SYMMETRIES OF THE CENTRAL FORCE PROBLEM IN CLASSICAL AND QUANTUM MECHANICS. THE OSCILLATOR AND SU(3).

A Thesis presented by Paul Barrie Guest, B.Sc., to the University of St. Andrews in application for the Degree of Master of Science.

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DECLARATION

I hereby declare that the accompanying thesis is my own composition, that it is based upon research carried out by me and that no part of it has previously been presented in application for a higher degree.

CERTIFICATE

I certify that Paul Barrie Guest, B.Sc., has spent four terms as a research student in the Department of Theoretical Physics of the United College of St. Salvator and St. Leonard in the University of St. Andrews, that he has fulfilled the conditions of Ordinance 51 (St. Andrews) and that he is qualified to submit the accompanying thesis in application for the degree of Master of Science.

Research Supervisor

INTRODUCTION

The introduction* of group theory into the study of quantum mechanics shows that the degeneracy of many quantum systems may be accounted for as a forced degeneracy that is due to some symmetry possessed by the system. For example, the spherical symmetry of the central force has as a consequence the conservation of angular momentum and gives rise to a degeneracy in the sense that many states, independent and corresponding to different values of the third component of angular momentum, have the same energy.

In addition, in some potentials (e.g. that of the three dimensional isotropic harmonic oscillator and of the hydrogen atom), the spherical symmetry alone is not enough to account for the observed degeneracy: an accidental degeneracy remains and it is tempting to think that there may be present a higher symmetry which has been overlooked and which will explain completely all the degeneracies present. It seems traditional to refer to these higher symmetries (if they exist) as 'hidden'. As Alliluev (1957) points out, such hidden symmetries actually exist in the two-dimensional oscillator. The study of several systems with accidental degeneracy, in particular the hydrogen atom (Fock, 1935), the three-dimensional oscillator (Demkov, 1953), and the n-dimensional oscillator (Baker, 1956), has shown that these systems possess in addition to the obvious symmetry, also a hidden symmetry.

The most serious problem in this respect is to identify the generators of the complete symmetry group with something that possesses physical significance -- constants of the motion, for instance. The fact that these accidental degeneracies are connected with the existence of constants of motion was recognized by Pauli (1926) who investigated the Kepler problem. In this problem, the commutation relations (on a classical level) of a new vector invariant, called the Lenz vector (Lenz, 1924), with the angular momentum components was recognized by Klein (Hulthen, 1933) as those of the four-dimensional rotation group (Fock, 1935; Bargmann, 1936). Numerous papers have been written on the degeneracy in the Kepler problem (Lenz, 1924; Pauli, 1926; Born, Jordan, 1930; Hulthen, 1933; Fock, 1935; Bargmann, 1936; Jauch, Hill, 1940; Pauli, 1956; Biedenharn, 1961; Schweiger, 1964). It has been shown that the invariance group of the Hamiltonian is isomorphic to the four-dimensional rotation group in the case of bound states and to the homogeneous Lorentz group for unbound states. For the three-dimensional isotropic harmonic oscillator, the relavent group has been shown to be SU(3) (Bargmann, 1936).

On a classical basis , an account of the Lenz vector in the Kepler problem is given by Sexl (1966) who also defines an analogous vector for the three-dimensional oscillator. For work on the oscillator and its connection with SU(3), a derivation of the generators is given by Fradkin (1965). For more general central potentials, Bacry, Ruegg and Souriau (1966) show the existence of a vector analogous to the Lenz vector. Indeed, it has recently been shown (Fradkin, 1967; Mukunda, 1967) that the symmetries of SO(4) and SU(3) exist for all classical central potential problems.

It seems that very little work has been done on the quantization of the various forms of the second vector invariant of the central potential (with the special cases of the hydrogen atom and oscillator) and the consequent determination of the corresponding wave functions. A rudimentary account of the SU(3) wave functions for the quantized oscillator is given by Elliott (1958).

* The statement that group theory is 'introduced' into quantum mechanics may be erroneous; to see the structure of quantum mechanics made wholly group-theoretic is, at present, more than a mere aesthetic vision.

PREFACE

These pages embody the results of work carried out in the first half of 1967 at the University of St. Andrews. As claims to originality, it may be said that the theory of the quantized oscillator given in Part III has, as far as I am aware, never before been attempted though the unearthing some day of some obscure manuscript purporting to give explicitly the oscillator wave functions cannot entirely be overruled at this stage. Part III, then, is a complete account of the quantized oscillator and incorporates a derivation of the classical SU(3) generators (whose final expressions are similar to those defined by Fradkin (1965)) and of the SU(3) wave functions; it may be that these functions, as the basic vectors of certain representations of SU(3), have also a purely group-theoretic interest.

The quantization of the vector $\underline{B}$, the second fundamental vector invariant of the central potential, in Part II, section 2, introduces new results, though there still remain problems concerning their interpretation. Portions of Part II, section 1, concerning the general problem of quantization, have evaded all my attempts to trace them in the literature.

The second vector invariant is derived in Part I and elsewhere no similar explicit expression has been found at the time of writing. Part I, then, is an account of the symmetry pertaining to the central force, with reference to the special cases of the Kepler problem, oscillator and free particle.

The tale throughout is of a non-relativistic single particle and no considerations are given to what modifications the introduction of general or special relativity or the two-body problem may produce.

To the reader who needs assistance through some of the more mathematical passages, I may refer him to Shephard (1966), Scott (1964), Chevalley (1946), Kobayashi and Nomizu (1963)--in that order.

CONTENTS

ACKNOWLEDGEMENTS

My deepest gratitude goes without question to Mr. A. Bors both for bringing to my attention in December 1966 the subject matter of which this thesis is a development and for his untiring surveillance of my work, without which a large part of this thesis would never have been completed.

My thanks go to Professor R. B. Dingle for accepting me as a prospective Advanced Course student in Spring 1966 and for helping me in many ways during the preparation of this work.

Finally, my gratitude goes to Dr. R. C. Clark for a discussion on cylindrical coordinates, to Mr. A. M. Brett for a discussion on infinities and, of course, to S. R. C. for an Advanced Course Studentship.

To be beautiful and to be calm, without mental fear, is the ideal of nature. If I cannot achieve it, at least I can think it.

RICHARD JEFFERIES

I

THE CLASSICAL PROBLEM

1. A Survey of Classical Mechanics

We take as configuration space, M, that of a three-dimensional differentiable manifold of zero curvature and endowed with a Euclidean metric of signature 3 and a system of global coordinates $\{q_1, q_2, q_3\}$. A point of M will be denoted by

$$q = (q_1, q_2, q_3).$$

Phase space, $\mathcal{M}_V$, is a six-dimensional differentiable manifold with global coordinates $\{q_1, q_2, q_3, p_1, p_2, p_3\}$ and is the cotangent bundle over M. A point of $\mathcal{M}_V$ will be denoted by

$$(q, p) = (q_1, q_2, q_3, p_1, p_2, p_3).$$

A classical observable is a real-valued function

$$f: \ \mathcal{M}_V \ni (q, p) \longrightarrow f(q, p) \in \mathcal{R}$$

on phase space. Any such function f can be taken as defining a one-parameter group* of diffeomorphisms of $\mathcal{M}_V$. In fact, if α is the single parameter, the one-parameter group describes a curve in $\mathcal{M}_V$ that is defined by the system of equations

$$\frac{dq_i}{d\alpha} = \frac{\partial f}{\partial p_i}, \qquad \frac{dp_i}{d\alpha} = -\frac{\partial f}{\partial q_i}. \tag{1}$$

Singled out from the infinity of functions is one called the Hamiltonian whose corresponding one-parameter group describes the evolution of the system in time ($\alpha = t$). The Hamiltonian has the form

$$H = \frac{1}{2m}\left(p_1^2 + p_2^2 + p_3^2\right) + V(q_1, q_2, q_3), \tag{2}$$

where V is a function on M, called the potential, and m is a constant called the mass of the particle.

The rate of change of f along the curve defined by g is

$$\frac{df}{d\alpha} = [f, g] \tag{3}$$

where

$$[f, g] = \sum_{i=1}^{3}\left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\cdot\frac{\partial g}{\partial q_i}\right) \tag{4}$$

is the Poisson Bracket of f and g. In particular, if g =H, then

$$\frac{df}{dt} = [f, H]. \tag{5}$$

Thus f is a constant of motion iff [f,H]=0.

Let

$$\mathcal{f} = \{f_1, f_2, \ldots, f_n\}$$

be a set of linearly independent classical observables that is closed with respect to Poisson Bracket. $\mathcal{f}$ is then a basis of a Lie algebra $\mathcal{a}$.

Each function f defines a covariant vector field $\tilde{df}$:

$$\tilde{df} = \sum_{i=1}^{3}\left(\frac{\partial f}{\partial q_i} dq_i + \frac{\partial f}{\partial p_i} dp_i\right). \tag{6}$$

It will be recalled that a covariant vector field $\tilde{\xi}$ is a mapping

$$\tilde{\xi} \quad M_V \ni (q, p) \longrightarrow \tilde{\xi}_{(q,p)} \in T^*_{(q,p)}(M_V),$$

where $T^*_{(q,p)}(M_V)$ is the dual of the tangent space $T_{(q,p)}(M_V)$ at the point (q,p) of M_V. The covariant vector fields dq_i, dp_i (i=1,2,3) map (q,p) onto the basic elements of $T^*_{(q,p)}(M_V)$ (which will also be denoted by dq_i, dp_i (i=1,2,3)), respectively, these being defined as the image of the basic elements $\frac{\partial}{\partial q_i}, \frac{\partial}{\partial p_i}$ (i=1,2,3) of $T_{(q,p)}(M_V)$ under the natural isomorphism that exists between $T_{(q,p)}(M_V)$ and $T^*_{(q,p)}(M_V)$.

The differential of the fundamental covariant vector field of M_V

is defined by

$$dW^o = \sum_{i=1}^{3} \left(dp_i \otimes dq_i - dq_i \otimes dp_i \right). \qquad (7)$$

Its value at (q,p) is a tensor in $T^*_{(q,p)}(M_v) \otimes T^*_{(q,p)}(M_v)$ and hence can be considered as a linear mapping of $T_{(q,p)}(M_v)$ into $T^*_{(q,p)}(M_v)$, whence it has the form

$$dW^o_{(q,p)} = \begin{pmatrix} 0 & \begin{matrix} -1 & & \\ & -1 & \\ & & -1 \end{matrix} \\ \begin{matrix} 1 & & \\ & 1 & \\ & & 1 \end{matrix} & 0 \end{pmatrix} \qquad (8)$$

with respect to the ordered basis $\left\{ \frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_2}, \frac{\partial}{\partial q_3}, \frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_2}, \frac{\partial}{\partial p_3} \right\}$ of $T_{(q,p)}(M_v)$.

We see that $dW^o_{(q,p)}$ is non-singular and so the inverse mapping exists. Thus if $\tilde{\xi}$ is a covariant vector field, we define the contravariant vector field ξ by

$$\xi_{(q,p)} = \left(dW^o_{(q,p)} \right)^{-1} \tilde{\xi}_{(q,p)} \qquad (9)$$

and thus, if

$$\tilde{\xi} = \sum_{i=1}^{3} \left(a^i dq_i + b^i dp_i \right), \qquad (10)$$

then

$$\xi = \sum_{i=1}^{3} \left(b^i \frac{\partial}{\partial q_i} - a^i \frac{\partial}{\partial p_i} \right) \qquad (11)$$

(Here $\frac{\partial}{\partial q_i}, \frac{\partial}{\partial p_i}$ (i=1,2,3) are contravariant vector fields that map (q,p) onto the basic elements of $T_{(q,p)}(M_v)$.)

Each function f, then, defines a contravariant vector field df:

$$df = \sum_{i=1}^{3} \left(\frac{\partial f}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial}{\partial p_i} \right). \qquad (12)$$

THEOREM (Jost,1964)

$$d[f,g]=[df,dg], \qquad (13)$$

where

$$[\xi_1, \xi_2](f') = \xi_1(\xi_2(f')) - \xi_2(\xi_1(f')),$$

and f', $\xi(f')$ are functions on M_v, the latter being defined by

$$(\xi(f'))_{(q,p)} = \xi_{(q,p)}(f') .$$

Phase space, being a Euclidean space, can be considered as a vector space and hence identifiable with its own tangent space at every point, the isomorphism being

$$(q,p) \leftrightarrow \sum_{i=1}^{3}\left(q_i\frac{\partial}{\partial q_i} + p_i\frac{\partial}{\partial p_i}\right). \qquad (14)$$

Under this isomorphism, each contravariant vector field is a mapping of phase space into itself.

An arbitrary C_∞ function g defines a one-parameter group, $\tau = \{h_\alpha\}$, of diffeomorphisms of M_V, given by (1). The infinitesimal generator, ξ , of the group is defined by

$$\xi_{(q,p)}(f) = \left[\frac{d}{d\alpha} f(h_\alpha(q,p))\right]_{\alpha=0} , \qquad (15)$$

for all C_∞ f. It can be shown that ξ is linear iff h_α is linear. Also

$$\xi = \left(\frac{dh_\alpha}{d\alpha}\right)_{\alpha=0} , \qquad (16)$$

which is a consequence of the definitions.

Now

$$\begin{aligned}\xi_{(q,p)} &= \left[\frac{d}{d\alpha}\left(h_\alpha(q,p)\right)\right]_{\alpha=0} = \left[\frac{d}{d\alpha}\sum_{i=1}^{3}\left(q_i(\alpha)\frac{\partial}{\partial q_i} + p_i(\alpha)\frac{\partial}{\partial p_i}\right)\right]_{\alpha=0} \\ &= \sum_{i=1}^{3}\left[\left(\frac{dq_i(\alpha)}{d\alpha}\right)_{\alpha=0}\frac{\partial}{\partial q_i} + \left(\frac{dp_i(\alpha)}{d\alpha}\right)_{\alpha=0}\frac{\partial}{\partial p_i}\right] \\ &= \sum_{i=1}^{3}\left(\frac{\partial g}{\partial p_i(0)}\frac{\partial}{\partial q_i} - \frac{\partial g}{\partial q_i(0)}\frac{\partial}{\partial p_i}\right),\end{aligned}$$

from (1), where $h_\alpha(q,p)=(q_1(\alpha), q_2(\alpha), q_3(\alpha), p_1(\alpha), p_2(\alpha), p_3(\alpha))$ and we have written $q_i(0), p_i(0) = q_i, p_i$ (i=1,2,3).

Thus

$$\xi = dg \qquad (17)$$

from (12).

From (13), the set $\mathcal{f}$ now defines a group of diffeomorphisms of

$\mathcal{M}_V$ whose linear closure of infinitesimal generators is a Lie algebra having the same structure constants as $\mathcal{Q}$.

For literature touching on aspects of classical mechanics similar to those of this section, see Hermann (1966), Jost (1964), Mackey (1963).

* Strictly speaking, we should say that f defines a <u>local</u> one-parameter group of diffeomorphisms, this including the case of those functions for which the solutions of (1) do not exist for all α .

2. The Symmetry Group of a Particle in a Central Potential

2.1 Definition of the Symmetry Group

The symmetry group of a particle described by the Hamiltonian (2) is defined by

$$\mathcal{G} = \{ \ell \mid H(q,p) = H(\ell(q,p))\}, \tag{18}$$

where

$$\ell : M_V \ni (q,p) \to \ell(q,p) \in M_V$$

is a mapping of M_V into itself. The subset $\mathcal{G}^c$ of $\mathcal{G}$ is defined by

$$\mathcal{G} \supset \mathcal{G}^c = \{ \ell^c \mid H(q,p) = H(\ell^c(q,p))\}, \tag{19}$$

where ℓ^c is a diffeomorphism. A one-parameter subgroup of $\mathcal{G}^c$ is

$$\mathcal{E}^c = \{ h_\alpha \mid h_{\alpha_1+\alpha_2} = h_{\alpha_1} h_{\alpha_2},\ a \le \alpha \le b;\ H(q,p) = H(h_\alpha(q,p))\}, \tag{20}$$

with $a, b \in \mathcal{R}$.

It can be shown that the infinitesimal generator of $\mathcal{E}^c$ is of the form dg, i.e. is derivable from a real-valued C_∞ function g. $\mathcal{E}^c$ can now be characterized by the fact that

$$\frac{dH}{d\alpha} = \frac{H(h_{\alpha+d\alpha}(q,p)) - H(h_\alpha(q,p))}{d\alpha} = 0,$$

i.e.

$$[H, g] = 0, \tag{21}$$

from (3). Thus g is a constant of the motion. Conversely, each constant of motion defines a one-parameter group of diffeomorphisms that is a subgroup of the full symmetry group, $\mathcal{G}$, of the system.

Given a set $\mathcal{g} = \{g_1, g_2, \cdots, g_r\}$ of r C_∞ real-valued constants of motion that is the basis of a Lie algebra $\mathcal{a}$, we can define an r-parameter Lie group $\mathcal{A}$ of diffeomorphisms of M_V that is a subgroup of $\mathcal{G}^c$.

