Computing with Permutations and Transformations

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Ph.D. Thesis

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March 22, 2006

to my parents

Contents

Declarations

I, Nelson Silva, hereby certify that this thesis, which is approximately 49000 words in length, has been written by me, that it is the record of work carried out by me and that it has not been submitted in any previous application for a higher degree.

Name Nelson Silva
Signature . **Date** 27 March 2006 ....

I was admitted as a research student in October, 2001 and as a candidate for the degree of Doctor of Philosophy in September, 2002; the higher study for which this is a record was carried out in the University of St Andrews between 2001 and 2005.

Name Nelson Silva
Signature . **Date** . 27 March 2006 .....

I hereby certify that the candidate has fulfilled the conditions of the Resolution and Regulations appropriate for the degree of Doctor of Philosophy in the University of St Andrews and that the candidate is qualified to submit this thesis in application for that degree.

Name Edmund Robertson
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Name Nikola Ruškuc
Signature **Date** 27/3/06

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In the next few lines I shall try to write something that probably cannot be described with mere words...

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Abstract

Inspired by the results of D. McAlister ([25]), we consider transformation semigroups generated by an n-cycle and a transformation of rank 2. We give structural properties of the generator of rank 2 which determine if the semigroup is regular or completely regular. We also show that there are no inverse semigroups of the type considered.

Using similar techniques, we determine all possible sizes for a given semigroup of this type. This is done via a complete description of the Green's relations of the semigroup.

In the next stage of this thesis we are concerned with the study of the isomorphism between two members of this class of semigroups. We give conditions in order to decide if two given semigroups are isomorphic. In the process of answering this question, we study presentations for these semigroups, which are then used as another tool for the study of isomorphisms.

Due to the combinatorial and algorithmic character of the properties defined in this thesis, the computer algebra system GAP ([12]) played an important role in our studies both as a tool for testing examples and as a tool for making conjectures. All the formal algorithms resulting from our work are also given.

In the last part of our thesis we study a different kind of problem. Having as a starting point the famously known "Sierpiński's Lemma" ([30]) and the proof of this lemma given by Banach ([4]), we give a generalisation of this result. We prove that every countable set of endomorphisms of an algebra $\mathcal{A}$ which has an

infinite basis in contained in a 2-generated subsemigroup of the semigroup of all endomorphisms of $\mathcal{A}$. Several corollaries of this result follow, among them the case where $\mathcal{A}$ is an independence algebra. This result was first obtained by K. D. Magill ([23]), using a very complicated and complex proof that makes no use of Banach's proof.

Chapter 1

Introduction

A *semigroup* is a non-empty set S endowed with a *binary operation* which is *associative.*

Maybe the beauty of the study of semigroup theory lies in the simplicity of its definition. And this simplicity implies that we are surrounded by obvious examples of semigroups, perhaps the most obvious being the set of the natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$ with the usual addition "+" which, for most of us, is the first contact with mathematics.

In the beginning of their history, semigroups were a mere generalisation of groups. The least that can be said is that things have changed quite radically in the last 50 years or so. In the present day, semigroup theory is a very broad and fertile area of mathematics, as one can deduce just from the number of results obtained by typing the word "semigroup" on any internet search engine. This is far from being a "formal proof" for the previous statement but it is nevertheless a persuasive non-mathematical argument.

In this chapter we present the definitions, notation and results necessary for a complete understanding of the work presented in this thesis. In the last section and with some detail, we outline the work presented in this thesis.

1 Definitions and known results

We ought to state some definitions so that the definition of a semigroup presented above makes sense.

One of the simplest algebraic structures is a *groupoid* or *magma*. A groupoid $(S, \circ)$ is a non-empty set S with "$\circ$", a binary operation. A binary operation is just a well defined map

$$\circ : S \times S \longrightarrow S.$$

A *semigroup* $(S, \circ)$ is a groupoid where the binary operation $\circ$ is associative. More formally, for all $x, y, z \in S$ the following holds

$$(x \circ y) \circ z = x \circ (y \circ z).$$

It is standard to write $x \circ y$ as the usual multiplication xy (or sometimes, but less frequently, $x.y$). Hence associativity is represented by the equality

$$(xy)z = x(yz).$$

Associativity can be generalised to any finite product of elements from S. We write xx as x^2 and more generally $\underbrace{xx \ldots x}_{k \text{ times}} = x^k$.

An element e in S is called an *idempotent* if $e^2 = e$.

If there is an element $u \in S$ such that for all $x \in S$ we have $ux = xu = x$ then u is the *identity* of S (it is easy to prove that u, if it exists, is unique). In this case, S is a semigroup with identity or a *monoid* and the element u is usually denoted by 1. Trivially, the identity of a semigroup is an idempotent.

In the case that a semigroup S does not have an identity, we can just simply adjoin an extra element 1 to S and impose the conditions $11 = 1$ and $1x = x1 = x$, for every element $x \in S$. It is trivial to prove that $S \cup \{1\}$ is a monoid. We can then define

$$S^1 = \begin{cases} S & \text{if } S \text{ has an identity} \\ S \cup \{1\} & \text{otherwise.} \end{cases}$$

The monoid S^1 is often used as a tool to prove results for semigroups from known results for monoids.

Let T be a non-empty subset of S. Then T is a *subsemigroup* of S if for all $x, y \in T$, $xy \in T$. We write $T \leq S$ to denote that T is a subsemigroup of S. In the case that T is a subsemigroup of S but $T \neq S$, we say that T is a *proper* subsemigroup of S and write $T < S$. Note that, in both cases, T inherits the multiplication of S, i.e. $\underbrace{xy}_{\text{in } T} = \underbrace{xy}_{\text{in } S}$.

As in most areas of Mathematics, the notion of homomorphism is a very important concept. Let S and T be semigroups, and let $\phi : S \longrightarrow T$ be a map. Then ϕ is a *homomorphism* if $(xy)\phi = (x\phi)(y\phi)$, for all $x, y \in S$. If S and T are monoids with identities 1_S and 1_T, respectively, then ϕ is a homomorphism if for all $x, y \in S$ we have $(xy)\phi = (x\phi)(y\phi)$ and $1_S\phi = 1_T$.

If ϕ is an injective homomorphism then ϕ is set to be a *monomorphism.* If ϕ is a surjective homomorphism then ϕ is called an *epimorphism.* If ϕ is both a monomorphism and an epimorphism then ϕ is an *isomorphism.* If there is an isomorphism $\phi : S \longrightarrow T$ then we say that S is *isomorphic* to T or that S and T are *isomorphic* and we write $S \cong T$.

Let A and B be non-empty subsets of S. We define

$$AB = \{ab : a \in A, b \in b\}.$$

It is clear that AB is a subset of S. In the particular case that $A = \{a\}$ it is traditional to write AB as aB, rather than $\{a\}B$.

If I is a non-empty subset of S such that $IS \subseteq I$ then we say that I is a *right ideal* of S. We define *left ideal* in a similar way. If I is both a left and right ideal of S then we say that I is a (two-sided) *ideal* of S. It is clear that, by definition, every (left, right or two-sided) ideal of S is a subsemigroup of S. It is a standard exercise to prove that aS^1 is the smallest right ideal of S which contains a. This ideal is called the *principal right ideal generated by a.* Analogously, we have the notion of *principal left ideal generated by a.* Also we can prove that the set S^1aS^1

is the smallest ideal which contains a. This is called the *principal ideal generated by* a.

It is not very hard to verify that the intersection of two subsemigroups of a semigroup is a subsemigroup, provided that this intersection is non-empty. More generally, we have the following result.

Proposition 1.1 *Let W be a non-empty set and S be a semigroup . Let T_i be a subsemigroup of S, for each $i \in W$. Then $T = \bigcap_{i \in W} T_i$ is a semigroup, provided that T is non-empty.*

Given a semigroup S and a non-empty subset X of S we have that the intersection of all subsemigroups T of S which contain X (formally, $\bigcap_{\substack{T \leq S \\ X \subseteq T}} T$) is a subsemigroup of S, called the *subsemigroup generated by* X and it is denoted by $\langle X \rangle$.

The following result is given without proof.

Proposition 1.2 *Let S be a semigroup and X a non-empty set of S. The subsemigroup generated by X is the set of all products of one or more elements of X. Formally, $\langle X \rangle = \{x_1 x_2 \ldots x_n : n \in \mathbb{N}, x_i \in X, i \in \{1, \ldots, n\}\}$.*

2 Green's relations

We shall now define *Green's relations.* We use [16] as the main guide for the following definitions.

Let S be a semigroup and take $a, b \in S$. We say that a is *$\mathcal{L}$-related to b* and we write $a\mathcal{L}b$ if a and b generate the same principal left ideal. In other words, $a\mathcal{L}b$ if and only if $S^1 a = S^1 b$.

Similarly, we say that a is *$\mathcal{R}$-related to b* and we write $a\mathcal{R}b$ if a and b generate the same principal right ideal, i.e. $a\mathcal{R}b$ if and only if $aS^1 = bS^1$.

The following result is a standard exercise.

Proposition 1.3 *Let S be a semigroup and $a, b \in S$. Then $a\mathcal{L}b$ if and only if there exist $x, y \in S^1$ such that $xa = b$ and $yb = a$. Similarly, $a\mathcal{R}b$ if and only if there exist $x, y \in S^1$ such that $ax = b$ and $by = a$*

We define $\mathcal{D}$ as the smallest equivalence relation which contains both $\mathcal{L}$ and $\mathcal{R}$. Since $\mathcal{L}$ and $\mathcal{R}$ commute (i.e. $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$), we have that $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$. The intersection of $\mathcal{L}$ with $\mathcal{R}$ is the relation $\mathcal{H}$.

The following holds.

Proposition 1.4 *The relations $\mathcal{D}, \mathcal{L}, \mathcal{R}$ and $\mathcal{H}$ are equivalence relations. Furthermore, $\mathcal{L}$ is a right congruence and $\mathcal{R}$ is a left congruence.*

We normally refer to the $\mathcal{L}$-, $\mathcal{R}$-, $\mathcal{D}$- and $\mathcal{H}$-class of an element $a \in S$ as L_a, R_a, D_a and H_a, respectively.

As described in [16], it is convenient to visualise a $\mathcal{D}$-class as an "eggbox", where each row represents an $\mathcal{R}$-class, each column represents an $\mathcal{L}$-class and each cell represents an $\mathcal{H}$-class. Note that we can have the case that there is only one $\mathcal{L}$- or $\mathcal{R}$-class or, in the case of infinite semigroups, we can have an "infinite eggbox".

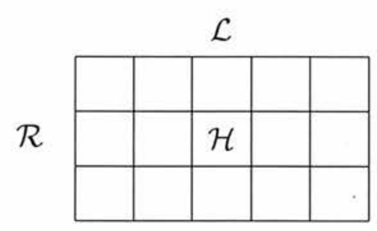

Figure 1.1: Illustration of a $\mathcal{D}$-class

The next proposition is a very important structural result, commonly known as "Green's Lemma".

Proposition 1.5 *Let S be a semigroup and let $a, b \in S$.*

(i) *If* $a\mathcal{R}b$ *and* $u, v \in S^1$ *are such that* $au = b$ *and* $bv = a$ *then the right translations*

$$\rho_u : L_a \longrightarrow L_b, x \mapsto xu \text{ and}$$

$$\rho_v : L_b \longrightarrow L_a, y \mapsto yv$$

are mutually inverse $\mathcal{R}$*-class preserving bijections.*

(ii) *If* $a\mathcal{L}b$ *and* $u, v \in S^1$ *are such that* $ua = b$ *and* $vb = a$ *then the left translations*

$$\lambda_u : R_a \longrightarrow R_b, x \mapsto ux \text{ and}$$

$$\lambda_v : R_b \longrightarrow R_a, y \mapsto vy$$

are mutually inverse $\mathcal{L}$*-class preserving bijections.*

Two important consequences of this result follow.

Proposition 1.6 *Let* S *be a semigroup and let* $a, b \in S$*. If* $a\mathcal{D}b$ *then* $|H_a| = |H_b|$*.*

Proposition 1.7 *Let* S *be a semigroup and let* H *be an* $\mathcal{H}$*-class of* S*. Then either* $H^2 \cap H = \emptyset$ *or* $H^2 = H$ *and, in this case,* H *is a group.*

Hence, if $e \in S$ is an idempotent then H_e is a group.

The next result describes the relation between the group $\mathcal{H}$-classes which lie in the same $\mathcal{D}$-class.

Proposition 1.8 *Let* S *be a semigroup. Take* $a, b \in S$*. If* H_a *and* H_b *are groups then* H_a *is isomorphic to* H_b*.*

The proof for this result can be found in [16] and uses a "clever choice" of the translations referred to in Proposition 1.5. Alternatively, the proof of this result (actually, of a more general result) can be found in [14] and uses the notion of the *Schützenberger group of an* $\mathcal{H}$*-class*, a notion which is not directly related to our work.

The next result we present in this section enables us to have some control of the multiplication within a certain $\mathcal{D}$-class of S.

Proposition 1.9 *Let S be a semigroup and let $a, b \in S$ be such that $a\mathcal{D}b$. Then $ab \in R_a \cap L_b$ if and only if $L_a \cap R_b$ contains an idempotent.*

Again, the proof for the above proposition can be found in [16].

3 Regular and inverse semigroups

An element a of the semigroup S is *regular* if there exists $x \in S$ such that $axa = a$. A semigroup is *regular* if all its elements are regular.

Let $a \in S$. We say that a' is an *inverse* of a if $aa'a = a$ and $a'aa' = a'$. It is clear that if $a \in S$ has an inverse than a is regular. The converse is also true because if a is regular then there is $x \in S$ such that $a = axa$. Therefore $a' = xax$ is an inverse of a. Note that an element may have more than one inverse (for example, in a rectangular band). If S is such that every element of S has one and only one inverse then S is called an *inverse semigroup.*

As a consequence of the above definitions one can easily remark that every inverse semigroup is a regular semigroup.

Regular semigroups can be looked at as being "one step closer" to groups. Next, we state a relation between groups and regular semigroups.

Proposition 1.10 *Let S be a regular semigroup. Then S is a group if and only if S contains exactly one idempotent.*

The following two results are used to decide if a certain semigroup is regular.

Proposition 1.11 *Let S be a semigroup. If a is regular then every element in the $\mathcal{D}$-class of a is regular.*

Proposition 1.12 *A semigroup S is regular if and only if every $\mathcal{L}$-class and every $\mathcal{R}$-class of S contain at least one idempotent.*

A semigroup is *completely regular* if every $\mathcal{H}$-class in S is a group, i.e. if for every $a \in S$, H_a is a group. It is easy to see that if S is completely regular then S is regular.

There are many equivalent ways of characterising inverse semigroups as one can observe in [16] or in [18], a much more specialised book on the theory of inverse semigroups. For our work we shall use the following characterisation.

Proposition 1.13 *The semigroup S is inverse if and only if every $\mathcal{L}$-class and every $\mathcal{R}$-class of S contains exactly one idempotent.*

There are some comments to make about Green's relations and regular semigroups.

Let S be a semigroup and T a subsemigroup of S. It is clear that if two elements a, b in T are $\mathcal{L}$, $\mathcal{R}$, $\mathcal{D}$ or $\mathcal{H}$-related in T then they are also $\mathcal{L}$, $\mathcal{R}$, $\mathcal{D}$ or $\mathcal{H}$-related in S. The converse is, in general, not true.

We do have the following useful result.

Proposition 1.14 *Let S be a semigroup. Let $T \leq S$ and $a, b \in T$. If T is regular then;*

(i) *$a\mathcal{R}b$ in T if and only if $a\mathcal{R}b$ in S,*

(ii) *$a\mathcal{L}b$ in T if and only if $a\mathcal{L}b$ in S,*

(iii) *$a\mathcal{H}b$ in T if and only if $a\mathcal{H}b$ in S.*

The proofs for the results in this section can be found in [16].

4 Transformation semigroups

Unavoidably, we shall probably repeat the statements made in the introduction of numerous manuals, books or theses on semigroup theory when we say that transformation semigroups are of utmost importance in this area of Mathematics.

The *full transformation semigroup on a set* X, which is normally denoted by $\mathcal{T}_X$, arises very naturally as the set of all maps from the set X into itself. We ought to say that the somehow trivial fact that the composition of functions is associative is an essential detail that makes the set of all selfmaps of X a semigroup.

In $\mathcal{T}_X$, as in the case of an abstract semigroup, we shall write the composition of maps as multiplication and the maps act on the right,

$$x(\tau \circ \beta) = x(\tau\beta) \quad (x \in X;\ \tau, \beta \in \mathcal{T}_X).$$

In the case that $X = \{1, 2, \dots, n\}$ then we shall write $\mathcal{T}_n$ rather than $\mathcal{T}_X$.

The full transformation semigroup $\mathcal{T}_X$ plays a role, in Semigroup Theory, equivalent to that of the symmetric group $\mathcal{S}_X$ in Group Theory, since every semigroup is isomorphic to a subsemigroup of $\mathcal{T}_X$.

At this point we will define some notation which will be used throughout this thesis.

Let α be a transformation in $\mathcal{T}_X$. The *kernel* of α is the equivalence relation $\mathsf{ker}\,(\alpha)$ defined by

$$\mathsf{ker}\,(\alpha) = \{(x, y) \in X \times X : x\alpha = y\alpha\}.$$

Whenever there is no risk of ambiguity we also denote the partition associated to the kernel of α by $\mathsf{ker}\,(\alpha)$. The *image* of α is the set

$$\mathsf{im}\,(\alpha) = \{x\alpha : x \in X\}.$$

The *rank* of α (which we denote by $\mathsf{rank}\,(\alpha)$) is the cardinal of the set $\mathsf{im}\,(\alpha)$.

Green's relations for the full transformation semigroup are described in the next result. This result emphasises the nice relationship between the theoretical and algebraic nature of the Green's relations for the particular case of $\mathcal{T}_X$.

Proposition 1.15 *Let $\alpha, \beta \in \mathcal{T}_X$. The following hold:*

(i) *$\alpha\mathcal{R}\beta$ if and only if $\mathsf{ker}(\alpha) = \mathsf{ker}(\beta)$;*

(ii) *$\alpha\mathcal{L}\beta$ if and only if $\mathsf{im}(\alpha) = \mathsf{im}(\beta)$;*

(iii) *$\alpha\mathcal{H}\beta$ if and only if $\mathsf{ker}(\alpha) = \mathsf{ker}(\beta)$ and $\mathsf{im}(\alpha) = \mathsf{im}(\beta)$;*

(iv) *$\alpha\mathcal{D}\beta$ if and only if $\mathsf{rank}(\alpha) = \mathsf{rank}(\beta)$.*

Proposition 1.16 *Let $\alpha, \beta \in S \leq \mathcal{T}_X$. Then the following hold:*

(i) *If $\alpha\mathcal{D}\beta$ in S then $\mathsf{rank}(\alpha) = \mathsf{rank}(\beta)$;*

(ii) *If $\alpha\mathcal{R}\beta$ in S then $\mathsf{ker}(\alpha) = \mathsf{ker}(\beta)$;*

(iii) *If $\alpha\mathcal{L}\beta$ in S then $\mathsf{im}(\alpha) = \mathsf{im}(\beta)$;*

(iv) *If $\alpha\mathcal{H}\beta$ in S then $\mathsf{ker}(\alpha) = \mathsf{ker}(\beta)$ and $\mathsf{im}(\alpha) = \mathsf{im}(\beta)$.*

The next result follows taking $S = \mathcal{T}_X$ and $T = S$ in Proposition 1.14 and applying Proposition 1.15.

Proposition 1.17 *Let $\alpha, \beta \in S \leq \mathcal{T}_X$. If S is regular then the following hold:*

(i) *$\alpha\mathcal{R}\beta$ in S if and only if $\mathsf{ker}(\alpha) = \mathsf{ker}(\beta)$;*

(ii) *$\alpha\mathcal{L}\beta$ in S if and only if $\mathsf{im}(\alpha) = \mathsf{im}(\beta)$;*

(iii) *$\alpha\mathcal{H}\beta$ in S if and only if $\mathsf{ker}(\alpha) = \mathsf{ker}(\beta)$ and $\mathsf{im}(\alpha) = \mathsf{im}(\beta)$.*

Sometimes we refer explicitly to these results but there are cases where they are simply implicitly assumed.

The next definition is maybe not the most standard but we find it to be appropriate in the context of this thesis. If $\tau \in \mathcal{T}_X$ is a transformation of rank two, we define the *conjugate* of τ as the only element α in $\mathcal{T}_X$ such that $\mathsf{ker}\,(\tau) = \mathsf{ker}\,(\alpha)$ and $\mathsf{im}\,(\tau) = \mathsf{im}\,(\alpha)$ but $\tau \neq \alpha$. We denote this element by $\overline{\tau}$. The fact that this is element is unique follows from having $\tau\mathcal{H}\overline{\tau}$ in $\mathcal{T}_X$ (from Proposition 1.15) and $|H_\tau| = 2$. The element $\overline{\tau}$ can also be looked at as the product $\tau(a\ b)$, where a, b are such that $\mathsf{im}\,(\tau) = \{a, b\}$.

We ought to introduce some notation regarding transformations which we sometimes use in this thesis. Given a transformation

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & 6 \\ a & a & b & a & \dots & b \end{pmatrix}$$

we write τ as $[\mathrm{a}, \mathrm{a}, \mathrm{b}, \mathrm{a}, \dots, \mathrm{b}]$. For example, if

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 1 & 1 & 3 \end{pmatrix},$$

we also write $\tau = [3, 1, 1, 1, 3]$.

In the case where X is finite, we have a result which gives us more information about the multiplication within a semigroup than Proposition 1.9.

Proposition 1.18 *Let $S \leq \mathcal{T}_n$. If $\alpha, \beta \in S$ are such that α, β and $\alpha\beta$ are $\mathcal{D}$-related then $\alpha\beta \in R_\alpha \cap L_\beta$ and $L_\alpha \cap R_\beta$ is a group.*

5 Presentations

Let A be a non-empty set. The set of all non-empty words $a_1 a_2 \dots a_n$, with $a_1, a_2, \dots, a_n \in A$ is denoted by A^+. Denoting by ϵ the empty word, then we define $A^* = A^+ \cup \{\epsilon\}$. Both sets A^+ and A^* are semigroups, where the binary operation is the concatenation of words. To be more accurate, A^* is a monoid since the concatenation of any word w with the empty word ϵ is w.

The semigroup A^+ is generated by A and it is called the *free semigroup on A*.

A *semigroup presentation* is an ordered pair $\langle A \mid R \rangle$, where $R \subseteq A^+ \times A^+$. A semigroup S is *defined by the presentation* $\langle A \mid R \rangle$ if $S \cong A^+/\rho$, where ρ is the smallest congruence on A^+ which contains R. This means, loosely speaking, that each word of A^+ represents an element of S.

Throughout this thesis we generally identify words w of A^+ with the corresponding element w/ρ of S. Sometimes we wish to distinguish these and then, given $w_1, w_2 \in A^+$, we write $w_1 \equiv w_2$ if w_1 and w_2 are identical words in A^+, and $w_1 = w_2$ if they represent the same element in S (i.e. $w_1/\rho = w_2/\rho$), in which case we say that S *satisfies the relation* $w_1 = w_2$.

Given two words $w_1, w_2 \in A^+$, we say that w_2 is obtained from w_1 by one application of one relation from R if there exist $\alpha, \beta \in A^*$ and $(u, v) \in R$ such that $w_1 \equiv \alpha u \beta$ and $w_2 \equiv \alpha v \beta$. We say that w_2 is a *consequence* of w_1 if there is a sequence $w_1 \equiv \alpha_1, \alpha_2, \ldots, \alpha_{k-1}, \alpha_k \equiv w_2$ of words from A^+ such that α^{i+1} is obtained from α_i by one application of one relation from R.

The next result can be found in [28] and it is frequently used in Chapter 4.

Proposition 1.19 *Let S be a finite semigroup, let A be a generating set for S, let $R \subseteq A^+ \times A^+$ be a set of relations, and let $W \subseteq A^+$. Assume that the following conditions are satisfied:*

(i) *the generators A of S satisfy all the relations from R;*

(ii) *for each word $w \in A^+$ there exists a word $\overline{w} \in W$ such that $w = \overline{w}$ is a consequence of R;*

(iii) $|W| \leq |S|$.

Then $\langle A \mid R \rangle$ is a presentation for S.

In the definitions above, if we replace A^+ by A^*, we obtain the notion of a *monoid presentation* and the consequent results.

6 Set theory

This section and the following one contain some definitions required in the last chapter of this thesis. In this section we give some definitions and results of set theory which can be found in [6] or [31].

Let A be a nonempty set. We say that A is *finite* if there is a bijection $f : \{1, 2, \ldots, n\} \longrightarrow A$, for some $n \in \mathbb{N}$. In this case we say that the size of A is n and we write $|A| = n$.

A set A is *countable* if there is a bijection between the set of all natural numbers and the set A, $f : \mathbb{N} \longrightarrow A$.

We say that A is *uncountable* if A is neither finite nor countable. The set of all real numbers $\mathbb{R}$ is an uncountable set. Another interesting example of an uncountable set is the set of all subsets of $\mathbb{N}$, denoted by $\mathcal{P}(\mathbb{N})$. This is proved by showing that

$$\begin{aligned} f: \mathcal{P}(\mathbb{N}) &\longrightarrow 2^{\mathbb{N}} \\ A &\longmapsto \chi_A : A \to \{0,1\} \\ &\qquad\qquad x \mapsto \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

is a bijection, knowing that the set $2^{\mathbb{N}}$ of all maps from $\mathbb{N}$ into $\{0, 1\}$ is in bijection with the set of all real numbers $\mathbb{R}$.

7 Universal algebra

For this section we shall use [5] as our main guide. Let A be a nonempty set. We define $A^0 = \{\emptyset\}$. Let $n \in \mathbb{N}$. We define A^n as the set of n-tuples of elements from

A. It is also standard to refer to A^n as the *Cartesian product* $A \times A \times A \times \cdots \times A$ with n terms. An element $a \in A^n$ is represented as $(a_1, a_2, \ldots, a_n)$, with $a_i \in A_i$, for each $i \in \{1, \ldots, n\}$.

An *n-ary operation* on A is any mapping from A^n to A. A *nullary operation* (or constant) is a mapping from A^0 to A. A nullary operation is thought as an element of A.

These operations are referred to as *finitary operations.*

There is also the notion of *infinitary operation*, which is a mapping from A^I to A, where I is infinite. More precisely, an infinitary operation is a mapping $\alpha : A^I \longrightarrow A$ where $A^I = \{(a_i)_{i \in I} : a_i \in A, \text{for each } i \in I\}$. Note that A^I is just the set of all sequences of elements of A, indexed on I. Nevertheless, we shall not give any further details about these kind of operations, since these are not directly related to the work in this thesis.

An *algebra* $\mathcal{A}$ is a pair (A, Ω), where A is a non-empty set and Ω is a set of (finitary or infinitary) operations on A. The operations in Ω are called *fundamental operations* of $\mathcal{A}$.

A semigroup $(S, \circ)$ is an example of a binary algebra, i.e. an algebra with a binary operation. We can find more interesting examples of algebras in the bibliography mentioned.

Let $\mathcal{A} = (A, \Omega)$ be an algebra. A map $\psi : A \longrightarrow A$ is an *endomorphism* if ψ is a selfmap of A which preserves the operations of $\mathcal{A}$. More formally, for every operation $\alpha \in \Omega$, $\alpha : A^I \longrightarrow A$ (I finite or infinite), we have that

$$\alpha(a)\psi = \alpha(a\psi), \text{ (for } a \in A^I).$$

Note that we write operations on the left and mappings on the right.

Remark 1.20 We are omitting the formal definition of homomorphism between two algebras $\mathcal{A} = (A, \Omega)$ and $\mathcal{B} = (B, \Lambda)$, although this concept is similar to the above, with the extra assumption that Ω and Λ must have the same *type*. The type of an algebra (loosely speaking) is the description of the arity of the fundamental operations of the algebra. As examples, semigroups $(S, .)$ and groupoids $(\mathcal{G}, .)$, and inverse semigroups $(S, ., ^{-1})$ and groups $(G, ., ^{-1})$ are algebras of the same type, their type being (2) and $(2, 1)$, respectively. For more details, see [5].

It is clear that the composition of endomorphisms is an endomorphism and that the composition of endomorphisms is associative. Therefore, the set of all endomorphisms of $\mathcal{A}$ is a semigroup, which we denote by $End(\mathcal{A})$.

An algebra $\mathcal{B} = (B, \Lambda)$ is a *subalgebra* of $\mathcal{A} = (A, \Omega)$ if $B \subseteq A$ and every fundamental operation α^B of $\mathcal{B}$ is the restriction of the corresponding operation of $\mathcal{A}$, i.e. $\alpha^B : B^I \longrightarrow B$ is the restriction of $\alpha^A : A^I \longrightarrow A$. Again, it is implicit in this argument that $\mathcal{A}$ and $\mathcal{B}$ have the same type.

Given $\emptyset \neq B \subseteq A$, we write $\langle B \rangle$ to denote the *subalgebra generated* by B. This is the smallest subalgebra of $\mathcal{A}$ (with respect to inclusion) that contains B. Note that when $\mathcal{A}$ has constants we may wish to consider the subalgebra $\langle \emptyset \rangle$, which will be the subalgebra generated by the constants of $\mathcal{A}$.

Let $\mathcal{A} = (A, \Omega)$ be an algebra with universe A and fundamental operations Ω. Let $X \subseteq A$. We say that X is *independent* if every mapping $f : X \longrightarrow \mathcal{A}$ can be extended to a homomorphism $\varphi_f : \langle X \rangle \longrightarrow \mathcal{A}$ such that the restriction of φ_f to X coincides with f, i.e. $x\varphi_f = xf$, for every $x \in X$.

For a mapping f from an independent set X to $\mathcal{A}$ we shall write φ_f to denote the extension of f from $\langle X \rangle$ to $\mathcal{A}$. Note that there are several different definitions of an independent set on an algebra $\mathcal{A}$. In this section we shall use the one found in [24].

Let $\mathcal{U}$ be a class of algebras of the same type. We say that $\mathcal{U}$ is a *variety* of algebras if,

(i) $\mathcal{U}$ is closed under subalgebras (if $\mathcal{A} \in \mathcal{U}$ and $\mathcal{B}$ is a subalgebra of $\mathcal{A}$ then $\mathcal{B} \in \mathcal{U}$),

(ii) $\mathcal{U}$ is closed under homomorphic images (if $\mathcal{A} \in \mathcal{U}$ and $\mathcal{B} = \mathcal{A}\phi$, for some homomorphism ϕ, then $\mathcal{B} \in \mathcal{U}$),

(iii) $\mathcal{U}$ is closed under direct products (finite or infinite).

Let $\mathcal{U}$ be a class of algebras of the same type (not necessarily a variety) and let X be a set such that the algebra generated by X is in $\mathcal{U}$. We say that $\mathcal{F}$ is a *free algebra* on the set X if for any mapping $f : X \longrightarrow \mathcal{A}$ there is a homomorphism $\phi : \langle X \rangle \longrightarrow \mathcal{A}$ which extends f (i.e. $x\phi = xf$, for every $x \in X$. We denote this algebra by $\mathcal{F}_X(\mathcal{U})$. For each mapping f, the extension ϕ is unique and we shall denote it by φ_f. One can verify that if $\langle X_1 \rangle$ and $\langle X_2 \rangle$ are algebras in a class $\mathcal{U}$ and $|X_1| = |X_2|$ then $\langle X_1 \rangle \cong \langle X_2 \rangle$. For details about these definitions and results see Chapter 10 in [5].

8 Motivation and overview

In [25] the author studies subsemigroups T of the full transformation semigroup $\mathcal{T}_n$, generated by a group of units G and any idempotent of rank $n-1$. In particular, the author studies the case where G is a permutation group on a set X of order n, the case where G is cyclic and the case where G is dihedral. The work developed in this thesis deals with what is perhaps, the next simplest case.

In this thesis we shall consider semigroups of transformations generated by the permutation $\sigma = (1\ 2\ \ldots\ n)$ and any transformation $\tau \in \mathcal{T}_n$ of rank two.

In Chapter 2 we study the general structure of such semigroups. This is done by determining the number of $\mathcal{D}$-classes and by describing the "normal forms" of such semigroups, that is a canonical decomposition of every element in S into a product of the generators σ and τ. We also give necessary and sufficient conditions on the generator of rank two for the semigroup to be regular and completely regular. Finally, we completely determine the number of $\mathcal{L}$- and $\mathcal{R}$-classes and also the size of the $\mathcal{H}$-classes of rank two of these semigroups. Again, we achieve this by giving some necessary and sufficient conditions on the generator of rank two. An immediate consequence of these results is a complete description of all possible sizes of the semigroup.

In Chapter 3 we study the isomorphism of two semigroups of the type under consideration. More accurately, we study the isomorphism in the case where these semigroups contain the constant maps. We give some properties which are invariant within the elements of rank two of the semigroup. We then conclude that these properties are also preserved by isomorphism, this implying that these properties are necessary conditions for the isomorphism of two such semigroups. In the final section of this chapter we give a necessary and sufficient condition for the isomorphism of two semigroups and an algorithm that explicitly determines an isomorphism. We also give an example which shows that these results do not hold in the case where the semigroups contain the constant maps.

In Chapter 4 we give general presentations for a semigroup with a certain type of generator of rank two. The aim of this chapter is to build a list of examples which will lead us to a method of finding a presentation for a semigroup, just by analysing its generator of rank two.

GAP being (see [12]) such a useful tool for the work presented in this thesis, we decided that it is only fair to give a short insight in Chapter 5 of the history of this computational tool. In this chapter we also present a very short note of the tools available in GAP for semigroup theory, where we look into some of the more relevant results and papers on this subject. In the last section of this chapter we implement the algorithms resulting from the theory developed in the previous chapters.

In Chapter 6 we study several particular cases. For each case, we compute all the semigroups generated by σ and $\tau \in \mathcal{T}_n$, a transformation of rank two. Then, for each semigroup and using the theory from the previous chapters, we deduce properties like the size, Green's structure and regularity. Also, we find a presentation for each of these semigroups. We then find all possible isomorphisms within the set of all these semigroups (for each n). This chapter constitutes an important database of examples that were thoroughly studied using the results found previously in this thesis.

In the last chapter of this thesis we generalise a famous result of Sierpiński for subsemigroups of the full transformation semigroup of a countable set (see [30]). We attain this by looking at the full transformation semigroup of a countable set X as the set of all endomorphisms of X. We then prove a similar result to Sierpiński's, where we replace the full transformation semigroup of a countable set by the set of all endomorphisms of a infinitely generated algebra. We use a similar proof to that of Banach in [4], making all the necessary adaptations. Examples of some known results that follow as an easy application of this result are given. The work in this chapter was motivated by the work of J. D. Mitchell in [26] and it has been published in [1].

A list of open problems can be found in Appendix A and all the GAP functions resulting from the algorithms found in Chapter 5 are in Appendix B.

Chapter 2

Structure

The main goal of this chapter is to find tools that allow us to study any semigroup generated by $\sigma = (1\ 2\ \ldots\ n)$ and a transformation $\tau \in \mathcal{T}_n$ of rank two, without having to compute all its elements. In the process of achieving this we shall find some rather interesting combinatorial properties of this kind of semigroup.

Throughout this chapter we consider S to be $\langle \sigma, \tau \rangle$, the transformation semigroup generated by the permutation $\sigma = (1\,2\ \ldots\ n)$ and a transformation $\tau \in \mathcal{T}_n$ of rank two, with $\mathsf{im}\,(\tau) = \{a, b\}$. We denote by K_a the kernel class of τ whose elements are mapped into a under τ. Similarly, we define K_b. Formally we have $K_a = a\tau^{-1}$ and $K_b = b\tau^{-1}$. We often denote by X the set $\{1, \ldots, n\}$.

In Section 1, we give some introductory results, the main ones being the description of the Green's structure and of the normal forms of the semigroups of this type. In particular, we prove that a semigroup $S = \langle \sigma, \tau \rangle$ has exactly one $\mathcal{D}$-class of rank two. This result allows us to reduce and focus our study on the $\mathcal{D}$-class of rank two, since it determines all the properties of the semigroup.

In Section 2, we give a necessary and sufficient condition on the generator of rank two for the semigroup to be regular. In this section we also prove that any element of rank two in the semigroup S generates S (together with σ). In the following section we analogously give a necessary and sufficient condition for a semigroup to be completely regular.

In Section 4 we determine the number of each of the Green's classes, namely the number of $\mathcal{L}$- and $\mathcal{R}$-classes and the size of the $\mathcal{H}$-classes of rank two. More precisely, we give necessary and sufficient conditions on the generator of rank two which completely determine the number of $\mathcal{L}$-classes, the number of $\mathcal{R}$-classes and the size of the $\mathcal{H}$-classes of rank two.

As a consequence of Section 4, in the last section we give a full description of the all possible sizes of S. In particular, we give necessary and sufficient conditions for each case to occur.

The results produced in this chapter are the basis for the later chapters and are used very often throughout this thesis.

1 Preliminary results

In this section we give some structural results of S which are frequently used in this thesis, explicitly or implicitly. These include the Green's structure, the normal forms and the type of the elements of S with rank two.

Lemma 2.1 *Let $\alpha, \beta \in \mathcal{T}_n$ be such that* $\mathsf{rank}\,(\alpha) = \mathsf{rank}\,(\beta) = 2$. *If $\alpha\mathcal{H}\beta$ then either $\beta = \alpha$ or $\beta = \overline{\alpha}$.*

PROOF. From Proposition 1.15 we know that $\alpha\mathcal{H}\beta$ if and only if $\mathsf{ker}\,(\alpha) = \mathsf{ker}\,(\beta)$ and $\mathsf{im}\,(\alpha) = \mathsf{im}\,(\beta)$. Because $\mathsf{rank}\,(\alpha) = 2$ then there are only two elements in $\mathcal{T}_n$ with the same image and kernel as α, which are α and its conjugate. $\square$

Based on the last result, we can describe the normal forms of the elements of rank two in S, that is a canonical decomposition of every element of rank two in S into a product of the generators σ and τ.

Lemma 2.2 *If $\alpha \in S$ is an element of rank 2, then α has one of the following two forms:*

(i) $\sigma^i \tau \sigma^j$;

(ii) $\sigma^i \tau \sigma^j \tau \sigma^k$,

where $i, j, k \in \{1, \ldots, n\}$.

PROOF. Every element of S with rank 2 is of the form

$$\sigma^{i_1} \tau \sigma^{i_2} \tau \sigma^{i_3} \ldots \tau \sigma^{i_m}, \text{ with } m \in \mathbb{N} \text{ and } i_j \geq 0, \text{ for every } j \in \{1, \ldots, m\}.$$

It is sufficient to show that

$$\sigma^{i_1} \tau \sigma^{i_2} \tau \sigma^{i_3} \tau \sigma^{i_4} \in \{\sigma^i \tau \sigma^j : i, j \in \{1, \ldots, n\}\} \cup \{\sigma^i \tau \sigma^j \tau \sigma^k : i, j, k \in \{1, \ldots, n\}\},$$

for all $i_1, i_2, i_3, i_4 \in \mathbb{N}$ such that $\sigma^{i_1} \tau \sigma^{i_2} \tau \sigma^{i_3} \tau \sigma^{i_4}$ has rank 2, since longer products are reduced to one of these in a similar way.

Note that we can have the case where $\tau \sigma^j \tau = \tau$, for all $j \in \{1, \ldots, n\}$ such that $\mathsf{rank}\,(\tau \sigma^j \tau) = 2$. In this case the two sets above are the same and $\sigma^{i_1} \tau \sigma^{i_2} \tau \sigma^{i_3} \tau \sigma^{i_4} = \sigma^{i_1} \tau \sigma^{i_3} \tau \sigma^{i_4} = \sigma^{i_1} \tau \sigma^{i_4}$ and our result holds.

