SOME ASPECTS OF GEOMETRIC QUANTIZATION AND THEIR PHYSICAL BASIS

A thesis submitted for the degree of

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by

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St. Leonard's College

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DECLARATIONS

I, John.Andrew.Pinto hereby certify that this thesis which is approximately 37,000 words in length has been written by me, that it is a record of work carried out by me, and that it has not been submitted in any previous application for a higher degree.

27 NOVEMBER 1986 John.Andrew.Pinto

I hereby certify that the candidate has fulfilled the conditions of the Resolution and regulations appropriate to the degree of Doctor of Philosophy in the University of St.Andrews and that he is qualified to submit this thesis in application for that degree.

27 November 1986 K.K.Wan

Research Supervisor

DECLARATION

I was admitted as a research student under Ordinance No. 12 on 1st October 1981, and as a candidate for a Ph.D on 1st October 1982; the higher study for which this is a record was carried out in the University of St.Andrews between 1981 and 1986.

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DECLARATION

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ABSTRACT

This thesis examines some outstanding problems in half-density geometric quantization.

It is well known that the half-density quantization scheme depends on the polarization employed: in general, the quantizations of the same observable in different polarizations leads to different physical results. Therefore, an outstanding problem in geometric quantization is to establish quantizations that are independent of the choice of polarization employed. We establish a scheme, based on physical reasoning, to render quantizations in certain canonically conjugate polarizations of 2-dimensional symplectic manifolds unitarily equivalent. The scheme we propose can handle examples on contractible and noncontractible 2-dimensional symplectic manifolds in a unified manner.

In the half-density quantization scheme, quantizations in a polarization with toroidal leaves give rise to what are known as BWS conditions. These BWS conditions depend on the choice of connection on the underlying Hermitian line bundle. Let ζ be an observable with closed integral curves on the phase space $T^{*}Q$ where Q is an open interval in $\mathbb{R}$. The eigenvalues of the bound states of the quantum observable corresponding to ζ are obtained by quantizing ζ in the polarization $\mathcal{P}_{c}$ spanned by X_{ζ}, the Hamiltonian vector field generated by ζ. This polarization has toroidal leaves, and so the bound states are obtained from the BWS conditions. However, in general, there is no formal procedure for constructing a unitary map between $H_{\mathcal{P}_c}$, the quantization Hilbert space associated with the polarization $\mathcal{P}_c$, and the Hilbert space $L^{2}(Q)$ associated with the usual

position representation. We construct an approximate unitary map by using a modified version of the Maslov-WKB method. This modified version of the Maslov-WKB method incorporates the BWS conditions by taking into account the fact that different choices of connection give rise to different BWS conditions. This thesis contains a study of the following observables:

(1) the Hamiltonian of a particle in a potential well,

(2) the Hamiltonian of a particle in a potential well localized in phase space, and

(3) certain multilinear momentum observables, (i.e. polynomials of the momentum p with functions of the position coordinate q as coefficients), with closed integral curves.

CONTENTS

CHAPTER 0

INTRODUCTION

INTRODUCTION

Many of the main contributors to the development of the theory of geometric quantization have had a strong bias towards pure mathematics. Therefore there has been a tendency to attempt the resolution of many of the outstanding problems in the theory by resorting to sophisticated mathematics. It could be argued that the theory as it stands is complicated enough, and so the introduction of even more sophisticated pure mathematical schemes to resolve minor problems in the theory is not always justifiable. We are not convinced that the introduction of the half-form quantization scheme to replace the half-density quantization scheme was worth the much increased mathematical complexity. Hence we shall restrict ourselves to the study of the half-density quantization scheme in this thesis. Whilst the above mentioned approach for resolving problems is perfectly valid, it seems reasonable to ask the following question: could some of the outstanding problems in the theory of geometric quantization be resolved by resorting to physical intuition?. This thesis is part of an on-going quantization programme at St.Andrews led by Dr.K.K.Wan. The underlying procedure involves:

(1) the application of geometric quantization schemes to some known examples in quantum mechanics to see what difficulties one may encounter,

(2) the study of the possible physical origin of the difficulties encountered, and

(3) attempts to formulate schemes based on physical reasoning to resolve the problems.

In this thesis we shall be concerned with the study of two outstanding problems in the theory of geometric quantization; these problems will be stated in the next three paragraphs.

Let $\mathcal{P}$ and $\mathcal{P}'$ be two arbitrary reducible polarizations of a symplectic manifold (M,ω), and let $H_{\mathcal{P}}$ and $H_{\mathcal{P}'}$ be the quantization Hilbert spaces associated with the polarizations $\mathcal{P}$ and $\mathcal{P}'$ respectively. The first problem is called the **pairing problem**, and it is stated as follows: how does one link quantizations in the Hilbert space $H_{\mathcal{P}}$ with quantizations in the Hilbert space $H_{\mathcal{P}'}$?. The link between the quantizations in the polarizations in $\mathcal{P}$ and $\mathcal{P}'$ is loosely referred to as **pairing**. In order to have a full quantum theory it is essential to be able to construct pairings between any two reducible real polarizations of the symplectic manifold (M,ω).

It is well known that the half-density quantization scheme depends on the polarization employed: in general, the quantizations of the same observable in different polarizationsleads to different physical results.

The second problem arises from the following question: how does one establish **unitarily equivalent quantizations** in different polarizations?.

In Chapter 1 we shall review the following background material:

(1) the geometric quantization scheme,

(2) the BWS conditions in the half-density quantization scheme, and

(3) the Maslov-WKB method for the one-dimensional Hamiltonian system of a particle in a potential well.

These reviews will be detailed and self-contained.

In Chapter 2 we shall attempt to establish a scheme to render quantizations in certain canonically conjugate polarizations of 2-dimensional symplectic manifolds unitarily equivalent. We shall study examples on both contractible and noncontractible symplectic manifolds. The work in Chapter 2 has been published [cf. Wan, McKenna and Pinto (1984); Wan, Pinto and McKenna (1984)].

Let H be the Hamiltonian of the one-dimensional Hamiltonian system of a particle in a potential well. Let Q and the cotangent bundle T^*Q be respectively the configuration space and phase space of the Hamiltonian system. Let X_H be the Hamiltonian vector field generated by H. Let M_0 be the maximal submanifold of T^*Q on which X_H spans a polarization with toroidal leaves, and let $\mathcal{P}_c$ be the polarization of M_0 spanned by X_H. In the half-density quantization scheme the eigenvalues of the bound states of the quantum observable corresponding to H are obtained by quantizing H in the polarization $\mathcal{P}_c$. (Note that quantization in a polarization with toroidal leaves gives rise to BWS conditions, and these BWS conditions are used to construct the eigenvalues of the observables that are being quantized.) Our task in Chapter 3 is to construct a pairing between the polarization $\mathcal{P}_c$ and the vertical polarization P (of the cotangent bundle T^*Q). In other words, we would like to construct a link between the quantization Hilbert space $\mathbf{H}_{\mathcal{P}_c}$ and the position representation $L^2(Q)$. (The quantization Hilbert space $\mathbf{H}_P$ is identifiable with $L^2(Q)$.)

We shall begin Chapter 4 with an attempt to establish unitarily equivalent quantizations of a general observable (of an arbitrary 2-dimensional symplectic manifold) in suitably chosen canonically conjugate polarizations.

Let the configuration space Q be an open interval in $\mathbb{R}$, T^*Q be the phase space and let P be the vertical polarization of T^*Q. Next we shall attempt to quantize the following observables in the vertical polarization P:

(1) the Hamiltonian (of a particle in a potential well) localized in the phase space T^*Q, and

(2) certain multilinear momentum observables with closed integral curves.

As in Chapter 3, we shall begin by quantizing each of the above mentioned observables in a suitably chosen polarization $\mathcal{P}_c$ which has toroidal leaves. Then we shall try to establish a pairing between $\mathcal{P}_c$ and the vertical polarization P. We are interested in quantizing the above mentioned observables because, in general, the standard canonical quantization scheme does not give unique quantum operators corresponding to these observables in the position representation $L^2(Q)$.

The following is a list on notation which will be adopted throughout the thesis. The symbols $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{C}$, $\mathbb{Z}$ denote respectively the set of real numbers, the set of positive real numbers, the set of complex numbers and the set of integers. The symbol $\not{i}$ denotes $i/\hbar$ where $i = (-1)^{1/2}$ and $\hbar$ is Planck's constant. The letters M, M_o, $\mathcal{M}$, Q, Q_c will represent real manifolds. Then TQ and T^*Q will represent respectively the tangent bundle and cotangent bundle of Q.

An index of symbols defined in the text is given at the end of the thesis.

CHAPTER 1

A REVIEW OF BACKGROUND MATERIAL

1.1 A REVIEW OF THE GEOMETRIC QUANTIZATION SCHEME

(1.1.1) Introduction

In this section we shall give a brief outline of the geometric quantization scheme. We shall follow, unless otherwise stated, the notation and conventions adopted by Woodhouse (1980).

(1.1.2) Hamiltonian mechanics

The definition of a symplectic manifold and the notation (M,ω), $C^{\infty}(M)$ and $V(M)$ are given in Appendix 1.1.

The basic model of the phase space of a (conservative) classical mechanical system is a symplectic manifold. The physical state of a classical mechanical system is represented by a point in the phase space.

(1.1.2.D1) Definitions [Campbell (1983); Woodhouse (1980), pp. 10-12]

Let (M,ω) be a 2k-dimensional symplectic manifold representing the phase space of a classical mechanical system.

(1) The real-valued functions in $C^{\infty}(M)$ are called **classical observables.**

(2) Let ζ be a classical observable; then the vector field $X_{\zeta} \in V(M)$ which is determined by

$$X_{\zeta} \lrcorner \omega + d\zeta = 0 \qquad (1.1.2.\text{Eq 1a})$$

is called the **Hamiltonian vector field generated by** ζ.

Let $\{p_1,\ldots,p_k,q_1,\ldots,q_k\}$ be local canonical coordinates on (M,ω); then locally X_ζ is given by

$$X_\zeta = \sum_{i=1}^{k} \{(\partial\zeta/\partial p_i)(\partial/\partial q_i)-(\partial\zeta/\partial q_i)(\partial/\partial p_i)\}. \qquad (1.1.2.\text{Eq 1b})$$

(3) Let ζ and η be any two classical observables; then the **Poisson bracket** of ζ and η is the classical observable $\{\zeta,\eta\}$ defined by

$$\{\zeta,\eta\} = X_\zeta(\eta) = 2\omega(X_\zeta,X_\eta). \qquad (1.1.2.\text{Eq 2a})$$

Locally ,we have

$$\{\zeta,\eta\} = \sum_{i=1}^{K} \{(\partial\zeta/\partial p_i)(\partial\eta/\partial q_i)-(\partial\zeta/\partial q_i)(\partial\eta/\partial p_i)\}. \qquad (1.1.2.\text{Eq 2b})$$

(4) The Poisson bracket makes $C^\infty(M)$ into an infinite-dimensional real Lie algebra called the **algebra of classical observables.**

For all $\zeta, \eta, \xi \in C^\infty(M)$ and $a,b \in \mathbb{R}$, we have

$$\{a\zeta+b\eta,\xi\} = a\{\zeta,\xi\}+b\{\eta,\xi\} \qquad (1.1.2.\text{Eq 3a})$$

and

$$\{\zeta,\{\eta,\xi\}\}+\{\xi,\{\zeta,\eta\}\}+\{\eta,\{\zeta,\xi\}\} = 0. \qquad (1.1.2.\text{Eq 3b})$$

(1.1.2.Ex 1) Example [Abraham and Marsden (1978), pp178-179; Campbell (1983); Woodhouse (1980), p7]

For most physical applications (M,ω) is the phase space with M being the cotangent bundle, T^*Q, of the configuration space Q of a classical mechanical system. Let Q be a k-dimensional manifold with local coordinates $q = (q_1,\ldots,q_k)$ and let $pr:T^*Q \dashrightarrow Q$ be the usual cotangent projection map. Each point in T^*Q is a covector at some point of Q: a covector at the point $q \in Q$ is a linear mapping $p:T_q^*Q \dashrightarrow \mathbb{R}$. Let $p_i = \partial/\partial q_i|_q \lrcorner\, p$, where $\partial/\partial q_i|_q \in T_qQ$: the p_is are the k-components of the covector p at q. Each covector p in T_q^*Q can be represented by the set of 2k functions

$\{p_1,\ldots,p_k,q_1,\ldots,q_k\}$; this set forms a collection of local coordinates on T^*Q. Hence we have given T^*Q a manifold structure, so we shall use M for T^*Q in what follows.

There exists a global one-form β_0 on M defined at each point $m = (p_1,\ldots,p_k,q_1,\ldots,q_k) \in M$ by

$$X_m \lrcorner \beta_0|_m = pr_* X_m \lrcorner\, p \text{ , for all } X_m \in T_mM; \qquad (1.1.2.\text{Eq } 4)$$

where $\beta_0|_m$ is the restriction of β_0 to the point m. β_0 is called the **canonical one-form** [cf. Abraham and Marsden (1978), pp178-179; Woodhouse (1980), p7].

The two-form ω defined by $\omega = d\beta_0$ is referred to as the **canonical two-form.**

Locally, we have

$$\beta_0 = \sum_{i=1}^{k} p_i dq_i \text{ and } \omega = \sum_{i=1}^{k} dp_i \wedge dq_i. \qquad (1.1.2.\text{Eq } 5)$$

The pair (M,ω) is an example of a symplectic manifold. Clearly, $\{p_1,\ldots,p_k,q_1,\ldots,q_k\}$ is a set of local canonical coordinates on (M,ω) [cf. Appendix 1.1 for a definition of local canonical coordinates].

(1.1.3) The Kostant-Souriau condition

The terms complex line-bundle, Hermitian structure on a complex line-bundle , connection on a line-bundle, the compatibility of Hermitian structure and connection, Hermitian line-bundle with (compatible) connection, curvature two-form of a connection on a line-bundle and the notation B, ∇, $(\cdot,\cdot)$, $(B,(\cdot,\cdot),\nabla)$ and $\mathrm{curv}(B,\nabla)$ are given in Appendix 1.1.

Let (M,ω) be an arbitrary symplectic manifold. The first step in the geometric quantization scheme is the construction of a Hermitian line-bundle with (compatible) connection $(B,(\cdot,\cdot),\nabla)$ over M, such that $\mathrm{curv}(B,\nabla) = \omega/\hbar$.

Such a Hermitian bundle does not always exist; the condition for its existence is given by the following

(1.1.3.T1) Theorem [Campbell (1983); Kostant (1970),p.133; Souriau (1970); Simms and Woodhouse (1976),p.37; Woodhouse (1980),pp.116-120]

Let (M,ω) be the given symplectic manifold; then there exists a Hermitian line-bundle with (compatible) connection $(B,(\cdot,\cdot),\nabla)$ over M such that $\mathrm{curv}(B,\nabla) = \omega/\hbar$ if and only if the integral

$$[2\pi]^{-1}\int_{\Sigma}\mathrm{curv}(B,\nabla) = [2\pi\hbar]^{-1}\int_{\Sigma}\omega \qquad (1.1.3.\mathrm{Eq}\ 1)$$

over any closed, oriented two-surface Σ in M is an integer.

We shall now give the following

(1.1.3.D1) Definition [Simms (1972)]

A symplectic manifold (M,ω) is said to satisfy the **Kostant-Souriau condition** if the integral $[2\pi\hbar]^{-1}\int_{\Sigma}\omega$ over any closed, oriented two-surface Σ in M is a integer.

Remark: (R1) We shall use the symbol $\not{i}$ to denote $i/\hbar$.

(1.1.3.Ex 1) Example

Let (M,ω) be a symplectic manifold and let β be a real one-form on M that satisfies the condition $d\beta = \omega$. Let $B = M \times \mathbb{C}$ be a trivial line-bundle over M and let $(\cdot,\cdot)$ be the (natural) Hermitian structure on B. Let s_o be a unit section of B: s_0 satisfies the condition $(s_o, s_o) = 1$. Then we can define a connection on B by

$$\nabla_X s_o = -\not{i}(X \lrcorner \beta) s_o \text{ , for all } X \in V_{\mathbb{C}}(M); \qquad (1.1.3.\text{Eq } 2)$$

(where $V_{\mathbb{C}}(M)$ is the space of smooth complex vector fields on M [cf. Appendix 1.1]). The one-form β is referred to as the **connection potential**, since it defines the connection ∇. Note that our definition of the connection potential differs from that given by Woodhouse (1980),[p116 and p297], by a factor $\hbar$. It can be shown that $\text{curv}(B,\nabla) = d\beta/\hbar = \omega/\hbar$ [cf. Appendix 1.1 or Woodhouse (1980), p116 and p297].

For every $X \in V_{\mathbb{C}}(M)$, we have

$$X(s_o, s_o) = X(1) = 0 \qquad (1.1.3.\text{Eq } 3a)$$

and

$$(\nabla_X s_o, s_o) + (s_o, \nabla_X s_o) = -\not{i}(X \lrcorner \beta - \bar{X} \lrcorner \beta)(s_o, s_o) = 0; \quad (1.1.3.\text{Eq } 3b)$$

the last result was obtained by using the following two facts: (i) the connection potential is real; and, (ii) the real part of X = the real part of $\overline{X}$. Hence the Hermitian structure $(\cdot,\cdot)$ and connection ∇ are compatible.

Let Σ be a closed, oriented, two-surface in M; then the boundary $\partial\Sigma$ of Σ is a zero-boundary: Σ is without a boundary. Then it follows from Stokes' Theorem that

$$[2\pi\hbar]^{-1}\int_{\Sigma}\omega = [2\pi\hbar]^{-1}\int_{\Sigma} d\beta = [2\pi\hbar]^{-1}\int_{\partial\Sigma}\beta = 0 \qquad (1.1.3.\text{Eq } 4)$$

[cf. Von Westenholz (1981),pp280-289 and p310; Woodhouse (1980),p293]. Hence the symplectic manifold (M,ω) satisfies the Kostant-Souriau condition and $(B,(\cdot,\cdot),\nabla)$ is a Hermitian line-bundle with (compatible) connection such that $\mathrm{curv}(B,\nabla) = \omega/\hbar$.

<u>Remark</u>: (R2) It is always possible to construct a Hermitian line-bundle with (compatible) connection over (M,ω) such that $\mathrm{curv}(B,\nabla) = \omega/\hbar$, when M is a cotangent bundle and ω is the canonical two-form [cf. example(1.1.2.Ex 1); and, Woodhouse (1980), p122]: we choose the connection potential β to be $\beta = \beta_0 + \alpha$ where β_0 is the canonical one-form and α is any closed one-form on M.

(1.1.4) <u>The prequantization procedure</u> [Campbell (1983); Simms and Woodhouse (1976), pp38-50; Sniatycki (1980), pp6-8 and pp51-59; Woodhouse (1980), pp113-122]

The geometric quantization scheme arose out of an attempt to solve the so-called **Dirac problem** in a given symplectic manifold (M,ω).

The attempt is as follows:

(Q1) We associate a Hilbert space **H** to (M,ω);

(Q2) Let S be a suitably chosen Lie subalgebra of $C^{\infty}(M)$. Then to each $\zeta \in S$ we assign a symmetric operator $\tilde{\zeta}$ on **H** such that:

(i) The map $\zeta \dashrightarrow \tilde{\zeta}$ is linear over $\mathbb{R}$;

(ii) $\tilde{\zeta} = \zeta$ when ζ is a constant function on M;

(iii) For each $\zeta, \eta \in S$ we have

$$[\tilde{\zeta},\tilde{\eta}] = -i\hbar\tilde{\xi}, \qquad (1.1.4.\text{Eq } 1)$$

where $[\tilde{\zeta},\tilde{\eta}] = \tilde{\zeta}\tilde{\eta} - \tilde{\eta}\tilde{\zeta}$ is the quantum commutator and $\xi = \{\zeta,\eta\}$ is the Poisson bracket of the two classical observables ζ and η.

In other words, the collection of operators $\mathbf{T} = \{\tilde{\zeta},\tilde{\eta},\ldots\}$ is an operator representation of the subalgebra S in H.

It turns out that the Dirac problem does not uniquely determine the quantum system associated with the symplectic manifold (M,ω). Not all the solutions to the Dirac problem provide physically reasonable results.

The prequantization procedure represents a first attempt at solving the Dirac problem geometrically, and it goes as follows.

(PQ1) At the outset we shall assume that (M,ω) satisfies the Kostant-Souriau condition. We choose a Hermitian line-bundle with (compatible) connection $(B,(\cdot,\cdot),\nabla)$ over M such that $\mathrm{curv}(B,\nabla) = \omega/\hbar$. We shall call the triple $(B,(\cdot,\cdot),\nabla)$ a **prequantization bundle.**

(PQ2) Let **W** be the space consisting of square-integrable smooth sections $s \in C^{\infty}_{B}(M)$ with respect to the inner product

$$\langle s,s\rangle = [2\pi\hbar]^{-1}\int_{M} (s,s)\,\mathcal{E}_{\omega} \qquad (1.1.4.\mathrm{Eq}\ 2)$$

where the notation $C^{\infty}_{B}(M)$ and $\mathcal{E}_{\omega}$ are given in Appendix 1.1. The **prequantization Hilbert space** H is defined to be the completion of **W.**

(PQ3) To each classical observable $\zeta \in C^{\infty}(M)$, we assign the symmetric operator $\tilde{\zeta}$ in H given by the formal expression

$$\tilde{\zeta}s = -i\hbar\nabla_{X_{\zeta}} s + \zeta s. \qquad (1.1.4.\mathrm{Eq}\ 3)$$

The domain of $\tilde{\zeta}$ is undetermined by this expression; however, it is usual to choose the domain to be

$$D_{\tilde{\zeta}} = \{s \in C^{\infty}_{B}(M)\text{: the support of } s \text{ is compact}\} \qquad (1.1.4.\mathrm{Eq}\ 4)$$

because $\tilde{\zeta}$ is a symmetric operator on $D_{\tilde{\zeta}}$. If the Hamiltonian vector field X_{ζ} is complete, then $\tilde{\zeta}$ is an essentially self-adjoint operator. Note that an essentially self-adjoint operator has a unique self-adjoint extension [cf. Hellwig (1964), pp.172-173]. When X_{ζ} is complete, then the unique self-adjoint extension will be referred to as the **prequantization operator.**

It is easy to check that the prequantization procedure is a solution to the Dirac problem [cf. Abraham and Marsden (1978), p.441].

(1.1.4.Ex 1) Example: The free particle

Consider the situation of a free particle in a configuration space $Q = \mathbb{R}$. The cotangent bundle, T^*Q, of Q can be identified with $\mathbb{R}^2$. Let q be the cartesian coordinate on Q and let (p,q) be the usual cartesian canonical coordinates on T^*Q, and let ω be the canonical two-form on T^*Q. The phase space of the free particle is the symplectic manifold (T^*Q,ω). Let $B = T^*Q \times \mathbb{C}$ be the trivial bundle over T Q, $(\cdot,\cdot)$ be the (natural) Hermitian structure on B and let s_0 be a unit section of B. Let ∇ be the connection defined by

$$\nabla_X s_0 = -\tfrac{i}{\hbar}(X \lrcorner\, pdq)s_0, \text{ for all } X \in V_{\mathbb{C}}(T^*Q). \qquad (1.1.4.\text{Eq } 5)$$

(Note that we have chosen the connection potential to be pdq.) Let $(B,(\cdot,\cdot),\nabla)$ be the chosen **prequantization bundle.**

The space **W** (which has been defined in (PQ2)) consists of smooth sections of the form $\Psi = \Psi(p,q)s_0 \in C^\infty_B(T^*Q)$, where $\Psi(p,q) \in C^\infty_{\mathbb{C}}(T^*Q)$, which are square-integrable with respect to the inner product

$$\langle\Psi,\Psi\rangle = [2\pi\hbar]^{-1}\int_{\mathbb{R}^2} |\Psi(p,q)|^2 dpdq. \qquad (1.1.4.\text{Eq } 6)$$

The **prequantization Hilbert space** H is the completion of **W**.

From a physical point of view, one is tempted to regard $|\Psi(p,q)|^2$ as the probability density of finding the particle in the classical state represented by the point (p,q) in the phase space. However, the prequantization Hilbert space **H** is physically unacceptable because of the following reasons:

(i) the elements of **H** with arbitrarily small support on T^*Q violate the Heisenberg uncertainty principle of quantum mechanics [cf. equation (1.1.4.Eq 6)]; and,

(ii) according to the Schrodinger prescription of quantum mechanics H should

be unitarily equivalent to $L^2(Q)$, which is $L^2(\mathbb{R})$; instead, it is unitarily equivalent to $L^2(T^*Q)$, which is $L^2(\mathbb{R}^2)$.

In other words, the prequantization Hilbert space **H** is "too large", and we need to reduce its size; the procedure for reducing the size of **H** (in order to obtain a physically acceptable solution of the Dirac problem) is referred to as quantization. However, before we give a prescription for the quantization scheme we shall introduce the following two geometric structures: real polarizations and half-densities.

(1.1.5) Real polarizations

The definitions of the following list of terms and notation are given in Appendix 1.1: real distribution, integrable distribution, Lie bracket [,] of two smooth vector fields, Frobenius Theorem, leaves of an integrable distribution, the space of leaves of an integrable distribution, $V(M;D)$, M/D, and the projection map $pr:M \dashrightarrow M/D$.

(1.1.5.D1) Definitions [Woodhouse (1980), p.73]

A **real polarization** $\mathcal{P}$ of a 2k-dimensional symplectic manifold (M,ω) is a k-dimensional smooth, real distribution which satisfies the following conditions:

(P1) $\mathcal{P}$ is integrable; and,

(P2) $\omega(X,Y) = 0$, for all $X,Y \in V(M;\mathcal{P})$.

(A k-dimensional distribution D that satisfies condition (P2) is called a **Lagrangian distribution.**)

Remarks: (R1) From now on we shall use the terms distribution and polarization to refer to a real distribution and a real polarization respectively; and, we shall use the letters $\mathcal{P}$ and P for polarizations.
(R2) It follows from Frobenius Theorem [cf. Appendix 1.1] that every point in M lies on a leaf of the polarization $\mathcal{P}$.

(1.1.5.D2) Definitions and notation [Woodhouse (1980), p73, p74-75, p291, p85]

Let $\mathcal{P}$ be a polarization of a symplectic manifold (M,ω).
(1) The polarization $\mathcal{P}$ is said to be **reducible** if the space of leaves $M/\mathcal{P}$ is a Hausdorff manifold with the projection map $pr: M \dashrightarrow M/\mathcal{P}$ being smooth.
(2) The map

$$\nabla : V(M;\mathcal{P}) \times V(M;\mathcal{P}) \dashrightarrow V(M;\mathcal{P}) : (X,Y) \dashrightarrow \nabla_X Y \qquad (1.1.5.\text{Eq } 1a)$$

determined by

$$(\nabla_X Y) \lrcorner \omega = X \lrcorner d(Y \lrcorner \omega) \qquad (1.1.5.\text{Eq } 1b)$$

is called the **partial connection defined by** $\mathcal{P}$.
(3) Then $C^{\infty}(M;\mathcal{P})$ and $C^{\infty}(M;\mathcal{P},1)$ are respectively the spaces of smooth real functions of M defined by

$$C^{\infty}(M;\mathcal{P}) = \{\zeta \in C^{\infty}(M) : X(\zeta) = 0, \text{ for all } X \in V(M;\mathcal{P})\}, \qquad (1.1.5.\text{Eq } 2a)$$

$$C^{\infty}(M;\mathcal{P},1) = \{\zeta \in C^{\infty}(M) : [X_{\zeta}, X] \in V(M;\mathcal{P}), \text{ for all } X \in V(M;\mathcal{P})\}, \qquad (1.1.5.\text{Eq } 2b)$$

where [,] is the Lie bracket of two vector fields [cf. Appendix 1.1].

(1.1.5.Ex 1) Example: The vertical and horizontal polarizations of a cotangent bundle [Woodhouse (1980), p7 and p73]

Let Q be a k-dimensional manifold with global coordinates $\mathbf{q} = \{q_1,..,q_k\}$ and let $(\mathbf{p},\mathbf{q}) = \{p_1,..,p_k,q_1,..,q_k\}$ be the usual canonical coordinates on the cotangent bundle, T^*Q, of Q. (Note that T^*Q is identifiable with $\mathbb{R} \times Q$.) Let ω be the canonical two-form on T Q and let $M = T^*Q$; then $\omega = \sum_{i=1}^{k} dp_i \wedge dq_i$, and (M,ω) is a 2k-dimensional symplectic manifold.

The set of vector fields $\{\partial/\partial p_1,...,\partial/\partial p_k,\partial/\partial q_1,...,\partial/\partial q_k\}$ is a basis of V(M). In fact, for all $i,j \in \{1,...k\}$, we have

$$\omega(\partial/\partial p_i,\partial/\partial p_j) = \omega(\partial/\partial q_i,\partial/\partial q_j) = 0, \qquad (1.1.5.\text{Eq } 3a)$$

$$\omega(\partial/\partial p_i,\partial/\partial q_j) = (1/2)\delta_{ij}. \qquad (1.1.5.\text{Eq } 3b)$$

(Note that we have followed the convention adopted by Woodhouse (1980), [cf.p2 and p289]of putting $\partial/\partial q_j \lrcorner (\partial/\partial p_i \lrcorner \omega) = 2\omega(\partial/\partial p_i,\partial/\partial q_j)$.)

Let P be the k-dimensional distribution on M spanned by the vector fields $\{\partial/\partial p_1,...,\partial/\partial p_k\}$. Then a typical element of V(M;P) is of the form $\sum_{i=1}^{k} \zeta_i(\mathbf{p},\mathbf{q})(\partial/\partial p_i)$ where $\zeta_i \in C^\infty(M)$. Let $X = \sum_{i=1}^{k} \zeta_i\, \partial/\partial p_i$ and $Y = \sum_{i=1}^{k} \eta_i\, \partial/\partial p_i$; then we have

$$[X,Y] = \sum_{j=1}^{k} \{\sum_{i=1}^{k} (\zeta_i\, \partial\eta_j/\partial p_i - \eta_i\, \partial\zeta_j/\partial p_i)\}\partial/\partial p_j \ . \qquad (1.1.5.\text{Eq } 4)$$

Clearly [X,Y] is an element of V(M;P). Thus P is an integrable distribution. Equation (1.1.5.Eq 3a) implies that P is a Lagrangian distribution. Therefore, P is a polarization of (M,ω). P is commonly referred to as the **vertical polarization,** and we shall reserve this particular symbol P to denote it. The leaves of P are surfaces $\{(\mathbf{p},\mathbf{q}) \in M: q_1=\text{constant},...,q_k=\text{constant}\}$. The polarization P is reducible because the space of leaves M/P is identifiable with Q.

Let ∇ be the partial connection defined by P and let $Y \in V(M;P)$ be given by $Y = \sum_{i=1}^{k} \zeta_i (\partial/\partial p_i)$. Then we have

$$\nabla_X Y = \sum_{i=1}^{k} X(\zeta_i)\,\partial/\partial p_i, \text{ for all } X \in V(M;P). \qquad (1.1.5.\text{Eq } 5)$$

A typical function χ in $C^{\infty}(M;P)$ is of the form

$$\chi = \xi(\mathbf{q}) \text{ where } \xi(\mathbf{q}) \in C^{\infty}(Q). \qquad (1.1.5.\text{Eq } 6a)$$

A typical function ζ in $C^{\infty}(M;P,1)$ is of the form

$$\zeta = \sum_{i=1}^{k} \xi_i(\mathbf{q})p_i + \eta(\mathbf{q}); \text{ where } \xi_i(\mathbf{q}), \eta(\mathbf{q}) \in C^{\infty}(Q). \qquad (1.1.5.\text{Eq } 6b)$$

Let P_c be the k-dimensional distribution spanned by the vector fields $\{\partial/\partial q_1, \ldots, \partial/\partial q_k\}$. We can show as we did above that P_c is a polarization of (M,ω). P_c is a polarization that is canonically conjugate to the vertical polarization P, so we have used the subscript c to indicate this relationship. (A definition of canonically conjugate polarizations is given by definition (2.1.D3) of Chapter 2.) P_c is commonly referred to as the **horizontal polarization.** The leaves of P_c are the surfaces $\{(\mathbf{p},\mathbf{q})\in M\colon p_1=\text{constant}, \ldots, p_k=\text{constant}\}$. The polarization P_c is reducible because the space of leaves M/P_c is identifiable with $\mathbb{R}^k$. Let $Q_c = M/P_c$; we shall refer to Q_c as the **effective configuration space with respect to the polarization P_c**, and we use the coordinates $\{p_1, \ldots, p_k\}$ to coordinatize Q_c.

