}$.*

Due to this corollary, we conjecture that for no N is it possible to deduce all the relations of L_{N+1} from Q_1 and $Q_{4,N}$. We appear to be tantalisingly close to a proof that Open Problem 5.9.7 has a negative answer, but we are unable to complete the investigation.

Another approach to the problem could be to utilise Theorem 4.7.2. That is, if S is a finitely generated monoid with a cyclic diagonal right act (which is the case if it satisfies properties (F1), (F2) and (F3)) then by Theorem 5.5.6 and Proposition 5.6.6 we know that $\mathcal{P}_f(S)$ is also a finitely generated monoid with a cyclic diagonal right act. Thus we may apply Theorem 4.7.2 and conclude that $\mathcal{P}_f(S)$ is a homomorphic image of a monoid defined by a presentation of the form (4.5), which appeared in Section 4.7. Perhaps a finite presentation for $\mathcal{P}_f(S)$ could be worked out from this observation, or a proof exhibited that one does not exist.

Chapter 6

Finite generation and presentability of Schützenberger products

In this chapter we are concerned with Schützenberger products. We start with the central definition, which first requires two acts to be defined.

Let S and T be semigroups and let $\mathcal{P}_f(S \times T)$ be the finitary power set of the direct product $S \times T$ (in this case we let $\emptyset \in \mathcal{P}_f(S \times T)$). Then we let S act on $\mathcal{P}_f(S \times T)$ from the left via $sQ = \{(sp, q) : (p, q) \in Q\}$, and we let T act on $\mathcal{P}_f(S \times T)$ from the right via $Qt = \{(p, qt) : (p, q) \in Q\}$.

The *Schützenberger product* $S \Diamond T$ is the set $S \times \mathcal{P}_f(S \times T) \times T$ with multiplication

$$(s_1, Q_1, t_1)(s_2, Q_2, t_2) = (s_1 s_2, s_1 Q_2 \cup Q_1 t_2, t_1 t_2). \tag{6.1}$$

Although this definition involves the finitary power set of a semigroup and the notion of actions, which was one of the initial reasons for undertaking this study as part of this project, none of the results in this chapter indicate that the Schützenberger product has any deeper connection with the topics that we covered in our earlier chapters.

The majority of this chapter, but not all of it, appeared in [14].

6.1 Some properties of the Schützenberger product

We begin by observing some basic properties of this construction. These will be frequently used, and will form a foundation on which more interesting statements will be proved.

(S1) $S\diamond T$ is a monoid if and only if S and T are monoids. If the identity of S is 1_S and the identity of T is 1_T then the identity of $S\diamond T$ is $(1_S, \emptyset, 1_T)$.

(S2) $S\diamond T$ is finite if and only if S and T are finite. Moreover, if $|S| = n$ and $|T| = m$ then $|S\diamond T| = nm2^{nm}$.

(S3) $S\diamond T$ is countable if and only if S and T are countable.

(S4) $S \times \{\emptyset\} \times T$ is a subsemigroup of $S\diamond T$ isomorphic to the direct product $S \times T$. Its complement $(S\diamond T) \setminus (S \times \{\emptyset\} \times T)$ is an ideal of $S\diamond T$.

(S5) S and T are homomorphic images of $S\diamond T$.

We immediately note a property that is unusual among semigroup products. By property (S2) we see that if T is trivial and S is finite then $|S\diamond T| > |S|$, so $S\diamond T \not\cong S$. Indeed, it is true in general that $S\diamond T \not\cong S$ where T is trivial. Similarly, in general if S is trivial then $S\diamond T \not\cong T$.

We now ask when the Schützenberger product $S\diamond T$ is in some of the standard classes of semigroups. The first result can also be deduced from Corollary 6.8 of [9].

Theorem 6.1.1 *Let S and T be semigroups. The Schützenberger product $S\diamond T$ is inverse if and only if S and T are groups.*

Proof. ($\Rightarrow$) Let us assume that $S\diamond T$ is inverse. Due to property (S5) (that is, S and T are homomorphic images of $S\diamond T$) it immediately follows that S and T are inverse.

Suppose that S is not a group. Then S contains two distinct but comparable idempotents e and f, say with $f < e$. Let t be arbitrary in T and define

$Q \in \mathcal{P}_f(S \times T)$ as $Q = \{(e,t)\}$. We consider the element $(f,Q,t) \in S\Diamond T$ and we suppose that its inverse is (s,P,u). Then

$$\begin{aligned} (f,Q,t)(s,P,u)(f,Q,t) &= (f,Q,t), \\ (s,P,u)(f,Q,t)(s,P,u) &= (s,P,u), \end{aligned} \tag{6.2}$$

so $sfs = s$, $fsf = f$ and hence s is the inverse of f. As f is idempotent it follows that $s = f$, so the middle co-ordinate of (6.2) becomes the equality $Qut \cup fPt \cup fQ = Q$. This means that $fQ \subseteq Q$, which is impossible as $fQ = \{(f,t)\}$. We have a contradiction, so we conclude that S is a group. An analogous argument shows that T is a group.

($\Leftarrow$) Let us assume that S and T are groups. We will use the fact that a semigroup is inverse if and only if it is regular and its idempotents commute (this is proven in [26]).

First, we let $(s,Q,t) \in S\Diamond T$ be arbitrary and we observe that

$$(s,Q,t)(s^{-1},\emptyset,t^{-1})(s,Q,t) = (s,Q,t),$$

so $S\Diamond T$ is regular.

Second, we suppose that (s,Q,t) is idempotent. Then s and t are each idempotent, so $s = 1_S$ and $t = 1_T$. Moreover, for any $Q \in \mathcal{P}_f(S \times T)$ we see that $(1_S,Q,1_S)$ is idempotent, so the set of idempotents of $S\Diamond T$ is precisely $\{(1_S,Q,1_T) : Q \in \mathcal{P}_f(S \times T)\}$. It is obvious that

$$(1_S,P,1_T)(1_S,Q,1_T) = (1_S,P \cup Q,1_T) = (1_S,Q,1_T)(1_S,P,1_T),$$

so idempotents commute. We conclude that $S\Diamond T$ is inverse. ■

The Schützenberger product of two groups has an interesting structure, which we describe in Corollary 6.4.2 and Theorem 6.5.3 below. We note that $S\Diamond T$ is never a group, as $(1_S,\{(1_S,1_T)\},1_T)$ and $(1_S,\emptyset,1_T)$ are distinct idempotents. We also observe the following result about Schützenberger products of groups.

Proposition 6.1.2 *Let G and H be groups. Then $G\Diamond H \cong H\Diamond G$.*

PROOF. For $Q \in \mathcal{P}_f(G \times H)$ we define $Q^* \in \mathcal{P}_f(H \times G)$ as

$$Q^* = \{(q^{-1}, p^{-1}) : (p, q) \in Q\}. \tag{6.3}$$

We now define $\phi : G \diamond H \to H \diamond G$ by $(g, Q, h)\phi = (h, hQ^*g, g)$ and claim that ϕ is an isomorphism.

Before we prove this, we make a further definition and some observations. For $Q \in \mathcal{P}_f(H \times G)$ we define $Q^* \in \mathcal{P}_f(G \times H)$ in exactly the same way as (6.3). We note that $(Q^*)^* = Q$ for all Q. Further, for all $g \in G, Q \in \mathcal{P}_f(G \times H)$ we see that

$$\begin{aligned}
(gQ)^* &= \{(gp, q) : (p, q) \in Q\}^* \\
&= \{(q^{-1}, p^{-1}g^{-1}) : (p, q) \in Q\} \\
&= \{(q^{-1}, p^{-1}) : (p, q) \in Q\}g^{-1} \\
&= Q^*g^{-1}.
\end{aligned}$$

Similarly, we can deduce that $(Qh)^* = h^{-1}Q^*$. It is also very easy to show that $(P \cup Q)^* = P^* \cup Q^*$.