Suppose that there is $j_0 \in \{1, \ldots, n\}$ such that $\tau \sigma^{j_0} \tau \neq \tau$ and $\mathsf{rank}\,(\tau \sigma^{j_0}) = 2$. Then we have that $\tau \sigma^{j_0} \tau = \overline{\tau}$, the conjugate of τ. It is clear that $\tau \sigma^{i_2} \tau \sigma^{i_3} \tau$ has rank 2 an therefore this element has the same kernel and the same image as τ. Thus we either have $\tau \sigma^{i_2} \tau \sigma^{i_3} \tau = \tau$ and therefore $\sigma^{i_1} \tau \sigma^{i_2} \tau \sigma^{i_3} \tau \sigma^{i_4} = \sigma^{i_1} \tau \sigma^{i_4}$ or $\tau \sigma^{i_2} \tau \sigma^{i_3} \tau = \overline{\tau} = \tau \sigma^{j_0} \tau$ and in this case $\sigma^{i_1} \tau \sigma^{i_2} \tau \sigma^{i_3} \tau \sigma^{i_4} = \sigma^{i_1} \overline{\tau} \sigma^{i_4} = \sigma^{i_1} \tau \sigma^{j_0} \tau \sigma^{i_4}$. In both cases our result holds. □

Without further thought, one would expect a semigroup of this form to have a rank n $\mathcal{D}$-class, several $\mathcal{D}$-classes of rank two and possibly a $\mathcal{D}$-class of rank one. This is due to the fact that the product of elements of rank two has rank two or one. Obviously, since transformations of rank n are permutations we have that "rank n × rank 2 = rank 2 and rank n × rank 1 = rank 1". The next figure illustrates this fact.

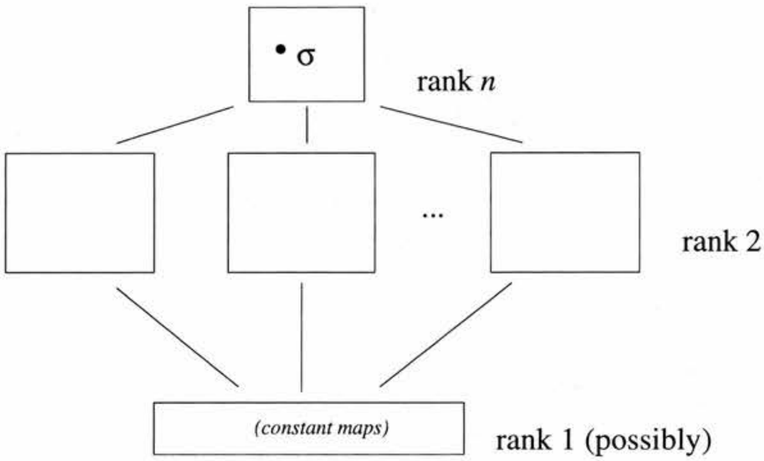

Figure 2.1: General structure

The next result tells us what exactly happens.

Lemma 2.3 *There is exactly one $\mathcal{D}$-class of elements of rank 2 in S.*

PROOF. It is clear that there is at least one $\mathcal{D}$-class of elements of rank 2, the one which contains τ. We will prove that every element $\alpha \in S$ of rank 2 is $\mathcal{D}$-related to τ. From Lemma 2.2, we just need to study two different cases.

Case 1: $\alpha = \sigma^i \tau \sigma^j$, *with* $i, j \in \{1, \ldots, n\}$.

To prove that $\alpha \mathcal{D} \tau$, we need to find $\beta \in S$ such that $\alpha \mathcal{R} \beta \mathcal{L} \tau$. Let $\beta = \sigma^i \tau$. Then if we take $\omega = \sigma^{n-j}, \nu = \sigma^j \in S$, we have $\alpha\omega = \beta$ and $\beta\nu = \alpha$. Hence $\alpha \mathcal{R} \beta$. Similarly, if we take $\omega = \sigma^{n-i}, \nu = \sigma^i \in S$, we have $\omega\beta = \tau$ and $\nu\tau = \beta$. Hence $\beta \mathcal{L} \tau$ and the result holds.

Case 2: $\alpha = \sigma^i \tau \sigma^j \tau \sigma^k$, *with* $i, j, k \in \{1, \ldots, n\}$.

Note that we are assuming that $\tau \sigma^j \tau \neq \tau$, otherwise we are again in Case 1. Therefore we have $\overline{\tau} = \tau \sigma^j \tau$. Similarly to Case 1, replacing τ by $\overline{\tau}$, it follows that

$\alpha\mathcal{D}\overline{\tau}$. So we are only left to prove that $\tau\mathcal{D}\overline{\tau}$. For this we shall prove that $\tau\mathcal{R}\overline{\tau}$. We have $\mathsf{im}\,(\tau\sigma^j) = \{a\sigma^j, b\sigma^j\}$. The elements $a\sigma^j$ and $b\sigma^j$ must be in different kernel classes of τ, otherwise we would have $(a\sigma^j)\tau = (b\sigma^j)\tau$ and $\mathsf{rank}\,(\overline{\tau}) = 1$. Because $\overline{\tau} \neq \tau$ we must have $b\sigma^j \in K_a$ and $a\sigma^j \in K_b$. Define $\tau' = \overline{\tau}\sigma^j\tau \in S$. Then $K_a\tau' = (K_a)\overline{\tau}\sigma^j\tau = (b\sigma^j)\tau = a$. Similarly, $K_b\tau' = b$. Hence $\mathsf{rank}\,(\tau') = 2$. Moreover, $\tau' = \tau$. Let $\omega, \nu = \sigma^j\tau \in S$. Then $\tau\nu = \overline{\tau}$, $\overline{\tau}\omega = \tau$, which proves our result. $\square$

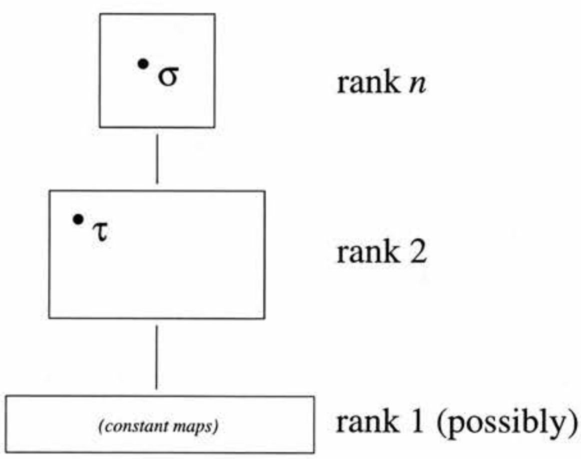

Figure 2.2: Illustration of Lemma 2.3

The following is a consequence of the last lemma.

Corollary 2.4 *Let $S = \langle \sigma, \tau \rangle$. Then for all $\alpha, \beta \in S$ we have $\alpha\mathcal{D}\beta$ if and only if* $\mathsf{rank}\,(\alpha) = \mathsf{rank}\,(\beta)$.

Notation: In the sequel of this thesis, we shall use the following notation:

- D_n is the $\mathcal{D}$-class of rank n of S.
- D_2 is the $\mathcal{D}$-class of rank 2 of S.
- D_1 is the $\mathcal{D}$-class of rank 1."

2 Regular semigroups

In this section we give a necessary and sufficient condition on τ such that the semigroup $\langle\sigma, \tau\rangle$ is regular.

The following definition is fundamental to fulfil this task. As considered at the beginning of this chapter, let $\tau \in \mathcal{T}_n$ be a transformation of rank two such that $\mathsf{im}\,(\tau) = \{a, b\}$.

Definition 2.5 We say that τ has *periodic image* if the following holds for all $x \in X$,

$$x \in K_a \text{ if and only if } x\sigma^{b-a} \in K_a.$$

Example 2.6 Let $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 1 & 3 \end{pmatrix} \in \mathcal{T}_4$. It is easy to check that this transformation has periodic image.

The following result shows us that in the definition of periodic image (Definition 2.5) we can take K_b instead of K_a.

Lemma 2.7 *The transformation τ has periodic image if and only if the following holds for all $x \in X$,*

$$x \in K_b \textit{ if and only if } x\sigma^{b-a} \in K_b.$$

PROOF. We know that

$$x \in K_a \text{ if and only if } x\sigma^{b-a} \in K_a$$

is equivalent to

$$x \notin K_a \text{ if and only if } x\sigma^{b-a} \notin K_a,$$

and, as $X = K_a \cup K_b$, with $K_a \cap K_b = \emptyset$, we have the required result. $\square$

A consequence of this definition is the following.

Lemma 2.8 *If τ has periodic image then* $\gcd(n, b-a) > 1$.

PROOF. Let $X = \{1, 2, \dots, n\}$. Take $x = n$ and assume, without loss of generality, that $n \in K_a$. Then $x\sigma^{b-a} = b - a \in K_a$, by definition of periodic image. Similarly $(b-a)\sigma^{b-a} = 2(b-a) \in K_a$. Inductively we have that $\langle b-a \rangle$, the cyclic subgroup of $(X, +)$, is contained in K_a. But since $\mathsf{rank}(\tau) = 2$ then $K_a \subset X$ and $K_a \neq X$. Note that $(\mathbb{Z}_n, +) \cong (X, +)$. Therefore $\langle b-a \rangle \neq \mathbb{Z}_n$ which is equivalent to having $\gcd(n, b-a) > 1$. □

This last lemma can be seen as a first test in the process of identification of a transformation which may have periodic image.

Definition 2.9 We say that the semigroup $S = \langle \sigma, \tau \rangle$ has *periodic image* if τ has periodic image.

We shall see later that this last definition is consistent, i.e. this definition is independent of the choice of the generator of rank two.

We state now the main theorem of this section.

Theorem 2.10 *The semigroup S is a regular semigroup if and only if S does not have periodic image.*

Throughout this section we shall gather all the auxiliary results in order to prove the above theorem.

Firstly, we give some alternative characterisations of the property of periodic image.

Lemma 2.11 *The transformation τ has periodic image if and only, for all $x \in X$,*

$$x \in K_a \text{ if and only if } x\sigma^{u(b-a)} \in K_a, \text{ for all } u \in \mathbb{N}.$$

PROOF. This result is proved by induction on u.

If $u = 1$, then the equivalence holds, by definition.

Take $u \in \mathbb{N}$. Suppose that for $x \in X$, we have $x \in K_a$ if and only if $x\sigma^{u(b-a)} \in K_a$. Let $y = x\sigma^{u(b-a)}$. Because τ has periodic image we have that $y \in K_a$ if and only if $y\sigma^{b-a} \in K_a$. Replacing y by $x\sigma^{u(b-a)}$ we have $x\sigma^{u(b-a)} \in K_a$ if and only if $x\sigma^{(u+1)(b-a)} \in K_a$. Using the inductive hypothesis we have $x \in K_a$ if and only if $x\sigma^{(u+1)(b-a)} \in K_a$. □

Remark 2.12 Let $\tau \in \mathcal{T}_n$ be as before. Then the following are equivalent:

(i) For all $x \in X$, $x \in K_a$ if and only if $x\sigma^{-(b-a)} \in K_a$;

(ii) for all $x \in X$, $x \in K_a$ if and only if $x\sigma^{u(-(b-a))} \in K_a$, for all $u \in \mathbb{N}$.

This is proved exactly as the previous lemma, just replacing $b-a$ by $-(b-a)$.

Corollary 2.13 *The transformation τ has periodic image if and only if, for all $x \in X$,*

$$x \in K_a \textit{ if and only if } x\sigma^{-(b-a)} \in K_a.$$

PROOF. Note that for all $x \in X$, we have $x = x\sigma^{b-a}\sigma^{-(b-a)}$.

Suppose that τ has periodic image. We have that $x \in K_a$ if and only if $x\sigma^{-(b-a)}\sigma^{b-a} \in K_a$. But, by assumption we have $x\sigma^{-(b-a)}\sigma^{b-a} \in K_a$ if and only if $x\sigma^{-(b-a)} \in K_a$. Therefore we have $x \in K_a$ if and only if $x\sigma^{-(b-a)} \in K_a$.

The converse follows using the same reasoning. □

This last corollary leads to a further generalisation of Lemma 2.11.

Corollary 2.14 *The transformation τ has periodic image if and only if, for all $x \in X$,*

$$x \in K_a \textit{ if and only if } x\sigma^{u(b-a)} \in K_a, \textit{ with } u \in \mathbb{Z}.$$

PROOF. It follows from the previous corollary, the remark that precedes it and Lemma 2.11. □

Remark 2.15 Let $\tau \in \mathcal{T}_n$ be a transformation of rank 2 and let $p \in \mathbb{N}$. It is easy to observe that the following are also equivalent:

(i) For all $x \in X$, $x \in K_a$ if and only if $x\sigma^p \in K_a$;

(ii) for all $x \in X$, $x \in K_a$ if and only if $x\sigma^{up} \in K_a$, for all $u \in \mathbb{Z}$.

This is proved using the same process as in the previous corollary.

A further generalisation of Definition 2.5 is presented in the next result.

Lemma 2.16 *Let $\tau \in \mathcal{T}_n$ such that* $\mathsf{rank}\,(\tau) = 2$ *and consider* $d = \gcd(n, b-a)$. *Then the following are equivalent:*

(i) *τ has periodic image;*

(ii) *for all $x \in X$, $x \in K_a$ if and only if $x\sigma^d \in K_a$.*

(iii) *for all $x \in X$, $x \in K_a$ if and only if $x\sigma^{ud} \in K_a$, for all $u \in \mathbb{Z}$.*

PROOF. Because of the previous remark we are only going to prove that (i) implies (ii) and (iii) implies (i). This is done as follows.
(i) $\Rightarrow$ (ii) Let $x \in K_a$. Then $x\sigma^{u(b-a)} \in K_a$, for every $u \in \mathbb{Z}$.

As $d = \gcd(n, b-a)$, using the Euclidean algorithm (see, for example, [22]) we can find $z_1, z_2 \in \mathbb{Z}$ such that $d = z_1(b-a) + z_2n$. Hence $x\sigma^d = x\sigma^{z_1(b-a)+z_2n} = x\sigma^{z_1(b-a)}\sigma^{z_2n} = x\sigma^{z_1(b-a)}$. Therefore, taking $u = z_1$, we conclude that $x\sigma^d = x\sigma^{u(b-a)} \in K_a$.

Conversely, let $x \in X$ such that $x\sigma^d \in K_a$. Using the same argument as before we have $x\sigma^d = x\sigma^{z_1(b-a)}$ and therefore, by the assumption with $u = z_1$, we get $x \in K_a$.

(iii) $\Rightarrow$ (i) Let $x \in K_a$. By the assumption, with $u = \frac{b-a}{d}$, we have $x\sigma^{(b-a/d)d} = x\sigma^{b-a} \in K_a$.

Conversely, let $x \in X$ such that $x\sigma^{b-a} \in K_a$. Since $x\sigma^{b-a} = x\sigma^{(b-a/d)d}$ we can conclude (by the assumption with $u = \frac{b-a}{d}$) that $x \in K_a$. $\square$

The first step towards the proof of Theorem 2.10 is to show that a transformation of rank two is not an idempotent if and only if it has periodic image. We need some preliminary results.

Lemma 2.17 *If τ has periodic image then τ is not an idempotent.*

PROOF. Note that, because $\mathsf{rank}\,(\tau) = 2$, we have that τ is an idempotent if and only if $a\tau = a$ and $b\tau = b$.

Suppose, without loss of generality, that $a \in K_a$. Then $b = a+b-a = a\sigma^{b-a} \in K_a$, because τ has periodic image. Hence $a, b \in K_a$, which implies that τ is not an idempotent. $\square$

The next result states that the property periodic image is an invariant within the elements of rank two in S.

Lemma 2.18 *If τ has periodic image then every element in S of rank 2 has periodic image.*

PROOF. Let $\alpha \in S = \langle \sigma, \tau \rangle$ be an element of rank 2. Suppose that τ has periodic image. From Lemma 2.2 we only need to prove that this result holds for two different cases.

Let us take $\alpha = \sigma^i\tau\sigma^j$, for some $i, j \in \{1, \dots, n\}$. We have $\mathsf{im}\,(\sigma^i\tau\sigma^j) = \{a', b'\}$, where $a' = a\sigma^j$ and $b' = b\sigma^j$. Note that $b' - a' = b - a$. Also $\mathsf{ker}\,(\sigma^i\tau\sigma^j) = \{K'_{a'}, K'_{b'}\}$ with $K'_{a'} = (a')(\sigma^i\tau\sigma^j)^{-1} = (a\sigma^j)(\sigma^i\tau\sigma^j)^{-1} = (a\sigma^j)(\sigma^{-j}\tau^{-1}\sigma^{-i}) = (a)\tau^{-1}\sigma^{-i} = K_a\sigma^{-i}$. Similarly, $K'_{b'} = K_b\sigma^{-i}$.

Take $x \in K'_{a'}$. Then there is $y \in K_a$ such that $x = y\sigma^{-i}$. Because $b' - a' = b - a$, our aim is to prove that $x\sigma^{b-a} \in K'_{a'}$, i.e. $(x\sigma^{b-a})\sigma^i\tau\sigma^j = a'$. For this, $(x\sigma^{b-a})\sigma^i\tau\sigma^j = (x)\sigma^i\sigma^{b-a}\tau\sigma^j = (y\sigma^{-i})\sigma^i\sigma^{b-a}\tau\sigma^j = (y\sigma^{b-a})\tau\sigma^j$. But because τ has periodic image, $y \in K_a \Leftrightarrow y\sigma^{b-a} \in K_a$, i.e. $y\sigma^{b-a}\tau = a$. Hence $(y\sigma^{b-a})\tau\sigma^j = a\sigma^j = a'$.

On the other hand, take $x \in X$ such that $x\sigma^{b-a} \in K'_{a'}$. Then there is $y \in K_a$ such that $x\sigma^{b-a} = y\sigma^{-i} \Leftrightarrow x = (y)\sigma^{-(b-a)}\sigma^{-i}$. Our aim is to prove that $x \in K'_{a'}$. For this, $(x)\sigma^i\tau\sigma^j = (y)\sigma^{-(b-a)}\sigma^{-i}\sigma^i\tau\sigma^j = (y)\sigma^{-(b-a)}\tau\sigma^j$. Because τ has periodic image and using Lemma 2.16, if $y\sigma^{-(b-a)} \in K_a$ then $y \in K_a$. Hence $y(\sigma^{-(b-a)}\tau\sigma^j) = a\sigma^j = a'$. Therefore $\sigma^i\tau\sigma^j$ has periodic image.

Let $\alpha = \sigma^i\tau\sigma^j\tau\sigma^k$, for some $i, j, k \in \{1, \ldots, n\}$. Let us suppose that $\tau\sigma^j\tau \neq \tau$ (otherwise we are in Case 1 again). We know that $\mathsf{ker}\,(\tau\sigma^j\tau) = \mathsf{ker}\,(\tau)$ and $\mathsf{im}\,(\tau\sigma^j\tau) = \mathsf{im}\,(\tau)$. So it is obvious that τ has periodic image if and only if $\tau\sigma^j\tau$ has periodic image. Hence, using Case 1, our result holds. □

A consequence of the last result is the following one.

Corollary 2.19 *Let $S = \langle \sigma, \tau \rangle$, where τ has periodic image. Then S has no idempotents of rank 2.*

PROOF. Let $\alpha \in S$ be an element of rank 2. Then α has periodic image (Lemma 2.18). It follows from Lemma 2.17 that α is not an idempotent. □

One of the main results follows.

Theorem 2.20 *The semigroup $S = \langle \sigma, \tau \rangle$ has no idempotents of rank 2 if and only if τ has periodic image.*

PROOF. Suppose that τ does not have periodic image. Let us fix and element $x \in X$ such that $x \in K_a$ and $x\sigma^{b-a} \in K_b$. We claim that there is a number $r \in \mathbb{Z}$

such that $a\sigma^r = x$ and $b\sigma^r = x\sigma^{b-a}$. If we take $r = x - a$, these two equalities hold. Therefore the element $\sigma^r\tau \in S$ is an idempotent, because $a(\sigma^r\tau) = x\tau = a$ and $b\sigma^r\tau = x(\sigma^{b-a}\tau) = b$.

The converse is just Corollary 2.19. □

This last result allows us to further describe the normal forms of a semigroup with periodic image.

Corollary 2.21 *Let $S = \langle\sigma, \tau\rangle$ be such that τ has periodic image. Then every element $\alpha \in S$ of rank 2 is of the form $\sigma^i\tau\sigma^j$, for some $i, j \in \{1, \ldots, n\}$.*

PROOF. Let us take an element $\alpha = \sigma^i\tau\sigma^k\tau\sigma^j \in S$, for some $i, k, j \in \{1, \ldots, n\}$. We shall see that $\mathsf{rank}(\alpha) = 1$. This is equivalent to proving that α is not $\mathcal{D}$-related to τ.

Define $\alpha_1 = \sigma^i\tau\sigma^k$ and $\alpha_2 = \tau\sigma^j$. It is clear that both these elements are in S and that $\mathsf{rank}(\alpha_1) = \mathsf{rank}(\alpha_2) = 2$. From Lemma 2.3, we know that $\alpha_1\mathcal{D}\alpha_2$. But, because τ has periodic image, the previous theorem allows us to conclude that there are no idempotents of rank 2 in S. By Proposition 1.18, this implies that $\alpha = \alpha_1\alpha_2$ is not $\mathcal{D}$-related to τ. □

Another important consequence of this result is the fact that any element of rank two of S, together with the n-cycle σ, generates S.

Corollary 2.22 *Let $S = \langle\sigma, \tau\rangle$. For every element $\alpha \in S$ of rank 2 we have $S = \langle\sigma, \alpha\rangle$.*

PROOF. Take $\alpha \in S$ and define $S' = \langle\sigma, \alpha\rangle$. Clearly $S' \leq S$. We can divide this study into two different cases:

Case 1: Suppose that α has periodic image. From the previous corollary, $\alpha = \sigma^i\tau\sigma^j$, for some $i, j \in \{1, \ldots, n\}$. Then $\sigma^{n-i}\alpha\sigma^{n-j} = \tau \in S'$. Hence $S' = S$.

Case 2: Suppose that α does not have periodic image. From Lemma 2.2, α can only take two possible different forms. If $\alpha = \sigma^i\tau\sigma^j$, for some $i, j \in \{1, \ldots, n\}$ then the result follows using the same argument as in Case 1. Suppose $\alpha = \sigma^i\tau\sigma^j\tau\sigma^k$, for some $i, j, k \in \{1, \ldots, n\}$ and let $\tau\sigma^j\tau \neq \tau$ (otherwise we are in the case already considered). Then we have $\overline{\tau} = \tau\sigma^j\tau$. Using the same reasoning as in Case 1, it is clear that $S' = \langle\sigma, \overline{\tau}\rangle$. Because α does not have periodic image then neither does $\overline{\tau}$ (follows from Lemma 2.18). As in the proof of Theorem 2.20, there is $x \in X$ such that $x \in K_a$ and $x\sigma^{b-a} \in K_b$. Note that $\mathsf{im}\,(\tau) = \mathsf{im}\,(\overline{\tau}) = \{a, b\}$ and $\mathsf{ker}\,(\tau) = \mathsf{ker}\,(\overline{\tau}) = \{K_a, K_b\}$. In the same way as in the proof of Theorem 2.20, it is easy to confirm that there is an integer r such that $a\sigma^r = x$ and $b\sigma^r = x\sigma^{b-a}$. Consequently, $\overline{\tau}\sigma^r\overline{\tau} = \tau$. To prove this equality, $(K_a)\overline{\tau}\sigma^r\overline{\tau} = (b)\sigma^r\overline{\tau} = (x)\sigma^{b-a}\overline{\tau} = a$. Analogously, $(K_b)\overline{\tau}\sigma^r\overline{\tau} = b$. Hence $\mathsf{rank}\,(\overline{\tau}\sigma^r\overline{\tau}) = 2$ and consequently $\overline{\tau}\sigma^r\overline{\tau} = \tau$. Thus $\tau \in S'$ and therefore we get $S' = S$. □

Remark 2.23 From the last result and Lemma 2.18, we have that Definition 2.9 is consistent.

From Proposition 1.12 we know that a semigroup S is regular if and only if each $\mathcal{L}$-class and each $\mathcal{R}$-class of S contains at least one idempotent. From Theorem 2.20 we know that there are no idempotents of rank two which implies that each $\mathcal{L}$-class and each $\mathcal{R}$-class in the $\mathcal{D}$-class of rank two does not have any idempotents and therefore S is not regular. In the next result we prove that the converse also holds.

Theorem 2.24 *If S does not have periodic image then S is a regular semigroup.*

PROOF. From Lemma 2.3, we know that S has, at most, three $\mathcal{D}$-classes;

(i) D_n, the $\mathcal{D}$-class of rank n which is the group $\langle\sigma\rangle$. Hence D_n is regular.

(ii) D_2, the $\mathcal{D}$-class of rank 2.

(iii) D_1, if it exists, the $\mathcal{D}$-class of rank 1, where every element is an idempotent. Hence D_1 is regular.

Hence, to prove our result we just need to prove that D_2 is regular. As S does not have periodic image, S has some idempotents of rank 2, by Theorem 2.20. Using Proposition 1.11 we conclude that D_2 is regular and we deduce that S is regular. □

PROOF. (of Theorem 2.10) It follows as a straightforward consequence of the previous theorem and the note that precedes it.

3 Completely regular semigroups

In this section we shall give several necessary and sufficient conditions for a semigroup to be completely regular. As mentioned in Section 3 of Chapter 1, a semigroup is completely regular if and only if every $\mathcal{H}$-class is a group if and only if every $\mathcal{H}$-class has an idempotent. By Proposition 1.18, it follows that a finite semigroup is completely regular if every multiplication within a certain $\mathcal{D}$-class remains in that $\mathcal{D}$-class.

Proposition 2.25 *A semigroup $T \leq \mathcal{T}_n$ is completely regular if and only if $\mathsf{rank}\,(\alpha^2) = \mathsf{rank}\,(\alpha)$, for every $\alpha \in T$.*

PROOF. Suppose that T is completely regular. Take $\alpha \in T$. Then H_α, the $\mathcal{H}$-class of α, is a group. Therefore, $\alpha^2 \in H_\alpha$ which implies that $\alpha \mathcal{D} \alpha^2$ and therefore $\mathsf{rank}\,(\alpha) = \mathsf{rank}\,(\alpha^2)$.

Conversely, suppose that $\mathsf{rank}\,(\alpha) = \mathsf{rank}\,(\alpha^2)$. Therefore, since $\mathsf{im}\,(\alpha^2) \subseteq \mathsf{im}\,(\alpha)$ and $\ker\,(\alpha) \subseteq \ker\,(\alpha^2)$, we can conclude the equality of these sets. This means that $\mathsf{im}\,(\alpha)$ is a transversal of $\mathsf{ker}\,(\alpha)$, i.e. $\mathsf{im}\,(\alpha)$ contains exactly one element from each of the kernel classes of α. Hence there exists $p \in \mathbb{N}$ such that $\alpha^{p+1} = \alpha$. If $p = 1$ then $\alpha^2 = \alpha$ and α is an idempotent. If $p > 1$ then α^p is an idempotent, since $\alpha^p\alpha^p = \alpha^{p+1}\alpha^{p-1} = \alpha\alpha^{p-1} = \alpha^p$. Thus $\alpha\mathcal{H}\alpha^p$ and so H_α contains an idempotent. Because α is arbitrary we have that T is completely regular. □

For further details about completely regular semigroups, see [16].

For the semigroups we have been studying we shall use the following characterisation of completely regular semigroup.

Lemma 2.26 *Let $S = \langle \sigma, \tau \rangle$. Then S is completely regular if and only if S does not contain constant maps.*

PROOF. Suppose that S is completely regular. Then there are three different cases.

(i) If $\alpha, \beta \in D_n$ then $\alpha\beta \in D_n$ (obvious). Hence $\mathsf{rank}\,(\alpha\beta) = n$.

(ii) If $\alpha \in D_n$ and $\beta \in D_2$ or $\alpha \in D_2$ and $\beta \in D_n$ then $\alpha\beta \in D_2$ (obvious). Hence $\mathsf{rank}\,(\alpha\beta) = 2$.

(iii) If $\alpha, \beta \in D_2$ then $\alpha\beta \in D_2$. This holds because every $\mathcal{H}$-class in D_2 has an idempotent. Therefore $\alpha\beta \in R_\alpha \cap L_\beta$, because $L_\alpha \cap R_\beta$ contains an idempotent (see Proposition 1.9). Hence $\alpha\beta \in D_2$ and consequently $\mathsf{rank}\,(\alpha\beta) = 2$.

Having in mind the normal forms of S, we conclude that every product in S has rank greater or equal than 2, that is, for all $\alpha \in S$, $\mathsf{rank}\,(\alpha) \neq 1$, i.e. S does not contain constant maps.

Conversely, suppose that S does not have any constant maps. Then we must have that for all $\alpha, \beta \in S$, $\mathsf{rank}\,(\alpha\beta) \neq 1$. Therefore $\mathsf{rank}\,(\alpha^2) = \mathsf{rank}\,(\alpha)$ and S is completely regular. □

The following definition is essential for the process of finding a necessary and sufficient condition (on the generator of rank two) for a semigroup to be completely regular.

As before $\tau \in \mathcal{T}_n$ is a transformation of rank two such that $\mathsf{im}(\tau) = \{a, b\}$ and $\mathsf{ker}(\tau) = \{K_a, K_b\}$, with $K_a = a\tau^{-1}$ and $K_b = b\tau^{-1}$.

Definition 2.27 We say that τ has *conjugate periodic image* if the following holds for all $x \in X$,

$$x \in K_a \text{ if and only if } x\sigma^{b-a} \in K_b.$$

Remark 2.28 It is easy to verify that this definition is equivalent to its "dual":

The transformation τ has conjugate periodic image if and only if the following holds for all $x \in X$,

$$x \in K_b \text{ if and only if } x\sigma^{b-a} \in K_a.$$

Example 2.29 The transformation $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 1 & 2 \end{pmatrix} \in \mathcal{T}_4$ and the transformation $\lambda = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 1 & 4 & 4 & 4 \end{pmatrix} \in \mathcal{T}_6$ have conjugate periodic image.

Definition 2.30 We say that the semigroup $S = \langle \sigma, \tau \rangle$ has *conjugate periodic image* if τ has conjugate periodic image.

In this section we shall prove the following:

Theorem 2.31 *S is a completely regular semigroup if and only if S has conjugate periodic image.*

Let us see some equivalent formulations of the definition of conjugate periodic image.

Remark 2.32 It is straightforward to check that if τ has conjugate periodic image then, for all $x \in X$,

$$x \in K_a \text{ if and only if } x\sigma^{v(b-a)} \in K_a,$$

for all $v \in \mathbb{N}$, v even.

Lemma 2.33 *The transformation τ has conjugate periodic image if and only if, for all $x \in X$,*

$$x \in K_a \textit{ if and only if } x\sigma^{u(b-a)} \in K_b,$$

for all $u \in \mathbb{N}, u$ odd.

PROOF. Suppose that τ has conjugate periodic image. This will be done by induction on $u \in \mathbb{N}$, with u odd.

If $u = 1$ then we have that the equivalence $x \in X$, $x \in K_a$ if and only if $x\sigma^{b-a} \in K_b$ holds, by definition of conjugate periodic image.

Suppose that $x \in K_a$ if and only if $x\sigma^{u(b-a)} \in K_b$, for a fixed u odd. We shall prove that this equivalence holds for $u+2$ (the next odd number). Let $x \in X$ be such that $x \in K_a$. By induction hypothesis, we have that $x \in K_a$ if and only if $x\sigma^{u(b-a)} \in K_b$. Because τ has conjugate periodic image we have that $x\sigma^{u(b-a)} \in K_b$ if and only if $x\sigma^{u(b-a)}\sigma^{b-a} \in K_a$. By the same reason, $x\sigma^{u(b-a)}\sigma^{b-a} \in K_a$ if and only if $x\sigma^{u(b-a)}\sigma^{b-a}\sigma^{b-a} \in K_b$. Since $x\sigma^{u(b-a)+(b-a)+(b-a)} = x\sigma^{(u+2)(b-a)}$, we have that $x \in K_a$ if and only if $x\sigma^{(u+2)(b-a)} \in K_b$.

The converse of the lemma is clear. $\square$

Remark 2.34 Let $\tau \in \mathcal{T}_n$ be as before. Then the following are equivalent:

(i) For all $x \in X$,

$$x \in K_a \text{ if and only if } x\sigma^{-(b-a)} \in K_b;$$

(ii) for all $x \in X$,

$$x \in K_a \text{ if and only if } x\sigma^{u(-(b-a))} \in K_b,$$

for all $u, v \in \mathbb{N}$, u odd.

This is proved exactly as the previous lemma, just replacing $b-a$ by $-(b-a)$.

Similarly to the previous remark, one can check that if τ satisfies (i) then for all $x \in X$ we have

$$x \in K_a \text{ if and only if } x\sigma^{v(-(b-a))} \in K_a,$$

for all $v \in \mathbb{N}$, v even.

Lemma 2.35 *The transformation τ has conjugate periodic image if and only if, for all $x \in X$,*

$$x \in K_a \textit{ if and only if } x\sigma^{-(b-a)} \in K_b.$$

PROOF. Suppose that τ has conjugate periodic image. By definition, we have that for all $x \in X$, $x \in K_a$ if and only if $x\sigma^{b-a} \in K_b$. This is equivalent to having for all $x \in X$, $x \in K_b$ if and only if $x\sigma^{b-a} \in K_a$. Thus $x = (x\sigma^{-(b-a)})\sigma^{b-a} \in K_a$ if and only if $x\sigma^{-(b-a)} \in K_b$.

The converse is analogous. □

Before proving the next corollary we need an auxiliary result.

Lemma 2.36 *Let $\tau \in \mathcal{T}_n$ be a transformation of rank 2 as before and let $d = \gcd(n, b-a)$. If τ has conjugate periodic image then n/d is even. Furthermore, n is even and $\frac{b-a}{d}$ is odd.*

PROOF. Let $x \in K_a$. We have that $x = x\sigma^{n\frac{b-a}{d}} = x\sigma^{\frac{n}{d}(b-a)}$. From Remark 2.32, we can conclude that n/d is even and so n is also even.

Suppose that $\frac{b-a}{d}$ is even. Then there is $q \in \mathbb{N}$ such that $\frac{b-a}{d} = 2q$. Therefore $b - a = 2qd$. As $n = 2ld$, for some $l \in \mathbb{N}$, we get that $2d$ is a common divisor of n and $b - a$. Since $d < 2d$, this contradicts the fact that $d = \gcd(n, b-a)$. Thus $\frac{b-a}{d}$ is odd. □

Using Lemma 2.35, we can generalise Lemma 2.33.

Corollary 2.37 *The transformation τ has conjugate periodic image if and only if, for all $x \in X$,*

$$x \in K_a \text{ if and only if } x\sigma^{u(b-a)} \in K_b,$$

for all $u \in \mathbb{Z}$, u odd.

PROOF. To prove this, we ought to observe the following. Let $w \in \mathbb{Z}$. If $w \in \mathbb{N}$, we have that $\sigma^{w(b-a)} = \sigma^{w'(b-a)}$, where $w' \in \{0, \ldots, n-1\}$. Furthermore, w is odd (even) if and only if w' is odd (even), because n is even. If $w \in \mathbb{Z} \setminus \mathbb{N}$ then we have that $-w \in \mathbb{N}$ and we have that w is odd (even) if and only if $-w$ is odd (even), because n is even. Therefore we can make the same statement as above, with w replaced by $-w$.

Concluding, if we take $w \in \mathbb{Z}$ then there exists $w' \in \{0, \ldots, n-1\}$ such that $\sigma^{w(b-a)} = \sigma^{w'(b-a)}$ and w and w' have the same parity.

Bearing in mind this observation and Lemma 2.35, this result is proved using Lemma 2.33, when $u \in \mathbb{N}$ (or $v \in \mathbb{N}$) or Remark 2.34, when $u \in \mathbb{Z} \setminus \mathbb{N}$ (or $v \in \mathbb{Z} \setminus \mathbb{N}$). □

Note that if τ has conjugate periodic image then for all $x \in X$,

$$x \in K_a \text{ if and only if } x\sigma^{v(b-a)} \in K_a,$$

for all $v \in \mathbb{Z}$, v even.

Remark 2.38

(a) Using the same process that lead us to the previous corollary, one can prove the following:

Let $\tau \in \mathcal{T}_n$ be a transformation of rank 2 and let $p \in \mathbb{N}$. The following are equivalent:

(i) *For all $x \in X$, $x \in K_a$ if and only if $x\sigma^p \in K_b$;*

(ii) *for all $x \in X$,*

$$x \in K_a \textit{ if and only if } x\sigma^{up} \in K_b,$$

for all $u \in \mathbb{Z}$, u odd.

(b) We also have that if τ satisfies (i) then for all $x \in X$,

$$x \in K_a \text{ if and only if } x\sigma^{vp} \in K_a,$$

for all $v \in \mathbb{Z}$, v even.

Lemma 2.39 *Let $\tau \in \mathcal{T}_n$ such that $\mathsf{rank}\,(\tau) = 2$ and consider $d = \gcd(n, b-a)$. The following are equivalent:*

(i) *τ has conjugate periodic image;*

(ii) *for all $x \in X$, $x \in K_a$ if and only if $x\sigma^d \in K_b$;*

(iii) *for all $x \in X$,*

$$x \in K_a \textit{ if and only if } x\sigma^{ud} \in K_b,$$

for all $u \in \mathbb{Z}$, u odd.

PROOF. In view of of Remark 2.38 we shall only prove that (i) implies (ii) and that (iii) implies (i).

(i) ⇒ (ii) Let $x \in K_a$. By Corollary 2.37, we have that $x\sigma^{u(b-a)} \in K_b$, for all u odd. We shall find u odd, such that $x\sigma^d = x\sigma^{u(b-a)}$, which will prove our result.

From $d = \gcd(n, b-a)$ and using the Euclidean algorithm, there are $z_1, z_2 \in \mathbb{Z}$ such that $d = z_1(b-a) + z_2 n$. Thus $1 = z_1\frac{(b-a)}{d} + z_2\frac{n}{d}$. From Lemma 2.36, we know that n/d is even. Therefore $z_2\frac{n}{d}$ is even and so $z_1\frac{(b-a)}{d}$ is odd. Because $\frac{(b-a)}{d}$ is odd (Lemma 2.36) we have that z_1 is odd.

It is clear that $x\sigma^d = x\sigma^{z_1(b-a)+z_2 n} = x\sigma^{z_1(b-a)}\sigma^{z_2 n} = x\sigma^{z_1(b-a)}$. Defining $u = z_1$, which is odd, our result follows.

Conversely, if $x \in X$ is such that $x\sigma^d \in K_b$ then, using the same argument we show that there is u odd such that $x\sigma^d = x\sigma^{u(b-a)}$. Hence, by Corollary 2.37, $x \in K_a$.

(iii) ⇒ (i) Let $x \in K_a$. Then $x\sigma^d \in K_b$. It is clear that $x = x\sigma^{\frac{n}{d}d}$. Because τ satisfies (iii) (and therefore it satisfies (i) as well) and using Remark 2.38, we have that $\frac{n}{d}$ is even. Therefore, as in the proof of Lemma 2.36, we can conclude that $\frac{b-a}{d}$ is odd. Hence, by the assumption, $x\sigma^{\frac{b-a}{d}d} = x\sigma^{b-a} \in K_b$.

Conversely, let $x \in X$ such that $x\sigma^{b-a} \in K_b$. We can write $x\sigma^{\frac{b-a}{d}d}$. By the assumption and again because $\frac{b-a}{d}$ is odd, we have that $x \in K_a$. □

A consequence of this lemma is the following.

Remark 2.40 If τ has conjugate periodic image then the elements of the image of τ are in different kernel classes. For this, suppose, without loss of generality, that $a \in K_a$. Then $b = a\sigma^{b-a} \in K_b$.

Similarly to the previous section, we shall prove that this property is an invariant within the elements of rank two in S.

Lemma 2.41 *If τ has conjugate periodic image then every element in S of rank 2 has conjugate periodic image.*

PROOF. Let $\alpha \in S$ such that $\mathsf{rank}\,(\alpha) = 2$. Suppose that τ has conjugate periodic image. There are only two different cases to study, due to Lemma 2.2.

Let us take $\alpha = \sigma^i\tau\sigma^j$, for some $i, j \in \{1, \ldots, n\}$. We have that $\mathsf{im}\,(\sigma^i\tau\sigma^j) = \{a', b'\}$, where $a' = a\sigma^j$, $b' = b\sigma^j$. Note that $b' - a' = b - a$. Also $\mathsf{ker}\,(\sigma^i\tau\sigma^j) = \{K'_{a'}, K'_{b'}\}$ with $K'_{a'} = (a')(\sigma^i\tau\sigma^j)^{-1} = (a)\sigma^j\sigma^{-j}\tau^{-1}\sigma^{-i} = (a)\tau^{-1}\sigma^{-i} = K_a\sigma^{-i}$. In the same way we have $K'_{b'} = K_b\sigma^{-i}$.