Let ∇^c be the partial connection defined by P_c and let $Y_c = \sum_{i=1}^{k} \zeta_{ci} (\partial/\partial q_i)$; then we have

$$\nabla^c_X Y_c = \sum_{i=1}^{k} X(\zeta_{ci})\,\partial/\partial q_i, \text{ for all } X \in V(M;P_c). \qquad (1.1.5.\text{Eq } 7)$$

A typical function in $C^{\infty}(M;P_c)$ is of the form

$$\chi_c = \xi_c(\mathbf{p}) \text{ where } \xi_c(\mathbf{p}) \in C^{\infty}(Q_c) \qquad (1.1.5.\text{Eq } 8a)$$

A typical function in $C^{\infty}(M;P_c,1)$ is of the form

$$\zeta_c = \sum_{i=1}^{k} \xi_{ci}(p)q_i + \eta_c(p); \quad \xi_{ci}(p), \eta_c(p) \in C^\infty(Q_c). \quad (1.1.5.\text{Eq } 8b)$$

Remark: (R3) We shall reserve the symbols p_is and q_is to denote cartesian coordinates.

The following proposition shows that an arbitrary polarization of a 2k-dimensional symplectic manifold has locally the structure of the vertical polarization of the cotangent bundle, T^*Q, of a k-dimensional manifold Q with global coordinates $\{q_1, \ldots, q_k\}$.

(1.1.5.T1) Theorem [Woodhouse (1980), p81]

Let $\mathcal{P}$ be a polarization of a 2k-dimensional symplectic manifold (M,ω). Then it is possible to find canonical coordinates $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_k, \mathfrak{q}_1, \ldots, \mathfrak{q}_k\}$ in some neighbourhood U of each $m \in M$ such that the leaves of $\mathcal{P}$ coincide locally with the surfaces $\{m \in U: \mathfrak{q}_1 = \text{constant}, \ldots, \mathfrak{q}_k = \text{constant}\}$. In other words, $\mathcal{P}$ is spanned locally by the vector fields $\{\partial/\partial\mathfrak{p}_1, \ldots, \partial/\partial\mathfrak{p}_k\}$, and the local canonical coordinates with this property are said to be **adapted to** $\mathcal{P}$.

Remark: (R4) Let (M,ω) be a 2k-dimensional symplectic manifold, $\mathcal{P}$ be a reducible polarization of (M,ω) and let $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_k, \mathfrak{q}_1, \ldots, \mathfrak{q}_k\}$ be a set of local canonical coordinates adapted to $\mathcal{P}$. Then the space of leaves $M/\mathcal{P}$ is a Hausdorff manifold which we shall denote by Q. Clearly Q is locally coordinatized by the set of coordinates $\{\mathfrak{q}_1, \ldots, \mathfrak{q}_k\}$. In future, the manifold Q shall be referred to as **the effective configuration space with respect to the polarization** $\mathcal{P}$.

(1.1.5.D3) Definition [Woodhouse (1980), p153]

Let $(B,(\cdot,\cdot),\nabla)$ be the prequantization bundle over a symplectic manifold (M,ω) and let $\mathcal{P}$ be a reducible polarization of (M,ω). We call a section $s \in C^{\infty}_{B}(M)$ that satisfies the condition

$$\nabla_X s = 0 \text{ for all } X \in V(M;\mathcal{P}), \qquad (1.1.5.\text{Eq } 9a)$$

a **polarized section of B.**

Remark: (R5) Suppose s is a polarized section (with respect to the polarization $\mathcal{P}$). Then the function (s,s) is constant along the leaves of the polarization $\mathcal{P}$ because we have

$$X(s,s) = (\nabla_X s,s) + (s,\nabla_X s) = 0, \text{ for all } X \in V(M;\mathcal{P}); \qquad (1.1.5.\text{Eq } 9b)$$

by the compatibility condition of the Hermitian structure $(\cdot,\cdot)$ and the connection ∇ [cf. Appendix 1.1].

(1.1.5.Eq 2) Example

Let $Q \subseteq \mathbb{R}$ be the configuration space with cartesian coordinate q, $M = T^{*}Q$ be the phase space with the usual cartesian canonical coordinates (p,q) and let ω be the canonical two-form on M. Let $B = M \times \mathbb{C}$, $(\cdot,\cdot)$ be the (natural) Hermitian structure on B, s_o be a unit section of B and let ∇ be the connection on B defined by $\nabla_X s_o = -i(X \lrcorner\, pdq)s_o$ for all $X \in V_{\mathbb{C}}(M)$. Let $(B,(\cdot,\cdot),\nabla)$ be the chosen prequantization bundle over (M,ω).

Let P be the vertical polarization; then P is spanned by the vector field $\partial/\partial p$. A typical polarized section of B (with respect to the polarization P) is of the form

$$s = \Psi(q)s_o,\ \Psi(q) \in C^{\infty}_{\mathbb{C}}(Q). \qquad (1.1.5.\text{Eq } 10)$$

(Note that we have used the fact that $\nabla_X s_o = 0$ for all $X \in V(M;P)$.)

Let P_c be the horizontal polarization; then P_c is spanned by the vector field $\partial/\partial q$. Let Q_c be the effective configuration space with respect to the polarization P_c. Then a typical polarized section of B (with respect to the polarization P_c) is of the form

$$\mathcal{S}_c = \phi_c(p)\{\exp \tfrac{i}{\hbar} pq\}s_o,\ \phi_c(p) \in C^\infty_{\mathbb{C}}(Q_c). \qquad (1.1.5.\text{Eq } 11)$$

(1.1.6) Half-densities

We shall split this section in two parts: in the first part we shall study what is referred to as r-D-densities (where D is a distribution on an arbitrary manifold), and in the second part we shall restrict ourselves to the study of so called half-densities.

(1.1.6.D1) Definitions [Woodhouse (1980), pp.150-154]

Let D be a distribution on an arbitrary l-dimensional manifold $\mathcal{M}$ and let $r \in \mathbb{R}$.

(1) For a given point m in $\mathcal{M}$ a **r-density on D_m** is a map ν_m that assigns to each basis $\{X_i\}_m$ in D_m a complex number $\nu_m\{X_i\}_m$ with the property that

$$\nu_m\{C^i{}_j X_i\}_m = |\det C|^r_m\ \nu_m\{X_j\}_m \qquad (1.1.6.\text{Eq } 1)$$

where the matrix $C = (C^i{}_j) \in GL(l,\mathbb{R})$, and the summation over the repeated index i is implied.

(2) The set of r-densities on D_m is a one-dimensional complex vector space to be denoted by $\Delta_r(D_m)$.

(3) A **r-D-density on $\mathcal{M}$** is a smooth section of the bundle

$$\Delta_r(D) = \bigcup_{m\in\mathcal{M}} \Delta_r(D_m). \qquad (1.1.6.\text{Eq } 2)$$

In other words, a r-D-density is a map ν which assigns an element ν_m of $\Delta_r(D_m)$ to each point $m \in \mathcal{M}$.

(1.1.6.Ex 1) <u>Example: The relationship of r-D-densities with volume forms</u>

Let $\mathcal{M}$ be an orientable manifold of dimension l and let ε be a volume form on $\mathcal{M}$. Let D be the distribution on $\mathcal{M}$ given by $D_m = T_m\mathcal{M}$ for each $m \in \mathcal{M}$ and let $\{Y_i\}$ be a field of bases for D.

Let $|\varepsilon|^r$ be the r-T$\mathcal{M}$-density defined by

$$|\varepsilon|^r_m\{Y_i\}_m = |l!\,\varepsilon(Y_1,\ldots,Y_l)|^r_m \qquad (1.1.6.\text{Eq } 3)$$

at each point $m \in \mathcal{M}$.

In general, there is no natural volume form on a orientable manifold except in certain special cases. Here are two of the special cases:

(i) A symplectic manifold (M,ω) is orientable with the Liouville volume form ε_ω [cf. Appendix 1.1], so $|\varepsilon_\omega|^r$ is generally chosen to be the natural r-TM-density on M.

(ii) A Riemmanian manifold Q with local coordinates $\{q_1,\ldots,q_k\}$ and metric (g_{ij}) has a natural volume form ε_g defined locally by

$$\varepsilon_g = g^{1/2}dq_1\wedge\ldots\wedge dq_k;\; g = \det(g_{ij}) \qquad (1.1.6.\text{Eq } 4)$$

[cf. Abraham and Marsden (1978), p152]. Therefore, $|\varepsilon_g|^r$ is generally chosen to be a natural r-TQ-density.

(1.1.6.Ex 2) <u>Example: The construction of -(1/2)-$\mathcal{P}$-densities</u> [Woodhouse (1980), pp151-152 and pp157-158]

The main purpose of this section is to introduce -(1/2)-$\mathcal{P}$-densities, so using the results given in the previous example we shall now show how -(1/2)-$\mathcal{P}$-densities may be constructed.

Let $\mathcal{P}$ be a reducible polarization of an arbitrary symplectic manifold (M,ω) of dimension 2k. Let Q be the effective configuration space with respect to the polarization $\mathcal{P}$, and let $pr:M \dashrightarrow Q$ be the usual projection map.

Then for each $m \in M$, we have

$$T_qQ = T_mM/\mathcal{P}_m;\ pr(m) = q. \qquad (1.1.6.\text{Eq } 5)$$

Let $\{X_i\}$ be a field of bases for $\mathcal{P}$ and let $\{Y_j\}$ be a field of bases for TQ; then by equation (1.1.6.Eq 5), we can take $\{X_i, Y_j\}$ as a field of bases for TM. Let μ be a -(1/2)-TM-density and let ρ be a +(1/2)-TQ-density. Let ν be a function of the field $\{X_1\}$ of bases for $\mathcal{P}$ defined by

$$\nu_m\{X_1\}_m = \mu_m\{X_i, Y_j\}_m\ \rho_q\{Y_\alpha\}_q\ ;\ pr(m) = q. \qquad (1.1.6.\text{Eq } 6)$$

We shall show that ν is a -(1/2)-$\mathcal{P}$-density as follows. Let $C = (C^i{}_j) \in GL(k,\mathbb{R})$. Let us replace $\{X_i\}$ by $\{C^i{}_j X_i\}$; then we need to replace $\{X_i, Y_j\}$ in equation (1.1.6.Eq 6) by $\{C^i{}_j X_i, Y_\beta\}$. Then

$$\begin{aligned}\nu_m\{C^i{}_j X_i\}_m &= \mu_m\{C^i{}_j X_i, Y_\beta\}_m\ \rho_q\{Y_\alpha\} \\ &= |\det C|^{-1/2}\mu_m\{X_j, Y_\beta\}_m\ \rho_q\{Y_\alpha\}_q. \qquad (1.1.6.\text{Eq } 7)\end{aligned}$$

Therefore, ν is a -(1/2)-$\mathcal{P}$-density [cf. equation (1.1.6.Eq 1)].

A -(1/2)-$\mathcal{P}$-density is commonly referred to as a **half-density.** We shall restrict ourselves to the study of half-densities for the rest of this section.

(1.1.6.D2) <u>Definitions</u> [Woodhouse (1980), pp154-155]

Let $\mathcal{P}$ be a polarization on an arbitrary symplectic manifold (M,ω).

(1) Let $\zeta \in C^\infty(M;\mathcal{P},1)$ such that X_ζ is a complete vector field, and let $\{F_t:M \dashrightarrow M \mid t \in \mathbb{R}\}$ be a one-parameter group of diffeomorphism generated by X_ζ. Let $\{X_i\}$ be a field of bases for $\mathcal{P}$; then $\{F_{t*}X_i\}$ is a field of bases

for $\mathcal{P}$ too [cf. the equation (1.1.5.Eq 2b)]. Let ν be a -(1/2)-$\mathcal{P}$-density. Then the **pull-back of ν (with respect to the diffeomorphism F_t)** is the -(1/2)-$\mathcal{P}$-density defined by

$$(F_t^*\nu)\{X_i\} = \nu\{F_{t*}X_i\}. \qquad (1.1.6.\text{Eq } 8)$$

The Lie derivative of ν along X_ζ is the -(1/2)-$\mathcal{P}$-density $L_{X_\zeta}\nu$ defined by

$$L_{X_\zeta}\nu = (d/dt)(F_t^*\nu)\big|_{t=0} \qquad (1.1.6.\text{Eq } 9)$$

(Note that the definition of the Lie derivative can trivially be extended to the case where X_ζ is incomplete.)

(2) Let ν be a smooth section of $\Delta_{-1/2}(\mathcal{P})$, and let $Y \in V(M;\mathcal{P})$. Then we can define a section $\nabla_Y\nu$ of $\Delta_{-1/2}(\mathcal{P})$ by

$$(\nabla_Y\nu)\{X_i\} = Y(\nu\{X_i\}) \qquad (1.1.6.\text{Eq } 10a)$$

where $\{X_i\}$ is any field of bases for $\mathcal{P}$ satisfying

$$\nabla_Y X_i = 0 \qquad (1.1.6.\text{Eq } 10b)$$

[cf. definition (1.1.5.D2), part (2)]. The section $\nabla_Y\nu$ is referred to as **the covariant derivative of the section ν along Y.**

<u>Remarks</u>: (R1) Note that the Lie derivative is only defined for a restricted class of vector fields. The covariant derivative too is only defined for a restricted class of vector fields.

(R2) We shall give a simple example to show that the condition given by equation (1.1.6.Eq 10b) is necessary to ensure that $\nabla_Y\nu$ is a -(1/2)-$\mathcal{P}$-density. Let $Q \subseteq \mathbb{R}$, $M = T^*Q$ and let ω be the canonical two-form on M. Let P be the vertical polarization of (M,ω), ν be a smooth section of $\Delta_{-1/2}(P)$, $\{X\}$ be a field of bases for P and let $Y \in V(M;P)$. Clearly X is of the form $X = \zeta\,\partial/\partial p$ where $\zeta \in C^\infty(M)$ and $\zeta \neq 0$. By equation (1.1.5.Eq 5), $\nabla_Y X = Y(\zeta)(\partial/\partial p)$. Thus, $\nu\{X\} = |\zeta|^{-1/2}\nu\{\partial/\partial p\}$. Therefore,

$$Y(\nu\{X\}) = |\zeta|^{-1/2}\, Y(\nu\{\partial/\partial p\}) + Y(|\zeta|^{-1/2})\nu\{\partial/\partial p\}. \qquad (1.1.6.\text{Eq } 11)$$

Clearly $Y(\nu\{X\})$ transforms like a -(1/2)-P-density if $\nabla_Y X = 0$ because then we have $Y(|\zeta|^{-1/2}) = 0$.

(1.1.6.D3) Definition [Woodhouse (1980), p153]

Let $\mathcal{P}$ be a reducible polarization of an arbitrary symplectic manifold (M,ω). Then the smooth section ν of the bundle $\Delta_{-1/2}(\mathcal{P})$ is said to be **covariantly constant along** $\mathcal{P}$ if it satisfies the condition

$$\nabla_X \nu = 0 \text{ for all } X \in V(M;\mathcal{P}). \qquad (1.1.6.\text{Eq } 12)$$

(1.1.6.Ex 3) Example: Half-densities of the vertical and horizontal polarizations of the cotangent bundle of a Riemmanian manifold

Let Q be a 2k-dimensional Riemmanian manifold with global coordinates $q = \{q_1,\ldots,q_k\}$ and metric (g_{ij}). Let $g = \det(g_{ij})$; then the natural volume form on Q is given explicitly by $\varepsilon_g = g^{1/2}\, dq_1\wedge\ldots\wedge dq_k$ [cf. equation (1.1.6.Eq 4)]. Let $M = T^*Q$, ω be the canonical two-form on M and let $(\mathbf{p},\mathbf{q}) = \{p_1,\ldots,p_k,q_1,\ldots,q_k\}$ be the usual global canonical coordinates on M. Then the Liouville volume form is given explicitly by $\varepsilon_\omega = dp_1\wedge\ldots\wedge dp_k\wedge dq_1\wedge\ldots\wedge dq_k$. Let P be the vertical polarization and let P_c be the horizontal polarization.

We shall now split the construction of half-densities into two parts which we shall denote by (i) and (ii), respectively, as follows: in part (i) we shall construct a (natural) -(1/2)-P-density that is covariantly constant along P, and in part(ii) we shall construct a -(1/2)-P_c-density that is covariantly constant along P_c.

(i) The space of leaves of P, M/P, is identifiable with Q (the actual configuration). Let $\{X_i\}$ be a field of bases for P and let $pr: M \dashrightarrow Q$ be the usual projection map.

Then a (natural) -(1/2)-P-density ν is defined by

$$\nu\{X_i\} = |\varepsilon_\omega|^{-1/2}\{X_i, \partial/\partial q_j\}\ |\varepsilon_g|^{1/2}\{\partial/\partial q_\alpha\}$$
$$= g^{1/4}\ |dp_1\wedge\ldots\wedge dp_k|^{-1/2}\{X_i\}. \qquad (1.1.6.\text{Eq } 13)$$

(We have referred to ν as natural because we constructed it using the natural +(1/2)-TQ-density $|\varepsilon_g|^{1/2}$ on Q.)

The section ν is covariantly constant along P because

$$\nabla_X\nu\{\partial/\partial p_i\} = X(g^{1/4}) = 0,\ \text{for all } X \in V(M;P) \qquad (1.1.6.\text{Eq } 14)$$

Let $\zeta \in C^\infty(M;P,1)$; then we have

$$\zeta = \sum_{i=1}^{k}\xi_i(q)p_i + \eta(q),\ \xi_i, \eta \in C^\infty(Q); \qquad (1.1.6.\text{Eq } 15a)$$
$$X_\zeta = \sum_{i=1}^{k}[\xi_i\partial/\partial q_i - \{\sum_{j=1}^{k}(\partial\xi_j/\partial q_i)p_j - \partial\eta/\partial q_i\}\partial/\partial p_i] \qquad (1.1.6.\text{Eq } 15b)$$
$$pr_*X_\zeta = \sum_{i=1}^{k}\xi_i\partial/\partial q_i. \qquad (1.1.6.\text{Eq } 15c)$$

Let $\mathrm{div}_{\varepsilon_g}(pr_*X_\zeta)$ be the divergence of the vector field pr_*X_ζ (on Q) with respect to the volume form ε_g [cf. Abraham and Marsden (1978), p152].The Lie derivative of ν along X_ζ is the section $L_{X_\zeta}\nu$ of $\Delta_{-1/2}(P)$ defined

$$L_{X_\zeta}\nu = (1/2)\mathrm{div}_{\varepsilon_g}(pr_*X_\zeta)\nu$$
$$= (1/2)g^{-1/2}[\sum_{i=1}^{k}\partial(g^{1/2}\xi_i)/\partial q_i]\nu \qquad (1.1.6.\text{Eq } 16)$$

[cf. McKenna (1982), p112; Woodhouse (1980), p158].

(ii) Let Q_c be the effective configuration space with respect to the polarization P_c and let $pr_c: M \dashrightarrow Q_c$ be the usual projection map. Then $p = \{p_1,\ldots,p_k\}$ are global coordinates on Q_c. There is no natural volume form on Q_c, so we choose the standard volume form (with respect to the coordinates $\{p_1,\ldots,p_k\}$) defined by

$$\varepsilon_c = dp_1\wedge\ldots\wedge dp_k. \qquad (1.1.6.\text{Eq } 17)$$

Let $\{Y_i^c\}$ be a field of bases for P_c.

Then let ν_c be the $-(1/2)$-P_c-density defined by

$$\nu_c\{Y_i^c\} = |\varepsilon_\omega|^{-1/2}\{Y_i^c, \partial/\partial p_j\}\ |\varepsilon|^{1/2}\{\partial/\partial p_\alpha\}$$
$$= |dq_1 \wedge \ldots \wedge dq_k|^{-1/2}\{Y_i^c\} \qquad (1.1.6.\text{Eq } 18)$$

[cf. equation (1.1.6.Eq 6)].

The section ν_c is covariantly constant along P_c because

$$\nabla_X \nu_c\{\partial/\partial q_i\} = X(1) = 0, \text{ for all } X \in V(M;P_c). \qquad (1.1.6.\text{Eq } 19)$$

Let $\zeta_c \in C\,(M;P_c,1)$; then we have

$$\zeta_c = \sum_{i=1}^{k} \xi_{ci}(p)q_i + \eta_c(p) \text{ where } \xi_{ci}, \eta_c \in C^\infty(Q_c); \qquad (1.1.6.\text{Eq } 20a)$$
$$X_{\zeta_c} = \sum_{i=1}^{k} [\{\sum_{j=1}^{k} (\partial\xi_{ci}/\partial p_j)q_i + \partial\eta_c/\partial p_i\}\partial/\partial q_i - \xi_{ci}\partial/\partial p_i]; \qquad (1.1.6.\text{Eq } 20b)$$
$$pr_{c*}X_{\zeta_c} = -\sum_{i=1}^{k} \xi_{ci}\,\partial/\partial p_i. \qquad (1.1.6.\text{Eq } 20c)$$

Let $div_{\varepsilon_c}(pr_{c*}X_{\zeta_c})$ be the divergence of the vector field $pr_{c*}X_{\zeta_c}$ (on Q_c) with respect to the volume form ε_c.

The Lie derivative of ν_c along the Hamiltonian vector field X_{ζ_c} is the $-(1/2)$-P_c-density defined by

$$L_{X_{\zeta_c}}\nu_c = (1/2)div_{\varepsilon_c}(pr_{c*}X_{\zeta})$$
$$= -(1/2)[\sum_{i=1}^{k}(\partial\xi_{ci}/\partial p_i)] \qquad (1.1.6.\text{Eq } 21)$$

[cf. Woodhouse (1980), p158].

(1.1.7) The half-density quantization scheme [Woodhouse (1980), pp156-157]

The half-density quantization scheme provides a physically reasonable solution to the Dirac problem. The main limitation of the scheme is that it only allows us to quantize a restricted class of observables.

The half-density quantization scheme consists of the following three steps:

(HDQ1) Let $\mathcal{P}$ be a reducible polarization of a 2k-dimensional symplectic manifold (M,ω) , Q be the effective configuration space with respect to the polarization and let pr:M ---> Q be the usual projection map. We first choose a prequantization bundle $(B,(\cdot,\cdot),\nabla)$ over (M,ω).

(HDQ2) A smooth section $\Psi = s\nu$ of the bundle B x $\Delta_{-1/2}(\mathcal{P})$ which satisfies

$$\nabla_X s = 0; \nabla_X \nu = 0 \text{ for all } X \in V(M;\mathcal{P}) \qquad (1.1.7.\text{Eq } 1)$$

is called a **$\mathcal{P}$-wave function.**

Let $\Psi = s\nu$ be a $\mathcal{P}$-wave function, $\{X_i\}$ be a field of bases for $\mathcal{P}$ on M and let $\{Y_j\}$ be a field of bases for TQ on Q. Then let (Ψ,Ψ) be the 1-TQ-density on Q defined by

$$(\Psi,\Psi)_q\{Y_i\}_q = (s,s)_m \nu_m\{X_j\}_m \overline{\nu}_m\{X_\alpha\} \; |2k!\, \varepsilon_\omega(X_1,\ldots,X_k,Y_1,\ldots,Y_k)|_m \qquad (1.1.7.\text{Eq } 2a)$$

where q = pr(m).

Let $\mathbf{W}_{\mathcal{P}}$ denote the space of square-integrable $\mathcal{P}$-wave functions with respect to the inner-product

$$\langle\Psi,\Psi\rangle_{\mathcal{P}} = [2\pi\hbar]^{-k/2}\int_Q (\Psi,\Psi) \qquad (1.1.7.\text{Eq } 2b)$$

[cf. Appendix 1.2 for the integration of a 1-TQ-density over Q]. We call $W_{\mathcal{P}}$ the **quantization pre-Hilbert space.** The **quantization Hilbert space** $H_{\mathcal{P}}$ is defined to be the completion of $W_{\mathcal{P}}$.

(HDQ3) Let $\zeta \in C^{\infty}(M;\mathcal{P},1)$; then we shall call $pr_*(X_\zeta)$ the **associated vector field generated by ζ on Q.**

Let $\zeta \in C^{\infty}(M;\mathcal{P},1)$ be a classical observable such that the associated vector field $pr_*(X_\zeta)$ is complete; then ζ determines a self-adjoint operator on $H_{\mathcal{P}}$ given formally by the expression

$$\tilde{\zeta}\Psi = -i\hbar[\nabla_{X_\zeta}\mathcal{S}+\zeta\mathcal{S}]\nu - i\hbar\mathcal{S}(L_{X_\zeta}\nu);\ \Psi = \mathcal{S}\nu \qquad (1.1.7.\mathrm{Eq}\ 3)$$

(The domain of $\tilde{\zeta}$ is yet to be determined; Wan and McFarlane (1983) have given the domain for the special case where $\mathcal{P}$ is the vertical polarization P of the cotangent bundle of a Riemmanian manifold.)
The operator $\tilde{\zeta}$ is called the **quantization operator (corresponding to the classical observable** ζ**).**

(1.1.7.Ex 1) <u>Example: Quantization in the vertical and horizontal polarizations of the cotangent bundle of a Riemmanian manifold</u>

Let Q be a 2k-dimensional Riemmanian manifold with global coordinates $\mathbf{q} = \{q_1,..,q_K\}$ and metric (g_{ij}). Let $M = T^*Q$ with canonical two-form ω and the usual global canonical coordinates $(\mathbf{p},\mathbf{q}) = \{p_1,..,p_K,q_1,..,q_K\}$. Let $\beta_o = \sum_{i=1}^{K} p_i dq_i$ (the canonical one-form), $B = M \times \mathbb{C}$ be the trivial line-bundle over M, $(\cdot,\cdot)$ be the usual Hermitian structure on B and let s_o be a unit section of B. Let ∇ be the connection on B defined by $\nabla_X s_o = -i(X \lrcorner \beta_o)s_o$ for all $X \in V_{\mathbb{C}}(M)$.

We shall split our presentation in two parts which we shall denote by (i) and (ii), respectively, as follows: in part (i) we shall quantize observables in the vertical polarization, and in part (ii) we shall quantize observables in the horizontal polarization.

(i) The effective configuration space with respect to the vertical polarization P is Q; let $pr: M \dashrightarrow Q$ be the usual projection map.

The **quantization pre-Hilbert space** W_P consists of square-integrable sections of $B \times \Delta_{-1/2}(P)$ of the form

$$\Psi = \psi(\mathbf{q}) s_o \nu; \qquad (1.1.7.\text{Eq } 4a)$$

where

$$\psi(\mathbf{q}) \in C^{\infty}(Q) \text{ and } \nu = g^{1/4} |dp_1 \wedge \ldots \wedge dp_K|^{-1/2} \qquad (1.1.7.\text{Eq } 4b)$$

[cf. equation (1.1.6.Eq 13)]. The inner-product on W_P is given by

$$\langle \Psi, \Psi \rangle = [2\pi\hbar]^{-k/2} \int_Q |\psi(\mathbf{q})|^2 g^{1/2} \, dq_1 \ldots dq_k. \qquad (1.1.7.\text{Eq } 5)$$

The **quantization Hilbert space** H_P is the completion of W_P.

An observable ζ in $C^{\infty}(M;P,1)$ is of the form $\zeta = \sum_{j=1}^{k} \xi_j(\mathbf{q}) p_j + \eta(\mathbf{q})$ where $\xi_j, \eta \in C^{\infty}(Q)$. If the associated vector field $pr_*(X_\zeta)$ is complete, then is quantizable and the **quantization operator** $\tilde{\zeta}$ is given formally by the expression

$$\tilde{\zeta}\Psi = -i\hbar [\sum_{j=1}^{k} \{\xi_j(\mathbf{q}) \ (\partial/\partial q_j) + (1/2) g^{-1/2} \partial(g^{1/2} \xi_j(\mathbf{q}))/\partial q_j\} \psi(\mathbf{q}) + \eta(\mathbf{q}) \psi(\mathbf{q})] s_o \nu; \qquad (1.1.7.\text{Eq } 6a)$$

[cf. Equations (1.1.6.Eq 16) and (1.1.7.Eq 3)]. The quantization operator $\tilde{\zeta}$ is self-adjoint with domain

$$D_{\tilde{\zeta}} = \{\Psi \in \mathbf{H} : \psi(\mathbf{q}) \in AC(Y_\zeta, Q) \text{ where } pr_*(X_\zeta) = Y_\zeta \text{ and } \tilde{\zeta}\Psi \in H_P\} \qquad (1.1.7.\text{Eq } 6b)$$

[cf. Wan and McFarlane (1983)]. Here $AC(Y_\zeta, Q)$ denotes the set of absolutely continuous functions on Q with respect to the vector field Y_ζ: $AC(Y_\zeta, Q)$ is the space of functions on Q that are differentiable with respect to Y_ζ almost everywhere.

In the case where $\chi = \eta(\mathbf{q})$ the **quantization operator** $\tilde{\chi}$ is the multiplication operator

$$\tilde{\chi}\Psi = \eta(\mathbf{q})\Psi; \qquad (1.1.7.\text{Eq } 7a)$$

and the domain of $\tilde{\chi}$ is

$$D_{\tilde{\chi}} = \{\Psi \in H_P : \eta(\mathbf{q})\Psi \in H_P\}. \qquad (1.1.7.\text{Eq } 7b)$$

(ii) Let Q_c be the effective configuration space with respect to the horizontal polarization P_c and let $pr_c : M \dashrightarrow Q_c$ be the usual projection map.

The **quantization pre-Hilbert space** W_{P_c} consists of square-integrable sections of $B \times \Delta_{-1/2}(P_c)$ of the form

$$\Phi_c = \phi_c(p) s_c \nu_c \qquad (1.1.7.\text{Eq } 8a)$$

where

$$\phi_c(p) \in C^\infty(Q_c), \quad s_c = \{\exp \pm \sum_{i=1}^{k} p_i q_i\} s_o \text{ and } \nu_c = |dq_1 \wedge \ldots \wedge dq_k|^{-1/2}. \qquad (1.1.7.\text{Eq } 8b)$$

The inner product on W_{P_c} is given by

$$\langle \Phi_c, \Phi_c \rangle_{P_c} = [2\pi\hbar]^{-k/2} \int_{Q_c} |\phi_c(p)|^2 dp_1 \ldots dp_k. \qquad (1.1.7.\text{Eq } 9)$$

The **quantization Hilbert space** H_{P_c} is the completion of W_{P_c}.

An observable in $C^\infty(M; P_c, 1)$ is of the form $\zeta_c = \sum_{i=1}^{k} \xi_{ci}(p) q_i + \eta_c(p)$ where $\xi_{ci}(p), \eta_c(p) \in C^\infty(Q_c)$. If the associated vector field $pr_{c*}(X_\zeta)$ is complete, then ζ is quantizable, and the corresponding **quantization operator** $\tilde{\zeta}_c$ is given formally by the expression

$$\tilde{\zeta}_c \Phi_c = i\hbar [\sum_{i=1}^{k} \{\xi_{ci}(p)\ (\partial/\partial p_i) + (1/2)\partial(\xi_{ci}(p))/\partial p_i\} \phi_c(p) + \eta_c(p)\, \phi_c(p)] s_c \nu_c; \qquad (1.1.7.\text{Eq } 10a)$$

[cf. equations (1.1.6.Eq 17) and (1.1.7.Eq 3)]. $\tilde{\zeta}_c$ is self-adjoint with domain

$$D_{\tilde{\zeta}_c} = \{\Phi_c \in H_{P_c} : \phi_c(p) \in AC(Y^c_{\zeta_c}, Q_c) \text{ where } Y^c_{\zeta_c} = pr_{c*}(X_{\zeta_c}) \text{ and } \tilde{\zeta}_c \Phi_c \in H_{P_c}\} \qquad (1.1.7.\text{Eq } 10b)$$

In the case where $\chi_c = \eta_c(p)$ the **quantization operator** is the multiplication operator

$$\tilde{\chi}_c \Phi_c = \eta_c(p) \Phi_c, \qquad (1.1.7.\text{Eq } 11a)$$

and the domain of $\tilde{\chi}_c$ is

$$D_{\tilde{\chi}_c} = \{\Phi_c \in H_{P_c} : \eta_c(p) \Phi_c \in H_{P_c}\}. \qquad (1.1.7.\text{Eq } 11b)$$

1.2 THE BOHR-WILSON-SOMMERFELD CONDITIONS IN THE HALF-DENSITY QUANTIZATION SCHEME

(1.2.1) Introduction

Historically, the Bohr-Wilson-Sommerfeld (BWS) conditions lay at the foundation of the old quantum theory; but after the formulation of quantum mechanics they became less important to physicists. However, they still remain of interest in problems that are intractable by the usual quantum mechanical methods; but which can be solved by the methods of Hamiltonian mechanics. In this section we shall derive BWS-like conditions in the framework of the half-density quantization scheme. We shall refer to these conditions as **BWS conditions** too, despite of the fact that they differ in several respects from the BWS conditions in the old quantum theory and the (corrected) BWS conditions in quantum mechanics. Our presentation is partially based on the work by Sniatycki and Toporowski (1977); Sniatycki (1980), [cf. pp8-9, pp71-76 and pp149-156] and Woodhouse (1980), [cf. pp185-187].