For an arbitrary $(h, Q, g) \in H \diamond G$ we can write

$$\begin{aligned}
(g, gQ^*h, h)\phi &= (h, h(gQ^*h)^*g, g) \\
&= (h, hh^{-1}(Q^*)^*g^{-1}g, g) \\
&= (h, Q, g),
\end{aligned}$$

so ϕ is surjective. To show that ϕ is injective, we suppose that the equality $(g_1, Q_1, h_1)\phi = (g_2, Q_2, h_2)\phi$ holds. Then $(h_1, h_1Q_1^*g_1, g_1) = (h_2, h_2Q_2^*g_2, g_2)$, so $h_1 = h_2, g_1 = g_2$ and $h_1Q_1^*g_1 = h_2Q_2^*g_2$. Thus $Q_1^* = Q_2^*$, so $Q_1 = Q_2$ and ϕ is injective.

Finally, the manipulation

$$\begin{aligned}
[(g_1, Q_1, h_1)(g_2, Q_2, h_2)]\phi &= (g_1g_2, Q_1h_2 \cup g_1Q_2, h_1h_2)\phi \\
&= (h_1h_2, h_1h_2(Q_1h_2 \cup g_1Q_2)^*g_1g_2, g_1g_2) \\
&= (h_1h_2, h_1h_2((Q_1h_2)^* \cup (g_1Q_2)^*)g_1g_2, g_1g_2) \\
&= (h_1h_2, h_1h_2(h_2^{-1}Q_1^* \cup Q_2^*g_1^{-1})g_1g_2, g_1g_2) \\
&= (h_1h_2, h_1Q_1^*g_1g_2 \cup h_1h_2Q_2^*g_2, g_1g_2) \\
&= (h_1, h_1Q_1^*g_1, g_1)(h_2, h_2Q^*g_2, g_2) \\
&= [(g_1, Q_1, h_1)\phi][(g_2, Q_2, h_2)\phi]
\end{aligned}$$

shows that ϕ is a homomorphism. ∎

At this point we ask whether it is always the case that $S\Diamond T \cong T\Diamond S$.

A similar question we could ask concerns the notion of the opposite of a semigroup. The *opposite* of S is the set S' which is in correspondence with S via the bijection $s \leftrightarrow s'$ and whose multiplication is defined as $s't' = (ts)'$. If S is inverse then $S \cong S'$ via the isomorphism $\phi : S \to S'$ which is specified by $(s)\phi = (s^{-1})'$. Therefore, another possible extension of Proposition 6.1.2 would be to show that, in general, $T\Diamond S \cong (S\Diamond T)'$.

We use the following result to show, in Example 6.1.4 below, that neither of these are the case.

Theorem 6.1.3 *Let S and T be semigroups. The Schützenberger product $S\Diamond T$ is a band if and only if S is a right zero semigroup and T is a left zero semigroup. In addition, if this is the case then $S\Diamond T$ is the direct product of the rectangular band $S \times T$ and the semilattice $\mathcal{P}_f(S \times T)$ with the operation of union.*

PROOF. ($\Rightarrow$) Assume that $S\Diamond T$ is idempotent. Let $s_1, s_2 \in S, t \in T$ be arbitrary and consider the element $(s_1, \{(s_2, t)\}, t) \in S\Diamond T$. This is idempotent, so

$$\begin{aligned}(s_1, \{(s_2,t)\}, t) &= (s_1, \{(s_2,t)\}, t)(s_1, \{(s_2,t)\}, t) \\ &= (s_1^2, \{(s_2, t^2), (s_1 s_2, t)\}, t^2).\end{aligned}$$

The middle co-ordinate gives $\{(s_2,t)\} = \{(s_2,t^2), (s_1s_2,t)\}$, so we see that $s_1 s_2 = s_2$. That is, S is a right zero semigroup. An analogous argument shows that T is a left zero semigroup.

($\Leftarrow$) Assume that S is a right zero semigroup and that T is a left zero semigroup. Then it is clear, for all $s \in S, t \in T$ and $Q \in \mathcal{P}_f(S \times T)$, that the equalities $sQ = Q$ and $Qt = Q$ hold. Therefore

$$(s, Q, t)(s, Q, t) = (s^2, sQ \cup Qt, t^2) = (s, Q, t),$$

so $S\Diamond T$ is a band. We also see, for $s_1, s_2 \in S, t_1, t_2 \in T, Q_1, Q_2 \in \mathcal{P}_f(S \times T)$, that

$$(s_1, Q_2, t_2)(s_2, Q_2, t_2) = (s_2, Q_1 \cup Q_2, t_1),$$

so $S\diamond T$ is the direct product of the rectangular band $S\times T$ and the semilattice $\mathcal{P}_f(S\times T)$ with respect to union. ■

Example 6.1.4 Let S be a non-trivial right zero semigroup and let T be a non-trivial left zero semigroup. By Theorem 6.1.3, $S\diamond T$ is a band but $T\diamond S$ is not. Hence $S\diamond T \not\cong T\diamond S$. Further, as the opposite of a band is also a band, we see that $(S\diamond T)' \not\cong T\diamond S$. ■

Theorem 6.1.5 *The Schützenberger product $S\diamond T$ is commutative if and only if S and T are trivial. In addition, if this is the case then $S\diamond T$ is the two-element semilattice.*

Proof. ($\Rightarrow$) Assume that $S\diamond T$ is commutative. By property (S5) it follows that S and T are also commutative. We now let $s_1, s_2 \in S, t\in T$ be arbitrary and observe that

$$\begin{aligned}(s_1^2, \{(s_1s_2,t)\}, t^2) &= (s_1,\emptyset,t)(s_1,\{(s_2,t)\},t)\\ &= (s_1,\{(s_2,t)\},t)(s_1,\emptyset,t)\\ &= (s_1^2,\{(s_2,t^2)\},t_2^2).\end{aligned}$$

Equating the middle co-ordinate gives $(s_1s_2,t)=(s_2,t^2)$, so $s_1s_2=s_2$ and S is a right zero semigroup. But S is commutative, so we conclude that S must be trivial. An analogous argument shows that T is trivial.

($\Leftarrow$) Assume that S and T are trivial. Then $S\diamond T$ has two elements, namely $(1_S,\emptyset,1_T)$ and $(1_S,\{(1_S,1_T)\},1_T)$, and it is a two-element semilattice. ■

The two element semilattice in Theorem 6.1.5 is also the only example in which $S\diamond T$ is zero-simple, as $(S\diamond T)\setminus(S\times\{\emptyset\}\times T)$ is an ideal (property (S4)) and has size greater than 1 in all other cases. The Schützenberger product is never simple for the same reason.

There is a range of open questions in this topic, of which the following is an example.

Open Problem 6.1.6 Can we classify the semigroups S and T for which the Schützenberger product $S\diamond T$ is regular? What about completely regular?

6.2 Finite generation

In our examination of the finite generation of the Schützenberger product we will use the connections with the direct product (properties (S4) and (S5)). We say that an element x of a semigroup S is *indecomposable* if there do not exist $y, z \in S$ such that $x = yz$. In [37] we find the following two results on the finite generation of direct products (that paper also contains results classifying the semigroups S and T for which $S \times T$ is finitely presented, but we do not use these).

Proposition 6.2.1 *Let S and T be infinite semigroups. The direct product $S \times T$ is finitely generated if and only if S and T are finitely generated and do not contain any indecomposable elements.*

Proposition 6.2.2 *Let S be an infinite semigroup and let T be a finite semigroup. The direct product $S \times T$ is finitely generated if and only if S is finitely generated and T does not contain any indecomposable elements.*

The property (S2) answers the question of finite generation of $S \Diamond T$ where S and T are finite. We now deal with the more difficult cases.

Theorem 6.2.3 *Let S and T be semigroups, at least one of which is infinite. The Schützenberger product $S \Diamond T$ is finitely generated if and only if the following conditions are satisfied:*

(i) *S and T are finitely generated;*

(ii) *S has a unique maximal $\mathcal{R}$-class R and there exists a finite $A \subseteq S$ such that $S = RA$;*

(iii) *T has a unique maximal $\mathcal{L}$-class L and there exists a finite $B \subseteq T$ such that $T = BL$.*

Proof. ($\Rightarrow$) Assume that X finitely generates $S \Diamond T$. By property (S5) it follows that S and T are finitely generated, so condition (i) holds.

Let $k = 1 + \max\{|Q| : (x, Q, y) \in X\}$. Then, for arbitrary $s \in S, t \in T$ and $P \in \mathcal{P}_f(S \times T)$ with $|P| = k$ there are generators $(x_i, Q_i, y_i) \in X$ such that

$$(s, P, t) = (x_1, Q_1, y_1) \ldots (x_n, Q_n, y_n).$$

We first note that $n > 1$. It is clear that $s = x_1 \dots x_n$ and $t = y_1 \dots y_n$, so neither S nor T contain any indecomposable elements.