Take $x \in K'_{a'}$. Then there is $y \in K_a$ such that $x = y\sigma^{-i}$. Our aim is to prove that $x\sigma^{b-a} \in K'_{b'}$, i.e. $(x\sigma^{b-a})\sigma^i\tau\sigma^j = b'$. For this, $(x\sigma^{b-a})\sigma^i\tau\sigma^j = (x)\sigma^i\sigma^{b-a}\tau\sigma^j = (y)\sigma^{-i}\sigma^i\sigma^{b-a}\tau\sigma^j = (y\sigma^{b-a})\tau\sigma^j$. But because τ has conjugate periodic image $y \in K_a$ if and only if $y\sigma^{b-a} \in K_b$. Therefore $(y\sigma^{b-a})\tau = b$ and consequently $(y\sigma^{b-a})\tau\sigma^j = b\sigma^j = b'$.

On the other hand, take $x \in X$ such that $x\sigma^{b-a} \in K'_{b'}$. Then there is $y \in K_b$ such that $x\sigma^{b-a} = y\sigma^{-i}$, so $x = (y)\sigma^{-(b-a)}\sigma^{-i}$. Our aim is to prove that $x \in K'_{a'}$. For this, $(x)\sigma^i\tau\sigma^j = (y)\sigma^{-(b-a)}\sigma^{-i}\sigma^i\tau\sigma^j = (y)\sigma^{-(b-a)}\tau\sigma^j$. Because τ has conjugate periodic image and using Lemma 2.35 we know that if $y \in K_b$ then $y\sigma^{-(b-a)} \in K_a$. Hence $(y)\sigma^{-(b-a)}\tau\sigma^j = a\sigma^j = a'$.

Therefore $\sigma^i\tau\sigma^j$ has conjugate periodic image.

Let $\alpha = \sigma^i\tau\sigma^j\tau\sigma^k$, for some $i, j, k \in \{1, \ldots, n\}$ and suppose that $\tau\sigma^j\tau \neq \tau$ (otherwise we are in Case 1 again). We know that $\mathsf{ker}\,(\tau\sigma^j\tau) = \mathsf{ker}\,(\tau)$ and $\mathsf{im}\,(\tau\sigma^j\tau) = \mathsf{im}\,(\tau)$. Thus $\tau\sigma^j\tau = \overline{\tau}$, the conjugate of τ. So it is obvious that τ has conjugate periodic image if and only if $\tau\sigma^j\tau$ has conjugate periodic image. Hence, using Case 1, our result holds. $\square$

Remark 2.42 From Lemma 2.41 and Corollary 2.22, we have that Definition 2.30 is consistent.

We can now prove the main result of this section, which gives us a necessary and sufficient condition on the generator of rank two for a semigroup to be completely regular.

PROOF. (of Theorem 2.31) From Proposition 2.26, it suffices to prove that S has conjugate periodic image if and only if it has no constant maps.

Suppose that S has conjugate periodic image and that there are elements of rank 1 in S. Then there must exist $\alpha, \beta \in S$, with $\mathsf{rank}\,(\alpha) = \mathsf{rank}\,(\beta) = 2$ such that $\mathsf{rank}\,(\alpha\beta) = 1$. Based on Lemma 2.2, we know that $\alpha = \sigma^i \tau \sigma^j \tau \sigma^k$ and $\beta = \sigma^l \tau \sigma^p \tau \sigma^q$, with $i, j, k, l, p, q \in \{1, \ldots, n\}$. Note that we assume the possibility of having $\tau\sigma^j\tau = \tau$ or $\tau\sigma^p\tau = \tau$ (or both or none). In any of these cases, $\tau\sigma^j\tau$ and $\tau\sigma^p\tau$ are $\mathcal{H}$-related with τ (in $\mathcal{T}_n$), because

$$\mathsf{im}\,(\tau\sigma^j\tau) = \mathsf{im}\,(\tau\sigma^p\tau) = \mathsf{im}\,(\tau) = \{a, b\} \text{ and}$$

$$\mathsf{ker}\,(\tau\sigma^j\tau) = \mathsf{ker}\,(\tau\sigma^p\tau) = \mathsf{ker}\,(\tau) = \{K_a, K_b\}.$$

It is easy to observe that $\mathsf{rank}\,(\alpha\beta) = 1$ if and only if $\mathsf{rank}\,(\tau\sigma^j\tau\sigma^r\tau\sigma^p\tau) = 1$, where r is such that $\sigma^r = \sigma^{k+l}$. Also, $\mathsf{rank}\,(\tau\sigma^j\tau\sigma^r\tau\sigma^p\tau) = 1$ if and only if $a\sigma^r$ and $b\sigma^r$ are in the same kernel class. But because $\tau\sigma^j\tau\sigma^r \in S$ and it has rank 2, by Lemma 2.41, $\tau\sigma^j\tau\sigma^r$ has conjugate periodic image and therefore, as referred to in Remark 2.40, $a\sigma^r$ and $b\sigma^r$ must be in different kernel classes, which is a contradiction.

Conversely, suppose that τ does not have conjugate periodic image. Then there must exist $x \in X$ such that $x \in K_a$ and $x\sigma^{b-a} \in K_a$. Because σ is an n-cycle, there is $r \in \mathbb{N}$ such that $a\sigma^r = x$. Furthermore, $b\sigma^r = (a\sigma^{b-a})\sigma^r = (a\sigma^r)\sigma^{b-a} = x\sigma^{b-a}$. This implies that $\tau\sigma^r\tau$ has rank 1 because

$$(K_a)\tau\sigma^r\tau = (a)\sigma^r\tau = x\tau = a \text{ and}$$

$$(K_b)\tau\sigma^r\tau = (b)\sigma^r\tau = (x\sigma^{b-a})\tau = a,$$

which proves our result. $\square$

4 Counting Green's classes

In this section we study the number of $\mathcal{L}$-classes and $\mathcal{R}$-classes, and the size of the $\mathcal{H}$-classes within D_2, the $\mathcal{D}$-class of rank two of the semigroup $S = \langle \sigma, \tau \rangle$.

Our aim is to find structural properties of τ that allow us to count the number of $\mathcal{L}$ and $\mathcal{R}$-classes and the size of the $\mathcal{H}$-classes which compose D_2. The ultimate goal is to compute the possible size of S.

Notation:

- $\mathfrak{L} = \{\mathcal{L}\text{-classes in } D_2\}$, $|\mathfrak{L}|$ = size of $\mathfrak{L}$.
- $\mathfrak{R} = \{\mathcal{R}\text{-classes in } D_2\}$, $|\mathfrak{R}|$ = size of $\mathfrak{R}$.
- $\mathfrak{H} = \{\mathcal{H}\text{-classes in } D_2\}$, $|[\mathfrak{H}]|$ = size of an $\mathcal{H}$-class in D_2. Note that this size is invariant within the set of all $\mathcal{H}$-classes in the same $\mathcal{D}$-class (see Lemma 1.6).

The following result gives us some insight on the Green's relations for this type of semigroups.

Lemma 2.43 *Let $\alpha \in S$ be an element of rank 2. If $\alpha = \sigma^i\tau\sigma^j$ or $\alpha = \sigma^i\tau\sigma^k\tau\sigma^j$, for some $i, j, k \in \{1, \ldots, n\}$, then $\alpha\mathcal{L}\tau\sigma^j$ and $\alpha\mathcal{R}\sigma^i\tau$.*

PROOF. Let $\alpha = \sigma^i\tau\sigma^j$. Choose $u = \sigma^{n-i}, v = \sigma^i \in S$. Then $u\alpha = \tau\sigma^j$ and $v\tau\sigma^j = \alpha$. Thus $\alpha\mathcal{L}\tau\sigma^j$. For the case where $\alpha = \sigma^i\tau\sigma^k\tau\sigma^j$ we can choose the same elements u and v.

Similarly we prove that $\alpha\mathcal{R}\sigma^i\tau$, by choosing $s = \sigma^{n-j}$ and $t = \sigma^j$ and noting that $\alpha s = \sigma^i\tau$ and $\sigma^i\tau t = \alpha$. $\square$

4.1 Number of $\mathcal{L}$-classes

We start our study by evaluating the number of $\mathcal{L}$-classes, i.e. the size of $\mathfrak{L}$.

We need an auxiliary result.

Lemma 2.44 *We have that $\overline{\tau} = \tau\sigma^l$, for some $l \in \mathbb{N}$ if and only if $b - a = n/2$. Furthermore, $l \equiv n/2 \pmod{n}$.*

PROOF. We ought to remember that the element $\overline{\tau}$ is such that $\tau \neq \overline{\tau}$ and

$$\mathsf{ker}\,(\tau) = \mathsf{ker}\,(\overline{\tau}) = \{K_a, K_b\},\ \mathsf{im}\,(\tau) = \mathsf{im}\,(\overline{\tau}) = \{a, b\}.$$

Suppose that $\overline{\tau} = \tau\sigma^l$. Then $K_a\overline{\tau} = b$ if and only if $(K_a)\tau\sigma^l = b$. This is equivalent to writing $a\sigma^l = b$. In a similar way we deduce $b\sigma^l = a$. Hence $a = b\sigma^l = a\sigma^{2l}$. Because σ is an n-cycle we can conclude that $2l \equiv n \pmod{n}$. So, we either have $l \equiv n/2 \pmod{n}$ or $l \equiv n \pmod{n}$. If the latter holds then $a\sigma^l = a\sigma^n = a$, which contradicts our assumption. Hence $l \equiv n/2 \pmod{n}$. So $a\sigma^{n/2} = b$. We also have that $a\sigma^{b-a} = b$ and therefore $b - a \equiv n/2 \pmod{n}$. Thus $b - a = n/2$.

Conversely, suppose that $b - a = n/2$. Then $a\sigma^{n/2} = b$, $b\sigma^{n/2} = a$, $\mathsf{im}\,(\tau\sigma^{n/2}) = \mathsf{im}\,(\tau)$, and $\mathsf{ker}\,(\tau\sigma^{n/2}) = \mathsf{ker}\,(\tau)$. Because $(K_a)\tau\sigma^{n/2} = a\sigma^{n/2} = b$ and $(K_b)\tau\sigma^{n/2} = a$, we deduce that $\overline{\tau} = \tau\sigma^{n/2}$. $\square$

The next result describes all the possible sizes of $|\mathfrak{L}|$.

Theorem 2.45 *Let $S = \langle \sigma, \tau \rangle$. Then the number of $\mathcal{L}$-classes of rank 2 in S is $n/2$ if and only if $b - a = n/2$ and the size of the $\mathcal{H}$-classes of rank 2 in S is 2. Otherwise, the number of $\mathcal{L}$-classes of rank 2 in S is n.*

PROOF. Let us denote by $\mathfrak{L}$ the set of all $\mathcal{L}$-classes of rank 2 in S. We first prove that the only possible sizes for $|\mathfrak{L}|$ are n and $n/2$.

From Lemma 2.43 we have that $\mathfrak{L} = \{L_\tau, L_{\tau\sigma}, L_{\tau\sigma^2}, \ldots, L_{\tau\sigma^{l-1}}\}$, with $l \in \{1, \ldots, n\}$, such that $\tau \mathcal{L} \tau\sigma^l$. Then $\mathsf{im}\,(\tau) = \mathsf{im}\,(\tau\sigma^l)$. Also $\mathsf{ker}\,(\tau) = \mathsf{ker}\,(\tau\sigma^l)$. As we have pointed out previously, there are two different elements of rank 2 which have the same kernel and the same image.

If $\tau = \tau\sigma^l$ then this means that $a = a\sigma^l$ and $b = b\sigma^l$. Hence $l = n$, because σ is an n-cycle, and therefore $|\mathfrak{L}| = n$.

If $\tau \neq \tau\sigma^l$ then we have $\overline{\tau} = \tau\sigma^l$. This means that $a = b\sigma^l$ and $b = a\sigma^l$. Analogously to the proof of the previous lemma, we can conclude that $2l \equiv n \pmod{n}$ and therefore $l \equiv n/2 \pmod{n}$. So $l = n/2$ (n is even) and consequently $|\mathfrak{L}| = n/2$.

We shall prove the necessary and sufficient condition stated in our theorem. Suppose that $|\mathfrak{L}| = n/2$. There exists $l < n$ such that $\tau \mathcal{L} \tau\sigma^l$. Because $l < n$ we have $\tau \neq \tau\sigma^l$. Trivially $\tau \mathcal{R} \tau\sigma^l$. Hence $\tau \mathcal{H} \tau\sigma^l$ and $|[\mathfrak{H}]| = 2$. Because $\tau \neq \tau\sigma^l$ we can conclude that $\overline{\tau} = \tau\sigma^l$. Therefore, using Lemma 2.44, we have $b - a = n/2 = l$.

Conversely, suppose that $b - a = n/2$ and $|[\mathfrak{H}]| = 2$. Then $\overline{\tau} \in S$. We have that $\mathsf{im}\ \tau = \mathsf{im}\ \tau\sigma^{n/2}$, because $b - a = n/2$. Also $\mathsf{ker}\ \tau = \mathsf{ker}\ \tau\sigma^{n/2}$. Note that $\tau \neq \tau\sigma^{n/2}$, because σ is an n-cycle. Since there are only two such elements with the same kernel and the same image, we conclude that $\overline{\tau} = \tau\sigma^{n/2}$. Consequently, $\tau \mathcal{L} \tau\sigma^{n/2}$. Because $\mathcal{L}$ is a right congruence we have $\tau\sigma^i \mathcal{L} \tau\sigma^{n/2}\sigma^i$, for all $i \in \{1, \ldots, n\}$. Therefore, $L_{\tau\sigma^i} = L_{\tau\sigma^{n/2}\sigma^i}$, for all $i \in \{1, \ldots, n\}$. This implies that $|\mathfrak{L}| \leq n/2$.

We have that $\mathsf{im}\ \tau\sigma^i \neq \mathsf{im}\ \tau\sigma^j$, for all $i, j \in \{1, \ldots, n/2\}, i \neq j$. For this, suppose that there is $l \in \{1, \ldots, \frac{n}{2} - 1\}$ such that $\mathsf{im}\ \tau = \mathsf{im}\ \tau\sigma^l$. Then $a = b\sigma^l$ and $b = a\sigma^l$, because $\tau \neq \tau\sigma^l$. Similarly to the previous proof, this implies that $2l \equiv n \pmod{n}$ which is impossible because $l \in \{1, \ldots, \frac{n}{2} - 1\}$. Thus, for all $i, j \in \{1, \ldots, n\}, i \neq j$ we have that $\tau\sigma^i$ is not $\mathcal{L}$-related with $\tau\sigma^j$, or equivalently, for all $i, j \in \{1, \ldots, n/2\}, i \neq j$, we have that $L_{\tau\sigma^i} \neq L_{\tau\sigma^j}$. Hence $\mathfrak{L} = \{L_\tau, L_{\tau\sigma}, L_{\tau\sigma^2}, \ldots, L_{\tau\sigma^{\frac{n}{2}-1}}\}$. $\square$

Observation 2.46 Let us enumerate the possible cases described in the previous theorem:

(i) If $|[\mathfrak{H}]| = 1$ then we have $|\mathfrak{L}| = n$ (regardless of the value of $b - a$).

(ii) If $|[\mathfrak{H}]| = 2$ and $b - a = n/2$ then $|\mathfrak{L}| = n/2$.

(iii) If $b - a \neq n/2$ then for all $i, j \in \{1, \dots, n\}, i \neq j$, we have $\tau\sigma^i \neq \tau\sigma^j$. Hence $\mathsf{im}\ \tau\sigma^i \neq \mathsf{im}\ \tau\sigma^j$. Therefore $|\mathfrak{L}| = n$.

We should remark that in this last case we cannot have $\overline{\tau} = \tau\sigma^l$, for some $l \in \{1, \dots, n\}$ (see Lemma 2.44), although it can happen that $\overline{\tau} \in S$. The next figure illustrates these three possibilities.

Case (i)	τ	$\tau\sigma$	$\dots$	$\tau\sigma^{n-1}$

Case (ii)	τ	$\tau\sigma$	$\dots$	$\tau\sigma^{\frac{n}{2}-1}$
	$\overline{\tau}$	$\overline{\tau}\sigma$	$\dots$	$\overline{\tau}\sigma^{\frac{n}{2}-1}$

Case (iii)	τ	$\tau\sigma$	$\dots$	$\tau\sigma^{n-1}$
	$\overline{\tau}$	$\overline{\tau}\sigma$	$\dots$	$\overline{\tau}\sigma^{n-1}$

Figure 2.3: Description of the possible cases for the size of $\mathcal{L}$-classes

There is another interesting consequence of Theorem 2.45.

Corollary 2.47 *Let $S = \langle \sigma, \tau \rangle$, where $\sigma = (1\ 2\ \dots\ n)$ and $\tau \in \mathcal{T}_n$ is a transformation of rank 2. If $n > 2$ then S is not an inverse semigroup.*

PROOF. Observe that if $n = 2$ then there is only one semigroup of this kind, $S = \{[1, 2], [2, 1]\}$ which is a group isomorphic to C_2 and therefore S is inverse.

Suppose that $n > 2$ and take $S = \langle \sigma, \tau \rangle$. From the previous result we know that $|\mathfrak{L}| = n/2$ or $|\mathfrak{L}| = n$. Therefore $|\mathfrak{L}| > 1$.

Suppose that S is inverse. Then S cannot have any idempotent elements of rank 1. Otherwise $D_1 \neq \emptyset$ and D_1 has only one $\mathcal{R}$-class which contains $|\mathfrak{L}| = n > 2$ idempotents. Therefore S cannot be inverse (by Proposition 1.13).

Thus, S must have only two $\mathcal{D}$-classes: the $\mathcal{D}$-class D_n of all elements of rank n and the $\mathcal{D}$-class D_2 of all elements of rank 2. But then we have that for all $\alpha \in S$, $\mathsf{rank}\,(\alpha^2) = \mathsf{rank}\,(\alpha)$ which means that S is completely regular and consequently each $\mathcal{R}$-class in D_2 has $|\mathfrak{L}|$ idempotents. Hence S cannot be inverse (again by Proposition 1.13). $\square$

4.2 Number of $\mathcal{R}$-classes

In this section we shall study the number of $\mathcal{R}$-classes in D_2. This is a more complex task than in the case of the $\mathcal{L}$-classes.

We need some essential definitions.

Definition 2.48 Let $w_0 \in \mathbb{N}$. Then τ is *w_0-block conjugate* if for every $x \in X$,

$$x \in K_a \text{ if and only if } x\sigma^{w_0} \in K_b.$$

Example 2.49 The transformation $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 1 & 3 & 1 & 3 \end{pmatrix} \in \mathcal{T}_6$ is 3-block conjugate. Nevertheless, this transformation is also 1-block conjugate.

Remark 2.50 Some relevant and straightforward consequences of this definition are:

(i) If $\tau \in \mathcal{T}_n$ is w_0-block conjugate then $|K_a| = |K_b|$. Hence n is even.

(ii) If τ has conjugate periodic image then τ is w_0-block conjugate, for some w_0 (we can take $w_0 = d$). The converse does not hold (see Example 2.49).

Analogously to the previous section (just replacing $b-a$ by w_0), one can check the following two results:

Lemma 2.51 *The following are equivalent:*

(i) *the transformation τ is w_0-block conjugate;*

(ii) *for every $x \in X$, $x \in K_b$ if and only if $x\sigma^{w_0} \in K_a$.*

(iii) *for every $x \in X$, $x \in K_a$ if and only if $x\sigma^{-w_0} \in K_b$.*

Lemma 2.52 *A transformation τ is w_0-block conjugate if and only if for every $x \in X$,*

$$x \in K_a \text{ if and only if } x\sigma^{uw_0} \in K_b,$$

for all $u \in \mathbb{Z}$, u odd.

Remark 2.53 It follows that if τ is w_0-block conjugate then for all $x \in X$,

$$x \in K_a \text{ if and only if } x\sigma^{sw_0} \in K_a,$$

for all $s \in \mathbb{Z}$, s even.

Another alternative characterisation of this property is presented in the next result.

Lemma 2.54 *A transformation τ is w_0-block conjugate if and only if $\overline{\tau} = \sigma^{w_0}\tau$.*

PROOF. We clearly have that $\overline{\tau} = \sigma^{w_0}\tau$ if and only if $K_a\sigma^{-w_0} = K_b$, $K_b\sigma^{-w_0} = K_a$ and $\mathsf{im}\,(\sigma^{w_0}\tau) = \mathsf{im}\,(\tau)$. This is equivalent to writing that, for all $x \in X$, $x \in K_a$ if and only if $x\sigma^{-w_0} \in K_b$ which means that τ is w_0-block conjugate (by Lemma 2.51 (iii)). $\square$

As in the case of the properties studied in the previous sections, the block conjugacy property is an invariant within the elements of rank two of the semigroup under consideration.

Lemma 2.55 *If τ is w_0-block conjugate then every element in S of rank 2 is w_0-block conjugate.*

PROOF. Let $\alpha \in S$ be an element of rank 2. Suppose that τ is w_0-block conjugate. From Lemma 2.2 we only need to prove that this property holds for two different cases.

If $\alpha = \sigma^i\tau\sigma^j$, for some $i, j \in \{1, \ldots, n\}$ then we have $\mathsf{im}\,(\sigma^i\tau\sigma^j) = \{a', b'\}$, where $a' = a\sigma^j$, $b' = b\sigma^j$. Also, $\mathsf{ker}\,(\sigma^i\tau\sigma^j) = \{K'_{a'}, K'_{b'}\}$ with $K'_{a'} = (a')(\sigma^i\tau\sigma^j)^{-1} = (a)\sigma^j\sigma^{-j}\tau^{-1}\sigma^{-i} = (a)\tau^{-1}\sigma^{-i} = K_a\sigma^{-i}$. Similarly, $K'_{b'} = K_b\sigma^{-i}$.

Take $x \in K'_{a'}$. Then there is $y \in K_a$ such that $x = y\sigma^{-i}$. Our aim is to prove that $x\sigma^{w_0} \in K'_{b'}$, i.e. $(x\sigma^{w_0})\sigma^i\tau\sigma^j = b'$. For this, $(x\sigma^{w_0})\sigma^i\tau\sigma^j = (x)\sigma^i\sigma^{w_0}\tau\sigma^j = (y)\sigma^{-i}\sigma^i\sigma^{w_0}\tau\sigma^j = (y\sigma^{w_0})\tau\sigma^j$. But because τ is w_0-block conjugate then $y \in K_a$ if and only if $y\sigma^{w_0} \in K_b$, i.e. $(y\sigma^{w_0})\tau = b$. Hence $(y\sigma^{w_0})\tau\sigma^j = b\sigma^j = b'$.

On the other hand, take $x \in X$ such that $x\sigma^{w_0} \in K'_{b'}$. Then there is $y \in K_b$ such that $x\sigma^{w_0} = y\sigma^{-i}$, and so $x = (y)\sigma^{-w_0}\sigma^{-i}$. Our aim is to prove that $x \in K'_{a'}$. For this, $(x)\sigma^i\tau\sigma^j = (y)\sigma^{-w_0}\sigma^{-i}\sigma^i\tau\sigma^j = (y)\sigma^{-w_0}\tau\sigma^j$. Because τ is w_0-block conjugate and using Lemma 2.51 we know that if $y \in K_b$ then $y\sigma^{-w_0} \in K_a$. Hence $(y)\sigma^{-w_0}\tau\sigma^j = a\sigma^j = a'$.

Therefore $\sigma^i\tau\sigma^j$ is w_0-block conjugate.

Conversely, suppose that $\sigma^i\tau\sigma^j$ is w_0-block conjugate, for all $i, j \in \{1, \ldots, n\}$. Then, from the implication already proved, we have that $\sigma^{-i}\sigma^i\tau\sigma^j\sigma^{-j} = \tau$ is also w_0-block conjugate.

Suppose now that $\alpha = \sigma^i\tau\sigma^j\tau\sigma^k$, for some $i, j, k \in \{1, \ldots, n\}$. Assume that $\tau\sigma^j\tau \neq \tau$ (otherwise we are again in Case 1). We know that $\mathsf{ker}\,(\tau\sigma^j\tau) = \mathsf{ker}\,(\tau)$ and $\mathsf{im}\,(\tau\sigma^j\tau) = \mathsf{im}\,(\tau)$. So it is obvious that τ is w_0-block conjugate if and only if $\tau\sigma^j\tau$ is w_0-block conjugate. Hence, again using Case 1, our result holds. $\square$

From the last result and Corollary 2.22, the following definition is consistent.

Definition 2.56 We say that $S = \langle\sigma, \tau\rangle$ is *w_0-block conjugate* if τ is w_0-block conjugate.

Another definition is required for the forthcoming study.

Definition 2.57 Let $v_0 \in \mathbb{N}$. We say that τ has *v_0-block structure* if for every $x \in X$,

$$x \in K_a \text{ if and only if } x\sigma^{v_0} \in K_a.$$

Example 2.58 The transformation

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 3 & 1 & 1 & 3 \end{pmatrix} \in \mathcal{T}_6$$

has 3-block structure. A more striking example is the transformation

$$\lambda = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 1 & 3 & 1 & 3 \end{pmatrix} \in \mathcal{T}_6,$$

which has 2-block structure and, as seen in Example 2.49, also is 1- and 3-block conjugate.

Remark 2.59 We ought to mention two important facts:

(i) For every $\tau \in \mathcal{T}_n$ we have that τ has n-block structure.

(ii) If τ has v_0-block structure, for some $v_0 \in \mathbb{N}$ then $v_0 \geq 2$. This is true because $\mathsf{rank}\,(\tau) = 2$.

Note that this definition is very similar to the property "periodic image" (see Definition 2.5). In fact, if τ has periodic image then it has v_0-block structure, for some $v_0 < n$.

Analogously to the case of the periodic image property (replacing $b - a$ by v_0), one can prove the following results.

Lemma 2.60 *The following are equivalent:*

(i) *the transformation τ has v_0-block structure;*

(ii) *for every $x \in X$, $x \in K_b$ if and only if $x\sigma^{v_0} \in K_b$.*

(iii) *for every $x \in X$, $x \in K_a$ if and only if $x\sigma^{-v_0} \in K_a$.*

Lemma 2.61 *A transformation τ has v_0-block structure if and only if for every $x \in X$,*

$$x \in K_a \text{ if and only if } x\sigma^{uv_0} \in K_a, \text{ for all } u \in \mathbb{Z}$$

Another alternative characterisation of this property.

Lemma 2.62 *A transformation τ has v_0-block structure if and only if $\tau = \sigma^{v_0}\tau$.*

PROOF. We have that $\tau = \sigma^{v_0}\tau$ if and only if $K_a\sigma^{-v_0} = K_a$, $K_b\sigma^{-v_0} = K_b$ and $\mathsf{im}\,(\sigma^{v_0}\tau) = \mathsf{im}\,(\tau)$. This is equivalent to saying that for all $x \in X$, $x \in K_a$ if and only if $x\sigma^{-v_0} \in K_a$. Hence τ has v_0-block structure. □

There is a relation between the block conjugate (Definition 2.48) and the block structure (Definition 2.57) properties, which is made obvious in Example 2.58. The next result clarifies this question.

Corollary 2.63 *If τ is w_0-block conjugate, for some $w_0 \in \mathbb{N}$ then τ has a v_0-block structure, where $v_0 = 2w_0$.*

PROOF. This is a consequence of the previous result, Lemma 2.54 and the fact that $\overline{\overline{\tau}} = \tau$. □

The property of block structure is invariant within the elements of rank two of the semigroup we are considering.

Lemma 2.64 *If τ has v_0-block structure then every element in S of rank 2 has v_0-block structure.*

PROOF. Let us take $\alpha \in S$ a transformation of rank 2. Suppose that τ has v_0-block structure. By Lemma 2.2 we only need to prove that this property holds for two cases.

Assume that $\alpha = \sigma^i\tau\sigma^j$, for some fixed $i, j \in \{1, \ldots, n\}$. We have $\mathsf{im}\,(\sigma^i\tau\sigma^j) = \{a', b'\}$, where $a' = a\sigma^j$, $b' = b\sigma^j$. Also, $\mathsf{ker}\,(\sigma^i\tau\sigma^j) = \{K'_{a'}, K'_{b'}\}$, with $K'_{a'} = K_a\sigma^{-i}$ and $K'_{b'} = K_b\sigma^{-i}$.

Take $x \in K'_{a'}$. Then there is $y \in K_a$ such that $x = y\sigma^{-i}$. Our aim is to prove that $x\sigma^{v_0} \in K'_{a'}$, i.e. $(x\sigma^{v_0})\sigma^i\tau\sigma^j = a'$. For this, $(x\sigma^{v_0})\sigma^i\tau\sigma^j = (x)\sigma^i\sigma^{v_0}\tau\sigma^j = (y)\sigma^{-i}\sigma^i\sigma^{v_0}\tau\sigma^j = (y\sigma^{v_0})\tau\sigma^j$. But because τ has v_0-block structure then $y \in K_a \Leftrightarrow y\sigma^{v_0} \in K_a \Leftrightarrow (y\sigma^{v_0})\tau = a$. Hence $(y\sigma^{v_0})\tau\sigma^j = a\sigma^j = a'$.

On the other hand, take $x \in X$ such that $x\sigma^{v_0} \in K'_{a'}$. Then there is $y \in K_a$ such that $x\sigma^{v_0} = y\sigma^{-i}$ and so $x = (y)\sigma^{-v_0}\sigma^{-i}$. Our aim is to prove that $x \in K'_{a'}$. For this, $(x)\sigma^i\tau\sigma^j = (y)\sigma^{-v_0}\sigma^{-i}\sigma^i\tau\sigma^j = (y)\sigma^{-v_0}\tau\sigma^j$. Because τ has v_0-block structure and using Lemma 2.61 we know that $y \in K_a$ if and only if $y\sigma^{-v_0} \in K_a$. Hence $(y)\sigma^{-v_0}\tau\sigma^j = a\sigma^j = a'$.

Therefore $\sigma^i\tau\sigma^j$ has v_0-block structure.

Conversely, suppose that $\sigma^i\tau\sigma^j$ has v_0-block structure, for all $i, j \in \{1, \ldots, n\}$. Then, from the implication already proved, we have that $\sigma^{-i}\sigma^i\tau\sigma^j\sigma^{-j} = \tau$ also has v_0-block structure.

Suppose that $\alpha = \sigma^i\tau\sigma^j\tau\sigma^k$, with $i, j, k \in \{1, \ldots, n\}$. Let us suppose that $\tau\sigma^j\tau \neq \tau$ (otherwise we are in Case 1 again). We know that $\mathsf{ker}\,(\tau\sigma^j\tau) = \mathsf{ker}\,(\tau)$ and $\mathsf{im}\,(\tau\sigma^j\tau) = \mathsf{im}\,(\tau)$. So it is obvious that τ has v_0-block structure if and only if $\tau\sigma^j\tau$ has v_0-block structure. Hence, again using Case 1, our result holds. □

From the last result and Corollary 2.22, the following definition is consistent.

Definition 2.65 We say that $S = \langle \sigma, \tau \rangle$ has *v_0-block structure* if τ has v_0-block structure.

As underlined in Examples 2.49 and 2.58, the values w_0 and v_0 for the definitions of block conjugate and block structure are not unique. The following definition will clarify this issue.

Definition 2.66 Let $\tau \in \mathcal{T}_n$. We define the following values:

- $w = \min\{w_0 : \tau \text{ is } w_0\text{-block conjugate}\}$.

  If τ is not block conjugate, we say that w is not defined.

- $v = \min\{v_0 : \tau \text{ has } v_0\text{-block structure}\}$.

  The value of v is always defined since τ has n-block structure.

The next two results show us an algorithm to determine the values of w and v described above.

Lemma 2.67 *Suppose that τ is w_0 and w_1-block conjugate, for some $w_0, w_1 \in \mathbb{N}$, $w_0 \neq w_1$. Let $w_2 = \gcd(w_0, w_1)$. Then τ is w_2-block conjugate.*

PROOF. By the definition of greatest common divisor, there are $r, k \in \mathbb{Z}$, $\gcd(k, r) = 1$ such that $w_0 = kw_2$ and $w_1 = rw_2$.

We have that k, r are both odd. To prove this, suppose first that k, r are both even numbers. Then $\gcd(k, r) \geq 2$, a contradiction.

The other case is if k is even and r is odd. Take $x \in K_a$. Then $x\sigma^{krw_2} = x\sigma^{kw_1} \in K_a$, because k is even and τ is w_1-block conjugate (see Remark 2.53). On the other hand, $x\sigma^{rkw_2} = x\sigma^{rw_0} \in K_b$, because r is odd and τ is w_0-block conjugate (see Lemma 2.52). We then have that $x\sigma^{krw_2} \in K_a$ and $x\sigma^{krw_2} \in K_b$ which is again a contradiction. Thus k and r are both odd numbers.

By the Euclidean algorithm, there are $z_0, z_1 \in \mathbb{Z}$ such that $w_2 = z_0 w_0 + z_1 w_1$. We have that z_0 and z_1 have different parity, that is, z_0 is even (odd) if and only if z_1 is odd (even). To prove this, suppose first that z_0 and z_1 are both even. If $x \in K_a$ we have that $x\sigma^{w_0} \in K_b$, because τ is w_0-block conjugate. But $w_0 = kw_2$, with k odd. Then $w_0 = k(z_0w_0 + z_1w_1) = (kz_0)w_0 + (kz_1)w_1$ and kz_0, kz_1 are both even. Hence $x\sigma^{w_0} = x\sigma^{(kz_0)w_0}\sigma^{(kz_1)w_1}$ and so $x\sigma^{w_0} \in K_a$ (by Remark 2.53), which is a contradiction. Suppose now that z_0 and z_1 are both odd. Using the same argument, we can conclude that $x\sigma^{w_0} \in K_b$ and $x\sigma^{w_0} \in K_a$. Thus z_0 and z_1 must have different parity.

Suppose that z_0 is even and z_1 is odd. Let us prove now that τ is w_2-block conjugate. Let $x \in K_a$. Then $x\sigma^{w_2} = x\sigma^{z_0w_0+z_1w_1}$. By Lemma 2.52, $x\sigma^{z_0w_0} \in K_a$ and $(x\sigma^{z_0w_0})\sigma^{z_1w_1} = x\sigma^{w_2} \in K_b$. Conversely, suppose that x is such that $x\sigma^{w_2} \in K_b$. Because $x\sigma^{w_2} = x\sigma^{z_0w_0+z_1w_1}$ and since z_0 is even and z_1 is odd (using Lemma 2.52 and Remark 2.53) we conclude that $x \in K_a$. The case where z_0 is odd and z_1 is even is similar. $\square$

Analogously, we have the following result.

Lemma 2.68 *Suppose that τ has v_0 and v_1-block structure, for some $v_0, v_1 \in \mathbb{N}$, $v_0 \neq v_1$. Take $v_2 = \gcd(v_0, v_1)$. Then τ has v_2-block structure.*

PROOF. By the Euclidean algorithm, there are $r, k \in \mathbb{Z}$ such that $v_2 = rv_0 + kv_1$. From Lemma 2.61, we have that for all $x \in X$, $x \in K_a$ if and only if $x\sigma^{rv_0} \in K_a$ and also $x \in K_a$ if and only if $x\sigma^{kv_1} \in K_a$. Therefore, for all $x \in X$, $x \in K_a$ if and only if $x\sigma^{rv_0+kv_1} \in K_a$, i.e. $x \in K_a$ if and only if $x\sigma^{v_2} \in K_a$. Hence τ has v_2-block structure. $\square$

Observation 2.69 From this point onwards, when we say that τ is w-block conjugate or has v-block structure, the numbers w and v are the values referred to in Definition 2.66.

The next result gives us a nice relation between the values w and v.

Lemma 2.70 *Suppose that τ is w-block conjugate for some w. Then τ has v-block structure with $v = 2w$.*

PROOF. From Corollary 2.63, we know that τ has $2w$-block structure.

Suppose that $v \neq 2w$. Then, by definition of v, we must have $v < 2w$. We have that v divides $2w$. For this, suppose that $2w$ is not divisible by v. Then $\gcd(2w, v) = r$, for some $r \in \mathbb{N}$ where $r < v$ (because $v \neq 2w$). But, according to Lemma 2.68, τ has r-block structure, which contradicts the fact that v is minimum.

We also have that w is not divisible by v. To prove this, suppose otherwise. Then $w = kv$, for some $k \in \mathbb{N}$. But for all $x \in X$, $x \in K_a$ if and only if $x\sigma^w \in K_b$, because τ is w-block conjugate. On the other hand for all $x \in X$, $x \in K_a$ if and only if $x\sigma^{kv} \in K_a$, because τ has v-block structure. Since we have $x\sigma^w = x\sigma^{kv}$, we get a contradiction.

We have that $2w = kv$, because v divides $2w$. Since $v \neq w$ (obviously) we have $k \neq 2$.

From $v \neq 2w$ we have that $k \neq 1$. Hence $k \geq 3$. Furthermore, k is odd because if it was even and from $2w = kv$, we would have that $w = \frac{k}{2}v$, where $\frac{k}{2} \in \mathbb{N}$. This would mean that v divides w which, as we already proved, is not true.

Concluding, we have $2w = kv$, with $k \geq 3$, k odd. This implies that $v < w$ and that v must be even.

Suppose that $k = 2k' + 1$, for some $k' \in \mathbb{N}$. Then $w - k'v = v/2$, because

$$w - k'v = v/2 \Leftrightarrow 2w - 2k'v = v \Leftrightarrow 2w - (k-1)v = v \Leftrightarrow 2w - kv = 0.$$

Let $x \in K_a$. Then $x\sigma^w \in K_b$, because τ is w-block conjugate and also $(x\sigma^w)\sigma^{-k'v} \in K_b$, because τ has v-block structure. Thus $x\sigma^{w-k'v} = x\sigma^{v/2} \in K_b$. Conversely, if $x\sigma^{v/2} \in K_b$ then we can conclude that $x \in K_a$, because τ is w-block conjugate and has v-block structure and also $x\sigma^{v/2} = x\sigma^{w-k'v}$.

This means that τ is $v/2$-block conjugate, which contradicts the fact that w is minimum. $\square$

Corollary 2.71 *Suppose that τ has v-block structure. If v is odd then, for all $w \in \mathbb{N}$, τ is not w-block conjugate.*

PROOF. Suppose that v is odd and that τ is w-block conjugate, for some $w \in \mathbb{N}$. From the previous result, $v = 2w$. But this is not possible, because v is odd. $\square$

We are now able to characterise all the possible sizes of $|\mathfrak{R}|$, for a given semigroup $S = \langle \sigma, \tau \rangle$. We ought to remember that for any given $\tau \in \mathcal{T}_n$, τ is either block conjugate or has block structure. Based on this fact, we deal with the next problem in two different stages.

Theorem 2.72 *Let $S = \langle \sigma, \tau \rangle$. Suppose that τ is w-block conjugate. Then:*

(i) *If $|[\mathfrak{H}]| = 2$ then $|\mathfrak{R}| = w$.*

(ii) *If $|[\mathfrak{H}]| = 1$ then $|\mathfrak{R}| = 2w$.*

PROOF. Consider w as referred to in Definition 2.66.

(i) If $|[\mathfrak{H}]| = 2$, then $\overline{\tau} \in S$. In particular, $\overline{\tau}\mathcal{R}\tau$. We know, because τ is w-block conjugate, that $\overline{\tau} = \sigma^w\tau$ (see Lemma 2.54).

We shall prove that $\mathfrak{R} = \{R_\tau, R_{\sigma\tau}, R_{\sigma^2\tau}, \ldots, R_{\sigma^{w-1}\tau}\}$. Note that for all $i, j \in \{1, \ldots, w\}$, with $i \neq j$, we have that $\mathsf{ker}\,(\sigma^i\tau) \neq \mathsf{ker}\,(\sigma^j\tau)$. To prove this, suppose that there are $i, j \in \{1, \ldots, w\}$, with $i < j$, such that $\mathsf{ker}\,(\sigma^i\tau) = \mathsf{ker}\,(\sigma^j\tau)$. It is clear that $\mathsf{im}\,(\sigma^i\tau) = \mathsf{im}\,(\sigma^j\tau)$. Therefore we either have $\sigma^i\tau = \sigma^j\tau$ or $\overline{\sigma^i\tau} = \sigma^j\tau$.

If $\sigma^i\tau = \sigma^j\tau$ then $\tau = \sigma^{j-i}\tau$ and therefore τ has $(j-i)$-block structure, with $0 < j - i < w$. This is a contradiction to the fact that $2w$ is the minimum value v_0 such that τ has v_0-block structure.

If $\overline{\sigma^i\tau} = \sigma^j\tau$ then $\overline{\tau} = \sigma^{j-i}\tau$ and therefore τ is $j-i$-block conjugate, with $0 < j-i < w$. This is a contradiction to the fact that w is the minimum value w_0 such that τ is w_0-block conjugate.