(1.2.2) Polarizations with compact leaves spanned by the Hamiltonian vector fields generated by a complete set of commuting observables

Let $\mathbf{q} = \{q_1,\ldots,q_k\}$ be a set of cartesian coordinates on $\mathbb{R}^k$, $\{\mathbf{p},\mathbf{q}\} = \{p_1,\ldots,p_k,q_1,\ldots,q_k\}$ be the usual cartesian canonical coordinates on $T^*\mathbb{R}^k$ and let ω be the canonical two-form on $T^*\mathbb{R}^k$. Let M be an contractible open subset of $T^*\mathbb{R}^k$; then (M,ω) is a 2k-dimensional symplectic manifold. (Here ω is taken to be a two-form on M.)

(1.2.2.D1) Definition

Let $\{\zeta_i\} = \{\zeta_1, \ldots, \zeta_k\}$ be a set k-observables on (M,ω). Then we shall call $\{\zeta_i\}$ a **complete set of commuting classical observables** if the following three conditions are satisfied:

(CCO1) The set

$$Z(\{\zeta_i\}) = \{m \in M:\ X_{\zeta_1}, \ldots, X_{\zeta_k} \text{ are linearly dependant}\} \qquad (1.2.2.\text{Eq } 1a)$$

has Lebesgue measure zero. Note that in the case where k = 1, we put

$$Z(\zeta_1) = \{m \in M:\ X_{\zeta_1} = 0\}. \qquad (1.2.2.\text{Eq } 1b)$$

(CCO2) Let $M_o = M - Z(\{\zeta_i\})$ and let ω_o be the restriction of the canonical two-form ω to M_o. Let $\mathcal{P}_c$ be the k-dimensional distribution on M_o spanned by the set of k linearly independent vector fields $\{X_{\zeta_1}, \ldots, X_{\zeta_k}\}$. The distribution $\mathcal{P}_c$ is a polarization.

(CCO3) The polarization $\mathcal{P}_c$ is reducible. Let Q_c denote the effective configuration space with respect to the polarization $\mathcal{P}_c$.

Remark: (R1) Condition (CCO2) implies that

$$\{\zeta_i, \zeta_j\} = 2\omega(X_{\zeta_i}, X_{\zeta_j}) = 0 \qquad (1.2.2.\text{Eq } 2)$$

[cf. equation (1.1.2.Eq 2a)]. Hence $\{\zeta_i\}$ constitutes a commuting set of observables with respect to the Poisson bracket.

In this section we shall be interested in a complete set of commuting observables $\{\zeta_i\}$ with the following property: the integral curves of the Hamiltonian vector fields $X_{\zeta_1}, \ldots, X_{\zeta_k}$ are closed in M_o. So in what follows we shall assume that the complete set of commuting observables $\{\zeta_i\}$ on (M_o, ω_o) also satisfy the following two additional conditions:

(CCO4) Let $R(\zeta_i)$ denote the range of the observable ζ_i and let $\{a_i\} = \{a_1, \ldots, a_k\}$ be a set of values in $R(\zeta_1) x \ldots x R(\zeta_k)$. Let $\Lambda(\{a_i\})$ be the submanifold in M_o defined by

$$\Lambda(\{a_i\}) = \{m \in M_o:\ \zeta_1(m) = a_1, \ldots, \zeta_k(m) = a_k\}. \quad (1.2.2.\text{Eq } 3)$$

Then for each $\{a_i\} \in R(a_1)x\ldots xR(a_k)$, the corresponding submanifold $\Lambda(\{a_i\})$ is diffeomorphic to the k-torus T^k.

(The submanifold $\Lambda(\{a_i\})$ is a leaf of the polarization $\mathcal{P}_c$; therefore, $\mathcal{P}_c$ is sometimes referred to either as a **polarization with compact leaves** or, as a **polarization with toroidal leaves.**)

(CCO5) Let $a \in R(\zeta_i)$ and let $\Omega_i(a)$ be the (2k-1)-dimensional submanifold in M_o defined by

$$\Omega_i(a) = \{m \in M_o: \zeta_i(m) = a\}; \qquad (1.2.2.\text{Eq } 4)$$

$\Omega_i(a)$ is called a **surface of constant value a generated by the observable** ζ_i. Then for each $a \in R(\zeta_i)$, the submanifold $\Omega_i(a)$ is connected.

Remarks: (R2) Let the map $\gamma_i : \mathbb{R} \dashrightarrow M_o$ given by $t_i \dashrightarrow m = \gamma_i(t_i)$ be the integral curve of the Hamiltonian vector field X_{ζ_i} that has originated from the point $m_{i_o} = \gamma_i(0)$. Then condition (CCO 4) implies that the curve γ_i has the topology of a circle.

(R3) Let β be any one-form on M_o that satisfies the condition $\omega_o = d\beta$. Then condition (CCO 5) implies that for each $i \in \{1,\ldots,k\}$, the closed integral

$$\oint_{\gamma_i} \beta \qquad (1.2.2.\text{Eq } 5)$$

over any integral curve γ_i on the submanifold $\Omega_i(a)$ has the same value [cf. Arnold (1978), p.283; Guillemin and Sternberg (1977), pp167-168]. In other words, the integral given by equation (1.2.2.Eq 5) depends only on the value of the observable ζ_i on $\Omega_i(a)$; so the integral can be treated as a function of ζ_i.

(R4) Finally, condition (CCO 5) implies that for each $i \in \{1,\ldots,k\}$ and $a \in R(\zeta_i)$, the period of the integral curves of the Hamiltonian vector field

X_{ζ_i} on which $\zeta_i = a_i$ are the same [cf. Guillemin and Sternberg (1977), pp167-168]. Therefore, we denote the period of the integral curves of X_{ζ_i} on which $\zeta_i = a$ by $T_i(a)$.

(1.2.3) <u>The construction of action-angle variables on (M_o, ω_o)</u>

[cf. Arnold (1978), pp279-285; Abraham and Marsden (1980), pp397-400]

We shall now coordinatize M_o by constructing action-angle variables in six steps as follows.

(AAV1) Let $\beta_o = \sum_{i=1}^{k} p_i dq_i$ be a one form on M_o;then we have $d\beta_o = \omega_o$. Then for each $\{a_i\} \in R(\zeta_1)x\ldots xR(\zeta_k)$, let $\gamma_1,\ldots,\gamma_k$ be the chosen set of integral curves on $\Lambda(\{a_i\})$ of the vector fields $X_{\zeta_1},\ldots,X_{\zeta_k}$ respectively.

The **action-angle variables** I_i are constants on the submanifold $\Lambda(\{a_i\})$ defined by

$$I_i = [2\pi]^{-1}\oint_{\gamma_i} \beta_o. \qquad (1.2.3.\text{Eq } 1)$$

For each $i \in \{1,\ldots,k\}$, let $R(I_i)$ be the range of the variable I_i. Let $\underline{I} = (I_1,\ldots,I_k)$ and let $R(\underline{I}) = R(I_1)x\ldots xR(I_k)$.

(<u>Remarks:</u> (R1) β_o should be confused with the canonical one-form on a cotangent bundle defined in example (1.1.2.Ex 1) (of section (1.1)).

(R2) It follows from remark (R3) of the last subsection that the value of each I_i on $\Lambda(\{a_i\})$ is independent of the set of integral curves $\gamma_1,\ldots,\gamma_k$ chosen. In fact for each $i \in \{1,\ldots,k\}$, the action-variable I_i is only only dependant on the value of the corresponding observable ζ_i. Similarly, each observable ζ_i is only dependant on the action variable I_i, i.e. $\zeta_i = \zeta_i(I_i)$. Hence each leaf of the polarization $\mathcal{P}_c$ is uniquely determined by the set of action variables $\underline{I}$, so we shall label the leaves by $\Lambda(\underline{I})$ instead of by

$\Lambda(\{a_i^\cdot\})$ from now on. The Hamiltonian vector fields $X_{I_1},\ldots,X_{I_k}$ also span the polarization $\mathcal{P}_c$.)

(AAV2) Let $F:M_o \dashrightarrow \mathbb{R}^k$ be the smooth function defined by

$$F(\mathbf{p},\mathbf{q}) = (F_1(\mathbf{p},\mathbf{q}),\ldots,F_k(\mathbf{p},\mathbf{q})) = \mathbf{I}, \qquad (1.2.3.\text{Eq } 3)$$

where the point $(\mathbf{p},\mathbf{q})$ lies on the leaf $\Lambda(\mathbf{I})$ and $F_i(\mathbf{p},\mathbf{q}) = I_i$.

Let $(\mathbf{p}_o,\mathbf{q}_o)$ be any point in M_o at which the (k x k)-matrix $(\partial F_i(\mathbf{p},\mathbf{q})/\partial p_j)$ is non-singular and let $\mathbf{I}_o = F(\mathbf{p}_o,\mathbf{q}_o)$. Then according to the Implicit function theorem [cf. Abraham and Marsden (1980), p.29] there are neighbourhoods $U_o \subset \mathbb{R}^k$ of $\mathbf{p}_o$, $V_o \subset \mathbb{R}^k$ of $\mathbf{q}_o$ where $U_o \times V_o \subset M_o$, and $W_o \subset R(\mathbf{I})$ of $\mathbf{I}_o$, and a unique smooth map

$$\mathbf{p} : W_o \times V_o \dashrightarrow U_o : (\mathbf{I},\mathbf{q}) \dashrightarrow \mathbf{p}(\mathbf{I},\mathbf{q}) = (p_1(\mathbf{I},\mathbf{q}),\ldots,p_k(\mathbf{I},\mathbf{q})) \qquad (1.2.3.\text{Eq } 4a)$$

such that

$$F(\mathbf{p}(\mathbf{I},\mathbf{q}),\mathbf{q}) = \mathbf{I}. \qquad (1.2.3.\text{Eq } 4b)$$

Hence $(\mathbf{I},\mathbf{q})$ can be considered as local coordinates on M_o; more precisely, they are coordinates on the open set

$$A_o = \{m \in M : m = (\mathbf{p}(\mathbf{I},\mathbf{q}),\mathbf{q}) \text{ for all } (\mathbf{I},\mathbf{q}) \in W_o \times U_o\}. \qquad (1.2.3.\text{Eq } 5)$$

(AAV3) Then for each $(\mathbf{I},\mathbf{q}) \in W_o \times V_o$, we define the function $S(\mathbf{I},\mathbf{q})$ on $\Lambda(\mathbf{I}) \cap A_o$ by

$$S(\mathbf{I},\mathbf{q}) = \int_{\mathbf{q}_o}^{\mathbf{q}} \sum_{i=1}^{k} p_i(\mathbf{I},\mathbf{q})\,dq_i; \qquad (1.2.3.\text{Eq } 6)$$

here the integration is just the usual integration of $f(q)\,d\mathbf{q}$ between the limits $\mathbf{q}_o$ and $\mathbf{q}$ [cf. Abraham and Marsden (1980), pp399-400; Arnold (1978), p284].

(AAV4) On the open submanifold A_o of M_o we define the **action variables** θ_i by

$$\theta_i = \partial S/\partial I_i \qquad (1.2.3.\text{Eq } 7)$$

Let $\boldsymbol{\theta} = (\theta_1,\ldots,\theta_k)$.

(AAV5) Then $(\underline{I},\underline{\theta}) = (I_1,\ldots,I_k,\theta_1,\ldots,\theta_k)$ are canonical coordinates on the open submanifold A_o, and they are referred to as **action-angle variables** [cf. Arnold (1978) , p.283]. Hence the symplectic two-form ω_o is $\sum_{i=1}^{k} dI_i \wedge d\theta_i$, and the Hamiltonian vector fields $X_{I_1},\ldots,X_{I_k}$ are respectively $\partial/\partial\theta_1,\ldots,\partial/\partial\theta_k$, on the submanifold A_o. In other words, the polarization $\mathcal{P}_c$ is spanned by the vector fields $\partial/\partial\theta_1,\ldots,\partial/\partial\theta_k$ on A_o.

(AAV6) Finally, we can coordinatize the symplectic manifold (M_o,ω_o) by constructing action-angle variables in the neighbourhood of each point in M_o.

Remarks: (R3) Let $(\underline{I},\underline{\theta'})$ and $(\underline{I},\underline{\theta''})$ be action-angle variables on the open submanifolds A'_o and A''_o respectively. Let $\mathbf{p}'(\underline{I},\mathbf{q})$ and $\mathbf{p}''(\underline{I},\mathbf{q})$ be the functions defined by the equations (1.2.3.Eq 4a) and (1.2.3.Eq 4b) on A'_o and A''_o respectively. Suppose $A'_o \cap A''_o \neq \phi$; then $\mathbf{p}'(\underline{I},\mathbf{q}) = \mathbf{p}''(\underline{I},\mathbf{q})$ on $A'_o \cap A''_o$. By using the latter result it can be shown that

$$d\theta'_i = d\theta''_i \text{ and } \partial/\partial\theta'_i = \partial/\partial\theta''_i \text{ on } A'_o \cap A''_o. \qquad (1.2.3.\text{Eq } 8)$$

Hence there exists k global one-forms on M_o which we shall denote by $d\theta_1,\ldots,d\theta_k$ such that if $(\underline{I},\underline{\theta'})$ are action-angle coordinates on some submanifold A'_o of M_o, then

$$d\theta_i = d\theta'_i \quad ,\ldots,d\theta_k = d\theta'_k \text{ on } A'_o. \qquad (1.2.3.\text{Eq } 9)$$

Similarly, we shall denote the k Hamiltonian vector fields $X_{I_1},\ldots,X_{I_k}$ by $\partial/\partial\theta_1,\ldots,\partial/\partial\theta_k$ respectively such that

$$\partial/\partial\theta_1 = \partial/\partial\theta'_1,\ldots,\partial/\partial\theta_k = \partial/\partial\theta'_k \text{ on } A'_o \qquad (1.2.3.\text{Eq } 10)$$

[cf. step (AAV5) in the construction of action-angle variables]. The vector fields $\partial/\partial\theta_1,\ldots,\partial/\partial\theta_k$ span the polarization $\mathcal{P}_c$ [cf. remark(R1) of this subsection].

It can be shown that

$$\oint_{\gamma_i} d\theta_j = 2\pi\delta_{ij} \qquad (1.2.3.\text{Eq } 11)$$

[cf. Abraham and Marsden (1980), p399; Arnold (1978), p284].

With these results in mind, we shall introduce the following notation:

(i) From now on let $(\underline{I},\underline{\theta}) = (I_1,\ldots,I_k,\theta_1,\ldots,\theta_k)$ denote a set of action-angle variables charts that cover M_o. In this notation let $\omega_o = \sum_{i=1}^{k} dI_i \wedge d\theta_i$.

(ii) We shall write formally

$$\oint_{\gamma_i} d\theta_i = 2\pi = \int_0^{2\pi} d\theta_i. \qquad (1.2.3.\text{Eq } 12)$$

(R4) Let $\gamma_i(t_i)$ be the integral curve of the vector field X_{ζ_i} that originates at the point $m_{io} = \gamma_i(0)$ and let $\zeta_i(m_{io}) = a_i$. Let $T_i(a_i)$ be the period of the integral curve $\gamma_i(t_i)$ [cf. remark (R4) of the last subsection]. Then it can be shown that

$$\theta_i = [2\pi/T_i(a_i)]t_i + \theta_{io}, \qquad (1.2.3.\text{Eq } 13)$$

where θ_{io} is a (real) constant [cf. Berry (1981), p9].

(R4) Let Q_c be the effective configuration space with respect to the polarization $\mathcal{P}_c$. Then Q_c is identifiable with $R(\underline{I})$ [cf. step (AAV1) for a definition of $R(\underline{I})$]; therefore, $\underline{I} = (I_1,\ldots,I_k)$ coordinatizes Q_c. The standard volume-form on Q_c is defined by

$$\varepsilon_c = dI_1\wedge\ldots\wedge dI_k. \qquad (1.2.3.\text{Eq } 14)$$

Let $\{X_i\}$ be a field of bases for $\mathcal{P}_c$.

Then let ν_c be the $-(1/2)$-$\mathcal{P}_c$-density defined by

$$\begin{aligned}\nu_c\{X_i\} &= |\varepsilon_\omega|^{-1/2}\{X_i, \partial/\partial I_j\}\ |\varepsilon_c|^{1/2}\{\partial/\partial I_\alpha\}\\ &= |d\theta_1\wedge\ldots\wedge d\theta_k|^{-1/2}\{X_i\} \qquad (1.2.3.\text{Eq } 15)\end{aligned}$$

[cf. equations (1.1.6.Eq 7) and (1.1.6.Eq 14)].

(1.2.4) <u>Quantization of the complete set of commuting observables $\{\zeta_i\}$ in the polarization</u>

[Sniatycki (1975); Sniatycki and Toporowski (1977); Sniatycki (1980), pp71-76 and pp149-156; Woodhouse (1980), pp185-187]

In this section we shall quantize the complete set of commuting observables $\{\zeta_i\}$, of (M_o,ω_o), in the polarization $\mathcal{P}_c$.

The first step of the half-density quantization scheme is as follows. Let $B_o = M_o \times \mathbb{C}$ be the trivial line-bundle over M_o, $(\cdot,\cdot)$ be the (natural) Hermitian structure on B_o and let s_o be a unit section of B_o. Let $c_i \in \mathbb{R}$ and let $f \in C^\infty(M_o)$; then let β be the one-form on M_o given by

$$\beta = \beta_o + \sum_{i=1}^{k} c_i d\theta_i + df \qquad (1.2.4.\text{Eq 1a})$$

(where the one-form β_o was defined in (AAV1)). Clearly β satisfies the condition $d\beta = \omega_o$. In terms of the action-angle variables $(\underline{I},\underline{\theta})$ we shall write

$$\beta = \sum_{i=1}^{k} \{\beta_{I_i} dI_i + \beta_{\theta_i} d\theta_i\} \qquad (1.2.4.\text{Eq 1b})$$

where $\beta_{I_i}, \beta_{\theta_i} \in C^\infty(M_o)$. Then let ∇ be the connection on B_o defined by

$$\nabla_X s_o = -i(X \lrcorner \beta)s_o, \text{ for all } X \in V_{\mathbb{C}}(M_o). \qquad (1.2.4.\text{Eq 2})$$

Let $(B_o,(\cdot,\cdot),\nabla)$ be the chosen prequantization bundle over (M_o,ω_o).

The next step of the half-density quantization scheme is the construction of non-trivial $\mathcal{P}_c$-wave functions, if they exist. The $\mathcal{P}_c$-wave functions, if they exist, should be smooth sections of the form $\Psi_c = s_c\nu_c$ which satisfy the conditions:

$$\nabla_{X_{I_1}} s_c = \ldots\ldots = \nabla_{X_{I_k}} s_c = 0 \qquad (1.2.4.\text{Eq 3a})$$

and

$$\nabla_{X_{I_1}} \nu_c = \ldots\ldots = \nabla_{X_{I_k}} \nu_c = 0. \qquad (1.2.4.\text{Eq 3b})$$

Let

$$S_{\gamma_i}(\underline{I},\underline{\theta}) = \int_{\gamma_i} \beta \quad \text{(integrating along } \gamma_i\text{)}$$
$$= \int_0^{\theta_i} \beta_{\theta_i} d\theta_i, \qquad (1.2.4.\text{Eq } 4a)$$

and let

$$\mathcal{S}_{co} = \{\exp \tfrac{i}{\hbar} S_{\gamma_1}(\underline{I},\underline{\theta})\} \times \ldots \times \{\exp \tfrac{i}{\hbar} S_{\gamma_k}(\underline{I},\underline{\theta})\} s_o. \qquad (1.2.4.\text{Eq } 4b)$$

Then a formal expression for $\mathcal{P}_c$-wave functions is of the form

$$\Psi_c = \psi_c(\underline{I})\mathcal{S}_{co}|d\theta_1 \wedge \ldots \wedge d\theta_k|^{-1/2}; \quad \psi_c(\underline{I}) \in C^{\infty}(Q_c) \qquad (1.2.4.\text{Eq } 5)$$

[cf. remark (R5) of the last subsection]. For this expression to be well-defined it is necessary that the section $\mathcal{S}_{co}$ is single-valued. The section $\mathcal{S}_{co}$ is single-valued if the following conditions are satisfied:

$$(1/\hbar)\oint_{\gamma_1} \beta = 2\pi n_1, \ldots\ldots, (1/\hbar)\int_{\gamma_k} \beta = 2\pi n_k; \; n_1, \ldots, n_k \in \mathbb{Z} \text{(the integers)}.$$
$$(1.2.4.\text{Eq } 6)$$

These conditions are very similar to the BWS conditions of the old quantum theory, and when $\beta = \beta_o$ then they are identical. Therefore, we shall call them **BWS conditions.**

(Since

$$(1/\hbar)\oint_{\gamma_i} \beta = (1/\hbar)[\oint_{\gamma_i} \beta_o + \oint_{\gamma_i} (\sum_{j=1}^{k} c_j d\theta_j + df)]$$
$$= (2\pi/\hbar)[I_i + c_i], \qquad (1.2.4.\text{Eq } 7a)$$

it follows that

$$I_i = [2\pi]^{-1}\oint_{\gamma_i} \beta - c_i.) \qquad (1.2.4.\text{Eq } 7b)$$

Hence the BWS conditions are only satisfied if the action variables $I_1, \ldots, I_k$ take the values:

$$I_1(n_1) = n_1\hbar - c_1, \ldots\ldots, I_k(n_k) = n_k\hbar - c_k; \quad I_i(n_i) \in R(I_i).$$
$$(1.2.4.\text{Eq } 8)$$

Let $\underline{n} = (n_1, \ldots, n_k)$, $\underline{I}(\underline{n}) = (I_1(n_1), \ldots, I_k(n_k))$ and let $\Lambda[\underline{n}] = \Lambda(\underline{I}(\underline{n}))$. We conclude that the formal expression of $\mathcal{P}_c$-wave functions given by equation (1.2.4.Eq 5) is only well defined on the isolated leaves $\Lambda[\underline{n}]$ of $\mathcal{P}_c$; these leaves are referred to as the **BWS leaves.** In other words, there are no non-trivial (smooth) $\mathcal{P}_c$-wave functions defined on the entire M_o; the only $\mathcal{P}_c$-wave-functions that exist are not smooth, and they take the value zero almost everywhere on M_o except on the isolated BWS leaves of $\mathcal{P}_c$. Therefore, the previous half-density quantization scheme cannot be applied to quantize the observables $\zeta_1,\ldots,\zeta_K$. If we want to quantize the complete set of commuting observables $\{\zeta_i\}$ in the polarization $\mathcal{P}_c$, we will need to modify the half-density quantization given in section (1.1).

Here is a modified version of the half-density quantization scheme, that will enables us to construct a non-empty Hilbert space associated with polarization $\mathcal{P}_c$, given in five steps:

(MHDQ1) Let $B_o\big|_{\Lambda[\underline{n}]}$ and $\Delta_{-1/2}(\mathcal{P}_c)\big|_{\Lambda[\underline{n}]}$ be the restriction of the bundles B_o and $\Delta_{-1/2}(\mathcal{P}_c)$ respectively to the BWS leaf $\Lambda[\underline{n}]$. Then $B_o\big|_{\Lambda[\underline{n}]} \times \Delta_{-1/2}(\mathcal{P}_c)\big|_{\Lambda[\underline{n}]}$ is a bundle on $\Lambda[\underline{n}]$.

(MHDQ2) A smooth section $\Psi_c^{\underline{n}} = \mathcal{S}_c^{\underline{n}}(m)\nu_c^{\underline{n}}(m)$ (where $m \in \Lambda[\underline{n}]$) of the bundle $B_o\big|_{\Lambda[\underline{n}]} \times \Delta_{-1/2}(\mathcal{P}_c)\big|_{\Lambda[\underline{n}]}$ that satisfies the conditions

$$\nabla_{\partial/\partial\theta_1}\mathcal{S}_c^{\underline{n}} = \ldots\ldots = \nabla_{\partial/\partial\theta_K}\mathcal{S}_c^{\underline{n}} = 0 \qquad (1.2.4.\text{Eq } 9a)$$

and

$$\nabla_{\partial/\partial\theta_1}\nu_c^{\underline{n}} = \ldots\ldots = \nabla_{\partial/\partial\theta_K}\nu_c^{\underline{n}} = 0 \qquad (1.2.4.\text{Eq } 9b)$$

is called a $\mathcal{P}_c$**-wave function on** $\Lambda[\underline{n}]$. (Here $\partial/\partial\theta_1,\ldots,\partial/\partial\theta_K$ are considered to be vector fields on $\Lambda[\underline{n}]$.)

Let $\mathcal{S}_{co}^{\underline{n}}$ be the restriction of the section $\mathcal{S}_{co}$ [cf. equation (1.2.4.Eq 4b)] to the BWS leaf $\Lambda[\underline{n}]$: for each $m = (\underline{I}(\underline{n}),\underline{\theta}) \in \Lambda[\underline{n}]$, we have $\mathcal{S}_{co}^{\underline{n}}(m) = \mathcal{S}_{co}(\underline{I}(\underline{n}),\underline{\theta})$.

Explicitly, the $\mathcal{P}_c$-wave functions on $\Lambda[\underline{n}]$ are of the form

$$\Psi_c^{\underline{n}} = b_{\underline{n}}\,\mathcal{S}_{co}^{\underline{n}}(m)\,|d\theta_1\wedge\ldots\wedge d\theta_K|^{-1/2}\,;\ m \in \Lambda[\underline{n}],\ b_{\underline{n}} \in \mathbb{C}. \qquad (1.2.4.\text{Eq } 10)$$

(MHDQ3) Let $H^{\underline{n}}_{\mathcal{P}_c}$ be the **one-dimensional Hilbert space associated with the BWS leaf** $\Lambda[\underline{n}]$ that consists of square-integrable $\mathcal{P}_c$-wave functions with respect to the inner-product

$$\langle \Psi^{\underline{n}}_c, \Psi^{\underline{n}}_c \rangle = b_{\underline{n}}\bar{b}_{\underline{n}} = |b_{\underline{n}}|^2. \qquad (1.2.4.\text{Eq } 11)$$

Then

$$\sigma_{c\underline{n}} = \mathcal{S}^{\underline{n}}_{co}(m)\,|d\theta_1\wedge\ldots\wedge d\theta_k|^{-1/2},\ m\in\Lambda[\underline{n}] \qquad (1.2.4.\text{Eq } 12)$$

is a normalized element in $H^{\underline{n}}_{\mathcal{P}_c}$.

(MHDQ4) The **quantization Hilbert space associated with the polarization** $\mathcal{P}_c$ is now defined by

$$H_{\mathcal{P}_c} = \bigoplus_{\Lambda[\underline{n}]} H^{\underline{n}}_{\mathcal{P}_c} \qquad (1.2.4.\text{Eq } 13)$$

where $\oplus$ is the direct sum over all the BWS leaves $\Lambda[\underline{n}]$.

The elements of $H_{\mathcal{P}_c}$ are given by

$$\Psi_c = \bigoplus_{\Lambda[\underline{n}]} b_{\underline{n}}\sigma_{c\underline{n}},\ \sum_{\underline{n}} b_{\underline{n}}\bar{b}_{\underline{n}} < \infty. \qquad (1.2.4.\text{Eq } 14)$$

(MHDQ5) Let ζ be an observable of (M_o,ω_o) such that $\zeta\in C^{\infty}(M_o;\mathcal{P}_c)$; then ζ is only dependant on the action variables $\underline{I}$ [cf. equation (1.1.5.Eq 2a)], i.e. $\zeta = \zeta(\underline{I})$.

The **quantization operator (corresponding to the observable** ζ**)** is postulated to be the self-adjoint operator $\tilde{\zeta}_c$ on $H_{\mathcal{P}_c}$ given by the expression

$$\tilde{\zeta}_c\Psi_c = \bigoplus_{\Lambda[\underline{n}]} \zeta(\underline{I}(\underline{n}))b_{\underline{n}}\sigma_{c\underline{n}}, \qquad (1.2.4.\text{Eq } 15a)$$

and the domain of $\tilde{\zeta}_c$ is given by

$$D_{\tilde{\zeta}_c} = \{\Psi_c\in H_{\mathcal{P}} : \tilde{\zeta}_c\Psi_c\in H_{\mathcal{P}_c}\}. \qquad (1.2.4.\text{Eq } 15b)$$

Let $R(\tilde{\zeta}_c)$ denote the spectrum of $\tilde{\zeta}_c$; then we have

$$R(\tilde{\zeta}_c) = \{\zeta(\underline{I}(\underline{n})) : \underline{n}\in\mathbb{Z}\times\ldots\times\mathbb{Z}\ (\text{k-times}) \text{ and } \underline{I}(\underline{n})\in R(\underline{I})\} \qquad (1.2.4.\text{Eq } 16)$$

The observables belonging to the complete set of commuting observables $\{\zeta_i\}$ are quantizable in the polarization $\mathcal{P}_c$ because they are all elements of $C^{\infty}(M_o;\mathcal{P}_c)$ [cf. remark (R1) of the last subsection].

<u>Remarks</u>: (R1) The values $I_1(n_1),\ldots,I_K(n_K)$ are dependant on the choice of the connection potential β. Hence the spectrum of the operators $\widetilde{\zeta}_{1c},\ldots,\widetilde{\zeta}_{Kc}$ in $\mathbf{H}_\rho$ also depends on the choice of β [cf. McKenna and Wan (1984)]. There appears to be no apriori rules provided by the geometric quantization scheme for picking a particular connection potential β. However, in most of the literature on BWS conditions in geometric quantization the connection potential is chosen to be $\beta_o = \sum_{i=1}^{k} p_i dq_i$ [cf. Simms (1972); Sniatycki (1975); Sniatycki (1980), pp71-72; Woodhouse (1980), pp185-187 and pp207-209]. Arens (1977) suggests the following criterion for choosing the connection potential β : β should be picked so that the BWS conditions obtained by using the modified version of half-density quantization scheme are the same as that given by the Maslov-WKB method [cf. Arnold (1967); Berry (1978), p26-29; Eckmann and Seneor (1976); Maslov and Fedoruk (1981), pp257-266]. (In the next section we shall study the Maslov-WKB method for a one-dimensional Hamiltonian system of a particle in a potential well.) McKenna and Wan (1984) have shown that for particular examples it is possible to choose a connection potential so that the BWS conditions give the physically correct results. To illustrate this point we shall present two new examples shortly.

(R2) The BWS conditions given by equation (1.2.4.7a) are exact quantization conditions in the half-density quantization scheme, unlike the BWS conditions given by the Maslov-WKB method which are approximate quantization conditions [cf. section (1.3) of this chapter].

(R3) In geometric quantization there are two main quantization schemes: the half-density quantization scheme and the half-form quantization scheme [cf. Woodhouse (1980), pp153-164 and pp188-202]. In our presentation we have restricted ourselves to the study of the half-density quantization scheme for the following reasons:

(i) The mathematical apparatus of the half-form quantization scheme is far more complex than that used by the half-density quantization scheme.

(ii) It has been shown by McKenna and Wan (1984) that with an appropriate choice of connection potential the so called **'corrected' BWS conditions** obtained using the half-form quantization scheme can be replicated, for many examples, by the BWS conditions in the half-density quantization scheme. The standard half-form quantization scheme with the connection potential chosen to be $\beta_0 = \sum_{i=1}^{K} p_i \, dq_i$ has been shown by McKenna and Wan (1984) to the produce the wrong spectra in many examples.

(1.2.4.Ex 1) <u>The one-dimensional isotonic oscillator</u> [Ter Haar (1964), pp69-72; Weissmann and Jortner (1979)]

Let $Q = \mathbb{R}^+ = (0,\infty)$ with cartesian coordinate q, $M = T^*Q$ with canonical two-form ω and the usual cartesian canonical coordinates (p,q). The Hamiltonian of an isotonic oscillator is given by

$$H = (1/2)p^2 + (q-1/q)^2. \qquad (1.2.4.\text{Eq } 17)$$

The Hamiltonian vector field X_H is given by

$$X_H = p\,\partial/\partial q - (1+1/q^2)(q-1/q)\,\partial/\partial p. \qquad (1.2.4.\text{Eq } 18)$$

The set Z(H) defined by equation (1.2.2.Eq 1b) is given by

$$Z(H) = \{(0,1)\}. \qquad (1.2.4.\text{Eq } 19)$$

Then let $M_0 = M - Z(H) = \mathbb{R} \times \mathbb{R}^+ - \{(0,1)\}$ and let ω_0 be the restriction of ω to M_0. Let $\mathcal{P}_c$ be the polarization of (M_0,ω_0) spanned by the vector field X_H.