As S is finitely generated it has only finitely many maximal $\mathcal{R}$-classes, and every other $\mathcal{R}$-class is below one of these. We let $R_1, \dots, R_l$ be the maximal $\mathcal{R}$-classes and we select representatives $r_i \in R_i$ for $i = 1, \dots, l$. Now we consider an arbitrary element $s \in S$. There are $u, v \in S$ such that $s = uv$ and there is $i \in \{1, \dots, l\}$ such that $R_u \leq R_i$. Hence there is $w \in S^1$ such that $u = r_i w$, so $wv \in S$ and $s = r_i wv$. We conclude that for each $s \in S$ there are $\alpha(s) \in \{1, \dots, l\}$ and $\mu(s) \in S$ such that $s = r_{\alpha(s)}\mu(s)$.

We define $A \subseteq S$ as the finite set $C \cup D$ where $C, D \subseteq S$ are given by:

$$C = \{s \in S : (s,t) \in Q \text{ for some } (x, Q, y) \in X \text{ and } t \in T\};$$

$$D = \{\mu(s) : s \in C\}.$$

We claim that

$$S = (\bigcup_{i=1}^{l} R_i)A. \tag{6.4}$$

To prove this assertion, we begin by letting $s \in S$ be arbitrary. Further, we pick $t, t_1 \in T$ and let $P \in \mathcal{P}_f(S \times T)$ be a k-set containing (s,t). We now consider the triple (r_1, P, t_1) and write it as

$$(r_1, P, t_1) = (x_1, Q_1, y_1) \dots (x_n, Q_n, y_n), \tag{6.5}$$

a product of generators from X. Again we note that $n > 1$. Now we see that $r_1 = x_1 \dots x_n$, so $x_1 \dots x_j \in R_1$ for every $j = 1, \dots, n$. Equating the middle co-ordinates of (6.5) gives that

$$P = \bigcup_{j=1}^{n} x_1 \dots x_{j-1} Q_j y_{j+1} \dots y_n.$$

Therefore there is $h \in \{1, \dots, n\}$ and $(a, b) \in Q_h$ (so $a \in C$) such that

$$(s,t) = (x_1 \dots x_{h-1}a, by_{h+1}, \dots, y_n).$$

We consider two cases concerning the equation $s = x_1 \dots x_{h-1}a$. If $h = 1$ then $s = a = r_{\alpha(a)}\mu(a) \in R_{\alpha(s)}A$. If $h > 1$ then, clearly, $s \in R_1 A$. This proves (6.4).

We now see that either S is infinite, in which case at least one of $R_1, \dots, R_l$ is infinite, or else S is finite and T is infinite. In any case, without loss of generality, we may assume that $R_1 \times T$ is infinite. We now consider an arbitrary set $P = \{(s_1, t_1), \ldots, (s_k, t_k)\} \subseteq R_1 \times T$ of size k and let $t \in T$ be arbitrary. We consider (r_2, P, t) and write it as another product of generators from X:

$$(r_2, P, t) = (x_1, Q_1, y_1) \ldots (x_n, Q_n, y_n).$$

Yet again, note that $n > 1$. For $(a, b) \in Q_2$ it follows that $(x_1 a, b y_3 \ldots y_n) \in P$, so there is some $m \in \{1, \ldots, k\}$ with $s_m = x_1 a$. Also, $r_2 = x_1 \ldots x_n$, so $x_1 \in R_2$. It now follows that $R_1 = R_{s_m} \leq R_{x_1} = R_2$, a contradiction. Therefore S has a unique maximal $\mathcal{R}$-class. Returning now to (6.4), we see that condition (ii) holds. By a symmetric argument, it can be shown that condition (iii) holds as well.

($\Leftarrow$) Assume that conditions (i), (ii) and (iii) hold. Neither S nor T contain any indecomposable elements (by (ii) and (iii)) and both S and T are finitely generated (by (i)). Hence, by Propositions 6.2.1 and 6.2.2, we see that $S \times T$ is finitely generated and we let Y be a finite generating set for $S \times \{\emptyset\} \times T$ (which is a subsemigroup isomorphic to $S \times T$). We fix arbitrary $r \in R, l \in L$ and let

$$X = Y \cup \{(r, \{(a, b)\}, l) : a \in A, b \in B\}.$$

We will complete this part of the proof by showing that $S \diamond T = \langle X \rangle$.

By induction on $|P|$ we will show that $(s, P, t) \in \langle X \rangle$ for all $P \in \mathcal{P}_f(S \times T)$ and all $s \in S, t \in T$. The base case $|P| = 0$ is obvious as X includes a generating set for $S \times \{\emptyset\} \times T$.

Next we consider the case $|P| = 1$, by letting $s, s_1 \in S$ and $t, t_1 \in T$ be arbitrary and examining the triple $(s, \{(s_1, t_1)\}, t)$. As $S = RA$ there are $r_1 \in R, a \in A$ such that $s_1 = r_1 a$. As R is the unique maximal $\mathcal{R}$-class of S there is $u \in S$ such that $s = r_1 u$. Further, there is $u_1 \in S$ such that $u = r u_1$. Therefore $s_1 = r_1 a$ and $s = r_1 r u_1$. A dual argument shows that there exist $b \in B, l_1 \in L$ and $v_1 \in T$ such that $t_1 = b l_1$ and $t = v_1 l l_1$. Hence

$$(s, \{(s_1, t_1)\}, t) = (r_1, \emptyset, v_1)(r, \{(a, b)\}, l)(u_1, \emptyset, l_1) \in \langle X \rangle.$$

For the inductive step we assume that X generates all of the triples in the set $\{(s, P, t) : |P| \leq k, s \in S, t \in T\}$ for some positive integer k. We let $s, s_i \in S, t, t_i \in T$ be arbitrary for $i = 1, \ldots, k+1$ and consider the triple

$(s, \{(s_1, t_1), \ldots, (s_{k+1}, t_{k+1})\}, t)$. There exist $r_1 \in R$ and $x_1, x_2 \in S$ such that $s_{k+1} = r_1 x_1$ and $s = r_1 x_2$. Also, there are $l_1 \in L$ and $y_1, \ldots, y_{k+1} \in T$ such that $t = y_{k+1} l_1$ and $t_i = y_i l_1$ for $i = 1, \ldots, k$. Then

$$\begin{aligned}
& (s, \{(s_1, t_1), \ldots, (s_{k+1}, t_{k+1})\}, t) \\
= \;& (r_1 x_2, \{(s_1, y_1 l_1), \ldots, (s_k, y_k l_1), (r_1 x_1, t_{k+1})\}, y_{k+1} l_1) \\
= \;& (r_1, \{(s_1, y_1), \ldots, (s_k, y_k)\}, y_{k+1})(x_2, \{(x_1, t_{k+1})\}, l_1)
\end{aligned}$$

and the claim follows by induction. ∎

We notice the similarity between this theorem and the following result, which appeared in [34] and concerned wreath products (the definition of which can also be found in that paper). However, this similarity does not extend as far as either set of conditions implying the other.

Proposition 6.2.4 *The restricted wreath product of two monoids A and B is finitely generated if and only if A and B are finitely generated and either A is trivial or $B = VG$ where $V \subseteq B$ is finite and $G \subseteq B$ is the group of units of B.*

We apply Theorem 6.2.3 to some examples. After this section we will mainly be interested in Schützenberger products of groups, so we state our first example as a formal result.

Corollary 6.2.5 *Let G and H be groups. The Schützenberger product $G \Diamond H$ is finitely generated if and only if G and H are finitely generated. Moreover,* $\mathrm{rank}(G \Diamond H) = \mathrm{rank}(G \times H) + 1$.

Proof. The first statement of the result follows directly from Theorem 6.2.3. From the proof of that result, we observe that

$$G \Diamond H \;=\; \langle G \times \{\emptyset\} \times H, (1_G, \{(1_G, 1_H)\}, 1_H)\rangle,$$

so $\mathrm{rank}(G \Diamond H) \leq \mathrm{rank}(G \times H) + 1$.