Therefore for all $i, j \in \{1, \ldots, w\}, i \neq j$ we have that $\sigma^i\tau$ is not $\mathcal{R}$-related with $\sigma^j\tau$, because $\mathsf{ker}\,(\sigma^i\tau) \neq \mathsf{ker}\,(\sigma^j\tau)$. Hence $|\{R_\tau, R_{\sigma\tau}, R_{\sigma^2\tau}, \ldots, R_{\sigma^{w-1}\tau}\}| = w$.

To prove that $\mathfrak{R} \subseteq \{R_\tau, R_{\sigma\tau}, R_{\sigma^2\tau}, \ldots, R_{\sigma^w\tau}\}$ we just need to prove that for all $i \in \{w+1, \ldots, n\}$ we have $R_{\sigma^i\tau} \in \{R_\tau, R_{\sigma\tau}, R_{\sigma^2\tau}, \ldots, R_{\sigma^{w-1}\tau}\}$. Note that $\sigma^w\tau = \overline{\tau}$ and $\overline{\tau}\mathcal{R}\tau$. This is equivalent to proving that for all $i \in \{1, \ldots, n-w\}$ we have $R_{\sigma^i\overline{\tau}} \in \{R_\tau, R_{\sigma\tau}, R_{\sigma^2\tau}, \ldots, R_{\sigma^{w-1}\tau}\}$. But this is obvious, because $\tau\mathcal{R}\overline{\tau}$ and $\mathcal{R}$ is a left congruence. Hence $|\mathfrak{R}| = w$.

(ii) Suppose that $|[\mathfrak{H}]| = 1$. From Lemma 2.54 and because τ is w-block conjugate, we know that $\tau = \sigma^{2w}\tau$, because τ has $2w$-block structure. But $\overline{\tau} = \sigma^w\tau$ and $\overline{\tau}$ is not $\mathcal{R}$-related with τ, because $|[\mathfrak{H}]| = 1$ and $\tau\mathcal{L}\overline{\tau}$. Thus we have $\mathfrak{R} \subseteq \{R_\tau, R_{\sigma\tau}, R_{\sigma^2\tau}, \ldots, R_{\sigma^{2w-1}\tau}\}$. Hence $|\mathfrak{R}| = 2w$. $\square$

Case (i)

τ	$\sigma\tau$	$\ldots$	$\sigma^{w-1}\tau$
$\overline{\tau}$	$\sigma\overline{\tau}$	$\ldots$	$\sigma^{w-1}\overline{\tau}$

Case (iii)

τ	$\sigma\tau$	$\ldots$	$\sigma^{w-1}\tau$	$\overline{\tau}$	$\sigma\overline{\tau}$	$\ldots$	$\sigma^{w-1}\overline{\tau}$

Figure 2.4: Description of Theorem 2.72

The remaining case is slightly more complicated.

Theorem 2.73 *Let $S = \langle \sigma, \tau \rangle$. Suppose that τ has v-block structure. Then:*

(i) *If $|[\mathfrak{H}]| = 2$ then there are two subcases:*

(a) *If $\overline{\tau} \neq \sigma^{v/2}\tau$ then $|\mathfrak{R}| = v$.*

(b) *If $\overline{\tau} = \sigma^{v/2}\tau$ then $|\mathfrak{R}| = v/2$.*

(ii) *If $|[\mathfrak{H}]| = 1$ then $|\mathfrak{R}| = v$.*

PROOF. Consider v as referred in Definition 2.66.

(i) Suppose that $|[\mathfrak{H}]| = 2$.

(a) We shall prove that $\mathfrak{R} = \{R_\tau, R_{\sigma\tau}, R_{\sigma^2\tau}, \ldots, R_{\sigma^{v-1}\tau}\}$. From Lemma 2.62 we know that $\tau = \sigma^v\tau$. Because $|[\mathfrak{H}]| = 2$ we know that $\overline{\tau} \in S$. We ought to remark that there is no $u \in \mathbb{N}$ such that $\sigma^u\tau = \overline{\tau}$. If there was such number u, because of the definition of v, we would have $u = v/2$, which goes against our assumption. Therefore, for all $i \in \{1, \ldots, v-1\}$, τ is not $\mathcal{R}$-related to $\sigma^i\tau$. Hence $|\{R_\tau, R_{\sigma\tau}, R_{\sigma^2\tau}, \ldots, R_{\sigma^{v-1}\tau}\}| = v$. To prove that $\mathfrak{R} \subseteq \{R_\tau, R_{\sigma\tau}, R_{\sigma^2\tau}, \ldots, R_{\sigma^{v-1}\tau}\}$, we only need to note that $\tau = \sigma^v\tau$ which implies that $R_\tau = R_{\sigma^v\tau}$. Thus the result holds.

(b) Because $\overline{\tau} = \sigma^{v/2}\tau$, by Lemma 2.54, we have that τ is $v/2$-block conjugate. Also, we have that $w = v/2$ (consequence of Lemma 2.70). Hence, by Theorem 2.72 (i), we have that $|\mathfrak{R}| = w = v/2$.

(ii) From Lemma 2.62 we know that $\tau = \sigma^v\tau$.

Because $|[\mathfrak{H}]| = 1$ we have that for all $i \in \{1, \ldots, v-1\}$, τ is not $\mathcal{R}$-related with $\sigma^i\tau$. Therefore $\mathfrak{R} = \{R_\tau, R_{\sigma\tau}, R_{\sigma^2\tau}, \ldots, R_{\sigma^{v-1}\tau}\}$. $\square$

Case (i)(a)

τ	$\sigma\tau$	$\ldots$	$\sigma^{v-1}\tau$
$\overline{\tau}$	$\sigma\overline{\tau}$	$\ldots$	$\sigma^{v-1}\overline{\tau}$

Case (i)(b)

τ	$\sigma\tau$	$\ldots$	$\sigma^{\frac{v}{2}-1}\tau$
$\overline{\tau}$	$\sigma\overline{\tau}$	$\ldots$	$\sigma^{\frac{v}{2}-1}\overline{\tau}$

Case (ii)

τ	$\sigma\tau$	$\ldots$	$\sigma^{v-1}\tau$

Figure 2.5: Description of Theorem 2.73

4.3 Size of $\mathcal{H}$-classes

The only information missing to complete this description regards the size of $[\mathfrak{H}]$.

We start with a sufficient condition for each $\mathcal{H}$-class in $\mathfrak{H}$ to have size two.

Lemma 2.74 *Let $S = \langle\sigma, \tau\rangle$. If S does not have periodic image then $|[\mathfrak{H}]| = 2$. Consequently, if $\alpha \in S$ and* $\mathsf{rank}\ \alpha = 2$ *then $\overline{\alpha} \in S$.*

PROOF. If $S = \langle\sigma, \tau\rangle$ does not have periodic image then, by Theorem 2.10, S is regular. Then there is $\xi \in S$ with $\mathsf{rank}\,(\xi) = 2$ such that $\xi^2 = \xi$. We have that $S = \langle\sigma, \xi\rangle$, according to Corollary 2.22.

Take $\overline{\xi} \in \mathcal{T}_n$ to be the conjugate of ξ and define $S' = \langle\sigma, \overline{\xi}\rangle$.

Note that, because S is regular, we have that $\mathcal{H}$ in S is the same as $\mathcal{H}$ in $\mathcal{T}_n$ (see Proposition 1.17). Because $\overline{\xi}\mathcal{H}\xi$ we have $\overline{\xi}^2 = \xi$. Therefore $\xi \in S'$. Then Corollary 2.22 implies that $S' = S$. In particular, $\overline{\xi} \in S$ and because $\overline{\xi}\mathcal{H}\xi$ we have $|[\mathfrak{H}]| = 2$. □

Observation 2.75 The converse of the previous lemma does not hold. Take, for example, $S = \langle\sigma, \tau\rangle$, where $\tau = [1, 3, 1, 3]$ $(n = 4)$. In this example, S is not regular and $|[\mathfrak{H}]| = 2$.

The following theorem describes all the possible sizes of $[\mathfrak{H}]$.

Theorem 2.76 *Consider $S = \langle\sigma, \tau\rangle$ as before.*

(i) *If τ does not have periodic image then $|[\mathfrak{H}]| = 2$.*

(ii) *If τ has periodic image then $|[\mathfrak{H}]| = 2$ if and only if $b - a = n/2$ and τ is w_0-block conjugate, for some w_0.*

PROOF. (i) This is just Lemma 2.74.

(ii) Because S has periodic image, every element $\alpha \in S$ of rank 2 can be written as $\sigma^i\tau\sigma^j$, for some $i, j \in \{1, \ldots, n\}$ (see Corollary 2.21). Also, there are no idempotents of rank 2 in S (see Corollary 2.19).

Suppose that $|[\mathfrak{H}]| = 2$. Hence $\overline{\tau} \in S$. This implies that $\tau \mathcal{R} \overline{\tau}$ which is equivalent to writing that there exist $\beta, \zeta \in S$ such that $\tau = \overline{\tau}\beta$ and $\overline{\tau} = \tau\zeta$. But S has periodic image and therefore we have that $\beta = \sigma^k, \zeta = \sigma^l$, for some $k, l \in \{1, \ldots, n-1\}$, from the considerations about the normal forms of the rank 2 elements of S and from the fact that there are no idempotents of rank 2. From the equality $\overline{\tau} = \tau\zeta = \tau\sigma^q$ and using Lemma 2.44, we can conclude that $b - a = n/2$.

Since $\tau \mathcal{H} \overline{\tau}$, we have $\tau \mathcal{L} \overline{\tau}$ and this is equivalent to saying that there are $\eta, \rho \in S$ such that $\tau = \eta\overline{\tau}$ and $\overline{\tau} = \rho\tau$. As before, we can conclude that $\eta = \sigma^r, \rho = \sigma^s$, for some $r, s \in \{1, \ldots, n-1\}$. From the equality $\overline{\tau} = \sigma^s\tau$ and using Lemma 2.54 we conclude that τ is s-block conjugate.

The converse of this statement is obvious. If $b - a = n/2$ then $\overline{\tau} = \tau\sigma^l$, for some $l \in \mathbb{N}$ (see Lemma 2.44) and so $\tau \mathcal{R} \overline{\tau}$. If τ is w_0-block conjugate then $\overline{\tau} = \sigma^{w_0}\tau$ (see Lemma 2.54). Therefore $\tau \mathcal{L} \overline{\tau}$. Hence $\tau \mathcal{H} \overline{\tau}$ and $|[\mathfrak{H}]| = 2$. □

5 Possible sizes

The next obvious question to ask is what are the possible sizes of $S = \langle \sigma, \tau \rangle$, where $\sigma = \begin{pmatrix} 1 & 2 & 3 & \ldots & n \end{pmatrix}$ and $\tau \in \mathcal{T}_n$ is a rank two transformation, for a given $n \in \mathbb{N}$.

In this section we shall answer this question.

We need a preliminary technical result.

Lemma 2.77 *Let $S = \langle \sigma, \tau \rangle$ such that S is $v/2$-block conjugate and satisfies $b - a = n/2$. Then S either has periodic image or conjugate periodic image.*

PROOF. Let $d = \gcd(n, b - a)$. Then $d = n/2$ and so $d > 1$, if we take $n > 3$ (which are the interesting cases, as we point out in the beginning of Chapter 6).

We have that v divides n. For this, note that τ has v-block structure and n-block structure. Therefore, by Lemma 2.68, it also has $\gcd(n, v)$-block structure.

But by definition of v, we have $\gcd(n, v) = v$ which means that v divides n. By assumption, n is even (because we assume that $b - a = n/2$). We also assume that τ is $v/2$-block conjugate and consequently v is even. Because v divides n, there is $k \in \mathbb{N}$ such that $n = kv$. Therefore, from the previous considerations about v and n, we can conclude that $v/2$ divides $n/2$, since $n/2 = k(v/2)$.

If k is even and because τ is $v/2$-block conjugate we have that $x \in K_a$ if and only if $x\sigma^{k\frac{v}{2}} \in K_a$ (see Remark 2.53). Hence $x \in K_a$ if and only if $x\sigma^{n/2} = x\sigma^d \in K_a$, which means that τ has periodic image (see Lemma 2.16).

If k is odd and because τ is $v/2$-block conjugate we have that $x \in K_a$ if and only if $x\sigma^{k\frac{v}{2}} \in K_b$ (see Lemma 2.52). Hence $x \in K_a$ if and only if $x\sigma^{n/2} = x\sigma^d \in K_b$, which means that τ has conjugate periodic image (see Lemma 2.39). $\square$

We should recall that every semigroup $S = \langle \sigma, \tau \rangle$ has v-block structure, for some v.

Theorem 2.78 *Let $S = \langle \sigma, \tau \rangle$ be such that S has v-block structure. Then we have $|S| \in \{2n(v+1), n(v+2), n(\frac{v}{2}+2), n(2v+1), n(v+1), n(\frac{v}{2}+1)\}$.*

PROOF. From Lemma 2.3, we know that S has, at most, three $\mathcal{D}$-classes;

- D_n, the $\mathcal{D}$-class of rank n which is the group $\langle \sigma \rangle$.
- D_2, the $\mathcal{D}$-class of rank 2.
- D_1, the $\mathcal{D}$-class of rank 1, where every element is an idempotent (if it exists).

We know that $|D_n| = |D_1| = n$. Therefore, to compute the size of S, we only need to decide if $D_1 \neq \emptyset$ and to compute the size of D_2.

We shall divide our study in three main cases.

(i) S has neither periodic image nor conjugate periodic image (so S is regular but not completely regular). Because S does not have periodic image, $|[\mathfrak{H}]| = 2$ (Lemma 2.74). Because S is not completely regular, $D_1 \neq \emptyset$ (Lemma 2.26).

Suppose that S is not $v/2$-block conjugate. Then we have two subcases.

(a) If $b - a \neq n/2$ then $|D_2| = |\mathfrak{L}| \times |\mathfrak{R}| \times |[\mathfrak{H}]| = n \times v \times 2 = 2nv$ (Theorems 2.76 (i), 2.45 and 2.73 (i)(a)). Therefore $|S| = |D_n| + |D_2| + |D_1| = n + 2nv + n = 2n(v+1)$.

(b) If $b - a = n/2$ then $|D_2| = |\mathfrak{L}| \times |\mathfrak{R}| \times |[\mathfrak{H}]| = \frac{n}{2} \times v \times 2 = nv$ (Theorems 2.76 (i), 2.45 and 2.73 (i)(a)). Therefore $|S| = |D_n| + |D_2| + |D_1| = n + nv + n = n(v+2)$.

Suppose now that S is $v/2$-block conjugate. Then:

(a) If $b - a \neq n/2$ then $|D_2| = |\mathfrak{L}| \times |\mathfrak{R}| \times |[\mathfrak{H}]| = n \times \frac{v}{2} \times 2 = nv$ (Theorems 2.76 (i), 2.45 and 2.72 (i)). Hence $|S| = |D_n| + |D_2| + |D_1| = n + nv + n = n(v+2)$.

(b) If $b - a = n/2$ then, using Lemma 2.77, we conclude that our premise is false, because if S is $v/2$-block conjugate and satisfies $b - a = n/2$ then S either has periodic image or conjugate periodic image.

(ii) S has conjugate periodic image. So S is completely regular. This implies that S is regular and this implies that S does not have periodic image. Therefore $|[\mathcal{H}]| = 2$ (Lemma 2.74). Also, we have $D_1 = \emptyset$ (Lemma 2.26).

If S has conjugate periodic image then S is w-block conjugate, for some w. But S has v-block structure. From Lemma 2.70, we know that $v = 2w$ and this implies $w = v/2$.

(a) If $b - a \neq n/2$ then $|D_2| = |\mathfrak{L}| \times |\mathfrak{R}| \times |[\mathfrak{H}]| = n \times \frac{v}{2} \times 2 = nv$ (Theorems 2.76 (i), 2.45 and 2.72 (i)). Therefore $|S| = |D_n| + |D_2| = n + nv = n(v+1)$.

(b) If $b - a = n/2$ then $|D_2| = |\mathfrak{L}| \times |\mathfrak{R}| \times |[\mathfrak{H}]| = \frac{n}{2} \times \frac{v}{2} \times 2 = n\frac{v}{2}$ (Theorems 2.76 (i), 2.45 and 2.72 (i)). Therefore $|S| = |D_n| + |D_2| = n + n\frac{v}{2} = n(\frac{v}{2} + 1)$.

(iii) S has periodic image (equivalently, S is not regular). Because S is not regular then surely is not completely regular. Thus $D_1 \neq \emptyset$ (Lemma 2.26). Note that, because S has periodic image we have that $|[\mathfrak{H}]| = 2$ if and only if τ is

w-block conjugate and $b - a = n/2$ (Theorem 2.76 (ii)).

(a) Suppose that either S is not w-block conjugate or $b - a \neq n/2$. There are two possibilities to consider.

(a1) If S is not w-block conjugate then $|D_2| = |\mathfrak{L}| \times |\mathfrak{R}| \times |[\mathfrak{H}]| = n \times v \times 1 = nv$ (Theorems 2.45 and 2.73 (ii)). Hence $|S| = |D_n| + |D_2| + |D_1| = n + nv + n = n(v+2)$.

(a2) S is w-block conjugate and $b-a \neq n/2$. Then, as we saw in (ii), $w = v/2$. Hence $|D_2| = |\mathfrak{L}| \times |\mathfrak{R}| \times |[\mathfrak{H}]| = n \times 2w \times 1 = nv$ (Theorems 2.45, 2.72 (ii)). Therefore $|S| = |D_n| + |D_2| + |D_1| = n + nv + n = n(v+2)$.

(b) If S is w-block conjugate and $b - a = n/2$ then $|D_2| = |\mathfrak{L}| \times |\mathfrak{R}| \times |[\mathfrak{H}]| = \frac{n}{2} \times \frac{v}{2} \times 2 = n\frac{v}{2}$ (Theorems 2.45 and 2.72 (i)). Therefore $|S| = |D_n| + |D_2| + |D_1| = n + n\frac{v}{2} + n = n(\frac{v}{2} + 2)$. $\square$

Some consequences of the previous proof.

Corollary 2.79 *Let $S = \langle \sigma, \tau \rangle$ be such that S has v-block structure. If S has neither periodic image nor conjugate periodic image then*

$$|S| = \begin{cases} 2n(v+1) & \text{if } S \text{ is not } v/2\text{-block conjugate and } b-a \neq n/2 \\ n(v+2) & \text{if } (S \text{ is not } v/2\text{-block conjugate and } b-a = n/2) \text{ or} \\ & \text{if } (S \text{ is } v/2\text{-block conjugate and } b-a \neq n/2) \end{cases}$$

Corollary 2.80 *Let $S = \langle \sigma, \tau \rangle$ be such that S has v-block structure. If S has conjugate periodic image then*

$$|S| = \begin{cases} n(v+1) & \text{if } b-a \neq n/2 \\ n(\frac{v}{2}+1) & \text{if } b-a = n/2 \end{cases}$$

Corollary 2.81 *Let $S = \langle \sigma, \tau \rangle$ be such that S has v-block structure. If S has* *periodic image then*

$$|S| = \begin{cases} n(v+2) & \textit{if } S \textit{ is not } v/2\textit{-block conjugate or } b-a \neq n/2 \\ n(\frac{v}{2}+2) & \textit{if } S \textit{ is } v/2\textit{-block conjugate and } b-a = n/2 \end{cases}$$

Next we give a set of examples that illustrate each of the corollaries above. We consider $S = \langle \sigma, \tau \rangle$, where $\sigma = (1\ 2\ 3\ \ldots\ n)$ and τ, a transformation of rank two.

Example 2.82 Let us see some examples that satisfy each of the cases in Corollary 2.79.

For each of the following examples, the transformation τ has neither periodic image nor conjugate periodic image.

(i) Let $\tau = [1, 2, 1, 1]$ (with $n = 4$). We have that τ has 4-block structure, it is not 2-block conjugate and $b - a = 2 - 1 = 1 \neq 2 = 4/2 = n/2$. Thus $|S| = 2.4(4 + 1) = 40$.

Another example; take $\tau = [1, 1, 3, 1, 1, 3, 1, 1, 3]$ (with $n = 9$). Here we have that τ has 3-block structure and therefore τ is not block conjugate (because 3 is odd). It does not satisfy $b - a = n/2$ either. Thus $|S| = 2.9(3 + 1) = 72$.

(ii) Let $\tau = [1, 1, 1, 4, 1, 1]$ (with $n = 6$). This transformation has 6-block structure. It is not block conjugate because $2\tau = 1$ and $(2\sigma^3)\tau = 1$. It satisfies $b - a = 4 - 1 = 3 = 6/2 = n/2$. Therefore $|S| = 6(6 + 2) = 48$.

Let us see a different example. Take $\tau = [1, 1, 1, 10, 1, 1, 1, 1, 1, 10, 1, 1, 1, 1, 1, 10, 1, 1]$ (with $n = 18$). In this case τ has 6-block structure and it is not block conjugate. It satisfies $b - a = n/2$ $(10 - 1 = 9 = 18/2)$. Therefore $|S| = 18(6 + 2) = 144$.

(iii) The transformation $\tau = [1, 2, 2, 1]$ (with $n = 4$) has 4-block structure, it is 2-block conjugate and it does not satisfy the condition $b - a = n/2$. In this case $|S| = 4(4 + 2) = 24$.

Example 2.83 Here we present some examples where we can apply Corollary 2.80.

Let us consider transformations which have conjugate periodic image. This implies that these transformations are $v/2$-block conjugate (see Remark 2.50 and Lemma 2.70).

(i) Let $\tau = [1, 2, 1, 2]$ (with $n = 4$). This transformation has 2-block structure and $b - a \neq n/2$. Therefore $|S| = 4(2 + 1) = 12$.

(ii) Let $\tau = [1, 4, 1, 4, 1, 4]$ (with $n = 6$). Then τ has 2-block structure and $b - a = 4 - 1 = 3 = 6/2 = n/2$. Therefore $|S| = 6(1 + 1) = 12$.

Example 2.84 The last set of examples illustrates Corollary 2.81.

For the next examples, all the transformations have periodic image.

(i) Take $\tau = [1, 1, 4, 1, 1, 4, 1, 1, 4]$ (with $n = 9$). This transformation has 3-block structure and it does not satisfy $b - a = n/2$, because n is odd. Then $|S| = 9(3 + 2) = 45$.

Another example for this case is the following: let $\tau = [1, 1, 4, 1, 1, 4]$ (with $n = 6$). Then τ has 3-block structure, satisfies $b-a = n/2$ but it is not block conjugate (because $3/2 \notin \mathbb{N}$). Therefore $|S| = 6(3+2) = 30$.

A last example in this case is $\tau = [1, 3, 1, 3, 1, 3]$ (with $n = 6$). Here τ has 2-block structure and it is 1-block conjugate, but it does not satisfy $b - a = n/2$. Hence $|S| = 6(2 + 2) = 24$.

(ii) Let $\tau = [1, 3, 1, 3]$ (with $n = 4$). The transformation τ has 2-block structure (and it is 1-block conjugate) and satisfies $b-a = n/2$. Therefore $|S| = 4(1 + 2) = 12$.

Chapter 3

Isomorphisms

The isomorphism of two algebraic structures is a well known and widely used concept. The study of isomorphisms between two structures is of great importance, since if two algebraic structures are isomorphic then they are "essentially" a copy of each other. Hence, when one studies a certain class of algebraic structures, it is possible to reduce this study, by finding all the subsets of this class, where in each of these subsets all the elements are isomorphic. Each of these subsets is called an isomorphism class.

The concept of isomorphism occurs and is studied in a great variety of contexts such as Numerical Functional Analysis, Differential Geometry, Group Theory, Semigroup, Graph and Finite Automata Theory.

Determining if two algebraic structures are isomorphic is generally a very difficult problem and the isomorphism of semigroups is no exception.

In the case of the semigroups we are considering, and based on Chapter 2, we know something more about their sets of possible generators. More precisely, we know that any element of rank two in S together with a power k of σ, where k is coprime to n, generates S (see Corollary 2.22). But even with all this information, supposing that we have p elements of rank two in S and that there are q powers k of σ where k is coprime to n, we still have $p \times q$ possible generators which is still quite a "big" number. For example, if $S = \langle\sigma, \tau\rangle$, where $\sigma = (1\ 2\ 3\ 4\ 5)$ and $\tau = [1, 1, 3, 3, 1]$, then there are 20 idempotents of rank two. Therefore we have that $p = 20$ and $q = 4$ which gives us 80 possibilities for generators of S. And this is only with $n = 5$... The large number of possible generators is one of the reasons why the study of the isomorphism classes of the set of all semigroups generated by $\sigma = (1\ 2\ \ldots\ n)$ and a transformation of rank two from $\mathcal{T}_n$, for a given $n \in \mathbb{N}$, is a very hard problem.

In this chapter we shall find some methods which allow us to study the isomorphism classes of the set of all semigroups of this type. Although this study does not cover all possible cases, we study the semigroups which appear "more frequently" and hopefully these methods will be the launching platform for future work that will include the remaining semigroups of this type.

In the first section we present some properties on the generator τ of rank two of a given semigroup $S = \langle\sigma, \tau\rangle$, which in fact, are invariant within the set of all elements of rank two within the semigroup. As a consequence, we have a necessary condition for the membership of elements of rank two.

In the second section, we give some necessary conditions for the isomorphism between two such non-completely regular semigroups (semigroups that contain constant maps).

In the third section we present a necessary and sufficient condition for the isomorphism of two such non-completely regular semigroups. Furthermore, we give a method of specifying this isomorphism and show that this condition does not hold in the completely regular case.

1 Invariants

We start by studying some properties which are invariant within the elements of rank two of a semigroup $S = \langle\sigma, \tau\rangle$.

Definition 3.1 Let $\tau \in \mathcal{T}_n$ be such that $\mathsf{rank}\,(\tau) = 2$ and $\mathsf{ker}\,(\tau) = \{K_a, K_b\}$. If $|K_a| < |K_b|$ then we say that τ has kernel type $\mathfrak{p}^1\mathfrak{q}^1$ if $|K_a| = \mathfrak{p}$ and $|K_b| = \mathfrak{q}$. If $|K_a| = |K_b|$ then the kernel type of τ is denoted by $\mathfrak{p}^2$.

In the next result we see that the kernel type is invariant within the semigroup, meaning that it is invariant for the elements of rank two of the semigroup.

Lemma 3.2 *Let $S = \langle \sigma, \tau \rangle$. For all $\alpha \in S$ such that $\mathsf{rank}\,(\alpha) = 2$ we have that the kernel type of α equals that of τ.*

Proof. By Lemma 2.2, as in many of the proofs before, we only need to study two different cases:

Case 1: $\alpha = \sigma^i \tau \sigma^j$, with $i, j \in \{1, \ldots, n\}$.

It is easy to observe that $\mathsf{ker}\,(\rho) = \mathsf{ker}\,(\rho\sigma^j)$, for any $\rho \in \mathcal{T}_n$. Therefore we only need to prove that $\sigma^i\tau$ has the same kernel type as τ. For this, note that $\mathsf{ker}\,(\tau) = \{K_a, K_b\}$ and therefore $\mathsf{ker}\,(\sigma^i\tau) = \{K_a\sigma^{-i}, K_b\sigma^{-i}\}$. Since $\sigma \in \mathcal{S}_n$ it is clear that $|K_a| = |K_a\sigma^{-i}|$ and $|K_b| = |K_b\sigma^{-i}|$ which proves our result.

Case 2: $\alpha = \sigma^i\tau\sigma^j\tau\sigma^k$, with $i, j, k \in \{1, \ldots, n\}$.

If we have $\tau\sigma^j\tau = \tau$ then we fall into the previous case. Let us suppose that $\tau\sigma^j\tau \neq \tau$. Then we have that $\tau\sigma^j\tau = \overline{\tau}$, which is the conjugate of τ. Since $\overline{\tau}$ has the same kernel (and image) as τ then we just use Case 1 to conclude our result. $\square$

Based on the last result and Corollary 2.22, the following definition is consistent.

Definition 3.3 Let $S = \langle \sigma, \tau \rangle$. We say that S has kernel type $\mathfrak{p}^1\mathfrak{q}^1$ if τ has kernel type $\mathfrak{p}^1\mathfrak{q}^1$.

Another property which will be important for the forthcoming study is the following.

Definition 3.4 We write $\mu_\sigma(\tau)$ to denote the least natural number l such that $a\sigma^l = b$ or $b\sigma l = a$. Formally, $\mu_\sigma(\tau) = \min\{l \in \mathbb{N} : a\sigma^l = b \text{ or } b\sigma^l = a\}$.

Remark 3.5 The value of $\mu_\sigma(\tau)$ is always defined. Supposing that $a < b$ then $\mu_\sigma(\tau) \leq b - a$.

We prove next that the value $\mu_\sigma(\tau)$ is an invariant within the elements of rank two of S.

Lemma 3.6 *For all $\alpha \in S$ such that $\mathsf{rank}\,(\alpha) = 2$ we have that $c\sigma^{\mu_\sigma(\tau)} = d$, where $\mathsf{im}\,(\alpha) = \{c, d\}$.*

PROOF. As before, we only need to study two possible cases:

Case 1: $\alpha = \sigma^i \tau \sigma^j$, with $i, j \in \{1, \ldots, n\}$.

We have that $c = a\sigma^j$ and $d = b\sigma^j$. Consequently $c\sigma^{\mu_\sigma(\tau)} = (a)\sigma^j\sigma^{\mu_\sigma(\tau)} = (a)\sigma^{\mu_\sigma(\tau)}\sigma^j = b\sigma^j = d$.

Case 2: $\alpha = \sigma^i \tau \sigma^j \tau \sigma^k$, with $i, j, k \in \{1, \ldots, n\}$.

The interesting case is when $\tau\sigma^j\tau \neq \tau$. Then $\tau\sigma^j\tau = \overline{\tau}$, the conjugate of τ, which has the same image (and kernel) as τ. Therefore the result trivially holds by using Case 1. □

Example 3.7 Consider the case when $n = 5$. Let $\tau = [1, 3, 3, 1, 1]$ and $S = \langle \sigma, \tau \rangle$. Take $\alpha = [4, 1, 1, 4, 4]$. We can easily check that $\alpha \in S$. We have that $\mu_\sigma(\tau) = 2$, because $1\sigma^2 = 3$ and $3\sigma^3 = 1$. Let us denote $c = 4$ and $d = 1$. Then $c\sigma^{\mu_\sigma(\tau)} = 4\sigma^2 = 1 = d$.

Another element of rank 2 in S is $\beta = [5, 5, 3, 3, 3]$. Denote $u = 3$ and $v = 5$. Then $u\sigma^{\mu_\sigma(\tau)} = 3\sigma^2 = 5 = v$.

The next definition is therefore consistent.

Definition 3.8 Let $S = \langle \sigma, \tau \rangle$. We write $\mu_\sigma(S)$ to denote the least natural number l such that $a\sigma^l = b$ or $b\sigma l = a$. Formally, $\mu_\sigma(S) = \min\{l \in \mathbb{N} : a\sigma^l = b \text{ or } b\sigma^l = a\}$.

Note that both this property and the kernel type are necessary conditions for the membership of an element of rank two in a semigroup S. More precisely, let $S = \langle \sigma, \tau \rangle$ be a semigroup and let $\alpha \in \mathcal{T}_n$ be a transformation of rank two. Then if either the kernel type of τ is not the same as that of α or if $\mu_\sigma(\tau) \neq \mu_\sigma(\tau)$ then, using Lemma 3.2 or Lemma 3.6, respectively, we can conclude that $\alpha \notin S$.

2 Isomorphisms - necessary conditions

In this section we give some necessary conditions for the isomorphism of two semigroups. For this we shall look at some concrete examples which give us some indications on these necessary conditions.

Lemma 3.9 *Let S, T be semigroups generated by σ and a transformation of rank 2. Suppose that $\phi : S \to T$ is an isomorphism. Then for every $v \in S$ we have* $\mathsf{rank}\,(v) = \mathsf{rank}\,((v)\phi)$.

PROOF. Let $\mathcal{C} = \{c_i : i \in \{1, \ldots, n\}\}$, where $c_i = [i, i, \ldots, i]$, for each $i \in \{1, \ldots, n\}$, i.e. $\mathcal{C}$ is the set of constant maps of $\mathcal{T}_n$. We shall prove that $(\mathcal{C})\phi = \mathcal{C}$. For this, let $\rho \in \mathcal{T}_n$ be a right zero. Then $c_i\rho = \rho$ implies that $\mathsf{rank}\,(c_i\rho) = \mathsf{rank}\,(\rho) = 1$, since $\mathsf{ker}\,(c_i\rho) = \mathsf{ker}\,(c_i)$ and $\mathsf{rank}\,(c_i) = 1$, for any $i \in \{1, \ldots, n\}$. Hence every right zero element in S is a constant map c_i, for some $i \in \{1, \ldots, n\}$. Because ϕ is an homomorphism we then have $(\mathcal{C})\phi \subseteq \mathcal{C}$. Because ϕ is a bijection the equality holds.

Let 1_S and 1_T be the identity maps of S and T, respectively. We shall see that $(\langle \sigma \rangle)\phi = \langle \sigma \rangle$. We have $(1_S)\phi = 1_T$ (because S and T are monoids and ϕ is a isomorphism). Then $(\sigma^i\sigma^{-i})\phi = 1_T$ and, because ϕ is a homomorphism,

$(\sigma^i)\phi(\sigma^{-i})\phi = 1_T$. Therefore $(\sigma^i)\phi$ is a permutation. Thus $(\langle\sigma\rangle)\phi \subseteq \langle\sigma\rangle$ and the equality follows from the fact that ϕ is a bijection and $\langle\sigma\rangle$ is finite.

Finally by exclusion, we have that every element of rank 2 in S is mapped into an element of rank 2 in T. □

Let us look at some concrete examples. We write Ω_5 to denote the set of all semigroups generated by $\sigma = (1\ 2\ 3\ 4\ 5)$ and a transformation $\tau \in \mathcal{T}_5$ of rank two. Some quite simple computations in GAP (see [12]) allow us to conclude that

$$\Omega_5 = \{S_i = \langle\sigma, \tau_i\rangle : \sigma = (1\ 2\ 3\ 4\ 5),\ \text{with } i \in \{1, \dots, 6\}\},$$

where

- $\tau_1 = [1, 2, 1, 1, 1]$,
- $\tau_2 = [1, 2, 2, 1, 1]$,
- $\tau_3 = [1, 2, 1, 2, 1]$,
- $\tau_4 = [1, 1, 3, 1, 1]$,
- $\tau_5 = [1, 1, 3, 3, 1]$ and
- $\tau_6 = [1, 1, 3, 1, 3]$.

Using our previous results (namely Corollary 2.79) we can deduce that $|S_i| = 60$, for every $i \in \{1, \dots, 6\}$ (this is explained thoroughly in Chapter 6, Section 3). Using GAP we can conclude that S_3 and S_5 have 20 idempotents of rank two and the remaining ones have 10 idempotents of rank two. Using a "brute force" method in GAP we can conclude that $S_1 \cong S_4$, $S_2 \cong S_6$ and $S_3 \cong S_5$, where all these isomorphisms are conjugations by an element of $\mathcal{S}_n$.

An argument based on the number of idempotents of rank two allows us to conclude that $S_3 \ncong S_1$ and $S_3 \ncong S_2$. The remaining question is if $S_1 \cong S_2$. If the answer is affirmative then this isomorphism cannot be a conjugation since these two semigroups have different kernel type.

The following figure shows the right Cayley graphs of the semigroups S_1 and S_2 over the generators σ and τ_i, $i = 1, 2$, restricted to the set $\mathcal{C}$ of the constant maps.

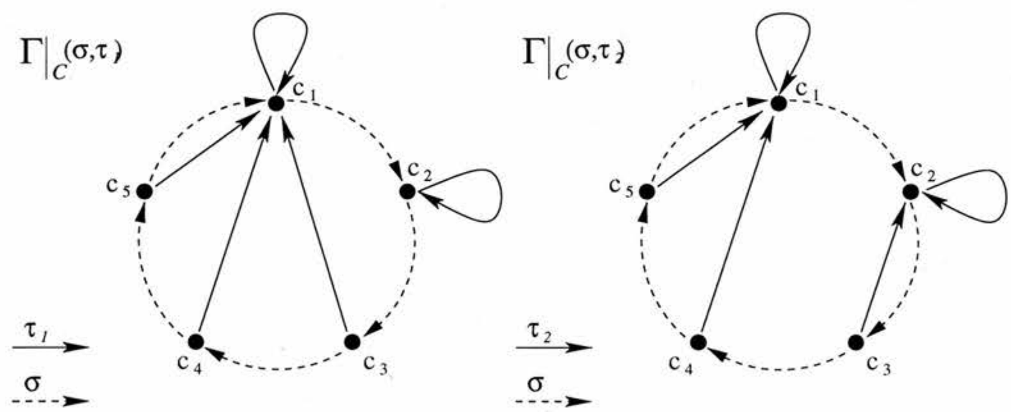

Figure 3.1: Right Cayley graphs of S_1 and S_2

Although this is not a formal proof, these graphs give us some fairly good hint that these two semigroups are not isomorphic. Since we know that within a semigroup every element of rank two has the same kernel type (Lemma 3.2), no matter which generator of rank two we take for the Cayley graph of S_2, we shall always have that the degrees of some vertices will be different from those of S_1. Let us formalise these ideas.

Suppose there is an isomorphism

$$\begin{aligned} \phi:\ S_1 &\to S_2 \\ \sigma &\mapsto \beta \\ \tau_1 &\mapsto \alpha, \end{aligned}$$

where $\langle\beta\rangle = \langle\sigma\rangle$ and $\mathsf{rank}\,(\alpha) = 2$ (see Lemma 3.9).

From Lemma 3.9 we also know that $(\mathcal{C})\phi = \mathcal{C}$. Consider $I \subset \{1, \ldots, n\}$ such that $\mathsf{im}\,(c_i\tau_1) = \{1\}$ (since $|K_1| > |K_2|$), for all $i \in I$. Note that this set I is actually K_1, but seen as a set of indexes. Then $|(\{c_i : i \in I\}\tau_1)\phi| = |(c_1\tau_1)\phi| = 1$. On the other hand, $|(\{c_i : i \in I\}\tau_1)\phi| = |\{(c_i)\phi : i \in I\}\alpha|$.

From Lemma 3.2 we have that α has the same kernel type as τ_2 which is different from that of τ_1. By the Pigeonhole Principle (see for example [22]) we must have some $j, k \in I$ such that $\mathsf{im}\,((c_j)\phi)$ and $\mathsf{im}\,((c_k)\phi)$ are in different kernel classes of α, i.e. $((c_j)\phi)\alpha \neq ((c_k)\phi)\alpha$.

Therefore $|\{(c_i)\phi : i \in I\}\alpha| = |\{(c_j)\phi, (c_k)\phi\}\alpha| = |\{((c_j)\phi), ((c_k)\phi)| = 2$. But this is a contradiction to the fact that ϕ is an homomorphism.

The general result follows quite easily and the proof is very similar to the above.

Lemma 3.10 *Let S and T be semigroups generated by σ and a rank 2 transformation of $\mathcal{T}_n$. Suppose that S and T contain the constant maps. If S and T are isomorphic then S and T have the same kernel type.*

PROOF. Let $S = \langle \sigma, \tau \rangle$ and let us assume that S is isomorphic to T. Then there is an isomorphism $\phi : S \to T, \sigma \mapsto \beta, \tau \mapsto \alpha$, where $\langle \beta \rangle = \langle \sigma \rangle$ and $\mathsf{rank}\,(\alpha) = 2$.

Suppose that S and T have different kernel types. Then we can choose $a \in \mathsf{im}\,(\tau)$ such that the size of K_a is maximum within the kernel classes of τ. We can also suppose, without loss of generality, that this size is greater than that of any of the kernel classes of α. Let us then consider $I = \{i \in \{1, \ldots, n\} : \mathsf{im}\,(c_i\tau) = \{a\}\}$. Then $|(\{c_i : i \in I\}\tau)\phi| = |(c_i\tau)\phi| = 1$. On the other hand, $|(\{c_i : i \in I\}\tau)\phi| = |\{(c_i)\phi : i \in I\}\alpha|$.

As in the previous proof, because S and T have different kernel types and using the pigeonhole principle (see [22]) we have some elements $j, k \in I$ such that $\mathsf{im}\,(((c_j)\phi)\alpha) \neq \mathsf{im}\,(((c_k)\phi)\alpha)$. Therefore $|\{(c_i)\phi : i \in I\}\alpha| = |\{(c_j)\phi, (c_k)\phi\}\alpha| = |\{((c_j)\phi), ((c_k)\phi)| = 2$.

This contradicts the fact that ϕ is an homomorphism. $\square$

We now have a necessary condition for two semigroups to be isomorphic. Next we illustrate with two examples the use of this condition and the fact that the converse of the previous lemma does not hold.