The action-angle variables (I,θ) are given by

$$I = H/8^{1/2}, \quad \theta = -\cos^{-1}(\{2q^2-H-2\}/\{E^2+4E\}^{1/2}) \qquad (1.2.4.\text{Eq } 20)$$

[cf. appendix 1.3]. The range of I is given by $R(I) = (0,\infty)$. Then the

Hamiltonian can be written the form

$$H(I) = 8^{1/2}\, I. \tag{1.2.4.Eq 21}$$

The polarization $\mathcal{P}_c$ is spanned by the vector field $\partial/\partial\theta$.

Let $(B,(\cdot,\cdot),\nabla)$ be the prequantization bundle defined at the beginning of this subsection, and let

$$\beta = pdq + cd\theta. \tag{1.2.4.Eq 22}$$

be the connection potential.

Then the BWS conditions given by equation (1.2.4.Eq 6) are satisfied if I takes the following values

$$I(n) = n\hbar - c;\ n \in \mathbb{Z} \text{ and } n\hbar - c \in R(I) \tag{1.2.4.Eq 23}$$

[cf. (1.2.4.Eq 8)]. The BWS leaves are

$$\Lambda[n] = \{m \in M:\ H(m) = 8^{1/2}(nh-c) \text{ and } nh-c \geqslant 0\}. \tag{1.2.4.Eq 24}$$

Let $\widetilde{H}_c$ be the quantized Hamiltonian in $\mathbf{H}_{\mathcal{P}_c}$; then the spectrum of $\widetilde{H}_c$ is given by

$$R(\widetilde{H}_c) = \{8^{1/2}(n\hbar - c):\ n \in \mathbb{Z} \text{ and } n\hbar - c \geqslant 0\}. \tag{1.2.4.Eq 25a}$$

If we put

$$c = -[(1/2)+(1/4)(8/\hbar^2+1)^{1/2} - (1/4)(8/\hbar^2)^{1/2}]\hbar; \tag{1.2.4.Eq 25b}$$

then $R(\widetilde{H}_c)$ coincides with the spectrum for the Hamiltonian of the isotonic oscillator determined by the usual Schrodinger equation [cf. Ter Haar (1964), pp69-72].

(1.2.4.Ex 2) Example: The two-dimensional Kepler problem [Abraham and Marsden (1980), pp619-631]

Let $Q = \mathbb{R}^2$ with cartesian coordinates $\mathbf{q} = (q_1, q_2)$, ω be the canonical two-form on T^*Q, $(\mathbf{p},\mathbf{q}) = (p_1, p_2, q_1, q_2)$ be the usual canonical coordinates on $T^*Q = \mathbb{R}^4$ and let $\| \ \|$ be the Euclidean norm on $\mathbb{R}^4$. Let

$$M = \{(\mathbf{p},\mathbf{q}) \in \mathbb{R}^4 : \mathbf{q} \neq (0,0) \text{ and } H(\mathbf{p},\mathbf{q}) < 0\} \qquad (1.2.4.\text{Eq } 26\text{a})$$

where

$$H(\mathbf{p},\mathbf{q}) = \|\mathbf{p}\|^2/2 + 1/\|\mathbf{q}\|. \qquad (1.2.4.\text{Eq } 26\text{b})$$

The manifold M is commonly referred to as the Kepler manifold, and the observable $H(\mathbf{p},\mathbf{q})$ restricted to M is called the Hamiltonian of the Kepler problem.

The angular momentum observable on M is

$$L = q_1 p_2 - q_2 p_1. \qquad (1.2.4.\text{Eq } 27)$$

The Hamiltonian vector fields generated by H and L are respectively

$$X_H = p_1 \partial/\partial q_1 + p_2 \partial/\partial q_2 - (q_1/\|\mathbf{q}\|^3)\partial/\partial p_1 - (q_2/\|\mathbf{q}\|^3)\partial/\partial p_2 \qquad (1.2.4.\text{Eq } 28\text{a})$$

and

$$X = -q_2 \partial/\partial q_1 + q_1 \partial/\partial q_2 - p_2 \partial/\partial p_1 + p_1 \partial/\partial p_2. \qquad (1.2.4.\text{Eq } 28\text{b})$$

Clearly the set Z(H,L) defined by equation (1.2.2.Eq 1a) is empty; hence we have $M_o = M$. Let ω_o be the restriction of the canonical two-form ω (on T^*Q) to M_o. The pair H,L constitute a complete set of commuting classical observables [cf. appendix 1.4]. Let $\mathcal{P}_c$ be the polarization of (M_o, ω_o) spanned by the vector fields X_H and X_L.

Let m_o be some point in M such that $H(m_o) = E$, and let $\gamma_1(t_1)$ be the integral curve of X_H that originates from m_o. Let $T_1(E)$ be the period of γ_1. Similarly, let $L(m_o) = L_o$, $\gamma_2(t)$ be the integral curve of X_L that originates at m_o and let $T_2(L_o)$ be the period of γ_2.

The action-angle variables $(I_1, I_2, \theta_1, \theta_2)$ are given by

$$I_1 = (-1/2E)^{1/2}, \quad \theta_1 = [2\pi/T_1(E)]t_1 + \theta_{1o}, \qquad (1.2.4.\text{Eq } 29a)$$

$$I_2 = L_o, \quad \theta_2 = [2\pi/T_2(L_o)]t_2 + \theta_{2o}; \qquad (1.2.4.\text{Eq } 29b)$$

where θ_{1o} and θ_{2o} are real constants [cf. appendix 1.4]. (Strictly speaking, θ_{1o} and θ_{2o} are constants depending on the choice of m_o.) The range of I_1 is $R(I_1) = (0,\infty)$ and the range of I_2 is $R(I_2) = \mathbb{R}$. (The integral curves of X_H in M_o are ellipses with eccentricity e given by $e = (1+2HL^2)^{1/2}$ [cf. Abraham and Marsden (1980), p625]. The range of I_1 is determined from the condition $0<e<1$.)

Then H and L can be written in the form

$$H(\underline{I}) = -(1/2I_1^2), \quad L(\underline{I}) = I_2. \qquad (1.2.4.\text{Eq } 30)$$

Let $(B,(\cdot,\cdot),\nabla)$ the chosen prequantization bundle over (M_o,ω_o) defined at the beginning of this subsection and let

$$\beta = \sum_{i=1}^{K} [p_i dq_i + c_i d\theta_i]. \qquad (1.2.4.\text{Eq } 31)$$

Then the BWS conditions are satisfied if the action variables I_1 and I_2 take the values:

$$I_i(n_i) = n_i\hbar - c \; ; \; n_i \in \mathbb{Z}, \; I_i(n_i) \in R(I_i) \; (i = 1,2). \qquad (1.2.4.\text{Eq } 32)$$

Let $\widetilde{H}_c$ and $\widetilde{L}_c$ be the quantization operators in $\mathbf{H}_{\mathcal{P}_c}$ corresponding to H and L respectively. Then the spectra of the quantization operators $\widetilde{H}_c$ and $\widetilde{L}_c$ in $H_{\mathcal{P}_c}$ are respectively

$$R(\widetilde{H}_c) = \{-[2(n_1\hbar - c_1)]^{-1/2} : n \in \mathbb{Z}, \; n_1\hbar - c_1 \in R(I_1)\} \quad (1.2.4.\text{Eq } 33a)$$

and

$$R(\widetilde{L}_c) = \{(n_2\hbar - c_2) : n \in \mathbb{Z}, \; n_2\hbar - c_2 \in R(I_2)\}. \quad (1.2.4.\text{Eq } 33b)$$

The physically correct spectra are obtained if we put

$$c_1 = -(1/2)\hbar \text{ and } c_2 = 0 \qquad (1.2.4.\text{Eq } 33c)$$

[cf. Arens (1977); Sommerfeld (1929), pp12-14; Sommerfeld (1930), pp67-68].

1.3 THE MASLOV-WKB METHOD

(1.3.1) Introduction

In this section we shall outline the Maslov-WKB method for the derivation of eigenvalues and eigenfunctions of the Hamiltonian operator (in the position representation) for the one-dimensional Hamiltonian system of a particle in a potential well; we shall follow the treatment given by Eckmann and Seneor (1976).

Let $Q = \mathbb{R}$ with cartesian coordinate q be the configuration space of the Hamiltonian system. Let $M = T^*Q = \mathbb{R}^2$ with the usual canonical cartesian coordinates (p,q) be the phase space and let ω be the canonical two-form on M.

Let $V(q) \in C^{\infty}(Q)$ be a potential well in Q that satisfies the following conditions:

$$V(q) = \sum_{r=0}^{\infty} A_r q^r, \quad A_r \text{ are real constants;} \qquad (1.3.1.Eq\ 1a)$$

$$0 \leqslant \lim_{q\to-\infty} V(q) = \lim_{q\to\infty} V(q) = E_0 \leqslant \infty \text{ and } q(\partial V/\partial q) \geqslant 0 \qquad (1.3.1.Eq\ 1b)$$

i.e., V(q) has a single minimum at q = 0.

Then let H(p,q) be the Hamiltonian of a particle in the potential well V(q) given by

$$H(p,q) = (p^2/2)+V(q) \qquad (1.3.1.Eq\ 2)$$

Let $m \in M$, $H(m) = E$ and let $\gamma_m^E(t) = (p(t),q(t))$ be the integral curve of X_H that originates at the point m. Here p(t) and q(t) satisfy the following differential equations:

$$(dq(t)/dt) = p(t), \ (dp(t)/dt) = -\sum_{r=1}^{\infty} rA\ [q(t)]^{r-1} \qquad (1.3.1.Eq\ 3a)$$

with constant of motion

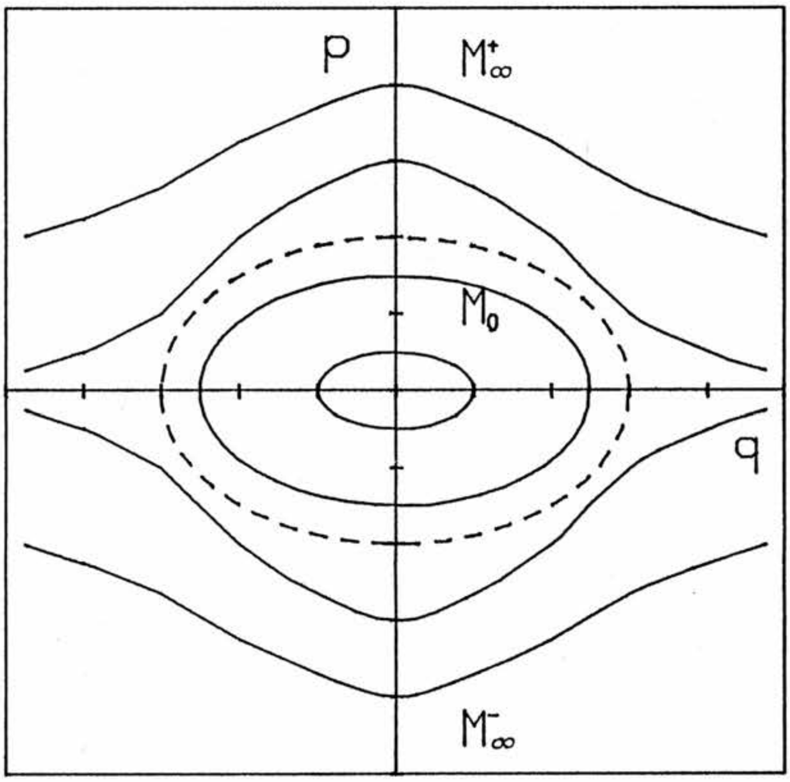

Fig 1-1: The curves are integral curves of X_H. The dashed line indicates the integral curve corresponding to the energy E_o. The regions M_o, M_∞^+ and M_∞^- are depicted.

$$H(p(t),q(t)) = E. \qquad (1.3.1.\text{Eq } 3b)$$

The integral curves of X_H split M into three distinct regions:

$$M_o = \{\text{closed integral curves of } X_H\}\text{--}\{(0,0)\}, \qquad (1.3.1.\text{Eq } 4a)$$

$$M_\infty^+ = \{\text{open integral curves of } X_H\text{: } p>0\}, \qquad (1.3.1.\text{Eq } 4b)$$

$$M_\infty^- = \{\text{open integral curves of } X_H\text{: } p<0\} \qquad (1.3.1.\text{Eq } 4c)$$

[cf. McKenna and Wan (1984)]; the regions are illustrated in Fig 1-1.

In this section we shall restrict ourselves to the study of integral curves in the region M_o. The range of H in the region M_o is $R_o(H) = (V(0),E_o]$.

Let ω_o be the restriction of ω to M_o; then (M_o,ω_o) is a symplectic manifold and H restricted to M_o is a **periodic Hamiltonian.** Let γ^E denote the integral curve of X_H that originates at the point $(p = (2E)^{1/2}, q = 0)$ in M_o. Let T(E) be the period of the curve γ^E.

Let (I,θ) be the action-angle variables on M given by

$$I = \oint_{\gamma^E} p\,dq, \quad \theta = [2\pi/T(E)]t. \qquad (1.3.1.\text{Eq } 5)$$

Let R(I) denote the classical range of I. Then the periodic Hamiltonian H on M_o can be expressed as a function of I, so let

$$H(I) = H \text{ on } M_o. \qquad (1.3.1.\text{Eq } 6)$$

The integral curve γ^E can be parameterized by θ instead of t as follows. Let

$$\gamma^E(\theta) = (p(\theta),q(\theta)) \qquad (1.3.1.\text{Eq } 7a)$$

where

$$p(\theta) = p(t),\ q(\theta) = q(t) \text{ with } t = [T(E)/2\pi]\theta. \qquad (1.3.1.\text{Eq } 7b)$$

Then $p(\theta)$ and $q(\theta)$ satisfy the following differential equations:

$$\partial q(\theta)/\partial\theta = [T(E)/2\pi]p(\theta), \qquad (1.3.1.\text{Eq } 8a)$$

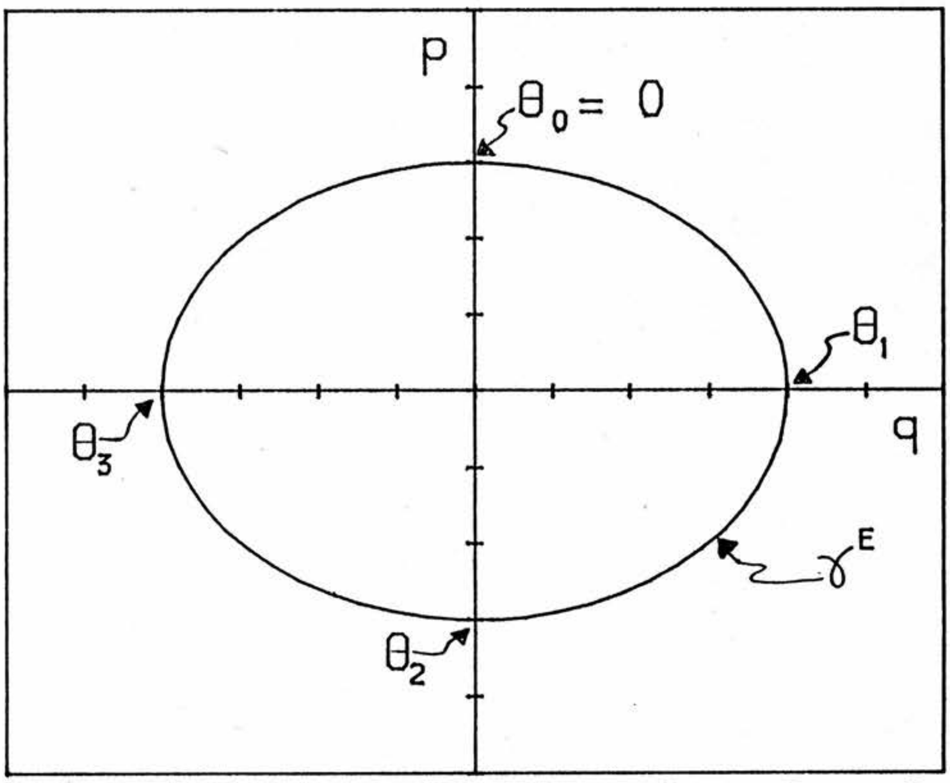

Fig 1-2: The angles θ_0, θ_1, θ_2 and θ_3 are depicted.

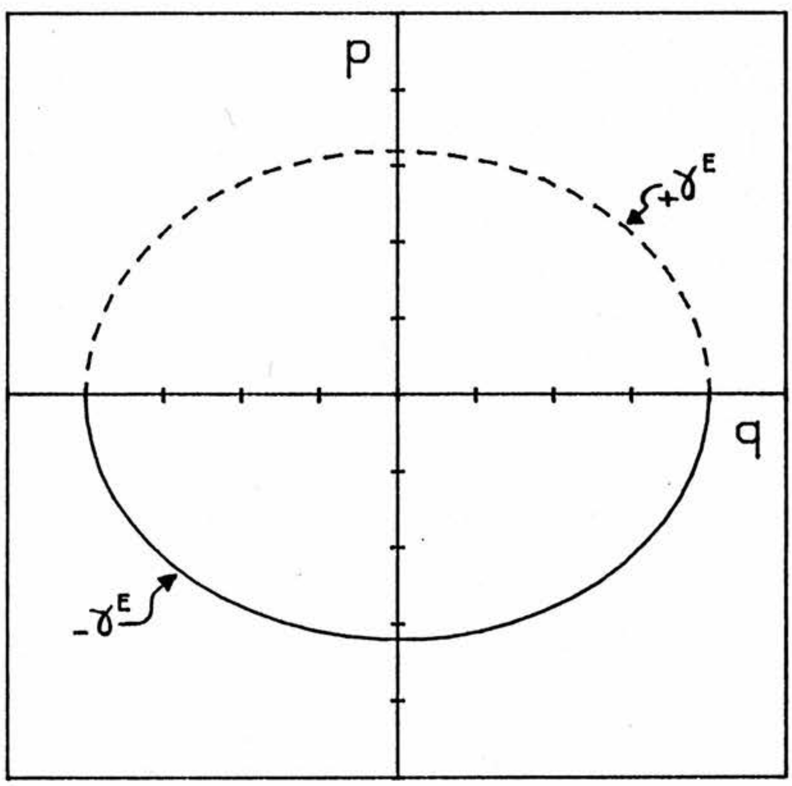

Fig 1-3a: The arcs $_+\gamma^E$ and $_-\gamma^E$ are depicted.

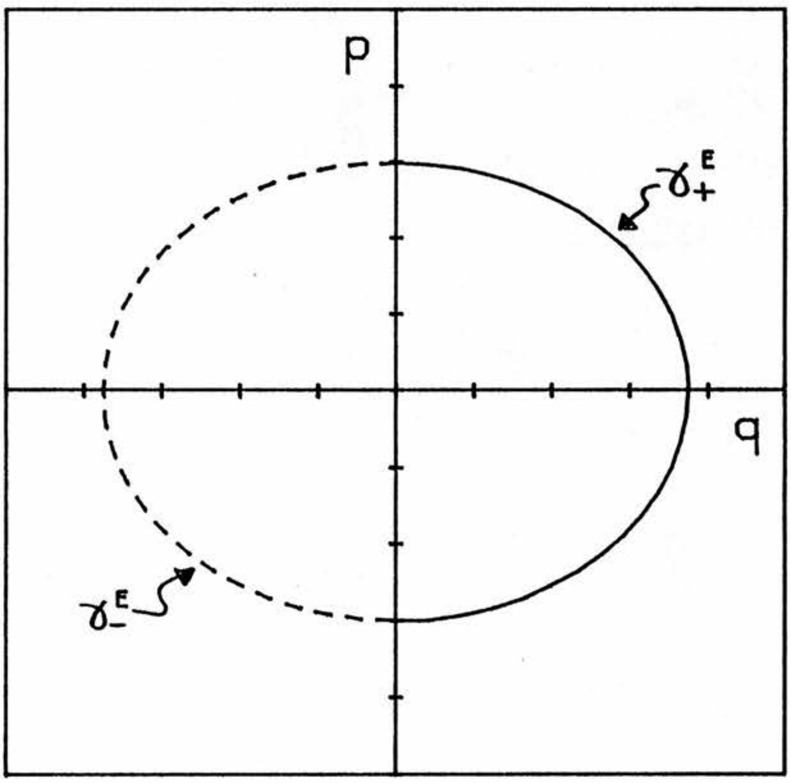

Fig 1-3b: The arcs γ_+^E and γ_-^E are depicted.

$$\partial p(\theta)/\partial\theta = -[T(E)/2\pi]\sum_{r=1}^{\infty} rA_r\{q(\theta)\}^{r-1} \qquad (1.3.1.\text{Eq } 8b)$$

with constant of motion

$$H(p(\theta),q(\theta)) = E. \qquad (1.3.1.\text{Eq } 8c)$$

Clearly we have $\partial q(\theta)/\partial\theta = 0$ when $p(\theta) = 0$, and we have $\partial p(\theta)/\partial\theta = 0$ when $q(\theta) = 0$ because $V(q)$ has a single minimum at $q(\theta) = 0$. We shall assume that $\partial q(\theta)/\partial\theta$ has exactly two stationary points in the range $[0,2\pi)$. Similarly, we shall assume that $\partial p(\theta)/\partial\theta$ has exactly two stationary points in the range $[0,2\pi)$.

Let $\theta_0,\theta_1,\theta_2,\theta_3 \in [0,2\pi)$ satisfy the following conditions [cf. Fig 1-2]:

(i) $\theta_0 = 0$;

(ii) $\theta_0<\theta_1<\theta_2<\theta_3<2\pi$;

(iii)

$$(\partial q(\theta)/\partial\theta) = 0 \text{ at } \theta = \theta_0,\ \theta_2; \qquad (1.3.1.\text{Eq } 9a)$$

(iv)

$$(\partial p(\theta)/\partial\theta) = 0 \text{ at } \theta = \theta_1,\ \theta_3. \qquad (1.3.1.\text{Eq } 9b)$$

Let ${}_{+}\gamma^E$, ${}_{-}\gamma^E$, γ^E_+, γ^E_- be the arcs on γ^E defined by

$${}_{+}\gamma^E = \{(p,q)\in\gamma^E : p\geqslant 0\}, \qquad (1.3.1.\text{Eq } 10a)$$

$${}_{-}\gamma^E = \{(p,q)\in\gamma^E : p\leqslant 0\}, \qquad (1.3.1.\text{Eq } 10b)$$

$$\gamma^E_+ = \{(p,q)\in\gamma^E : q\geqslant 0\}, \qquad (1.3.1.\text{Eq } 10c)$$

$$\gamma^E_- = \{(p,q)\in\gamma^E : q\leqslant 0\} \qquad (1.3.1.\text{Eq } 10d)$$

respectively; the arcs are illustrated in Fig 1-3a and Fig 1-3b.

Remark: (R1) By the Implicit Function Theorem [cf. Abraham and Marsden (1980), p29], p can be considered a function of q on the arcs ${}_{+}\gamma^E$ and ${}_{-}\gamma^E$, and q can be considered a function of p on the arcs γ^E_+ and γ^E_-.

(1.3.1.D1) Definition: [Abraham and Marsden (1980), p409; Woodhouse (1980), p13]

Let (M,ω) be a 2k-dimensional symplectic manifold and let $N \subset M$ be a k-dimensional submanifold in M. N is said to be a **Lagrangian submanifold** in M if at every $m \in N$ the following condition is satisfied:

$$\omega(X_m, Y_m) = 0 \text{ for all } X_m, Y_m \in T_mN. \qquad (1.3.1.\text{Eq } 11)$$

Remark: (R2) The curve γ^E is an example of a Lagrangian submanifold in M_0.

(1.3.2) The Hamilton-Jacobi equations [Abraham and Marsden (1980), p381; Woodhouse (1980); pp66-69]

The **Hamilton-Jacobi equations** are

$H((\partial S/\partial q),q) = E$, (here $(\partial S/\partial q)$ replaces p in the expression for H);

(1.3.2.Eq 1a)

$H(p,(-\partial W/\partial p)) = E$,(here $(-\partial W/\partial p)$ replaces q in the expression for H);

(1.3.1.Eq 1b)

where S(q) and W(p) are **generating functions of the Lagrangian submanifold** γ^E.

We shall now show how S(q) and W(p) may be constructed. Let **s**(q,I) and **w**(p,I) be local functions on M that satisfy the following equations:

$$ds(q,I) = pdq - Id\theta, \qquad (1.3.2.\text{Eq } 2a)$$

$$dw(p,I) = -qdp - Id\theta. \qquad (1.3.2.\text{Eq } 2b)$$

Then the **generating function** S(q) is defined to be the restriction of **s**(q,I) to γ^E. Similarly, the **generating function** W(p) is defined to be the restriction of **w**(p,I) to γ^E. Note that S(q) and W(p) are only defined locally on the Lagrangian submanifold γ^{E}.

Let $m_o = (p = (2E)^{1/2}, q = 0)$. Then a set of solutions of $S(q)$ and $W(p)$ are:

$$_{+}S^{E}(q) = \int_{m_o}^{m} pdq \text{ on } {}_{+}\gamma^{E},\ m \in {}_{+}\gamma^{E}; \qquad (1.3.2.\text{Eq } 3a)$$

$$_{-}S^{E}(q) = \int_{m_o}^{m} pdq \text{ on } {}_{-}\gamma^{E},\ m \in {}_{-}\gamma^{E}; \qquad (1.3.2.\text{Eq } 3b)$$

$$W_{+}^{E}(p) = -\int_{m_o}^{m} qdp \text{ on } \gamma_{+}^{E},\ m \in \gamma_{+}^{E}; \qquad (1.3.2.\text{Eq } 3c)$$

$$W_{-}^{E}(p) = -\int_{m_o}^{m} qdp \text{ on } \gamma_{-}^{E},\ m \in \gamma_{-}^{E}; \qquad (1.3.2.\text{Eq } 3d)$$

(here all the integrals are along γ^{E}).

(1.3.3) <u>The WKB method</u> [Eckmann and Seneor (1976)]

The usual WKB method consists of constructing approximate solutions of the Schrodinger equation as $\hbar \to 0$. In the case of our example, we shall see that these approximate solutions of the bound states of the Schrodinger equation in both the position representation and momentum representation are not square-integrable functions. Therefore, these approximate bound state solutions do not belong to the domain of the respective Hamiltonian operators in the position representation and momentum representation.

The usual **WKB method for the Schrodinger equation in the position representation** is given as follows. The Schrodinger equation of the one-dimensional Hamiltonian system of a particle with energy E in the potential well V(q) is

$$(\hat{H}-E)\Psi(q) = 0, \qquad (1.3.3.\text{Eq } 1a)$$

where

$$\hat{H} = -(\hbar^2/2)(d^2/dq^2)+V(q) \text{ and } \Psi(q) \in L^2(\mathbb{R}). \qquad (1.3.3.\text{Eq } 1b)$$

The WKB method consists of solving equation (1.3.3.Eq 1a) as $\hbar \to 0$ by putting

$$\phi_w^E(q) = f(q)\{\exp \tfrac{i}{\hbar}S(q)\}. \qquad (1.3.3.\text{Eq } 2)$$

(Here we have use the superscript E and the subscript w to highlight the fact that $\phi_w^E(q)$ is a **WKB-wave function corresponding to the energy** E.)

Then expanding equation (1.3.3.Eq 1a) in terms of $\hbar$ we get

$$(\hat{H}-E)\phi_w^E(q) = \hbar^0\{(1/2)(\partial S/\partial q)^2+V(q)-E\}\phi_w^E(q)$$
$$+ \hbar(-i)\{(1/2)(\partial^2 S/\partial q^2)f+(\partial f/\partial q)(\partial S/\partial q)\}\{\exp \tfrac{i}{\hbar}S\}$$
$$+ \text{higher order terms of } \hbar \qquad (1.3.3.\text{Eq } 3)$$

[cf. Appendix 1.5]. Now put the coefficients of $\hbar^0$ and $\hbar$ equal to zero; then we get the following pair of equations:

$H((\partial S/\partial q),q) = E$, (this is a Hamilton-Jacobi equation); (1.3.3.Eq 4a)

$(1/2)(\partial^2 S/\partial q^2)f+(\partial f/\partial q)(\partial S/\partial q) = 0.$ (1.3.3.Eq 4b)

So the generating functions ${}_+S^E(q)$ and ${}_-S^E(q)$ are independent solutions of S(q) [cf. equations (1.3.2.Eq 3a) and (1.3.2.Eq 3b)].

An independent pair of solutions of the function f(q) are [cf. Appendix 1.5]:

$${}_+f^E(q) = (\text{constant})|\partial{}_+S^E/\partial q|^{-1/2}$$
$$= (\text{constant})|(\partial q/\partial\theta)|^{-1/2}\ (q) \text{ on } {}_+\gamma^E; \qquad (1.3.3.\text{Eq } 5a)$$

and

$${}_-f^E(q) = (\text{constant})|\partial{}_-S^E/\partial q|^{-1/2}$$
$$= (\text{constant})|(\partial q/\partial\theta)|^{-1/2}\ (q) \text{ on } {}_-\gamma^E; \qquad (1.3.3.\text{Eq } 5b)$$

(Here $(\partial q/\partial\theta) = [T(E)/2\pi]p$ is considered a function of q on each on the arcs ${}_+\gamma^E$ and ${}_-\gamma^E$ [cf. remark (R1) of subsection (1.3.1)].)

Then **the general WKB-approximation of the eigenfunction** (**corresponding to the eigenvalue** E) of the Hamiltonian operator $\hat{H}$ is of the form

$$\phi_w^E(q) = {}_+K\ {}_+f^E(q)\{\exp \tfrac{i}{\hbar}{}_+S^E(q)\}+{}_-K\ {}_-f^E(q)\{\exp \tfrac{i}{\hbar}{}_-S^E(q)\}. \qquad (1.3.3.\text{Eq } 6)$$

where ${}_+K$ and ${}_-K$ are constants. Since $(\partial q/\partial\theta) = 0$ on the set $\{q: V(q) = E\}$, it follows that the WKB-wave function $\phi_{W}^{E}(q)$ is singular on this set. Therefore, $\phi_{W}^{E}(q)$ is not an element of $L^2(\mathbb{R})$.

The **WKB method for the Schrodinger equation in the momentum representation** is given as follows.

In the momentum representation, the Schrodinger equation of the one-dimensional Hamiltonian system of a particle in a potential well V(q) is

$$(\hat{H}_c - E)\Psi_c(p) = 0, \qquad (1.3.3.\text{Eq } 7a)$$

where

$$H_c = (p^2/2) + \sum_{r=0}^{\infty} (i\hbar)^r A_r (\partial/\partial p)^r \text{ and } \Psi_c(p) \in L^2_c(\mathbb{R}). \qquad (1.3.3.\text{Eq } 7b)$$

(Here the subscript c is used to indicate the fact that the mathematical objects are associated with the coordinate p which is canonically conjugate to the coordinate q.)

The WKB method consists of solving equation (1.3.3.Eq 7a) as $\hbar \to 0$ by putting

$$\phi_{c,W}^{E}(p) = g(p)\{\exp \pm W(p)\}. \qquad (1.3.3.\text{Eq } 8)$$

Then expanding equation in terms of $\hbar$ we get

$$(\hat{H}_c - E)\phi_{c,W}^{E}(p) = \hbar^0[(p^2/2) + \{\sum_{r=1}^{\infty} A_r(-\partial W/\partial p)^r\} - E]\,\phi_{c,W}^{E}(p)$$
$$- i\hbar[\sum_{r=1}^{\infty}(i)^{2r} A_r r\{[(r-1)/2](\partial W/\partial p)^{r-2}(\partial^2 W/\partial p^2)g + (\partial W/\partial p)^{r-1}(\partial g/\partial p)\}]\{\exp \pm W\}$$
$$+ \text{ higher order terms in } \hbar \qquad (1.3.3.\text{Eq } 9)$$

[cf. Appendix 1.5].

Now if we put the coefficients $\hbar^0$ and $\hbar$ equal to zero, then we get the following equations:

$H(p, (-\partial W/\partial p)) = E$, (this is a Hamilton-Jacobi equation); (1.3.3.Eq 10a)

and

$\sum_{r=1}^{\infty}(-1)^r rA_r(\partial W/\partial p)^{r-2}\{[(r-1)/2](\partial^2 W/\partial p^2)g+(\partial W/\partial p)(\partial g/\partial p)\} = 0$ (1.3.3.Eq 10b)

So the generating functions $W_+^E(p)$ and $W_-^E(p)$ are independent solutions of W(p) [cf. equations (1.3.2.Eq 3c) and (1.3.2.Eq 3d)].