From the property (S4) it follows that $\mathrm{rank}(G \Diamond H) > \mathrm{rank}(G \times H)$, completing the proof. ∎

We can make similar statements for certain constructions on groups.

Example 6.2.6 Let the infinite finitely generated groups G_β and G_γ be isomorphic (under the isomorphisms $g_\beta \leftrightarrow g_\gamma$). Let Y be the semilattice $\{\beta, \gamma\}$ with $\gamma < \beta$. Let $\phi_{\beta,\gamma} : G_\beta \to G_\gamma$ be defined as $(g_\beta)\phi_{\beta,\gamma} = g_\gamma$, and let $\phi_{\beta,\beta}$ and $\phi_{\gamma,\gamma}$ be the identity transformations on G_β and G_γ respectively. We let $S = (G_\alpha, Y, \phi_{\alpha,\beta})$ be the usual Clifford semigroup construction with these ingredients, and let H be another infinite finitely generated group.

Clearly G_β is both the unique maximal $\mathcal{R}$-class and $\mathcal{L}$-class of S. We also see that there is a finite set $A = \{1_{G_\beta}, 1_{G_\gamma}\} \subseteq S$ which satisfies the equality $G_\beta A = AG_\beta = S$. Also, H is its own unique maximal $\mathcal{R}$-class and $\mathcal{L}$-class, and there is a finite set $B = \{1_H\}$ which satisfies $H = BH = HB$. By Theorem 6.2.3 we conclude that $S \Diamond H$ and $H \Diamond S$ are both finitely generated. ■

We also note that the conditions of Theorem 6.2.3 are not symmetric, in the sense shown by the following example.

Example 6.2.7 Let G and H be infinite finitely generated groups, let Z be a finite non-trivial right zero semigroup and let $S = G \times Z$ (which is also a Rees matrix semigroup). Letting X be a finite generating set for G, it is clear that S is finitely generated by $X \times Z$.

Further, S has a unique maximal $\mathcal{R}$-class, which is S itself, and there is a finite set $A = \{(1_G, z) : z \in Z\} \subseteq S$ satisfying $S = SA$. Also, H has a unique $\mathcal{L}$-class, which is H itself and there is a finite set $B = \{1_H\} \subseteq H$ satisfying $H = BH$. Therefore $S \Diamond H$ is finitely generated by Theorem 6.2.3.

However, S consists of $|Z|$ incomparable $\mathcal{L}$-classes, so $H \Diamond S$ is not finitely generated, again by Theorem 6.2.3. It is also clear that $S \Diamond H \ncong H \Diamond S$. ■

6.3 Schützenberger products of two groups: an infinite presentation

In this section we lay foundations for proofs of Theorems B and C by obtaining an infinite, but 'nicely behaved', presentation ((6.7) in Theorem 6.3.2) for the

Schützenberger product of two groups. A very similar presentation (which we label here as (6.9)) appears in [16], where it is derived from a presentation for a more general structure.

We show another way to derive it, beginning with the following theorem from [25] and using Tietze transformations.

Theorem 6.3.1 *If $S = \mathrm{Mon}\langle X \mid R_S\rangle$ and $T = \mathrm{Mon}\langle Y \mid R_T\rangle$ then*

$$\begin{aligned} S\Diamond T &= \mathrm{Mon}\langle X, Y, z_{s,t}(s \in S, t \in T) \mid R_S,\ R_T,\ xy = yx, \\ &\quad z_{s,t}^2 = z_{s,t},\ z_{s,t}z_{u,v} = z_{u,v}z_{s,t},\ xz_{s,t} = z_{xs,t}x, \\ &\quad z_{s,t}y = yz_{s,ty}\ (x \in X,\ y \in Y,\ s,u \in S,\ t,v \in T)\rangle. \end{aligned} \tag{6.6}$$

We now transform presentation (6.6) in the case where the monoids considered are groups G and H.

Theorem 6.3.2 *Let G and H be groups and let $G \times H = \mathrm{Mon}\langle A \mid R\rangle$. For each $(g, h) \in G \times H$ we let $w_{g,h} \in A^*$ be a fixed word representing (g, h) and let $z_{g,h} = w_{g,h^{-1}}zw_{g^{-1},h}$. Then*

$$\langle A, z \mid R,\ z^2 = z,\ z_{g_1,h_1}z_{g_2,h_2} = z_{g_2,h_2}z_{g_1,h_1}(g_1, g_2 \in G, h_1, h_2 \in H)\rangle \tag{6.7}$$

presents $G\Diamond H$ as a monoid.

Proof. Let $\langle X \mid R_G\rangle$ and $\langle Y \mid R_H\rangle$ present G and H, respectively, as monoids. By Theorem 6.3.1,

$$\begin{aligned} G\Diamond H &= \mathrm{Mon}\langle X, Y, z_{g,h}(g \in G, h \in H) \mid R_G,\ R_H,\ xy = yx, \\ &\quad z_{g,h}^2 = z_{g,h},\ z_{g_1,h_1}z_{g_2,h_2} = z_{g_2,h_2}z_{g_1,h_1},\ xz_{g,h} = z_{xg,h}x, \\ &\quad z_{g,h}y = yz_{g,hy}\ (x \in X,\ y \in Y,\ g, g_1, g_2 \in G,\ h, h_1, h_2 \in H)\rangle. \end{aligned} \tag{6.8}$$

For every $g \in G$ we let $w_g \in X^*$ be a fixed word representing g. For every $h \in H$ we let $w_h \in Y^*$ be a fixed word representing h. By repeated application of the relations $xz_{g,h} = z_{xg,h}x$ we can deduce all relations of the form $w_{g_1}z_{g,h} = z_{g_1g,h}w_{g_1}$, so we use the Tietze transformation (T2) to add these to the presentation (6.8). Likewise, from $z_{g,h}y = yz_{g,hy}$ we can deduce all relations of the form $z_{g,h}w_{h_1} = w_{h_1}z_{g,hh_1}$, so we use (T2) to add these to (6.8) as well.

Now, from $w_{g_1}z_{g,h} = z_{g_1g,h}w_{g_1}$, $z_{g,h}w_{h_1} = w_{h_1}z_{g,hh_1}$, R_G and R_H we can deduce all relations of the form $w_g w_{h^{-1}} z_{1_G,1_H} w_h w_{g^{-1}} = z_{g,h}$, so we use (T2) to add these to our presentation.

We now use (T3) to remove all $z_{g,h}$, except for $z_{1_G,1_H}$, from the generating set, and delete all relations $z_{g,h} = w_g w_{h^{-1}} z_{1_G,1_H} w_h w_{g^{-1}}$ from the set of defining relations. For the sake of brevity we will rename $z_{1_G,1_H}$ as z. We will also continue to use the symbols $z_{g,h}$ with the understanding that they represent the corresponding words $w_g w_{h^{-1}} z w_h w_{g^{-1}}$.

From the relations R_G, R_H and $z^2 = z$ we can deduce all relations of the form $z_{g,h}^2 = z_{g,h}$ so we use (T4) to remove these from our presentation, except for $z^2 = z$.

Similarly, from R_G, R_H and $xy = yx$ we can deduce all relations of the forms $w_{g_1}z_{g,h} = z_{g_1g,h}w_{g_1}$ and $z_{g,h}w_{h_1} = w_{h_1}z_{g,hh_1}$ so we use (T4) to delete these from our presentation. Thus we have transformed the original presentation (6.8) for $G\diamond H$ into

$$\begin{aligned} \mathrm{Mon}\langle X, Y, z \ \mid \ & R_G,\ R_H,\ xy = yx,\ z^2 = z,\ z_{g_1,h_1}z_{g_2,h_2} = z_{g_2,h_2}z_{g_1,h_1} \\ & (x \in X,\ y \in Y,\ g_1, g_2 \in G, h_1, h_2 \in H)\rangle. \end{aligned} \tag{6.9}$$

Since both $\langle X, Y \mid R_G, R_H, xy = yx(x \in X, y \in Y)\rangle$ and $\langle A \mid R\rangle$ define $G\diamond H$, there is a sequence of Tietze transformations which converts the former into the latter. Applying the same sequence to (6.9) yields the presentation (6.7), as desired. ∎

6.4 Finite presentability of $G\diamond H$ as an inverse monoid

From Theorem 6.1.1 there arises the question of whether $G\diamond H$ is finitely presented as an inverse monoid. We use Theorem 6.3.2 as a starting point.