Example 3.11 (i) Let $\tau = [1, 2, 1, 1, 1]$ and $\alpha = [1, 1, 3, 1, 3]$ and let $S = \langle \sigma, \tau \rangle$, $T = \langle \sigma, \alpha \rangle$, with $\sigma = (1\ 2\ 3\ 4\ 5)$. Both these semigroups have the same size 60, the same Green's structure and the same number of idempotents (details on how these are determined can be found in Chapter 6, Section 3).

But S has kernel type $1^1 4^1$ and T has kernel type $2^1 3^1$. By the previous result, $S \not\cong T$.

(ii) Let $\tau = [1, 2, 1, 1, 1, 1]$ and $\alpha = [1, 1, 3, 1, 1, 1]$ and let $S = \langle \sigma, \tau \rangle$, $T = \langle \sigma, \alpha \rangle$, with $\sigma = (1\ 2\ 3\ 4\ 5\ 6)$.

These semigroups have the same size 84, the same Green's structure, the same number of idempotents and the same kernel type (details in Chapter 6, Section 4).

Nevertheless, $S \not\cong T$, because S is isomorphic to the semigroup with presentation

$$\begin{aligned}\langle s, t \mid\ & s^6 = 1, t^2 = t, (tst)^2 = t, sts^2t = ts^2t, ts^2tst = ts^2ts, \\ & ts^it = ts^2t\ (i = 3, \ldots, 6)\, \rangle\end{aligned}$$

and this is not a presentation for T, since none of the possible generators of T satisfies all the relations in this presentation. We prove this fact using the GAP implementation of Algorithm 14 found in Chapter 5 (see Appendix B).

Having as motivation the last example, we are going to investigate semigroups like the above, which have the same kernel type and Proposition 3.10 does not apply. Let $\mu_\sigma(\tau)$ denote the least natural number l such that $a\sigma^l = b$, where $\mathsf{im}\,(\tau) = \{a, b\}$ (see Definition 3.8).

Definition 3.12 The number $\mu(S) = min\{\mu_\beta(S) : \beta \in \langle \sigma \rangle, \langle \beta \rangle = \langle \sigma \rangle\}$ is called the *image distance* of S.

Remark 3.13 We can state that $\mu(S)$ is always defined since $\mu(S) \leq \mu_\sigma(S) \leq b-a$. Also, the previous definition is consistent because of Lemma 3.6.

The next result gives us a further necessary condition for the isomorphism of two semigroups.

Lemma 3.14 *Let S and T be semigroups generated by σ and a rank 2 transformation of $\mathcal{T}_n$. Suppose that S and T contain the constant maps. If S is isomorphic to T then $\mu(S) = \mu(T)$.*

PROOF. Let us choose $\varrho \in \langle\sigma\rangle$ and $\tau \in S$, where $\langle\varrho\rangle = \langle\sigma\rangle$ and $\mathsf{rank}\,(\tau) = 2$ such that $\mu(S) = \mu_\varrho(\tau)$. Because S is isomorphic to T, we have $\phi : S \to T$, $\varrho \mapsto \beta, \tau \mapsto \alpha$, where $\langle\beta\rangle = \langle\sigma\rangle$, $\mathsf{rank}\,(\alpha) = 2$ and $T = \langle\beta, \alpha\rangle$. Therefore $\mu(T) \leq \mu_\beta(\alpha) \leq \mu_\varrho(\tau) = \mu(S)$.

The other inequality follows taking $\phi^{-1} : T \to S$. $\square$

3 Isomorphisms - a necessary and sufficient condition

In this section we shall give a necessary and sufficient condition for the isomorphism of two semigroups, which contain the set $\mathcal{C}$ of all constant maps.

Definition 3.15 Let $n \in \mathbb{N}$ and $S, T \leq \mathcal{T}_n$. We say that S and T are *conjugate* if there exists $\pi \in \mathcal{S}_n$ such that $\pi^{-1}S\pi = T$.

A necessary and sufficient condition for the isomorphism of two semigroups which contain the constant maps follows.

Theorem 3.16 *Let S and T be semigroups generated by σ and a rank 2 transformation of $\mathcal{T}_n$. Suppose that S and T contain the constant maps. Then S is isomorphic to T if and only if S is conjugate to T.*

PROOF. Suppose that $S \cong T$. Let $\phi : S \to T$ be an isomorphism. From Lemma 3.9, we know that $\mathcal{C}\phi = \mathcal{C}$. Therefore ϕ induces a permutation $\pi \in \mathcal{S}_n$, given by

$$i\pi = j \Leftrightarrow (c_i)\phi = c_j, \text{ for all i, j} \in \{1, \ldots, \mathrm{n}\}.$$

Take $\upsilon \in T$ and $c_j \in \mathcal{C}$, for some $j \in \{1, \ldots, n\}$. Then there is $\omega \in S$ and $c_i \in \mathcal{C}$ such that $(\omega)\phi = \upsilon$ and $(c_i)\phi = c_j$. Therefore $c_j\upsilon = (c_i)\phi(\omega)\phi = (c_i\omega)\phi = c_i\omega\pi = (c_j)\phi^{-1}\omega\pi = (c_j)\pi^{-1}\omega\pi$.

Hence $(j)\upsilon = (j)\pi^{-1}\omega\pi$, for all $j \in \{1, \ldots, n\}$. Thus $\upsilon \in \pi^{-1}S\pi$ and consequently, because $\upsilon \in T$ is arbitrary, $T \subseteq \pi^{-1}S\pi$.

But, because $S \cong T$ we have that $|S| = |T|$ and then $|T| = |\pi^{-1}S\pi|$. It then follows that $T = \pi^{-1}S\pi$. □

As we can easily observe, Lemma 3.10 is an obvious consequence of this last result.

What is also very interesting, from a computational point of view, is that we can actually determine the element $\pi \in \mathcal{S}_n$ that conjugates S into T.

There are some relevant facts to state before we undertake this task.

Remark 3.17 Let S and T be semigroups generated by $\sigma = (1\ 2\ 3\ \ldots\ n)$ and a transformation of rank 2 in $\mathcal{T}_n$.

(i) We can always choose $\tau \in S$ of rank 2 such that $1\tau = 1$ and $S = \langle \sigma, \tau \rangle$. For this suppose that $S = \langle \sigma, \tau' \rangle$, where τ' is such that $1\tau' = a$. Then $(1)\tau'\sigma^k = 1$, where k is such that $a\sigma^k = 1$, and we can choose $\tau = \tau'\sigma^k$.

(ii) Let $S = \langle \sigma, \tau \rangle$ be such such that $1\tau = 1$. Suppose that there is $\pi \in \mathcal{S}_n$ such that $\pi^{-1}S\pi = T$. Take $i \in \{1, \ldots, n\}$ such that $1\pi = i$. Then $(i)\pi^{-1}\tau\pi = (1)\tau\pi = 1\pi = i$. Hence $\pi^{-1}\tau\pi$ fixes i.

We need a further technical result, which allows us to have an extra assumption over the permutation which conjugates the two semigroups.

Lemma 3.18 *Let S and T be semigroups generated by $\sigma = (1\ 2\ 3\ \ldots\ n)$ and a transformation of rank 2 of $\mathcal{T}_n$. Take $S = \langle \sigma, \tau \rangle$, where τ is such that $1\tau = 1$. Suppose that there is $\pi \in \mathcal{S}_n$ such that $\pi^{-1} S \pi = T$. Then there is $\gamma \in \mathcal{S}_n$ such that $\gamma^{-1} S \gamma = T$, $(1)\gamma^{-1}\tau\gamma = 1$ and $\gamma^{-1}\sigma\gamma = \pi^{-1}\sigma\pi$.*

PROOF. It is easy to see that $\pi^{-1}\sigma\pi = \sigma^k$, where $k \in \mathbb{N}$ and $\gcd(n,k) = 1$, because conjugation is an isomorphism. Suppose $\mathsf{im}\,(\pi^{-1}\tau\pi) = \{a, b\}$, with $1\pi = a$. Then $(a)\pi^{-1}\tau\pi = a$, by (ii) of the previous remark. Take $l \in \mathbb{N}$ such that $a\sigma^l = 1$ and define $\gamma = \pi\sigma^l$.

Let us take $\upsilon \in \gamma^{-1} S \gamma$. Then $\upsilon = \gamma^{-1}\omega\gamma$, for some $\omega \in S$. Hence $\upsilon \in T$ because $\upsilon = \gamma^{-1}\omega\gamma = \sigma^{-l}\pi^{-1}\omega\pi\sigma^l$ and $\pi^{-1}\omega\pi \in T$.

Conversely, take $\upsilon \in T$. Then there is $\omega \in S$ such that $\upsilon = \pi^{-1}\omega\pi$. Hence $\upsilon \in \gamma^{-1} S \gamma$, because $\upsilon = \gamma^{-1}\gamma\pi^{-1}\omega\pi\gamma^{-1}\gamma = \gamma^{-1}\pi\sigma^l\pi^{-1}\omega\pi\sigma^{-l}\pi^{-1}\gamma$, where $\pi^{-1}\omega\pi \in T$. Hence $\sigma^l\pi^{-1}\omega\pi\sigma^{-l} \in T$ and so $\pi\sigma^l\pi^{-1}\omega\pi\sigma^{-l}\pi^{-1} \in S$.

Also $(1)\gamma^{-1}\tau\gamma = (1)\sigma^{-l}\pi^{-1}\tau\pi\sigma^l = (a)\pi^{-1}\tau\pi\sigma^l = a\sigma^l = 1$.

The remaining equality from the lemma also holds: $\gamma^{-1}\sigma\gamma = \sigma^{-l}\pi^{-1}\sigma\pi\sigma^l = \sigma^{-l}\sigma^k\sigma^l = \sigma^k = \pi^{-1}\sigma\pi$. □

We now have all the requirements necessary to compute an element of $\mathcal{S}_n$ which conjugates S into T.

Theorem 3.19 *Let S and T be semigroups generated by $\sigma = (1\ 2\ 3\ \ldots\ n)$ and a transformation of rank 2 of $\mathcal{T}_n$. Suppose that $S = \langle \sigma, \tau \rangle$, with $1\tau = 1$. Let $\Sigma_\tau = \{i \in \{1, \ldots, n\} : i\tau = i\}$. If S conjugates into T then there exists $k \in \mathbb{N}$, with $\gcd(n,k) = 1$, such that π defined as $(\epsilon\sigma^i)\pi = 1\sigma^{ik}$, for some $\epsilon \in \Sigma_\tau$ and for all $i \in \{1, \ldots, n\}$, conjugates S into T.*

Proof. Since S conjugates into T and using the previous lemma, we have that there is $\gamma \in \mathcal{S}_n$ such that $\gamma^{-1}S\gamma = T$ and $(1)\gamma^{-1}\tau\gamma = 1$. Let $k \in \mathbb{N}$ be such that $\gamma^{-1}\sigma\gamma = \sigma^k$. Obviously $\gcd(n, k) = 1$ and therefore the map π is defined on every element of $\{1, \ldots, n\}$. Also, we have that π is injective, because $|\langle\sigma\rangle| = n$. Hence $\pi \in \mathcal{S}_n$.

We shall prove that $\gamma = \pi$. First note that if we define $\epsilon = 1\gamma^{-1}$ then we have $\epsilon \in \Sigma_\tau$, since $(1)\gamma^{-1}\tau\gamma = 1$ and so $(1\gamma^{-1})\tau = 1\gamma^{-1}$.

Let $i \in \{1, \ldots, n\}$. Then $1\sigma^{ik} = (1)(\gamma^{-1}\sigma\gamma)^i = (1)\gamma^{-1}\sigma^i\gamma = (\epsilon)\sigma^i\gamma$.

Hence $(\epsilon\sigma^i)\gamma = (\epsilon\sigma^i)\pi$, for all $i \in \{1, \ldots, n\}$. Thus $\gamma = \pi$. □

Remark 3.20 Given $S = \langle\sigma, \tau\rangle$, we know that we can always choose $\tau \in S$ such that $|\Sigma_\tau| \geq 1$. It is also obvious, since $\mathsf{rank}\,(\tau) = 2$, that $|\Sigma_\tau| \leq 2$ (Remark 3.17 (i)).

The last theorem is of utmost importance for the study of the isomorphism of two semigroups. Given two semigroups S, T generated by $\sigma = (1\ 2\ 3\ \ldots\ n)$ and a transformation of rank two of $\mathcal{T}_n$, such that $\mathcal{C} \subset S, T$, we can decide if S is isomorphic to T. Furthermore, we can find an element of $\mathcal{S}_n$ which conjugates S into T.

The next example illustrates the fact that Theorem 3.16 does not apply when $\mathcal{C} \not\subset S, T$, or equivalently, when S and T are completely regular (see Lemma 2.26).

Example 3.21 Let $n = 12$. Consider $\tau = [1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2]$ and $\alpha = [1, 4, 1, 4, 1, 4, 1, 4, 1, 4, 1, 4]$. Both these maps have conjugate periodic image (see Definition 2.27). Define $S = \langle\sigma, \tau\rangle$ and $T = \langle\sigma, \alpha\rangle$. From Theorem 2.31, the semigroups S and T are completely regular. We shall prove that S is not conjugate to T. For each case, we use the implementation in GAP of Theorem 3.19 to compute π (see Appendix B).

In this example we shall use the notation of the previous theorem. We clearly have $\Sigma_\tau = \{1, 2\}$.

Because $\mu_\sigma(\tau) = 1 \neq 3 = \mu_\sigma(\alpha)$ we conclude that $\alpha \notin S$ (Lemma 3.6) and therefore $S \neq T$, since $\alpha \in T$. Hence we do not consider the case $k = 1$.

- Let $k = 5$.

  Take $\epsilon = 1$. We have $\pi = (2,6)(3,11)(5,9)(8,12)$. By construction, $\pi^{-1}\sigma\pi = \sigma^5$. We then have $\pi^{-1}\tau\pi = [1,6,1,6,1,6,1,6,1,6,1,6] \notin T$, because $\mu_\sigma(\pi^{-1}\tau\pi) = 5$.

  Take $\epsilon = 2$. Then $\pi = (1,8,7,2)(3,6,9,12)(4,11,10,5)$ and $\pi^{-1}\tau\pi = [1,8,1,8,1,8,1,8,1,8,1,8]$. We have that $\mu_\sigma(\pi^{-1}\tau\pi) = 5$ and therefore $\pi^{-1}\tau\pi \notin T$.

- Let $k = 7$.

  Take $\epsilon = 1$. Then we have that $\pi = (2,8)(4,10)(6,12)$ and $\pi^{-1}\tau\pi = [1,8,1,8,1,8,1,8,1,8,1,8]$. Hence, as before, $\pi^{-1}\tau\pi \notin T$.

  Take $\epsilon = 2$. Then $\pi = (1,6,5,10,9,2)(3,8,7,12,11,4)$ and thus we have $\pi^{-1}\tau\pi = [1,6,1,6,1,6,1,6,1,6,1,6] \notin T$.

- Let $k = 11$.

  Take $\epsilon = 1$. Then $\pi = (2,12)(3,11)(4,10)(5,9)(6,8)$ and therefore $\pi^{-1}\tau\pi = [1,12,1,12,1,12,1,12,1,12,1,12]$. We have that $\pi^{-1}\tau\pi \in S$ and obviously is not in T.

  Take $\epsilon = 2$. Then $\pi = (1,2)(3,12)(4,11)(5,10)(6,9)(7,8)$ and $\pi^{-1}\tau\pi = [1,2,1,2,1,2,1,2,1,2,1,2] = \tau \notin T$

Therefore, there is no $\pi \in \mathcal{S}_{12}$ such that $\pi^{-1}S\pi = T$.

But we actually have that S is isomorphic to T, since they are both represented by the following presentation

$$\mathcal{P} = \langle s, t \mid s^{12} = 1, t^2 = t, (st)^2 = t \rangle.$$

Again, we used GAP to check that both S and T are represented by $\mathcal{P}$ (see Appendix B).

It is easy to check that both transformations τ and α from the previous example have conjugate periodic image (see Definition 2.27) and therefore, by Theorem 2.31, we have that S and T are completely regular semigroups, so $\mathcal{C} \not\subset S, T$. Consequently, this example proves that Theorem 3.16 does not apply to completely regular semigroups, meaning that there are completely regular semigroups S, T which are isomorphic but not conjugate.

Another interesting point is that this example proves that Proposition 3.14 does not hold for semigroups that do not contain the set of constant maps, because, as seen in the example, $S \cong T$ but $\mu(S) = 1 \neq 3 = \mu(T)$.

Nevertheless, the example above illustrates the algorithm to check whether two semigroups that contain constant maps (equivalently, that do not have conjugate periodic image) are isomorphic or otherwise. The GAP implementation of this algorithm can be found in Appendix B.

Chapter 4

Presentations

In this chapter we give presentations for semigroups generated by the n-cycle σ and a specific kind of transformation of rank two. In order to find these presentations we use with frequency some results mentioned in Chapter 2.

The study of presentations arises quite frequently in different areas of group theory and, with more interest for us, in semigroup theory. Methods to find semigroup presentations and to study semigroups defined by presentations can be found for example in [7], [8], [10] and [28].

1 Generalisation of some cases

Consider the following transformation, for n even,

$$\tau = \begin{pmatrix} 1 & 2 & 3 & \dots & \frac{n}{2}-1 & \frac{n}{2} & \frac{n}{2}+1 & \dots & n \\ 1 & 1 & 1 & \dots & 1 & \frac{n}{2}+1 & \frac{n}{2}+1 & \dots & \frac{n}{2}+1 \end{pmatrix}.$$

Clearly τ is an idempotent, because $1\tau = 1$ and $(\frac{n}{2}+1)\tau = \frac{n}{2}+1$.

Is is easy to verify that τ has conjugate periodic image (see Definition 2.39). Therefore $S = \langle \sigma, \tau \rangle$ is a completely regular semigroup (see Theorem 2.27). The transformation τ is $n/2$-block conjugate (see Definition 2.48). Thus, using Lemma

2.54, we know that $\overline{\tau} = \sigma^{n/2}\tau$. We also have that τ satisfies the condition $b - a = n/2$. Consequently, using Lemma 2.44, we have $\overline{\tau} = \tau\sigma^{n/2}$. With these two facts, we have the equality $\sigma^{n/2}\tau = \tau\sigma^{n/2}$. Furthermore, using Corollary 2.80, we have that $|S| = n(\frac{n}{2} + 1)$, since $v = n$.

Let $i \in \{1, 2, \ldots, \frac{n}{2} - 1\}$ and take $x \in X$ such that $x\tau = 1$. We then have that $1\sigma^i \in \{2, \ldots, \frac{n}{2}\}$, because $i \in \{1, 2, \ldots, \frac{n}{2} - 1\}$. Therefore $(x)\tau\sigma^i\tau = (1)\sigma^i\tau = 1 = x\tau$. The same holds if we take $x \in X$ such that $x\tau = \frac{n}{2} + 1$. Hence, for all $x \in X$, $(x)\tau\sigma^i\tau = \tau$, i.e. $\tau\sigma^i\tau = \tau$ (for all $i \in \{1, 2, \ldots, \frac{n}{2} - 1\}$).

This is part of the proof of the following result.

Lemma 4.1 *Let $\tau \in \mathcal{T}_n$ be as above. A presentation for $S = \langle \sigma, \tau \rangle$ is the following*

$$\begin{aligned} \mathcal{P} = \langle s, t \mid\ & s^n = 1, t^2 = t, ts^{n/2} = s^{n/2}t, \\ & ts^it = t, i \in \{1, 2, \ldots, \tfrac{n}{2} - 1\} \rangle. \end{aligned}$$

PROOF. Let us associate s with σ and t with τ. Clearly $s^n = 1$ holds, because σ has order n. The equalities $t^2 = t, ts^{n/2} = s^{n/2}t, ts^it = t$, for every $i \in \{1, 2, \ldots, \frac{n}{2} - 1\}$ hold, from the discussion above.

Consider the set $W = \{s^i : i \in \{0, \ldots, n-1\}; s^jts^k : j \in \{0, \ldots, n-1\}, k \in \{0, \ldots, \frac{n}{2} - 1\}\}$ and let $A = \{s, t\}$. We shall prove that for every word $u \in A^*$ there is a word $u' \in W$, such that $u = u'$ is a consequence of the relations in $\mathcal{P}$. This will be done by induction on the length of u (denoted by $|u|$).

• If $|u| = 0$ then $u = 1 = s^0 \in W$.

• Suppose that for every $m \in \mathbb{N}$, if we take $u \in A^*$ such that $|u| < m$ then u can be reduced to some $u' \in W$, by means of the relations of $\mathcal{P}$. Take $v \in A^*$ such that $|v| = m$. Then $v = uw$, where $|u| = m - 1$ and $w \in A$. By assumption, there is $u' \in W$ such that $u = u'$ is a consequence of the relations of $\mathcal{P}$. We aim to prove that $u'w \in W$, using the relations of $\mathcal{P}$. There are several different possibilities:

(a) Suppose that $u' = s^i$, for some $i \in \{0, \ldots, n-1\}$.

(1) If $w = s$ then $u'w = s^is = s^{i+1} \in W$.

(2) If $w = t$ then $u'w = s^it = s^its^0 \in W$.

(b) Suppose that $u' = s^jts^k$, for some $j \in \{0, \dots, n-1\}, k \in \{0, \dots, \frac{n}{2}-1\}$.

(1) If $w = s$ then $u'w = s^jts^{k+1}$. If $k \neq \frac{n}{2} - 1$ then $u'w = s^jts^{k+1} \in W$ because $k + 1 \in \{0, \dots, \frac{n}{2} - 1\}$. If $k = \frac{n}{2} - 1$ then $k + 1 = \frac{n}{2}$ and therefore $u'w = s^jts^{n/2} = s^js^{n/2}t = s^{j+\frac{n}{2}}t \in W$ (using $ts^{n/2} = s^{n/2}t$).

(2) If $w = t$ then $u'w = s^jts^kt$. If $k = 0$ then $u'w = s^jt^2 = s^jt \in W$ (using $t^2 = t$). If $k \neq 0$ then $s^jts^kt = s^jt \in W$ (using $ts^it = t$, for all $i \in \{1, \dots, \frac{n}{2} - 1\}$).

We have that $|W| \leq n + n\frac{n}{2} = n(\frac{n}{2} + 1) = |S|$. Thus, using Proposition 1.19, we conclude that $\mathcal{P}$ is a presentation for the semigroup S. □

A further generalisation of the previous result is possible.

Lemma 4.2 *For n even, let*

$$\alpha = \begin{pmatrix} 1 & 2 & \dots & \frac{n}{2}-1 & \frac{n}{2} & \frac{n}{2}+1 & \dots & n \\ a & a & \dots & a & b & b & \dots & b \end{pmatrix},$$

where $a, b \in \mathbb{N}$ are such that $b - a = n/2$. Then

$$\mathcal{P} = \langle s, t \mid s^n = 1, t^2 = t, ts^{n/2} = s^{n/2}t, \\ ts^it = t, i \in \{1, 2, \dots, \tfrac{n}{2} - 1\} \rangle$$

is a presentation for $S = \langle \sigma, \alpha \rangle$.

PROOF. All we need to prove is that $\langle \sigma, \tau \rangle = \langle \sigma, \alpha \rangle$.

For this, we ought to note that $\alpha = \tau\sigma^{a-1} \in \langle \sigma, \tau \rangle$ and $\tau = \alpha\sigma^{a'}$, where $a' \in \mathbb{N}$ is such that $a\sigma^{a'} = 1$. Therefore $\tau \in \langle \sigma, \alpha \rangle$ and the result holds. □

Another interesting case is the following one.

Lemma 4.3 *For n even, let*

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n-1 & n \\ 1 & \frac{n}{2}+1 & 1 & \frac{n}{2}+1 & \dots & 1 & \frac{n}{2}+1 \end{pmatrix},$$

and consider $S = \langle \sigma, \tau \rangle$.

(i) *If $n = 4k$, for some $k \in \mathbb{N}$, then*

$$\mathcal{P} = \langle s, t \mid s^n = 1, t^3 = t^2, ts^{n/2} = st \rangle$$

is a presentation for S.

(ii) *If $n = 4k+2$, for some $k \in \mathbb{N}$, then*

$$\mathcal{P} = \langle s, t \mid s^n = 1, t^2 = t, ts^{n/2} = st \rangle$$

is a presentation for S.

PROOF. There are a few facts that are relevant to state at this point. The first is that if $n \in \mathbb{N}$, with $n \geq 2$, is even then either $n = 4k$ or $n = 4k+2$, for some $k \in \mathbb{N}$. Secondly, we have that, by construction, the kernel classes of τ are $K_1 = 1\tau^{-1} = \{1\sigma^{2m} : m \geq 1\}$ and $K_{\frac{n}{2}+1} = (\frac{n}{2}+1)\tau^{-1} = \{2\sigma^{2m} : m \geq 1\}$.

(i) Suppose that $n = 4k$, for some $k \in \mathbb{N}$. We associate s with σ and t with τ. The transformation τ has periodic image. To prove this, let $x \in K_1$. Then $x = 1\sigma^{2m}$, for some $m \geq 1$. We also have that $1\sigma^{2m} \in K_1$ if and only if $1\sigma^{2(m+k)} \in K_1$, by definition of K_1. Therefore

$$x \in K_1 \Leftrightarrow 1\sigma^{2m} \in K_1 \Leftrightarrow 1\sigma^{2(m+k)} \in K_1 \Leftrightarrow$$

$$1\sigma^{2m}\sigma^{2k} \in K_1 \Leftrightarrow 1\sigma^{2m}\sigma^{n/2} \Leftrightarrow 1\sigma^{2m}\sigma^{b-a} \in K_1 \Leftrightarrow x\sigma^{b-a} \in K_1,$$

which is the required condition in Definition 2.5. Consequently, there are no idempotents of rank 2 in S (see Corollary 2.19) and therefore $\mathsf{rank}\,(\tau^2) = 1$ (see Proposition 1.18). Thus $\tau^3 = \tau^2$ and then the relation $t^3 = t^2$ holds.

Clearly τ satisfies the condition $b - a = n/2$ and therefore $\overline{\tau} = \tau\sigma^{n/2}$ (see Lemma 2.44). The transformation τ is 1-block conjugate and therefore $\overline{\tau} = \sigma\tau$ (see Lemma 2.54). These two facts allow us to conclude that the condition $ts^{n/2} = st$ holds. It is clear that $s^n = 1$ holds.

There are some relevant consequences that follow from these relations. From $ts^{n/2} = st$ it follows that, for every $p \in \mathbb{N}$ we have

$$s^p t = ts^{p\frac{n}{2}}. \tag{1}$$

Furthermore, $ts^{p\frac{n}{2}} = ts^n = t$, if p is even or $ts^{p\frac{n}{2}} = ts^{n/2} = st$, if p is odd. Resuming,

$$s^p t = \begin{cases} t, & \text{if } p \text{ is even.} \\ st, & \text{otherwise.} \end{cases} \tag{2}$$

Let $W = \{s^i : i \in \{0, \ldots, n-1\}; s^j t s^k : j \in \{0,1\}, k \in \{0, \ldots, \frac{n}{2} - 1\}; t^2 s^l : l \in \{0, \ldots, n-1\}\}$ and consider $A = \{s, t\}$. We shall prove that for every word $u \in A^*$ there exists a word $u' \in W$ such that $u = u'$ is a consequence of the relations in $\mathcal{P}$.

• If $|u| = 0$ then $u = 1 = s^0 \in W$.

• Suppose that for every $m \in \mathbb{N}$, if $u \in A^*$ is such that $|u| < m$ then u can be reduced to $u' \in W$, by means of the relations of $\mathcal{P}$. Take $v \in A^*$ such that $|v| = m$. Then $v = uw$, where $|u| = m - 1$ and $w \in A$. By assumption, there is $u' \in W$ such that $u = u'$ is a consequence of the relations of $\mathcal{P}$. We shall prove that there is $v' \in W$ such that $u'w = v'$ is a consequence of the relations of $\mathcal{P}$. There are several different possibilities:

(a) Suppose that $u' = s^i$, for some $i \in \{0, \ldots, n-1\}$.

(1) If $w = s$ then $u'w = s^{i+1} \in W$.

(2) If $w = t$ then (using the relations in (2))

$$u'w = s^i t = \begin{cases} t, & \text{if } i \text{ is even.} \\ ts^{n/2} = st, & \text{otherwise.} \end{cases}$$
In either case we have that $u'w \in W$.

(b) Suppose that $u' = s^j ts^k$, for some $j \in \{0, 1\}$, $k \in \{0, \ldots, \frac{n}{2} - 1\}$.

(1) If $w = s$ then $u'w = s^j ts^{k+1}$. If $k \neq \frac{n}{2} - 1$ then $u'w = s^j ts^{k+1} \in W$, because $k + 1 \in \{1, \ldots, \frac{n}{2} - 1\}$. If $k = \frac{n}{2} - 1$ then $k + 1 = n/2$. Thus $u'w =$
$$s^j ts^{n/2} = s^j st = s^{j+1}t = \begin{cases} st\ (\in W), & \text{if } j = 0. \\ s^2 t = t\ (\in W), & \text{if } j = 1. \end{cases}$$

(2) If $w = t$ then $u'w = s^j ts^k t = s^j tts^{k\frac{n}{2}} = ts^{j\frac{n}{2}} ts^{k\frac{n}{2}} = tts^{(j\frac{n}{2})\frac{n}{2}} s^{k\frac{n}{2}} = t^2 s^l$, for some $l \in \{0, \ldots, n-1\}$, using the relation (1). Therefore $u'w \in W$.

(c) Suppose that $u' = t^2 s^l$, for some $l \in \{0, \ldots, n-1\}$.

(1) If $w = s$ then $u'w = t^2 s^{l+1} \in W$.

(2) If $w = t$ then $u'w = t^2 s^l t = t^2 ts^{l\frac{n}{2}} = t^2 s^{l\frac{n}{2}} \in W$.

As seen above, S has periodic image, satisfies $b - a = n/2$ and is 1-block conjugate. Hence $|S| = n(\frac{2}{2} + 2) = 3n$ (see Corollary 2.81). Because $|W| \leq n + 2(n/2) + n = 3n = |S|$ we can deduce that S is represented by $\mathcal{P}$ (using Proposition 1.19).

(ii) Let $n = 4k+2$, for some $k \in \mathbb{N}$. If we take $x \in K_1$ then, as before, we have $x\sigma^{2m} \in K_1$, for any $m \in \mathbb{N}$. But for any $k, m \in \mathbb{N}$ we have $2k + 1 \neq 2m$. Hence $x\sigma^{b-a} = x\sigma^{n/2} = x\sigma^{2k+1} \notin K_1$, so $x\sigma^{b-a} \in K_{\frac{n}{2}+1}$. This allows us to conclude that τ has conjugate periodic image and therefore $S = \langle \sigma, \tau \rangle$ is completely regular (see Theorem 2.31). As in (i), we have that τ is 1-block conjugate and it satisfies $b - a = n/2$. Hence $|S| = n(\frac{2}{2} + 1) = 2n$ (see Corollary 2.80).

Let us associate σ with s and τ with t. We have that τ is an idempotent since $1\tau = 1$ and $\frac{n}{2}\tau = \frac{n}{2}$. Thus $t^2 = t$ must hold.

Similarly to (i), we can prove that the relation $ts^{n/2} = st$ holds.

Trivially $s^n = 1$ must hold too.

In the same way as in (i), we can deduce the following relations, for every $p \in \mathbb{N}$,

$$s^p t = t s^{p\frac{n}{2}} \tag{3}$$

and

$$s^p t = \begin{cases} t, & \text{if } p \text{ is even.} \\ st, & \text{otherwise.} \end{cases} \tag{4}$$

Let us define $W = \{s^i : i \in \{0, \ldots, n-1\}; s^j t s^k : j \in \{0,1\}, k \in \{0, \ldots, \frac{n}{2}-1\}\}$ and $A = \{s, t\}$. As before, we shall prove that every word $u \in A^*$ can be reduced to a word in W by means of the relations in $\mathcal{P}$.

• If $|u| = 0$ then $u = 1 = s^0 \in W$.

• Suppose that for every $m \in \mathbb{N}$, if $u \in A^*$ is such that $|u| < m$ then u can be reduced to $u' \in W$, by means of the relations of $\mathcal{P}$. Take $v \in A^*$ such that $|v| = m$. Then $v = uw$, where $|u| = m - 1$ and $w \in A$. By assumption, there is $u' \in W$ such that $u = u'$ is a consequence of the relations of $\mathcal{P}$. We shall prove that there is $v' \in W$ such that $u'w = v'$ is a consequence of the relations of $\mathcal{P}$. There are several different possibilities:

(a) Suppose that $u' = s^i$, for some $i \in \{0, \ldots, n-1\}$.

(1) If $w = s$ then $u'w = s^{i+1} \in W$.

(2) Suppose that $w = t$. Then, using the relations in (4), we have that $u'w = s^i t = \begin{cases} t \ (\in W), & \text{if } i \text{ is even.} \\ ts^{n/2} = st \ (\in W), & \text{otherwise.} \end{cases}$

(b) Suppose that $u' = s^j t s^k$, for some $j \in \{0,1\}, k \in \{0, \ldots, \frac{n}{2} - 1\}$.

(1) If $w = s$ then $u'w = s^j t s^{k+1}$. If $k \neq \frac{n}{2} - 1$ then $u'w = s^j t s^{k+1} \in W$,

because $k+1 \in \{1, \ldots, \frac{n}{2}-1\}$. If $k = \frac{n}{2}-1$ then $k+1 = n/2$. Thus $u'w =$
$s^j ts^{n/2} = s^j st = s^{j+1}t = \begin{cases} st \ (\in W), & \text{if } j = 0. \\ s^2t = t \ (\in W), & \text{otherwise.} \end{cases}$

(2) If $w = t$ then $u'w = s^j ts^k t = s^j tts^{k\frac{n}{2}} = s^j ts^{k\frac{n}{2}} = s^j s^k t$, using the relation in (3) and $t^2 = t$. Then we have $u'w = \begin{cases} s^j t \ (\in W), & \text{if } k \text{ is even.} \\ s^j st, & \text{otherwise.} \end{cases}$

If k is odd then we have $u'w = s^j st = \begin{cases} st \ (\in W), & \text{if } j = 0. \\ s^2t = t \ (\in W), & \text{if } j = 1. \end{cases}$

Because $|S| = 2n$, we only need to prove that $|W| \leq |S|$. But this holds because $|W| \leq n + 2(n/2) = 2n = |S|$. Thus $\mathcal{P}$ is a presentation for S. □

A further generalisation of this result is possible.

Lemma 4.4 *For n even, let*

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & n-1 & n \\ a & b & a & b & \ldots & a & b \end{pmatrix},$$

where $b - a = n/2$ and consider $S = \langle \sigma, \alpha \rangle$.

(i) *If $n = 4k$, for some $k \in \mathbb{N}$ then*

$$\mathcal{P} = \langle s, t \mid s^n = 1, t^3 = t^2, ts^{n/2} = st \rangle$$

is a presentation for S.

(ii) *If $n = 4k+2$, for some $k \in \mathbb{N}$ then*

$$\mathcal{P} = \langle s, t \mid s^n = 1, t^2 = t, ts^{n/2} = st \rangle$$

is a presentation for S.

PROOF. As in Lemma 4.2, we only need to prove that $\alpha \in \langle \sigma, \tau \rangle$ and $\tau \in \langle \sigma, \alpha \rangle$. For this, it suffices to note that $\mathsf{ker}(\tau) = \mathsf{ker}(\alpha)$ and that $1\sigma^{a-1} = a$ and that there is $a' \in \mathbb{N}$ such that $a\sigma^{a'} = 1$. □

Let us look now at a different kind of semigroup.

Lemma 4.5 *Let*

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n \\ 1 & 2 & 1 & 1 & \dots & 1 \end{pmatrix} \in \mathcal{T}_n, \text{ where } n \geq 4.$$

The semigroup $S = \langle \sigma, \tau \rangle$ is represented by

$$\begin{aligned} \mathcal{P} = \langle s, t \mid\ & s^n = 1, t^2 = t, (tst)^2 = t, sts^2t = ts^2t, ts^2tst = ts^2ts, \\ & ts^it = ts^2t, i \in \{3, \dots, n-1\} \rangle. \end{aligned}$$

PROOF. This semigroup is regular, since τ is an idempotent but has neither conjugate periodic image (since τ has n-block structure) nor periodic image (because τ is an idempotent). It is also obvious that τ does not satisfy $b - a = n/2$. Therefore, by Corollary 2.79, we have that $|S| = 2n(n+1)$.

We associate σ with s and τ with t. The relations $s^n = 1$ and $t^2 = t$ clearly must hold. Also, $\sigma\tau = [2, 1, 1, 1, 1, \dots, 1]$ and therefore $\tau\sigma\tau = [2, 1, 2, 2, 2, \dots, 2] = \overline{\tau}$, which is the conjugate of τ. Since S is regular we have that $|[\mathfrak{H}]| = 2$ (see Lemma 2.74) and therefore $\tau \mathcal{H} \overline{\tau}$. Because τ is an idempotent we have that $\overline{\tau}^2 = \tau$, hence the relation $(tst)^2 = t$ must hold. We have that $\tau\sigma^2 = [3, 4, 3, 3, 3, \dots, 3]$ and then $\tau\sigma^2\tau = [1, 1, 1, 1, 1, \dots, 1] = \sigma\tau\sigma^2\tau$. Therefore the relation $sts^2t = ts^2t$ hold. We also have that $\tau\sigma^2\tau\sigma = [2, 2, 2, 2, 2, \dots, 2]$ and $\tau\sigma^2\tau\sigma\tau = [2, 2, 2, 2, 2, \dots, 2]$ which implies that $ts^2tst = ts^2ts$ holds. We have that $\tau\sigma^i = [a, b, a, a, a, \dots, a]$ with $a, b \neq 2$, for all $i \in \{3, \dots, n-1\}$. Then $a\tau = b\tau = 1$ and consequently $\tau\sigma^i\tau = [1, 1, 1, 1, 1, \dots, 1]$. This implies that the relation $ts^it = ts^2t$ holds, for all $i \in \{3, \dots, n-1\}$.

Let $W = \{s^j : j \in \{0, \dots, n-1\}; s^kts^l : k, l \in \{0, \dots, n-1\}; s^mtsts^p : m, p \in \{0, \dots, n-1\}; ts^2ts^q : q \in \{0, \dots, n-1\}\}$. Let $A = \{s, t\}$. We shall prove by induction over the length of words that every word of A^* can be reduced to one of W, by means of the relations of $\mathcal{P}$.

Take $u \in A^*$.

• If $|u| = 0$ then $u = 1 = s^0 \in W$.

• Suppose that for every $r \in \mathbb{N}$, if $u \in A^*$ is such that $|u| < r$ then u can be reduced to $u' \in W$, by means of the relations of $\mathcal{P}$. Take $v \in A^*$ such that $|v| = r$. Then $v = uw$, where $|u| = r - 1$ and $w \in A$. By assumption, there is $u' \in W$ such that $u = u'$ is a consequence of the relations of $\mathcal{P}$. We shall prove that there is $v' \in W$ such that $u'w = v'$ is a consequence of the relations of $\mathcal{P}$. There are several different possibilities:

(a) Suppose that $u' = s^j$, for some $j \in \{0, \dots, n-1\}$.

(1) If $w = s$ then $u'w = s^{j+1} \in W$.

(2) If $w = t$ then $u'w = s^j t = s^j t s^0 \in W$.

(b) Suppose that $u' = s^k t s^l$, for some $k, l \in \{0, \dots, n-1\}$.

(1) If $w = s$ then $u'w = s^k t s^{l+1} \in W$.

(2) If $w = t$ then

$$u'w = s^k t s^l t = \begin{cases} s^k t^2 = s^k t \ (\in W), & \text{if } l = 0. \\ s^k tst \ (\in W), & \text{if } l = 1. \\ s^k ts^2 t = ts^2 t \ (\in W), & \text{if } l = 2. \\ s^k ts^2 t = ts^2 t \ (\in W), & \text{otherwise.} \end{cases}$$

(c) Suppose that $u' = s^m tsts^p$, for some $m, p \in \{0, \dots, n-1\}$.

(1) If $w = s$ then $u'w = s^m tsts^{p+1} \in W$.

(2) If $w = t$ then

$$u'w = s^m tsts^p t = \begin{cases} s^m tst^2 = s^m tst \ (\in W), & \text{if } p = 0. \\ s^m tstst = s^m tsttst = s^m t \ (\in W), & \text{if } p = 1. \\ s^m tsts^2 t = s^m t^2 s^2 t = s^m ts^2 t \ (\in W), & \text{if } p = 2. \\ s^m tsts^2 t = s^m tts^2 t = ts^2 t \ (\in W), & \text{otherwise.} \end{cases}$$

(d) Suppose that $u' = ts^2ts^q$, for some $q \in \{0, \ldots, n-1\}$.