2.2 The Constants of Motion for a Central Potential

The equation [f,H]=0 or

$$\sum_{i=1}^{3}\left(\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i}-\frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i}\right)=0 \tag{22}$$

is a partial differential equation whose solutions are exactly calculable for a general central potential, when expressed in spherical polar coordinates. We have

$$H=\frac{1}{2m}\left(p_r^2+\frac{p_\theta^2}{r^2}+\frac{p_\varphi^2}{r^2\sin^2\theta}\right)+V(r),$$

with

$$p_r=m\frac{dr}{dt},\quad p_\theta=mr^2\frac{d\theta}{dt},\quad p_\varphi=mr^2\sin^2\theta\frac{d\varphi}{dt},$$

$$\begin{aligned}q_1&=r\sin\theta\cos\varphi\\ q_2&=r\sin\theta\sin\varphi\\ q_3&=r\cos\theta\end{aligned}\quad,\qquad \begin{aligned}r&=\sqrt{q_1^2+q_2^2+q_3^2}\\ \theta&=\arccos\left(\frac{q_3}{\sqrt{q_1^2+q_2^2+q_3^2}}\right)\\ \varphi&=\arctan\left(\frac{q_2}{q_1}\right),\end{aligned}$$

$$\begin{aligned}rp_r&=p_1q_1+p_2q_2+p_3q_3,\\ \sqrt{q_1^2+q_2^2}\,p_\theta&=q_3rp_r-r^2p_3,\\ p_\varphi&=q_1p_2-q_2p_1.\end{aligned}$$

(22) becomes

$$p_r\frac{\partial f}{\partial r}+\frac{p_\theta}{r^2}\frac{\partial f}{\partial\theta}+\frac{p_\varphi}{r^2\sin^2\theta}\frac{\partial f}{\partial\varphi}+\left(\frac{p_\theta^2}{r^3}+\frac{p_\varphi^2}{r^3\sin^2\theta}-m\frac{dV}{dr}\right)\frac{\partial f}{\partial p_r}$$
$$+\quad\frac{p_\varphi^2\cos\theta}{r^2\sin^3\theta}\frac{\partial f}{\partial p_\theta}=0. \tag{23}$$

THEOREM (Forsyth,1954)

The equation

$$R_1(x_1,x_2,\dots,x_n)\frac{\partial z}{\partial x_1}+R_2(x_1,x_2,\dots,x_n)\frac{\partial z}{\partial x_2}+\dots$$
$$+R_n(x_1,x_2,\dots,x_n)\frac{\partial z}{\partial x_n}=0 \tag{24}$$

has the general solution

$$z=z(u_1,u_2,\dots,u_{n-1}), \tag{25}$$

where

$$u_1 = u_1(x_1, x_2, \ldots, x_n) = a_1,$$
$$u_2 = u_2(x_1, x_2, \ldots, x_n) = a_2,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$u_{n-1} = u_{n-1}(x_1, x_2, \ldots, x_n) = a_{n-1}$$

is a complete system of (n-1) distinct and independent integrals of the (n-1) simultaneous equations

$$\frac{dx_1}{R_1} = \frac{dx_2}{R_2} = \ldots = \frac{dx_n}{R_n} . \qquad (26)$$

Moreover, every solution of (24) is contained in (25) (i.e. there are no 'special' integrals).

(26) is, in our case,

$$\frac{dr}{p_r} = \frac{r^2 d\theta}{p_\theta} = \frac{r^2 \sin^2\theta \, d\varphi}{p_\varphi} = \frac{dp_r}{\left(\frac{p_\theta^2}{r^3} + \frac{p_\varphi^2}{r^3 \sin^2\theta} - m\frac{dV}{dr}\right)}$$
$$= \frac{r^2 \sin^3\theta \, dp_\theta}{p_\varphi^2 \cos\theta} = \frac{dp_\varphi}{0} , \qquad (27)$$

whose five independent solutions are

$$L^2 = p_\theta^2 + \frac{p_\varphi^2}{\sin^2\theta} ,$$
$$L_2 = p_\theta \cos\varphi - p_\varphi \sin\varphi \cot\theta ,$$
$$L_3 = p_\varphi ,$$
$$E = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2 \sin^2\theta}\right) + V(r), \qquad (28)$$
$$S = G + \arcsin\left(\frac{L\cos\theta}{\sqrt{L_1^2 + L_2^2}}\right) ,$$

for $p_r \neq 0$, and

$$L^2, L_2, L_3, E, r , \qquad (29)$$

for $p_r = 0$, with

$$G = \int_{r_0}^{r} \frac{1}{r'}\left[\frac{2m}{L^2}(E-V)r'^2 - 1\right]^{-1/2} dr' \qquad (30)$$

(E and L being considered as constants in the integration).
Writing $L^2 = L_1^2 + L_2^2 + L_3^2$, we see that L_1, L_2, L_3, are the components of angular momentum; E is just the total energy; the physical interpretation of the fifth constant of motion, S, will be discussed below. The general solution of (23) can now be written

$$f = f_1(L^2, L_2, L_3, E, S) = f_2(L_1, L_2, L_3, E, S);$$

thus any constant of motion can be expressed in terms of the five quantities L_1, L_2, L_3, E, S. We shall, however, introduce a quantity B_3, instead of S.

Define

$$B_i = \frac{1}{r}[q_i L\sin G - (q_j L_k - q_k L_j)\cos G] \tag{31}$$

$$= r\cos G p_i - (p_r \cos G - \frac{L}{r}\sin G)q_i, \tag{32}$$

where (i,j,k) are in cyclic order and we have used identity 4 in writing (32). From (28), we see that

$$B_3 = -\sqrt{L^2 - L_3^2}\cos S, \tag{33}$$

and we may write

$$f = f(L_1, L_2, L_3, E, B_3). \tag{34}$$

The quantities B_1 and B_2 are also constants of motion, for the following relations may be easily derived from the definition (32):

$$B^2 \equiv B_1^2 + B_2^2 + B_3^2 = L^2, \tag{35}$$

$$B_1 L_1 + B_2 L_2 + B_3 L_3 = 0. \tag{36}$$

From (31), it is seen that B_1, B_2, B_3 are the components of a 3-vector

$$\underline{B} = \frac{1}{r}[\underline{r}L\sin G - (\underline{r} \times \underline{L})\cos G], \tag{37}$$

which, from (36) and (35), is perpendicular to the angular momentum vector $\underline{L}$ and equal in magnitude.

Define

$$C_i = \frac{1}{r}[(q_j L_k - q_k L_j)\sin G + q_i L\cos G] \tag{38}$$

$$= (p_r \sin G + \frac{L}{r}\cos G)q_i - r\sin G p_i, \tag{39}$$

where (i,j,k) are in cyclic order and we have used identity 4. From (28), we see that

$$C_3 = \sqrt{L^2 - L_3^2}\sin S, \tag{40}$$

and from the definition (39),

$$C^2 \equiv C_1^2 + C_2^2 + C_3^2 = L^2, \tag{41}$$

$$C_1 L_1 + C_2 L_2 + C_3 L_3 = 0. \tag{42}$$

From (38), it is seen that C_1, C_2, C_3 are the components of a 3-vector

$$\underline{C} = \frac{1}{r}[(\underline{r} \times \underline{L})\sin G + \underline{r}L\cos G], \tag{43}$$

which, from (42) and (41), is perpendicular to the angular momentum vector $\underline{L}$ and equal in magnitude. From (40)-(42), it is seen that $\underline{C}$ is a constant of motion. (37) and (43) show that

$$\underline{B}.\underline{C} = \underline{0}, \tag{44}$$

so that $\underline{B}$ and $\underline{C}$ are perpendicular, and that

$$\begin{aligned} \frac{1}{L}(\underline{C} \times \underline{B}) &= \underline{L}, \\ \frac{1}{L}(\underline{B} \times \underline{L}) &= \underline{C}, \\ \frac{1}{L}(\underline{L} \times \underline{C}) &= \underline{B}, \end{aligned} \tag{45}$$

showing that the set

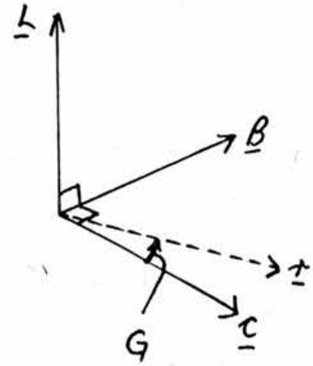
$$\{ \underline{C}/L,\ \underline{B}/L,\ \underline{L}/L \}$$

is an orthonormal positive triad. Further,

$$rL\cos G = \underline{C}.\underline{r}, \qquad rL\sin G = \underline{C} \times \underline{r}, \tag{46}$$

which follow from the definitions (37) and (43). Thus G is the angle between $\underline{C}$ and $\underline{r}$, measured in the positive direction, $\underline{B}$, $\underline{C}$ and $\underline{r}$

lying in the plane of motion:



We give below the explicit form of the component B_i, calculated from (32), in the cases of the hydrogen atom, oscillator and free particle, for each of which G can be evaluated exactly.

$$B_i^{HA} = \frac{L\left(r p_r p_i - p^2 q_i + \frac{e^2 m}{r} q_i\right)}{\sqrt{m(e^4 m + 2EL^2)}}, \qquad (V = -e^2/r) \qquad (47)$$

$$B_i^{HO} = \frac{1}{\sqrt{2}}\left[p_i\sqrt{r^2 + \frac{r p_r L}{\sqrt{m(mE^2 - 2kL^2)}}} - q_i\sqrt{p^2 - \frac{2mk r p_r L}{\sqrt{m(mE^2 - 2kL^2)}}}\right],$$
$$(V = kr^2) \qquad (48)$$

$$B_i^{FP} = \frac{r p_r p_i}{\sqrt{2mE}} - \sqrt{2mE}\, q_i \; . \qquad (V = 0) \qquad (49)$$

It is seen that B_i^{HA} is proportional to the component of the Lenz vector (Lenz, 1924), and that B_i^{HO} is very similar to that of the axial vector as used by Sexl (1966).

For those motions for which $p_r = 0$, the vectors $\underline{B}$ and $\underline{C}$ do not exist. Consequently, we are led to define $\underline{B} = \underline{C} = \underline{0}$ for such motions. These motions are, of course, defined by

$$p_r = m\frac{dr}{dt} = 0,$$

or $\qquad r = \text{constant},$

and are seen to be the circular motions. The Hamilton equations of motion are

$$m\frac{dr}{dt} = p_r, \qquad m\frac{dp_r}{dt} = \frac{p_\theta^2}{r^3} + \frac{p_\phi^2}{r^3\sin^2\theta} - m\frac{dV}{dr},$$

$$m\frac{d\theta}{dt} = \frac{p_\theta}{r^2}, \qquad m\frac{dp_\theta}{dt} = \frac{p_\phi^2\cos\theta}{r^2\sin^3\theta}, \qquad (50)$$

$$m\frac{d\phi}{dt} = \frac{p_\phi}{r^2\sin^2\theta}, \qquad \frac{dp_\phi}{dt} = 0,$$

and, for $p_r = 0$, we have

$$m\frac{dV}{dr} = \frac{L^2}{r^3}, \qquad (51)$$

so that, for circular motions to be possible, there must exist a region $r_1 \leq r \leq r_2$ for which $dV/dr > 0$.

2.3 The Symmetry Groups SO(3,1) and SO(4)

For $p_r \neq 0$, it can be shown, by a direct and somewhat tedious calculation, that the following commutation relations hold:

$$[B_1, B_2] = -L_3, \quad [B_1, L_2] = B_3, \quad [L_1, B_2] = B_3,$$

$$[B_2, B_3] = -L_1, \quad [B_2, L_3] = B_1, \quad [L_2, B_3] = B_1,$$

$$[B_3, B_1] = -L_2, \quad [B_3, L_1] = B_2, \quad [L_3, B_1] = B_2, \qquad (52)$$

$$[B_1, L_1] = [B_2, L_2] = [B_3, L_3] = 0.$$

Add to these the relations

$$[L_1, L_2] = L_3, \quad [L_2, L_3] = L_1, \quad [L_3, L_1] = L_2, \qquad (53)$$

and we have the commutation relations of the Lie algebra of the homogeneous Lorentz group.

However, we note that B_i (i = 1,2,3)is not uniquely determined by the relations (52), for let

$$B_i' = \psi B_i, \qquad (i=1,2,3) \tag{54}$$

where $\psi = \psi(E,L)$, be another function satisfying (52)*. Then a straightforward calculation, making use of (35) and (36), shows that ψ must satisfy the equation

$$L\psi\frac{\partial\psi}{\partial L} + \psi^2 = 1, \tag{55}$$

the solution of which is

$$\psi = \sqrt{1 + \frac{\chi(E)}{L^2}}, \tag{56}$$

where $\chi(E)$ is an arbitrary function.

Let V(r) be such that there exists a region $r_1 \leq r \leq r_2$ for which $dV/dr > 0$ and such that there exists $E = E_0$ for which the motion is circular. We require B_i' (i=1,2,3) to be a continuous function of the coordinates $\{q_1, q_2, q_3, p_1, p_2, p_3\}$, so that we must have

$$\lim_{E \to E_0} B\ (L, E, \ldots) = 0. \qquad (i=1,2,3) \tag{57}$$

This condition determines ψ:

$$\psi(E_0, L) = 0$$

or $$\chi(E_0) = -L^2. \tag{58}$$

This is so because (51) permits a solution

$$r = r(L), \tag{59}$$

and the equation

$$E_0 = \frac{L^2}{2m[r(L)]^2} + V[r(L)] \tag{60}$$

permits a solution

$$L = L(E_o). \tag{61}$$

We write

$$\chi(E) = -[L(E)]^2. \tag{62}$$

Consider the hydrogen atom, with $V = -e^2/r$:

(51) gives

$$\frac{L^2}{r^3} = \frac{\pi e^2}{r^2},$$

giving, for (59),

$$r = \frac{L^2}{\pi e^2},$$

whence (60) becomes

$$E_o = -\frac{\pi e^4}{2L^2},$$

and (61)

$$L = \sqrt{\frac{-\pi e^4}{2E_o}}.$$

Thus

$$\chi^{HA}(E) = \frac{\pi e^4}{2E}, \tag{63}$$

and

$$\gamma^{HA}(E,L) = \sqrt{1 + \frac{\pi e^4}{2EL^2}}. \tag{64}$$

A similar argument applied to the oscillator ($V=kr^2$) gives

$$\chi^{HO}(E) = -\frac{\pi E^2}{2k} \tag{65}$$

and

$$\gamma^{HO}(E,L) = \sqrt{1 - \frac{\pi E^2}{2kL^2}}. \tag{66}$$

Using (47) and (48), we now find

$$\mathcal{B}_i'^{HA} = \frac{1}{\sqrt{2\pi E}} \left(r p_r p_i - p^2 q_i + \frac{e^2 \pi}{r} q_i \right), \tag{67}$$

$$\mathcal{B}_i'^{HO} = \frac{i(\pi^2E^2 - 2\pi k L^2)^{1/4}}{2\sqrt{\pi k}\, L} \Bigg[p_i \sqrt{r^2\sqrt{\pi(\pi E^2 - 2kL^2)} + r p_r L}$$

$$- q_i \sqrt{p^2 \sqrt{\pi(\pi E^2 - 2kL^2)} - 2\pi k r p_r L} \Bigg] . \tag{68}$$

For a potential $V = a\, r^n$, it is straightforward to verify that

$$\begin{aligned} \chi(E) &= -\pi (n a)^{-2/n} \left(\frac{2nE}{n+2} \right)^{1+\frac{2}{n}}, && (n \neq 0, -2) \\ \chi(E) &= 0, && (n = 0) \\ \chi(E) &= 2\pi a . && (n = -2) \end{aligned} \tag{69}$$

It will be noted that B_i $(i=1,2,3)$ is a real-valued function of the coordinates, for the integrand of G is just $L/(r^2 p_r)$, which is real. For potentials V(r) such that there exists no region, $r_1 \le r \le r_2$, for which $dV/dr > 0$, circular motion is not possible and χ is undetermined. Indeed, B_i $(i=1,2,3)$ is a real-valued continuous function of the coordinates for all such potentials. For other potentials, when B_i $(i=1,2,3)$ is pure imaginary, write

$$B_j' = i B_j^{\times}, \qquad (j=1,2,3) \tag{70}$$

with $B_j^{\times}$ $(j=1,2,3)$ real. (52) now shows that the set $\{L_i, B_i^{\times};\ i=1,2,3\}$ satisfy the commutation relations of the four-dimensional rotation group. With these considerations, then, we can make the following observations:

For entirely repulsive potentials, i.e. those for which $dV/dr \leq 0$ for all r, the Lorentz group is a symmetry group for all values of the energy. No general statements at this stage can be made for other potentials; however, for the hydrogen atom,(67) shows thata symmetry group is SO(4) for E $<$ 0 and SO(3,1) for E $>$ 0 and for the oscillator, (68) shows that a symmetry group is SO(4) for all energies.

2.4 SO(3,1) as a Group of Transformations of Phase Space

To find explicitly the transformations generated by the set $\{L_i, B_i' ;\ i=1,2,3\}$ we have to integrate equations (1). In the case of our six-parameter group this can be accomplished by an easier, indirect method. The integration to find the group generated by B_3' is carried out as follows:

$$\frac{dL_1}{d\alpha_3} = [L_1, B_3'] = -B_2', \qquad \frac{dL_2}{d\alpha_3} = [L_2, B_3'] = B_1',$$

$$\frac{dL_3}{d\alpha_3} = 0 \qquad , \qquad \frac{dB_1'}{d\alpha_3} = [B_1', B_3'] = L_2,$$

$$\frac{dB_2'}{d\alpha_3} = [B_2', B_3'] = -L_1 \quad , \qquad \frac{dB_3'}{d\alpha_3} = 0,$$

which leads to

$$L_1 = L_{10} \cosh\alpha_3 - B_{20}' \sinh\alpha_3,$$

$$L_2 = L_{20} \cosh\alpha_3 + B_{10}' \sinh\alpha_3,$$

$$L_3 = L_{30}, \qquad (71)$$

$$B_1' = B_{10}'\cosh\alpha_3 + L_{20}\sinh\alpha_3,$$
$$B_2' = B_{20}'\cosh\alpha_3 - L_{10}\sinh\alpha_3,$$
$$B_3' = B_{30}',$$

where L_{i0}, B_{i0}' are the values of L_i, B_i' when $\alpha_3 = 0$. The same argument can be applied to the parameters α_1 (B_1') and α_2 (B_2') and similar expressions result. The set of equations (71) can be written in matrix form

$$\begin{pmatrix} L_1 \\ L_2 \\ L_3 \\ B_1' \\ B_2' \\ B_3' \end{pmatrix} = \begin{pmatrix} \cosh\alpha_3 & 0 & 0 & 0 & -\sinh\alpha_3 & 0 \\ 0 & \cosh\alpha_3 & 0 & \sinh\alpha_3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \sinh\alpha_3 & 0 & \cosh\alpha_3 & 0 & 0 \\ -\sinh\alpha_3 & 0 & 0 & 0 & \cosh\alpha_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} L_{10} \\ L_{20} \\ L_{30} \\ B_{10}' \\ B_{20}' \\ B_{30}' \end{pmatrix} \qquad (72)$$

or $$x = Y_3 x_0. \qquad (73)$$

For the group generated by L_i, with parameter θ_i, we find, by a procedure exactly analogous to that which led to (71),

$$x = X_i x_0, \qquad (74)$$

with

$$X_i = \begin{pmatrix} d_i & 0 \\ 0 & d_i \end{pmatrix}, \qquad (75)$$

$$d_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{pmatrix}, \quad d_2 = \begin{pmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{pmatrix},$$

$$d_3 = \begin{pmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For an arbitrary infinitesimal transformation specified by parameters $d\theta_1$, $d\theta_2$, $d\theta_3$, $d\alpha_1$, $d\alpha_2$, $d\alpha_3$, it is found that

$$x = (\mathcal{I} + d\Delta)x_o, \tag{76}$$

with

$$\Delta = \begin{pmatrix} 0 & -\theta_3 & \theta_2 & 0 & -\alpha_3 & \alpha_2 \\ \theta_3 & 0 & -\theta_1 & \alpha_3 & 0 & -\alpha_1 \\ -\theta_2 & \theta_1 & 0 & -\alpha_2 & \alpha_1 & 0 \\ 0 & \alpha_3 & -\alpha_2 & 0 & -\theta_3 & \theta_2 \\ -\alpha_3 & 0 & \alpha_1 & \theta_3 & 0 & -\theta_1 \\ \alpha_2 & -\alpha_1 & 0 & -\theta_2 & \theta_1 & 0 \end{pmatrix}. \tag{77}$$

$\mathcal{I} + d\Delta$ is, of course, just

$$\lim_{\substack{d\theta_i \to 0 \\ d\alpha_i \to 0}} (X_1 X_2 X_3 Y_1 Y_2 Y_3)_{\substack{\theta_i = d\theta_i \\ \alpha_i = d\alpha_i}} . \tag{78}$$

The explicit finite transformations, in the form of a set of six simultaneous algebraic equations in the variables $q_1, q_2, q_3, p_1, p_2, p_3$, can now be obtained from

$$x = e^{\Delta} x_o. \tag{79}$$

Alternatively, the explicit infinitesimal transformations can be obtained by writing $L_i = L_{io} + dL_{io}$, $B'_i = B'_{io} + dB'_{io}$ (i=1,2,3), giving

$$dx = d\Delta\, x, \tag{80}$$

whereby, since

$$dL_i = \sum_{j=1}^{3}\left(\frac{\partial L_i}{\partial q_j}dq_j + \frac{\partial L_i}{\partial p_j}dp_j\right),$$

$$dB_i' = \sum_{j=1}^{3}\left(\frac{\partial B_i'}{\partial q_j}dq_j + \frac{\partial B_i'}{\partial p_j}dp_j\right), \qquad (i=1,2,3)$$

we obtain a set of six linear simultaneous equations from which dq_1, dq_2, dq_3, dp_1, dp_2, dp_3 can be calculated.