Theorem 6.4.1 *If* $G \times H$ *is presented by* $\mathrm{Mon}\langle A \mid R\rangle$ *then* $G\diamond H$ *is presented by* $\mathrm{InvMon}\langle A, z \mid R, z^2 = z\rangle$.

PROOF. We consider presentation (6.7) from Theorem 6.3.2 as an inverse monoid presentation for $G\Diamond H$. Let $B = A \cup \{z\}$. By definition, we have

$$\begin{aligned} G\Diamond H &= \mathrm{Inv}\langle B \mid R,\ z^2 = z,\ z_{g_1,h_1}z_{g_2,h_2} = z_{g_2,h_2}z_{g_1,h_1}(g_1, g_2 \in G, h_1, h_2 \in H)\rangle \\ &= \mathrm{Mon}\langle B^{\pm 1} \mid \Re_B,\ R,\ z^2 = z, \\ &\qquad z_{g_1,h_1}z_{g_2,h_2} = z_{g_2,h_2}z_{g_1,h_1}(g_1, g_2 \in G, h_1, h_2 \in H)\rangle \end{aligned}$$

But $\Re_B$ includes $uu^{-1}vv^{-1} = vv^{-1}uu^{-1}$ for all $u, v \in (B^{\pm 1})^*$. For $u = w_{g_1,h_1^{-1}}z$ and $v = w_{g_2,h_2^{-1}}z$, this relation becomes $z_{g_1,h_1}z_{g_2,h_2} = z_{g_2,h_2}z_{g_1,h_1}$. Hence

$$G\Diamond H = \mathrm{Mon}\langle B^{\pm 1} \mid \Re_B,\ R,\ z^2 = z\rangle = \mathrm{Inv}\langle A, z \mid R, z^2 = z\rangle, \quad (6.10)$$

as desired. ∎

We note an immediate consequence of this result.

Corollary 6.4.2 *Let G and H be groups. The Schützenberger product $G\Diamond H$ is the inverse monoid free product of $G \times H$ and the two-element semilattice.*

More importantly, our second main result follows from Theorem 6.4.1.

Corollary 6.4.3 *The Schützenberger product $G\Diamond H$ of two groups is finitely presented as an inverse monoid if and only if G and H are finitely presented.*

PROOF. ($\Rightarrow$) Assume that $G\Diamond H$ is finitely presented as an inverse monoid. Again we use property (S4). Corollary 5.4 of [21] states that a subsemigroup with an ideal complement inherits finite presentability (this also appears in [38] and is considered folklore), so it follows that $G \times H$ is finitely presented. In turn this implies that G and H are each finitely presented.

($\Leftarrow$) If G and H are finitely presented, then $G \times H$ is finitely presented and (6.10) in Theorem 6.4.1 is a finite presentation for $G\Diamond H$ as an inverse monoid. ∎

6.5 A coincident structure

At this point we briefly leave our study of presentability in order to include an interesting observation. Namely, the Schützenberger product of two groups is isomorphic to the semi-direct product of a semilattice and a group, in which the group acts on the semilattice by automorphisms. This is particularly useful because it leads to an alternative proof of Corollary 6.4.3. In this and the following section we will write mappings on the left in order to conform with the usual notation for semi-direct products.

Let G and H be groups. Also let $K = G \times H$ and $Y = \mathcal{P}_f(K)$ with the operation of union. For each $g \in G, h \in H$ we define $\phi_{g,h} : Y \to Y$ by $\phi_{g,h}(P) = gPh$.

Lemma 6.5.1 *For all $g \in G, h \in H$, we have that $\phi_{g,h} \in \mathrm{Aut}(Y)$.*

PROOF. Let $P, Q \in Y$ be arbitrary. The manipulation

$$\begin{aligned} \phi_{g,h}(P \cup Q) &= g(P \cup Q)h \\ &= gPh \cup gQh \\ &= \phi_{g,h}(P) \cup \phi_{g,h}(Q) \end{aligned}$$

shows that $\phi_{g,h}$ is a homomorphism.

Further, an arbitrary $Q \in Y$ may be written as $\phi_{g,h}(g^{-1}Qh^{-1}) = Q$, so $\phi_{g,h}$ is a surjection.

Finally, if $\phi_{g,h}(P) = \phi_{g,h}(Q)$ then $gPh = gQh$. It quickly follows that $P = Q$, so $\phi_{g,h}$ is an injection. ■

We now define $\theta : K \to \mathrm{Aut}(Y)$ by $\theta(g, h) = \phi_{g,h^{-1}}$.

Lemma 6.5.2 *The mapping θ is a homomorphism.*

PROOF. By Lemma 6.5.1, we note that θ is well-defined. For arbitrary $Q \in Y$ and $(g_1, h_1), (g_2, h_2) \in K$, we may perform the following manipulation:

$$\begin{aligned} \theta[(g_1, h_1)(g_2, h_2)](Q) &= \theta(g_1g_2, h_1h_2)(Q) \\ &= \phi_{g_1g_2, h_2^{-1}h_1^{-1}}(Q) \\ &= g_1g_2Qh_2^{-1}h_1^{-1} \\ &= \phi_{g_1, h_1^{-1}}(g_2Qh_2^{-1}) \\ &= \phi_{g_1, h_1^{-1}}\phi_{g_2, h_2^{-1}}(Q) \\ &= \theta(g_1, h_1)\theta(g_2, h_2)(Q). \end{aligned}$$

Therefore $\theta[(g_1,h_1)(g_2,h_2)] = \theta(g_1,h_1)\theta(g_2,h_2)$, so θ is a homomorphism. ■

Theorem 6.5.3 *The Schützenberger product $G\diamond H$ is isomorphic to the semi-direct product $Y \rtimes_\theta K$.*

Proof. Lemma 6.5.2 shows that $\theta : K \to \mathrm{Aut}(Y)$ is a well-defined homomorphism, which is the necessary condition for its role in $Y \rtimes_\theta K$.

We define the mapping $\pi : Y \rtimes_\theta K \to G\diamond H$ by $\pi[P,(g,h)] = (g, Ph, h)$ and we claim that this is an isomorphism. To show this, we examine arbitrary elements $g, g' \in G, h, h' \in H$ and $P, Q \in \mathcal{P}_f(G \times H)$.

We may write $(g, P, h) = \pi[Ph^{-1}, (g,h)]$, so π is surjective.

If $\pi[P,(g,h)] = \pi[Q,(g',h')]$ then $(g, Ph, h) = (g', Qh', h')$. It follows, in this case, that $g = g', h = h'$ and $P = Q$, so $[P,(g,h)] = [Q,(g',h')]$. We conclude that π is injective.

Finally, the manipulation

$$\begin{aligned}
\pi([P,(g,h)][Q,(g',h')]) &= \pi[P \cup \theta(g,h)(Q), (g,h)(g',h')] \\
&= \pi[P \cup gQh^{-1}, (gg', hh')] \\
&= (gg', Phh' \cup gQh', hh') \\
&= (g, Ph, h)(g', Qh', h') \\
&= \pi[P,(g',h')]\pi[Q,(g',h')]
\end{aligned}$$

shows that π is a homomorphism. ■

6.6 An alternative proof of Corollary 6.4.3

In [6, 7] there appears the result labelled below as Theorem 6.6.1. This gives necessary and sufficient conditions for a semi-direct product of a semilattice and a group, in which the group acts on the semilattice by automorphisms, to be finitely presented as an inverse semigroup. By Theorem 6.5.3 this clearly applies directly to the question of finite presentability of $G\diamond H$ as an inverse monoid. We require the following technical definition, which also appeared in [6, 7].

Let K be a group that is finitely presented as a monoid by $\text{Mon}\langle A \mid P\rangle$. Let Y be a semilattice and let $\theta : K \to \text{Aut}(Y)$ be a homomorphism. For $k \in K, y \in Y$, we may regard $[\theta(k)](y)$ as the element k acting on y, which we may also write as ${}^k y$.