An independent pair of solutions of g(p) are [cf. Appendix 1.5]:

$$g_+^E(p) = (\text{constant})\left|\sum_{r=1}^{\infty} rA_r(-\partial W_+^E/\partial p)^{r-1}\right|^{-1/2}$$
$$= (\text{constant})|(\partial p/\partial\theta)|^{-1/2}(p) \text{ on } \gamma_+^E; \qquad (1.3.3.\text{Eq } 11a)$$

and

$$g_-^E(p) = (\text{constant})\left|\sum_{r=1}^{\infty} rA_r(-\partial W_-^E/\partial p)^{r-1}\right|^{-1/2}$$
$$= (\text{constant})|(\partial p/\partial\theta)|^{-1/2}(p) \text{ on } \gamma_-^E; \qquad (1.3.3.\text{Eq } 11b)$$

(Here $(\partial p/\partial\theta)$ which is given by equation (1.3.1.Eq 8b) is considered a function of p [cf. remark (R1) of subsection (1.3.1)].)

Then **the general WKB-approximation of the eigenfunctions (corresponding to the eigenvalue** E) of the operator $\hat{H}_c$ is of the form

$$\phi_{c,W}^E(p) = K_+ g_+^E(p)\{\exp \tfrac{i}{\hbar} W_+^E(p)\} + K_- g_-^E(p)\{\exp \tfrac{i}{\hbar} W_-^E(p)\} \qquad (1.3.3.\text{Eq } 12)$$

where K_+ and K_- are constants. The WKB-wave function $\phi_{c,W}^E(p)$ has singularities at the points belonging to $\{p: p = [2E-V(0)]^{1/2}\}$. Hence $\phi_{c,W}^E(p)$ is not an element of $L_C^2(\mathbb{R})$.

The momentum representation and position representation are related by the **Fourier transform F**: $L_c^2(\mathbb{R}) \dashrightarrow L^2(\mathbb{R})$ which is given by

$$(\mathbf{F}\Psi_c)(q) = [2\pi\hbar]^{-1/2}\int_{\mathbb{R}} \{\exp \tfrac{i}{\hbar} pq\}\Psi_c(p)dp;\ \Psi_c(p) \in L_c^2(\mathbb{R}). \qquad (1.3.3.\text{Eq } 13)$$

<u>Remark</u>: (R1) It can be shown that

$$\hat{H}_c = \mathbf{F}^{-1} H\mathbf{F}. \qquad (1.3.3.\text{Eq } 14)$$

So roughly speaking, we may expect $(\mathbf{F}\phi_{c,W}^E)(q)$ to be a WKB-approximation of the eigenfunction (corresponding to the eigenvalue E) of the Hamiltonian operator $\hat{H}$ [cf. equation (1.3.3.Eq 6)]. A corresponding statement applies to the function $(F^{-1}\phi_{w}^{E})(p)$ in the momentum representation. In the next subsection, we shall pursue this line of thought further.

(1.3.4) The Maslov-WKB method [Eckmann and Seneor (1976)]

The Maslov-WKB method consists of constructing square-integrable bound state solutions of the Schrodinger equation given by equation (1.3.3.Eq 1a), such that these solutions belong to domain of the Hamiltonian operator $\hat{H}$. Roughly speaking, the Maslov-WKB method may be summarized as follows. Let $\phi_{W}^{E}(q)$ be a suitably chosen WKB-approximation (corresponding to the energy E) of the Hamiltonian operator $\hat{H}$. Similarly, let $\phi_{c,w}^{E}(p)$ be a suitably chosen WKB-approximation (corresponding to the energy E) of the Hamiltonian operator $\hat{H}_{c}$. Then one chooses $\phi_{W}^{E}(q)$ as the approximate solution (corresponding to the energy E) of the Schrodinger equation at the points belonging to the set $\{q: V(q)<E\}$, and one chooses $(F\phi_{cw}^{E})(q)$ as the approximate solution at the points belonging to the set $\{q: V(q) = E\}$ (on which $\phi_{W}^{E}(q)$ is singular). The procedure is complicated by the fact that $\phi_{c,w}^{E}(p)$ does not belong to the domain of the Fourier transform F because it is not a square-integrable function of p, so to overcome this technicality we need to replace $\phi_{c,w}^{E}(p)$ with a suitably chosen 'WKB-like function' that is square-integrable.

A brief outline of the Maslov-WKB method is as follows. We shall begin by listing the notation and defining the various functions we shall need to use.

(1.3.4.ND 1) Notation and Definitions

(ND 1.1) From now on all functions of the angle variable θ will be considered to be functions on the integral curve γ^E and not functions on M_o.

(ND 1.2) For each positive integer j, let

$$\Delta_j = [-2j\pi-(2\pi-\theta_2), 2j\pi+\theta_2]. \qquad (1.3.4.\text{Eq } 1)$$

(ND 1.3) Let $\Psi(\theta)$ be a function on γ^E; then we define the operation $\ulcorner\urcorner^j$ by

$$(\ulcorner\Psi\urcorner^j)(q) = \sum_{\substack{\theta\in\Delta_j \\ q(\theta)=q}} \Psi(\theta) \text{ if } q \in \{q: V(q)\leqslant E\}$$
$$= 0 \text{ otherwise.} \qquad (1.3.4.\text{Eq } 2)$$

(ND 1.4) Similarly, we define the operation $\ulcorner\urcorner^{c,j}$ by

$$(\ulcorner\Psi\urcorner^{c,j})(p) = \sum_{\substack{\theta\in\Delta_j \\ p(\theta)=p}} \Psi(\theta) \text{ if } p \in \{p: \; p^2\leqslant[2E-V(0)]\}$$
$$= 0 \text{ otherwise.} \qquad (1.3.4.\text{Eq } 3)$$

(ND 1.5) For each positive integer k, let $_+A_k$ and $_-A_k$ be a pair of real constants. Then let $J(\theta)$ be a function on γ^E defined by

$$J(\theta) = {}_+A_k \text{ for } \theta \in (2k\pi-(2\pi-\theta_3), 2k\pi+\theta_1]$$
$$= {}_-A_k \text{ for } \theta \in (2k\pi+\theta_1, 2k\pi+\theta_3]. \qquad (1.3.4.\text{Eq } 4a)$$

Similarly, for each positive integer k', let $B^+_{k'}$ and $B^-_{k'}$ be a pair of real constants. Then let $J_c(\theta)$ be a function on γ^E defined by

$$J_c(\theta) = B^+_{k'} \text{ for } \theta \in (2k'\pi, 2k'\pi+\theta_2]$$
$$= B^-_{k} \text{ for } \theta \in (2k'\pi+\theta_2, 2k'\pi+2\pi]. \qquad (1.3.4.\text{Eq } 4b)$$

(ND 1.6) Let $\phi(\theta)$ and $\phi_c(\theta)$ be the two **WKB-like functions** on γ^E defined by

$$\phi(\theta) = |(\partial q/\partial\theta)|^{-1/2}\{\exp \tfrac{i}{\hbar}\int_0^\theta p(\theta)[\partial q(\theta)/\partial\theta]d\theta\}\{\exp iJ(\theta)\} \qquad (1.3.4.\text{Eq } 5a)$$

and

$$\phi_c(\theta) = |(\partial p/\partial\theta)|^{-1/2}\{\exp -\tfrac{i}{\hbar}\int_0^\theta q(\theta)[\partial p(\theta)/\partial\theta]d\theta\}\{\exp iJ_c(\theta)\}. \qquad (1.3.4.\text{Eq } 5b)$$

respectively.

(Remark: (R1) Note that $\Phi^j(q)$ is a WKB-wave function corresponding to the energy E, for each positive integer j. Similarly, $(\Phi_c^{c,j})(p)$ is a WKB-wave function corresponding to the energy E, for each positive integer j.)

(ND 1.7) Let

$\Pi = \{\theta \in \mathbb{R} : \exists k \in \mathbb{Z}$ with either $\theta = \theta_1 + 2k\pi$ or $\theta = \theta_3 + 2k\pi\}$. (1.3.4.Eq 6a)

Then Π is the set of singularities of the function $\phi(\theta)$.

Let

$\Pi_c = \{\theta \in \mathbb{R} : \exists\, k \in \mathbb{Z}$ with either $\theta = 2k\pi$ or $\theta = \theta_2 + 2k\pi\}$. (1.3.4.Eq 6b)

Then Π_c is the set of singularities of the function $\phi_c(\theta)$.

(ND 1.8) Let

$$\Pi = \{q: V(q) = E\}. \qquad (1.3.4.\text{Eq 7a})$$

Then Π is the set of singularities of the WKB-wave function $\Phi^j(q)$ where j is an arbitrary positive integer. Let

$$\Pi_c^c = \{p: p^2 = [2E-V(0)]\}. \qquad (1.3.4.\text{Eq 7b})$$

Then Π_c^c is the set of singularities of the WKB-wave function $(\Phi_c^{c,j})(p)$ where j is an arbitrary positive integer.

(ND 1.9) Let $e(\theta)$ and $e_c(\theta)$ be two smooth real-valued functions on γ^E that satisfy the following three conditions:

(i) $e(\theta)+e_c(\theta) = 1$ for all $\theta \in \mathbb{R}$;

(ii) For all $\theta \in \mathbb{R}$ and $k \in \mathbb{Z}$, we have

$e(\theta) = e(\theta+2k\pi)$ and $e_c(\theta) = e_c(\theta+2k\pi)$ (periodic conditions).

(iii) $e(\theta) = 0$ in the neighbourhood of the points belonging to the set Π. Similarly, $e_c(\theta) = 0$ in the neighbourhood of points belonging to the set Π_c.

Remark: (R2) Clearly the functions $\phi(\theta)e(\theta)$ and $\phi_c(\theta)e_c(\theta)$ are free of singularities. Hence it follows that for each positive integer j, the functions $(\overline{\phi e}^{\,j})(q)$ and $(\overline{\phi_c e_c}^{\,c,j})(p)$ are singularity-free and have compact support. We shall use the following theorem to show the link between the functions $(\overline{\phi e}^{\,j})(q)$ and $(\overline{\phi_c e_c}^{\,c,j})(p)$.

(1.3.4.T1) Theorem [Eckmann and Seneor (1976)]

For each $q \in \{q : \exists\ p \text{ such that } (p,q) \in \gamma^{\varepsilon}\}$ and positive integer j, we have

$$(F\overline{\phi_c e_c}^{\,c,j})(q) = \sum_{\substack{\theta\in\Delta_j \\ q(\theta)=q}} \phi(\theta)e_c(\theta)+O(\hbar) \qquad \text{(1.3.4.Eq 8a)}$$

if $J(\theta)$ and $J_c(\theta)$ satisfy the following conditions:

$$J_c(\theta)-J(\theta) = -\pi/4 \pmod{2\pi} \text{ for } \theta \in \bigcup_{k\in\mathbb{Z}} (2k\pi, 2k\pi+\theta_1);$$

$$J_c(\theta)-J(\theta) = +\pi/4 \pmod{2\pi} \text{ for } \theta \in \bigcup_{k\in\mathbb{Z}} (2k\pi+\theta_1, 2k\pi+\theta_2);$$

$$J_c(\theta)-J(\theta) = -\pi/4 \pmod{2\pi} \text{ for } \theta \in \bigcup_{k\in\mathbb{Z}} (2k\pi+\theta_2, 2k\pi+\theta_3);$$

$$J_c(\theta)-J(\theta) = +\pi/4 \pmod{2\pi} \text{ for } \theta \in \bigcup_{k\in\mathbb{Z}} (2k\pi+\theta_3, 2k\pi+2\pi);$$

(1.3.4.Eq 8b)

Proof. See Appendix 1.6.■

(1.3.4.C1) Corollary

If we put

$$J(\theta) = +A_k = -k\pi \text{ for } \theta \in (2k\pi-(2\pi-\theta_3), 2k\pi+\theta_1],$$
$$J(\theta) = -A_k = -k\pi-\pi/2 \text{ for } \theta \in (2k\pi+\theta_1, 2k +\theta_3],$$
$$J_c(\theta) = B_k^+ = -k'\pi-\pi/4 \text{ for } \theta \in (2k'\pi, 2k'\pi+\theta_2],$$
$$J_c(\theta) = B_k^- = -k'\pi-3\pi/4 \text{ for } \theta \in (2k'\pi+\theta_2, 2k'\pi+2\pi];$$

(1.3.4.Eq 9a)

then for each $q \in \{q: V(q)<E\}$ and positive integer j, we have:

$$(F\,\overline{\phi_c e_c}^{c,j})(q) = \sum_{\substack{\theta\in\Delta_j \\ q(\theta)=q}} \phi(\theta)e_c(\theta)+O(\hbar) \qquad (1.3.4.\text{Eq } 9b)$$

and

$$(F\,\overline{\phi_c e_c}^{c,j})(q)+(\overline{\phi e}^{j})(q) = (\overline{\Phi}^{j})(q)+O(\hbar) \qquad (1.3.4.\text{Eq } 9c)$$

Proof. The proof follows from Theorem (1.3.4.T1).■

Remarks: (R3) The functions $J(\theta)$ and $J_c(\theta)$ are referred to as **Maslov indices** by Eckmann and Seneor (1976) if they satisfy the conditions given by equation (1.3.4.Eq 8b). Therefore, the functions $J(\theta)$ and $J_c(\theta)$ defined by equation (1.3.4.Eq 9a) are Maslov indices, so from now on we shall assume that $J(\theta)$ and $J_c(\theta)$ are given by equation (1.3.4.Eq 9a).

(R4) Note that above theorem and its corollary does not deal with the behaviour of the functions $(F\,\overline{\phi_c e_c}^{c,j})(q)$ and $\{(F\,\overline{\phi_c e_c}^{c,j})(q)+(\overline{\phi e}^{j})(q)\}$ at the points belonging to the set $\overline{\Pi}$. However, in the neighbourhood of points belonging to the set $\overline{\Pi}$ we do have the following results:

(i) $(F\,\overline{\phi_c e_c}^{c,j})(q)$ is finite because **F** is a unitary map, and $(\overline{\phi_c e_c}^{c,j})(p)$ is a bounded and has compact support [cf. definition of e_c in (ND 1.9)];

(ii) $(\overline{\phi e}^{j})(q) = 0$ [cf.definition of e in (ND 1.9)].

(iii) It follows from (i) and (ii) that

$$(F\overline{\phi_c e_c}^{c,j})(q)+(\overline{\phi e}^{j})(q) = (F\overline{\phi_c e_c}^{c,j})(q) \text{ at } q \in \Pi.$$

With the above results in mind, we shall now define the Maslov-WKB wave-function (corresponding to the energy E) of the Schrodinger equation given by equation (1.3.4.Eq 1a) as follows.

(1.3.4.D2) Definition [Eckmann and Seneor (1976)]

Let

$$\Phi^{E}(q) = \lim_{j\to\infty} \{[1/j][(F\overline{\phi_c e_c}^{c,j})(q)+(\overline{\phi e_j}^{j})(q)]\}. \qquad (1.3.4.\text{Eq } 10)$$

We shall refer to $\Phi^{E}(q)$ as the **Maslov-WKB wave function (corresponding to the energy E) of the Schrodinger equation.** (Here the Schrodinger equation referred to is given by equation (1.3.4.Eq 1a) and $E \in (V(0),E_0)$.)

(1.3.4.T2) Theorem [Eckmann and Seneor (1976)]

(i) If

$$[2\pi]^{-1}\oint_{\gamma^E} pdq \neq (n+1/2)\hbar, \text{ for all } n \in \mathbb{Z}, \qquad (1.3.4.\text{Eq } 11)$$

then the Maslov-WKB wave function (corresponding to the energy E) is the zero-function; i.e. $\Phi^{E} = 0$ everywhere on the configuration space Q which is $\mathbb{R}$. Otherwise, $\Phi^{E}(q)$ is a non-trivial function.

(ii) The Maslov-WKB wave function (corresponding to the energy E) satisfies the following condition

$$\|(\hat{H}-E)\,\Phi^{E}(q)\| = O(\hbar^2). \qquad (1.3.4.\text{Eq } 12)$$

Here $\| \;\|$ is the norm in $L^2(\mathbb{R})$ given by

$$\|\Psi(q)\| = \int_{\mathbb{R}} |\Psi(q)|^2 dq;\ \Psi(q) \in L^2(\mathbb{R}). \qquad (1.3.4.\text{Eq } 13)$$

(iii) In the limit as $\hbar \to 0$, the Maslov-WKB wave function $\Phi^{E}(q)$ is square-integrable.

Proof.

The proof of assertion (i) is given in Appendix 1.7 and the proof of assertion (ii) is given in Appendix 1.8. Assertion (iii) has been proved by Eckmann and Seneor (1976). ■

Remarks: (R5) According to assertion (i) of the last theorem, the Maslov-WKB wave function $\Phi^E(q)$ is non-trivial only if the following condition is satisfied:

$$[2\pi]^{-1}\oint_{\gamma^E} p\,dq = (n+1/2)\hbar, \text{ for some integer n.} \qquad (1.3.4.\text{Eq } 14)$$

We shall refer to these conditions as the **Maslov-WKB conditions.** A critical comparison of the BWS conditions (in the half-density quantization scheme and the Maslov-WKB conditions will be made in Chapter 3.

Hence according to the Maslov-WKB conditions the allowed values of the action variable I are:

$$I(n) = (n+1/2)\hbar, \text{ where } n \in \mathbb{Z} \text{ and } I(n) \in R(I). \qquad (1.3.4.\text{Eq } 15)$$

Therefore, the allowed values of H are:

$$E(n) = H(I(n)), \text{ where } E(n) \in (V(0), E_o). \qquad (1.3.4.\text{Eq } 16)$$

We shall refer to E(n) as the **approximate eigenvalues of the Hamiltonian operator $\hat{H}$ (predicted by the Maslov-WKB conditions).**

(R6) It follows from assertions (ii) and (iii) of the last theorem that for each $E(n) \in (V(0), E_o)$, the Maslov-WKB wave function corresponding to the energy E(n) is an approximate eigenfunction of the Hamiltonian operator $\hat{H}$. Let $\Phi_n(q)$ denote the **Maslov-WKB wave function corresponding to the energy** E(n).

APPENDICES A1.1-A1.8

APPENDIX 1.1

Notation and definitions

(ND1) Notation [Woodhouse (1980), pp288-289]

We shall assume that the term **manifold** refers to a smooth real manifold.

Let M be an arbitrary **manifold** and let U be an open set of M. Then:

(i) $C^{\infty}(M)$ ($C^{\infty}_{\mathbb{C}}(M)$) is the **space of smooth, real (complex)-valued functions on M**;

(ii) $V(M)$ ($V_{\mathbb{C}}(M)$) is the **space of smooth, real (complex) vector fields on M.** ($C^{\infty}(U)$, $C^{\infty}_{\mathbb{C}}(U)$, $V(U)$ and $V_{\mathbb{C}}(U)$ are the corresponding spaces on U.)

(ND2) Definitions and Theorem [Woodhouse (1980), p1]

(ND2.1) A **symplectic manifold** is a pair (M,ω) in which

(i) M is a manifold;

(ii) ω is a closed, non-degenerate two-form defined everywhere on M:

(a) $d\omega = 0$ on M, and

(b) the one form $X \lrcorner \omega$, where $X \in V(M)$, is everywhere zero on M if and only if $X = 0$.

(The two-form ω is called the **symplectic two-form.**)

(Remark: (R1) Every symplectic manifold is even-dimensional [cf. Abraham and Marsden (1980), p165].)

(ND2.2) **Darboux's theorem** [Abraham and Marsden (1980), p175; Woodhouse (1980), p7]

Let (M,ω) be a 2k-dimensional symplectic manifold and let m M. Then there is a neighbourhood U of m and coordinates $\{p_1,\ldots,p_k,q_1,\ldots,q_k\}$ on U such that $\omega = \sum_{i=1}^{k} dp_i \wedge dq_i$.
The coordinates $\{p_1,\ldots,p_k,q_1,\ldots,q_k\}$ are called **local canonical coordinates.**

(ND3) Definitions

(ND3.1) A **volume-form** on a k-dimensional manifold $\mathcal{M}$ is a nowhere-zero k-form on $\mathcal{M}$ [cf. Abraham and Marsden (1980), p123].

(ND3.2) A manifold $\mathcal{M}$ is said to be **orientable** if there exists a volume-form on $\mathcal{M}$ [cf. Abraham and Marsden (1980), p123].

(ND3.3) A 2k-dimensional symplectic manifold (M,ω) is orientable and it carries a natural volume-form ε_ω which is given by

$$\varepsilon_\omega = (-1)^{k(k-1)/2}\,\omega^k/k!$$

[cf. Woodhouse (1980), p3]. ε_ω is called the **Liouville volume-form.**

In terms of local canonical coordinates $\{p_1,\ldots,p_k,q_1,\ldots,q_k\}$ on a neighbourhood of U of a point $m \in M$, we have

$$\varepsilon_\omega = dp_1\wedge\ldots\wedge dp_k\wedge dq_1\wedge\ldots\wedge dq_k .$$

[cf. Woodhouse (1980), p114].

(ND4) <u>Definitions and notation</u> [Campbell (1983); Woodhouse (1980), p294]

(ND4.1) A **complex line-bundle** B over a manifold M (called the base space) is defined to consist of:

(i) A manifold B (called the total space);

(ii) A smooth map $\pi: B \dashrightarrow M$ (called the projection map) such that for each $m \in M$, $B_m = \pi^{-1}(m)$ is a vector space over $\mathbb{C}$ of dimension one (B_m is called the fibre over m);

(iii) For each $m \in M$, there exists a neighbourhood U of m and a diffeomorphism

$$\phi : U \times \mathbb{C} \dashrightarrow \pi^{-1}(U)$$

with $\pi \circ \phi(m', z) = m'$ for all $m' \in U$ and $z \in \mathbb{C}$.

(The pair (U, ϕ) is called a **local trivialization** for B.)

(<u>Remark</u>: (R2) A complex line-bundle is often called **line-bundle** for short.)

(ND4.2) Let B be a line-bundle over an arbitrary manifold M and let U be an open subset of M.

(i) A map $s: U \dashrightarrow B$ such that $\pi(s(m)) = m$ for every $m \in U$ is called a **section** over U or simply a **local section**. (In the case where U = M, s is simply referred to as a **section** of B.)

(ii) Let $\Gamma_B(M)$ be the set of all sections (over M).

(iii) Let $C^\infty_B(M)$ be the set of all smooth sections over M.

(Similarly , let $\Gamma_B(U)$ and $C^\infty_B(U)$ be the corresponding sets on U.)

(<u>Remarks</u>: (R3) Let B be a line-bundle over an arbitrary manifold M and let (U, ϕ) be a local trivialization for B. Then any section $s \in C^\infty_B(U)$ can be written in the form

$$s(m) = f(m)\, \phi(m, z)$$

where $m \in U$, $f(m) \in C^{\infty}_{\mathbb{C}}(U)$ and z is a non-zero (complex) constant.

(R4) It follows from remark (R3) that $C^{\infty}_{B}(M)$ is empty unless a local trivialization (M,ϕ) exists.

(R5) The simplest example of a line-bundle is the **trivial bundle** $B = M \times \mathbb{C}$. In this case, each point in B is given by the pair (m,z) where $m \in M$ and $z \in \mathbb{C}$ and the projection map $\pi: B \dashrightarrow M$ is given by $\pi(m,z) = m$. The set $C^{\infty}_{B}(M)$ is not empty because one can choose as a local trivialization (M,ϕ) where

$$\phi : M \times \mathbb{C} \dashrightarrow B : m \times z \dashrightarrow (m,z).$$

Then any $s \in C^{\infty}_{B}(M)$ is given by

$$s(m) = f(m)\phi(m,z) = (m, zf(m))$$

where $m \in M$, $f \in C^{\infty}_{\mathbb{C}}(M)$ and z is non-zero (complex) constant.

(ND5)<u>Definitions</u> [Simms and Woodhouse (1976), p25; Woodhouse (1980), p295]

(ND5.1) A **Hermitian structure** $(\cdot,\cdot)$ on a line-bundle B (over an arbitrary manifold M) is inner-product $(\cdot,\cdot)_m$ in each fibre B_m, $m \in M$, with the following property: for every s, $t \in \Gamma_B(M)$, the function defined by

$$(s,t) : M \dashrightarrow \mathbb{C} : m \dashrightarrow (s(m), t(m))_m$$

is smooth.

(ND5.2) Let B be a line-bundle over M with Hermitian structure $(\cdot,\cdot)$. A section $s_0 \in C^{\infty}_{B}(M)$ that satisfies the condition $(s_0, s_0) = 1$ everywhere on M is called a **unit section**. (Similarly, a **unit section** over $U \subseteq M$ can be defined.)

(<u>Remark</u>: (R6) The trivial bundle $B = M \times \mathbb{C}$ has a natural Hermitian structure given by

$$((m,z'),(m,z'')) = z'\overline{z''};\ (m,z'),\ (m,z'') \in B_m;$$

[cf. Simms and Woodhouse (1976), p26]. A unit section of B is given by

$$s_o(m) = (m,f(m));\ f(m) \in C^\infty_{\mathbb{C}}(M),\ |f(m)| = 1 \text{ everywhere on } M.)$$

(ND6) <u>Definitions</u> [Campbell (1983); Simms and Woodhouse (1976), pp25-26, p31; Woodhouse (1980), pp294-297]

Let B be a line-bundle over a manifold M.

(ND6.1) A **connection** ∇ is a map that assigns to each $X \in V_{\mathbb{C}}(M)$ an operator ∇_X on $C^\infty_B(M)$ such that:

(i) $\nabla_{fX+gY} = f\nabla_X + g\nabla_Y$;

(ii) $\nabla_X(fs) = X(f)s + f\nabla_X s$;

(iii) $\nabla_X(s+t) = \nabla_X s + \nabla_X t$;

for each $s,\ t \in C^\infty_B(M)$, $f,\ g \in C^\infty(M)$ and $X,\ Y \in V_{\mathbb{C}}(M)$. ($\nabla_X$ is called the **covariant derivative along** X.)

Let B be a line-bundle over M with connection ∇.

(ND6.2) A section $s \in C^\infty_B(M)$ is said to be **covariantly constant along the vector field** $X \in V_{\mathbb{C}}(M)$ if $\nabla_X s = 0$ everywhere on M.

(ND6.3) The **curvature two-form** of the connection is defined to be the complex two-form on M determined by

$$\mathrm{curv}(B,\nabla)(X,Y)s = (i/2)([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})s$$

where $X,\ Y \in V_{\mathbb{C}}(M)$, $s \in \Gamma_B(M)$ and $[\nabla_X, \nabla_Y] = \nabla_X(\nabla_Y s) - \nabla_Y(\nabla_X s)$ is the commutator of ∇_X and ∇_Y.

(ND6.4) Let B be a line bundle over M with Hermitian structure $(\cdot,\cdot)$ and connection ∇. Then the Hermitian structure and connection are said to be **compatible** if

$$X(s,t) = (\nabla_X s,t)+(s,\nabla_X t)$$

for every $X \in V_{\mathbb{C}}(M)$ and for every s, $t \in C^{\infty}(M)$.

The line-bundle B with Hermitian structure $(\cdot,\cdot)$ and connection ∇ is called a **Hermitian line-bundle with connection** if the Hermitian structure and connection are compatible. We shall denote a Hermitian line-bundle with connection by the triple $(B,(\cdot,\cdot),\nabla)$.

(<u>Remark</u>: (R7) Let $B = M \times \mathbb{C}$ be a trivial line-bundle over M with (natural) Hermitian structure $(\cdot,\cdot)$ [cf. remark (6)] and connection ∇. Let s_o be a unit section of B [cf. remark (R6)]. Then it follows from the definition of the connection ∇ that for each $X \in V_{\mathbb{C}}(M)$, there exists $g(m) \in C^{\infty}(M)$ such that $\nabla_X s = g(m)s_o$. The function g(m) can be rewritten in the form $-i(X \lrcorner \beta)$ where β is a complex one-form on M. Therefore, it is usual to define the connection on B by

$$\nabla_X \ s_o = -i(X \lrcorner \beta)s_o; \text{ for all } X \in V_{\mathbb{C}}(M).$$

The one-form β is called the **connection potential** [cf. Woodhouse (1980), p297].

It can be shown that

$$\mathrm{curv}(B,\nabla) = d\beta/\hbar$$

[cf. Woodhouse (1980), 297].)

(ND7) <u>Definitions, notation and theorem</u> [cf. Woodhouse (1980), p290-291]

(ND7.1) A (l-dimensional) **real distribution** on a manifold M is a map D that assigns to each $m \in M$ a subspace D_m of the tangent space at m, T_mM, such that

(i) $l = \dim D_m$ for all m;

(ii) In some neighbourhood U' of m' it is possible to find l smooth vector fields that span D_m, at each $m \in U'$.

(ND7.2) Let

$$V(M;D) = \{X \in V(M) \colon X \in D_m \text{ for all } m \in M\}.$$

The vector fields in V(M;D) are said to be **tangent** to D.

(ND7.3) A connected submanifold $\mathcal{N} \subset M$ is called an **integral surface** of a real distribution D if $T_m\mathcal{N} = D_m$, at every $m \in \mathcal{N}$.

We shall assume that M is a k-dimensional manifold.

(ND7.4) Let X, $Y \in V(M)$; then the **Lie bracket** of X and Y is defined to be the unique vector field $[X,Y] \in V(M)$ determined by

$$[X,Y]f = X(Y(f))-Y(X(f)), \text{ for all } f \in C^{\infty}(M).$$

Let U be an open subset of M with coordinates $(x_1,\ldots,x_k)$, and let $X = \sum_{i=1}^{k} a_i(\partial/\partial x_i)$ and let $Y = \sum_{j=1}^{k} b_j(\partial/\partial x_j)$ on U; (here $a_1,\ldots a_k, b_1,\ldots,b_k$ are smooth functions on U). Then on U, we have

$$[X,Y] = \sum_{j=1}^{k}\sum_{i=1}^{k} \{a_i(\partial b_j/\partial x_i)-b_i(\partial a_j/\partial x_i)\}(\partial/\partial x_i).$$

(ND7.5) A real distribution is said to be **involutive** if $[X,Y] \in V(M;D)$ for all X, $Y \in V(M;D)$.

(ND7.6) A (l-dimensional) real distribution D on M is said to be **integrable** if it is possible to find local coordinates $(x_1,\ldots,x_k)$ in some neighbourhood U of each point $m \in M$ such that surfaces

$$\{m \in U: x_{l+1} = \text{constant}, \ldots, x_k = \text{constant}\}$$

are integral surfaces of D. (In other words, ever point of M lies on an integral surface of the distribution D).

(ND7.7) **Frobenius Theorem:** A real distribution D is integrable if and only if it is involutive.

(ND7.8) The maximal integral surfaces of an integrable distribution are called **leaves of the distribution.**

(ND7.9) The **space of leaves of an integrable real distribution** D on M is denoted by M/D.

(ND7.10) An integrable distribution D on M is said to be **reducible** if M/D is a Hausdorff manifold with the projection map pr:M --> M/D being a smooth map.

APPENDIX 1.2

Integration of one-densities

Let Q be a k-dimensional manifold and let ρ be a one-TQ-density [cf. definition (1.1.6.D1)]. Let $(q_1,\ldots,q_k)$ be a set of coordinates on an open set U of Q. Then the **integral of ρ over U** is defined by

$$\int_U \rho = \int_U \rho\{\partial/\partial q_i\}dq_1\ldots dq_k$$

[cf. Loomis and Sternberg (1968), p409; Woodhouse (1980), p152].

Remarks: (R1) Suppose $(y_1,\ldots,y_k)$ is also a set of coordinates on U; then

$$\int_U \rho\{\partial/\partial y_i\}dy_1\ldots dy_k = \int_U \rho\{\partial/\partial q_j\}dq_1\ldots dq_k$$

because

$$\rho\{\partial/\partial q_j\} = |(\partial y_\alpha/\partial q_\beta)| \quad \{\partial/\partial y_i\} \quad \text{(by definition of one-TQ-density),}$$

and

$$dq_1\ldots dq_k = |(\partial q_\alpha/\partial y_\beta)|\ dy_1\ldots dy_k.$$

Thus the integral $\int_U \rho$ is independent of the choice of coordinates.

(R2) If U is not covered by a single coordinate chart, then the integral is built up using a partition of unity.