* I am grateful to Mr. A. Bors for first suggesting this possibility.

II

THE QUANTUM PROBLEM

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1. The Quantization of a Classical System

The quantum mechanical substitute for phase space is the infinite-dimensional Hilbert space, $\mathcal{b}_M$, of square-summable complex-valued functions on M, with respect to Lebesque measure (or the analogue of M in the case of two or more particles).

The problem of quantizing a classical system may be stated as, given a set $\mathcal{f} = \{ f_1, f_2, \ldots, f_n \}$ of linearly independent classical observables that is the basis of a Lie algebra $\mathfrak{a}$ with respect to Poisson Bracket, we require a mapping

$$\Lambda : \mathcal{f} \ni f_i \longrightarrow \Lambda f_i \in \mathcal{P} \subset L(\mathcal{b}_M ; \mathcal{b}_M),$$

where $\mathcal{P} = \{ \Lambda f_i ;\ i=1,2,3,\ldots,n \}$ and $L(\mathcal{b}_M ; \mathcal{b}_M)$ is the space of linear mappings of $\mathcal{b}_M$ into itself, that is a Lie algebra homomorphism, i.e.

$$\Lambda [f_i , f_j] = (\Lambda f_i)(\Lambda f_j) - (\Lambda f_j)(\Lambda f_i) .$$

We must concentrate on a <u>subset</u> of the set of all classical observables because of a general theorem (Van Hove, 1951; Amiet, 1963) asserting that it is impossible to define the required homomorphism for Lie algebras of arbitrary large dimension.

The quantum observable $\mathcal{F}$ corresponding to the classical observable f is defined as (not necessarily uniquely) the self-adjoint extension of Λ f (Mackey, 1963).

A quantization procedure having certain desirable features is that which extends the representation of the Heisenberg algebra given by the Stone-von Neumann theorem (von Neumann, 1931). The Heisenberg algebra is that algebra (for a one-dimensional configuration space) having as its basis the set of the three basic classical observables q, p, and 1, with the commutation relations

$$[q, p] = 1, \quad [q, 1] = [p, 1] = 0. \tag{1}$$

The corresponding linear operators on $\mathcal{b}_M$ are given by (Kirilov, 1964)

$$\Lambda q = q, \quad \Lambda p = -\frac{\partial}{\partial q}, \quad \Lambda 1 = 1. \tag{2}$$

THEOREM (Hermann, 1966)

Given a set $f = \{f_1, f_2, \ldots, f_n\}$ of classical observables that is the basis of a Lie algebra $\mathcal{U}$, with

$$f_i = g_i(q_1, q_2, q_3) + \sum_{n_1, n_2, n_3} a_i^{n_1, n_2, n_3} p_1^{n_1} p_2^{n_2} p_3^{n_3}, \tag{3}$$

where n_1, n_2, n_3 are non-negative integers, then the mapping Λ, with

$$\Lambda f_i = g_i(q_1, q_2, q_3) + \sum_{n_1, n_2, n_3} (-1)^{n_1+n_2+n_3} a_i^{n_1, n_2, n_3} \frac{\partial^{n_1+n_2+n_3}}{\partial q_1^{n_1} q_2^{n_2} q_3^{n_3}} \tag{4}$$

defines a representation of $\mathcal{U}$ that extends that of the Heisenberg algebra.

The self-adjoint extension (with respect to the usual inner product

$$\langle \psi_1 | \psi_2 \rangle = \int_M \psi_1 \bar{\psi}_2 \, dq_1 dq_2 dq_3 \tag{5}$$

defined in $\mathcal{b}_M$) of an operator of the form (4) will be taken to be

$$\mathcal{F}_j = g_j(q_1, q_2, q_3) + \sum_{n_1, n_2, n_3} (-i\hbar)^{n_1+n_2+n_3} a_j^{n_1, n_2, n_3} \frac{\partial^{n_1+n_2+n_3}}{\partial q_1^{n_1} q_2^{n_2} q_3^{n_3}}, \tag{6}$$

where $\hbar$ is a positive real number called Planck's constant.

An alternative procedure of quantization which need not extend the representation of the Heisenberg algebra, is based on more group theoretic arguments. It was shown in I.1 that a C_∞ classical observable f defined a one-parameter group $\tau = \{h_\alpha;\ a \le \alpha \le b\}$ of transformations of M_V, given by (I.1). Under the mapping

$$\omega:\ M_V \ni (q,p) \longrightarrow \omega(q,p) = q \in M\ , \tag{7}$$

the group τ defines a curve in M via the mapping

$$h_\alpha':\ M \ni q \longrightarrow h_\alpha'(q) = \omega h_\alpha(q,0) \in M\ . \tag{8}$$

The condition for $\tau' = \{h_\alpha';\ a \le \alpha \le b\}$ to be a representation (not necessarily linear) of τ is that

$$\omega h_{\alpha_1}(\omega h_{\alpha_2}(q,0),0) = \omega h_{\alpha_1} h_{\alpha_2}(q,0)\ , \tag{9}$$

which can be seen by applying (8) twice, for the L.H.S. is just $h_{\alpha_1}' h_{\alpha_2}'(q)$. That (9) is not always satisfied can be seen from the example $f = q^2 + p^2$ (for one-dimensional M), which generates a curve in M_V defined by

$$q = q_0 \cos 2\alpha + p_0 \sin 2\alpha\ ,$$

$$p = p_0 \cos 2\alpha - q_0 \sin 2\alpha\ .$$

The L.H.S. of (9) is seen to be $q\cos 2\alpha_1 \cos 2\alpha_2$ and the R.H.S. $q\cos 2(\alpha_1 + \alpha_2)$.

When (9) is satisfied, we have a homomorphism between the one-parameter groups, τ, τ', of transformations of M_V, M, respectively. Given n functions $f_1, f_2, \ldots, f_n$ that are the basic elements of a Lie algebra $\mathfrak{a}$, and for each of whose one-parameter groups, $\tau_1, \tau_2, \ldots, \tau_n$, (9) is satisfied, we can construct the infinitesimal generators $\xi_1', \xi_2', \ldots, \xi_n'$ of $\tau_1', \tau_2', \ldots, \tau_n'$ which will then span a Lie algebra homomorphic to $\mathfrak{a}$. The determination of the linear operators Λf_1, Λf_2,

$\ldots, \Lambda f_n$ is then attained using the theory of induced representations (Kirilov, 1964):

For $\psi \in \mathcal{B}_M$, $h_\alpha' \in \tau'$, $v(\alpha) \in \mathcal{C}$, define the mapping

$$T : \tau' \ni h_\alpha' \longrightarrow T_\alpha \in L(\mathcal{B}_M;\mathcal{B}_M), \tag{10}$$

by

$$(T_\alpha \psi)(q_0) = v(\alpha)\psi(h_\alpha'^{-1} q_0). \tag{11}$$

It is easily seen that T is a representation of τ' iff $v(\alpha_1)v(\alpha_2) = v(\alpha_1 + \alpha_2)$ or $v(\alpha) = e^{x\alpha}$, where x is a number independent of α.

The infinitesimal generator, I, of $\mathcal{T} = \{T_\alpha\}$ is given by

$$\begin{aligned}
(I\psi)(q_0) &= \left(\frac{\partial T_\alpha}{\partial \alpha}\psi\right)_{\alpha=0}(q_0) \\
&= \left[v(\alpha)\, \frac{\partial \psi(h_\alpha'^{-1} q_0)}{\partial \alpha}\right]_{\alpha=0} + \left[\frac{d v(\alpha)}{d\alpha}\psi(h_\alpha'^{-1}(q_0))\right]_{\alpha=0} \\
&= \sum_{i=1}^{3}\left[\frac{\partial \psi(h_\alpha'^{-1} q_0)}{\partial q_i(-\alpha)}\, \frac{\partial q_i(-\alpha)}{\partial \alpha}\right]_{\alpha=0} + \left[x\psi(h_\alpha'^{-1} q_0)\right]_{\alpha=0} \\
&= -\sum_{i=1}^{3}\left(\frac{\partial q_i(\alpha)}{\partial \alpha}\right)_{\alpha=0} \frac{\partial \psi(q_0)}{\partial q_{i0}} + x\psi(q_0).
\end{aligned}$$

Thus

$$I = x - \sum_{i=1}^{3}\left[\frac{\partial q_i(\alpha)}{\partial \alpha}\right]_{\alpha=0} \frac{\partial}{\partial q_i}. \tag{12}$$

$\left[\frac{\partial q_i(\alpha)}{\partial \alpha}\right]_{\alpha=0}$ is, of course, $\left[\left(\frac{\partial h_\alpha}{\partial \alpha}\right)_{\alpha=0} q\right]_i = (\xi' q)_i$, from (I.16), and carrying out the above argument for each of the groups $\tau_1', \tau_2', \ldots, \tau_n'$, we find

$$I_k = x_k - \sum_{i=1}^{3} \gamma_k^i \frac{\partial}{\partial q_i}, \tag{13}$$

where $$\gamma_k^i = \left[\frac{\partial q_i(\alpha_k)}{\partial \alpha_k}\right]_{\alpha_k=0}. \tag{14}$$

The x_k (k=1,2,...,n) are determined from the fact that a necessary and sufficient condition for two groups to be (locally) homomorphic is that their Lie algebras shall be homomorphic. The group whose one-parameter subgroups are $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_n$ is defined to be a representation of the group whose one-parameter subgroups are $\tau_1', \tau_2', \ldots, \tau_n'$. With this definition, the x_k are found in terms of the structure constants of $\mathcal{A}$. Let

$$[f_k, f_\ell] = \sum_\rho c_{k\ell}^\rho f_\rho \ . \tag{15}$$

We then require that

$$[I_k, I_\ell] = \sum_\rho c_{k\ell}^\rho I_\rho \ . \tag{16}$$

This leads to the set of equations

$$c_{k\ell}^\rho x_\rho = \sum_{i=1}^3 \left(\gamma_\ell^i \frac{\partial x_k}{\partial q_i} - \gamma_k^i \frac{\partial x_\ell}{\partial q_i} \right) . \tag{17}$$

That this is so can be seen as follows. Write $Z_k = -\sum_{i=1}^3 \gamma_k^i \frac{\partial}{\partial q_i}$; then we must have

$$[x_k + Z_k, x_\ell + Z_\ell] = \sum_\rho c_{k\ell}^\rho (x_\rho + Z_\rho) \ .$$

or
$$[x_k, x_\ell] + [x_k, Z_\ell] + [Z_k, x_\ell] + [Z_k, Z_\ell] = \sum_\rho c_{k\ell}^\rho x_\rho + \sum_\rho c_{k\ell}^\rho Z_\rho \ . \tag{18}$$

Now a special case of the definition (11) is when $v(\alpha) = 1$ or $x = 0$, (18) then becoming

$$[Z_k, Z_\ell] = \sum_\rho c_{k\ell}^\rho Z_\rho \ .$$

Since Z_k does not involve x_i, this last equation is true always. (18) thus reduces to

$$[x_k, x_\ell] + [x_k, Z_\ell] + [Z_k, x_\ell] = \sum_\rho c_{k\ell}^\rho x_\rho \ .$$

On noticing that the first bracket is zero and on substituting the explicit form for Z_k, we have

$$\sum_\rho c_{k\ell}^\rho x_\rho = \sum_{i=1}^3 \left[x_k \gamma_\ell^i \frac{\partial}{\partial q_i} + \gamma_\ell^i \frac{\partial}{\partial q_i} x_k - \gamma_k^i \frac{\partial}{\partial q_i} x_\ell + x_\ell \gamma_k^i \frac{\partial}{\partial q_i} \right] ,$$

which easily reduces to (17), from which x_k as a function of q_1, q_2, q_3, can be determined. Then

$$\Lambda\, f_k = I_k . \tag{19}$$

In some special cases, the $\Lambda\, f_k$ obtained by this method are the same as those found by using (4). Indeed, with x = 0, the foregoing method can be used to derive the equation $\Lambda\, p = -\frac{\partial}{\partial q}$.

The discussions of this section, culminating in equations (4) and (19), are helpful in obtaining explicitly, as differential operators, expressions for the quantum observables. When these methods fail, the quantum operators can usually be obtained abstractly as matrices, using pure Lie algebra representation theory. Such a case will be considered in the next section.

It is of interest to note that quantization procedures involving bracket relations other than the Poisson Bracket have been suggested (Jordan, Sudarshan, 1961; Sudarshan 1961; Shankara, 1967).

2. Degeneracy and the Group Theoretic Classification of Eigenvalues

Let T be a representation of a group Y; then T is a mapping

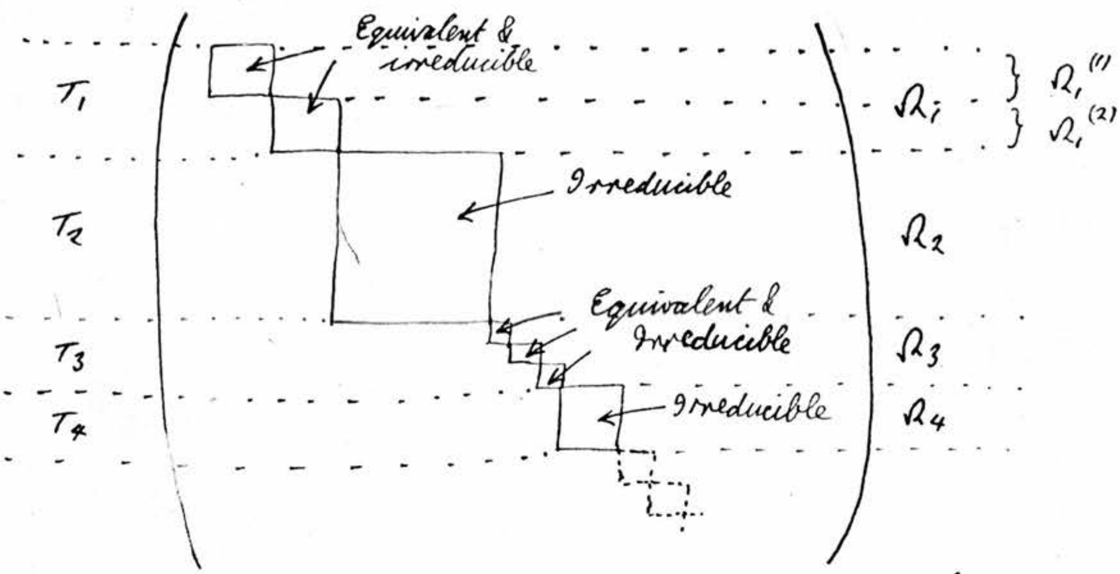
$$T : Y \ni y \longrightarrow T(y) \in L(\Omega;\Omega),$$

where Ω is the representation space.

Let Π be a self-adjoint linear operator (with respect to an inner product defined in Ω) in Ω that commutes with all T(y). Let Y be such that all representations are decomposable.

Let $T = T_1 + T_2 + \ldots$

be the primary decomposition of T, i.e., each T_i is the direct sum of mutually equivalent, irreducible representations:



Let $\Omega = \Omega_1 \oplus \Omega_2 \oplus \Omega_3 \oplus \ldots$ be the decomposition of Ω induced by this primary decomposition of T. Write

$$\Omega_i = \Omega_i^{(1)} \oplus \Omega_i^{(2)} \oplus \ldots, \qquad (i=1,2,3,\ldots)$$

where $\Omega_i^{(1)}, \Omega_i^{(2)}, \ldots$ are subspaces, irreducible with respect to T, each of dimension s_i.

In all of what follows, we shall be concerned with representations T such that the T_i are irreducible, i.e. the extra index '(j)' will not be necessary.

T_i , where $T_i(y) = T(y)/\Omega_i$, is an irreducible representation of Y of degree s_i with values in $L(\Omega_i ; \Omega_i)$ and Π/Ω_i commutes with all $T_i(y)$ (it can be shown that Π leaves Ω_i invariant). Schur's lemma states that a linear operator that commutes with each element of a group of linear operators defined on a space that is irreducible with respect to the group is a multiple of the identity operator when defined on that space. Thus

$$\Pi \psi = \pi_i \psi , \tag{20}$$

for all $\psi \in \Omega_i$, where π_i is a real number, and so Ω_i is an eigenspace of Π with eigenvalue π_i .