We say that $\text{InvAct}_{\langle A \mid P\rangle}\langle B \mid Q\rangle$ *finitely presents* Y *as an inverse semigroup with respect to the action* of K if

$$Y = \text{Inv}\langle B' \mid Q'\rangle$$

where $B \subseteq Y$ is finite and $B' = \{{}^w b : w \in A^*, b \in B\}$, Q is a finite set of relations on B and Q' is defined as

$$\{{}^{\alpha u \beta}w = {}^{\alpha v \beta}w,\ {}^{\alpha u \beta}p = {}^{\alpha v \beta}q,\ {}^{\alpha}p = {}^{\alpha}q :$$

$$(p = q) \in Q \cup \Re_{B'}, (u = v) \in P, \alpha, \beta \in A^*,\ w \in B^+\}.$$

There then appears the following result.

Theorem 6.6.1 *Let Y be a semilattice, let K be a group and let the mapping $\theta : K \to \text{Aut}(Y)$ be a homomorphism. The semi-direct product $Y \rtimes_\theta K$ is finitely presented as an inverse semigroup if and only if the following conditions hold:*

(i) K is finitely presented;

(ii) Y has only finitely many maximal elements and every element of Y is below one of these;

(iii) Y is finitely presented as an inverse semigroup with respect to the action of K.

From this there follows another way to show when $G \diamond H$ is finitely presented as an inverse semigroup (or monoid).

Proof of Corollary 6.3.2. We consider $G \diamond H$ as the semi-direct product stated in Theorem 6.5.3 and we apply Theorem 6.6.1.

($\Rightarrow$) If $G \diamond H$ is finitely presented as an inverse semigroup then condition (i) of Theorem 6.6.1 holds. That is, $K(= G \times H)$ is finitely presented, so G and H are each finitely presented.

($\Leftarrow$) Assume that G and H are finitely presented. We claim that conditions (i), (ii) and (iii) of Theorem 6.6.1 holds. It is obvious that K is finitely presented, say by $\text{Mon}\langle A \mid P\rangle$, so condition (i) holds.

As $Y = \mathcal{P}_f(G \times H)$ with the operation of union, the natural order $\leq$ on the semilattice Y follows the rule that $P \leq Q$ if and only if $Q \subseteq P$. Therefore $\emptyset$ is the unique maximal element of Y and every element of Y is below it, so condition (ii) holds.

We note that $\{\{(g,h)\} : g \in G, h \in H\} = \{^k(1_G, 1_H) : k \in K\}$ and that Y is the free semilattice on this set, with an identity $\emptyset$ adjoined. Therefore Y is generated with respect to the action by $\emptyset$ and $\{(1_G, 1_H)\}$. We let $e_0 \equiv \emptyset$ and $e_1 \equiv \{(1_G, 1_H)\}$. We define R as the set of relations

$$\{e_0^2 = e_0, e_1^2 = e_1, e_1e_0 = e_1, {}^ae_0 = e_0 : a \in A\}$$

and we claim that $\mathrm{InvAct}_{\langle A \mid P \rangle}\langle e_0, e_1 \mid R \rangle$ finitely presents Y as an inverse semigroup with respect to the action. The relations R, together with the standard inverse semigroup relations, imply that all elements of the forms ${}^ke_0$ and ${}^ke_1$, with $k \in K$, are idempotent and commute with each other. Also, e_0 is the identity, but the relations give no more information. This agrees with the fact that Y is the free semilattice on $\{{}^ke_1 : k \in K\}$ with an identity adjoined. Therefore our claim is shown. ■

The inverse monoid presentation for $G \diamond H$ which was deduced in Theorem 6.4.1 also follows from the proof of Theorem 6.6.1; we omit the details.

6.7 (Non) finite presentability of $G \diamond H$ as a monoid

We return to the question of the finite presentability of $G \diamond H$ as a monoid. Again we use Theorem 6.3.2.

Theorem 6.7.1 *The Schützenberger product $G \diamond H$ of two groups is finitely presented as a monoid if and only if both G and H are finite.*

PROOF. Assume that G and H are not both finite but that $G \diamond H$ is finitely presented. Therefore, by the same argument as the proof of Corollary 6.4.3, but applied to monoid presentations, G and H are finitely presented. We can now let $G \times H$ be finitely presented as a monoid by $\langle A \mid R \rangle$. Remembering the definition of $z_{g,h}$ in Theorem 6.3.2, we rearrange those relations using $g = g_1^{-1}g_2$ and $h = h_1h_2^{-1}$ to see that

$$\langle A, z \mid R,\ z^2 = z,\ w_{g,h}zw_{g^{-1},h^{-1}}zw_{g,h} = zw_{g,h}z\ ((g,h) \in G \times H) \rangle \qquad (6.11)$$

presents $G \diamond H$ as a monoid. As $G \diamond H$ is finitely presented, the generators of (6.11) and a finite subset of the relations of (6.11) will suffice to present it. Therefore, for some finite $W \subseteq G \times H$, we have

$$G \diamond H = \mathrm{Mon}\langle A, z \mid R,\ z^2 = z,\ w_{g,h}zw_{g^{-1},h^{-1}}zw_{g,h} = zw_{g,h}z\ ((g,h) \in W) \rangle. \qquad (6.12)$$

We will complete the proof by showing that this is not possible.

Consider a word of the form $zw_{g,h}z$ for some fixed $g \in G, h \in H$ such that $(g,h) \notin W \cup W^{-1} \cup \{(1_G, 1_H)\}$. We claim that any word obtained from $zw_{g,h}z$ by applying the relations from (6.12) is of the form

$$\alpha_1 z \alpha_2 \ldots \alpha_r z \beta z \alpha_{r+1} \ldots \alpha_{r+l-1} z \alpha_{r+l} \qquad (6.13)$$

for some $\alpha_i \in A^*$ $(i = 1, \ldots, r+l)$, all of which represent $(1_G, 1_H)$, and some $\beta \in A^*$ which represents (g,h). Intuitively, a word of the form (6.13) has the 'centre' $z\beta z$, which is surrounded by 'redundant' factors which do not change the element of $G \diamond H$ that the word represents. Clearly the word $zw_{g,h}z$ is of this form. We proceed to show that applying any relation from presentation (6.12) to a word of the form (6.13) yields another word of the same form.

A relation from R contains no occurrences of z. Hence any application of such a relation is wholly within γ, where $\gamma = \alpha_i$ or $\gamma = \beta$, and it does not change the element of $G \times H$ that γ represents.

The result of applying the relation $z^2 = z$ to a word of the form (6.13) is that, depending on whether the right hand side is substituted for the left or vice versa, the number of occurrences of z is decreased or increased by one and an empty α_i is removed or inserted.

We now consider the effect of applying the relation

$$w_{g_1,h_1}zw_{g_1^{-1},h_1^{-1}}zw_{g_1,h_1} = zw_{g_1,h_1}z,$$

for some particular $(g_1, h_1) \in W$, to a word of the form (6.13). We first consider the case where this relation is applied by replacing the left hand side by the right. Consider the particular occurrence of $w_{g_1,h_1}zw_{g_1^{-1},h_1^{-1}}zw_{g_1,h_1}$ as a subword, which is to be replaced by $zw_{g_1,h_1}z$. If this occurrence is in $\alpha_{i-1}z\alpha_i z\alpha_{i+1}$, then $\alpha_{i-1} \equiv cw_{g_1,h_1}, \alpha_i \equiv w_{g_1^{-1},h_1^{-1}}, \alpha_{i+1} \equiv w_{g_1,h_1}d$ for some $c, d \in A^*$. But α_i represents $(1_G, 1_H)$, so all of the words $w_{g_1,h_1}, w_{g_1^{-1},h_1^{-1}}, c$ and d in fact represent $(1_G, 1_H)$. Applying the relation as stated transforms the subword $cw_{g_1,h_1}zw_{g_1^{-1},h_1^{-1}}zw_{g_1,h_1}d$ into $czw_{g_1,h_1}zd$ and leaves the rest of the word unchanged. In particular, the newly obtained word also has the form (6.13). If $w_{g_1,h_1}zw_{g_1^{-1},h_1^{-1}}zw_{g_1,h_1}$ appears as a subword in $\alpha_{r-1}z\alpha_r z\beta$ or $\beta z\alpha_{r+1}z\alpha_{r+2}$, then w_{g_1,h_1} again represents $(1_G, 1_H)$ and a similar argument as above shows that applying the relation as stated produces a new word which also has form (6.13). If $w_{g_1,h_1}zw_{g_1^{-1},h_1^{-1}}zw_{g_1,h_1}$ is a subword in $\alpha_r z\beta z\alpha_{r+1}$, then $\beta \equiv w_{g_1^{-1},h_1^{-1}}$. This leads to the contradiction $(g, h) = (g_1, h_1)^{-1} \in W^{-1}$, and so this case cannot arise.