APPENDIX 1.3

The one-dimensional isotonic oscillator

Here is a list of results we shall use later:

$$\int[a+by+cy^2]^{-1/2}\,dy = -(-1/c)^{1/2}\sin^{-1}(2cy+b/[b^2-4ac]^{1/2}) \quad [c>0,\ b-4ac>0]; \qquad \text{(A1.3.Eq 1)}$$

$$\sin^{-1}A-\sin^{-1}B = -\cos^{-1}([1-A^2]^{1/2}[1-B^2]^{1/2}+AB) \quad [A<B]; \qquad \text{(A1.3.Eq 2)}$$

and

$$\int_0^{2\pi}[\sin^2\theta/(A+B\cos\theta)]d\theta = 2\int_0^{\pi}[\sin^2\theta/A+B\cos\theta]d\theta$$
$$= (2\pi A/B^2)[1-\{1-(B^2/A^2)\}^{1/2}] \qquad \text{(A1.3.Eq 3)}$$

[cf. Gradshteyn and Ryhzik (1980), p81, p49 and p379].

Let $Q = \mathbb{R}^+ = (0,\infty)$ with cartesian coordinate q, $M = T^*Q = \mathbb{R}\times\mathbb{R}^+$ with the usual cartesian canonical coordinates (p,q), and let ω be the canonical two-form on M. The Hamiltonian of the one-dimensional isotonic oscillator is given by

$$H = (p^2/2)+(q-1/q)^2$$

The Hamiltonian vector field X_H is given by

$$X = p(\partial/\partial q)-(1+1/q^2)(q-1/q)(\partial/\partial p)$$

Then we have $Z(H) = \{(0,1)\}$ and $M_0 = M - -Z(H) = \mathbb{R}\times\mathbb{R}^+ - -\{(0,1)\}$. Let $\gamma^E(t) = (p(t),q(t))$ be the integral curve of X_H that originates at the point $(p_0,q_0) = (0,\{[E+2+(E^2+4E)^{1/2}]/2\}^{1/2})$. Here p(t) and q(t) are solutions of the following differential equations:

$$dq(t)/dt = p(t) = 2^{1/2}[E+2-q^2-1/q^2]^{1/2} \qquad \text{(A1.3.Eq 4a)}$$

$$dp(t)/dt = -[1+(1/q(t))^2][q(t)-1/q(t)] \qquad \text{(A1.3.Eq 4b)}$$

with constant of motion

$$H(p(t),q(t)) = E. \qquad \text{(A1.3.Eq 4c)}$$

We shall split our presentation into two parts which we shall denote by (i) and (ii), respectively, as follows: in part (i) we shall solve for p(t) and q(t), and in part (ii) we shall construct action-angle variables on (M_0, ω_0).

(i) Integrating equation (A1.3.Eq 4a) we get

$$t = (1/2)\int_{q_0}^{q} [E+2-q^2-(1/q)^2]^{-1/2}dq$$

Put $q^2 = y$, $q_o = y_o$ and $dq = dy/2y^{1/2}$; then we get

$$t = (1/8)^{1/2}\int_{Y_o}^{Y} [(E+2)y-y^2-1]^{-1/2}\,dy$$

$$= (1/8)^{1/2}\,[-\sin^{-1}(\{-2y+(E+2)\}/\{E^2+4E\}^{1/2})]$$

[by equation (A1.3.Eq 1)]

$$= (1/8)^{1/2}\,[-\sin^{-1}(\{-2q^2+(E+2)\}/\{E^2+4E\}^{1/2})+\sin^{-1}(-1)]$$

$$= -(1/8)^{1/2}\,\cos^{-1}(\{2q^2-E-2\}/\{E^2+4E\}^{1/2})$$

[by equation (A1.3.Eq 2)]

Thus

$$q(t) = (1/2)[(E^2+4E)^{1/2}\cos(-8^{1/2}t)+E+2]^{1/2}$$

and

$$p(t) = dq(t)/dt = [\{E^2+4E\}/\{2(E^2+4E)^{1/2}\cos(-8^{1/2}t)+2(E+2)\}]^{1/2}\sin(-8^{1/2}t).$$

(ii) Clearly the period of $\gamma^E(t)$ is $T(E) = 2\pi/8^{1/2} = \pi/2^{1/2}$. The action-angle variables on (M_0,ω_0) are given by

$$I = [2\pi]^{-1}\oint_{\gamma E} p\,dq, \quad \theta = 2\pi t/T(E).$$

Explicitly, we have

$$\theta = -\cos^{-1}(\{2q^2-E-2\}/\{E^2+4E\}^{1/2})$$

and

$$\begin{aligned} I &= [2\pi]^{-1}\int_0^{T(E)} p(t)dq(t) \\ &= [2\pi]^{-1}\int_0^{T(E)}[\{E^2+4E\}/\{2(E^2+4E)^{1/2}\cos(-8^{1/2}t)+2(E+2)\}]\sin(-8^{1/2}t)\,dt \\ &= [2\pi]^{-1}\int_0^{2\pi}[\{E^2+4E\}/\{2(E^2+4E)^{1/2}\cos\theta+2(E+2)\}]\sin^2\theta\ (dt/d\theta)d\theta \\ &= E/8^{1/2} \qquad \text{(by equation (A1.3.Eq 3) and } (dt/d\theta) = (1/8)^{1/2}). \end{aligned}$$

APPENDIX 1.4

<u>The two-dimensional Kepler problem</u> [Abraham and Marsden (1980), pp622-625]

Here are two integrals that we shall use later [cf. Pierce (1929), pp41-42]:

$\int[(A+B\cos x)/(C+D\cos x)^2]dx$

$= [(BC-AD)/(C^2-D^2)][\sin x/\{C+D\cos x\}] + [\{AC-BD\}/\{C^2-D^2\}]\int\{C+D\cos x]^{-1}dx$

(A1.4.Eq 1a)

and

$\int[C+D\cos x]^{-1}dx = [2/(C^2-D^2)^{1/2}]\tan^{-1}([(C^2-D^2)^{1/2}\tan(x/2)/(C+D)])$

(where $-\pi<x<\pi$). (A1.4.Eq 1b)

We have:

$Q = \mathbb{R}^2$ with cartesian coordinates $\mathbf{q} = (q_1,q_2)$, $T^*Q = \mathbb{R}^4$ with usual canonical cartesian coordinates $(\mathbf{p},\mathbf{q}) = (p_1,p_2,q_1,q_2)$, $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^4$, $H(\mathbf{p},\mathbf{q}) = (\|\mathbf{p}\|^2/2)+(1/\|\mathbf{q}\|)$ (the Hamiltonian of the Kepler problem), $L(\mathbf{p},\mathbf{q}) = (q_1p_2-q_2p_1)$ (the angular momentum observable), $M_o = M = \{(\mathbf{p},\mathbf{q})\in\mathbb{R}^4\colon q \neq (0,0) \text{ and } H(\mathbf{p},\mathbf{q})<0\}$, $\omega_o = \sum_{i=1}^{2} dp_i\wedge dq_i$, and

$$X_H = p_1(\partial/\partial q_1)+p_2(\partial/\partial q_2)-(q_1/\|\mathbf{q}\|^3)(\partial/\partial p_1)-(q_2/\|\mathbf{q}\|^3)(\partial/\partial p_2),$$

$$X_L = -q_2(\partial/\partial q_1)+q_1(\partial/\partial q_2)-p_2(\partial/\partial p_1)+p_1(\partial/\partial p_2)$$

L is a constant of motion of the Hamiltonian system because

$$\{H,L\} = 2\omega(X_H,X_L) = 0.$$

Let $\mathcal{P}_c$ be the polarization of (M_o,ω_o) spanned by the vector fields X_H and X_L. Let m_o be a some point in M_o such that $H(m_o) = E$ and $L(m_o) = L_o$. Let $\gamma_1(t_1)$ be the integral curve of X_H that originates at the point $m_o\in M_o$, and let $\gamma_2(t_2)$ be the integral curve of X_L that originates at m_o. Let $T_1(E)$ and

$T_2(L_o)$ be the period of γ_1 and γ_2 respectively. Let $(I_1, I_2, \theta_1, \theta_2)$ be action-angle variables on (M_o, ω_o) given by

$$I_1 = [2\pi]^{-1}\oint_{\gamma_1} \beta_o\ , \quad \theta_1 = [2\pi/T_1(E)] + \theta_{1o}\ ,$$

$$I_2 = [2\pi]^{-1}\oint_{\gamma_2} \beta_o\ , \quad \theta_2 = [2\pi/T_2(L_o)] + \theta_{2o}\ ,$$

where $\beta_o = \sum_{i=1}^{2} p_i dq_i$, and θ_{1o} and θ_{2o} are real constants.

We shall split our this presentation into the two parts (i) and (ii), respectively, as follows: in part (i) we shall evaluate the action variable I_1, and in part (ii) we shall evaluate the action variable I_2. (We shall not evaluate the angle-variables explicitly because it is tedious and messy, and we do not need the explicit expressions.)

(i) For the sake of brevity, we shall replace t_1 by t in what follows. Let $\gamma_1(t) = (\mathbf{p}(t), \mathbf{q}(t))$. Here $\mathbf{p}(t)$ and $\mathbf{q}(t)$ are solutions of differential equations:

$$(dq_i(t)/dt) = p_i(t),\ (dp_i(t)/dt) = -[q_i(t)/\|\mathbf{q}(t)\|^3]\ ;\ (i = 1,\ 2) \tag{A1.4.Eq 2a}$$

with constants of motion

$$H(\mathbf{p}(t), \mathbf{q}(t)) = E\ ,\ L(\mathbf{p}(t), \mathbf{q}(t)) = L_o. \tag{A1.4.Eq 2b}$$

It has been shown by Abraham and Marsden (1980) [cf. pp624-625] that the solutions for $q_1(t)$ and $q_2(t)$ are given by

$$q_1(t) = r(\alpha(t))\cos\alpha(t),\ q_2(t) = r(\alpha(t))\sin\alpha(t) \tag{A1.4.Eq 3a}$$

here $\alpha(t)$ is a solution of the differential equation

$$(d\alpha/dt) = L_o/[r(\alpha(t))]^2, \tag{A1.4.Eq 3b}$$

where $r(\alpha(t))$ is given by

$$r(\alpha(t)) = [L_o^2/\{1+K\cos(\alpha-\lambda)\}];\ K = (1+2EL_o^2)^{1/2},\ \lambda \text{ is a constant.} \tag{A1.4.Eq 3c}$$

(Strictly speaking λ is a constant dependant on m_o.)

Here is a list of a few more results that we shall use later:

$$p_1(t) = (dq_1(t)/dt) = -r\sin\alpha + (dr/d\alpha)\cos\alpha, \qquad \text{(A1.4.Eq 4a)}$$

$$p_2(t) = (dq_2(t)/dt) = r\cos\alpha + (dr/d\alpha)\sin\alpha, \qquad \text{(A1.4.Eq 4b)}$$

$$(1/r)(dr/d\alpha) = -[\{K\sin(\alpha-\lambda)\}/\{1+K\cos(\alpha-\lambda)\}] \qquad \text{(A1.4.Eq 4c)}$$

The action variable I_1 can now be evaluated as follows. (We shall choose the domain of integration with respect to $(\alpha-\lambda)$ to be $(-\pi,\pi)$.)

Then

$$I_1 = (2\pi)^{-1}\oint_{\gamma E}[\sum_{i=1}^{2} p_i(\alpha)\{dq_i(\alpha)/d\alpha\}]d\alpha$$

$$= (2\pi)^{-1}\int_{-\pi}^{\pi}[\sum_{i=1}^{2}(dq_i/d\alpha)^2(d\alpha/dt)]d(\alpha-\lambda) \qquad \text{(by equation (A1.4.Eq 2a))}$$

$$= (2\pi)^{-1}\int_{-\pi}^{\pi}[\{r^2+(dr/d\alpha)^2\}\{L_o/r^2\}]d(\alpha-\lambda)$$

(by equations (A1.4.Eq 3b), (A1.4.Eq 4a) and (A1.4.Eq 4b))

$$= (2\pi)^{-1}L_o\int_{-\pi}^{\pi}[1+(1/r)(dr/d\alpha)^2]d(\alpha-\lambda)$$

$$= (2\pi)^{-1}L_o\int_{-\pi}^{\pi}[\{1+K^2+2K\cos(\alpha-\lambda)\}/\{1+K\cos(\alpha-\lambda)\}^2]d(\alpha-\lambda)$$

(by equation (A1.4.Eq 4c))

$$= (2\pi)^{-1}\{[L_oK/(1-K)][\sin x/(1+K\cos x)]_{\pi}^{-\pi}\} + (2\pi)^{-1}L_o\int_{-\pi}^{\pi}[1+K\cos x]^{-1}dx$$

(by equation (A1.4.Eq 1a), when $\alpha-\lambda = x$)

$$= (2\pi)^{-1}L_o\{2/(1-K^2)^{1/2}\}[\tan^{-1}\{(1-K^2)^{1/2}\tan(x/2)/(1+K)\}]_{-\pi}^{\pi}$$

(by equation (A1.4.Eq 1b))

$$= (-2E)^{-1/2}.$$

(ii) Let $\gamma_2(t_2) = (\mathbf{p}(t_2),\mathbf{q}(t_2))$. For the sake of brevity, we shall write t for t_2. Then $\mathbf{p}(t)$ and $\mathbf{q}(t)$ are solutions of the following differential equations:

$$(dq_1(t)/dt) = -q_2(t),\ (dq_2(t)/dt) = q_1(t), \qquad \text{(A1.4.Eq 5a)}$$

$$(dp_1(t)/dt) = -p_2(t),\ (dp_2(t)/dt) = p_1(t), \qquad \text{(A1.4.Eq 5b)}$$

with constant of motion

$$L(\mathbf{p}(t),\mathbf{q}(t)) = L_o. \qquad \text{(A1.4.Eq 5c)}$$

The differential equations can be rewritten in the form

$$(d^2q_1(t)/dt^2) = -q_1(t) \ ,(d^2p_1(t)/dt^2) = -p_1(t).$$

Let $m_o = (p_{1o}, p_{2o}, q_{1o}, q_{2o})$; then

$$q_1(t) = q_{1o}\cos t - q_{2o}\sin t,$$

$$q_2(t) = q_{2o}\cos t + q_{1o}\sin t,$$

$$p_1(t) = p_{1o}\cos t - p_{2o}\sin t,$$

$$p_2(t) = p_{2o}\cos t + p_{1o}\sin t,$$

Therefore, the period of $\gamma_2(t)$ is $T_2(L_o) = 2\pi$.

The action variable I can now be evaluated as follows. Then

$$I_2 = [2\pi]^{-1}\int_0^{2\pi} [\sum_{i=1}^{2} p_i(t)(dq_i(t)/dt)]dt$$

$$= [2\pi]^{-1}\int_0^{2\pi} [p_1(t)q_2(t)-p_2(t)q_1(t)]dt \quad \text{[by equation (A1.4.Eq 5a)]}$$

$$= [2\pi]^{-1}\int_0^{2\pi} L_o dt \quad \text{[by equation (A1.4.Eq 5c)]}$$

$$= L_o.$$

APPENDIX 1.5

The WKB solutions of the Schrodinger equations in the position representation and momentum representation

(A1.5.1) Certain formulas from differential calculus [Gradshetyn and Ryzhik (1980), p19]

(i) Leibnitz rule for the r-derivative of a product of two functions

Let u(p) and v(p) be two r-times-differentiable functions of p. Then

$$(d^r(uv)/dp^r) = u(d^r v/dp^r) + {}^rC_1(du/dp)(d^{r-1}v/dp^{r-1}) + \ldots$$
$$+ {}^rC_{r-1}(d^{r-1}u/dp^{r-1})(dv/dp) + {}^rC_r(d^r u/dp^r)v \qquad \text{(A1.5.Eq 1)}$$

where ${}^rC_k = [r!/k!(r-k)!]$.

(ii) the k-th derivative of a composite function

Let f(p) = F(y) and let y = G(p), then the k-th derivative of the composite function f(p) is given by

$$(\partial^k f(p)/\partial p^k) = \sum \{k!/(a!b!\ldots c!)\}(\partial^j F/\partial y^j)\{y'/1!\}^a\{y''/2!\}^b\ldots\{y^{(t)}/t!\}^c$$
$$\text{(A1.5.Eq 2)}$$

where $y^{(r)} = (\partial^r y/\partial p^r)$, and the symbol $\sum$ indicates the summation over all solutions in positive integers of the following equations:

$$a+b+\ldots+c = j \text{ and } a+2b+\ldots+tc = k.$$

(A1.5.T1) Theorem

Let $\phi_W^E(q) = f(q)\{\exp \pm S(q)\}$ and let $\phi_{C,W}^E(p) = g(p)\{\exp \pm W(p)\}$. Then:

(i) $$(\hat{H}-E)\phi_W^E(q) = \hbar^0\{(1/2)(\partial S/\partial q)^2 + V(q) - E\}\phi_W^E(q)$$
$$+ \hbar(-i)\{(1/2)(\partial^2 S/\partial q^2)f + (\partial f/\partial q)(\partial S/\partial q)\}\{\exp \pm S\}$$
$$+ \text{higher order terms of } \hbar$$

(ii) $(\hat{H}_c - E)\,\phi^{\epsilon}_{c,w}(p) = \hbar^{0}[(p^2/2)+\{\sum_{k=0}^{\infty} A_k(-\partial W/\partial p)^k\}-E]\,\phi^{\epsilon}_{c,w}(p)$

$- i\hbar[\sum_{r=1}^{\infty}(i)^{2r} A_r r\{[(r-1)/2](\partial W/\partial p)^{r-2}(\partial^2 W/\partial p^2)g+(\partial W/\partial p)^{r-1}(\partial g/\partial p)\}]\{\exp \not{i} W\}$

+ higher order terms in $\hbar$

Proof.

(i) $[-(\hbar^2/2)(\partial^2/\partial q^2)+V(q)-E][\{\exp \not{i} S(q)\}f(q)\}]$

$= -(\hbar^2/2)(\partial/\partial q)[\{\not{i}(\partial S/\partial q)f+(\partial f/\partial q)\}\{\exp \not{i} S\}] + [V-E]\{\exp \not{i} S\}f$

$= -(\hbar^2/2)[\not{i}(\partial^2 S/\partial q^2)f+\not{i}(\partial S/\partial q)(\partial f/\partial q)+(\partial^2 f/\partial q^2)+\not{i}^2(\partial S/\partial q)^2 f$

$+(\partial S/\partial q)(\partial f/\partial q)]\{\exp \not{i} S\} + [V-E]f\{\exp \not{i} S\}$ (here $\not{i} = (i/\hbar)$)

$= \hbar^{0}[(1/2)(\partial S/\partial q)^2+V-E]\{\exp \not{i} S\}f - i\hbar[(1/2)(\partial^2 S/\partial q^2)f+(\partial S/\partial q)(\partial f/\partial q)]\{\exp \not{i} S\}$

+ higher order terms of $\hbar$.

(ii) We shall start our proof by deriving the following result:

$(i\hbar)^r(\partial^r/\partial p^r)[g(p)\{\exp \not{i} W(p)\}]$

$= (i\hbar)^r[\{\exp \not{i} W\}(\partial^r g/\partial p^r)+r(\partial\{\exp \not{i} W\}/\partial p)(\partial^{r-1}g/\partial p^{r-1})+\ldots.$

$+{}^rC_{r-1}(\partial^{r-1}\{\exp \not{i} W\}/\partial p^{r-1})(\partial g/\partial p) + {}^rC_r(\partial^r\{\exp iW\}/\partial p^r)g$

(by equation (A1.5.Eq 1))

$= \hbar\,[i^{2r}\{\exp \not{i} W\}g\{\partial W/\partial p\}^r] + \hbar[i^{2r-1}\{\exp \not{i} W\}\{\partial W/\partial p\}^{r-2}g\{r!/(r-1)!2!\}$

$+ {}^rC_{r-1}i^{2r-1}\{\exp \not{i} W\}\{\partial W/\partial p\}^{r-1}\{\partial g/\partial p\}]$ + higher order terms in $\hbar$.

The last line was obtained by using

$(\partial^k/\partial p^k)\{\exp \not{i} W\}$

$= [\sum\{k!/a!b!\ldots c!\}\not{i}^r(\partial W/\partial p)^a\{(\partial^2 W/\partial p^2)/2!\}^b\ldots\{(\partial^t W/\partial p^t)/t!\}^c]\{\exp \not{i} W\}$

[cf. equation (A1.5.Eq 2)].

Thus

$[(p^2/2)+\sum_{k=1}^{\infty}(i\hbar)^r A_r(\partial/\partial p)^r-E]g(p)\{\exp \not{i} W\}$

$= \hbar^{0}[(p^2/2)+\{\sum_{k=0}^{\infty} A_k(-\partial W/\partial p)^k\}-E][g\{\exp \not{i} W\}]$

$- i\hbar[\sum_{r=1}^{\infty}(i)^{2r} A_r r\{[(r-1)/2](\partial W/\partial p)^{r-2}(\partial^2 W/\partial p^2)g+(\partial W/\partial p)^{r-1}(\partial g/\partial p)\}]\{\exp \not{i} W\}$

+ higher order terms in $\hbar$. ∎

(A1.5.T2) Theorem

If $(-\partial W/\partial p) = q$, then

$$g(p) = |\sum_{k=1}^{\infty} kA_k(-\partial W/\partial p)^{k-1}|^{-1/2} = |\sum_{k=1}^{\infty} kA_k q^{k-1}|^{-1/2} \quad \text{(A1.5.Eq 3a)}$$

is a solution of the following equation

$$[\sum_{r=1}^{\infty} (-1)^r A_r r\{((r-1)/2)(\partial W/\partial p)^{r-2}(\partial^2 W/\partial p^2)g+(\partial W/\partial p)^{r-1}(\partial g/\partial p)\}] = 0. \quad \text{(A1.5.Eq 3b)}$$

(Here q is treated a a local function of p on integral curve γ^E.)

Proof.

We shall split the proof into two cases according to whether $\sum_{k=1}^{\infty} kA_k(-\partial W/\partial p)^{k-1}$ is positive or negative as follows.

Case 1: $\sum_{k=1}^{\infty} kA_k(-\partial W/\partial p)^{k-1} > 0$

In this case, we have

$$g(p) = [\sum_{k=1}^{\infty} kA_k(-\partial W/\partial p)^{k-1}]^{-1/2}$$

and

$$(\partial g/\partial p) = [(g^3/2)(\partial^2 W/\partial p^2)\{\sum_{k=1}^{\infty} k(k-1)A_k(-\partial W/\partial p)^{k-2}\}]$$

Let

$$G = [\sum_{k=1}^{\infty} \{k(k-1)/2\}(-\partial W/\partial p)^{k-2}].$$

Then evaluating the left hand side of equation (A1.5.Eq 3) we get

$$\sum_{r=1}^{\infty}(-1)^r A_r[\{r(r-1)/2\}(\partial W/\partial p)^{r-2}(\partial^2 W/\partial p^2)g+r(\partial W/\partial p)^{r-1}(\partial^2 W/\partial p^2)(g^3/2)\mathbf{G}$$

$$= (\partial^2 W/\partial p^2)g[\sum_{r=1}^{\infty}\{r(r-1)/2\}A_r(-\partial W/\partial p)^{r-2}]+Gg^3(\partial^2 W/\partial p^2)[-\sum_{j=1}^{\infty} jA_j(-\partial W/\partial p)^{j-1}]$$

(by using $(-1)^r = (-1)^{r-2} = -(-1)^{r-1}$)

$$= (\partial^2 W/\partial p^2)gG[1-g^2\{\sum_{j=1}^{\infty} jA_j(-\partial W/\partial p)^{j-1}]$$

= 0 (by definition of g).

Case 2: $\sum_{r=1}^{\infty} rA_r(-\partial W/\partial p)^{r-1} < 0$

In this case, we have

$$g(p) = [-\sum_{r=1}^{\infty} rA_r(-\partial W/\partial p)^{r-2}]^{-1/2}$$

and

$$(\partial g/\partial p) = [(g^3/2)(\partial^2 W/\partial p^2)\{-\sum_{k=1}^{\infty} k(k-1)A_k(-\partial W/\partial p)^{k-2}\}]$$

As in the last case one can show that g(p) defined by equation (A1.5.Eq 3a) is a solution of the differential equation given by equation (A1.5.Eq 3b).■

APPENDIX 1.6

(A1.6.N1) Notation

(ND1) Let j be a positive integer, then Δ_j is the interval given by

$$\Delta_j = [2j\pi-(2\pi-\theta_2), 2j\pi+\theta_2].$$

Let us subdivide Δ_j into the subintervals:

$$[2k\pi, 2k\pi+\theta_2], \ k\in\mathbb{Z} \text{ and } -j\leqslant k\leqslant j;$$

and

$$[2r\pi-(2\pi-\theta_2), 2j\pi], \ r\in\mathbb{Z} \text{ and } -j\leqslant r\leqslant j.$$

(ND2) Let us fix k and r. Then on the interval $[2k\pi, 2k\pi+\theta_2]$ the map $\theta \dashrightarrow p(\theta)$ has a unique inverse which we shall denote by $\theta^{+}_{k}(p)$. Similarly, on the interval $[2r\pi-(2\pi-\theta_2), 2r\pi]$ the map $\theta \dashrightarrow p(\theta)$ has a unique inverse which we shall denote by $\theta^{-}_{r}(p)$.

Then we can write $(\overline{\phi_c\, e_c}^{c,j})(p)$ in the form

$$\begin{aligned}(\overline{\phi_c\, e_c}^{c,j})(p) &= \sum_{\substack{\theta\in\Delta_j \\ p(\theta)=p}} \phi_c(\theta)e_c(\theta) \\ &= \sum_{\pm}\sum_{k=-j}^{k=+j} \phi_c(\theta^{\pm}_{k}(p))e\ (\theta^{\pm}_{k}(p)).\end{aligned}$$

(A1.6.L1) Lemma [Woodhouse (1980), p294]

Let y be a cartesian coordinate on an open interval (a,b) in $\mathbb{R}$, f(y) be a compactly supported smooth function and let A(y) be a real-valued smooth function. Then as $\hbar \dashrightarrow 0$, we have

$$[2\pi\hbar]^{-1/2}\int_a^b\{\exp \pm A\}f\,dy = [\sum_{A'=0} |(\partial^2 A/\partial y^2)|^{-1/2}\{\exp i(\pi/4)\text{sign}(A'')\}\{\exp \pm A\}f]+O(\hbar).$$

Here $A'' = (\partial^2 A/\partial y^2)$, sign(A") is the signature of A", and $\sum_{A'=0}$ is the summation over the critical points of A(y). The set of critical points is

$\{y: (\partial A/\partial y) = 0\}$; (these critical points are assumed to be nondegenerate, and hence isolated.)

(A1.6.T1) Theorem [Eckmann and Seneor (1976)]

For each $q \in \{q: \exists\, p$ such that $(p,q) \in \gamma^{\epsilon}\}$ and positive integer j, we have

$$(F\overline{\phi_c e_c}^{c,j})(q) = \sum_{\substack{\theta \in \Delta_j \\ q(\theta)=q}} \phi(\theta)e_c(\theta)+O(\hbar)$$

if $J(\theta)$ and $J_c(\theta)$ satisfy the following conditions:

$$J_c(\theta)-J(\theta) = -\pi/4 (\mathrm{mod}\ 2\pi) \text{ for } \theta \in \bigcup_{k\in\mathbb{Z}} (2k\pi, 2k\pi+\theta_1),$$

$$J_c(\theta)-J(\theta) = +\pi/4 (\mathrm{mod}\ 2\pi) \text{ for } \theta \in \bigcup_{k\in\mathbb{Z}} (2k\pi+\theta_1, 2k\pi+\theta_2),$$

$$J_c(\theta)-J(\theta) = -\pi/4 (\mathrm{mod}\ 2\pi) \text{ for } \theta \in \bigcup_{k\in\mathbb{Z}} (2k\pi+\theta_2, 2k\pi+\theta_3),$$

$$J_c(\theta)-J(\theta) = +\pi/4 (\mathrm{mod}\ 2\pi) \text{ for } \theta \in \bigcup_{k\in\mathbb{Z}} (2k\pi+\theta_3, 2k\pi+2\pi).$$

Proof.

Let $\sigma_c = \mathrm{supp}(\overline{\phi_c\ e_c}^{c,j})$ (the support of $(\overline{\phi_c\ e_c}^{c,j})(p)$). Then for each $q \in \{q: \exists\, p \text{ s.t } (p,q) \in \gamma^{\epsilon}\}$, we have

$$(F\overline{\phi_c e_c}^{c,j})(q) =$$

$$= [2\pi\hbar]^{-1/2} \sum_{\pm} \sum_{k=-j}^{k=j} [\int_{\sigma_c} dp\{\exp \pm(pq-\int_0^{\theta_k^{\pm}} q(\theta)(\partial p(\theta)/\partial\theta)d\theta)\} |\partial p(\theta_k^{\pm}(p))/\partial\theta|^{-1/2}$$
$$\times\ e_c(\theta_k^{\pm}(p))\{\exp iJ_c(\theta_k^{\pm}(p))\}]$$

Let $A_k^{\pm}(p)$ be the phase given by

$$A_k^{\pm}(p) = pq - \int_0^{\theta_k^{\pm}} q(\theta)(\partial p(\theta)/\partial\theta)d\theta.$$

The first and second derivative of $A^{\pm}(p)$ with respect to p are

$$(\partial A_k^{\pm}/\partial p) = q - q(\theta_k^{\pm}(p))$$

and

$$(\partial^2 A_k^{\pm}/\partial p^2) = -(\partial q/\partial p)(\theta_k^{\pm}(p))$$

respectively. The phases $A_k^{\pm}(p)$ have isolated critical points on the sets $\{p: q(\theta_k^{\pm}(p)) = q\}$.

Then for each $q \in \{q: \exists\ p \text{ s.t } (p,q) \in \gamma^{\epsilon}\}$, we get

$$(F\,\overline{\phi_c e_c}^{c,j})(q) =$$

$$= [\sum_{\substack{\theta \in \Delta_j \\ q(\theta)=q}} |(\partial q(\theta)/\partial\theta)|^{-1/2} |(\partial p(\theta)/\partial\theta)|^{-1/2} \{\exp \pm(p(\theta) - \int_0^{\theta} q(\theta)(\partial p(\theta)/\partial\theta)d\theta)\}$$

$$\times \{\exp iJ_c(\theta)\}\{\exp -i(\pi/4)\operatorname{sign}(-(\partial q/\partial p)(\theta))\} e_c(\theta)] + O(\hbar)$$

(by Lemma (A1.6.L1))

$$= \sum_{\substack{\theta \in \Delta_j \\ q(\theta)=q}} \phi_c(\theta) e_c(\theta) \{\exp i(J_c(\theta) - J(\theta) - (\pi/4)\operatorname{sign}((\partial q(\theta)/\partial p))\} + O(\hbar).$$

The assertion follows from this result. ■

APPENDIX 1.7

(A1.7.L1) Lemma

If $(A/2\pi) \notin \mathbb{Z}$, then we have

$$\lim_{r\to\infty} \left[(1/r)\sum_{k=-r}^{r} \{\exp ikA\}\right] = 0.$$

Proof

We get

$$(1/r)\sum_{k=-r}^{r} \{\exp ikA\} = (1/r)\sum_{k=-r}^{r} [\cos(kA)+\sin(kA)]$$

$$= (1/r)\sum_{k=-r}^{k} \cos(kA) \text{ (since } \sin(kA) = -\sin(-kA))$$

$$= (1/r)\sum_{k=-r}^{k} [2\cos(kA)-1] \text{ (since } \cos(kA) = \cos(-kA))$$

$$= (1/r)[2\cos((r+1/2)A)\sin(rA/2)\mathrm{cosec}(A/2)+1-1]$$

[cf. Gradshteyn and Ryzhik (1980), p30]

$$= (1/r)[2\cos((r+1/2)A)\sin(rA/2)\mathrm{cosec}(A/2)]$$

Since $(A/2\pi) \notin \mathbb{Z}$, it follows that:

(i) $\mathrm{cosec}(A/2)$ is finite;

(ii) $(1/r)\left|\sum_{r=-k}^{k} \{\exp ikA\}\right| < (1/r)\mathrm{cosec}(A/2)$.