We therefore see that when T contains irreducible components that are not one-dimensional, some eigenspaces of Π are forced to be greater than one-dimensional. This occurrence of multiple eigenvalues is called degeneracy. Accidental degeneracy occurs when $\pi_{i_1} = \pi_{i_2}$ for some $i_1 \neq i_2$.

When $\Pi = \mathcal{H}$, the Hamiltonian, we are led to the most important case of degeneracy.

3. The Quantization of B'

3.1 General

We require a representation Λ of the algebra spanned by L_i, B_i' (i=1,2,3) and having the structure given by (I.52) and (I.53). With regard to representation theory, the results are independent of the nature of the basic elements; we can, therefore, consider B_i'(i=1,2,3) to be, in general, complex-valued functions on phase space. Our problem thus reduces to that of finding the representations of the Lie algebra of the homogeneous Lorentz group. We shall merely quote the results. For convenience, ΛL_i, $\Lambda B_i'$ will also be denoted by L_i, B_i', respectively.

The algebra is a rank two semi-simple algebra and in the standard notation (Racah, 1951) we define

$$\begin{aligned}
H_1 &= \tfrac{1}{2}(B_3' + iL_3),\\
H_2 &= \tfrac{1}{2}(B_3' - iL_3),\\
E_{0,1} &= \tfrac{1}{\sqrt{8}}(iL_1 + L_2 - B_1' + iB_2'),\\
E_{0,-1} &= \tfrac{1}{\sqrt{8}}(iL_1 - L_2 - B_1' - iB_2'),\\
E_{1,0} &= \tfrac{1}{\sqrt{8}}(iL_1 - L_2 + B_1' + iB_2'),\\
E_{-1,0} &= \tfrac{1}{\sqrt{8}}(iL_1 + L_2 + B_1' - iB_2'),
\end{aligned} \tag{21}$$

with the commutation relations

$$\begin{gathered}
[H_1, H_2] = [H_1, E_{0,1}] = [H_1, E_{0,-1}] = [H_2, E_{1,0}] = [H_2, E_{-1,0}] = 0,\\
[H_1, E_{1,0}] = E_{1,0}, \qquad [H_1, E_{-1,0}] = -E_{-1,0},\\
[H_2, E_{0,1}] = E_{0,1}, \qquad [H_2, E_{0,-1}] = -E_{0,-1},\\
[E_{1,0}, E_{-1,0}] = H_1, \qquad [E_{0,1}, E_{0,-1}] = H_2,
\end{gathered} \tag{22}$$

all relations not derivable from these being zero.

The irreducible representations are characterized by a two-component highest weight (j,k), with $j,k = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots,$ the representation space $\Omega_{j,k}$ having dimension $s = (2j + 1)(2k + 1)$. Let $\{e_{\mu,\nu};\ \mu = -j,-j+1,\ldots,j;\ \nu = -k,-k+1,\ldots,k\}$ be a normalized basis of $\Omega_{j,k}$. Then

$$
\begin{aligned}
H_1 e_{\mu,\nu} &= \mu e_{\mu,\nu}, \\
H_2 e_{\mu,\nu} &= \nu e_{\mu,\nu}, \\
E_{0,1} e_{\mu,\nu} &= \sqrt{\tfrac{1}{2}[k(k+1)-\nu(\nu+1)]}\, e_{\mu,\nu+1}, \\
E_{0,-1} e_{\mu\nu} &= \sqrt{\tfrac{1}{2}[k(k+1)-\nu(\nu-1)]}\, e_{\mu,\nu-1}, \\
E_{1,0} e_{\mu\nu} &= \sqrt{\tfrac{1}{2}[j(j+1)-\mu(\mu+1)]}\, e_{\mu+1,\nu}, \\
E_{-1,0} e_{\mu\nu} &= \sqrt{\tfrac{1}{2}[j(j+1)-\mu(\mu-1)]}\, e_{\mu-1,\nu},
\end{aligned}
\tag{23}
$$

Let us consider the possible self-adjoint extensions of the operators L_i, B_i' $(i=1,2,3)$. Since H_1 and H_2 are self-adjoint, we have

$$
\begin{aligned}
B_3'^{+} - iL_3^{+} &= B_3' + iL_3, \\
B_3'^{+} + iL_3^{+} &= B_3' - iL_3,
\end{aligned}
$$

from (21). Hence

$$
B_3'^{+} = B_3, \qquad L_3^{+} = -L_3.
$$

We consequently define the quantum observables

$$
\begin{aligned}
\mathcal{L}_1 &= i\hbar L_1, & \mathcal{L}_2 &= i\hbar L_2, & \mathcal{L}_3 &= i\hbar L_3, \\
\mathcal{B}_1 &= \hbar B_1', & \mathcal{B}_2 &= \hbar B_2', & \mathcal{B}_3 &= \hbar B_3'.
\end{aligned}
\tag{24}
$$

(23) now gives

$$
\begin{aligned}
\mathcal{L}_3 e_{\mu\nu} &= \hbar(\mu-\nu) e_{\mu\nu}, \\
\mathcal{B}_3 e_{\mu\nu} &= \hbar(\mu+\nu) e_{\mu\nu},
\end{aligned}
\tag{25}
$$

$$
(\mathcal{L}^2 + \mathcal{B}^2) e_{\mu\nu} = 2[j(j+1)+k(k+1)]\hbar^2 e_{\mu\nu}, \tag{26}
$$

where $\mathcal{L}^2 = \mathcal{L}_1^2 + \mathcal{L}_2^2 + \mathcal{L}_3^2$, $\mathcal{B}^2 = \mathcal{B}_1^2 + \mathcal{B}_2^2 + \mathcal{B}_3^2$; (25) and (26) are derived from the relations inverse to (21):

$$L_1 = \frac{1}{\sqrt{2}i}(E_{0,1} + E_{0,-1} + E_{1,0} + E_{-1,0}),$$
$$L_2 = \frac{1}{\sqrt{2}}(E_{0,1} - E_{0,-1} - E_{1,0} + E_{-1,0}),$$
$$L_3 = \frac{1}{i}(H_1 - H_2),$$
$$B_1' = \frac{-1}{\sqrt{2}}(E_{0,1} + E_{0,-1} - E_{1,0} - E_{-1,0}), \tag{27}$$
$$B_2' = \frac{-1}{\sqrt{2}i}(E_{0,1} - E_{0,-1} + E_{1,0} - E_{-1,0}),$$
$$B_3' = H_1 + H_2.$$

(26) shows that $L^2 - B'^2$ is a Casimir operator of SO(3,1).

Decompose $\Omega_{j,k}$ into subspaces ω_ℓ, irreducible with respect to the subalgebra spanned by L_i (i=1,2,3); then

$$\ell = |j-k|,\ |j-k|+1, \ldots,\ j+k.$$

There will be 2min(j,k)+1 such subspaces. As an operator on $\Omega_{j,k}$, the Hamiltonian H is a multiple of the identity. $\Omega_{j,k}$ will be an eigenspace, then, of H, with eigenvalue $\zeta_{j,k}$, say, of multiplicity (2j+1)(2k+1). For $\psi \in \omega_\ell \subset \Omega_{j,k}$, we shall have

$$\mathcal{L}^2\psi = \ell(\ell+1)\hbar^2\psi, \tag{28}$$

and since

$$\mathcal{H} = \frac{\mathcal{L}^2}{2m r^2} - \frac{\hbar^2}{2m r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + V(r), \tag{29}$$

which follows from the classical expressions for H and L and (6), we have

$$\left[\frac{\ell(\ell+1)\hbar^2}{2m r^2} - \frac{\hbar^2}{2m r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + V(r)\right]\psi = \zeta_{j,k}\psi, \tag{30}$$

considering $\Omega_{j,k}$ to be a subspace of $\mathcal{B}_M$. With the condition that

$$\mathcal{L}_3\psi = m\hbar\psi, \tag{31}$$

with $m=-\ell, -\ell+1, \ldots, \ell$, the normalized solutions of (30) can be written

$$\mathcal{Y}^{(j,k)}_{\ell,m} = \mathcal{R}^{(j,k)}_{\ell}(r)\, Y_{\ell,m}(\theta,\varphi), \qquad (m=-\ell,-\ell+1,\ldots,\ell) \tag{32}$$

where $Y_{\ell,m}$ are the spherical harmonics:

$$Y_{\ell,m}(\theta,\varphi) = P^{m}_{\ell}(\cos\theta)\, e^{im\varphi} . \tag{33}$$

The $e_{\mu\nu}$ can be expressed in terms of the $\mathcal{Y}^{(j,k)}_{\ell,m}$, after noting that the representation (j,k) of SO(3,1) is the tensor product of representations (j) and (k) of SO(3). We have

$$e^{(j,k)}_{\mu\nu} = \sum_{|j-k|\le \ell \le j+k} (jk\,\mu\nu/\ell\, m)\, \mathcal{R}^{(j,k)}_{\ell}\, Y_{\ell,m}, \tag{34}$$

where $(jk\,\mu\nu/\ell\, m)$ are the Clebsch-Gordan coefficients. The inverse relation also exists:

$$\mathcal{R}^{(j,k)}_{\ell}\, Y_{\ell,m} = \sum_{\mu+\nu=m} (jk\,\mu\nu/\ell\, m)\, e^{(j,k)}_{\mu\nu} . \tag{35}$$

An analysis of the solutions of (30) is needed to find a relation between j and k (usually, the condition of square-integrability is necessary).

From (31) and (34), we have

$$\mathcal{L}_3\, e_{\mu\nu} = m\hbar\, e_{\mu\nu}, \tag{36}$$

which, from (25), gives

$$m = \mu - \nu . \tag{37}$$

Since ℓ is an integer, j+k is an integer and so $\mu + \nu$ is an integer, showing that the eigenvalues of $\mathcal{B}_3$ are all multiples of $\hbar$ in the range -(j+k),-(j+k)+1,...,j+k.

The foregoing arguments depend upon the assumption that the classical B'_i exists and that either B'_i or $B^{\times}_i$ defined by (I.70) is real. When these requirements are satisfied, the quantum observable corresponding to the classical B_i' or B_i^* (whichever is real) is $\mathcal{B}_i$, since $\mathcal{B}_i$ is self-adjoint. In practice, when considering a particular form of V(r), having obtained the eigenvalues ζ of $\mathcal{H}$, a consideration of the corresponding classical expression for B_i' (with E replaced by ζ) should be sufficient to indicate the existence, or otherwise, of its quantum counterpart.

3.2 The Hydrogen Atom and Oscillator

With $V = -e^2/r$, the eigenvalues of $\mathcal{H}$ are

$$\zeta = -\frac{me^4}{2\hbar^2 n^2}, \quad (n = 1,2,3,\ldots) \tag{38}$$

for $\psi \in \mathcal{C}_M$, with

$$\ell = 0,1,2,\ldots,n-1. \tag{39}$$

Since the maximum value of ℓ is j+k, we must have j+k=n-1; its minimum value is $|j-k|$, so that $|j-k|=0$. Thus

$$j=k, \qquad n=2j+1. \tag{40}$$

Since E is negative, we see from (I.67) that B_i^* exists and is real; moreover, (40) shows that the representation of SO(4) on the energy eigenstates is the tensor product of the same representation of SO(3); a further fact is that j can be integral or half-integral:

$$j=0,\tfrac{1}{2},1,\tfrac{3}{2},\ldots. \tag{41}$$

The eigenvalues of $\mathcal{B}_3$ are

$$-(n-1)\hbar, -(n-2)\hbar, \ldots, (n-1)\hbar. \tag{42}$$

The eigenvalues of $\mathcal{L}^2 + \mathcal{B}^2$ are $4j(j+1)\hbar^2 = (n^2-1)\hbar^2$ and we see that the quantum number n is in fact related to the eigenvalue of this operator.

With $V=kr^2$, the eigenvalues of $\mathcal{H}$ are

$$\zeta = (N+\tfrac{3}{2})\sqrt{\frac{2k}{m}}\,\hbar, \qquad (N=0,1,2,\ldots) \tag{43}$$

for $\psi \in \mathcal{C}_M$, with

$$\ell = \begin{cases} N, N-2, \ldots, 0 & \text{if } N \text{ is even} \\ N, N-2, \ldots, 1 & \text{if } N \text{ is odd.} \end{cases} \tag{44}$$

It is easy to see that the representations (j,k) that are relevant are those that are also irreducible with respect to the rotation group, i.e. we must have

$$j+k=|j-k|,$$

which is

$$j=0 \text{ or } k=0. \tag{45}$$

Writing $k=0$, we have

$$j=\ell, \tag{46}$$

and, from (34),

$$e_{\mu\nu} = \psi_{\ell,m} \tag{47}$$

The eigenvalues of $\mathcal{B}_3$ are

$$-\ell\hbar, -(\ell-1)\hbar, \ldots, \ell\hbar, \tag{48}$$

and those of $\mathcal{L}^2+\mathcal{B}^2$ $2\ell(\ell+1)\hbar^2$, $B_i^\times$ being real and finite for all energies (see (I.68)).

From (47), we see that

$$\mathcal{L}^2 e_{\mu\nu} = \mathcal{L}^2\psi_{\ell,m} = \ell(\ell+1)\hbar^2\psi_{\ell,m} = \ell(\ell+1)\hbar^2 e_{\mu\nu},$$

so that

$$\mathcal{B}^2 e_{\mu\nu} = \ell(\ell+1)\hbar^2 e_{\mu\nu};$$

thus $\mathcal{B}$ has the same eigenvalues as $\mathcal{L}$.

It remains to remark that, while for the hydrogen atom the group SO(4) explains completely all the degeneracies present, the same group fails to do so in the case of the oscillator: a degeneracy of magnitude $\frac{1}{2}(N+1)(N+2)$ remains unaccounted for. The problem of relating the quantum number N to the eigenvalue of some Casimir operator is still unsolved at this stage. The complete solution will be forthcoming in Part III.

III

THE OSCILLATOR

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1. The Constants of Motion

It was shown in Part II that the vector $\underline{B}$ is of no great significance in the case of the oscillator. Accordingly, we are led to seek new constants of motion, whose existence depends upon the explicit form of the oscillator potential.

With $V = k(q_1^2 + q_2^2 + q_3^2)$, (I.22) becomes

$$p_1\frac{\partial f}{\partial q_1} + p_2\frac{\partial f}{\partial q_2} + p_3\frac{\partial f}{\partial q_3} - 2\pi k\left(q_1\frac{\partial f}{\partial p_1} + q_2\frac{\partial f}{\partial p_2} + q_3\frac{\partial f}{\partial p_3}\right) = 0, \tag{1}$$

the analogue of (I.26) then being

$$\frac{dq_1}{p_1} = \frac{dq_2}{p_2} = \frac{dq_3}{p_3} = \frac{-dp_1}{2\pi kq_1} = \frac{-dp_2}{2\pi kq_2} = \frac{-dp_3}{2\pi kq_3} \tag{2}$$

with solutions

$$\begin{aligned} A_{11} &= p_1^2 + 2\pi kq_1^2, & L_2 &= q_3p_1 - q_1p_3, \\ A_{22} &= p_2^2 + 2\pi kq_2^2, & L_3 &= q_1p_2 - q_2p_1. \\ A_{33} &= p_3^2 + 2\pi kq_3^2, \end{aligned} \tag{3}$$

These are the five constants of motion; from them we can get other constants (dependent on these). In fact, we can introduce the components of a second rank tensor:

$$A_{ij} = p_ip_j + \gamma^2q_iq_j, \qquad (i,j=1,2,3) \tag{4}$$

with

$$\gamma = \sqrt{2\pi k}. \tag{5}$$

The following commutation relations can be derived from the definitions (3) and (4) (Fradkin, 1965):

$$[A_{ii}, L_j] = 2\epsilon_{ijk} A_{ik}, \quad \text{(i,j,k all different if } i \neq j)$$

$$\left.\begin{aligned} [A_{ij}, L_k] &= \epsilon_{ijk} (A_{ii} - A_{jj}), \\ [A_{ij}, L_i] &= \epsilon_{ik} A_{ik}, \\ [A_{ii}, A_{ij}] &= 2\epsilon_{ijk} \gamma^2 L_k, \\ [A_{ij}, A_{ik}] &= \epsilon_{jki} \gamma^2 L_i, \\ [A_{ii}, A_{jk}] &= 0, \end{aligned}\right\} \begin{pmatrix}\text{i,j,k all}\\ \text{different}\end{pmatrix} \qquad (6)$$

$$[A_{ii}, A_{jj}] = 0. \qquad \text{(all i,j)}$$

2. The Symmetry Group SU(3)

2.1 Derivation of the Generators

We attempt to define an eight-dimensional subalgebra $\mathcal{A}$ of the nine-dimensional algebra $\mathcal{A}'$ spanned by A_{ij}, L_i (i,j=1,2,3) and such that $D = A_{11} + A_{22} + A_{33} \notin \mathcal{A}$ (in other words, we try to eliminate the Hamiltonian). Define

$$\begin{aligned} A_0 &= a_1 A_{11} + a_2 A_{22} + a_3 A_{33}, \\ A_1 &= b_1 A_{11} + b_2 A_{22} + b_3 A_{33}, \\ D &= A_{11} + A_{22} + A_{33}, \end{aligned} \tag{7}$$

with

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 1 & 1 & 1 \end{vmatrix} \neq 0 \quad . \tag{8}$$

Then

$$\begin{aligned} A_{11} &= x_1 A_0 + x_2 A_1 + x_3 D, \\ A_{22} &= y_1 A_0 + y_2 A_1 + y_3 D, \\ A_{33} &= z_1 A_0 + z_2 A_1 + z_3 D, \end{aligned} \tag{9}$$

where

$$\begin{aligned} x_1 &= \frac{b_2 - b_3}{\Delta}, & x_2 &= \frac{-(a_2 - a_3)}{\Delta}, & x_3 &= \frac{a_2 b_3 - a_3 b_2}{\Delta}, \\ y_1 &= \frac{b_3 - b_1}{\Delta}, & y_2 &= \frac{-(a_3 - a_1)}{\Delta}, & y_3 &= \frac{a_3 b_1 - a_1 b_3}{\Delta}, \\ z_1 &= \frac{b_1 - b_2}{\Delta}, & z_2 &= \frac{-(a_1 - a_2)}{\Delta}, & z_3 &= \frac{a_1 b_2 - a_2 b_1}{\Delta}. \end{aligned} \tag{10}$$

A direct application of (6) gives

$$\left.\begin{aligned} [A_0, L_i] &= -2(a_j - a_k)A_{jk}, \\ [A_1, L_i] &= -2(b_j - b_k)A_{jk}, \end{aligned}\right\} \text{(i,j,k) in cyclic order}$$

$$\left.\begin{aligned} [A_0, A_{ij}] &= 2(a_i - a_j)\gamma^2 L_k, \\ [A_1, A_{ij}] &= 2(b_i - b_j)\gamma^2 L_k, \end{aligned}\right\} \begin{array}{l}\text{(i,j) and (i,j,k)} \\ \text{in cyclic order}\end{array}$$

$$[A_{12}, L_3] = (x_1 - y_1)A_0 + (x_2 - y_2)A_1 + (x_3 - y_3)D,$$

$$[A_{23}, L_1] = (y_1 - z_1)A_0 + (y_2 - z_2)A_1 + (y_3 - z_3)D,$$

$$[A_{31}, L_2] = (z_1 - x_1)A_0 + (z_2 - x_2)A_1 + (z_3 - x_3)D,$$

$$[A_0, A_1] = 0. \tag{11}$$

Obviously, for $D \notin \mathcal{A}$, we must have

$$x_3 = y_3 = z_3,$$

or, from (10) and (8),

$$a_2b_3 - a_3b_2 = a_3b_1 - a_1b_3 = a_1b_2 - a_2b_1 = \Delta/3. \tag{12}$$

Anticipating the maximal Abelian subalgebra $\mathcal{A}_s$ of $\mathcal{A}$ to contain A_0 and L_3 as elements, we put

$$a_1 = a_2, \tag{13}$$

so that A_0 will commute with L_3, and

$$x_1 = y_1,$$

or, from (10),

$$b_2 - b_3 = b_3 - b_1, \tag{14}$$

so that $[A_{12}, L_3] \notin \mathcal{A}_s$.