We now consider the case where the same relation can be applied by replacing the right hand side by the left hand side. Then $zw_{g_1,h_1}z$ appears as a subword of v. If this subword is $z\alpha_m z$, then $\alpha_m \equiv w_{g_1,h_1}$ and hence $(g_1, h_1) = (1_G, 1_H) = (g_1^{-1}, h_1^{-1})$. Applying the relation as stated keeps the overall word in the given form. If $zw_{g_1,h_1}z$ appears in the word as $z\beta z$ then $\beta \equiv w_{g_1,h_1}$, which leads to the contradiction $(g, h) = (g_1, h_1) \in W$. Our claim is shown.

To complete the proof of the theorem we note that since $(g, h) \neq (1_G, 1_H)$, the word $w_{g,h}zw_{g^{-1},h^{-1}}zw_{g,h}$ is not of the form (6.13). Thus the relation $zw_{g,h}z = w_{g,h}zw_{g^{-1},h^{-1}}zw_{g,h}$ holds in $G\Diamond H$ but is not a consequence of presentation (6.12), a contradiction. ■

There is an interesting dichotomy, reminiscent of the free inverse monoid, that the Schützenberger product $G\Diamond H$, where G and H are finitely presented infinite groups (or one infinite and one finite), is finitely presented as an inverse monoid but not finitely presented as a monoid.

There remains an obvious unsolved question.

Open Problem 6.7.2 Let S and T be monoids, or even semigroups. Can $S\Diamond T$ be finitely presented (without S and T both being finite) and, if so, when?

6.8 A generalisation of Theorem 6.7.1

In fact we may show that the inverse monoid free product of any infinite group and non-trivial semilattice with an identity is not finitely presented as a monoid.

To begin, we let $K = \mathrm{Mon}\langle A \mid R\rangle$ and $Y = \mathrm{Mon}\langle B \mid Q\rangle$ be inverse monoids and let S be the inverse monoid free product of K and Y.

Lemma 6.8.1 *We have:*

$$S \;=\; \mathrm{Mon}\langle A, B \mid R, Q, ww^{-1}zz^{-1} = zz^{-1}ww^{-1} \ (w, z \in \{A, B\}^*) \ \rangle. \tag{6.14}$$

PROOF. It is clear that $K = \mathrm{InvMon}\langle A \mid R\rangle$ and $Y = \mathrm{InvMon}\langle B \mid Q\rangle$, so $S = \mathrm{InvMon}\langle A, B \mid R, Q\rangle$. By the definition it follows that

$$\begin{aligned} S &= \mathrm{Mon}\langle A^{\pm 1}, B^{\pm 1} \mid R, Q, \mathfrak{R}_{A,B}\rangle \\ &= \mathrm{Mon}\langle A, A^{-1}, B, B^{-1} | R, Q, ww^{-1}w = w, \\ &\quad ww^{-1}zz^{-1} = zz^{-1}ww^{-1} \ (w, z \in \{A, A^{-1}, B, B^{-1}\}^*)\rangle. \end{aligned} \tag{6.15}$$

We will now perform Tietze transformations on (6.15) to turn it into (6.14).

As K is an inverse monoid defined by a monoid presentation $\mathrm{Mon}\langle A \mid R\rangle$, it follows that for all $a \in A$ there is a word $w_a \in A^*$ such that the equality $w_a = a^{-1}$ holds as a consequence of R. Therefore, we may use the Tietze transformation (T3) to remove A^{-1} from the set of generators and replace each occurrence of a^{-1} by w_a in the relations. For ease of notation we continue to refer to these subwords w_a as a^{-1}. Similarly, we may use (T3) to remove B^{-1} from the set of generators.

We select an arbitrary word $w \in (A \cup B)^*$ and write $w = k_1 y_1 \dots k_n y_n$ where $k_1, \dots, k_n \in A^*$ and $y_1, \dots, y_n \in B^*$. We refer to the smallest possible value of n as the length of the word w, and also say that the length of the empty word 1 is 0. We will use induction on n to show that if w is a word of length n then $ww^{-1}w = w$ is a consequence of the other relations in (6.15).

We first consider the base case $n = 0$; that is, in which $w = 1 \in (A \cup B)^*$. Applying the relations R and Q, we can easily deduce that $ww^{-1}w = 11^{-1}1 = 1 = w$, so the base case is finished.

For the inductive step we assume, for some $n \in \mathbb{N}$, that if w is a word of length less than n then $ww^{-1}w = w$ is a consequence of the other relations in (6.15). We consider the word $w = k_1y_1 \ldots k_ny_n$ of length n. We let $v = k_2y_2 \ldots k_ny_n$ and we note that $w = k_1y_1v$. The length of v is $n - 1$ so, by our inductive assumption, we have that $vv^{-1}v = v$. Using this and the relations R, Q and those of the form $ww^{-1}zz^{-1} = zz^{-1}ww^{-1}$, we observe that

$$\begin{aligned}
ww^{-1}w &= (k_1y_1v)(k_1y_1v)^{-1}(k_1y_1v) \\
&= k_1y_1vv^{-1}y_1^{-1}k_1^{-1}k_1y_1v \\
&= k_1y_1[vv^{-1}][(k_1y_1)^{-1}(k_1y_1)]v \\
&= k_1y_1[(k_1y_1)^{-1}(k_1y_1)][vv^{-1}]v \\
&= k_1(y_1y_1^{-1})(k_1^{-1}k_1)y_1vv^{-1}v \\
&= k_1k_1^{-1}k_1y_1y_1^{-1}y_1v \\
&= k_1y_1v \\
&= w,
\end{aligned}$$

which finishes the inductive part.

It is now straightforward to complete the proof: because all the relations of the form $ww^{-1}w = w$ are consequences of the other relations, we may use (T4) to remove these from the presentation. This gives us (6.14). ■

From this point we consider the more specific case in which K is a group and that Y is a semilattice with an identity.

Lemma 6.8.2 *We have:*

$$S = \mathrm{Mon}\langle A, B \mid R, Q, bkc = kck^{-1}bk \ (k \in A^*, b, c \in B)\rangle. \quad (6.16)$$

Proof. We will show this by reducing the relations of (6.14).

Denoting the semilattice of idempotents of S as $E(S)$, we claim that $E(S) = \langle k^{-1}yk : k \in A^*, y \in B^*\rangle$. To show this, we define $\phi : S \to K$ as the extension to a homomorphism of the mapping $\phi : A \cup B \to K$ given by $(a)\phi = a, (b)\phi = 1_K$ (for each $a \in A, b \in B$). The image of an idempotent under a homomorphism is idempotent, so $(e)\phi = 1_K$ for all $e \in E(S)$. That is, if e is idempotent and we write $e = k_1 y_1 \ldots y_n k_{n+1}$ with $k_1, \ldots, k_{n+1} \in A^*$ and $y_1, \ldots, y_n \in B^*$, then $k_1 \ldots k_{n+1} = 1_K$. Therefore

$$\begin{aligned} e &= k_1 y_1 \ldots y_n k_{n+1} \\ &= (k_1 y_1 k_1^{-1})(k_1 k_2 y_2 k_2^{-1} k_1^{-1}) \ldots (k_1 \ldots k_n y_n k_n^{-1} \ldots k_1^{-1})(k_1 \ldots k_{n+1}) \\ &= (k_1 y_1 k_1^{-1}) \ldots (k_1 \ldots k_n y_n k_n^{-1} \ldots k_1^{-1}), \end{aligned}$$

and this is contained in $\langle k^{-1} y k : k \in A^*, y \in B^* \rangle$, so our claim is shown.