Thus

$$\left|\lim_{r\to\infty} (1/r)\left[\sum_{k=-r}^{k} \{\exp ikA\}\right]\right| < \lim_{r\to\infty} [(1/r)\mathrm{cosec}(A/2)] = 0. \blacksquare$$

List of results:

We have:

$$\oint_{\gamma_E} pdq = -\oint_{\gamma_E} qdp;$$

$$\int_0^{2k\pi} p(\theta)(\partial q(\theta)/\partial\theta)d\theta = k\oint_{\gamma_E} pdq,\ k \in \mathbb{Z};$$

$$-\int_0^{2k\pi} q(\theta)(\partial p(\theta)/\partial\theta)d\theta = k\oint_{\gamma E} pdq,\ k \in \mathbb{Z}\ ;$$

$$J(2k\pi)-J(0) = -k\pi \text{ (by equation (1.3.4.Eq 9a))};$$

$$J_c(2k\pi)-J_c(0) = -k\pi \text{ (by equation (1.3.4.Eq 9a))}.$$

Thus

$$\phi(\theta+2k\pi)e(\theta+2k\pi) = \phi(\theta)e(\theta)\{\exp ik(\hbar^{-1}\oint_{\gamma E} pdq-\pi)\} \qquad \text{(A1.7.Eq 1a)}$$

and

$$\phi_c(\theta+2k\pi)e_c(\theta+2k\pi) = \phi_c(\theta)e_c(\theta)\{\exp ik(\hbar^{-1}\oint_{\gamma E} pdq-\pi)\} \qquad \text{(A1.7.Eq 1b)}.$$

(A1.7.T1) <u>Theorem</u> [Eckmann and Seneor (1976)]

If

$$[2\pi]^{-1}\oint_{\gamma E} pdq \neq (n+1/2)\hbar, \text{ for all } n \in \mathbb{Z},$$

then $\Phi^E(q)$ the Maslov-WKB wave function (corresponding to the energy E) is the zero-function; i.e., $\Phi^E = 0$ everywhere on the configuration space $\mathbb{R}$. Otherwise, $\Phi^E(q)$ is a non-trivial function.

<u>Proof</u>.

We have

$$\Phi^E(q) = \lim_{J\to\infty} (1/j)[(\mathbf{F}\,\overline{\phi_c e_c}^{c,j}\ )(q)+(\overline{\phi\, e}^{j}\ )(q)]$$

by equation (1.3.4.Eq 10). Let $A = \hbar^{-1}\oint_{\gamma E} pdq-\pi$. Then by equations (A1.7.Eq 1a) and (A1.7.Eq 1b), we get

$$(\overline{\phi\, e}^{j}\ )(q) = [\sum_{\substack{\theta\in[2\pi-\theta_3,\theta_1]\\ q(\theta)=q}} \phi\, e][\sum_{k=-j}^{+j} \{\exp ikA\}]+[\sum_{\substack{\theta\in[\theta_1,\theta_3]\\ q(\theta)=q}}\phi\, e][\sum_{k=-j}^{j} \{\exp ikA\}] \qquad \text{(A1.7.Eq 2a)}$$

and

$$(\overline{\phi_c e_c}^{c,j}\ )(p) = [\sum_{\substack{\theta\in[0,\theta_2]\\ p(\theta)=p}} \phi_c\, e_c][\sum_{k=-j}^{j} \{\exp ikA\}]+[\sum_{\substack{\theta\in[\theta_2,2\pi]\\ p(\theta)=p}} \phi_c\, e_c][\sum_{k=-j}^{j} \{\exp ikA\}]. \qquad \text{(A1.7.Eq 2b)}$$

Suppose $(A/2\pi) \notin \mathbb{Z}$, then we have

$$\lim_{j\to\infty} |(1/j)(\overline{\phi\, e}^{j})(q)| < 2[\max_{\theta\in[0,2\pi)} |\phi(\theta)e(\theta)|][\lim \{(1/j)|\sum_{k=-j}^{k} \{\exp ikA\}|$$

(by equation (A1.7.Eq 2a)

$$= 0 \text{ (by Lemma (A1.7.L1)).}$$

We shall now prove that $[\lim_{j\to\infty} (1/j)(F\overline{\phi_c\, e_c}^{c,j})(q)] = 0$ when $(A/2\pi) \notin \mathbb{Z}$ as follows. The function $(1/j)(\overline{\phi_c\, e_c}^{c,j})(p)$ is uniformly bounded because $e_c(\theta) = 0$ in the neighbourhood of points belonging to the set $\Pi_c = \{\theta: \exists\, k \in \mathbb{Z}$ with either $\theta = 2k\pi$ or $\theta = \theta_2+2k\pi\}$. Then by equation (A1.7.Eq 2b), we get the following inequality

$$(1/j)(\overline{\phi_c\, e_c}^{c,j})(p) < 2 \max_{\theta\in[0,2\pi)} |\phi_c(\theta)e_c(\theta)|.$$

Therefore, we can interchange the Fourier transform F and the limit symbol lim in the expression lim $[(1/j)(F\overline{\phi_c e_c}^{c,j})(q)]$. Then,

$$\lim_{j\to\infty} |(1/j)(F\overline{\phi_c\, e_c}^{c,j})(q)| < 2[\max |\phi_c(\theta)e_c(\theta)|][\lim \{(1/j)|\sum_{k=-j}^{j} \{\exp ikA\}|$$

$$= 0 \text{ (by Lemma (A1.7.L1)).}$$

Now $\Phi^{\epsilon}(q)$ is clearly a non-trivial function if $(A/2\pi)$ is an integer. Therefore, $\Phi^{\epsilon}(q) = 0$ everywhere on the configuration space $\mathbb{R}$ if $(A/2\pi)$ is not an integer. ■

APPENDIX 1.8

(A1.8.T1) Theorem [Eckmann and Seneor (1976)]

The Maslov-WKB wave function (corresponding to the energy E) $\Phi^{E}(q)$ satisfies the following condition

$$\|(\hat{H}-E)\,\Phi^{E}(q)\| = O(\hbar^2).$$

Here $\|\ \|$ is the norm in $L^2(\mathbb{R})$ given by

$$\|\Psi(q)\| = \int_{\mathbb{R}} |\Psi(q)|^2 dq;\ \Psi(q) \in L^2(\mathbb{R}).$$

Proof.

If we have $[2\pi]^{-1}\oint_{\gamma E} pdq \neq (n+1/2)\hbar$ for all $n \in \mathbb{Z}$, then $\Phi^{E}(q) = 0$ and the assertion is trivially true.

In the case where $[2\pi]^{-1}\oint_{\gamma E} pdq = (n+1/2)\hbar$ for some integer n, $\Phi^{E}(q)$ is not a zero-function and it can be written in the form

$\Phi^{E}(q) = (\overline{\phi e}^{1,0}) + (F\,\overline{\phi_c e_c}^{c,0})$ (by equations (A1.7.Eq 1a) and (A1.7.Eq 1b)).

In Appendix 3.2 of Chapter 3 we shall prove that

$$\|\,(\hat{H}-E)[(\overline{\phi e}^{1,0})(q)+(F\,\overline{\phi_c e_c}^{c,0})(q)]\| = O(\hbar^2). \blacksquare$$

CHAPTER 2

THE HALF-DENSITY QUANTIZATIONS IN CANONICALLY CONJUGATE POLARIZATIONS AND THEIR UNITARY EQUIVALENCE IN 2-DIMENSIONAL SYMPLECTIC MANIFOLDS

HALF-DENSITY QUANTIZATIONS IN CANONICALLY CONJUGATE POLARIZATIONS AND THEIR UNITARY EQUIVALENCE IN 2-DIMENSIONAL SYMPLECTIC MANIFOLDS

(2.1) Introduction

We shall start by giving a few definitions.

(2.1.D1) Definition [Sniatycki (1980), p1]

Let $\mathcal{P}$ and $\mathcal{P}'$ be any two reducible polarizations of a symplectic manifold (M,ω), and let H and H' be any two Hilbert spaces associated with the polarizations $\mathcal{P}$ and $\mathcal{P}'$ respectively. (Note that in the case of the standard half-density quantization scheme the Hilbert spaces H and H' are taken to be the quantization Hilbert spaces $H_{\mathcal{P}}$ and $H_{\mathcal{P}'}$ respectively.) Let $\{\zeta_1,\zeta_2,\ldots.\}$ be a set of classical observables on (M,ω). Then the quantizations of $\{\zeta_1,\zeta_2,\ldots.\}$ in $\mathcal{P}$ and $\mathcal{P}'$ are said to be **unitarily equivalent** (or **unitarily related**) if the following two conditions are satisfied:

(UEQ1) ζ_j is quantizable in H if and only if ζ_j is quantizable in H';

(UEQ2) For each quantizable $\zeta_j \in \{\zeta_1,\zeta_2,\ldots\}$, let $\widetilde{\zeta}_j$ and $\widetilde{\zeta}_j'$ be the corresponding quantized operators in H and H' respectively. There exists a unitary map $\mathbf{U}:\mathbf{H} \dashrightarrow \mathrm{H}'$ such that for each quantizable observable $\zeta_j \in \{\zeta_1,\zeta_2,\ldots.\}$, we have

$$U\widetilde{\zeta}_jU^{-1} = \widetilde{\zeta}_j'. \qquad (2.1.\text{Eq } 1).$$

Alternatively, we say that the quantizations of $\{\zeta_1,\zeta_2,\ldots.\}$ in H and H' are **unitarily equivalent (or unitarily related)** if conditions (UEQ1) and (UEQ2) are satisfied.

(2.1.D2) Definition [Woodhouse (1980), p290]

Two polarizations $\mathcal{P}$ and $\mathcal{P}'$ of a symplectic manifold (M,ω) are said to be **transverse** if

$$\mathcal{P}_m + \mathcal{P}'_m = T_m M, \text{ for every } m \in M. \qquad (2.1.\text{Eq } 2a)$$

(Note that the condition given by equation (2.1.Eq 2a) is satisfied if and only if

$$\mathcal{P}_m \cap \mathcal{P}'_m = \{0\}, \text{ for every } m \in M. \qquad (2.1.\text{Eq } 2b))$$

(2.1.D3) Definitions [Blattner (1973); Guillemin and Sternberg (1977), p271]

Two polarizations $\mathcal{P}$ and $\mathcal{P}_c$ of a 2k-dimensional symplectic manifold (M,ω) are said to be **canonically conjugate (or Heisenberg related)** if there exists in a neighbourhood of each point of M (local) canonical coordinates $\{p_1,\ldots,p_k,q_1,\ldots,q_k\}$ such that $\mathcal{P}$ is spanned locally by the vector fields $\{\partial/\partial p_1,\ldots,\partial/\partial p_k\}$, and $\mathcal{P}_c$ is spanned locally by the vector fields $\{\partial/\partial q_1,\ldots,\partial/\partial q_k\}$.

Remarks: (R1) Clearly canonically conjugate polarizations are transverse.
(R2) The most common example of canonically conjugate polarizations is the vertical polarization P and horizontal polarization P of the cotangent bundle [cf. example (1.1.5.Ex 1)].
(R3) Throughout this chapter we shall use the term polarization to refer to a reducible polarization.

We shall now give the motivation behind this chapter. One of the outstanding problems confronting geometric quantization is the failure to establish the unitary equivalence of quantizations in different polarizations. This is closely linked to the difficulty to establish a unitary link between the different quantization Hilbert spaces.

In the case where $\mathcal{P}$ and $\mathcal{P}'$ are transverse polarizations there is a formal procedure for establishing a pairing (or link) between $\mathcal{P}$ and $\mathcal{P}'$ which is given as follows. Suppose $\mathcal{P}$ and $\mathcal{P}'$ are transverse; then one can write down a formal sesquilinear map from the product space $H_{\mathcal{P}} \times H_{\mathcal{P}'}$ to $\mathbb{C}$ (where $H_{\mathcal{P}}$ and $H_{\mathcal{P}'}$ are the quantization Hilbert spaces associated with the polarizations $\mathcal{P}$ and $\mathcal{P}'$ respectively); this map is referred to as the **pairing map between $H_{\mathcal{P}}$ and $H_{\mathcal{P}'}$**. The pairing map determines a linear map between $H_{\mathcal{P}}$ and $H_{\mathcal{P}'}$ called the **linear map (induced by the pairing map)**. Some details on these two maps are given in Appendix 2.1. However, in general the linear map (induced by the pairing map) is not unitary. Our object here is two-fold. Firstly, we shall highlight this unitary inequivalence with a number of examples. Secondly, as a contribution to tackling this outstanding problem, we shall propose a scheme based on physical reasoning for establishing unitary equivalence applicable at least to the examples considered. The main results presented here have been published [cf. Wan , McKenna and Pinto (1984); Wan, Pinto and McKenna (1984)]. We shall restrict ourselves to the study of simple examples of canonically conjugate polarizations of 2-dimensional symplectic manifolds. It follows from remark (R1) that we can construct a linear map (induced by the pairing map) between the quantization Hilbert spaces of canonically conjugate polarizations. In section (2.2) we shall concentrate on examples in contractible symplectic manifolds. In section (2.3) we shall extend the results obtained in the previous section to examples in noncontractible symplectic manifolds.

2.2 HALF-DENSITY QUANTIZATIONS IN CANONICALLY CONJUGATE POLARIZATIONS AND THEIR UNITARY EQUIVALENCE IN 2-DIMENSIONAL CONTRACTIBLE SYMPLECTIC MANIFOLDS

(2.2.1) Two simple one-dimensional problems in quantum mechanics

Let Q be an open interval of $\mathbb{R}$ and let q be the usual cartesian coordinate on Q. Let (p,q) be the usual cartesian canonical coordinates on T^*Q. Let β_0 and ω be respectively the canonical one-form and canonical two-form on T^*Q [cf. example (1.1.2.Ex 1)]. Let P and P_c be respectively the vertical and horizontal polarizations of (T^*Q,ω) [cf. remark (R2) of section (2.1)]. Let $pr:T^*Q \dashrightarrow Q$ be the usual projection map of the cotangent bundle T^*Q. Let $Q_c = M/P_c = \mathbb{R}$ be the effective configuration space with respect to the polarization P_c and let $pr_c:T^*Q \dashrightarrow Q_c$ be the corresponding projection map.

Let $B = T^*Q \times \mathbb{C}$ be the trivial line-bundle over T^*Q. Let $(\cdot,\cdot)$ be the natural Hermitian structure on B and let s_0 be a unit section of B. Let be the connection on B defined by

$$\nabla_X s_0 = -\frac{i}{\hbar}(X \lrcorner\ pdq)s_0, \text{ for all } X \in V_{\mathbb{C}}(T^*Q). \qquad (2.2.1.\text{Eq } 1)$$

i.e., β_0 is the chosen connection potential. Let $(B,(\cdot,\cdot),\nabla)$ be the chosen prequantization bundle over (T^*Q,ω).

In this subsection, we shall study two simple one-dimensional problems in quantum mechanics in which the configuration and phase spaces of the corresponding classical mechanical system are respectively Q and T^*Q. We shall use these examples to illustrate the following point: the standard half-density quantizations of the canonical variables p and q in the canonically conjugate polarizations P and P_c are only unitarily related when $Q = \mathbb{R}$.

(2.2.1.Ex 1) Example: The free particle

Consider the physical situation of a particle free to move on the real line; then the configuration space is $Q = \mathbb{R}$ and the phase space is $T^*Q = \mathbb{R}^2$. In quantum mechanics, the spectrum of the quantum momentum observable and the spectrum of the position observable are both $(-\infty,\infty)$ [cf. Messiah (1961), pp63-65].

We shall now quantize the canonical variables p and q in the canonically conjugate polarizations P and P_c using the standard half-density quantization scheme and compare the predicted physical results in each polarization with the corresponding results of quantum mechanics. We shall start by quantizing p and q in the vertical polarization P.

The quantization Hilbert space H_P consists of square-integrable sections of $B \times \Delta_{-1/2}(P)$ of the form $\Psi = \psi(q)\rho$ where $\rho = s_0|dp|^{-1/2}$.

The position variable is quantizable in H_P, and the quantization operator corresponding to q is given by the multiplication operator

$$\tilde{q}\Psi = q\Psi \qquad (2.2.1.\text{Eq } 2a)$$

with domain

$$D_{\tilde{q}} = \{\Psi \in H_P : q\Psi \in H_P\} \qquad (2.2.1.\text{Eq } 2b)$$

[cf. equations (1.1.7.Eq 7a) and (1.1.7.Eq 7b)]. In particular, the spectrum of $\tilde{q}$ is $(-\infty,\infty)$.

The associated vector field $pr_*(X_p)$ generated by p is given explicitly by the expression $pr_*(X_p) = (\partial/\partial q)$; for short, let $pr_*(X_p) = Y_p$. The momentum observable p is quantizable because Y_p is complete on Q. The quantization operator $\tilde{p}$ corresponding to p is given by the expression

$$\tilde{p}\Psi = -i\hbar(\partial\psi(q)/\partial q)\rho, \qquad (2.2.1.\text{Eq } 3a)$$

and the domain of $\widetilde{p}$ is given by

$$D_{\widetilde{p}} = \{\Psi = \psi(q)\rho \in H_p: \psi(q) \in AC(Y_p, Q),\ \widetilde{p}\,\Psi \in H_p\} \qquad (2.2.1.\text{Eq } 3b)$$

[cf. equations (1.1.7.Eq 6a) and (1.1.7.Eq 6b)]. The spectrum of $\widetilde{q}$ and the spectrum of $\widetilde{p}$ agree with the results of quantum mechanics.

We shall now quantize p and q in the horizontal polarization P_c.

The quantization Hilbert space H_{P_c} consists of square-integrable sections of $B \times \Delta_{-1/2}(P_c)$ of the form $\Phi_c = \varphi_c(p)\rho_c$ where $\rho_c = \{\exp \pm pq\} s_0 |dq|^{-1/2}$.

The associated vector field $pr_{c*}(X_q)$ generated by q is given explicitly by the expression $pr_{c*}(X_q) = -(\partial/\partial p)$; for short, let $pr_{c*}(X_q) = Y^c_q$. The position variable q is quantizable in the polarization P_c because Y^c_q is complete on Q_c. The quantization operator $\widetilde{q}_c$ corresponding to q in H_{P_c} is given by the expression

$$\widetilde{q}_c\Phi_c = i\hbar(\partial\varphi_c(p)/\partial p)\rho_c, \qquad (2.2.1.\text{Eq } 4a)$$

and the domain of $\widetilde{q}_c$ is given by

$$D_{\widetilde{q}_c} = \{\Phi_c = \varphi_c(p)\rho_c \in H_{P_c}: \varphi_c(p) \in AC(Y^c_q, Q_c),\ \widetilde{q}_c\Phi_c \in H_{P_c}\}$$
$$(2.2.1.\text{Eq } 4b)$$

[cf. equations (1.1.7.Eq 10a) and (1.1.7.Eq 10b)]. The spectrum of $\widetilde{q}_c$ is $(-\infty,\infty)$.

The momentum variable p is quantizable in the polarization P_c. The quantization operator $\widetilde{p}_c$ corresponding to the classical variable p is given by the multiplication operator

$$\widetilde{p}_c\Phi_c = p\Phi_c \qquad (2.2.1.\text{Eq } 5a)$$

and the domain of $\widetilde{p}_c$ is given by

$$D_{\widetilde{p}_c} = \{\Phi_c \in H_{P_c}: p\Phi_c \in H_{P_c}\} \qquad (2.2.1.\text{Eq } 5b)$$

[cf. equations (1.1.7.Eq 11a) and (1.1.7.Eq 11b)]. The spectrum of $\widetilde{p}_c$ is

$(-\infty,\infty)$.

The spectrum of $\widetilde{q}_c$ and the spectrum of $\widetilde{p}_c$ agree with the results of quantum mechanics.

The linear map (induced by the pairing map) $U_{PP_c}: H_P \dashrightarrow H_{P_c}$ is given by

$$U_{PP_c}\Psi = (2\pi\hbar)^{-1/2}\left[\int_{-\infty}^{\infty} \Psi(q)\{\exp -\tfrac{i}{\hbar}pq\}dq\right]\rho_c \qquad (2.2.1.\text{Eq } 6)$$

[cf. Appendix 2.1, example (A2.1.Ex 1)]. The map U_{PP_c} is the identifiable with the inverse Fourier transform [cf. equation (1.3.3.Eq 13) for the Fourier transform]; therefore, U_{PP_c} is unitary.

Clearly, in this case the half-density quantizations of the canonical variables p and q in the canonically conjugate polarizations P and P_c are unitarily equivalent.

(2.2.1.Ex 2) Example: The one-dimensional infinitely high potential barrier

Consider the one-dimensional quantum mechanical problem of a particle encountering an infinitely high potential barrier [cf. Messiah (1961), p86]. The special feature of this problem is that the wave functions vanish at the edge of the barrier.

We shall, for definiteness, choose $Q = \mathbb{R}^+ = (0,\infty)$: the particle is constrained to move on the positive part of the real line. Then $T^*Q = \mathbb{R} \times \mathbb{R}^+$.

We shall start by quantizing p and q in the polarization P, as we did in the last example.

The quantization Hilbert space H_{ρ} consists of square-integrable sections of $B \times \Delta_{-1/2}(P)$ of the form $\Psi = \psi(q)\rho$ where $q \in (0,\infty)$ and $\rho = s_o |dp|^{-1/2}$.

The position variable is quantizable in H_P, and the quantization operator corresponding to q is given by the multiplication operator

$$\tilde{q}\Psi = q\Psi \qquad (2.2.1.\text{Eq } 7a)$$

with domain

$$D_{\tilde{q}} = \{\Psi \in H_P : q\Psi \in H_P\}. \qquad (2.2.1.\text{Eq } 7b)$$

The spectrum of $\tilde{q}$ is $(0,\infty)$ as it should be according to quantum mechanics.

The associated vector field $pr_*(X_p)$ generated by p is $(\partial/\partial q)$. The momentum variable p is not quantizable in H_ρ because $pr_*(X_p)$ is not complete on Q.

We shall now quantize p and q in the horizontal polarization P_c.

The quantization Hilbert space H_{P_c} consists of square-integrable sections of $B \times \Delta_{-1/2}(P_c)$ of the form $\Phi_c = \varphi_c(p)\rho_c$ where $\rho_c = \{\exp \frac{i}{\hbar} pq\} s_o |dq|^{-1/2}$.

The associated vector field $pr_{c*}(X_q)$ generated by q is given explicitly by the expression $pr_{c*}(X_q) = -(\partial/\partial p)$; for short, let $pr_{c*}(X_q) = Y^c{}_q$. The position variable q is quantizable in the polarization P_c because $Y^c{}_q$ is complete on Q_c. The quantization operator $\tilde{q}_c$ corresponding to q in H_{P_c} is given by the expression

$$\tilde{q}_c \Phi_c = i\hbar(\partial\varphi_c(p)/\partial p)\rho_c, \qquad (2.2.1.\text{Eq } 8a)$$

and the domain of $\tilde{q}_c$ is given by

$$D_{\tilde{q}_c} = \{\Phi_c = \varphi_c(p)\rho_c \in H_{P_c} : \varphi_c(p) \in AC(Y^c{}_q, Q_c),\ \tilde{q}_c\Phi_c \in H_{P_c}\}.$$
$$(2.2.1.\text{Eq } 8b)$$

The spectrum of $\widetilde{q}_c$ is $(-\infty,\infty)$; this result is in disagreement with the result predicted by quantum mechanics for the spectrum of the quantum position observable.

The momentum variable is quantizable in the polarization P_c. The quantization operator $\widetilde{p}_c$ corresponding to the classical variable p is given by the multiplication operator

$$\widetilde{p}_c \Phi_c = p\,\Phi_c \qquad (2.2.1.\text{Eq } 9a)$$

and the domain of $\widetilde{p}_c$ is given by

$$D_{\widetilde{p}_c} = \{\Phi_c \in H_{P_c} : p\Phi_c \in H_{P_c}\}. \qquad (2.2.1.\text{Eq } 9b)$$

The spectrum of $\widetilde{p}_c$ is $(-\infty,\infty)$.

The linear maps (induced by the pairing map) $U_{PP_c} : H_P \dashrightarrow H_{P_c}$ and $U_{P_cP} : H_{P_c} \dashrightarrow H_P$ are given by

$$U_{PP_c}\Psi = (2\pi\hbar)^{-1/2}\left[\int_0^{\infty} \Psi(q)\{\exp -\tfrac{i}{\hbar}pq\}dq\right]\rho_c \qquad (2.2.1.\text{Eq } 10a)$$

and

$$U_{P_cP}\Phi_c = (2\pi\hbar)^{-1/2}\left[\int_{\mathbb{R}} \phi_c(p)\{\exp \tfrac{i}{\hbar}pq\}dp\right]\rho \qquad (2.2.1.\text{Eq } 10b)$$

respectively [cf. Appendix (2.1)]. These maps are not unitary.

The standard half-density quantizations of the canonical variables p and q in the canonically conjugate polarizations P and P_c are not unitarily equivalent because of the following three reasons:

(i) the variable p is only quantizable in H_{P_c};

(ii) the spectra of the operators $\widetilde{q}$ and $\widetilde{q}_c$ do not coincide;

(iii) the linear maps (induced by the pairing map) U_{PP_c} and U_{P_cP} are not unitary.

This example illustrates an important feature of the standard half-density quantization scheme: the physical results predicted by the scheme depends on the choice of polarization employed.

Let us examine this problem more closely to pinpoint the reason why the spectrum of the operator $\widetilde{q}_c$ in H_{ρ_c} disagrees with the result predicted by quantum mechanics for the spectrum of the quantum position observable.

We recall that the inner-product on H and H are given by

$$\langle \Psi, \Psi \rangle_\rho = [2\pi\hbar]^{-1/2} \int_0^\infty |\psi(q)|^2 dq, \ \Psi = \psi(q)\rho \in H_\rho; \qquad (2.2.1.\text{Eq } 11a)$$

and

$$\langle \Phi_c, \Phi_c \rangle_{\rho_c} = [2\pi\hbar]^{-1/2} \int_{-\infty}^\infty |\varphi_c(p)|^2 dp; \ \Phi_c = \varphi_c(p)\rho_c \in H_{\rho_c}; \qquad (2.2.1.\text{Eq } 11b)$$

[cf. equations (1.1.7.Eq 5) and (1.1.7.Eq 9)]. It follows from these inner-products that H_ρ is identifiable with $L^2(\mathbb{R}^+)$, and H_{ρ_c} is identifiable with $L^2(\mathbb{R})$. Let $F: L^2_c(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ be Fourier transform given by equation (1.3.3.Eq 13) and let F^{-1} be the inverse Fourier transform. (Here $L^2_c(\mathbb{R})$ is the space of square-integrable functions of p, and $L^2(\mathbb{R})$ is the space of square-integrable functions of q.) Let F_+^{-1} be the restriction of the inverse Fourier transform F^{-1} to $L^2(\mathbb{R}^+)$.

As $L^2(\mathbb{R}^+)$ is a proper subspace of $L^2(\mathbb{R})$ it follows that $F_+^{-1} L^2(\mathbb{R}^+)$ is a proper subspace of $L^2_c(\mathbb{R})$. Clearly the maps $U_{\rho\rho_c}$ and F_+^{-1} are identical. Therefore, $U_{\rho\rho_c} H_\rho$ is a proper subspace of H_{ρ_c}.

Now let us extend the range of q from $\mathbb{R}^+$ to $\mathbb{R}$ in the expression for $U_{\rho_c\rho}$ given by equation (2.2.1.Eq 10b). Then for each $\Phi_c = \varphi_c(p)\rho_c \in H_{\rho_c}$, we have $U_{\rho_c\rho}\Phi_c = (F\varphi_c)(q)$; so the maps $U_{\rho_c\rho}$ and F are identical. Clearly H_{ρ_c} contains elements of the form $\Phi_c = \varphi_c(p)\rho_c$ such that the support of $(F\varphi_c)(q)$ is not wholly contained in $\mathbb{R}^+$, so $U_{\rho_c\rho}H_{\rho_c}$ contains elements that do not belong to H_ρ.

From a physical point of view, we have already established that H_P is the correct space to use for the quantization of the variables p and q. It is now clear that the reason that the quantization of q in H_{P_c} leads to physically incorrect results is that the quantization Hilbert space H_{P_c} is "too large", so the quantization operator $\widetilde{q}_c$ admits too big a spectrum.

We shall now propose a way, based on physical reasoning, of rendering the quantizations of the variables p and q in the polarizations P and P_c unitarily equivalent.

Since we have established that the quantization Hilbert space H_{P_c} is "too large" it seems reasonable, from a physical point of view, to choose the (proper) subspace of H_{P_c} spanned by the generalized eigensections of $\widetilde{q}_c$ with positive eigenvalues as the physically correct Hilbert associated with the polarization P_c. We shall denote such a subspace by $H^+_{P_c}$. An alternative definition of $H^+_{P_c}$ is given as follows. The Hilbert space $H^+_{P_c}$ consists of elements of the form $\Phi_c = \varphi_c(p)\rho_c$ that satisfy the condition

$$[2\pi\hbar]^{-1/2}\int_{-\infty}^{\infty}\varphi_c(p)\{\exp \tfrac{i}{\hbar}pq\}dp = 0 \text{ if } q<0, \qquad (2.1.2.\text{Eq } 12a)$$

or alternatively,

$$\varphi_c(p) = [2\pi\hbar]^{-1/2}\int_0^{\infty}\Psi(q)\{\exp -\tfrac{i}{\hbar}pq\}dq \text{ for some } \Psi = \Psi(q)\rho \in H_P. \qquad (2.2.1.\text{Eq } 12b)$$

The restriction of $\widetilde{p}_c$ to H_{P_c} is not self-adjoint.

Clearly the quantizations of the canonical variables p and q in H_P and H_{P_c} are unitarily related. So by choosing $H^+_{P_c}$ instead of H_{P_c} as the physically correct Hilbert space associated with the polarization P_c we have rendered the quantizations of the canonical variables p and q in the polarizations P and P_c unitarily equivalent.

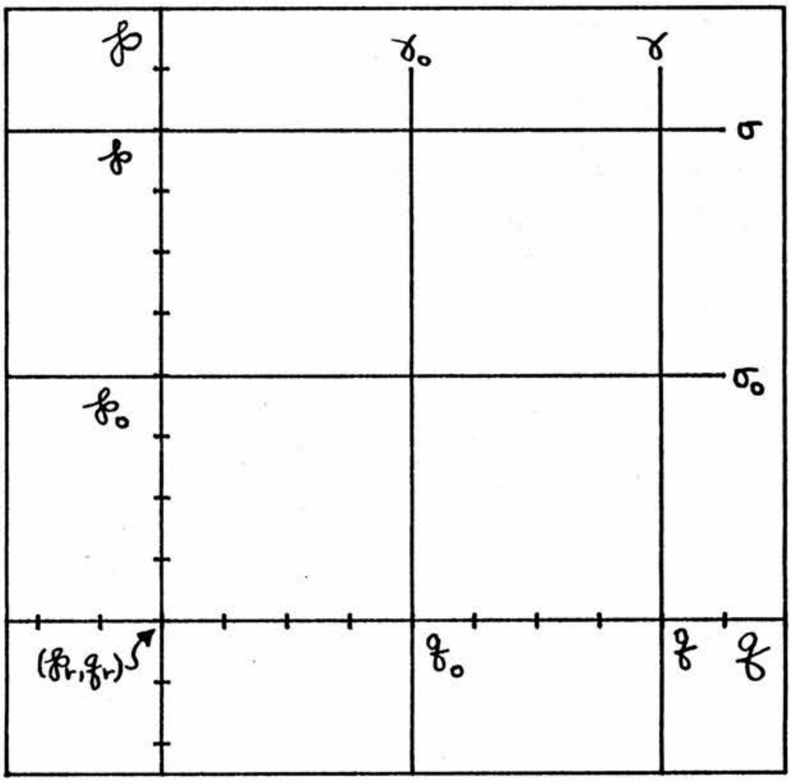

Fig 2-1: The coordinate curves

In the next subsection we shall present a general scheme for rendering quantizations in canonically conjugate polarizations unitarily equivalent.

(2.2.2) Quantum Hilbert spaces and unitarily equivalent quantizations

Let (M,ω) be a two-dimensional contractible symplectic manifold with global canonical coordinates (p,q), i.e. $\omega = dp\wedge dq$.

Let γ_m denote the p-coordinate curve passing through the point m in M, i.e. the integral curve of the vector field $(\partial/\partial p)$ that originates at m. Similarly, let σ_m be the q-coordinate curve through m, i.e. the integral of the vector field $(\partial/\partial q)$ that originates at m. Let $R_m(p)$ be the range of values of p along γ_m and let $R_m(q)$ be the range of values of q along σ_m. For the sake of simplicity we shall assume that $R_m(p)$ and $R_m(q)$ are independent of m, and so for brevity we shall drop the subscript m and write $R(p)$ and $R(q)$.

All the notation we shall introduce in this paragraph are illustrated in Fig 2-1. Let (p_r, q_r) be the chosen reference point in $R(p) \times R(q)$ and let (p_0, q_0) be an arbitrary point in $R(p) \times R(q)$. Then let:

(i) γ denote the p-coordinate curve through the point (p_r, q);

(ii) γ_0 denote the p-coordinate curve through the point (p_r, q_0);

(iii) σ denote the q-coordinate curve through the point (p, q_r);

(iv) σ_0 denote the -coordinate curve through the point (p_0, q_r).