(12), with the conditions (13) and (14), gives

$$a_1b_3 = a_3b_3 \tag{15}$$

and

$$a_1(b_1 - b_2) = a_3(b_2 - b_3). \tag{16}$$

Since we must have $a_1 \neq a_3$ (for $\Delta = 0$ if $a_1 = a_3$), (15) now gives

$$b_3 = 0, \tag{17}$$

(14) then becoming

$$b_2 = -b_1, \tag{18}$$

and consequently (16) shows that

$$a_3b_1 = -2a_1b_1. \tag{19}$$

Since we must have $b_1 \neq 0$ (for $\Delta = 0$ if $b_1 = 0$), (19) gives

$$a_3 = -2a_1 , \tag{20}$$

and (12)

$$\Delta = -6a_1 b_1 . \tag{21}$$

The numbers a_1 and b_1 are arbitrary, except that they must each be non-zero. It is convenient to write

$$a_1 = -\frac{1}{6}, \quad b_1 = \frac{1}{2}, \tag{22}$$

whence a_2, a_3, b_2, b_3 are calculated from (13), (20), (18), (17), respectively.

The definitions (7) and (9) give

$$\begin{aligned} A_0 &= \tfrac{1}{6}(2A_{33} - A_{11} - A_{22}), & A_{11} &= -A_0 + A_1 + \tfrac{1}{3}D, \\ A_1 &= \tfrac{1}{2}(A_{11} - A_{22}), & A_{22} &= -A_0 - A_1 + \tfrac{1}{3}D, \\ D &= A_{11} + A_{22} + A_{33}, & A_{33} &= 2A_0 + \tfrac{1}{3}D, \end{aligned} \tag{23}$$

the relations (11) becoming

$$\begin{aligned} [A_0, L_1] &= A_{23}, & [A_1, L_1] &= A_{23}, \\ [A_0, L_2] &= -A_{31}, & [A_1, L_2] &= A_{31}, \\ [A_0, L_3] &= 0, & [A_1, L_3] &= -2A_{12}, \\ [A_0, A_{12}] &= 0, & [A_1, A_{12}] &= 2\gamma^2 L_3, \\ [A_0, A_{23}] &= -\gamma^2 L_1, & [A_1, A_{23}] &= -\gamma^2 L_1, \\ [A_0, A_{31}] &= \gamma^2 L_2, & [A_1, A_{31}] &= -\gamma^2 L_2, \\ [A_{12}, L_3] &= 2A_1, \\ [A_{23}, L_1] &= -3A_0 - A_1, \\ [A_{31}, L_2] &= 3A_0 - A_1, \\ [A_0, A_1] &= 0. \end{aligned} \tag{24}$$

We have now constructed the eight-dimensional algebra $\mathcal{A}$ spanned by L_1, L_2, L_3, A_{12}, A_{23}, A_{31}, A_0, A_1. It is easily shown that $\mathcal{A}_s$ has dimension two. The choice of its basic elements is arbitrary -- we take them to be A_0 and L_3.

We note that $\mathfrak{A}$ is semi-simple (Cartan's criterion, Racah, 1951) and attempt to define a new basis such that the commutation relations between its elements are in the standard form.

Define

$$H_i = c_i A_0 + d_i L_3, \qquad (i=1,2) \tag{25}$$

with

$$c_1 d_2 - c_2 d_1 \neq 0. \tag{26}$$

We have to solve the equation

$$\begin{aligned}[H_i,\ &\kappa_1 L_1 + \kappa_2 L_2 + \kappa_3 A_{12} + \kappa_4 A_{23} + \kappa_5 A_{31} + \kappa_6 A_1] \\ &= t_i(\kappa_1 L_1 + \kappa_2 L_2 + \kappa_3 A_{12} + \kappa_4 A_{23} + \kappa_5 A_{31} + \kappa_6 A_1)\end{aligned} \tag{27}$$

for t_i and the constants κ_1, κ_2, κ_3, κ_4, κ_5, κ_6.
Using (6) and (24), (27) reduces to

$$\begin{aligned} c_i\kappa_1 + d_i\kappa_5 &= t_i\kappa_4, \\ -c_i\kappa_2 - d_i\kappa_4 &= t_i\kappa_5, \\ d_i\kappa_1 + \gamma^2 c_i\kappa_5 &= t_i\kappa_2, \\ -d_i\kappa_2 - \gamma^2 c_i\kappa_4 &= t_i\kappa_1, \\ 2d_i\kappa_6 &= t_i\kappa_3, \\ -2d_i\kappa_3 &= t_i\kappa_6. \end{aligned} \tag{28}$$

For a non-trivial solution, we must have

$$\begin{vmatrix} 0 & c_i & d_i & -t_i & 0 & 0 \\ c_i & 0 & t_i & d_i & 0 & 0 \\ -t_i & d_i & \gamma^2 c_i & 0 & 0 & 0 \\ d_i & t_i & 0 & \gamma^2 c_i & 0 & 0 \\ 0 & 0 & 0 & 0 & 2d_i & -t_i \\ 0 & 0 & 0 & 0 & t_i & 2d_i \end{vmatrix} = 0, \tag{29}$$

giving the following six solutions for t_j:

$$t_j = \pm i(d_j \pm \gamma c_j), \quad \pm 2id_j, \tag{30}$$

whence (28) gives the relations

$$\kappa_3 = \kappa_6 = 0, \quad \frac{\kappa_2}{\kappa_1} = \mp i, \quad \frac{\kappa_4}{\kappa_1} = \frac{\mp \pm i}{\gamma}, \quad \frac{\kappa_5}{\kappa_1} = \frac{\pm 1}{\gamma}, \tag{31}$$

for $t_j = \pm i(d_j \pm \gamma c_j)$; and

$$\kappa_1 = \kappa_2 = \kappa_4 = \kappa_5 = 0, \quad \frac{\kappa_3}{\kappa_6} = \mp i, \tag{32}$$

for $t_j = \pm 2id_j$.

Write

$$\alpha = (\alpha_1, \alpha_2), \quad \beta = (\beta_1, \beta_2), \tag{33}$$

where

$$\alpha_j = i(d_j + \gamma c_j), \quad \beta_j = i(d_j - \gamma c_j), \tag{34}$$

with the inverse

$$c_j = \frac{1}{2i\gamma}(\alpha_j - \beta_j), \quad d_j = \frac{1}{2i}(\alpha_j + \beta_j). \tag{35}$$

The theory of semi-simple Lie algebras shows that α_j, β_j can be chosen to be real numbers. α, β, then, are elements of $\mathcal{R}_2$. Since $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$ (from (34) and (26)), we can define

$$\begin{matrix}\alpha^1\\ \beta^1\end{matrix} = \frac{\mp 8\kappa_1^2}{(\alpha_1\beta_2 - \alpha_2\beta_1)}\left(\begin{matrix}\alpha_2\\ \beta_2\end{matrix} + 2\begin{matrix}\beta_2\\ \alpha_2\end{matrix}\right), \qquad \begin{matrix}\alpha^2\\ \beta^2\end{matrix} = \frac{\pm 8\kappa_1^2}{(\alpha_1\beta_2 - \alpha_2\beta_1)}\left(\begin{matrix}\alpha_1\\ \beta_1\end{matrix} + 2\begin{matrix}\beta_1\\ \alpha_1\end{matrix}\right). \tag{36}$$

Further, the non-vanishing of $\alpha_1\beta_2 - \alpha_2\beta_1$ shows that α and β are linearly independent and hence they may be taken as basic elements of $\mathcal{R}_2$.

Define the bilinear functional $\diamond$ in $\mathcal{R}_2$ by

$$\diamond \quad : (x,y) \rightarrow \diamond(x,y) = \langle x,y\rangle \in \mathcal{R},$$

for all $x,y \in \mathcal{R}_2$, where

$$\begin{aligned} \langle\alpha\,,\alpha\rangle &= \alpha^1\alpha_1 + \alpha^2\alpha_2 &&= -16\kappa_1{}^2, \\ \langle\beta\,,\beta\rangle &= \beta^1\beta_1 + \beta^2\beta_2 &&= -16\kappa_1{}^2, \\ \langle\alpha\,,\beta\rangle &= \alpha^1\beta_1 + \alpha^2\beta_2 &&= 8\,\kappa_1{}^2, \\ \langle\beta\,,\alpha\rangle &= \beta^1\alpha_1 + \beta^2\alpha_2 &&= 8\,\kappa_1{}^2, \end{aligned} \tag{37}$$

the equalities on the R.H.S. following from (36). The last two members of (37) show that $\diamond$ is symmetric, and hence from the first two we see that by choosing κ_1 to be pure imaginary $\diamond$ may be made into a real inner product.

(31) and (32) enable us to define six functions

$$\begin{aligned} E_{\text{\textcircled{$\pm$}}\substack{\alpha\\ \beta}} &= \kappa_1\Big(L_1 \text{\textcircled{$\mp$}}\, iL_2 \mp \text{\textcircled{$\pm$}}\frac{i}{\gamma}A_{23} \pm \frac{1}{\gamma}A_{31}\Big), \\ E_{\pm(\alpha+\beta)} &= \kappa_6\big(A_1 \mp iA_{12}\big). \end{aligned} \tag{38}$$

(27) and (30) now show that

$$\begin{aligned} &[H_i, E_{\text{\textcircled{$\pm$}}\substack{\alpha\\ \beta}}] = \text{\textcircled{$\pm$}}\,{}^{\alpha_i}_{\beta_i}\,E_{\text{\textcircled{$\pm$}}\substack{\alpha\\ \beta}}\,, \quad [H_i, E_{\pm(\alpha+\beta)}] = \pm(\alpha_i+\beta_i)E_{\pm(\alpha+\beta)}, \\ &[E_{\substack{\alpha\\ \beta}}, E_{-\substack{\alpha\\ \beta}}] = {}^{\alpha^1}_{\beta^1}H_1 + {}^{\alpha^2}_{\beta^2}H_2\,, \quad [E_{\alpha+\beta}, E_{-(\alpha+\beta)}] = \sigma^1H_1 + \sigma^2H_2\,, \end{aligned} \tag{39}$$

where

$$\sigma^1 = \frac{4\gamma^2\kappa_6^2(\alpha_2-\beta_2)}{(\alpha_1\beta_2-\alpha_2\beta_1)}\,, \qquad \sigma^2 = \frac{-4\gamma^2\kappa_6^2(\alpha_1-\beta_1)}{(\alpha_1\beta_2-\alpha_2\beta_1)}\,, \tag{40}$$

and we have used (35). We note that $\sigma^i = \alpha^i + \beta^i$ iff $\gamma^2\kappa_6{}^2 = 2\kappa_1{}^2$.

Define

$$\kappa_6 = \frac{-\sqrt{2}}{\gamma}\kappa_1\,, \tag{41}$$

enabling (39) and other relations to be written

$$[H_1, H_2] = 0,$$

$$[H_i, E_{\pm\frac{\alpha}{\beta}}] = \pm \tfrac{\alpha_i}{\beta_i} E_{\pm\frac{\alpha}{\beta}} ,$$

$$[H_i, E_{\pm(\alpha+\beta)}] = \pm(\alpha_i+\beta_i) E_{\pm(\alpha+\beta)} ,$$

$$[E_{\frac{\alpha}{\beta}}, E_{-\frac{\alpha}{\beta}}] = \tfrac{\alpha^1}{\beta^1} H_1 + \tfrac{\alpha^2}{\beta^2} H_2 ,$$

$$[E_{\alpha+\beta}, E_{-(\alpha+\beta)}] = (\alpha^1+\beta^1) H_1 + (\alpha^2+\beta^2) H_2 ,$$

$$[E_{\pm\alpha}, E_{\pm\beta}] = N_{\pm\alpha,\pm\beta} E_{\pm(\alpha+\beta)} ,$$

$$[E_{\pm\frac{\alpha}{\beta}}, E_{\mp(\alpha+\beta)}] = N_{\pm\frac{\alpha}{\beta},\mp(\alpha+\beta)} E_{\mp\frac{\beta}{\alpha}} ; \tag{42}$$

$$H_j = \frac{-i}{2\gamma}(\alpha_j-\beta_j)A_0 - \frac{i}{2}(\alpha_j+\beta_j)L_3 ,$$

$$E_{\pm\frac{\alpha}{\beta}} = \frac{i\lambda}{4}\left(L_1 \pm i L_2 \mp \pm \frac{i}{\gamma} A_{23} \pm \frac{1}{\gamma} A_{31}\right),$$

$$E_{\pm(\alpha+\beta)} = \frac{-i\lambda}{\sqrt{8}\,\gamma}\left(A_1 \mp i A_{12}\right), \tag{43}$$

where

$$N_{\pm\alpha,\pm\beta} = -N_{\pm\alpha,\mp(\alpha+\beta)} = N_{\pm\beta,\mp(\alpha+\beta)} = \frac{\pm\lambda}{\sqrt{2}} ; \tag{44}$$

$$\lambda = -4i\kappa_1 . \tag{45}$$

Choosing κ_1 to be pure imaginary and non-zero, we have, from (37),

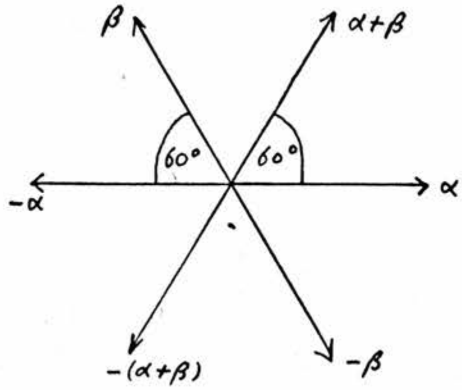
$$\langle\alpha , \alpha\rangle = \langle\beta , \beta\rangle = -2\langle\alpha , \beta\rangle = \lambda^2, \tag{46}$$

where λ is real. $|\lambda|$ is the length of each of the vectors α, β. The angle, $\theta_{\alpha,\beta}$, between α and β is defined by

$$\cos\theta_{\alpha,\beta} = \frac{\langle\alpha,\beta\rangle}{\sqrt{\langle\alpha,\alpha\rangle\langle\beta,\beta\rangle}} = -\frac{1}{2}, \tag{47}$$

from (46).

Our new basis of $\mathfrak{a}$ is now $\{H_1 , H_2 , E_{\pm\frac{\alpha}{\beta}} , E_{\pm(\alpha+\beta)}\}$. The vectors $\pm\alpha$, $\pm\beta$, $\pm(\alpha+\beta)$ are called the roots of the algebra, α, β being the simple roots. The root diagram is



and the Schouten diagram

enabling $\mathfrak{a}$ to be identified with A_2 or the Lie algebra of SU(3) (Behrends, Dreitlein, Fronsdal, Lee, 1962; Dynkin, 1957 a, b; Kleima, 1965; Racah, 1951).

Henceforward we shall simplify matters by defining values for α, β, λ. Define

$$\alpha = (1,0), \qquad \beta = (0,1); \tag{48}$$

and

$$\lambda = \sqrt{2}, \tag{49}$$

or

$$\begin{aligned} \alpha_1 &= \beta_2 = 1, \quad \alpha_2 = \beta_1 = 0, \\ \alpha^1 &= \beta^2 = -2\beta^1 = -2\alpha^2 = 2, \end{aligned} \tag{50}$$

from (33), (36), (45).