(1) If $w = s$ then $u'w = ts^2ts^{q+1} \in W$.

(2) If $w = t$ then

$$u'w = ts^2ts^q t = \begin{cases} ts^2t^2 = ts^2t \ (\in W), & \text{if } q = 0. \\ ts^2tst = ts^2ts \ (\in W), & \text{if } q = 1. \\ ts^2ts^2t = tts^2t = ts^2t \ (\in W), & \text{if } q = 2. \\ ts^2ts^2t = ts^2t \ (\in W), & \text{otherwise.} \end{cases}$$

Because $|W| \leq n + n^2 + n^2 + n = 2n(n+1) = |S|$, we can conclude that $\mathcal{P}$ is a presentation for S. $\square$

A case similar to the previous is the following one.

Lemma 4.6 *Let*

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \ldots & n \\ 1 & 1 & 3 & 1 & 1 & \ldots & 1 \end{pmatrix} \in \mathcal{T}_n, \ \textit{where } n \geq 5.$$

The semigroup $S = \langle \sigma, \tau \rangle$ is represented by

$$\mathcal{P} = \langle s, t \mid s^n = 1, t^2 = t, (ts^2t)^2 = t, stst = tst, tsts^2t = tsts^2, \\ ts^it = tst, i \in \{3, 4, \ldots, n-1\} \rangle.$$

PROOF. In an analogous way to the previous proof, we can verify that $|S| = 2n(n+1)$.

Note that $\mathsf{ker}\,(\tau) = \{K_1, K_3\}$, where $K_3 = \{3\}$ and $K_1 = X \setminus K_3$.

Associating σ with s and τ with t we can easily check that the relations $s^n = 1$ and $t^2 = t$ must hold. We have that $\tau\sigma = [2, 2, 4, 2, 2, \dots, 2]$ and therefore $\tau\sigma\tau = [1, 1, 1, 1, 1, \dots, 1]$. Hence $stst = tst$ also holds. Also, $\tau\sigma^2 = [3, 3, 5, 3, 3, \dots, 3]$ which implies that $\tau\sigma^2\tau = [3, 3, 1, 3, 3, \dots, 3] = \overline{\tau}$, the conjugate of τ. Thus, the relation $(ts^2t)^2 = t$ holds. To prove that the relation $tsts^2t = tsts^2$ holds, note that $\tau\sigma\tau\sigma^2\tau = [3, 3, 3, 3, \dots, 3] = \tau\sigma\tau\sigma^2$. Taking $i \in \{3, \dots, n-1\}$, we have that $\mathsf{im}\,(\tau\sigma^i) = \{1\sigma^i, 3\sigma^i\} \subseteq K_1$. Therefore $(1\sigma^i)\tau = (2\sigma^i)\tau = 1$ which implies that $\tau\sigma^i\tau = [1, 1, 1, 1, \dots, 1]$. This implies that the relation $ts^it = tst$ must hold, for all $i \in \{3, \dots, n-1\}$.

Define $W = \{s^j : j \in \{0, \dots, n-1\}; s^kts^l : k, l \in \{0, \dots, n-1\}; s^mts^2ts^p : m, p \in \{0, \dots, n-1\}; tsts^q : q \in \{0, \dots, n-1\}\}$ and let $A = \{s, t\}$.

As before, we shall prove that for every word $u \in A^*$ there is a word $u' \in W$ such that $u = u'$ is a consequence of the relations of $\mathcal{P}$. This will be done by induction over the length of u.

Take $u \in A^*$.

• If $|u| = 0$ then $u = 1 = s^0 \in W$.

• Suppose that for every $r \in \mathbb{N}$, if $u \in A^*$ is such that $|u| < r$ then u can be reduced to $u' \in W$, by means of the relations of $\mathcal{P}$. Take $v \in A^*$ such that $|v| = r$. Then $v = uw$, where $|u| = r - 1$ and $w \in A$. By assumption, there is $u' \in W$ such that $u = u'$ is a consequence of the relations of $\mathcal{P}$. We shall prove that there is $v' \in W$ such that $u'w = v'$ is a consequence of the relations of $\mathcal{P}$. There are several different possibilities:

(a) Suppose that $u' = s^j$, for some $j \in \{0, \dots, n-1\}$.

(1) If $w = s$ then $u'w = s^{j+1} \in W$.

(2) If $w = t$ then $u'w = s^jt = s^jts^0 \in W$.

(b) Suppose that $u' = s^kts^l$, for some $k, l \in \{0, \dots, n-1\}$.

(1) If $w = s$ then $u'w = s^kts^{l+1} \in W$.

(2) If $w = t$ then

$$u'w = s^k ts^l t = \begin{cases} s^k t^2 = s^k t \ (\in W), & \text{if } l = 0. \\ s^k tst = tst \ (\in W), & \text{if } l = 1. \\ s^k ts^2 t \ (\in W), & \text{if } l = 2. \\ s^k tst = tst \ (\in W), & \text{otherwise.} \end{cases}$$

(c) Suppose that $u' = s^m ts^2 ts^p$, for some $m, p \in \{0, \ldots, n-1\}$.

(1) If $w = s$ then $u'w = s^m ts^2 ts^{p+1} \in W$.

(2) If $w = t$ then

$$u'w = s^m ts^2 ts^p t = \begin{cases} s^m ts^2 t^2 = s^m ts^2 t \ (\in W), & \text{if } p = 0. \\ s^m ts^2 tst = s^m ttst = s^m tst = tst \ (\in W), & \text{if } p = 1. \\ s^m ts^2 ts^2 t = s^m ts^2 tts^2 t = s^m t \ (\in W), & \text{if } p = 2. \\ s^m ts^2 tst = tst \ (\in W), & \text{otherwise.} \end{cases}$$

(d) Suppose that $u' = tsts^q$, for some $q \in \{0, \ldots, n-1\}$.

(1) If $w = s$ then $u'w = tsts^{q+1} \in W$.

(2) If $w = t$ then

$$u'w = tsts^q t = \begin{cases} tstt = tst \ (\in W), & \text{if } q = 0. \\ tstst = ttst = tst \ (\in W), & \text{if } q = 1. \\ tsts^2 t = tsts^2 \ (\in W), & \text{if } q = 2. \\ tstst = tst \ (\in W), & \text{otherwise.} \end{cases}$$

Clearly $|W| \leq n + n^2 + n^2 + n = 2n(n+1) = |S|$ and therefore $\mathcal{P}$ is a presentation for S. □

It is relevant to make an informal remark at this stage. When $n = 4$, the last transformation τ satisfies $b - a = n/2$. Therefore, the presentation from the previous lemma does not work for this specific value of n.

Let us study another example.

Lemma 4.7 *Let*

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n-1 & n \\ 1 & 2 & 1 & 2 & \dots & 1 & 2 \end{pmatrix} \in \mathcal{T}_n, \text{ with } n \geq 4.$$

Then $S = \langle \sigma, \tau \rangle$ *has the following presentation*

$$\mathcal{P} = \langle s, t \mid s^n = 1, t^2 = t, (st)^2 = t \rangle.$$

PROOF. The transformation τ has conjugate periodic image (i.e. for each $x \in X$, the following equivalence holds,

$$x \in K_1 \text{ if and only if } x\sigma \in K_2.$$

Therefore S is completely regular. We do not have $b - a = n/2$ and S is 1-block conjugate. Consequently we have $|S| = n(2+1) = 3n$.

Let us associate σ with s and τ with t. The relations $s^n = 1$ and $t^2 = t$ clearly must hold. Because τ is 1-block conjugate we have $\overline{\tau} = \sigma\tau$ (see Lemma 2.54) and since $\overline{\tau}^2 = \tau$, we have that the relation $(st)^2 = t$ holds.

We can deduce some further relations which we shall use in this proof. From $(st)^2 = t$ we can deduce that $tst = s^{-1}t$ (because s is invertible). Multiplying by t on the right side and having in mind that $t^2 = t$ we get $tst = ts^{-1}t$. On the other hand, $(s^{-1}t)^2 = (tst)^2 = tstst = t(st)^2 = tt = t$, that is $s^{-1}ts^{-1}t = t$. Multiplying by s^2 on the right we get $s^2s^{-1}ts^{-1}t = s^2t$, which is equivalent to $sts^{-1}t = s^2t$. From the second relation deduced we get $stst = s^2t$, which is $(st)^2 = s^2t$ and therefore we conclude that $t = s^2t$.

We can generalise this relation.

$$s^p t = \begin{cases} t, & \text{if } p \text{ is even.} \\ st, & \text{otherwise.} \end{cases} \tag{1}$$

Define $W = \{s^i : i \in \{0, \ldots, n-1\}; s^j t s^k : j \in \{0,1\}, k \in \{0, \ldots, n-1\}\}$ and $A = \{s, t\}$. As in the previous proofs, we shall see that for every $u \in A^*$ there is $u' \in W$ such that $u = u'$ is a consequence of $\mathcal{P}$.

Let $u \in A^*$.

• If $|u| = 0$ then $u = 1 = s^0 \in W$.

• Suppose that for every $m \in \mathbb{N}$, if $u \in A^*$ is such that $|u| < m$ then u can be reduced to $u' \in W$, by means of the relations of $\mathcal{P}$. Take $v \in A^*$ such that $|v| = m$. Then $v = uw$, where $|u| = m - 1$ and $w \in A$. By assumption, there is $u' \in W$ such that $u = u'$ is a consequence of the relations of $\mathcal{P}$. We will prove that there is $v' \in W$ such that $u'w = v'$ is a consequence of $\mathcal{P}$. There are several different possibilities:

(a) Suppose that $u' = s^i$, for some $i \in \{0, \ldots, n-1\}$.

(1) If $w = s$ then $u'w = s^{i+1} \in W$.

(2) If $w = t$ then $u'w = s^i t = s^i t s^0 \in W$.

(b) Suppose that $u' = s^j t s^k$, for some $j, k \in \{0, \ldots, n-1\}$.

(1) If $w = s$ then $u'w = s^j t s^{k+1} \in W$.

(2) If $w = t$ then, using the relations in (1),

$$u'w = s^j t s^k t = \begin{cases} s^j tt = s^j t \ (\in W), & \text{if } k \text{ is even.} \\ s^j tst, & \text{otherwise.} \end{cases}$$

If k is odd, $u'w = s^j tst = \begin{cases} tst = s^{n-1}t, & \text{if } j \text{ is } 0. \\ stst = t \ (\in W), & \text{if } j \text{ is } 1. \end{cases}$

Because n is even we have that $n-1$ is odd. Using the relations in (1), we deduce that $s^{n-1}t = st \in W$.

Clearly $|W| \le n + 2n = 3n = |S|$ and therefore $\mathcal{P}$ is a presentation for S. $\square$

The next example follows.

Lemma 4.8 *For n even, let*

$$\tau = \begin{pmatrix} 1 & 2 & 3 & \dots & \frac{n}{2}+1 & \frac{n}{2}+2 & \frac{n}{2}+3 & \dots & n \\ 1 & 2 & 2 & \dots & 2 & 1 & 1 & \dots & 1 \end{pmatrix} \in \mathcal{T}_n, \text{ with } n \geq 4,$$

and consider $S = \langle \sigma, \tau \rangle$. We have that

$$\mathcal{P} = \langle s, t \mid s^n = 1, t^2 = t, (s^{\frac{n}{2}}t)^2 = t, stst = tst, ts^{\frac{n}{2}+1}ts = tst, \\ ts^it = tst, i \in \{2, 3, \dots, \tfrac{n}{2} - 1\} \rangle$$

is a presentation for S.

PROOF. The element τ is an idempotent ($1\tau = 1, 2\tau = 2$) and therefore τ does not have periodic image and consequently S is regular. We have $\mathsf{ker}(\tau) = \{K_1, K_2\}$, where $K_2 = \{2\sigma^m, m \in \{0, \dots, \frac{n}{2} - 1\}$ and $K_1 = X \setminus K_2$. This transformation clearly does not have periodic image ($2 \in K_2$ but $3 = 2\sigma^{2-1} \notin K_1$). Thus S is not completely regular. S is $n/2$-block conjugate and it does not satisfy $b - a = n/2$. Se we can conclude that $|S| = n(n+2) = n^2 + 2n$.

Associating the generators σ and τ with s and t, respectively, we have that $s^n = 1$ and $t^2 = t$ clearly must hold. Also, because τ is $n/2$-block conjugate, we have $\sigma^{n/2}\tau = \overline{\tau}$, which is the conjugate of τ. Thus $(s^{n/2}t)^2 = t$ holds. We have $\mathsf{im}(\tau\sigma) = \{2, 3\}$ and this implies that $\tau\sigma\tau = [2, 2, 2, \dots, 2]$. Consequently $stst = tst$ holds. For the last relations, we have that $\mathsf{im}(\tau\sigma^i) = \{1\sigma^i, 2\sigma^i\} = \{2\sigma^{i-1}, 2\sigma^i\}$, for $i \in \{2, 3, \dots, \frac{n}{2} - 1\}$ and $2\sigma^{i-1}, 2\sigma^i \in K_2$, for any $i \in \{2, \dots, \frac{n}{2} - 1\}$. Thus $\tau\sigma^i\tau = [2, 2, 2, \dots, n]$ which means that $ts^it = tst$ holds, for all $i \in \{2, \dots, \frac{n}{2} - 1\}$.

Let us deduce some more relations which we will use during this proof. We shall prove that $ts^it = tsts^{n-1}$, for all $i \in \{\frac{n}{2}+1, \dots, n-1\}$. Note that $s^{n/2}ts^{n/2}t = t$ is equivalent to $ts^{n/2}t = s^{n/2}t$.

If $i = \frac{n}{2}+1$ then $ts^{\frac{n}{2}+1}t = tsts^{-1} = tsts^{n-1}$. Let i be such that $\frac{n}{2}+2 \leq i \leq n-1$. We have that $i = (\frac{n}{2} + 1) + j$ with $1 \leq j \leq \frac{n}{2} - 1$. Then $ts^it = ts^{j+1}s^{n/2}t = ts^{j+1}ts^{n/2}t = tsts^{n/2}t = tss^{n/2}t = ts^{\frac{n}{2}+1}t = tsts^{n-1}$.

Hence we have the relations $ts^it = tsts^{n-1}$, for $i \in \{\frac{n}{2}+1, \ldots, n-1\}$.

Let $W = \{s^j : j \in \{0, \ldots, n-1\}; s^kts^l : k, l \in \{0, \ldots, n-1\}; tsts^m : m \in \{0, \ldots, n-1\}\}$ and $A = \{s, t\}$.

Take $u \in A^*$.

• If $|u| = 0$ then $u = 1 = s^0 \in W$.

• Suppose that for all $r \in \mathbb{N}$, if $u \in A^*$ is such that $|u| < r$ then there is $u' \in W$ such that $u = u'$ follows from the relations of $\mathcal{P}$. Take $v \in A^*$ such that $|v| = r$. Then $v = uw$, where $|u| = r-1$ and $w \in A$. By assumption, there is $u' \in W$ such that $u = u'$ is a consequence of the relations of $\mathcal{P}$. We shall prove that there is $v' \in W$ such that $u'w = v'$ is a consequence of the relations of $\mathcal{P}$. There are several different possibilities:

(a) Suppose that $u' = s^j$, for some $j \in \{0, \ldots, n-1\}$.

(1) If $w = s$ then $u'w = s^{j+1} \in W$.

(2) If $w = t$ then $u'w = s^jt = s^jts^0 \in W$.

(b) Suppose that $u' = s^kts^l$, for some $k, l \in \{0, \ldots, n-1\}$.

(1) If $w = s$ then $u'w = s^kts^{l+1} \in W$.

(2) If $w = t$ then

$$u'w = s^kts^lt = \begin{cases} s^ktt = s^kt \ (\in W), & \text{if } l = 0. \\ s^ktst = tst \ (\in W), & \text{if } l = 1. \\ s^ktst \ (\in W), & \text{if } l \in \{2, \ldots, \frac{n}{2}-1\}. \\ s^kts^{n/2}t = s^ks^{n/2}t = s^{k+\frac{n}{2}}t \ (\in W), & \text{if } l = n/2. \\ s^ktsts^{n-1} = tsts^{n-1} \ (\in W) & \text{if } l \in \{\frac{n}{2}+1, \ldots, n-1\}. \end{cases}$$

(c) Suppose that $u' = tsts^m$, for some $m \in \{0, \ldots, n-1\}$.

(1) If $w = s$ then $u'w = tsts^{m+1} \in W$.

(2) If $w = t$ then $u'w =$

$$tsts^m t = \begin{cases} tstt = tst\ (\in W), & \text{if } m = 0. \\ tstst = ttst = tst\ (\in W), & \text{if } m = 1. \\ tstst\ (\in W), & \text{if } m \in \{2, \dots, \frac{n}{2} - 1\}. \\ tsts^{n/2}t = tss^{n/2}t = tsts^{n-1}\ (\in W), & \text{if } m = n/2. \\ tststs^{n-1} = ttsts^{n-1} = tsts^{n-1}\ (\in W) & \text{if } m \in \{\frac{n}{2} + 1, \dots, n-1\}. \end{cases}$$

Because $|W| \leq n + 2n = 3n = |S|$ we have that $\mathcal{P}$ is a presentation for S. $\square$

Next we have the last case of this section.

Lemma 4.9 *For n even, let*

$$\tau = \begin{pmatrix} 1 & 2 & 3 & \dots & \frac{n}{2} & \frac{n}{2}+1 & \frac{n}{2}+2 & \frac{n}{2}+3 & \dots & n \\ 1 & 1 & 1 & \dots & 1 & \frac{n}{2}+1 & 1 & 1 & \dots & 1 \end{pmatrix} \in \mathcal{T}_n, \ \textit{for } n \geq 4.$$

The semigroup $S = \langle \sigma, \tau \rangle$ is represented by

$$\mathcal{P} = \langle s, t \mid s^n = 1, t^2 = t, (ts^{\frac{n}{2}})^2 = t, stst = tst, \\ ts^i t = tst, i \in \{2, 3, \dots, \tfrac{n}{2} - 1\} \rangle.$$

PROOF. The semigroup S is regular but not completely regular because τ has neither periodic image nor conjugate periodic image. S satisfies $b - a = n/2$. Therefore $|S| = n(n+2)$.

Let us associate σ with s and τ with t. The relations $s^n = 1$ and $t^2 = t$ hold. We have that τ satisfies $b - a = n/2$. Thus $\tau\sigma^{n/2} = \overline{\tau}$ (see Lemma 2.44) and then $(ts^{n/2})^2 = t$ holds. Because $\mathsf{im}\,(\tau\sigma) = \{2, \frac{n}{2} + 2\}$ we have $\tau\sigma\tau = [1, 1, 1, \dots, 1]$ and therefore $stst = tst$ holds. Let $i \in \{2, \dots, \frac{n}{2} - 1\}$. Then $\mathsf{im}\,(\tau\sigma^i) = \{1\sigma^i, (\frac{n}{2} + 1)\sigma^i\} \subset K_1$. Thus $\tau\sigma^i\tau = [1, 1, 1, \dots, 1]$ and this implies that $ts^i t = tst$ holds.

Note that $ts^{n/2}ts^{n/2} = t$ is equivalent to $ts^{n/2}t = ts^{n/2}$.

Let $i \in \{\frac{n}{2} + 1, \dots, n-1\}$. Then $i = \frac{n}{2} + j$, with $j \in \{1, \dots, \frac{n}{2} - 1\}$. Therefore

$ts^it = ts^{n/2}s^jt = ts^{n/2}ts^jt = ts^{n/2}tst = ttst = tst$. Thus $ts^it = tst$ holds, for all $i \in \{1, \ldots, n-1\} \setminus \{\frac{n}{2}\}$.

Let $W = \{s^j : j \in \{0, \ldots, n-1\}; s^kts^l : k, l \in \{0, \ldots, n-1\}; tsts^m : m \in \{0, \ldots, n-1\}\}$ and $A = \{s, t\}$.

Take $u \in A^*$.

• If $|u| = 0$ then $u = 1 = s^0 \in W$.

• Suppose that for all $r \in \mathbb{N}$, if $u \in A^*$ is such that $|u| < r$ then there is $u' \in W$ such that $u = u'$ follows from the relations of $\mathcal{P}$. Take $v \in A^*$ such that $|v| = r$. Then $v = uw$, where $|u| = r - 1$ and $w \in A$. By assumption, there is $u' \in W$ such that $u = u'$ is a consequence of the relations of $\mathcal{P}$. We shall prove that there is $v' \in W$ such that $u'w = v'$ is a consequence of the relations of $\mathcal{P}$. There are several different possibilities:

(a) Suppose that $u' = s^j$, for some $j \in \{0, \ldots, n-1\}$.

(1) If $w = s$ then $u'w = s^{j+1} \in W$.

(2) If $w = t$ then $u'w = s^jt = s^jts^0 \in W$.

(b) Suppose that $u' = s^kts^l$, for some $k, l \in \{0, \ldots, n-1\}$.

(1) If $w = s$ then $u'w = s^kts^{l+1} \in W$.

(2) If $w = t$ then

$$u'w = s^kts^lt = \begin{cases} s^ktt = s^kt \ (\in W), & \text{if } l = 0. \\ s^ktst = tst \ (\in W), & \text{if } l \in \{1, \ldots, n-1\} \setminus \{\frac{n}{2}\}. \\ s^kts^{n/2}t = s^kts^{n/2} \ (\in W), & \text{if } l = \frac{n}{2}. \end{cases}$$

(c) Suppose that $u' = tsts^m$, for some $m \in \{0, \ldots, n-1\}$.

(1) If $w = s$ then $u'w = tsts^{m+1} \in W$.

(2) If $w = t$ then

$$u'w = tsts^m t = \begin{cases} tstt = tst \ (\in W), & \text{if } m = 0. \\ tstst = ttst = tst \ (\in W), & \text{if } m \in \{1, \ldots, n-1\} \setminus \{\frac{n}{2}\}. \\ tsts^{n/2}t = tsts^{n/2} \ (\in W), & \text{if } m = n/2. \end{cases}$$

From $|W| \leq n + n^2 + n = n(n+2) = |S|$ we conclude that $\mathcal{P}$ is a presentation for S. □

2 Future work

The set of cases/examples of the previous section constitute some sort of database which will allow us to answer the obvious question: *"Given $S = \langle \sigma, \tau \rangle$, can we provide, immediately and in an algorithmic way, a presentation for S?"*

It is clear that, at this stage, given S, we can decide if it has periodic image, if it has conjugate periodic image, if it is block conjugate, if it has block structure and we can also determine its kernel type and its image distance. With all this information and possibly a further description of its kernel, we would like to find an algorithm that gives us a presentation for S.

For example, if S is regular (in which case S does not have periodic image) then we can take its generator of rank two to be an idempotent and therefore $t^2 = t$ holds. If S contains the constant maps (in which case S does not have conjugate periodic image) then we must have the relation $sts^u t = ts^u t$, for some $u \in \{0, \ldots, n-1\}$. If S has v-block structure, for some v, then the relation $s^v t = t$ holds. If we have taken the generator of rank two to be an idempotent and if S is w-block conjugate,then we have that the relation $(s^w t)^2 = t$. These are just some examples of relations that must hold in a presentation for S, if this semigroup satisfies certain properties that we have described throughout this document.

Nonetheless, we still need to further describe and understand certain (still to be defined) properties in order to achieve this goal.

Chapter 5

GAP and applications

Maybe the most common description of GAP (which stands for "Groups, Algorithms and Programing") is that it is a computer algebra system for computational discrete algebra with particular emphasis on, but not restricted to, computational group theory.

GAP was originally designed to be a computational tool that dealt with groups and group theory. Consequently, it was only later that theory from other areas of algebra was implemented in GAP and semigroup theory was no exception. Up to the present day, the most efficient functions that deal with semigroups can be found in the share package MONOID (see [29]) which is only available in an older version of GAP , more precisely, GAP 3, release 4 (see [19]).

Lallement and McFadden in [17] give computational methods for studying transformation semigroups. These algorithms compute structural properties, through the computation of all Green's relations and a natural partial order on the set of the $\mathcal{D}$-classes. This document is still a reference for the structural study of finite semigroups and its implications.

Linton, Pfeiffer, Robertson and Ruškuc in [20] and [21] give the theory and the algorithms, respectively, which lead to the GAP 3 package MONOID.

In this chapter we shall describe some of the existing GAP facilities for semigroup theory which we used in the research described in this thesis. We also describe the algorithms that we implemented to increase GAP functionality and are a consequence of our research. The GAP code for these algorithms is given in the Appendix B.

1 Historical note

The GAP system was developed at the Lehrstuhl D für Mathematik (LDFM), in RWTH Aachen, Germany from 1986 until the year of 1997. On the 31st of July, 1997, Prof J. Neubüser retired from the LDFM and the development and maintenance of GAP was transferred to the (what was then called) School of Mathematical and Computational Sciences at the University of St Andrews.

In March, 2005, with the agreement of the GAP council, chaired then by Prof E. F. Robertson, and the GAP developers, the status and responsibilities of the "GAP headquarters" was passed to an equal collaboration of four different "GAP centres"; Aachen, Braunschweig, Fort Collins and St Andrews.

2 General semigroup theory and GAP

> *"Semigroups and, even more, monoids are not far away from being like groups. But, surprisingly, they have not received much attention yet in the form of GAP programs."*

We can find this statement at the beginning of the first chapter of the on-line manual of MONOID ([19]). Just from looking at the (on-line) reference manual of GAP ([12]), one will quite easily realise the vast difference between the resources available for semigroup theory and for group theory.

It is unquestionable that [20] is, in the study of structure of semigroup theory, one of the most relevant documents today. This document is also the basis for

[21], where we find all the formal algorithms which are implemented in the GAP 3 share package, MONOID.

In this section we shall briefly describe these documents [20] and [21].

It is known that, within a $\mathcal{D}$-class, all $\mathcal{R}$-classes have the same size, all $\mathcal{L}$-classes have the same size (Proposition 1.5), all $\mathcal{H}$-classes have the same size (Proposition 1.6) and that the intersection of any $\mathcal{L}$-class with any $\mathcal{R}$-class is an $\mathcal{H}$-class. These results are described in detail in [16].

Maybe one of the most surprising results in [20] is the following: "Let $S \leq \mathcal{T}_n$ be a transformation semigroup. If we take two $\mathcal{R}$-classes (resp., $\mathcal{L}$-classes) in S which have the same set of images (resp., set of kernels) then these $\mathcal{R}$-classes (resp. $\mathcal{L}$-classes) have the same size."

In these two papers the authors developed a new paradigm for effective computation with finite transformation semigroups. As is standard, the building blocks of semigroups are its $\mathcal{D}$-classes. In this work, the authors take the building blocks to be the $\mathcal{R}$-classes of the semigroup. The basic idea is to represent an $\mathcal{R}$-class by a *data structure* which is composed by a representative of the class, the list of images of each of the elements of the class, a list of multipliers (which are partial bijections) that allow us to go from the set of images of a transformation to any other and a group (which is represented as a permutation group) called the *right generalised Scützenberger group*. This data structure allows us to compute the size of the $\mathcal{R}$-class and all its elements.

There is also an analogous data structure for $\mathcal{L}$-classes.

As a consequence, an $\mathcal{H}$-class has a data structure composed of a representative of the class and a permutation group, which is the intersection of the right and the left generalised Schützenberger groups (from the data structure of the $\mathcal{R}$- and $\mathcal{L}$-class, respectively).

A data structure for a $\mathcal{D}$-class is composed of a representative and the data structures of the $\mathcal{R}$- and $\mathcal{L}$-classes which contain this element.

These data structures allow us to answer all the usual questions, for example, the size of an $\mathcal{H}$-,$\mathcal{L}$-,$\mathcal{R}$- or $\mathcal{D}$-class, compute their elements, test for membership, regularity of a $\mathcal{D}$-class, etc.

The efficiency of these algorithms when implemented in GAP is due to the already existing and very efficient machinery for dealing with permutation groups and also to the efficiency of the algorithms for semigroup actions introduced in [21]. The study of semigroups with GAP using these data structures is much more efficient than any other method known to this date.

3 Algorithms

3.1 GAP and structural study

In this section we present the formal algorithms which result from the theory developed in Chapter 2.

Throughout this section we have that $S = \langle \sigma, \tau \rangle$, where $\sigma = (1\ 2\ \ldots\ n)$ and $\tau \in \mathcal{T}_n$, such that $\mathsf{im}\,(\tau) = \{a, b\}$ (with $a < b$).

ALGORITHM 1 *(v-block structure)*: Given τ and a non-zero integer v which divides n, we determine if τ has v-block structure (See Definition 2.57).

1.1 [Initialise I.] Set $x \leftarrow 1$ and $q \leftarrow n/v$.

1.2 [Initialise II.] Set $y \leftarrow x\tau$ and $i \leftarrow 1$.

1.3 [Evaluate.] If $(x\sigma^{id})\tau \neq y$ then return false.

1.4 [Loop I.] Set $i \leftarrow i + 1$. If $i < q$ then go to 1.3.

1.5 [Loop II.] Set $x \leftarrow x + 1$. If $x < d$ then go to 1.2.

1.6 [Terminate.] Otherwise, return true.

Remark 5.1 Note that we are choosing v such that v divides n, because, ultimately, we are only interested in the minimum v such that τ has v-block structure. In the case where v is this minimum, we have that v divides n. For this, note that τ has v-block structure and n-block structure. Therefore, by Lemma 2.68, it also has $\gcd(n, v)$-block structure. But by definition of v, we have $\gcd(n, v) = v$ which means that v divides n.

Therefore we can just test natural numbers greater than 1 (because $\mathsf{rank}(\tau) = 2$) which divide n.

ALGORITHM 2 *(w-block conjugate)*: Given τ and a non-zero integer w which divides n such that n/w is even, we determine if τ is w-block conjugate (See Definition 2.48).

2.1 [Initialise I.] Set $x \leftarrow 1$ and $q \leftarrow n/w$.

2.2 [Initialise II.] Set $y \leftarrow x\tau$ and $i \leftarrow 1$.

2.3 [Evaluate.] If $(x\sigma^{(2i-1)w})\tau = y$ or $(x\sigma^{2iw})\tau \neq y$ then return false.

2.4 [Loop I] Set $i \leftarrow i + 1$. If $2i < q$ then go to 2.3.

2.5 [Loop II.] Set $x \leftarrow x + 1$. If $x < w$ then go to 2.2.

2.6 [Terminate.] Otherwise, return true.

Remark 5.2 Similarly to the previous remark, we ought to note that we are only choosing w such that w divides n and n/w is even, because, ultimately, we are only interested in the minimum w such that τ is w-block conjugate. In the case where w is this minimum, we have that τ has $2w$-block structure (by Lemma 2.70). By the previous remark, we have that $2w$ divides n and consequently w divides n. Because τ is w-block conjugate and noting that $x = x\sigma^{\frac{n}{w}n}$, we can conclude that n/w is even, by Remark 2.53.

Therefore we can just test natural numbers which divide n, where this quotient is even.

ALGORITHM 3 *(Periodic image)*: Given a transformation τ, we determine if it has periodic image (See Definition 2.5).

3.1 [Trivial cases.] If n is prime then return false.

Set $d \leftarrow \gcd(n, b-a)$. If $d = 1$ then return false.

If $a\tau = a$ and $b\tau = b$ (and so τ is idempotent) then return false.

3.2 [Delegate.] Using Algorithm 1, with $v = d$, return the result.

ALGORITHM 4 *(Conjugate periodic image)*: Given τ, we determine if τ has conjugate periodic image.

4.1 [Trivial cases.] If n is odd then return false.

Set $d \leftarrow \gcd(n, b-a)$. If n/d is odd then return false.

4.2 [Delegate.] Using Algorithm 2, with $w = d$, return the result.

In the next algorithm we shall determine if a transformation has v-block structure (see Definition 2.57). More precisely, since any transformation $\tau \in \mathcal{T}_n$ of rank two has n-block structure, we determine (if it exists) the minimum v such that $1 < v < n$ and τ has v-block structure (see Definition 2.66 and Remark 2.59).

ALGORITHM 5 *(Block structure)*: Given τ, we determine the minimum number v such that τ has v-block structure.

5.1 [Trivial cases.] If n is prime then return true and $v = n$.

5.2 [Definitions.] Set $D = \{d_1, d_2, \ldots, d_r\} \setminus \{1\} \leftarrow$ "all divisors of n excluding 1". Note that D is ordered with the usual order of natural numbers.

5.3 [Delegate.] Loop over each element $d \in D$, using Algorithm 1 (with $v = d$). When true, return d.

The next algorithm investigates if a transformation is w-block conjugate and finds the minimum value of w. As referred before, there is a relation between the value v (from the previous algorithm) and w. If a transformation has v-block structure and it is w-block conjugate then $v = 2w$ (see Lemma 2.70).

ALGORITHM 6 *(Block conjugate)*: Given τ, we determine if this transformation is w-block conjugate and if so, we determine the minimum number w such that τ has w-block structure.

6.1 [Trivial cases.] If n is odd then return false.

If Algorithm 5 has been computed then return the output of Algorithm 2, with $w = v/2$.

6.2 [Definitions.] Set $D = \{d_1, d_2, \ldots, d_r\} \setminus \{n\} \leftarrow$ "all divisors of $n/2$". Note that D is ordered with the usual order of natural numbers.

6.3 [Delegate.] Loop over each element $d \in D$, using Algorithm 2 (with $w = d$). If this algorithm returns true, then also return d.

The next algorithm is just the application of Theorem 2.76. We ought to remember that the size of the $\mathcal{H}$-classes of rank two in S is either 1 or 2. Also, this size is invariant within the set of all $\mathcal{H}$-classes of rank two in S.

ALGORITHM 7 *(Size of $\mathcal{H}$-classes)*: Given τ, we determine the size of the $\mathcal{H}$-classes of rank 2 in the semigroup $S = \langle \sigma, \tau \rangle$.

7.1 [Delegate.] If Algorithm 3 returns false, then return 2. If Algorithm 3 returns true then, if Algorithm 6 returns true and $b - a = n/2$, return 2. Otherwise, return 1.

The next algorithm is based on Theorem 2.45.

ALGORITHM 8 *(Number of $\mathcal{L}$-classes)*: Given $S = \langle \sigma, \tau \rangle$, we determine the number of $\mathcal{L}$-classes of rank 2 in S.

8.1 [Trivial cases.] If n is odd then return n.

8.2 [Delegate.] If Algorithm 7 returns 2 and $b - a = n/2$, then return $n/2$. Otherwise return n.

The next procedure makes use of Theorems 2.72 and 2.73.

ALGORITHM 9 *(Number of $\mathcal{R}$-classes)*: Given $S = \langle \sigma, \tau \rangle$, we determine the number of $\mathcal{R}$-classes of rank 2 in S.

9.1 [Trivial cases.] If n is prime then return n.

9.2 [Delegate.] If Algorithm 6 returns true (and consequently w) and Algorithm 7 returns 2 then return w. If Algorithm 6 returns true (and consequently w) and Algorithm 7 returns 1 then return $2w$.

If Algorithm 6 returns false then, using Algorithm 5, return v.

The next algorithm allows us to compute the size of the semigroup S, making use of the previous algorithms. We implicitly use Corollaries 2.79, 2.80 and 2.81.

ALGORITHM 10 *(Size)*: Given $S = \langle \sigma, \tau \rangle$, we determine its size.

10.1 [Trivial cases.] If n is prime then return $2n(n + 1)$.

10.2 [Delegate.] If Algorithms 3 and 4 return false and Algorithm 5 returns v then,

(i) if Algorithm 2 returns false for $w = v/2$ and $b - a \neq n/2$ then return $2n(v+1)$;

(ii) if Algorithm 2 returns false for $w = v/2$ and $b - a = n/2$ or if Algorithm 2 returns true for $w = v/2$ and $b - a = n/2$ then return $n(v+2)$.

If Algorithm 4 returns true and Algorithm 5 returns v then return $n(v+1)$, if $b - a \neq n/2$ or return $n(\frac{v}{2}+1)$, if $b - a = n/2$.

If Algorithm 3 returns true and Algorithm 5 returns v then,

(i) if Algorithm 2 returns false for $w = v/2$ or $b - a \neq n/2$ then return $n(v+2)$;

(ii) if Algorithm 2 returns true for $w = v/2$ and $b - a = n/2$ then return $n(\frac{v}{2}+2)$.

Using Theorem 2.10, we describe the process of determining if a semigroup is regular.

ALGORITHM 11 *(Regular semigroup)*: Given $S = \langle \sigma, \tau \rangle$, we determine if S is regular.

11.1 [Delegate.] If Algorithm 3 returns true then return false. Otherwise, return true.

Based on Theorem 2.31, we specify the algorithm to verify if a semigroup is completely regular.

ALGORITHM 12 *(Completely regular semigroup)*: Given $S = \langle \sigma, \tau \rangle$, we determine if S is completely regular.

12.1 [Delegate.] For τ, return the result of Algorithm 4.

3.2 GAP and isomorphism study

In this section we look at the applications in GAP resulting from the theory developed in Chapter 3 and the methods used in Chapter 6.

As described in Theorem 3.16, if S and T are semigroups generated by the permutation $\sigma = (1\,2\,3\ \ldots\ n)$ and a transformation of rank two in $\mathcal{T}_n$ such that S and T contain constant maps (so they are not completely regular, or equivalently, do not have conjugate periodic image) then S is isomorphic to T if and only if S is conjugate to T. Furthermore, Theorem 3.19 gives us a way of finding that conjugating element (if it exists).

The next algorithm allows us to decide if two given semigroups S and T (as above) are isomorphic.

In order to have a non-trivial problem, we take S and T such that these semigroups have the same size, same Green's structure, same number of idempotents, same kernel type and same image distance (see Definition 3.12).

ALGORITHM 13 *(Conjugator)*: Given two semigroups S and T as described above, we determine if S is isomorphic to T and, if so, we provide an element $\pi \in \mathcal{S}_n$ such that $\pi^{-1}S\pi = T$. We assume that the transformation $\tau \in S$ of rank 2 is such that $1\tau = 1$.

13.1 [Initialise I.] Set $\Sigma_\tau \leftarrow \{x \in \{1,\ldots,n\} : x\tau = x\}$ and $P \leftarrow \{k \in \mathbb{N} : k \leq n, \gcd(n,k) = 1\}$. Define $k \leftarrow$ "the first element in P".

13.2 [Initialise II.] Define $x \leftarrow$ "the first element in Σ_τ".

13.3 [Evaluate] Define $\pi \in \mathcal{S}_n$ as follows,

$$(x\sigma^i)\pi = 1\sigma^{ik}, \text{for all } i \in \{1,\ldots,n\}.$$

If $\pi^{-1}\tau\pi \in T$ then return true.

13.4 [Loop I] Set $x \leftarrow$ "the next element of Σ_τ". Go to 13.3.

13.5 [Loop II] Set $k \leftarrow$ "the next element of P". Go to 13.2.

13.6 [Terminate.] Otherwise, return false.

The next algorithm describes the process of deciding if a (given) presentation represents a (given) semigroup. This algorithm is based on Corollary 2.22 and it will be heavily used to analyse the examples in Chapter 6.

ALGORITHM 14 *(Presentation)*: Given a semigroup $S = \langle \sigma, \tau \rangle$ and a presentation $\mathcal{P} = \langle A \mid R \rangle$, we determine if $\mathcal{P}$ is a presentation for S.

14.1 [Initialise I.] Set $I \leftarrow$ "the list of all elements of rank 2 in S" and $P \leftarrow \{k \in \mathbb{N} : k \leq n, \gcd(n, k) = 1\}$. Set $k \leftarrow$ "the first element in P".

14.2 [Initialise II.] Set $\alpha \leftarrow$ "the first element in I".

14.3 [Evaluate.] If the pair (σ^k, α) satisfies all the relations in R then return true and return the pair (σ^k, α).

14.4 [Loop I.] Set $\alpha \leftarrow$ "the next element in I". Go to 14.3.

14.5 [Loop II] Set $k \leftarrow$ "the next element in P". Go to 14.2.

14.6 [Terminate.] Otherwise, return false.

We ought to note that all these algorithms were implemented in GAP and these implementations can be found in Appendix B. It is obvious that these algorithms are only applicable to these very specific semigroups. Still, there is a remarkable difference of efficiency in GAP between the general methods for semigroups and these methods.

Chapter 6

Description of cases of small degree

In this chapter we describe some of the particular cases which we have studied very thoroughly. This study was done in a very early stage of our work and therefore it used some "brute force" methods, which implied the computation of all the elements of the semigroups in question.