For all $y \in Y$ we may write $y = b_1 \ldots b_r$ with each $b_i \in B$. Then

$$k^{-1} y k = (k^{-1} b_1 k) \ldots (k^{-1} b_r k),$$

so in fact $E(S) = \langle k^{-1} b k : b \in B, k \in A^* \rangle$. Those relations of the form $ww^{-1}zz^{-1} = zz^{-1}ww^{-1}$ (which state that idempotents commute) may be deduced from only those of the form

$$(k_1^{-1} b k_1)(k_2^{-1} c k_2) = (k_2^{-1} c k_2)(k_1^{-1} b k_1)$$

with $k_1, k_2 \in A^*, b_1, b_2 \in B$ (which state that the generators of the semilattice of idempotents commute). If we let $k = k_1 k_2^{-1}$ then a simple rearrangement gives (6.16). ■

At this stage we note that if $B = \{z\}$ and $Q = \{z^2 = z\}$ then (6.16) is within a simple alteration of (6.7). We are also in a position to generalise Theorem 6.7.1 as follows.

Theorem 6.8.3 *Let K be an infinite group, let Y be a non-trivial semilattice with an identity and let S be the inverse monoid free product of K and Y. Then S is not finitely presented as a monoid.*

We note that the proof of this result is simply a generalisation of the proof of Theorem 6.7.1.

PROOF. To derive a proof by contradiction, we begin by assuming that S is finitely presented as a monoid. It follows that K and Y are each finitely presented as monoids, say by $\mathrm{Mon}\langle A \mid R\rangle$ and $\mathrm{Mon}\langle B \mid Q\rangle$, respectively. In particular, Y is finite.

By Lemmas 6.8.1 and 6.8.2 we conclude that S is presented by (6.16). As we have assumed that S is finitely presented as a monoid, it follows that the generators and a finite subset of the relations of (6.16) suffice to present this semigroup. That is, for some finite $W \subseteq A^*$,

$$S \;=\; \mathrm{Mon}\langle A, B \mid R, Q, bwc = wcw^{-1}bw \ (w \in W, b, c \in B)\rangle. \quad (6.17)$$

We will complete the proof by showing that this is not possible.

Consider a word puq where $p, q \in B, u \in A^*$ and u does not represent the same element of K as any word in $W \cup W^{-1} \cup \{1_K\}$. We claim that any word which may be obtained from puq by applying relations from (6.17) is of the form

$$\alpha_1 b_1 \ldots \alpha_r b_r \beta b_{r+1} \alpha_{r+1} \ldots b_{r+l} \alpha_{r+l} \quad (6.18)$$

for some $\alpha_i \in A^*$ $(i = 1, \ldots, r+l)$ all of which represent 1_K, $\beta \in A^*$ which represents the same word as u, and some $b_j \in B$ $(j = 1, \ldots, r+l)$. Intuitively, a word of the form (6.18) has the 'centre' $b_r \beta b_{r+1}$, which is surrounded by 'redundant' factors which do not change the element of S that the word represents. Clearly the word puq is of this form. We proceed to show that applying any relation from presentation (6.17) to a word of the form (6.18) yields another word of the same form.

A relation in R contains no occurrences of any element of B. Hence any application of such a relation is wholly within γ, where $\gamma = \alpha_i$ or $\gamma = \beta$, and it does not change the element of K that γ represents.

The result of applying a relation from Q to a word of the form (6.18) is that the b_i may change and some empty α_i may be removed or inserted.

We now examine the effect of applying the relation $wcw^{-1}bw = bwc$, for some particular $b, c \in B$ and $w \in W$, to a word of the form (6.18). We look first at the case where this relation is applied by replacing the left hand side by the right. Consider the particular occurrence of $wcw^{-1}bw$, as a subword, which is to be replaced by bwc. If this is in $\alpha_{i-1} b_i \alpha_i b_{i+1} \alpha_{i+1}$ then $\alpha_{i-1} \equiv dw, \alpha_i \equiv w^{-1}$ and $\alpha_{i+1} \equiv we$ for some $d, e \in A^*$. But α_i represents 1_K, so in fact w, w^{-1}, d

and e all represent 1_K. Applying this relation as stated transforms the subword $dwb_iw^{-1}b_{i+1}we$ into $db_{i+1}wb_ie$ and leaves the rest of the word unchanged. In particular, the newly obtained word has the form (6.18). If $wcw^{-1}bw$ appears as a subword of $\alpha_{r-1}b_{r-1}\alpha_r b_r\beta$ or $\beta b_{r+1}\alpha_{r+1}b_{r+2}\alpha_{r+2}$ then it can similarly be shown that w represents 1_K, and hence that applying the relation as stated produces a word which also has form (6.18). If $wcw^{-1}bw$ is a subword in $\alpha_r b_r \beta b_{r+1}\alpha_{r+1}$ then β represents the same element as w^{-1}, which contradicts our assumptions; thus this case cannot arise.

We now consider the case where the same relation can be applied by replacing the right hand side by the left hand side. Then bwc appears as a subword in our word of form (6.18). If this is in $b_{i-1}\alpha_i b_i$ then $\alpha_i \equiv w$, so w represents 1_K. Applying the relation as stated keeps the overall word in form (6.18). If bwc appears in the word as $b_r\beta b_{r+1}$ then β represents the same element as w, which again contradicts our assumptions.

To complete the proof we note that since w does not represent 1_K, the word $wcw^{-1}bw$ is not of form (6.18). However, bwc clearly is of this form, so $bwc = wcw^{-1}bw$ does not hold in the presentation (6.17). This equality does hold in S, so we have a contradiction and we conclude that S is not finitely presented as a monoid. ■

Because of Corollary 6.4.2, Theorem 6.7.1 follows as a direct consequence of Theorem 6.8.3.

It is well-known that the monoid free product $\mathrm{Mon}(G * H)$ of two finitely presented groups G and H (which is also the group free product of these groups) is a finitely presented group. As this is inverse, it coincides with the inverse monoid free product $\mathrm{InvMon}(G * H)$. So Theorem 6.8.3 cannot be generalised to state that if S and T are inverse monoids, at least one of which is infinite, then the inverse monoid free product $\mathrm{InvMon}(S * T)$ is not finitely presented as a monoid. However, the following question remains unsolved.

Open Problem 6.8.4 Let S and T be inverse monoids, at least one of which is infinite, and at least one of which is not a group. Can the inverse monoid free product of S and T be finitely presented as a monoid and, if so, when?

The main open problems

As so many unsolved questions have arisen, we list those that seem most important. They are almost uniformly distributed over the covered topics, although the most promise appears in those regarding the finitary power semigroup, particularly Open Problem H, which concerns finite presentability. However, there is also strong potential for future research in several others.

Open Problem A Is it possible to classify all of the infinite inverse semigroups that have finitely generated diagonal bi-acts? What about completely regular?

Open Problem B Does there exist an infinite totally ordered set X such that the diagonal bi-act of $\mathcal{O}_X$, the semigroup of monotonic transformations on X, is finitely generated, or even cyclic? What about the semigroup $\mathcal{Q}_X$ of strictly monotonic transformations on X?

Open Problem C Let R be an infinite ring, let R^* be the multiplicative semigroup of R and let S be the semigroup of $n \times n$ matrices over R. Can the diagonal right, left or bi-act of S, or of R^*, be finitely generated or cyclic?

Open Problem D Let $\mathbf{A}$ be an independence algebra. Is it always the case that the direct square $\mathbf{A} \times \mathbf{A}$ is also an independence algebra?

Open Problem E For which choices of the ingredients are the semigroups $\mathcal{S}(X, A, C, D, E, F)$ and $\mathcal{M}(X, A, C, D, (a', b'), r)$ (both of which are defined in Section 4.7) infinite?

Open Problem F Does there exist an infinite inverse semigroup S for which the finitary power semigroup $\mathcal{P}_f(S)$ is finitely generated?

Open Problem G If S is finitely generated and the diagonal right act of S is finitely generated, does it always follow that $\mathcal{P}_f(S)$ is finitely generated?

Open Problem H Does there exist an infinite semigroup S for which the finitary power semigroup $\mathcal{P}_f(S)$ is finitely presented?

Open Problem I Let S and T be semigroups, at least one of which is infinite. Can the Schützenberger product $S \Diamond T$ be finitely presented and, if this is possible, can we classify all pairs of semigroups for which this is the case?

Open Problem J Let S and T be inverse monoids, at least one of which is infinite, and at least one of which is not a group. Can the inverse monoid free product $\mathrm{InvMon}(S * T)$ be finitely presented and, if so, when?

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