For ease of reference, we shall rewrite equations (42) and (43) (together with its inverse), using the values (49), (50):

$$[H_1, H_2] = 0,$$

$$[H_1, E_{\pm\alpha}] = \pm E_{\pm\alpha},$$

$$[H_2, E_{\pm\beta}] = \pm E_{\pm\beta},$$

$$[H_1, E_{\pm\beta}] = [H_2, E_{\pm\alpha}] = 0,$$

$$[H_i, E_{\pm(\alpha+\beta)}] = \pm E_{\pm(\alpha+\beta)},$$

$$[E_\alpha, E_{-\alpha}] = 2H_1 - H_2,$$

$$[E_\beta, E_{-\beta}] = -H_1 + 2H_2,$$

$$[E_{\alpha+\beta}, E_{-(\alpha+\beta)}] = H_1 + H_2 ,$$

$$[E_{\pm\alpha}, E_{\pm\beta}] = \pm E_{\pm(\alpha+\beta)},$$

$$[E_{\pm\alpha}, E_{\mp(\alpha+\beta)}] = \mp E_{\mp\beta},$$

$$[E_{\pm\beta}, E_{\mp(\alpha+\beta)}] = \pm E_{\mp\alpha}; \tag{51}$$

$$H_{\substack{1\\2}} = \frac{\mp i}{2}\left(\frac{A_0}{\gamma} \pm L_3\right),$$

$$E_{\pm\substack{\alpha\\ \beta}} = \frac{i}{\sqrt{8}}\left(L_1 \pm i L_2 \mp \pm \frac{i}{\gamma} A_{23} \pm \frac{1}{\gamma} A_{31}\right),$$

$$E_{\pm(\alpha+\beta)} = \frac{-i}{2\gamma}\left(A_1 \mp i A_{12}\right); \tag{52}$$

$$
\begin{aligned}
A_0 &= i\gamma\,(H_1 - H_2),\\
L_3 &= i(H_1 + H_2),\\
L_1 &= \frac{-i}{\sqrt{2}}\,(E_\alpha + E_{-\alpha} + E_\beta + E_{-\beta}),\\
L_2 &= \frac{1}{\sqrt{2}}\,(E_\alpha - E_{-\alpha} + E_\beta - E_{-\beta}),\\
A_{23} &= \frac{\gamma}{\sqrt{2}}\,(E_\alpha - E_{-\alpha} - E_\beta + E_{-\beta}),\\
A_{31} &= \frac{-i\gamma}{\sqrt{2}}\,(E_\alpha + E_{-\alpha} - E_\beta - E_{-\beta}),\\
A_1 &= i\gamma\,(E_{\alpha+\beta} + E_{-(\alpha+\beta)}), \qquad A_{12} = -\gamma\,(E_{\alpha+\beta} - E_{-(\alpha+\beta)}).
\end{aligned} \tag{53}
$$

2.2 Determination of the Representations

Much of the discussion in this and subsequent sections is concerned with the formal theory of SU(3) and many of the results thereof will be stated without proof, the interested reader being referred to Behrends, Dreitlein, Fronsdal, Lee, (1962); Dynkin, (1957 b); Racah, (1951); Weyl, (1925).

We assume the following facts:

(i) The eigenvectors of the oscillator Hamiltonian, for a given eigenvalue ζ_N, span a vector space Ω_N of dimension $\frac{1}{2}(N+1)(N+2)$ $(N=0,1,2,\ldots)$. (54)

(ii) Ω_N decomposes into spaces ω_ℓ, each of dimension $(2\ell+1)$ and irreducible with respect to SO(3), where

$$\ell = \begin{cases} N, N-2, \ldots, 0 & \text{if N is even} \\ N, N-2, \ldots, 1 & \text{if N is odd}. \end{cases} \tag{55}$$

We require an irreducible representation

$$\Lambda \; : \mathfrak{a} \ni f \longrightarrow \Lambda f \in \mathcal{P} \subset L(\Omega_N ; \Omega_N); \tag{56}$$

for convenience, Λf will also be denoted by f.

THEOREM 1

The dimensions, s, of the irreducible representations of SU(3) are given by

$$s = \frac{1}{2}(j+1)(k+1)(j+k+2) \qquad (j,k=0,1,2,\ldots). \tag{57}$$

THEOREM 2

$\exists$ a set $\mathcal{V}$ of linearly independent simultaneous eigenvectors of H_1, H_2 such that

(i) the eigenvalue of each $v \in \mathcal{V}$ is real,

(ii) $\mathcal{V}$ is a basis of Ω_N.

DEFINITION

1. Let $v \in \mathcal{V}$ and $H_i v = \chi_i v$ (i=1,2). Then v is said to be a vector of weight

$$\chi = (\chi_1, \chi_2) \in \mathcal{R}_2 . \tag{58}$$

2. χ is said to be positive if either

$$\chi_1 > 0$$

or

$$\chi_1 = 0, \quad \chi_2 > 0. \tag{59}$$

3. $\chi^{(1)}$ is said to be greater than $\chi^{(2)}$ if $\chi^{(1)} - \chi^{(2)}$ is positive.

THEOREM 3

$\chi^{(0,0)} \in \mathcal{R}_2$ is the greatest weight of the irreducible representation (j,k) of SU(3) iff

$$\frac{2\langle \chi^{(0,0)}, \alpha\rangle}{\langle\alpha,\alpha\rangle} = j, \qquad \frac{2\langle \chi^{(0,0)}, \beta\rangle}{\langle\beta,\beta\rangle} = k. \tag{60}$$

Using (46), (60) gives

$$\chi^{(0,0)} = \frac{1}{3}(2j+k)\alpha + \frac{1}{3}(j+2k)\beta \tag{61}$$

as the greatest weight of the irreducible representation (j,k).

THEOREM 4

If χ is a weight, then

$$\chi - \alpha, \ \chi - \beta$$

are weights iff

$$\frac{2\langle \chi, \alpha\rangle}{\langle\alpha,\alpha\rangle} + Q(\chi, \alpha) > 0,$$
$$\frac{2\langle \chi, \beta\rangle}{\langle\beta,\beta\rangle} + Q(\chi, \beta) > 0,$$

where $\chi + Q(\chi, \alpha)\alpha$, $\chi + Q(\chi, \beta)\beta$ are weights, while $\chi + [Q(\chi, \alpha) + 1]\alpha$, $\chi + [Q(\chi, \beta) + 1]\beta$ are not, respectively.

With the values (50), Theorem 4 shows that, if χ is an arbitrary weight, then

$$\chi_1^{(0,0)} \geqslant \chi_1, \quad \chi_2^{(0,0)} \geqslant \chi_2. \tag{62}$$

For any vector $v \in \mathcal{V}$, of weight χ, we have

$$(H_1 + H_2)v = \sigma v, \tag{63}$$

where

$$\sigma = \chi_1 + \chi_2 \leq \chi_1^{(0,0)} + \chi_2^{(0,0)} = j + k,$$

from (61) and (50). Hence, from (53),

$$L_3 v = i\sigma v, \tag{64}$$

where

$$\sigma \leq j + k. \tag{65}$$

Now any $v \in \mathcal{V}$ can be written

$$v = \sum_{\ell,m} c^{\ell m} \psi_{\ell m}, \tag{66}$$

where $\psi_{\ell m}$ ($m=-\ell, -\ell+1, \ldots, \ell$) are the basic elements of $\mathcal{W}_\ell$, and the summation over ℓ is given by (55). A representation of the subalgebra SO(3) spanned by L_1, L_2, L_3 can be defined in which

$$L_3 \psi_{\ell m} = -im\, \psi_{\ell m}, \tag{67}$$

and therefore

$$\begin{aligned} L_3 v &= \sum_{\ell,m} -im\, c^{\ell m} \psi_{\ell m} \\ &= i\sigma \sum_{\ell,m} c^{\ell m} \psi_{\ell m}, \end{aligned}$$

from (66) and (64). Thus $c^{\ell m}$ is non-zero only if $m = -\sigma$. Consequently, $\max(\sigma) = -\min(m) = N$, from (55). (65) now gives

$$j + k = N, \tag{68}$$

and, together with (57) and (54), we deduce that

$$j = 0, \quad k = N \tag{69}$$

or

$$j = N, \quad k = 0. \tag{70}$$

The next section will be concerned with the calculation of the irreducible representations of type (N,0).

2.3 Calculation of the Representations

When j = N, k = 0, the weights can be calculated from Theorem 4. They are

$$\begin{array}{l}
\chi^{(0,0)}, \\
\chi^{(0,0)}-\alpha,\ \chi^{(0,0)}-\alpha-\beta, \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\chi^{(0,0)}-\mu\alpha,\ \ldots,\ \chi^{(0,0)}-\mu\alpha-\nu\beta,\ \ldots\ldots,\ \chi^{(0,0)}-\mu\alpha-\mu\beta, \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\chi^{(0,0)}-N\alpha,\ \chi^{(0,0)}-N\alpha-\beta,\ \ldots\ldots\ldots,\ \chi^{(0,0)}-N\alpha-N\beta,
\end{array}$$

or

$$\chi^{(\mu,\nu)} = \chi^{(0,0)}-\mu\alpha-\nu\beta, \tag{71}$$

$(\mu = 0,1,2,\ldots,N;\ \nu = 0,1,2,\ldots,\mu)$.

It is easily seen that there are $\frac{1}{2}(N+1)(N+2)$ of these, there consequently being no multiplicity of weights and hence a 1-1 correspondence between the set of weights and $\mathcal{V}$. The element of $\mathcal{V}$ of weight $\chi^{(\mu,\nu)}$ will be denoted by $v_{\mu\nu}$, the operators H_1, H_2, $E_{\pm\beta}^{\alpha}$ being defined by

$$\begin{aligned}
H_1 v_{\mu\nu} &= \tfrac{1}{3}(2N - 3\mu)v_{\mu\nu}, \\
H_2 v_{\mu\nu} &= \tfrac{1}{3}(N - 3\nu)v_{\mu\nu}, \\
E_{-\alpha} v_{\mu\nu} &= v_{\mu+1,\nu}, \\
E_{-\beta} v_{\mu\nu} &= v_{\mu,\nu+1}, \\
E_{\alpha} v_{\mu\nu} &= R_{\mu\nu} v_{\mu-1,\nu}, \\
E_{\beta} v_{\mu\nu} &= S_{\mu\nu} v_{\mu,\nu-1}.
\end{aligned} \tag{72}$$

$R_{\mu\nu}$ arises and is calculated as follows:

Assume $E_\alpha v_{\mu\nu} = R_{\mu\nu} v_{\mu-1,\nu}$. Then

$$\begin{aligned} E_\alpha v_{\mu\nu} &= E_\alpha E_{-\alpha} v_{\mu-1,\nu} \\ &= ([E_\alpha, E_{-\alpha}] + E_{-\alpha} E_\alpha) v_{\mu-1,\nu} \\ &= (2H_1 - H_2 + E_{-\alpha} E_\alpha) v_{\mu-1,\nu} \\ &= [\tfrac{2}{3}(2N - 3(\mu-1)) - \tfrac{1}{3}(N - 3\nu) + R_{\mu-1,\nu}] v_{\mu-1,\nu} \end{aligned}$$

where we have used (72) and (51). Thus

$$R_{\mu\nu} = R_{\mu-1,\nu} + N - 2\mu + \nu + 2. \tag{73}$$

(73) gives a difference equation whereby $R_{\mu\nu}$ may be calculated if $R_{\nu\nu}$ is known. Iteration gives

$$R_{\mu\nu} = R_{\nu\nu} + (\mu - \nu)(N - \mu + 1). \tag{74}$$

Since, from (71), $\chi^{(0,0)} - (\nu - 1)\alpha - \nu\beta$ is not a weight, we must have $E_\alpha v_{\nu\nu} = 0$: hence $R_{\nu\nu} = 0$. Thus

$$R_{\mu\nu} = (\mu - \nu)(N - \mu + 1). \tag{75}$$

An analogous argument gives

$$S_{\mu\nu} = \nu(\mu - \nu + 1). \tag{76}$$

To effect normalization of the basic vectors, we make use of the fact that

$$E_{\substack{\alpha \\ \beta}}^{+} = E_{-\substack{\alpha \\ \beta}}, \tag{77}$$

where X^+ is the adjoint of X with respect to the inner product $\langle / \rangle$ defined in Ω_N, and proceed as follows:

Let

$$\tilde{\Xi}^N_{\mu\nu} = \lambda_{\mu\nu} v_{\mu\nu} \tag{78}$$

be normalized; then $\lambda_{\mu\nu}$ is real ($\lambda_{\mu\nu}^{-1}$ is just the length of $v_{\mu\nu}$).

We have

$$\langle E_\alpha v_{\mu\nu} / v_{\mu-1,\nu}\rangle = \langle v_{\mu\nu} / E_{-\alpha} v_{\mu-1,\nu}\rangle ,$$

i.e. $$R_{\mu\nu} \langle v_{\mu-1,\nu} / v_{\mu-1,\nu}\rangle = \langle v_{\mu\nu} / v_{\mu\nu}\rangle ,$$

from (77) and (72). Thus

$$|\lambda_{\mu\nu}| = \frac{|\lambda_{\mu-1,\nu}|}{\sqrt{R_{\mu\nu}}} . \tag{79}$$

Similarly,

$$|\lambda_{\mu\nu}| = \frac{|\lambda_{\mu,\nu-1}|}{\sqrt{S_{\mu\nu}}} . \tag{80}$$

With respect to this normalized basis, the operators are defined, from (72), by

$$\begin{aligned}
H_1 \, \tilde{\Xi}^{N}_{\mu\nu} &= \tfrac{1}{3}(2N-3\mu)\, \tilde{\Xi}^{N}_{\mu\nu} ,\\
H_2 \, \tilde{\Xi}^{N}_{\mu\nu} &= \tfrac{1}{3}(N-3\nu)\, \tilde{\Xi}^{N}_{\mu\nu} ,\\
E_\alpha \, \tilde{\Xi}^{N}_{\mu\nu} &= \sqrt{(\mu-\nu)(N-\mu+1)}\; \tilde{\Xi}^{N}_{\mu-1,\nu} ,\\
E_{-\alpha} \, \tilde{\Xi}^{N}_{\mu\nu} &= \sqrt{(\mu-\nu+1)(N-\mu)}\; \tilde{\Xi}^{N}_{\mu+1,\nu} ,\\
E_\beta \, \tilde{\Xi}^{N}_{\mu\nu} &= \sqrt{\nu(\mu-\nu+1)}\; \tilde{\Xi}^{N}_{\mu,\nu-1} ,\\
E_{-\beta} \, \tilde{\Xi}^{N}_{\mu\nu} &= \sqrt{(\nu+1)(\mu-\nu)}\; \tilde{\Xi}^{N}_{\mu,\nu+1} ,\\
E_{\alpha+\beta} \, \tilde{\Xi}^{N}_{\mu\nu} &= \sqrt{\nu(N-\mu+1)}\; \tilde{\Xi}^{N}_{\mu-1,\nu-1} ,\\
E_{-(\alpha+\beta)} \, \tilde{\Xi}^{N}_{\mu\nu} &= \sqrt{(\nu+1)(N-\mu)}\; \tilde{\Xi}^{N}_{\mu+1,\nu+1}
\end{aligned} \tag{81}$$

($\mu = 0,1,2,\ldots,N$; $\nu = 0,1,2,\ldots,\mu$), and we have used (75), (76) and (51).

We shall see that (81) contains all we need to construct explicitly the square-integrable functions $\tilde{\Xi}^{N}_{\mu\nu}$.

2.4 A Casimir Operator

A direct application of (81) shows that

$$J \, \overset{\sim N}{\omega}_{\mu\nu} = \frac{2N}{3}(N + 3) \, \overset{\sim N}{\omega}_{\mu\nu}, \tag{82}$$

where

$$J = E_\alpha E_{-\alpha} + E_{-\alpha} E_\alpha + E_\beta E_{-\beta} + E_{-\beta} E_\beta + E_{\alpha+\beta} E_{-(\alpha+\beta)}$$
$$+ E_{-(\alpha+\beta)} E_{\alpha+\beta} + 2H_1^2 + 2H_2^2 - 2H_1 H_2. \tag{83}$$

Using the defining relations (52), we obtain

$$J = \frac{-1}{2\gamma^2}[(3A_0^2 + A_1^2 + A_{12}^2 + A_{23}^2 + A_{31}^2) + \gamma^2(L_1^2 + L_2^2 + L_3^2)]. \tag{84}$$

From (23), we find that

$$3A_0^2 + A_1^2 = \frac{1}{3}D^2 - (A_{11} A_{22} + A_{22} A_{33} + A_{33} A_{11}). \tag{85}$$

The quantities A_{ij}, L_i (i,j=1,2,3) may be expressed as differential operators:

We have, from (II.4),

$$A_{ij} = \frac{\partial^2}{\partial x_i x_j} + \gamma^2 x_i x_j \qquad (i,j=1,2,3) \tag{86}$$

The expressions for L_i (i=1,2,3) may be obtained by means of the second procedure of quantization given in II.1. The result is

$$L_i = x_k \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_k} \qquad ((i,j,k) \text{ in cyclic order}) \tag{87}$$

A_{ij}, L_i are defined on the space of differentiable functions of the complex variables x_1, x_2, x_3.

With (86) and (87), the following identities result:

$$A_{12}^2 + A_{23}^2 + A_{31}^2 = \frac{\partial^4}{\partial x_1^2 x_2^2} + \frac{\partial^4}{\partial x_2^2 x_3^2} + \frac{\partial^4}{\partial x_3^2 x_1^2}$$
$$+ 2\gamma^2\left(x_1 x_2 \frac{\partial^2}{\partial x_1 x_2} + x_2 x_3 \frac{\partial^2}{\partial x_2 x_3} + x_3 x_1 \frac{\partial^2}{\partial x_3 x_1}\right)$$
$$+ 2\gamma^2\left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}\right) + \gamma^4(x_1^2 x_2^2 + x_2^2 x_3^2 + x_3^2 x_1^2)$$
$$+ 3\gamma^2, \tag{88}$$

$$A_{11}A_{22} + A_{22}A_{33} + A_{33}A_{11} = \frac{\partial^4}{\partial x_1^2 x_2^2} + \frac{\partial^4}{\partial x_2^2 x_3^2} + \frac{\partial^4}{\partial x_3^2 x_1^2}$$

$$+ \gamma^2 x^2\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right) - \gamma^2\left(x_1^2\frac{\partial^2}{\partial x_1^2} + x_2^2\frac{\partial^2}{\partial x_2^2} + x_3^2\frac{\partial^2}{\partial x_3^2}\right)$$

$$+ \gamma^4\left(x_1^2x_2^2 + x_2^2x_3^2 + x_3^2x_1^2\right), \qquad (89)$$

$$L^2 = L_1^2 + L_2^2 + L_3^2 = x^2\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)$$

$$- \left(x_1^2\frac{\partial^2}{\partial x_1^2} + x_2^2\frac{\partial^2}{\partial x_2^2} + x_3^2\frac{\partial^2}{\partial x_3^2}\right) - 2\left(x_1x_2\frac{\partial^2}{\partial x_1 x_2} + x_2x_3\frac{\partial^2}{\partial x_2 x_3} + x_3x_1\frac{\partial^2}{\partial x_3 x_1}\right)$$

$$- 2\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}\right). \qquad (90)$$

The last three equations and (85) give, from (84),

$$J = \frac{-1}{6\gamma^2} D^2 - \frac{3}{2}, \qquad (91)$$

or

$$D^2 = -3\gamma^2(2J + 3). \qquad (92)$$

We may now write our fundamental differential equation (82) as

$$[D^2 + (2N + 3)^2\gamma^2]\, \tilde{\omega}^N_{\mu\nu} = 0. \qquad (93)$$

Using the first two equations of (53) and (81) and the expression for A_0, we have

$$6A_0 \tilde{\omega}^N_{\mu\nu} \equiv (2A_{33} - A_{11} - A_{22})\tilde{\omega}^N_{\mu\nu} = 2i(N - 3\mu + 3\nu)\gamma\, \tilde{\omega}^N_{\mu\nu}, \qquad (94)$$

and

$$L_3 \tilde{\omega}^N_{\mu\nu} = i(N - \mu - \nu)\tilde{\omega}^N_{\mu\nu}. \qquad (95)$$

At this stage, it is interesting to note that, since

$$D = 2\pi H, \tag{96}$$

the square of the Hamiltonian is directly related to the Casimir operator J of SU(3), the equation being (92).