Our aim is to make use of the theory developed so far in a way that illustrates the potential of the results presented up to this stage. The study of these cases of small degree was the starting point for conjecturing and proving most of the results in our work.

We shall use some results of Chapter 2 very frequently. More precisely, we will use Theorems 2.10, 2.31, 2.45, 2.72, 2.73, 2.76 and 2.78 (and its corollaries). We shall also use some of the results from Chapter 2, namely Lemmas 3.10 and 3.14, Theorem 3.16 and the GAP implementation of Theorem 3.19 (described in Appendix B). We also provide presentations for each of these semigroups. As an informal note, we ought to remark that some of these presentations are not the most efficient ones, but are written in a way such that each of the relations from a presentation tries to be as "informative" as possible. Although we do not dive into any considerations about this information, these redundant presentations gave us some insight on how to generalise onto some presentations presented in Chapter 4.

We need to introduce some notation for this chapter. For a given $n \in \mathbb{N}$, we write Ω_n as the set of all semigroups generated by $\sigma = (1\ 2\ \ldots\ n)$ and a transformation $\tau \in \mathcal{T}_n$ of rank two. When picturing an eggbox diagram (see Section 2 in Chapter 1) we write a 1 (one) in a cell if the associated $\mathcal{H}$-class contains an idempotent and we write a 0 (zero) otherwise.

1 Study of Ω_3

In this section we describe the simplest non-trivial case. When computing all the semigroups of Ω_3 in GAP we conclude that

$$\Omega_3 = \{S_1 = \langle \sigma, [1,2,1] \rangle, \text{ with } \sigma = (1\ 2\ 3)\}.$$

This semigroup has neither periodic image nor conjugate periodic image, so, it is regular but not completely regular. It has 3-block structure and it does not satisfy the condition $b - a = n/2$. We can then conclude that $|S_1| = 24$ and the Green's structure of S_1 is shown in Figure 6.1.

1

3 elements

1	1	0
1	0	1
0	1	1

18 elements

1	1	1

3 elements

Figure 6.1: Green's structure of S_1

A presentation for S_1 is

$$\mathcal{P}_1 = \langle s, t \mid s^3 = 1, t^2 = t, (tst)^2 = t, sts^2t = ts^2t, ts^2tst = ts^2ts \rangle.$$

As mentioned before, this is a very simple case and the only interesting fact about this case is that this semigroup is regular but is neither completely regular nor inverse.

2 Study of Ω_4

Doing some straightforward computations with GAP , we get

$$\Omega_4 = \{S_i = \langle \sigma, \tau_i \rangle, \text{ where } \sigma = (1\ 2\ 3\ 4) \text{ and } i \in \{1, \ldots, 6\}\},$$

where

- $\tau_1 = [1, 2, 1, 1]$,
- $\tau_2 = [1, 2, 2, 1]$,
- $\tau_3 = [1, 2, 1, 2]$,
- $\tau_4 = [1, 1, 3, 1]$,
- $\tau_5 = [1, 1, 3, 3]$ and
- $\tau_6 = [1, 3, 1, 3]$.

We consider each of these semigroups in turn.

Take $S_1 = \langle \sigma, \tau_1 \rangle$. This semigroup has neither periodic image nor conjugate periodic image. Thus it is regular but not completely regular. It has 4-block structure and $b - a \neq n/2$. We can then deduce that $|S_1| = 2.4(4 + 1) = 40$. In Figure 6.2 we can find the Green's structure of S_1.

1

4 elements

1	1	0	0
1	0	0	1
0	0	1	1
0	1	1	0

32 elements

1	1	1	1

4 elements

Figure 6.2: Green's structure of S_1

A possible presentation for this semigroup is

$$\mathcal{P}_1 = \langle s, t \mid s^4 = 1, t^2 = t, sts^2t = ts^2t, ts^3t = ts^2t, ts^2tst = ts^2ts \rangle.$$

Take $S_2 = \langle \sigma, \tau_2 \rangle$. The semigroup S_2 has neither periodic image nor conjugate periodic image. Therefore it is regular but not completely regular. It is 2-block conjugate and $b - a \neq n/2$. Hence $|S_2| = 4(4+2) = 24$. Its Green's structure is in the next figure.

1

4 elements

1	0	1	0
0	1	0	1

16 elements

1	1	1	1

4 elements

Figure 6.3: Green's structure of S_2

This semigroup has the following presentation

$$\mathcal{P}_2 = \langle s, t \mid s^4 = 1, t^2 = t, (s^2t)^2 = t, sts^3t = ts^3t, ts^3tst = ts^3ts \rangle.$$

Take $S_3 = \langle \sigma, \tau_3 \rangle$. This semigroup does not have periodic image but it has conjugate periodic image. Hence S_3 is completely regular. It is 1-block conjugate (hence it has 2-block structure) and $b - a \neq n/2$. Therefore we have $|S_3| = 4(2+1) = 12$. The Green's structure of S_3 is below.

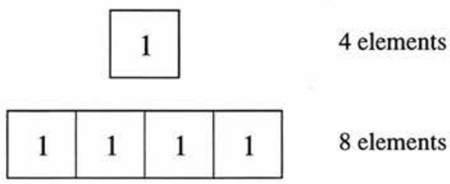

Figure 6.4: Green's structure of S_3

A presentation for S_3 is

$$\mathcal{P}_3 = \langle s, t \mid s^4 = 1, t^2 = t, (st)^2 = t \rangle.$$

Take $S_4 = \langle \sigma, \tau_4 \rangle$. S_4 has neither periodic image nor conjugate periodic image (it is regular but not completely regular). It has 4-block structure and $b-a = n/2$. Thus $|S_4| = 4(4+2) = 24$ and its Green's structure is represented in Figure 6.5.

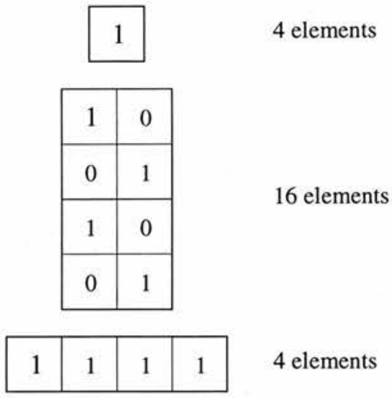

Figure 6.5: Green's structure of S_4

This semigroup has the following presentation

$$\mathcal{P}_4 = \langle s, t \mid s^4 = 1, t^2 = t, (ts^2)^2 = t, stst = tst, ts^3t = tst, tsts^2t = tsts^2 \rangle.$$

Take $S_5 = \langle \sigma, \tau_5 \rangle$. This semigroup does not have periodic image but it has conjugate periodic image. It is 2-block conjugate (hence it has a 4-block structure) and $b-a = n/2$. Consequently, $|S_5| = 4(2+1) = 12$. Figure 6.6 shows the Green's structure of S_5.

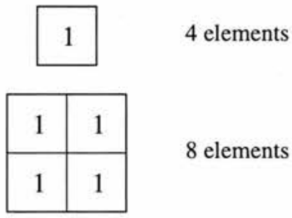

Figure 6.6: Green's structure of S_5

The semigroup S_5 is represented by

$$\mathcal{P}_5 = \langle s, t \mid s^4 = 1, t^2 = t, (s^2t)^2 = t, (st)^2 = st, ts^2 = s^2t \rangle.$$

Take $S_6 = \langle \sigma, \tau_6 \rangle$. It has periodic image and therefore it is not regular. It is 1-block conjugate (thus is has 2-block structure) and $b-a = n/2$. As a consequence we have $|S_6| = 4(1+2) = 12$. The Green's structure of S_6 is shown in Figure 6.7.

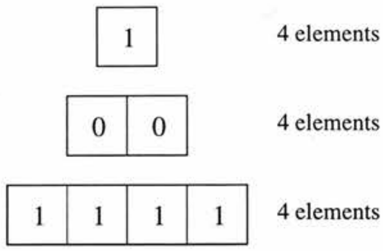

Figure 6.7: Green's structure of S_6

A presentation for S_6 is

$$\mathcal{P}_6 = \langle s, t \mid s^4 = 1, t = s^2t, sts^2t = ts^2t, st = ts^2 \rangle.$$

Just based on the size, the display of the Green's structure and the regularity of each of the previous semigroups, it is easy to conclude that there are no isomorphic semigroups in Ω_4. The next table sums up all the information we have got about the semigroups in Ω_4.

	Size	Number of $\mathcal{D}$-classes	$\lvert\mathfrak{L}\rvert$	$\lvert\mathfrak{R}\rvert$	Regular	Completely Regular
S_1	40	3	4	4	yes	no
S_2	24	3	4	2	yes	no
S_3	12	3	4	1	yes	yes
S_4	24	3	2	4	yes	no
S_5	12	2	2	2	yes	yes
S_6	12	3	2	1	no	no

3 Study of Ω_5

Using GAP we deduce that

$$\Omega_5 = \{S_i = \langle \sigma, \tau_i \rangle : \sigma = (1,2,3,4,5), i \in \{1,\ldots,6\}\},$$

where

- $\tau_1 = [1, 2, 1, 1, 1]$,
- $\tau_2 = [1, 2, 2, 1, 1]$,
- $\tau_3 = [1, 2, 1, 2, 1]$,
- $\tau_4 = [1, 1, 3, 1, 1]$,
- $\tau_5 = [1, 1, 3, 3, 1]$ and
- $\tau_6 = [1, 1, 3, 1, 3]$.

We can easily observe that none of these semigroups has periodic image nor conjugate periodic image. This means that all the semigroups of Ω_5 are regular but not completely regular. All these semigroups have 5-block structure. Also, since $n = 5$, the condition $b - a = n/2$ does not hold. Consequently, all the semigroups in Ω_5 have size 60.

It is not hard to check that the semigroups S_1, S_2, S_4 and S_6 have the Green's structure shown in Figure 6.8 (for a "convenient rearrangement" of $\mathcal{L}$ and $\mathcal{R}$-classes).

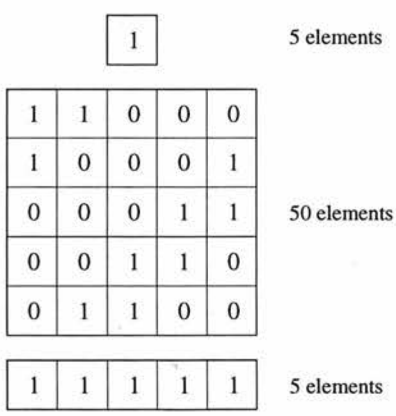

Figure 6.8: Green's structure of S_1, S_2, S_4 and S_6

The Green's structure of the semigroups S_3 and S_5 is pictured in Figure 6.9.

1	5 elements

1	1	1	1	0	
1	1	1	0	1	
1	1	0	1	1	50 elements
1	0	1	1	1	
0	1	1	1	1	

1	1	1	1	1	5 elements

Figure 6.9: Green's structure of S_3 and S_5

The following are presentations for S_1, S_2, S_3, S_5 and S_6.

(i) $\mathcal{P}_1 = \langle s, t \mid s^5 = 1, t^2 = t, (ts^2t)^2 = t, sts^2t = ts^2t, ts^3t = ts^2t, ts^4t = ts^2t, ts^2tst = ts^2ts \rangle$;

(ii) $\mathcal{P}_2 = \langle s, t \mid s^5 = 1, t^2 = t, (ts^2t)^2 = t, sts^3t = ts^3t, tst = ts^3ts, ts^4t = ts^3t, ts^3tst = ts^3ts \rangle$;

(iii) $\mathcal{P}_3 = \langle s, t \mid s^5 = 1, t^2 = t, (ts^2)^2 = ts^2, (tst)^2 = t, sts^4t = ts^4t, ts^3t = tst, ts^4tst = ts^4ts \rangle$;

(iv) $\mathcal{P}_4 = \langle s, t \mid s^5 = 1, t^2 = t, (ts^2t)^2 = t, stst = tst, ts^3t = tst, ts^4t = tst, tsts^2t = tsts^2 \rangle$;

(v) $\mathcal{P}_5 = \langle s, t \mid s^5 = 1, t^2 = t, (ts^4)^2 = ts^4, (ts^2t)^2 = t, sts^3t = ts^3t, ts^2t = tst, ts^3ts^2t = ts^3ts^2 \rangle$;

(vi) $\mathcal{P}_6 = \langle s, t \mid s^5 = 1, t^2 = t, (ts^4t)^2 = t, stst = tst, ts^2t = tsts^2, ts^3t = tst, tsts^2t = tsts^2 \rangle$.

With these presentations we could prove which of these semigroups are isomorphic, but our aim is to use the theory developed.

The next table shows us all the relevant data about the semigroups of Ω_5. Recall we know that all these semigroups have size 60, have 3 $\mathcal{D}$-classes and are regular but not completely regular.

	Kernel type	#{idemp of rank 2}	Possibly isomorphic to...
S_1	$1^1 4^1$	10	S_4
S_2	$2^1 3^1$	10	S_6
S_3	$2^1 3^1$	20	S_5
S_4	$1^1 4^1$	10	S_1
S_5	$2^1 3^1$	20	S_3
S_6	$2^1 3^1$	10	S_2

We used Lemma 3.10 to conclude that the only possible isomorphisms are between S_1 and S_4, S_2 and S_6, S_3 and S_5.

The GAP implementation of the method illustrated in Example 3.21 (see Appendix B) allows us to conclude that $S_1 \cong S_4$, $S_2 \cong S_4$ and $S_3 \cong S_5$, all these isomorphisms being conjugations by $\pi = (2\ 3\ 5\ 4) \in \mathcal{S}_5$.

4 Study of Ω_6

Using GAP we can conclude that Ω_6 has 21 semigroups. We shall describe each of these semigroups individually so that some of the results from previous chapters become clearer. Throughout this section we have $\sigma = (1\ 2\ 3\ 4\ 5\ 6)$.

Let $S_1 = \langle \sigma, [1,4,1,4,1,4] \rangle$. This semigroup has conjugate periodic image, hence it is completely regular. It is 1-block conjugate (thus it has 2-block structure) and it satisfies $b - a = n/2$. Hence $|S_1| = 6(1+1) = 12$. The Green's structure for this semigroup is represented in Figure 6.10.

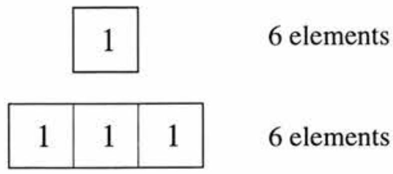

Figure 6.10: Green's structure of S_1

A presentation for S_1 is

$$\mathcal{P}_1 = \langle s, t \mid s^6 = 1, t^2 = t, (st)^2 = t, ts^3 = st \rangle.$$

Let $S_2 = \langle \sigma, [1, 2, 1, 2, 1, 2] \rangle$. The semigroup S_2 has conjugate periodic image (it is completely regular), it is 1-block conjugate and $b - a \neq n/2$. Hence $|S_2| = 6(2 + 3) = 18$. The Green's structure figure for S_2 follows.

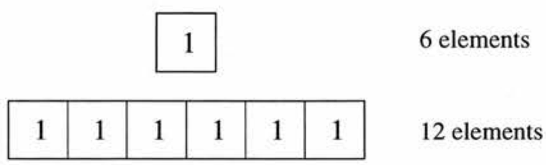

Figure 6.11: Green's structure of S_2

A presentation for this semigroup is

$$\mathcal{P}_2 = \langle s, t \mid s^6 = 1, t^2 = t, (st)^2 = t \rangle.$$

Let $S_3 = \langle \sigma, [1, 3, 1, 3, 1, 3] \rangle$. This semigroup has periodic image (it is not regular) and it is 1-block conjugate. This semigroup does not satisfy the condition $b - a = n/2$. Hence $|S_3| = 6(2 + 2) = 24$. Note that this semigroup has $|[\mathcal{H}]| = 1$ (see Theorem 2.76). Figure 6.12 shows the Green's structure of S_3.

1	6 elements

0	0	0	0	0	0
0	0	0	0	0	0

12 elements

1	1	1	1	1	1

6 elements

Figure 6.12: Green's structure of S_3

A presentation for S_3 is

$$\mathcal{P}_3 = \langle s, t \mid s^6 = 1, s^2t = t, st^2 = t^2, tst = t^2s^2 \rangle.$$

Let $S_4 = \langle \sigma, [1, 1, 1, 4, 4, 4] \rangle$. The semigroup S_4 has conjugate periodic image (it is completely regular), it is 3-block conjugate and it satisfies $b - a = n/2$. Then we can conclude that $|S_4| = 6(3 + 1) = 24$.

1	6 elements

1	1	1
1	1	1
1	1	1

18 elements

Figure 6.13: Green's structure of S_4

A presentation for this semigroup is

$$\mathcal{P}_4 = \langle s, t \mid s^6 = 1, t^2 = t, (s^3t)^2 = t, (st)^2 = st, (s^2t)^2 = s^2t, ts^3 = s^3t \rangle.$$

Let $S_5 = \langle \sigma, [1, 1, 4, 1, 1, 4] \rangle$. This semigroup has periodic image and therefore it is not regular. It has a 3-block structure and it satisfies $b - a = n/2$. Hence

$|S_5| = 6(3+2) = 30$. Note that $|[\mathcal{H}]| = 1$. The Green's structure for S_5 is in Figure 6.14.

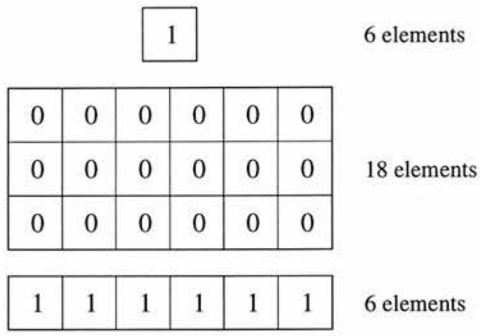

Figure 6.14: Green's structure of S_5

The following is a presentation for this semigroup:

$$\mathcal{P}_5 = \langle s, t \mid s^6 = 1, st^2 = t^2, s^3t = t, tst = t^2, ts^2t = t^2s^3, t^2s^2t = t^2s^3 \rangle.$$

The next table resumes all the information we have about Ω_6, up to this stage.

	Size	Number of $\mathcal{D}$-classes	$\lvert\mathfrak{L}\rvert$	$\lvert\mathfrak{R}\rvert$	Regular	Completely Regular
S_1	12	2	3	1	yes	yes
S_2	18	2	6	1	yes	yes
S_3	24	3	6	2	no	no
S_4	24	2	3	3	yes	yes
S_5	30	3	6	3	no	no

It is clear from the information on this table that, so far, there are no isomorphic semigroups.

Let $S_6 = \langle \sigma, [1, 2, 2, 2, 1, 1] \rangle$. This semigroup has neither periodic image nor conjugate periodic image (it is regular but not completely regular). It is 3-block conjugate (hence it has a 6-block structure) and $b - a \neq n/2$. Thus $|S_6| = 6(6+2) = 48$ and its Green's structure is represented in Figure 6.15.

			1			6 elements
1	0	0	0	1	0	
0	0	0	1	0	1	36 elements
0	0	1	0	1	0	
1	1	1	1	1	1	6 elements

Figure 6.15: Green's structure of S_6

A possible presentation for S_6 is

$$\mathcal{P}_6 = \langle s, t \mid s^6 = 1, t^2 = t, (s^3t)^2 = t, sts^4t = ts^4, ts^2t = ts^4ts, ts^4tst = ts^4ts \rangle.$$

Let $S_7 = \langle \sigma, [1, 2, 1, 1, 2, 1] \rangle$. The semigroup S_7 has neither periodic image nor conjugate periodic image. It has a 3-block structure and $b - a \neq n/2$. As a consequence we have $|S_7| = 2.6(3 + 1) = 48$. In Figure 6.16 we have the Green's structure for S_7.

			1			6 elements
1	1	0	1	1	0	
1	0	1	1	0	1	36 elements
0	1	1	0	1	1	
1	1	1	1	1	1	6 elements

Figure 6.16: Green's structure of S_7

A presentation for S_7 is

$$\mathcal{P}_7 = \langle s, t \mid s^6 = 1, t^2 = t, (tst)^2 = t, s^3t = t, sts^2t = ts^2t, ts^2tst = ts^2ts \rangle.$$

Let $S_8 = \langle \sigma, [1, 1, 3, 3, 3, 1] \rangle$. This semigroup has neither periodic image nor conjugate periodic image. It is 3-block conjugate and $b - a \neq n/2$. Consequently, $|S_8| = 2.6(3 + 1) = 48$.

1	6 elements

1	1	0	1	1	0	
1	0	1	1	0	1	36 elements
0	1	1	0	1	1	

1	1	1	1	1	1	6 elements

Figure 6.17: Green's structure of S_8

The following is a presentation for this semigroup;

$$\mathcal{P}_8 = \langle s, t \mid s^6 = 1, t^2 = t, (s^3t)^2 = t, tst = t, sts^2t = ts^2t, ts^5ts^2 = ts^2t \rangle.$$

Let $S_9 = \langle \sigma, [1, 1, 3, 1, 1, 3] \rangle$. S_9 has neither periodic image nor conjugate periodic image. It has 3-block structure and $b - a \neq n/2$. As before, $|S_9| = 48$. The Green's structure of semigroup S_9 is shown in Figure 6.18.

1	6 elements

1	0	1	1	0	1
0	1	1	0	1	1
1	1	0	1	1	0

36 elements

1	1	1	1	1	1

6 elements

Figure 6.18: Green's structure of S_9

A presentation for S_9 is

$$\mathcal{P}_9 = \langle s, t \mid s^6 = 1, t^2 = t, (ts^2t)^2 = t, stst = tst, s^3t = t, tsts^2t = tsts^2 \rangle.$$

Let $S_{10} = \langle \sigma, [1, 1, 1, 4, 1, 1] \rangle$. This semigroup has neither periodic image nor conjugate periodic image. It has 6-block structure and it satisfies the condition $b - a = n/2$. Thus $|S_{10}| = 6(6+2) = 48$. The Green's structure of S_{10} is different than that of the last four examples, as illustrated in Figure 6.19.

1	6 elements

1	0	0
0	0	1
0	1	0
1	0	0
0	0	1
0	1	0

36 elements

1	1	1	1	1	1

6 elements

Figure 6.19: Green's structure of S_{10}

A presentation for this semigroup is

$$\mathcal{P}_{10} = \langle s,t \mid \; s^6 = 1, t^2 = t, (ts^3)^2 = t, stst = tst, ts^2t = tst, \\ ts^3t = ts^3, ts^4t = tst, ts^5t = tst \,\rangle.$$

Let $S_{11} = \langle \sigma, [1,1,4,4,1,1] \rangle$. This semigroup has neither periodic image nor conjugate periodic image. It has 6-block structure and it satisfies the condition $b - a = n/2$. Thus $|S_{11}| = 6(6+2) = 48$. Its Green's structure can be found in Figure 6.20.

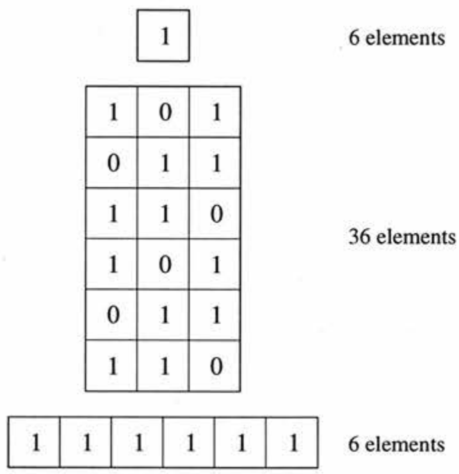

Figure 6.20: Green's structure of S_{11}

A presentation for S_{11} is

$$\mathcal{P}_{11} = \langle s,t \mid \; s^6 = 1, t^2 = t, (ts^3)^2 = t, stst = tst, ts^2t = ts^3, \\ ts^3t = ts^3, ts^4t = tst, ts^5t = t \,\rangle.$$

Let $S_{12} = \langle \sigma, [1,1,1,4,1,4] \rangle$. This semigroup has neither periodic image nor conjugate periodic image. It has 6-block structure and it satisfies the condition $b - a = n/2$. Therefore $|S_{12}| = 6(6+2) = 48$. Figure 6.21 represents the Green's structure of this semigroup.

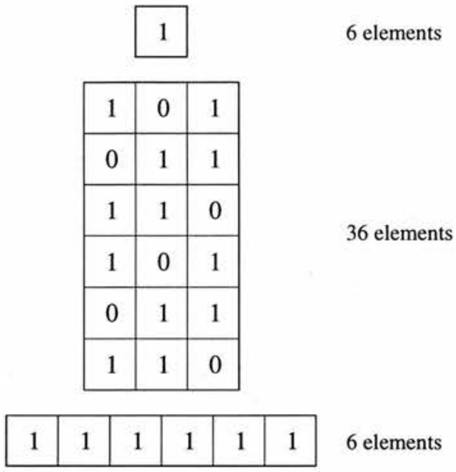

Figure 6.21: Green's structure of S_{12}

A presentation for this semigroup is

$$\mathcal{P}_{12} = \langle s, t \mid s^6 = 1, t^2 = t, (ts^3)^2 = t, stst = tst, (ts^2)^2 = ts^2 \rangle.$$

Let $S_{13} = \langle \sigma, [1, 4, 1, 4, 4, 1] \rangle$. This semigroup has neither periodic image nor conjugate periodic image. It has 6-block structure and it satisfies the condition $b - a = n/2$. Then we have $|S_{13}| = 6(6 + 2) = 48$. The Green's structure of this semigroup is pictured in Figure 6.22.

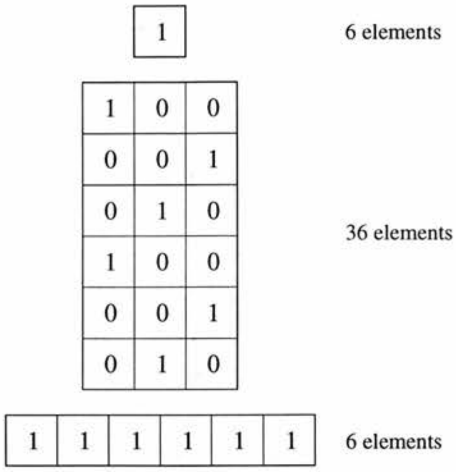

Figure 6.22: Green's structure of S_{13}

This semigroup is represented by

$$\mathcal{P}_{13} = \langle s, t \mid s^6 = 1, t^2 = t, (ts^3)^2 = t, stst = tst, ts^2ts^3 = tst \rangle.$$

We summarise all the information we have gathered so far in the next table. Note that all the semigroups $S_6, \ldots, S_{13}$ have size 48, are regular but not completely regular. Therefore they all have 3 $\mathcal{D}$-classes.

Also, because all these semigroups have size 48 we can conclude that none of these is isomorphic to $S_1, \ldots, S_5$.

	$\lvert\mathfrak{L}\rvert$	$\lvert\mathfrak{R}\rvert$	#{idemp of rank 2}	Kernel type	$\mu(S_i)$
S_6	6	3	6	3^2	1
S_7	6	3	12	$2^1 4^1$	1
S_8	6	3	12	3^2	2
S_9	6	3	12	$2^1 4^1$	2
S_{10}	3	6	6	$1^1 5^1$	3
S_{11}	3	6	12	$2^1 4^1$	3
S_{12}	3	6	12	$2^1 4^1$	3
S_{13}	3	6	6	3^2	3

The information in the table above allows us to exclude the existence of any possible isomorphism between $S_6, \ldots, S_{13}$, just based on the display of D_2, the kernel type (using Lemma 3.10) and the image distance μ (using Lemma 3.14). The only exception is the pair S_{11} and S_{12}. Both these semigroups have the same main properties and, so far, we have not developed any condition similar to the properties in Definitions 3.1 and 3.8. Using the notation of Theorem 3.19, we have $\Sigma_\tau = \{1, 4\}$, where $\tau = [1, 1, 4, 4, 1, 1]$ is the generator of rank two of S_{11}. We also have that $k = 5$ is the only number $1 < k < n$, coprime to n, where $n = 6$. Therefore, according to Theorem 3.19, or more precisely, to its GAP implementation (see Appendix B), we have that the permutations $(2\ 6)(3\ 5)$ and $(1\ 4)(2\ 3)(5\ 6)$ of $\mathcal{S}_6$ are the only possible $\pi \in \mathcal{S}_6$ such that $\pi^{-1} S_{11} \pi = S_{12}$. If $\pi = (2\ 6)(3\ 5)$ then $\pi^{-1} \tau \pi = [1, 1, 1, 4, 4, 1]$. Since $\sigma^5 \tau = [1, 1, 1, 4, 4, 1] \in S_{11}$ we conclude that this permutation π conjugates S_{11} into itself. If $\pi = (1\ 4)(2\ 3)(5\ 6)$ then $\pi^{-1} \tau \pi = [1, 1, 4, 4, 4, 4]$. But we also have $(\sigma\tau)^2 = [1, 1, 4, 4, 4, 4] \in S_{11}$ and therefore π also conjugates S_{11} into itself. Since these two semigroups do not have conjugate periodic images, they both contain the constant maps. Hence we can apply Theorem 3.16 to conclude that $S_{11} \cong S_{12}$ if and only if they are conjugate. However we have proved that there is no $\pi \in \mathcal{S}_6$ which conjugates S_{11} into S_{12} and therefore $S_{11} \ncong S_{12}$.

In summary, up to this point, there are no isomorphic semigroups in Ω_6.

The next eight semigroups have neither periodic image nor conjugate periodic image. Consequently, they are regular but not completely regular. They all have 6-block structure and satisfy $b - a \neq n/2$. Therefore they all have 84 elements.

We shall describe the Green's structure and give a presentation for each individual case.

Let $S_{14} = \langle \sigma, [1, 2, 1, 1, 1, 1] \rangle$. The Green's structure of this semigroup is shown in Figure 6.23.

1	6 elements

1	1	0	0	0	0
1	0	0	0	0	1
0	0	0	0	1	1
0	0	0	1	1	0
0	0	1	1	0	0
0	1	1	0	0	0

72 elements

1	1	1	1	1	1	6 elements

Figure 6.23: Green's structure of S_{14}

A presentation for this semigroup is

$$\begin{aligned}\mathcal{P}_{14} = \langle s, t \mid\ & s^6 = 1, t^2 = t, (tst)^2 = t, sts^2t = ts^2t, ts^3t = ts^2t, \\ & ts^4t = ts^2t, ts^5t = ts^2t, ts^2tst = ts^2ts \,\rangle.\end{aligned}$$

Let $S_{15} = \langle \sigma, [1, 2, 2, 1, 1, 1] \rangle$. This semigroup's Green's structure is in Figure 6.24.

1	6 elements

1	1	1	1	0	0
1	1	1	0	0	1
1	1	0	0	1	1
1	0	0	1	1	1
0	0	1	1	1	1
0	1	1	1	1	0

72 elements

1	1	1	1	1	1	6 elements

Figure 6.24: Green's structure of S_{15}

A possible presentation for S_{15} is

$$\begin{aligned}\mathcal{P}_{15} = \langle s,t \mid\ & s^6=1, t^2=t, (ts^2t)^2 = ts^2t, stst = tst, ts^3ts = tst,\\ & ts^4t = ts^3t, ts^5t = ts^3t, ts^3tst = ts^3ts\,\rangle.\end{aligned}$$

Let $S_{16} = \langle \sigma, [1,2,1,2,1,1]\rangle$. This semigroup's Green's structure is illustrated Figure 6.25.

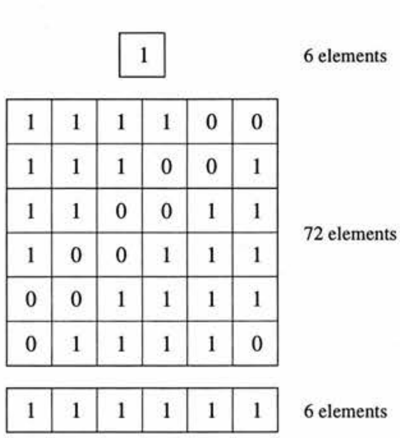

Figure 6.25: Green's structure of S_{16}

The semigroup S_{16} is represented by

$$\begin{aligned}\mathcal{P}_{16} = \langle s,t \mid\ & s^6=1, t^2=t, (tst)^2 = t, (ts^2)^2 = ts^2, sts^4t = ts^4t,\\ & ts^3t = tst, ts^5t = ts^4t, ts^4tst = ts^4ts\,\rangle.\end{aligned}$$

Let $S_{17} = \langle \sigma, [1,2,1,2,2,1]\rangle$. We represent in Figure 6.26 the Green's structure of this semigroup.

			1			6 elements
1	1	1	0	1	0	
1	1	0	1	0	1	
1	0	1	0	1	1	72 elements
0	1	0	1	1	1	
1	0	1	1	1	0	
0	1	1	1	0	1	
1	1	1	1	1	1	6 elements

Figure 6.26: Green's structure of S_{17}

A presentation for S_{17} is

$$\mathcal{P}_{17} = \langle s, t \mid s^6 = 1, t^2 = t, (ts^2)^2 = ts^2, sts^5t = ts^5t, ts^3t = ts^5ts, ts^4t = tst \rangle.$$

Let $S_{18} = \langle \sigma, [1, 1, 3, 1, 1, 1] \rangle$. The Green's structure for this semigroup is shown in Figure 6.27.

			1			6 elements
1	0	1	0	0	0	
0	1	0	0	0	1	
1	0	0	0	1	0	72 elements
0	0	0	1	0	1	
0	0	1	0	1	0	
0	1	0	1	0	0	
1	1	1	1	1	1	6 elements

Figure 6.27: Green's structure of S_{18}

This semigroup is represented by

$$\mathcal{P}_{18} = \langle s, t \mid s^6 = 1, t^2 = t, (ts^2t)^2 = t, stst = tst, ts^3t = tst, \\ ts^4t = tst, ts^5t = tst, tsts^2t = tsts^2 \rangle.$$

Let $S_{19} = \langle \sigma, [1, 1, 3, 3, 1, 1] \rangle$.

1

6 elements

1	1	1	1	0	0
1	1	1	0	0	1
1	1	0	0	1	1
1	0	0	1	1	1
0	0	1	1	1	1
0	1	1	1	1	0

72 elements

1	1	1	1	1	1

6 elements

Figure 6.28: Green's structure of S_{19}

A presentation for S_{19} is

$$\mathcal{P}_{19} = \langle s, t \mid s^6 = 1, t^2 = t, (ts^2t)^2 = t, (ts)^2 = ts, sts^4t = ts^4t, \\ ts^3t = ts^2t, ts^5t = ts^4t, ts^4ts^2t = ts^4ts^2 \rangle.$$

Let $S_{20} = \langle \sigma, [1, 1, 3, 1, 3, 1] \rangle$. Its Green's structure is in Figure 6.29.

1	6 elements

1	0	0	0	1	0
0	0	0	1	0	1
0	0	1	0	1	0
0	1	0	1	0	0
1	0	1	0	0	0
0	1	0	0	0	1

72 elements

1	1	1	1	1	1	6 elements

Figure 6.29: Green's structure of S_{20}

A possible presentation for S_{20} is

$$\mathcal{P}_{20} = \langle s,t \mid s^6=1, t^2=t, (ts^4t)^2=t, stst=tst, ts^2t=tsts^2, ts^3t=tst, ts^5t=tst \rangle.$$

Let $S_{21} = \langle \sigma, [1,1,3,1,3,3] \rangle$.

1	6 elements

1	0	0	1	1	1
0	0	1	1	1	1
0	1	1	1	1	0
1	1	1	1	0	0
1	1	1	0	0	1
1	1	0	0	1	1

72 elements

1	1	1	1	1	1	6 elements

Figure 6.30: Green's structure of S_{21}

A presentation for S_{21} is

$$\mathcal{P}_{21} = \langle s,t \mid s^6=1, t^2=t, (ts^4t)^2=t, (ts^3)^2=ts^3, stst=tst, ts^2t=tsts^2, ts^5t=ts^4t \rangle.$$

The next table shows us all the relevant details about $S_{14}, \ldots, S_{21}$, keeping in mind that all these semigroups are regular but not completely regular, have size 84 and have three $\mathcal{D}$-classes.

	#{idemp of rank 2}	Kernel type	$\mu(S_i)$
S_{14}	12	$1^1 5^1$	1
S_{15}	12	$2^1 4^1$	1
S_{16}	24	$2^1 4^1$	1
S_{17}	24	3^2	1
S_{18}	12	$1^1 5^1$	2
S_{19}	24	$2^1 4^1$	2
S_{20}	12	$2^1 4^1$	2
S_{21}	24	3^2	2

Similarly to the previous cases, it is easy to check, from the information in this table, that there are no isomorphic semigroups within $S_{14}, \ldots, S_{21}$. Consequently we can deduce that there are no isomorphic semigroups in Ω_6.

Chapter 7

On embedding countable sets of endomorphisms

In this chapter we delve into the domain of universal algebra, although motivated by some results from semigroup theory.

Having as a starting point the known result of Sierpiński (see [30]), which we state and prove, we will generalise it to a broader range of algebras. In the last section we give some examples of algebras to which we can apply this result. We also give an example which shows that some conditions in our generalisation cannot be dropped.

1 Sierpiński's Lemma

We start this section by stating and proving a famous lemma of Sierpiński. The proof we present is the one given by Banach in [4], almost immediately after Sierpiński published his result. This proof will be the model we follow to generalise Sierpiński's Lemma.

Let X be an infinite set and let $\mathcal{T}_X$ be the set of all selfmaps of the infinite set X.

Lemma 7.1 *Any countable subset S of the semigroup $\mathcal{T}_X$ of all transformations on X is contained in a 2-generated subsemigroup of $\mathcal{T}_X$.*

PROOF. Let $S = \{\theta_1, \theta_2, \theta_3, \dots\}$ be a countable set of elements of $\mathcal{T}_X$. Our aim is to find two maps $\alpha, \beta \in \mathcal{T}_X$ such that $S \subseteq \langle \alpha, \beta \rangle$.

We first partition X into a countable disjoint union of infinitely many sets such that each of these sets has the same cardinality as X. Formally,

$$X = X_0 \cup X_1 \cup \cdots \cup X_n \cup \cdots, \text{ such that } |X_i| = |X|, \text{for all } i.$$

In a similar way, we partition X_0 into a countable disjoint union of infinitely many sets such that each of these sets has the same cardinality as X_0 (and therefore, as X). That is,

$$X_0 = X_{0,1} \cup X_{0,2} \cup \cdots \cup X_{0,n} \cup \cdots, \text{ such that } |X_{0,i}| = |X|, \text{for all } i.$$

We then define $\alpha \in \mathcal{T}_X$ to be such that $\alpha|_{X_i}$ is a bijection between X_i and X_{i+1}, for all $i \in \mathbb{N} \cup \{0\}$

$$\alpha : X_i \longrightarrow X_{i+1}, \ i \in \mathbb{N} \cup \{0\}.$$

The second mapping $\beta \in \mathcal{T}_X$ is defined for $X \setminus X_0$ as any mapping which maps X_i bijectively onto $X_{0,i}$, for all $i \geq 1$

$$\beta : X_i \longrightarrow X_{0,i}, \ i \geq 1.$$

It is clear that $\delta_i = \alpha\beta\alpha^i\beta$, for $i \geq 1$, is a well defined bijection from X onto $X_{0,i}$. We have

$$\delta_i|_{X_k} : X_k \xrightarrow{\alpha} X_{k+1} \xrightarrow{\beta} X_{0,k+1} \xrightarrow{\alpha^i} X_i \xrightarrow{\beta} X_{0,i}, \text{ for some } k \in \mathbb{N} \cup \{0\}.$$

Let us now complete the definition of β, since we have not done so for X_0. Because δ_i is a bijection, every element of $X_{0,i}$ is the image of a unique element of $x \in X$. Thus, for each $i \in \mathbb{N}$ and $x\delta_i \in X_{0,i}$, we define $x\delta_i\beta = x\theta_i$. Since $\theta_i = \delta_i\beta$, we have $\theta_i = \alpha\beta\alpha^i\beta^2$ and consequently $\theta_i \in \langle \alpha, \beta \rangle$, which proves our result. $\square$

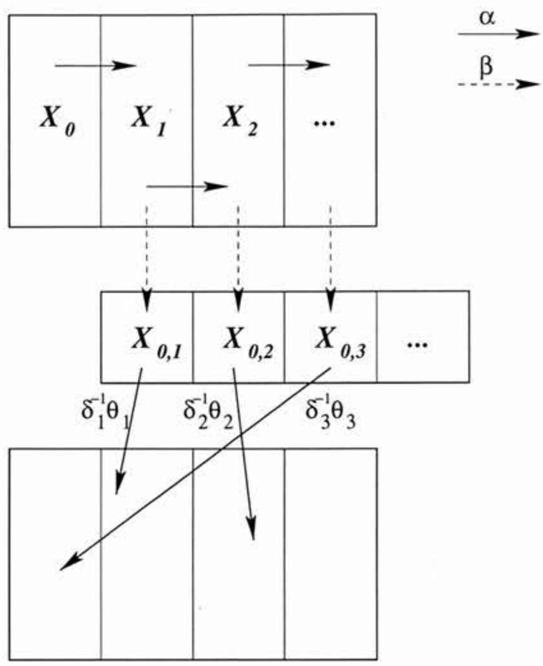

Figure 7.1: Description of Lemma 7.1

In [15], the authors, using a slightly modified version of the proof above, proved that any countable subset of the semigroup of all partial bijections $\mathcal{I}_X$ can be embedded in a two-generated subsemigroup of $\mathcal{I}_X$.