The analysis below is concerned with finding the simultaneous solutions $\tilde{\omega}^{N}_{\mu\nu}$ of equations (93), (94) and (95) above.

2.5 The Wave Functions

Write

$$\tilde{\omega}^{N}_{\mu\nu}(x_1, x_2, x_3) = \Theta^{N}_{\mu\nu}(x_1, x_2)\, F^{N}_{\mu\nu}(x_3). \tag{97}$$

(94) becomes

$$\left\{2\left(\frac{\partial^2}{\partial x_3^2} + \gamma^2 x_3^2\right) - \left[\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \gamma^2(x_1^2 + x_2^2)\right]\right\} \tilde{\omega}^{N}_{\mu\nu}$$
$$= 2i(N - 3\mu + 3\nu)\gamma\, \tilde{\omega}^{N}_{\mu\nu}$$

or

$$2\,\Theta^{N}_{\mu\nu}\left(\frac{d^2}{dx_3^2} + \gamma^2 x_3^2\right)F^{N}_{\mu\nu} - F^{N}_{\mu\nu}\left[\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \gamma^2(x_1^2 + x_2^2)\right]\Theta^{N}_{\mu\nu}$$
$$= 2i(N - 3\mu + 3\nu)\gamma\,\Theta^{N}_{\mu\nu} F^{N}_{\mu\nu}. \tag{98}$$

The last equation, (98), can be separated to give

$$\left(\frac{d^2}{dx_3^2} + \gamma^2 x_3^2\right) F^{N}_{\mu\nu} = i\gamma b\, F^{N}_{\mu\nu}, \tag{99}$$

$$\left[\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \gamma^2(x_1^2 + x_2^2)\right]\Theta^{N}_{\mu\nu} = i\gamma a\, \Theta^{N}_{\mu\nu}, \tag{100}$$

with

$$2b - a = 2(N - 3\mu + 3\nu). \tag{101}$$

Since

$$D = 3A_{33} - 6A_0,$$

from (23), (93) can be written

$$[(3A_{33} - 6A_0)^2 + (2N+3)^2\gamma^2]\,\tilde{\Xi}^N_{\mu\nu} = 0,$$

which becomes, on observing that

$$\begin{aligned} A_{33}\tilde{\Xi}^N_{\mu\nu} &= A_{33}\,\Theta^N_{\mu\nu}F^N_{\mu\nu} = \Theta^N_{\mu\nu}A_{33}F^N_{\mu\nu} \\ &= ib\gamma\,\Theta^N_{\mu\nu}F^N_{\mu\nu} = ib\gamma\,\tilde{\Xi}^N_{\mu\nu}, \end{aligned}$$

from (99), and that

$$A_0\tilde{\Xi}^N_{\mu\nu} = \frac{i}{3}(N - 3\mu + 3\nu)\gamma\,\tilde{\Xi}^N_{\mu\nu},$$

from (94),

$$9b^2 - 12(N-3\mu+3\nu)b + 4(N-3\mu+3\nu)^2 - (2N+3)^2 = 0, \quad (102)$$

i.e.

$$b = \begin{cases} (2\nu - 2\mu - 1) & = b' \\ \frac{1}{3}(4N - 6\mu + 6\nu + 3) & = b'', \end{cases} \quad (103)$$

giving for the respective values of a, from (101),

$$a = \begin{cases} 2(\mu - \nu - N - 1) & = a' \\ \frac{2}{3}(N + 3\mu - 3\nu + 3) & = a''. \end{cases} \quad (104)$$

We thus have two linearly independent solutions which we shall denote by

$${}'\tilde{\Xi}^N_{\mu\nu} = {}'\Theta^N_{\mu\nu}\,{}'F^N_{\mu\nu} \qquad \text{, with } b = b',\ a = a',$$

and

$${}''\tilde{\Xi}^N_{\mu\nu} = {}''\Theta^N_{\mu\nu}\,{}''F^N_{\mu\nu} \qquad \text{, with } b = b'',\ a = a''.$$

When we do not wish to specify the solution with which we are dealing, we shall simply omit all primes, as in (97).

Write

$$\Theta^N_{\mu\nu} = P^N_{\mu\nu}(\rho)\,\Phi^N_{\mu\nu}(\varphi), \quad (105)$$

where

$$x_1 = \rho\cos\varphi, \qquad x_2 = \rho\sin\varphi. \quad (106)$$

Now

$$\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2} = \frac{\partial^2}{\partial \rho^2}+\frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2}+\frac{1}{\rho}\frac{\partial}{\partial \rho},$$

so that (100) gives

$$\left(\frac{\partial^2}{\partial \rho^2}+\frac{1}{\rho^2}\frac{\partial^2}{\partial \varphi^2}+\frac{1}{\rho}\frac{\partial}{\partial \rho}+\gamma^2\rho^2\right)P_{\mu\nu}^N\Phi_{\mu\nu}^N = ia\gamma P_{\mu\nu}^N\Phi_{\mu\nu}^N. \qquad (107)$$

We have

$$L_3 = x_2\frac{\partial}{\partial x_1}-x_1\frac{\partial}{\partial x_2} = -\frac{\partial}{\partial \varphi},$$

so that (95) reads

$$\frac{\partial \tilde{\Xi}_{\mu\nu}^N}{\partial \varphi} = i(\mu+\nu-N)\tilde{\Xi}_{\mu\nu}^N$$

or

$$\frac{d\Phi_{\mu\nu}^N}{d\varphi} = i(\mu+\nu-N)\Phi_{\mu\nu}^N; \qquad (108)$$

with this, (107) becomes

$$\left(\rho^2\frac{d^2}{d\rho^2}+\rho\frac{d}{d\rho}+\gamma^2\rho^4\right)P_{\mu\nu}^N = \left[ia+(\mu+\nu-N)^2\right]P_{\mu\nu}^N, \qquad (109)$$

the solution of (108) being

$$\Phi_{\mu\nu}^N = C_3\, e^{i(\mu+\nu-N)\varphi}, \qquad (110)$$

C_3 being a constant.

Our equations have now been reduced to (99), (109) and (110). We shall first solve (99):

Write

$$z = i\gamma x_3^2, \qquad (111)$$

to give

$$\left(4z\frac{d^2}{dz^2}+2\frac{d}{dz}-z\right)F_{\mu\nu}^N = bF_{\mu\nu}^N, \qquad (112)$$

which, with the substitution

$$F_{\mu\nu}^{N} = e^{-z/2}\, u_{\mu\nu}^{N}(z)\,, \tag{113}$$

becomes

$$\left[z\frac{d^2}{dz^2} + \left(\frac{1}{2} - z\right)\frac{d}{dz} - \frac{1}{4}(\bar{b}+1)\right]u_{\mu\nu}^{N} = 0\ . \tag{114}$$

(114) is recognized as Kummer's equation, whose general solution is given in terms of confluent hypergeometric functions as (H.M.F.,1965,504)

$$u_{\mu\nu}^{N} = c_1 M\left[\tfrac{1}{4}(b+1), \tfrac{1}{2}; z\right] + c_2\, U\left[\tfrac{1}{4}(b+1), \tfrac{1}{2}; z\right]\ . \tag{115}$$

We require $F_{\mu\nu}^{N}$ to be square-integrable along some infinite curve in the complex plane. Consequently, we investigate the behaviour of $u_{\mu\nu}^{N}$ as $|z| \to \infty$.

For $|z|$ large,

$$\begin{aligned} M(\alpha,\beta;z) &= \frac{e^{\pm i\pi\alpha}\Gamma(\beta)}{\Gamma(\beta-\alpha)}\, z^{-\alpha}\left[1 + O\left(\frac{1}{|z|}\right)\right] \\ &\quad + \frac{\Gamma(\beta)}{\Gamma(\alpha)}\, e^{z} z^{\alpha-\beta}\left[1 + O\left(\frac{1}{|z|}\right)\right], \end{aligned} \tag{116}$$

$$U(\alpha,\beta;z) = z^{-\alpha}\left[1 + O\left(\frac{1}{|z|}\right)\right],$$

$$\left(\pm \quad \text{for} \begin{cases} -\pi/2 < \arg z < 3\pi/2 \\ -3\pi/2 < \arg z \leq -\pi/2 \end{cases}\right)$$

so that

$$F_{\mu\nu}^{N} \to c_2\, e^{-iy_2/2}\, e^{-y_1/2}\, (y_1 + iy_2)^{-\frac{1}{4}(b+1)}$$

$$+ \; c_1 \left[\frac{e^{\pm \frac{i\pi}{4}(b+1)} \Gamma(\frac{1}{2})}{\Gamma[\frac{1}{4}(1-b)]} \, e^{-iy_2/2} \, e^{-y_1/2} (y_1 + iy_2)^{-\frac{1}{4}(b+1)} \right.$$

$$\left. + \frac{\Gamma(\frac{1}{2})}{\Gamma[\frac{1}{4}(b+1)]} \, e^{iy_2/2} \, e^{y_1/2} (y_1 + iy_2)^{\frac{1}{4}(b-1)} \right] \tag{117}$$

where $z \to y_1 + iy_2$ and y_1 and y_2 are real, as $|z| \to \infty$.

We take the two cases separately:

(i) $b = b'$; then $b' \leq -1$.

If $y_1 = 0$, we must have

$$c_2 = - \frac{e^{\pm \frac{i\pi}{4}(b+1)} \Gamma(\frac{1}{2})}{\Gamma[\frac{1}{4}(1-b)]} \, c_1$$

in order to make the offending terms vanish. Further, for $b' = -1$, $F^N_{\mu\nu} \overline{F^N_{\mu\nu}}$ varies as y_2^{-1} for $|z|$ large and its integral therefore diverges. Hence we must have $c_1 = 0$.

If $y_1 < 0$, we must have

$$c_2 = - \frac{e^{\pm \frac{i\pi}{4}(b+1)} \Gamma(\frac{1}{2})}{\Gamma[\frac{1}{4}(1-b)]} \, c_1 .$$

If $y_1 > 0$, the only offending term is that which has $\Gamma\,[\frac{1}{4}(b+1)]$ in its denominator. From (103) we see that, in general, $\Gamma\,[\frac{1}{4}(b+1)]$ is not infinite and hence the offending term can be made to vanish only by making c_1 zero.

(ii) $b = b''$.

If $y_1 = 0$, then since b'' can take arbitrary large positive and negative values, it is impossible for $F^N_{\mu\nu}$ to remain finite at infinity for all b''.

If $y_1 < 0$, we must have

$$c_2 = \frac{-e^{\pm \frac{i\pi}{4}(b+1)} \Gamma(\tfrac{1}{2})}{\Gamma[\tfrac{1}{4}(1-b)]} c_1$$

If $y_1 > 0$, we must have $c_1 = 0$.

For $|z|$ small,

$$M(\alpha, \beta; 0) = 1; \qquad (\beta \text{ not equal to a negative integer or zero})$$

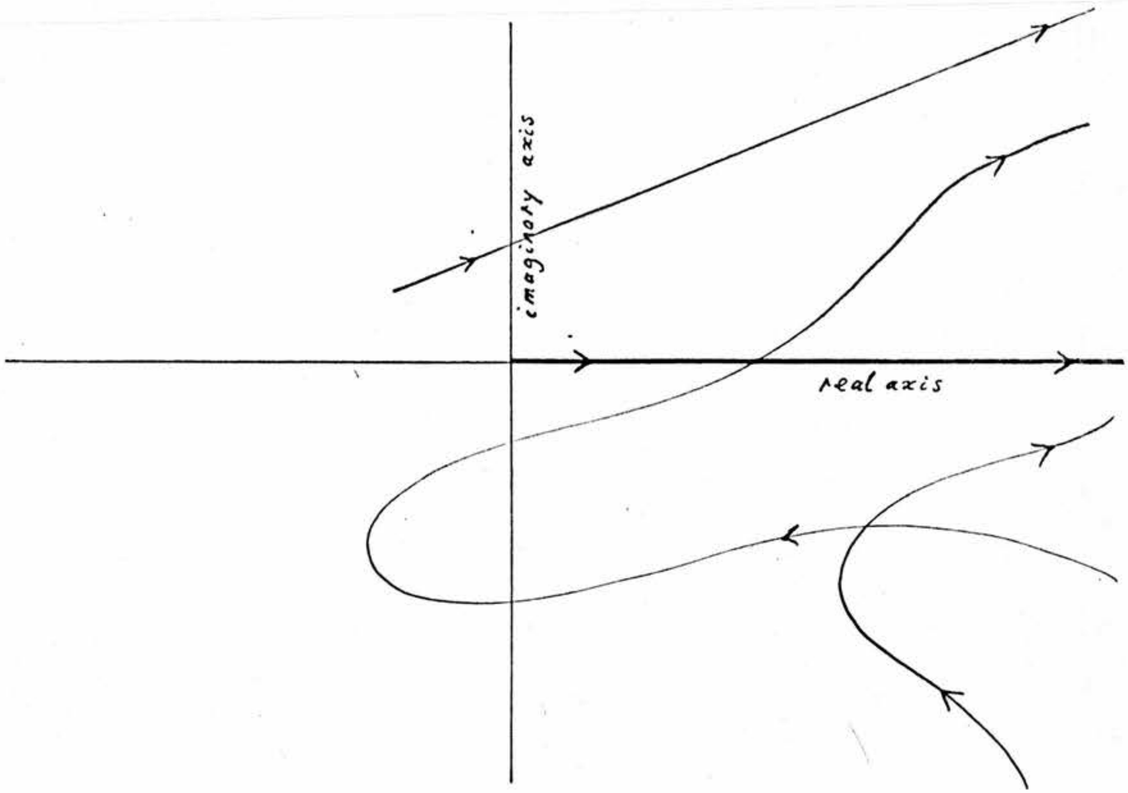
$$U(\alpha, \tfrac{1}{2}; z) = \frac{\Gamma(\frac{1}{2})}{\Gamma(\alpha + \frac{1}{2})} + O(|z|^{1/2}) \qquad (118)$$

Hence $F^N_{\mu\nu}$ remains finite as $|z| \to 0$ for all c_1, c_2.

From the preceding argument we conclude that for $F^N_{\mu\nu}$ to be square-integrable along some infinite curve in the complex plane, it is necessary that $R(z) \neq 0$ as $|z| \to \infty$ and that

$$u^N_{\mu\nu} = \begin{cases} c_1 \left\{ M[\tfrac{1}{4}(b+1), \tfrac{1}{2}; z] - \dfrac{e^{\frac{i\pi}{4}(b+1)}\sqrt{\pi}}{\Gamma[\frac{1}{4}(1-b)]} U[\tfrac{1}{4}(b+1), \tfrac{1}{2}; z] \right\} \\ \qquad\qquad\qquad (\text{for } \mathcal{R}(z) < 0), \\ c_2\, U[\tfrac{1}{4}(b+1), \tfrac{1}{2}; z] \quad (\text{for } \mathcal{R}(z) > 0). \end{cases} \qquad (119)$$

To obtain a single functional element for all z, we must restrict both end points of the curve (when at infinity) to be in either the positive or negative half-planes. The latter is ruled out by the fact that the first solution of (119) continually vanishes when b = -1, this value being taken by both b' and b'' — $\{ \Xi^N_{\mu\nu} \}$ cannot then be taken as a basis of Ω_N. The acceptable curve may be of the form of any of those shown below:



The simplest is that which coincides with the non-negative real axis. Hence write

$$z = \gamma q_3^2/\hbar , \tag{120}$$

where q_3 is real; i.e., from (111),

$$x_3 = \frac{(1 - i)}{\sqrt{2\hbar}} q_3 \quad . \tag{121}$$

Finally,

$$F_{\mu\nu}^{N}(q_3) = C_2 e^{-\gamma q_3^2/2\hbar} U\left[\tfrac{1}{4}(b+1), \tfrac{1}{2}; \gamma q_3^2/\hbar\right] . \tag{122}$$

We next solve (109):

Write

$$\eta^2 = \gamma\rho^2 , \tag{123}$$

to give

$$\left\{\eta^2 \frac{d^2}{d\eta^2} + \eta\frac{d}{d\eta} - \left[(\mu+\nu-N)^2 + i a\eta^2 - \eta^4\right]\right\} P_{\mu\nu}^{N} = 0 , \tag{124}$$

which, with the substitution

$$P_{\mu\nu}^{N} = \frac{1}{\eta} w_{\mu\nu}^{N}(\xi), \tag{125}$$

$$\xi = i\eta^{2}, \tag{126}$$

becomes

$$\left\{4\xi^{2}\frac{d^{2}}{d\xi^{2}} + \left[1-(\mu+\nu-N)^{2} - a\xi - \xi^{2}\right]\right\} \omega_{\mu\nu}^{N} = 0. \tag{127}$$

Write

$$\omega_{\mu\nu}^{N} = \xi^{\frac{1}{2}(|m|+1)} e^{-\xi/2} f_{\mu\nu}^{N}(\xi), \tag{128}$$

where

$$m = \mu + \nu - N, \tag{129}$$

to give

$$\left[\xi\frac{d^{2}}{d\xi^{2}} + (|m|+1-\xi)\frac{d}{d\xi} - s\right] f_{\mu\nu}^{N} = 0, \tag{130}$$

where

$$s = \frac{1}{2}\left(|m| + 1 + \frac{a}{2}\right), \tag{131}$$

i.e., from (129) and (104),

$$s' = \begin{cases} -\nu & \text{if } m \leqslant 0 \\ -(N-\mu) & \text{if } m \geqslant 0 \end{cases}$$

$$s'' = \begin{cases} \frac{1}{3}(2N - 3\nu + 3) & \text{if } m \leqslant 0 \\ \frac{1}{3}(3\mu - N + 3) & \text{if } m \geqslant 0. \end{cases} \tag{132}$$

The general solution of (130) is

$$f_{\mu\nu}^{N}(\xi) = c_1 M(s,|m|+1;\xi) + c_2 U(s,|m|+1;\xi). \tag{133}$$

Writing

$$x_1 = \frac{(1-i)}{\sqrt{2k}} q_1, \quad x_2 = \frac{(1-i)}{\sqrt{2k}} q_2; \quad \rho^{2} = q_1^{2} + q_2^{2}, \quad \xi = \gamma\rho^{2}/k, \tag{134}$$

in analogy with (121), we require

$$\int_0^\infty P_{\mu\nu}^N \bar{P}_{\mu\nu}^N \rho \, d\rho$$

to be finite. We are thus led to question the convergence of

$$\rho P_{\mu\nu}^N \bar{P}_{\mu\nu}^N = \left(\frac{\gamma}{\hbar}\right)^{|m|} \rho^{2|m|+1} e^{-\gamma\rho^2/\hbar} f_{\mu\nu}^N \bar{f}_{\mu\nu}^N \tag{135}$$

as $\rho \to 0$ and $\rho \to \infty$.

A procedure analogous to that which led to (119) can be carried out for the upper limit for $s = s''$ to give $c_1 = 0$. For $s = s'$, we note that the offending term in $M(s', |m|+1; \gamma\rho^2/\hbar)$ as $\rho \to \infty$ does not appear: it vanishes because of an infinite denominator. Thus both c_1 and c_2 are arbitrary in this case as far as convergence at infinity is concerned. With regard to the lower limit, let us make the following expansion:

For ξ small, $|m| > 1$,

$$\mathcal{U}(s, |m|+1; \xi) = \frac{\Gamma(|m|)}{\Gamma(s)} \xi^{-|m|} + O(\xi^{|m|-1}), \tag{136}$$

while

$$M(s, |m|+1; \xi) \to 1 \text{ as } \xi \to 0.$$

(135) shows that

$$\rho P_{\mu\nu}^N \bar{P}_{\mu\nu}^N \to \left(\frac{\gamma}{\hbar}\right)^{|m|} \rho^{2|m|+1} e^{-\gamma\rho^2/\hbar} \left[c_1^2 + c_2^2 \left[\frac{\Gamma(|m|)}{\Gamma(s)}\right]^2 \left(\frac{\hbar}{\gamma}\right)^{2|m|} \rho^{-4|m|} + \frac{1}{\Gamma(s)} O(\rho^{-2|m|}) \right]$$

as $\rho \to 0$.

Immediately we deduce that the integral exists iff $s = s'$, for only then is $1/\Gamma(s)$ zero.

For ζ small and $|m| = 1$,

$$U(s',2;\zeta) = O(|\ln \zeta|)$$

and hence is not well defined as $\zeta \to 0$; the same applies for $|m|=0$. Hence we write $c_2 = 0$.

Summarizing, then, the only functions $\tilde{\omega}^N_{\mu\nu}(\rho, q_3, \varphi)$ such that

$$\int_0^\infty \int_{-\infty}^\infty \int_0^{2\pi} \tilde{\omega}^N_{\mu\nu} \, \bar{\tilde{\omega}}^N_{\mu\nu} \, \rho \, d\rho \, dq_3 \, d\varphi$$

exists are given by

$$'\tilde{\omega}^N_{\mu\nu}(\rho, q_3, \varphi) = {}'P^N_{\mu\nu}(\rho) \, {}'F^N_{\mu\nu}(q_3) \, \Phi^N_{\mu\nu}(\varphi), \tag{137}$$

where

$$'P^N_{\mu\nu}(\rho) = c_1 \rho^{|m|} e^{-\gamma\rho^2/2\hbar} M(-n, |m|+1; \gamma\rho^2/\hbar), \tag{138}$$

$$'F^N_{\mu\nu}(q_3) = c_2 \, e^{-\gamma q_3^2/2\hbar} \, U\left[-\tfrac{1}{2}(\mu-\nu), \tfrac{1}{2}; \gamma q_3^2/\hbar\right], \tag{139}$$

$$\Phi^N_{\mu\nu}(\varphi) = c_3 \, e^{im\varphi}, \tag{140}$$

with

$$m = \mu + \nu - N, \tag{141}$$

$$n = \min(\nu, N-\mu), \tag{142}$$

where we have used (125), (128), (133), (126), (134), (132), (122), (103).

Henceforth we shall drop the primes and write $M(-n, |m|+1; \gamma\rho^2/\hbar)$ and $U[-\frac{1}{2}(\mu-\nu), \frac{1}{2}; \gamma q_3^2/\hbar]$ in terms of more familiar functions:

$$M(-n, |m|+1; \gamma\rho^2/\hbar) = \frac{n!\,|m|!}{(|m|+n)!} L_n^{|m|}\left(\frac{\gamma\rho^2}{\hbar}\right), \tag{143}$$

where

$$L_n^{\alpha}(x) = \frac{e^x x^{-\alpha}}{n!} \frac{d^n}{dx^n}\left(e^{-x} x^{\alpha} x^n\right) \tag{144}$$

is the Generalized Laguerre polynomial.

$$U\left[-\tfrac{1}{2}(\mu-\nu), \tfrac{1}{2}; \gamma q_3^2/\hbar\right] = 2^{-(\mu-\nu)} H_{\mu-\nu}\left(\sqrt{\frac{\gamma}{\hbar}}\, q_3\right), \tag{145}$$

where

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right) \tag{146}$$

is the Hermite polynomial. The normalized functions are now

$$P_{\mu\nu}^N = \delta_1(\mu,\nu)\left(\frac{\gamma}{\hbar}\right)^{\frac{1}{2}(|m|+1)} \sqrt{\frac{2n!}{(|m|+n)!}}\, \rho^{|m|} e^{-\gamma\rho^2/2\hbar} L_n^{|m|}(\gamma\rho^2/\hbar), \tag{147}$$

$$F_{\mu\nu}^N = \delta_2(\mu,\nu)\left(\frac{\gamma}{\pi\hbar}\right)^{1/4} \sqrt{\frac{1}{2^{\mu-\nu}(\mu-\nu)!}}\, e^{-\gamma q_3^2/2\hbar} H_{\mu-\nu}\left(\sqrt{\frac{\gamma}{\hbar}}\, q_3\right), \tag{148}$$

$$\Phi_{\mu\nu}^N = \delta_3(\mu,\nu) \frac{1}{\sqrt{2\pi}} e^{im\varphi}, \tag{149}$$

i.e.

$$\tilde{\omega}_{\mu\nu}^N = C_{\mu\nu}^N \rho^{|m|} e^{-\gamma r^2/2\hbar} L_n^{|m|}\left(\frac{\gamma\rho^2}{\hbar}\right) H_{\mu-\nu}\left(\sqrt{\frac{\gamma}{\hbar}}\, q_3\right) e^{im\varphi}, \tag{150}$$

with

$$C_{\mu\nu}^N = \delta(\mu,\nu)\left(\frac{\gamma}{\hbar}\right)^{\frac{1}{2}(|m|+\frac{3}{2})} \sqrt{\frac{n!}{2^{\mu-\nu}\pi^{3/2}(\mu-\nu)!(|m|+n)!}}\,, \tag{151}$$

where δ_1, δ_2, δ_3, δ are phase factors of unit modulus; they can, in principle, be determined from (81).

We give below the explicit form of the functions for N = 0,1,2:

$$\tilde{\omega}^0_{00} = \left(\frac{\gamma}{\pi\hbar}\right)^{3/4} e^{-\gamma r^2/2\hbar} \; ; \tag{152}$$

$$\tilde{\omega}^1_{00} = \left(\frac{\gamma^5}{\pi^3\hbar^5}\right)^{1/4} \rho\, e^{-\gamma r^2/2\hbar} e^{-i\varphi} ,$$

$$\tilde{\omega}^1_{10} = \sqrt{2}\left(\frac{\gamma^5}{\pi^3\hbar^5}\right)^{1/4} q_3\, e^{-\gamma r^2/2\hbar} ,$$

$$\tilde{\omega}^1_{11} = \left(\frac{\gamma^5}{\pi^3\hbar^5}\right)^{1/4} \rho\, e^{-\gamma r^2/2\hbar} e^{i\varphi} \; ; \tag{153}$$

$$\tilde{\omega}^2_{00} = \frac{1}{\sqrt{2}}\left(\frac{\gamma^7}{\pi^3\hbar^7}\right)^{1/4} \rho^2 e^{-\gamma r^2/2\hbar} e^{-2i\varphi} ,$$

$$\tilde{\omega}^2_{10} = \sqrt{2}\left(\frac{\gamma^7}{\pi^3\hbar^7}\right)^{1/4} \rho\, q_3\, e^{-\gamma r^2/2\hbar} e^{-i\varphi} ,$$

$$\tilde{\omega}^2_{11} = \left(\frac{\gamma}{\pi\hbar}\right)^{3/4} \left(1 - \frac{\gamma\rho^2}{\hbar}\right) e^{-\gamma r^2/2\hbar} ,$$

$$\tilde{\omega}^2_{20} = \frac{1}{\sqrt{2}}\left(\frac{\gamma}{\pi\hbar}\right)^{3/4} \left(\frac{2\gamma}{\hbar} q_3^2 - 1\right) e^{-\gamma r^2/2\hbar} ,$$

$$\tilde{\omega}^2_{21} = \sqrt{2}\left(\frac{\gamma^7}{\pi^3\hbar^7}\right)^{1/4} \rho\, q_3\, e^{-\gamma r^2/2\hbar} e^{i\varphi} ,$$

$$\tilde{\omega}^2_{22} = \frac{1}{\sqrt{2}}\left(\frac{\gamma^7}{\pi^3\hbar^7}\right)^{1/4} \rho^2 e^{-\gamma r^2/2\hbar} e^{2i\varphi} . \tag{154}$$

3. The Oscillator and U(3)

The Lie algebra of U(3) is the space of skew-adjoint 3×3 matrices over the real field. A representation of the algebra $\mathcal{U}'$ spanned by A_{ij}, L_i $(i,j=1,2,3)$ may be taken as:

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$A_{12} = \pm\gamma \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_{23} = \pm\gamma \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}, \quad A_{31} = \pm\gamma \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$A_{11} = \pm 2\gamma \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_{22} = \pm 2\gamma \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_{33} = \pm 2\gamma \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}.$$

We thus identify $\mathcal{U}'$ with the Lie algebra of U(3).

The development of 2.1 was concerned with finding a new basis of $\mathcal{U}'$ that contained a basis of $\mathcal{U}$ as a subset. The new basis was

$$\{ L_1, L_2, L_3, A_{12}, A_{23}, A_{31}, A_0, A_1, D \}$$

with the subset

$$\{ L_1, L_2, L_3, A_{12}, A_{23}, A_{31}, A_0, A_1 \}$$

as a basis of $\mathcal{U}$.

It is a theorem that a representation of $\mathcal{U}'$ is irreducible iff the induced representation of $\mathcal{U}$ is irreducible. A representation Λ' of $\mathcal{U}'$ determines a unique representation Λ of $\mathcal{U}$. However, the converse is not true. We shall see that the space Ω_N of functions $\tilde{\omega}^N_{\mu\nu}$ defined by (150), realizing the representation (N,0) of SU(3), also realizes a definite representation of U(3).

(23) gives

$$D = 3A_{33} - 6A_0. \tag{155}$$

Hence, from (99), (94) and (103),

$$\begin{aligned} D\,\tilde{\omega}^N_{\mu\nu} &= i[3b' - 2(N - 3\mu + 3\nu)]\,\gamma\,\tilde{\omega}^N_{\mu\nu} \\ &= -i(2N + 3)\gamma\,\tilde{\omega}^N_{\mu\nu}. \end{aligned} \tag{156}$$

(156) defines the representation Λ' of U(3) that is realized by Ω_N.

As a differential operator, D is written

$$\begin{aligned} D &= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} + \gamma^2(x_1^2 + x_2^2 + x_3^2) \\ &= i\hbar\left(\frac{\partial^2}{\partial q_1^2} + \frac{\partial^2}{\partial q_2^2} + \frac{\partial^2}{\partial q_3^2}\right) - \frac{i\gamma^2}{\hbar}(q_1^2 + q_2^2 + q_3^2), \end{aligned}$$

by the definitions (134), (121). Thus

$$\frac{i\hbar}{2\pi} D = \frac{-\hbar^2}{2\pi}\nabla^2 + kr^2 \equiv \mathcal{H}, \tag{157}$$

from (5), where $\mathcal{H}$ is the Hamiltonian. (156) now gives

$$\mathcal{H}\,\tilde{\omega}^N_{\mu\nu} = \left(N + \frac{3}{2}\right)\sqrt{\frac{2k}{\pi}}\,\hbar\,\tilde{\omega}^N_{\mu\nu}. \tag{158}$$

(158) is the Schrodinger equation for the oscillator and (150) is recognized as its solution in cylindrical polar coordinates.

As regards pure representation theory, the solutions with $b = b''$ are infinite series and are not square-integrable over M. However, they are of interest in their own right and determine a representation of U(3) defined by

$$D\,{}''\tilde{\omega}^N_{\mu\nu} = i(2N + 3)\gamma\,{}''\tilde{\omega}^N_{\mu\nu}. \tag{159}$$

It remains to remark that, had we considered the second solution (69), i.e. representations of type (0,N), we would have been led along a parallel path to two sets of functions neither of which would have been square-integrable over M. Perhaps a closer perusal of the theory will explain the apparently singular features of the representation (N,0) in this respect.

To conclude: the oscillator energy eigenstates transform according to an irreducible representation of type (N,0) of SU(3), with the Hamiltonian proportional to the ninth infinitesimal generator D of U(3) and its square a function of the Casimir operator J of SU(3).

SOME IDENTITIES

$$r p_r = p_1 q_1 + p_2 q_2 + p_3 q_3, \tag{1}$$

$$q_i p^2 - p_i r p_r = p_j L_k - p_k L_j, \quad ((i,j,k) \text{ in cyclic order}) \tag{2}$$

$$L^2 = r^2 (p^2 - p_r^2), \tag{3}$$

$$r(q_i p_r - p_i r) = q_j L_k - q_k L_j; \quad ((i,j,k) \text{ in cyclic order}) \tag{4}$$

Expression of confluent hypergeometric functions as Parabolic Cylinder functions, Hermite and Generalized Laguerre polynomials:

$$U(c, \tfrac{1}{2}; z) = 2^c e^{z/2} U(2c - \tfrac{1}{2}, \sqrt{2z}) \tag{5}$$

$$= 2^c e^{z/2} D_{-2c}(\sqrt{2z}), \tag{6}$$

$$U(-\tfrac{1}{2}n, \tfrac{1}{2}; z) = 2^{-n} H_n(\sqrt{z}), \tag{7}$$

(n a non-negative integer)

$$M(-n, \alpha+1; z) = \frac{n!\,\alpha!}{(\alpha+n)!} L_n^{\alpha}(z); \tag{8}$$

$$\frac{d}{dx} H_n(x) = 2n H_{n-1}(x), \tag{9}$$

$$\frac{d}{dx} L_n^{\alpha}(x) = -L_{n-1}^{\alpha+1}(x); \tag{10}$$

An integral:

$$\int^x \frac{dx}{x\sqrt{ax^2 + bx - 1}} = \arcsin\left(\frac{bx - 2}{x\sqrt{b^2 + 4a}}\right). \tag{11}$$

The angle G in four special cases:

$$\text{If } V = ar^2, \quad G = \frac{1}{2} \arcsin\left(\frac{mEr^2 - L^2}{r^2\sqrt{m^2E^2 - 2maL^2}}\right), \tag{12}$$

$$\text{If } V = a, \quad G = -\arcsin\left(\frac{L}{\sqrt{2m(E-a)}\, r}\right), \tag{13}$$

$$\text{If } V = a/r, \quad G = -\arcsin\left(\frac{mar + L^2}{r\sqrt{m^2a^2 + 2mEL^2}}\right), \tag{14}$$

$$\text{If } V = a/r^2, \quad G = \frac{-L}{\sqrt{L^2 + 2ma}} \arcsin\left(\frac{\sqrt{L^2 + 2ma}}{\sqrt{2mE}\, r}\right). \tag{15}$$

NOTATION

$\mathcal{C}$	space of complex numbers over complex field
$\mathcal{C}_M$	Hilbert space of square-summable complex functions on M
C_∞	infinitely differentiable
H	Hamiltonian (classical observable or Λ H)
$\mathcal{H}$	Hamiltonian (self-adjoint quantum observable)
$\mathcal{I}$	unit matrix
iff	if and only if
L(S;S)	space of linear mappings of S into S
M	configuration space
$\mathcal{M}_V$	phase space
oscillator	three-dimensional isotropic harmonic oscillator
q	point of M
(q,p)	point of $\mathcal{M}_V$
$\mathcal{R}$	space of real numbers over real field
$\mathcal{R}_2$	space of all ordered pairs of real numbers over real field
SO(n)	n-dimensional rotation group
SO(3,1)	Lorentz group
X/S	X (a mapping) restricted to S (a vector subspace)
ε_{ij}	$\begin{cases} \left.\begin{matrix} +1 \\ -1 \end{matrix}\right\} & \text{if } i,j \text{ are } \begin{cases} \text{in} \\ \text{not in} \end{cases} \text{ cyclic order} \\ 0 & \text{if } i = j \end{cases}$
ε_{ijk}	$\begin{cases} \left.\begin{matrix} +1 \\ -1 \end{matrix}\right\} & \text{if } (i,j,k) \text{ is an } \begin{cases} \text{even} \\ \text{odd} \end{cases} \text{ permutation of } (123) \\ 0 & \text{if any two of } i,j,k \text{ are equal} \end{cases}$
$\in$	is an element of

$\subset$	is a subset of
$\exists$	there exists

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