As one would expect, the case of $\mathcal{S}_X$, the set of all bijections over an infinite set X, is more complicated. Nevertheless, the analogue of Sierpiński's result for $\mathcal{S}_X$ was proved to hold in [11].

Some important corollaries of Sierpiński's Lemma appeared later, one of these being the fact that every countable semigroup can be embedded in a two-generated semigroup. This was proved by Evans in [9], although without using the lemma above.

Looking at the full transformation semigroup as the set of endomorphisms of the unstructured set X, the obvious question arises: what happens in the case where X is endowed with some structure?

In [23] it is proved that every countable set of endomorphisms of an infinite dimensional vector space $\mathcal{V}$, over an arbitrary field, is contained in a two-generated subsemigroup of the semigroup of all endomorphisms of $\mathcal{V}$.

The analogous result holds if we take the semigroup of all continuous selfmaps of a topological space X. Namely, when X is the rationals, the irrationals, the countable discrete space, an m-dimensional closed unit cube or the Cantor set. For more details, see [32].

2 Generalising Sierpiński's Lemma

In this section we prove a result analogous to Sierpiński's Lemma for the semigroup of endomorphisms of a wider class of algebras. The notation and definitions used can be found in Section 7 of Chapter 1.

Let $\mathcal{A} = (A, \Omega)$ be an algebra with universe S and set of fundamental operations Ω and let $End(\mathcal{A})$ be the semigroup of all endomorphisms over the algebra $\mathcal{A}$.

Let $B \subseteq A$. We say that B is a *basis* for $\mathcal{A}$ if B is an independent set which generates $\mathcal{A}$. If B is a basis for $\mathcal{A}$ then it is clear that for any $\alpha, \beta \in End(\mathcal{A})$ we have that

$$\alpha = \beta \text{ if and only if } \beta_{|B} = \alpha_{|B}.$$

There is a broad range of algebraic structures which have a basis, among them vector spaces. Nonetheless, not all algebraic structures have a basis, as we show in the next example.

Example 7.2 Let $p \in \mathbb{N}$ be a prime and let $\mathbb{Z}_p$ be the cyclic group of order p. Inductively, let $\mathbb{Z}_{p^i}$ be the cyclic group of order p^i. Note that $\mathbb{Z}_{p^{i-1}}$ is a subgroup of $\mathbb{Z}_{p^i}$. Let $\mathbb{Z}_\infty$ be the countable union of this ascending chain of groups, $\bigcup_{i=1}^{\infty} \mathbb{Z}_{p^i}$.

Let us see that the group $\mathbb{Z}_\infty$ is not finitely generated. Suppose that the set $\{x_1, x_2, \ldots, x_n\}$ generates $\mathbb{Z}_\infty$. Then, because we have an ascending chain of groups, there must be $i \in \mathbb{N}$ such that $\{x_1, x_2, \dots, x_n\} \subseteq \mathbb{Z}_{p^i}$. Take $y \in \mathbb{Z}_{p^{i+1}} \setminus \mathbb{Z}_{p^i}$. Then it is not possible to generate y by means of the elements $\{x_1, x_2, \dots, x_n\}$.

The maximum size of an independent set of $\mathbb{Z}_\infty$ is 1. For this, let $\{x, y\} \subset \mathbb{Z}_\infty$ be an independent set. As before, there must be an $i \in \mathbb{N}$ such that $\{x, y\} \subseteq \mathbb{Z}_{p^i}$. Since $\mathbb{Z}_{p^i}$ is 1-generated (for any $i \in \mathbb{N}$) we have that $x = y$ and our statement holds. These two facts allow us to conclude that $\mathbb{Z}_\infty$ does not have a basis.

Let $\mathcal{A} = (A, \Omega)$ be an algebra with an infinite basis B. Let us partition this basis B in an analogous way to the one used to partition X in the proof of Lemma 7.1. Take countably many disjoint sets $B_0, B_1, B_2, \dots$ such that each of these has the same cardinality as B and their union equals B. In the same way, we partition B_0 in countably many disjoint sets $B_{0,1}, B_{0,2}, B_{0,3}, \dots$ such that each $B_{0,i}$ has cardinality equal to that of B_0 (and therefore, of B).

At this point, we need to make a slightly technical remark. Let $\alpha \in End(\mathcal{A})$, let i be a natural number and let f be any bijection from B into $B_{0,i}$. Then the mapping

$$\begin{array}{rccl} \delta_i : & B_{0,i} & \longrightarrow & \mathcal{A} \\ & x & \longmapsto & (x)f^{-1}\alpha|_B \end{array}$$

satisfies $f\delta_i = \alpha|_B$. Therefore $\varphi_{f\delta_i} = \alpha$, since B is a basis of $\mathcal{A}$.

Let us then state and prove the main result of this chapter.

Theorem 7.3 *Let $\mathcal{A}$ be an algebra which has an infinite basis. Then any countable subset of $End(\mathcal{A})$ is contained in a 2-generated subsemigroup of $End(\mathcal{A})$.*

Proof. Let $\{\alpha_1, \alpha_2, \alpha_3, \dots\}$ be a countable set of elements of $End(\mathcal{A})$. We aim to find two elements of $End(\mathcal{A})$ such that the subsemigroup of $End(\mathcal{A})$ generated by those two elements contains the above set.

Let g be any mapping from B to $B \setminus B_0$ which maps B_i to B_{i+1} bijectively, for each $i \in \mathbb{N} \cup \{0\}$. Because B is independent, g is uniquely extended to an endomorphism φ_g of $\mathcal{A}$. We define the other mapping h from B to $\mathcal{A}$ in two steps.

Firstly, let us define h on $B \setminus B_0$ to be any mapping from $B \setminus B_0$ to B_0 which takes B_i to $B_{0,i}$ bijectively, for $i \geq 1$. Note that $\phi_i = ghg^ih$ is a well defined bijection from B to $B_{0,i}$. We have

$$\phi_i|_{B_k} : B_k \xrightarrow{g} B_{k+1} \xrightarrow{h} B_{0,k+1} \xrightarrow{g^i} B_i \xrightarrow{h} B_{0,i}, \text{ for some } k \in \mathbb{N} \cup \{0\}.$$

Using the technical remark stated before this theorem, with $f = \phi_i$ and $\alpha = \alpha_i$, it follows that for each $i \in \mathbb{N}$ there is a mapping δ_i such that $\phi_i\delta_i = \alpha_i|_B$.

The second step of the definition of h consists of defining this map on B_0. For each $x \in B_{0,i}$ we define $xh = x\delta_i$. Thus, for each $i \in \mathbb{N}$ we have $\alpha_i|_B = \phi_i\delta_i = \phi_ih = ghg^ih^2$. Therefore (and again, because B is a basis) we have $\alpha_i = \varphi_g\varphi_h\varphi_g^i\varphi_h^2$ and this implies that $\{\alpha_1, \alpha_2, \alpha_3, \dots\} \subseteq \langle\varphi_g, \varphi_h\rangle$. $\square$

3 Examples

In this section we give some examples, under the form of corollaries, of algebraic structures to which we can apply Theorem 7.3. We also give an example of a finitely generated algebra for which it is not true that every countable subset of the semigroup of all endomorphisms of this algebra is contained in a two-generated subsemigroup of this semigroup.

For the next result we use the notion of free algebra (see Section 7, Chapter 1).

Corollary 7.4 *Let $\mathcal{A}$ be a non-finitely generated free $\mathcal{U}$-algebra. Then every countable subset of $End(\mathcal{A})$ is contained in a 2-generated subsemigroup of $End(\mathcal{A})$.*

PROOF. Suppose that $\mathcal{A} = \mathcal{F}_X(\mathcal{U})$, where X is an infinite basis of $F_X(\mathcal{U})$. By definition of basis, X generates $\mathcal{A}$ and it is independent. Thus we can apply Theorem 7.3 to conclude the result. $\square$

Some examples of such algebras are non-finitely generated free algebras of any variety.

Let $\mathcal{A}$ be an algebra with universe A and let X be a subset of A. We say that X is *T-independent* if for each $x \in X$, $x \notin \langle X \setminus \{x\} \rangle$.

Definition 7.5 Let $\mathcal{A}$ be an algebra with universe A. Then $\mathcal{A}$ is an *independence algebra* if it satisfies the following properties:

(i) for every $X \subseteq A$ and every $u \in A$, if X is T-independent and $u \notin \langle X \rangle$, then $X \cup \{u\}$ is T-independent;

(ii) for any basis X of $\mathcal{A}$ and any function $f : X \longrightarrow \mathcal{A}$ there is an endomorphism $\varphi_f : \mathcal{A} \longrightarrow \mathcal{A}$ such that $\varphi_f|_X = f$.

The cardinality of a basis of an independence algebra $\mathcal{A}$ is called the *dimension* of $\mathcal{A}$. That this value is well defined is a consequence of the first axiom of the previous definition.

Some examples of independence algebras are (finite and infinite dimensional) vector spaces.

Remark 7.6 The notions of independence and T-independence are not equivalent (in general). But in the case of independence algebras these two notions coincide.

The concept of independence algebra has been introduced in [27]. Later, the basic structure of the endomorphism semigroup of an independence algebra was given in [13]. Some more examples of the study of both finite and infinite independence algebras can be found in [2] and [11]. A good historical overview of independence algebras can be found in [3].

The next case where we can apply Theorem 7.3 follows.

Corollary 7.7 *Let $\mathcal{A}$ be an infinite dimensional independence algebra. Then any countable subset of $End(\mathcal{A})$ is contained in a 2-generated subsemigroup of $End(\mathcal{A})$.*

Proof. This result is a trivial application of our theorem, bearing in mind the definition of independence algebra. $\square$

Since every vector space is an independence algebra, we have given a shorter proof of Theorem 3.1 in [23].

To end this section and chapter, we give an example of a finitely generated free-algebra $\mathcal{A}$ for which it is not true that every countable subset of $End(\mathcal{A})$ is contained in a two-generated subsemigroup of $End(\mathcal{A})$. This example allows us to conclude that, in Theorem 7.3, the condition of $\mathcal{A}$ being non-finitely generated cannot be dropped.

To provide this example, we return to the world of transformation semigroups. Let X be a non-empty set and let $\mathcal{T}_X$ be the full transformation semigroup on X. Let α be an element of $\mathcal{T}_X$. The element α is called a *proper idempotent* if α is an idempotent and $\alpha \neq 1_X$, the identity map on X.

Let T be a subsemigroup of $\mathcal{T}_X$ and let Y be a subset of X. We say that X is *T-isomorphic* to Y if there exist transformations $\delta, \gamma \in T$ such that $X\delta \subseteq Y$, $Y\gamma \subseteq X$, $\delta\gamma = 1_X$ and $\gamma\delta|_Y = 1_Y$, where 1_X and 1_Y are the identity maps on X and Y, respectively.

We need two auxiliary results.

Lemma 7.8 *Let X be an infinite set and let T be a subsemigroup of $\mathcal{T}_X$ which satisfies the following conditions;*

(i) *the transformation 1_X is in T;*

(ii) *there exists $\alpha \in T$ such that α is not injective;*

(iii) *there exists* $\beta \in T$ *such that* β *is not surjective but it is injective.*

If every countable subset of T *can be embedded in a 2-generated subsemigroup of* T *then* X *is* T*-isomorphic to the image of a proper idempotent in* T.

For a proof of this result, see Theorem 2.4 in [23].

Let $A = \{a, b\}$ be a two element alphabet and let A^+ be the free semigroup on A (see Section 5).

Lemma 7.9 *Let* $\alpha \in End(A^+)$. *If* α *is a proper idempotent then the image of* α *is either* $\{a\}^+$ *or* $\{b\}^+$.

PROOF. Let $u \in A^+$ be an element of $\mathsf{im}\,(\alpha)$. Then because α is an idempotent we have $u\alpha = u$. Suppose that $u = x_1 x_2 \ldots x_n$, where $x_i \in A$, for all $i \in \{1, \ldots, n\}$. Then we have $(x_1 x_2 \ldots x_n)\alpha = x_1 x_2 \ldots x_n$, Since α is a homomorphism, this is equivalent to $(x_1\alpha)(x_2\alpha)\ldots(x_n\alpha) = x_1 x_2 \ldots x_n$, where each $x_i\alpha \in A^+$, and therefore we have that $x_i\alpha = x_i$, for every $i \in \{1, \ldots n\}$, because $|x_i\alpha| \geq 1$. Suppose that there are $i, j \in \{1, \ldots, n\}$ such that $x_i \neq x_j$. Then this implies that $x_i\alpha = x_i$ and $x_j\alpha = x_j$ (since the mapping defined as $a\alpha = b$ and $b\alpha = a$ is not an idempotent). Therefore $\alpha = id_{A^+}$, which contradicts the assumption that α is a proper idempotent. Thus $u = \epsilon\epsilon\ldots\epsilon$, where $\epsilon \in \{a, b\}$, i.e. either $u \in \{a\}^+$ or $u \in \{b\}^+$.
□

Next we give an example of a finitely generated free-algebra $\mathcal{A}$ for which it is not true that every countable subset of $End(\mathcal{A})$ is contained in a two-generated subsemigroup of $End(\mathcal{A})$.

Proposition 7.10 *Let* $A = \{a, b\}$ *be a two letter alphabet and let* A^+ *be the free semigroup on* A. *It is not possible to embed every countable set of endomorphisms of* A^+ *in a 2-generated subsemigroup of* $End(\mathcal{A})$.

PROOF. It is clear that $End(\mathcal{A})$ is a proper subsemigroup of $\mathcal{T}_{A^+}$, since not every map $f : A^+ \longrightarrow A^+$ is a homomorphism. For example, the map $uf = ab$, for every $u \in A^+$, is not a homomorphism.

Let us see that $End(A^+)$ satisfies the three conditions of Lemma 7.8.

(i) Trivially the map $1_{A^+} : A^+ \longrightarrow A^+$, $x \mapsto x$ is an element of $End(A^+)$.

(ii) Let $\alpha \in End(A^+)$ defined as $a\alpha = b$ and $b\alpha = b$. Then α is not injective.

(iii) Let $\beta \in End(A^+)$ be defined as $a\beta = ab$ and $b\beta = ab^2$. Suppose that $u, v \in A^+$ are such that $u\beta = v\beta$, where $u = x_1 \dots x_n$, with $x_i \in A$, for all $i \in \{1, \dots, n\}$, and $v = y_1 \dots y_m$, with $y_j \in A$, for all $j \in \{1, \dots, m\}$, for some $m, n \in \mathbb{N}$. Because β is an homomorphism, we have $u\beta = (x_1\beta)\dots(x_n\beta)$ and $v\beta = (y_1\beta)\dots(y_m\beta)$. By definition of β, we have that $u\beta$ and $v\beta$ both finish in ab or ab^2, so u and v both finish in a or b, respectively. Thus $x_n = y_m$. Similarly, because $u\beta = v\beta$ and $x_n\beta = y_m\beta$, we get $(x_1 \dots x_{n-1})\beta = (y_1 \dots y_{m-1})\beta$. Repeating the process, we conclude that $n = m$ and $x_i = y_i$, for each $i \in \{1, \dots, n\}$. Hence $u = v$ and β is injective.

Also, β is not surjective since every element of $\mathsf{im}\,(\beta)$ has occurrences of a's and b's. Therefore $\{a\}^+, \{b\}^+ \not\subseteq \mathsf{im}\,(\beta)$.

We now use Lemma 7.9 to note that every proper idempotent of $End(A^+)$ has image $\{a\}^+$ or $\{b\}^+$.

Suppose that every countable subset of $End(A^+)$ can be embedded in a 2-generated subsemigroup of $End(A^+)$. Using Lemma 7.8, there exist $\delta, \gamma \in End(A^+)$ such that $A^+\delta \subseteq \{a\}^+$, $\{a\}^+\gamma \subseteq A^+$, $\delta\gamma = 1_{A^+}$ and $\gamma\delta|_{\{a\}^+} = 1_{\{a\}^+}$.

Since no subset of $\{a\}^+$ can have A^+ has a homomorphic image, we have a contradiction. $\square$

Appendix A

Open problems

In this appendix we can find some of the open questions that arise from each of the chapters in this thesis.

As before, we consider S to be a semigroup $\langle \sigma, \tau \rangle$, where $\sigma = (1\ 2\ \dots\ n)$ and $\tau \in \mathcal{T}_n$ has rank two. Also, τ is such that $\mathsf{im}\,(\tau) = \{a, b\}$.

From the results in Chapter 2, we are able to determine when a given semigroup $S = \langle \sigma, \tau \rangle$ has no idempotents of rank two or it has $p \times q$ idempotents of rank two (where $p =$"the number of $\mathcal{L}$-classes of rank 2" and $q =$"the number of $\mathcal{R}$-classes of rank 2". These two cases correspond to the case where S is non-regular (see Theorem 2.10) and S is completely regular (see Theorem 2.31), respectively.

The next step is to completely determine the number of idempotents of rank two for any semigroup $S = \langle \sigma, \tau \rangle$.

Question A.1 Given a transformation $\tau \in \mathcal{T}_n$ of rank 2, find a necessary and sufficient condition on τ such that the semigroup $\langle \sigma, \tau \rangle$ has k idempotents of rank 2 ($k \in \mathbb{N}$).

As one can observe, this condition will also be used to decide if two given semigroups are isomorphic.

The next question follows from Chapter 3.

Question A.2 Given two completely regular semigroups find a necessary and sufficient condition to decide if these two semigroups are isomorphic.

An answer to this problem is given in the next conjecture which involves a presentation for $S = \langle \sigma, \tau \rangle$, when S is completely regular.

Conjecture A.3 *Let $S = \langle \sigma, \tau \rangle$ be such that $\tau \in \mathcal{T}_n$ has conjugate periodic image. Suppose that $w \in \mathbb{N}$ is such that τ is w-block conjugate (see Definition 2.66).*

(i) *If $b - a \neq n/2$ then*

$$\mathcal{P} = \langle s, t \mid s^n = 1, t^2 = t, (s^w t)^2 = t, \\ (s^i t)^2 = s^i t,\ i \in \{1, \ldots, w-1\} \rangle$$

is a presentation for S.

(ii) *If $b - a = n/2$ then*

$$\mathcal{P} = \langle s, t \mid s^n = 1, t^2 = t, (ts^{n/2})^2 = t, (s^w t)^2 = t, \\ (s^i t)^2 = s^i t,\ i \in \{1, \ldots, w-1\} \rangle$$

is a presentation for S.

If this conjecture proves to be true then the following (pseudo)theorem follows.

Theorem A.4 *Let $S = \langle \sigma, \tau \rangle$ and $T = \langle \sigma, \tau \rangle$ be completely regular semigroups. Suppose that $\mathsf{im}\,(\tau) = \{a, b\}$ and $\mathsf{im}\,(\alpha) = \{a', b'\}$ and S and T are w-block conjugate. Then S is isomorphic to T if and only if $b - a, b' - a' \neq n/2$ or $b - a = b' - a' = n/2$.*

There are some more modest conjectures which give us necessary conditions for the isomorphism of ANY two semigroups of the given type.

Conjecture A.5 *Let S and T be semigroups generated by σ and a transformation of rank 2. If S is isomorphic to T then S and T have the same kernel type.*

For the next conjecture we use the value defined in Definition 3.8.

Conjecture A.6 *Let S and T be semigroups generated by σ and a transformation of rank 2. If S is isomorphic to T then $\mu(S) = \mu(T)$.*

As we noticed in the last section of Chapter 4, our aim is to find a systematic method of determining presentations for this type of semigroup. More precisely,

Question A.7 Given a semigroup $S = \langle \sigma, \tau \rangle$ which has v-block structure (and possibly is w-block conjugate) and $\mathsf{im}\,(\tau) = \{a, b\}$, determine a presentation $\mathcal{P}$ for S.

As an example, we state the following conjecture.

Conjecture A.8 *Let S be a semigroup such that its kernel type is $1^1(n-1)^1$ and $\mu(S) = 1$. Then*

$$\begin{aligned}\mathcal{P} = \langle s, t \mid\ & s^n = 1, t^2 = t, (tst)^2 = t, sts^2t = ts^2t, ts^2tst = ts^2ts, \\ & ts^it = ts^2t, i \in \{3, \ldots, n-1\} \rangle\end{aligned}$$

is a presentation for S.

The work in Chapter 6 is done with the aim of building a set of examples which hopefully will lead to finding such a method of computing presentations.

Question A.9 What happens when we replace the cyclic group generated by σ by an arbitrary subgroup G of the symmetric group $\mathcal{S}_n$ and consider semigroups $S = \langle G, \tau \rangle$, where $\tau \in \mathcal{T}_n$ has rank two?

This is perhaps the next obvious case to study...

The final question we shall pose is related to the work done in Chapter 7.

After Sierpiński proved that every countable subset of the semigroup $\mathcal{T}_X$ of all transformations on an infinite set X is contained in a two-generated subsemigroup of $\mathcal{T}_X$, it was proved in [11] that a similar result for the symmetric group of an infinite set X also holds. Following this line of results and having in mind Theorem 7.3, the obvious question follows.

Question A.10 Find algebras $\mathcal{A}$ for which every countable subset of the group of automorphisms $Aut(\mathcal{A})$ of $\mathcal{A}$ is contained in a two-generated subgroup of $Aut(\mathcal{A})$.

Appendix B

GAP programs

In Chapter 5, Section 3 we presented the algorithms resulting from the theory contained in the previous chapters. Here we give the GAP code associated to these algorithms. These functions are listed in the same order as the corresponding algorithms.

In some of these functions we call some other different functions (corresponding to the step *"Delegate"* in the respective algorithm). These calls are made via the command

```
Read("<filename.g>");
```

where $< filename.g >$ self-explains the function used.

We ought to note that we are aware that some of these functions are maybe not the most efficient. There is definitely still room for improvement and it is part of our future work to refine this code.

ALGORITHM 1 *(v-block structure)*

```
#################################################################
#
#   Given a transformation <t> and a non-zero integer <v>
```

```
#   which divides <n>, we check if <t> has <v>-block structure.
#
##################################################################


hasXBlockStructure:= function (t,v)

    local  bol, n, temp, s, x, q, i, y;

    bol:=true;
    n:=DegreeOfTransformation(t);
    temp:=[2..n];
    Add(temp,1);
    s:=PermList(temp);
    if n=v then
        return true;   # Every transformation has n-block structure.
    fi;
    if IsInt(n/v)=false then
        return false;
    fi;
    x:=0;
    q:=n/v;
    while bol=true and x<v do
        x:=x+1;
        i:=1;
        y:=x^t;
        while bol=true and i<q do
            bol:=(x^(s^(i*v)))^t=y;
            i:=i+1;
        od;
    od;
```

```
    return bol;

end;
```

ALGORITHM 2 *(w-block conjugate)*

```
  #####################################################
  #
  #  Given a transformation <t> and a non-zero integer
  #  <w> such that n/w is even, we check if <t> is
  #  <w>-block conjugate.
  #
  #####################################################


isXBlockConjugate:= function (t,w)

    local  bol, n, temp, s, x, y, i, q;

    # To avoid some problems later
    if not IsInt(w) then
        return false;
    fi;

    bol:=true;
    n:=DegreeOfTransformation(t);
    temp:=[2..n];
    Add(temp,1);
    s:=PermList(temp);
```

```
    q:=n/w;
    # This condition is required if <t> is block conjugate
    if IsEvenInt(q)=false then
        return false;
    fi;

    x:=0;
    while x<w and bol=true do
        x:=x+1;
        y:=x^t;         #y is the image of x under t
        i:=0;           #i is the exponent of sigma
        while 2*i<q and bol=true do
            i:=i+1;
            bol:=not((x^(s^((2*i-1)*w)))^t=y or
                     not((x^(s^((2*i)*w)))^t=y));
        od;
    od;
    return bol;
end;
```

The next function calls a slightly modified version of the GAP function in Algorithm 1 ("hasXblockstructure").

ALGORITHM 3 *(Periodic image)*

```
############################################
#
#  Given a transformation <t> we check
#  if it has periodic image.
#
```

```
  ###########################################
  # to get rid of the warning message


    hasXblockstructure:= "2Bdefined";

    DeclareGlobalVariable("s");
    MakeReadWriteGlobal("s");
    DeclareGlobalVariable("n");
    MakeReadWriteGlobal("n");


hasPeriodicImage:= function (t)

    local  temp, a,  b,  d;


    n:=DegreeOfTransformation(t);
    temp:=[2..n];
    Add(temp,1);
    s:=PermList(temp);

    if IsPrimeInt(n)=true then
        return false;
    fi;

    a:=ImageSetOfTransformation(t)[1];
    b:=ImageSetOfTransformation(t)[2];

    if a^t=a and b^t=b then
        return false;
```

```
    fi;

    d:=GcdInt(n,b-a);
    if d=1 then
        return false;
    fi;

    Read("hasXblockstructure.g");

    return hasXblockstructure(t,d);

end;

UnbindGlobal("s");
UnbindGlobal("n");
```

The function in the file called “hasXblockstructure.g” follows.

```
  ##############################################
  #
  #  Given a transformation <t> and a non-zero
  #  integer <v>, which divides <n>
  #  we check if <t> has <v>-block structure.
  #
  ##############################################


hasXblockstructure:= function (t,v)

    local   bol,  x,  q,  i,  y;
```

```
    bol:=true;
    x:=0;
    q:=n/v;
    while bol=true and x<v do
        x:=x+1;
        i:=1;
        y:=x^t;
        while bol=true and i<q do
            bol:=(x^(s^(i*v)))^t=y;
            i:=i+1;
        od;
    od;

    return bol;

end;
```

The next function calls a slightly modified version of the GAP function in Algorithm 2 ("isXblockconjugate.g").

ALGORITHM 4 *(Conjugate periodic image)*

```
  ##########################################
  #
  #   Given a transformation <t> we check
  #   if it has conjugate periodic image.
  #
  ###########################################
```

```
  # to get rid of the warning message


    isXblockconjugate:= "2Bdefined";


    DeclareGlobalVariable("s");
    MakeReadWriteGlobal("s");
    DeclareGlobalVariable("n");
    MakeReadWriteGlobal("n");



hasConjugatePeriodicImage:=function(t)

    local  temp, d,  a,  b;


    n:=DegreeOfTransformation(t);
    temp:=[2..n];
    Add(temp,1);
    s:=PermList(temp);

    # If <t> has CPI then <n> is even
    if IsEvenInt(n)=false then
        return false;
    fi;


    a:=ImageSetOfTransformation(t)[1];
    b:=ImageSetOfTransformation(t)[2];

    # If <t> has CPI then <n/d> is even
    d:=GcdInt(n,b-a);
```

```
    if IsEvenInt(n/d)=false then
        return false;
    fi;

    Read("isXblockconjugate.g");

    return isXblockconjugate(t,d);

end;

    UnbindGlobal("s");
    UnbindGlobal("n");
```

The function in the file called "isXblockconjugate.g" follows.

```
#####################################################
#
#   Given a transformation <t> and a non-zero integer
#   <w> such that n/w is even, we check if <t> is
#   <w>-block conjugate.
#
######################################################


isXblockconjugate:= function (t,w)

    local  bol,  x,  y,  i,  q;

    bol:=true;
    x:=0;
```

```
    q:=n/w;
    while x<w and bol=true do
        x:=x+1;
        y:=x^t;         #y is the image of x under t
        i:=0;           #i is the exponent of sigma
        while 2*i<q and bol=true do
            i:=i+1;
            bol:=not((x^(s^((2*i-1)*w)))^t=y or
                     not((x^(s^((2*i)*w)))^t=y));
        od;
    od;
    return bol;
end;
```

For the next function we also need to call the function from the file "hasXblock-structure.g".

ALGORITHM 5 *(Block structure)*

```
  ##########################################################
  #
  #  Given a transformation <t> we determine the
  #  minimum <v> such that <t> has <v>-block structure.
  #
  #  NOTE: Any transformation of degree <n>
  #  has <n>-block structure!
  ##########################################################
  #to get rid of the warning message
```

```
    hasXblockstructure:= "2Bdefined";

    DeclareGlobalVariable("s");
    MakeReadWriteGlobal("s");
    DeclareGlobalVariable("n");
    MakeReadWriteGlobal("n");


minimumBlockStructure:= function (t)

    local   temp,  bol,  divisors,  numberdivisors,  i;


    n:=DegreeOfTransformation(t);
    temp:=[2..n];
    Add(temp,1);
    s:=PermList(temp);


    # If n is prime then it is obvious
    if IsPrimeInt(n) then
        return n;
    fi;

    bol:=false;
    divisors:=DivisorsInt(n);
    numberdivisors:=Length(divisors);

    Read("hasXblockstructure.g");

    i:=1;
```

```
    while bol=false and i<numberdivisors do
        i:=i+1;
        bol:=hasXblockstructure(t,divisors[i]);
    od;


    return divisors[i];

end;

UnbindGlobal("s");
UnbindGlobal("n");
```

For the next function we also need to call the function from the file "isXblock-conjugate.g".

Algorithm 6 *(Block conjugate)*

```
  ##########################################################
  #
  #  Given a transformation <t> we determine
  #  the minimum <w> such that <t> is <w>-block conjugate.
  #
  ##########################################################
  #to get rid of the warning message


    isXblockconjugate:= "2Bdefined";

    DeclareGlobalVariable("s");
```

```
MakeReadWriteGlobal("s");
DeclareGlobalVariable("n");
MakeReadWriteGlobal("n");


minimumBlockConjugate:= function (t)

    local   temp, bol, divisors, nicedivisors, i,
            numbernicedivisors;


    n:=DegreeOfTransformation(t);
    temp:=[2..n];
    Add(temp,1);
    s:=PermList(temp);


     # If n is odd then it is not block conjugate
    if IsOddInt(n) then
        return false;
    fi;


    bol:=false;
    divisors:=DivisorsInt(n);

    #We only want divisors d s.t. n/d is even
    nicedivisors:=Filtered( divisors, x->IsEvenInt(n/x) );
    numbernicedivisors:=Length(nicedivisors);
```

```
    Read("isXblockconjugate.g");

    i:=0;
    while bol=false and i<numbernicedivisors do
        i:=i+1;
        bol:=isXblockconjugate(t,divisors[i]);
    od;

    if bol=false then
        return bol;
    else
        return divisors[i];
    fi;


end;

UnbindGlobal("s");
UnbindGlobal("n");
```

ALGORITHM 7 *(Size of $\mathcal{H}$-classes)*

```
#############################################
#
#  Given a transformation <t> we determine
#  the size of the H-classes of rank 2
#  in the semigroup generated by <s> and <t>.
#
#############################################
#to get rid of the warning message


   hasPeriodicImage:= "2Bdefined";
   minimumBlockConjugate:= "2Bdefined";

   DeclareGlobalVariable("s");
   MakeReadWriteGlobal("s");
   DeclareGlobalVariable("n");
   MakeReadWriteGlobal("n");


sizeRank2HClasses:=function(t)

   local  temp,  a,  b;


   n:=DegreeOfTransformation(t);
   temp:=[2..n];
   Add(temp,1);
   s:=PermList(temp);

   a:=ImageSetOfTransformation(t)[1];
```

```
    b:=ImageSetOfTransformation(t)[2];


    Read("hasPeriodicImage.g");
    Read("minimumBlockConjugate.g");
    if hasPeriodicImage(t)=false then
        return 2;
    elif IsPosInt(minimumBlockConjugate(t)) and b-a=n/2 then
        return 2;
    else
        return 1;
    fi;

end;


UnbindGlobal("s");
UnbindGlobal("n");
```

ALGORITHM 8 *(Number of $\mathcal{L}$-classes)*

```
  ##############################################
  #
  #  Given a transformation <t> we determine
  #  the number of L-classes of rank 2 in the
  #  semigroup generated by <s> and <t>.
  #
  ##############################################
  #to get rid of the warning message
```

```
    sizeRank2HClasses:= "2Bdefined";

    DeclareGlobalVariable("s");
    MakeReadWriteGlobal("s");
    DeclareGlobalVariable("n");
    MakeReadWriteGlobal("n");


numberRank2LClasses:=function(t)

    local   temp,  a,  b;

    Read("sizeRank2HClasses.g");

    n:=DegreeOfTransformation(t);

    # The trivial case
    if IsOddInt(n) then
        return n;
    fi;

    a:=ImageSetOfTransformation(t)[1];
    b:=ImageSetOfTransformation(t)[2];


    if b-a=n/2 and sizeRank2HClasses=2 then
        return n/2;
    else
        return n;
    fi;
```

```
end;


UnbindGlobal("s");
UnbindGlobal("n");
```

ALGORITHM 9 *(Number of $\mathcal{R}$-classes)*

```
  ##############################################
  #
  #  Given a transformation <t> we determine
  #  the number of R-classes of rank 2 in the
  #  semigroup generated by <s> and <t>.
  #
  ##############################################
  #to get rid of the warning message

    sizeRank2HClasses:= "2Bdefined";
    minimumBlockConjugate:= "2Bdefined";
    minimumBlockStructure:= "2Bdefined";



numberRank2RClasses:=function(t)

    local  n;


    # Trivial case
```

```
    n:=DegreeOfTransformation(t);
    if IsPrimeInt(n) then
        return n;
    fi;

    Read("sizeRank2HClasses.g");
    Read("minimumBlockConjugate.g");
    Read("minimumBlockStructure.g");


    if IsInt(minimumBlockConjugate(t)) then
        if sizeRank2HClasses(t)=2 then
            return minimumBlockConjugate(t);
        else
            return 2*minimumBlockConjugate(t);
        fi;
    else
        return minimumBlockStructure(t);
    fi;

end;
```

ALGORITHM 10 *(Size)*

```
####################################################
#
#  Given a transformation <t> we determine the
#  size of the semigroup generated by <s> and <t>.
#
####################################################
#to get rid of the warning message

   hasConjugatePeriodicImage:= "2Bdefined";
   hasPeriodicImage:= "2Bdefined";
   minimumBlockStructure:= "2Bdefined";
   isXBlockConjugate:= "2Bdefined";



mySize:=function(t)

   local  n,  a,  b,  v;



   # Trivial case
   n:=DegreeOfTransformation(t);
   if IsPrimeInt(n) then
       return n;
   fi;

   Read("hasConjugatePeriodicImage.g");
   Read("hasPeriodicImage.g");
```

```
Read("minimumBlockStructure.g");
Read("isXBlockConjugate.g");

# Trivial case
n:=DegreeOfTransformation(t);
if IsPrimeInt(n) then
    return n;
fi;

a:=ImageSetOfTransformation(t)[1];
b:=ImageSetOfTransformation(t)[2];


v:=minimumBlockStructure(t);

if not hasPeriodicImage(t) and not hasConjugatePeriodicImage(t) then
    if not isXBlockConjugate(t,v/2) and not b-a=n/2 then
        return 2*n*(v+1);
    elif (not isXBlockConjugate(t,v/2) and b-a=n/2) or
      (isXBlockConjugate(t,v/2) and not b-a=n/2) then
        return n*(v+2);
    fi;
elif hasConjugatePeriodicImage(t) then
    if b-a=n/2 then
        return n*(v/2+1);
    else
        return n*(v+1);
    fi;
elif hasPeriodicImage(t)
  then
    if IsInt(v/2) and isXBlockConjugate(t,v/2) and b-a=n/2 then
```

```
            return n*(v/2+2);
        else
            return n*(v+2);
        fi;
    fi;

end;
```

ALGORITHM 11 *(Regular semigroup)*

```
  #####################################################
  #
  #  Given a transformation <t> we determine if the
  #  semigroup generated by  <s> and <t> is regular.
  #
  #####################################################
  #to get rid of the warning message


hasPeriodicImage:= "2Bdefined";



isRegular:=function(t)

    Read("hasPeriodicImage.g");

    return not hasPeriodicImage(t);

end;
```

ALGORITHM 12 *(Completely regular semigroup)*

```
  ##############################################
  #
  #  Given a transformation <t> we determine
  #  if the semigroup generated by
  #  <s> and <t> is completely regular.
  #
  ##############################################
  #to get rid of the warning message


hasConjugatePeriodicImage:= "2Bdefined";


isCompletelyRegular:=function(t)

    Read("hasConjugatePeriodicImage.g");

    return hasConjugatePeriodicImage(t);

end;
```

ALGORITHM 13 *(Conjugator)*

```
######################################################
#
# Finds a permutation which conjugates
# <transf1> into <transf2> and <sigma>
# into <sigma^k>, for some k such
# <k> is coprime with <n>
# The transformations to input shoud have image={1,c}
#
# NOTE: n=DegreeOfTransformation(<transf*>)
#
######################################################
#
#Auxiliary function


candidate:= function( n, k, x)


    local   temp,  list,  i;


    list:=[];
    for i in [1..n] do
        list[x^(sigma^i)]:=1^(sigma^(i*k));
    od;

    return PermList(list);

end;
```

```
conjugator:= function( transf1, transf2)

    local  n,  temp,  sigma,  T,  coprimes,  image,
           bol,  k,  i,  x,  coprime;


    n:=DegreeOfTransformation(transf1);
    temp:=[2..n];
    Add(temp,1);
    sigma:=PermList(temp); # define the n-cycle
    T:=Monoid(AsTransformation(sigma), transf2);

     #list of numbers that are coprime with n
    coprimes:=PrimeResidues(n);

    #image of transf1
    image:=ImageSetOfTransformation(transf1);

    bol:=false;
    k:=0;

    while bol = false and k < Length(coprimes) do
        k:=k+1;
        i:=0;
        while bol = false and i < 2 do
            i:=i+1;
            x:=image[i];
            coprime:=coprimes[k];
            bol:=(candidate(n, coprime, x))^-1*transf1*
                 (candidate(n, coprime, x)) in T;
```

```
        od;
    od;

    if bol=true then
        Print(bol,"\n");
        return candidate(n, k, x);
    else
        return bol;
    fi;


end;
```

ALGORITHM 14 *(Presentation)*

```
##################################################
#
# Given a finitely presented semigroup <fp_semi>
# and a "concrete" semigroup <semi> (input
# must be in this order), we check
# if the two semigroups are isomorphic
#
# NOTE: This function relies on the ordering of
# the generators of both semigroups
#
##################################################


isPresentation := function (fp_semi, semi)

    local  rels, one, idemp, n, p, s, t,
           powers, bol, i, power, j,
           trans, k, bol1, perm, transf;


    rels:=RelationsOfFpMonoid(fp_semi);
    one:=Identity(semi);
    #this element must be a transf of rank 2
    transf:=GeneratorsOfMonoid(semi)[1];
    n:=DegreeOfTransformation(transf);

    #check if the semigroup is regular
    Read("isRegular.g");
```

```
if isRegular then
    idemp:=Filtered(Idempotents(semi),
               x->RankOfTransformation(x)=2);
             else
    idemp:=Filtered(Elements(semi),
                   x->RankOfTransformation(x)=2);
fi;

p:=GeneratorsOfMonoid(semi)[1];
s:=FreeGeneratorsOfFpMonoid(fp_semi)[1];
t:=FreeGeneratorsOfFpMonoid(fp_semi)[2];
powers:=[];

bol:=false;


#loop over the powers of the permutation
#that can be taken as generators of the semigroup

powers:=PrimeResidues(n);
i:=0;

while bol=false and i<Length(powers) do

    i:=i+1;

    power:=powers[i];

    j:=0;
```

```
# loop over the idempotents that can be taken
# as generators of the semigroup

while bol=false and j<Length(idemp) do

    j:=j+1;
    trans:=idemp[j];

    k:=0;
    bol1:=true;


    #bol all the relations of the fp semigroup

    while bol1=true and k<Length(rels) do

        k:=k+1;
        perm:=p^power;

        bol1:=MappedWord(rels[k][1], [s,t],[perm,trans])=
             MappedWord(rels[k][2], [s,t],[perm,trans]);

    od;


    #are all the relations satisfied?

    if bol1=true then
        bol:=bol1;
        Print(perm,trans,"\n");
    fi;
```

```
        od;

    od;

    return bol;

end;
```

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