Let β be a global one-form on M that satisfies the condition $d\beta = \omega$. Since M is contractible it follows from Poincare's lemma [cf. Von Westenholz (1981), pp165-167] that the closed one-form $(\beta - p\,dq)$ is globally exact. Therefore, there exists a function $f(p,q) \in C^{\infty}(M)$ such that

$$\beta = \mathfrak{p}\,d\mathfrak{q} + df(\mathfrak{p},\mathfrak{q}). \qquad (2.2.2.\text{Eq 1a})$$

We shall write

$$\beta = \beta_{\mathfrak{p}}(\mathfrak{p},\mathfrak{q})\,d\mathfrak{p} + \beta_{\mathfrak{q}}(\mathfrak{p},\mathfrak{q})\,d\mathfrak{q} \qquad (2.2.2.\text{Eq 1b})$$

where

$$\beta_{\mathfrak{p}} = (\partial f/\partial \mathfrak{p}) \text{ and } \beta_{\mathfrak{q}} = \mathfrak{p} + (\partial f/\partial \mathfrak{q}). \qquad (2.2.2.\text{Eq 1c})$$

Here is a list of line integral of along the coordinate curves that we shall need:

$$S_\sigma(\mathfrak{p},\mathfrak{q}) = \int_\sigma \beta = \Big(\int_{\mathfrak{q}_r}^{\mathfrak{q}} \beta_{\mathfrak{q}}\, d\mathfrak{q}\Big)_{\mathfrak{p}=\text{const.}};\; S_{\sigma_0}(\mathfrak{p}_0,\mathfrak{q}) = \int_{\sigma_0} \beta = \Big(\int_{\mathfrak{q}_r}^{\mathfrak{q}} \beta_{\mathfrak{q}}\, d\mathfrak{q}\Big)_{\mathfrak{p}=\text{const. } \mathfrak{p}_0}; \qquad (2.2.2.\text{Eq 2a})$$

$$S_\gamma(\mathfrak{p},\mathfrak{q}) = \int_\gamma \beta = \Big(\int_{\mathfrak{p}_r}^{\mathfrak{p}} \beta_{\mathfrak{p}}\, d\mathfrak{p}\Big)_{\mathfrak{q}=\text{const.}};\; S_{\gamma_0}(\mathfrak{p},\mathfrak{q}_0) = \int_{\gamma_0} \beta = \Big(\int_{\mathfrak{p}_r}^{\mathfrak{p}} \beta_{\mathfrak{p}}\, d\mathfrak{p}\Big)_{\mathfrak{q}=\text{const. } \mathfrak{q}_0}; \qquad (2.2.2.\text{Eq 2b})$$

$$S_\sigma(\mathfrak{p},\mathfrak{q}_0) = \Big(\int_{\mathfrak{q}_r}^{\mathfrak{q}} \beta_{\mathfrak{q}}\, d\mathfrak{q}\Big)_{\mathfrak{p}=\text{const.}};\; S_\gamma(\mathfrak{p}_0,\mathfrak{q}) = \Big(\int_{\mathfrak{p}_r}^{\mathfrak{p}_0} \beta_{\mathfrak{p}}\, d\mathfrak{p}\Big)_{\mathfrak{q}=\text{const.}}. \qquad (2.2.2.\text{Eq 2c})$$

These are given explicitly by:

$$S_\sigma(\mathfrak{p},\mathfrak{q}) = \mathfrak{p}(\mathfrak{q}-\mathfrak{q}_r) + f(\mathfrak{p},\mathfrak{q}) - f(\mathfrak{p},\mathfrak{q}_r);\; S_{\sigma_0}(\mathfrak{p}_0,\mathfrak{q}) = \mathfrak{p}_0(\mathfrak{q}-\mathfrak{q}_r) + f(\mathfrak{p}_0,\mathfrak{q}) - f(\mathfrak{p}_0,\mathfrak{q}_r); \qquad (2.2.2.\text{Eq 2d})$$

$$S_\gamma(\mathfrak{p},\mathfrak{q}) = f(\mathfrak{p},\mathfrak{q}) - f(\mathfrak{p}_r,\mathfrak{q});\; S_{\gamma_0}(\mathfrak{p},\mathfrak{q}_0) = f(\mathfrak{p},\mathfrak{q}_0) - f(\mathfrak{p}_r,\mathfrak{q}_0); \qquad (2.2.2.\text{Eq 2e})$$

$$S_\sigma(\mathfrak{p},\mathfrak{q}_0) = \mathfrak{p}(\mathfrak{q}_0-\mathfrak{q}_r) + f(\mathfrak{p},\mathfrak{q}_0) - f(\mathfrak{p},\mathfrak{q}_r);\; S_\gamma(\mathfrak{p}_0,\mathfrak{q}) = f(\mathfrak{p}_0,\mathfrak{q}) - f(\mathfrak{p}_r,\mathfrak{q}). \qquad (2.2.2.\text{Eq 2f})$$

In the case where (0,0) is a point in $R(\mathfrak{p}) \times R(\mathfrak{q})$ one normally chooses $(\mathfrak{p}_r,\mathfrak{q}_r) = (0,0)$.

Let $B = M \times \mathbb{C}$ be the trivial line bundle over M, $(\cdot,\cdot)$ be the natural Hermitian structure, s_o be a unit section of B and let ∇ be the connection on B defined by

$$\nabla_X s_o = -i(X \lrcorner \beta)s_o \text{ for all } X \in V_{\mathbb{C}}(M). \qquad (2.2.2.\text{Eq 3})$$

Let $(B,(\cdot,\cdot),\nabla)$ be the chosen prequantization bundle over (M,ω).

Let $\mathcal{P}$ and $\mathcal{P}_c$ be the canonically conjugate polarizations of (M,ω) spanned by the vector fields $(\partial/\partial \mathfrak{p})$ and $(\partial/\partial \mathfrak{q})$ respectively. Let Q and Q_c be the effective configuration spaces with respect to the polarizations $\mathcal{P}$ and $\mathcal{P}_c$ respectively. Then Q is identifiable with $R(\mathfrak{q})$, and Q_c is identifiable with $R(\mathfrak{p})$. Let $pr:M \dashrightarrow Q$ and $pr_c:M \dashrightarrow Q_c$ be the usual projection maps from M onto Q and Q_c respectively.

We shall restrict ourselves to the study of four cases as follows:

(1) $R(\mathfrak{p}) = \mathbb{R}$, $R(\mathfrak{q}) \neq \mathbb{R}$;

(2) $R(\mathfrak{p}) \neq \mathbb{R}$, $R(\mathfrak{q}) = \mathbb{R}$;

(3) $R(\mathfrak{p}) = \mathbb{R}$, $R(\mathfrak{q}) = \mathbb{R}$;

(4) $R(\mathfrak{p}) \neq \mathbb{R}$, $R(\mathfrak{q}) \neq \mathbb{R}$.

Our objective is to establish unitarily equivalent quantizations of the canonical variables $\mathfrak{p}$ and $\mathfrak{q}$ in the canonically conjugate polarizations $\mathcal{P}$ and $\mathcal{P}_c$ for the above four cases.

Case (1)

Let (p,q) be the usual cartesian canonical coordinates on the phase space of a particle confined to an infinitely deep one-dimensional potential well and let R(q) be the range of the position variable q. Then according to quantum mechanics, the wave function in the position representation vanishes at the edges of the potential well, i.e. the spectrum of the quantum position observable is R(q) [cf. Messiah (1961), pp86-88]. We shall make use of this result when we make physical assumptions.

We shall start by quantizing p and q in the quantization Hilbert space H_{P_c}.

The quantization pre-Hilbert space W_{P_c} consists of square-integrable sections of the bundle B x $\Delta_{-1/2}(P_c)$ of the form $\Psi_c = s_c \nu_c$ which obey the following conditions:

$$\nabla_{X_p} s_c = 0 \text{ and } \nabla_{X_p} \nu_c = 0 \text{ where } X_p = (\partial/\partial q). \qquad (2.2.2.\text{Eq } 4)$$

Explicitly, we have

$$\Psi_c = \psi_c(p)\rho_c, \; p \in \mathbb{R}; \qquad (2.2.2.\text{Eq } 5\text{a})$$

where

$$\rho_c = s_{co}|dq|^{-1/2}; \; s_{co} = \{\exp \pm S_\sigma(p,q)\} s_0. \qquad (2.2.2.\text{Eq } 5\text{b})$$

The inner-product on W_{P_c} is given by

$$\langle \Psi_c, \Psi_c \rangle_{P_c} = [2\pi\hbar]^{-1/2} \int_{\mathbb{R}} |\psi_c(p)|^2 dp. \qquad (2.2.2.\text{Eq } 6)$$

The quantization Hilbert space H_{P_c} is the completion of W_{P_c}.

The associated vector field $pr_{c*}(X_q) = -(\partial/\partial p)$; for short, let $Y^c_q = pr_{c*}(X_q)$. The canonical variable q is quantizable in H_{P_c} because Y^c_q is complete on Q_c. The quantization operator $\tilde{q}_c$ in H_{P_c} is given by

$$\tilde{q}_c \Psi_c = [\{i\hbar(\partial/\partial p) + (\partial f(p,q_r)/\partial p) + q_r\} \psi_c(p)]\rho_c \qquad (2.2.2.\text{Eq } 7\text{a})$$

[cf. Appendix 2.3], acting on the domain

$$D_{\tilde{q}_c} = \{\Psi_c \in H_{P_c} : \psi_c(p) \in AC(Y^c_q, Q_c), \tilde{q}_c \Psi_c \in H_{P_c}\}. \qquad (2.2.2.\text{Eq } 7\text{b})$$

The spectrum of q_c is $(-\infty,\infty)$.

The variable p can be quantized in H_{P_c} with quantization operator $\tilde{p}_c$ the multiplication operator

$$\tilde{p}_c \Psi_c = p\Psi_c, \qquad (2.2.2.\text{Eq } 8\text{a})$$

and the domain of $\tilde{p}_c$ is given by

$$D_{\tilde{p}_c} = \{\Psi_c \in H_{P_c} : \tilde{p}_c \Psi_c \in H_{P_c}\}. \qquad (2.2.2.\text{Eq } 8\text{b})$$

The spectrum of $\tilde{p}_c$ is $(-\infty,\infty)$.

We shall now construct explicit expressions for the generalized eigensections of $\tilde{q}_c$.

(2.2.2.P1) Proposition

Each value q_0 of the classical variable q is a **generalized eigenvalue of the operator** $\tilde{q}_c$ in H_{ρ_c} corresponding to the **generalized eigensections**

$$\Psi_{cq_0} = \psi_{cq_0}(p)\rho_c \qquad (2.2.2.\text{Eq } 9a)$$

where

$$\psi_{cq_0}(p) = [2\pi\hbar]^{-1/4}\{\exp \tfrac{i}{\hbar}S_{\gamma_0}(p,q_0)\}\{\exp -\tfrac{i}{\hbar}S_{\sigma}(p,q_0)\} \qquad (2.2.2.\text{Eq } 9b)$$

The generalized eigensections satisfy

$$\langle \Psi_{cq_0}, \Psi_{cq_0'}\rangle_{\rho_c} = \delta(q_0 - q_0'). \qquad (2.2.2.\text{Eq } 10)$$

(Note that our expression for Ψ_{cq_0} is $[2\pi\hbar]^{1/4}$ times the expression given in the paper by Wan, McKenna, Pinto (1984); the reason for this that our choice of inner-product on H_{ρ_c} is different from that given in the above-mentioned paper.)

Proof

We have:

$$S_{\gamma_0}(p,q_0) = f(p,q_0)-f(p_r,q_0);\ S_{\sigma}(p,q_0) = p(q_0-q_r)+f(p,q_0)-f(p,q_r);$$

$$S_{\gamma_0}(p,q_0)-S_{\sigma}(p,q_0) = -f(p_r,q_0)-p(q_0-q_r)+f(p,q_r);$$

$$[(\partial/\partial p)\{S_{\gamma_0}(p,q_0)-S_{\sigma}(p,q_0)\}] = -(q_0-q_r)+\{\partial f(p,q_r)/\partial p\};$$

$$\begin{aligned}\{\partial\psi_{cq_0}(p)/\partial p\} &= \tfrac{i}{\hbar}[(\partial/\partial p)\{S_{\gamma_0}(p,q_0)-S_{\sigma}(p,q_0)\}]\psi_{cq_0}(p)\\ &= \tfrac{i}{\hbar}[-(q_0-q_r)+\{\partial f(p,q_r)/\partial p\}]\psi_{cq_0}(p).\end{aligned}$$

Thus

$$\tilde{q}_c\Psi_{cq_0} = [\{i\hbar(\partial/\partial p)+\partial f(p,q_r)/\partial p+q_r\}\psi_{cq_0}(p)]\rho_c = q_0\Psi_{cq_0}. \qquad (2.2.2.\text{Eq } 11)$$

Thus Ψ_{cq_0} is a generalized eigensection of q_c corresponding to the generalized eigenvalue q_0.

The crucial point now is to observe that for each $q_0 \in R(q)$, the generalized eigensection can be constructed in three steps as follows.

(i) Let γ_0 be the p-coordinate curve originating at the point (p_r, q_0); (note that $q = q_0$ on γ_0). Let $L_{\gamma_0}: B_{(p_r, q_0)} \dashrightarrow B_{(p, q_0)}$ be the parallel transport along the curve γ_0 [cf. Appendix 2.2 for definitions on parallel transport and parallel sections along a curve].

Then $L_{\gamma_0}([2\pi\hbar]^{-1/4} s_0(p_r, q_0))$ is a parallel section along γ_0 given explicitly by

$$L_{\gamma_0}([2\pi\hbar]^{-1/4} s_0(p_r, q_0)) = [2\pi\hbar]^{-1/4}\{\exp \tfrac{i}{\hbar}(\int_{p_r}^{p} \beta_p \, dp)_{\gamma_0}\} s_0(p, q_0)$$
$$= [2\pi\hbar]^{-1/4}\{\exp \tfrac{i}{\hbar} S_{\gamma_0}(p, q_0)\} s_0(p, q_0). \quad (2.2.2.\text{Eq } 12)$$

(ii) Let σ be the q-coordinate curve originating at the point (p, q_0); (note that p is constant on σ). Let $L_\sigma: B_{(p, q_0)} \dashrightarrow B_{(p, q)}$ be parallel transport along σ.

We shall extend the section along the curve γ_0 defined by equation (2.2.2.Eq 12) to the entire manifold M as follows.

For each $p \in \mathbb{R}$, we define a parallel section along the curve σ (which is determined by the value of p) by

$$L_\sigma([2\pi\hbar]^{-1/4}\{\exp \tfrac{i}{\hbar} S_{\gamma_0}(p, q_0)\} s_0(p, q_0))$$
$$= [2\pi\hbar]^{-1/4}\{\exp \tfrac{i}{\hbar} S_{\gamma_0}(p, q_0)\}\{\exp \tfrac{i}{\hbar}(\int_{q_0}^{q} \beta_q \, dq)_\sigma\} s_0(p, q)$$
$$= [2\pi\hbar]^{-1/4}\{\exp \tfrac{i}{\hbar} S_{\gamma_0}(p, q_0)\}\{\exp -\tfrac{i}{\hbar}(\int_{q_r}^{q_0} \beta_q \, dq)_\sigma\}\{\exp \tfrac{i}{\hbar}(\int_{q_r}^{q} \beta_q \, dq)_\sigma\} s_0(p, q)$$
$$= [2\pi\hbar]^{-1/4}\{\exp \tfrac{i}{\hbar} S_{\gamma_0}(p, q_0)\}\{\exp -\tfrac{i}{\hbar} S_\sigma(p, q_0)\} \mathcal{S}_{c_0}(p, q). \quad (2.2.2.\text{Eq } 13)$$

Let $\Psi_{c q_0}(p) \mathcal{S}_{c_0}(p, q)$ be the global section of B that is given on each curve σ by equation (2.2.2.Eq 13). By construction, $\Psi_{c q_0}(p) \mathcal{S}_{c_0}(p, q)$ is a polarized section of B (with respect to the polarization $\mathcal{P}_c$), i.e., $\nabla_{X_p}\{\Psi_{c q_0}(p) \mathcal{S}_{c_0}(p, q)\} = 0$ [cf. definition (1.1.5.D3)].

(iii) Let $\Psi_{cq_0} = \psi_{cq_0}(p)\rho_c$ be the corresponding $\mathcal{P}_c$-wave function. This construction shows up the extent to which the generalised eigensection Ψ_{cq_0} is anchored to the curve γ_0 (on which $q = q_0$).

Finally, the inner-product of Ψ_{cq_0} and $\Psi_{cq'_0}$ is given by

$$\begin{aligned}\langle \Psi_{cq_0}, \Psi_{cq'_0}\rangle_{\mathcal{P}_c} &= [2\pi\hbar]^{-1/2}\int_{\mathbb{R}} \psi_{cq_0}(p)\,\overline{\psi_{cq'_0}}(p)\,dp \\ &= [2\pi\hbar]^{-1}\{\exp \tfrac{i}{\hbar}(f(p_r,q_0)-f(p_r,q'_0))\}\int_{\mathbb{R}}\{\exp -\tfrac{i}{\hbar}p(q_0-q'_0)\}\,dp \\ &= \{\exp \tfrac{i}{\hbar}(f(p_r,q_0)-f(p_r,q'_0))\}\,\delta(q_0-q'_0) \\ &= \delta(q_0-q'_0) \quad \blacksquare \qquad (2.2.2.\text{Eq } 14)\end{aligned}$$

So far we have only constructed generalized eigensections of $\tilde{q}_c$ corresponding to eigenvalues that lie in $R(q)$ (the classical range of q). One can use the formal expressions for the generalized eigenfunctions of $\tilde{q}_c$ corresponding to eigenvalues in $R(q)$ to construct generalized eigenfunctions of q_c corresponding to eigenvalues that lie outside $R(q)$ in three steps as follows.

(i) For each $q_0 \in R(q)$, the function $\psi_{cq_0}(p)$ defined by equation (2.2.2.Eq 9b) can be written in the form

$$\psi_{cq_0}(p) = [2\pi\hbar]^{-1/4}\{\exp \tfrac{i}{\hbar}[-p(q_0-q_r)-f(p_r,q_0)+f(p,q_r)]\}. \qquad (2.2.2.\text{Eq } 15)$$

(ii) Let us formally extend the range of q from $R(q)$ to the whole of $\mathbb{R}$. Let $f_\infty(p,q)$ be a smooth function on $\mathbb{R}^2$ that satisfies the condition

$$f_\infty(p,q) = f(p,q) \text{ on } R(p) \times R(q). \qquad (2.2.2.\text{Eq } 16)$$

(iii) Then for each $q_0 \in \mathbb{R} - R(q)$, the generalized eigensection corresponding to the eigenvalue q_0 is given by

$$\Psi_{cq_0} = \psi_{cq_0}(p)\rho_c \qquad (2.2.2.\text{Eq } 17a)$$

where

$$\psi_{cq_0}(p) = [2\pi\hbar]^{-1/4}\{\exp \tfrac{i}{\hbar}[-p(q_0-q_r)-f_\infty(p_r,q_0)+f_\infty(p,q_r)]\}. \qquad (2.2.2.\text{Eq } 17b)$$

It follows from equation (2.2.2.Eq 11) that Ψ_{cq_0} is a generalized eigenfunction of $\tilde{q}_c$ corresponding to the eigenvalue q_0.

We then have

$$[2\pi\hbar]^{-1/2}\int_{\mathbb{R}} dq_0\, \Psi_{cq_0}(p)\overline{\Psi}_{cq_0}(p')$$
$$= \{\exp \tfrac{i}{\hbar}(f_\infty(p,q_r)-f_\infty(p',q_r))\}[2\pi\hbar]^{-1}\int_{\mathbb{R}}\{\exp -\tfrac{i}{\hbar} q_0(p-p')\}dq_0$$
$$= \delta(p-p'_0). \qquad (2.2.2.\text{Eq } 18)$$

(This property must not be confused with the inner-product of any two generalized eigensections of $\tilde{q}_c$ given by equation (2.2.2.Eq 14).)

Clearly the quantization Hilbert space $H_{\mathcal{P}_c}$ is spanned by $\{\Psi_{cq_0} : q_0 \in \mathbb{R}\}$.

Bearing the examples of the infinite potential barrier and the infinite square potential well in mind we shall make the following physical assumption.

Physical assumption PA1

The values assumed by the **quantum observable** corresponding to the classical variable q should be contained in $R(q)$ (the classical range of q).

Clearly the spectrum of $\tilde{q}_c$ is physically incorrect because the quantization Hilbert space $H_{\mathcal{P}_c}$ is "too big". Therefore, to be consistent with this physical assumption we shall make the following quantization assumption.

Quantization assumption QA1

When quantizing q in $\mathcal{P}_c$ the appropriate Hilbert space, to be referred to as the **quantum Hilbert space associated with** $\mathcal{P}_c$ and denoted by $H(\mathcal{P}_c)$, should be spanned by all the generalized eigensections of $\tilde{q}_c$ corresponding to eigenvalues consistent with physical assumption PA1. In other words, the quantum Hilbert space $H(\mathcal{P}_c)$ is spanned by the set $\{\Psi_{c q_0} : q_0 \in R(q)\}$. The restriction of the operator $\tilde{q}_c$ to $H(\mathcal{P})$ will be the **quantum observable corresponding to the variable** q and will be denoted by $\hat{q}_c$. A corresponding statement also applies when quantizing q in $\mathcal{P}$.

The quantum Hilbert space $H(\mathcal{P}_c)$ is clearly a proper subspace of the quantization Hilbert space $H_{\mathcal{P}_c}$.

Let us now quantize the variables p and q in the polarization $\mathcal{P}$.

The quantization pre-Hilbert space $W_{\mathcal{P}}$ consists of square-integrable sections of the bundle $B \times \Delta_{-1/2}(\mathcal{P})$ of the form $\Phi = s\nu$ which satisfy the following conditions

$$\nabla_{X_q} s = 0 \text{ and } \nabla_{X_q} \nu = 0 \text{ where } X_q = -(\partial/\partial p). \qquad (2.2.2.\text{Eq } 19)$$

Explicitly, we have

$$\Phi = \varphi(q)\rho, \quad q \in R(q); \qquad (2.2.2.\text{Eq } 20a)$$

where

$$\rho = s_0 |dp|^{-1/2}; \quad s_0 = \{\exp \pm i S_\gamma(p,q)\} s_0. \qquad (2.2.2.\text{Eq } 20b)$$

The inner product on $W_{\mathcal{P}}$ is given by

$$\langle \Phi, \Phi \rangle_{\mathcal{P}} = [2\pi\hbar]^{-1/2} \int_{R(q)} |\varphi(q)|^2 dq. \qquad (2.2.2.\text{Eq } 21)$$

The quantization Hilbert space $H_{\mathcal{P}}$ is the completion of $W_{\mathcal{P}}$.

The variable q can be quantized in $H_{\mathcal{P}}$, and the quantization operator is the multiplication operator

$$\tilde{q}\,\Phi = q\,\Phi \tag{2.2.2.Eq 22a}$$

acting on the domain

$$D_{\tilde{q}} = \{\Phi \in H_{\mathcal{P}} : q\Phi \in H_{\mathcal{P}}\}. \tag{2.2.2.Eq 22b}$$

The associated vector field $pr_*(X_p)$ is $(\partial/\partial q)$. Clearly $pr_*(X_p)$ is incomplete on Q. Hence the variable p is not quantizable in $H_{\mathcal{P}}$.

Since the quantization operator $\tilde{q}$ and the quantization Hilbert space $H_{\mathcal{P}}$ satisfy the physical assumption PA1 and the quantum assumption QA1 it follows that they are respectively $\hat{q}$ (the quantum operator corresponding to the variable q) and $H(\mathcal{P})$ (the quantum Hilbert space associated with the polarization $\mathcal{P}$).

The pairing map between $H_{\mathcal{P}}$ and $H_{\mathcal{P}_c}$ is given by

$$\langle \Phi, \Psi_c \rangle_{\mathcal{P}\mathcal{P}_c} = [2\pi\hbar]^{-1} \int_{R(q)} \int_{R(p)=\mathbb{R}} \varphi(q)\overline{\psi_c}(p)\{\exp \tfrac{i}{\hbar} S_\gamma(p,q)\}\{\exp -\tfrac{i}{\hbar} S_\sigma(p,q)\}\, dp\, dq \tag{2.2.2.Eq 23}$$

[cf. Appendix 2.6, equation (A2.6.Eq 1)].

It follows from equation (2.2.2.Eq 20a) that $\varphi(q)$ is only defined for $q \in R(q)$. Let the range of q be formally extended to $\mathbb{R}$. Then the function $\varphi(q)$ can be formally defined on $\mathbb{R}$ by putting

$$\varphi(q) = 0, q \in \mathbb{R} - R(q). \tag{2.2.2.Eq 24}$$

Let $\Psi_{cq} = \psi_{cq}(p)\rho_c$ be the generalized eigensections of the quantum operator $\hat{q}_c$ in $H(\mathcal{P}_c)$, i.e. for each $q \in R(q)$, we have

$$\psi_{cq}(p) = [2\pi\hbar]^{-1/4}\{\exp \tfrac{i}{\hbar} S_\gamma(p,q)\}\{\exp -\tfrac{i}{\hbar} S_\sigma(p,q)\} \tag{2.2.2.Eq 25}$$

[cf. equation (2.2.2.Eq 9b)].

Then the pairing map given by equation (2.2.2.Eq 23) can be rewritten as

$$\langle \Phi, \Psi_c \rangle_{\mathcal{P}\mathcal{P}_c} = [2\pi\hbar]^{-3/4} \int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(q)\overline{\Psi}_c(p)\Psi_{cq}(p)\,dp\,dq. \qquad (2.2.2.\text{Eq } 26)$$

The pairing map determines a linear map $\mathbf{U}:\mathbf{H}(\mathcal{P}) \dashrightarrow \mathbf{H}(\mathcal{P}_c)$ which maps $\Phi = \varphi(q)\rho$ (where $\varphi(q) = 0$ for $q \in \mathbb{R}-R(q)$) to $\Psi_c = \Psi_c(p)\rho_c$ given by

$$\Psi_c(p) = [2\pi\hbar]^{-1/4}\int_{\mathbb{R}} \varphi(q)\,\Psi_{cq}(p)\,dq \qquad (2.2.2.\text{Eq } 27a)$$

with the inverse map $\mathbf{U}^{-1}$ given by

$$\phi(q) = [2\pi\hbar]^{-1/4}\int_{\mathbb{R}} \Psi_c(p)\,\overline{\Psi}_{cq}(p)\,dp \qquad (2.2.2.\text{Eq } 27b)$$

[cf. Appendix 2.6, equations (A2.6.Eq 3) and (A2.6.Eq 4)].

Let us digress for a moment to give the reason for formally extending the range of q and the domain of $\varphi(q)$ to $\mathbb{R}$, by using the following simple analogy. Let $F:L^2(\mathbb{R}) \dashrightarrow L^2(\mathbb{R})$ be the Fourier transform, (a_0, b_0) be an open interval of $\mathbb{R}$ and let $L^2_0 = L^2(a_0, b_0)$. The Hilbert space L^2_0 can formally be interpreted as the subspace of $L^2(\mathbb{R})$ consisting of functions of $L^2(\mathbb{R})$ that vanish outside the interval (a_0, b_0). Then $L''^2 = FL^2_0$ is a proper subspace of $L^2(\mathbb{R})$ [cf. example (2.2.2.Ex 2)]. Hence the restriction of the Fourier transform F to L^2_0 is a unitary map from L^2_0 to L''^2 and will be denoted by F_0. To evaluate $F_0\varphi_0$, for each $\varphi_0 \in L^2_0$, one can choose to perform the Fourier integration of φ_0 formally over the entire $\mathbb{R}$. The main advantage of performing the Fourier integration over $\mathbb{R}$ instead of over (a_0, b_0) is that one can make use of the integral definition of the Dirac delta function when evaluating combinations of F_0 and F_0^{-1}. The same justification holds for extending the range of q from $R(q)$ to the entire $\mathbb{R}$.

Now returning to the problem at hand, in Appendix 2.6 we show that $\mathbf{U}:\mathbf{H}(\mathcal{P}) \dashrightarrow \mathbf{H}(\mathcal{P}_c)$ is a unitary map, and that $\mathbf{U}\hat{q}\mathbf{U}^{-1} = \hat{q}_c$. In Appendix 2.4 we show that the restriction of the quantization operator $\widetilde{p}_c$ in $\mathbf{H}_{\mathcal{P}_c}$ to the quantum Hilbert space $H(\mathcal{P}_c)$ is not an essentially self-adjoint operator in $H(\mathcal{P}_c)$. This means that $\mathfrak{p}$ is not quantizable in $H(\mathcal{P}_c)$; this result is consistent with the fact that $\mathfrak{p}$ cannot be quantized in $H(\mathcal{P})$ either. Thus the quantizations of the canonical variables $\mathfrak{p}$ and $\mathfrak{q}$ in $H(\mathcal{P})$ and $H(\mathcal{P}_c)$ are unitarily equivalent. Therefore, we have rendered the quantizations in the canonically conjugate polarizations $\mathcal{P}$ and $\mathcal{P}_c$ unitarily equivalent.

Case (2)

Let $\mathbb{R}$ be the configuration space of a free particle and let q be the usual cartesian coordinate on $\mathbb{R}$. Let $M = T^*\mathbb{R} = \mathbb{R}^2$, ω be the canonical two-form on M and let (p,q) be the usual cartesian coordinates on M.

Remark: (R1) We have not used Q to denote the configuration space as we usually do because Q has already been reserved to denote the effective configuration space with respect to the polarization $\mathcal{P}$.

(R2) The cartesian coordinates (p,q) should not be confused with the canonical variables $(\mathfrak{p},\mathfrak{q})$.

Our object here is to quantize the Hamiltonian of the free particle $H = p^2$. The general method in geometric quantization scheme of quantizing an observable ζ consists of two steps:

(i) effect a canonical transformation from (p,q) to $(\mathfrak{p},\mathfrak{q})$ such that $\mathfrak{p} = \zeta$; and

(ii) quantize ζ in the polarization spanned by $(\partial/\partial\mathfrak{p})$.

We shall carry out this scheme for the case $\zeta = H$, and check the result with that obtained by quantizing H in the vertical polarization P. The quantization operator corresponding to H in the vertical polarization P has been explicitly worked out in Appendix 2.7; the result is well known. We shall establish unitarily equivalent quantizations in the canonically conjugate polarizations $\mathcal{P}$ and $\mathcal{P}_c$ for a case (2) situation in the process.

The first complication to arise is the nonglobal nature of the above canonical transformation. We have to split M into two disjoint submanifolds M_1 and M_2 where

$$M_1 = \{(p,q) \in M: p>0\} \qquad (2.2.2.\text{Eq } 29a)$$

and

$$M_2 = \{(p,q) \in M: p<0\}. \qquad (2.2.2.\text{Eq } 28b)$$

Let ω_1 and ω_2 be the restrictions of the canonical two-form ω to M_1 and M_2 respectively. Then (M_1,ω_1) and (M_2,ω_2) are symplectic manifolds.

Introduce canonical coordinates $(\mathfrak{p}_1,\mathfrak{q}_1)$ on M_1 and $(\mathfrak{p}_2,\mathfrak{q}_2)$ on M_2 given by

$$\mathfrak{p}_1 = p^2,\ \mathfrak{q}_1 = (q/2p) \qquad (2.2.2.\text{Eq } 29a)$$

and

$$\mathfrak{p}_2 = p^2,\ \mathfrak{q}_2 = (q/2p). \qquad (2.2.2.\text{Eq } 29b)$$

Clearly $R(\mathfrak{p}_1) = R(\mathfrak{p}_2) = (0,\infty)$ and $R(\mathfrak{q}_1) = R(\mathfrak{q}_2) = \mathbb{R}$, so M_1 with canonical coordinates $(\mathfrak{p}_1,\mathfrak{q}_1)$ and M_2 with canonical coordinates $(\mathfrak{p}_2,\mathfrak{q}_2)$ are both examples of case (2) situations.

Before we proceed we shall clarify the notation that we shall use. We shall, unless otherwise stated, adopt the notation given at the beginning of this subsection. We shall use the subscripts 1 and 2 to differentiate between structures on M_1 and M_2 respectively.

In addition, we shall assume that the prequantization bundles over (M_1,ω_1) and (M_2,ω_2) are the restrictions of a chosen prequantization bundle over (M,ω). This assumption makes it easier to compare the quantization of p^2 in the vertical polarization P with the quantization of $\mathfrak{p} = \mathfrak{p}_i$ (on $M_1 \cup M_2$) in the polarization $\mathcal{P}$ spanned by $(\partial/\partial\mathfrak{p}_i)$ on $M_1 \cup M_2$. The details of the prequantization bundles are given as follows. Let $B = M \times \mathbb{C}$ be a trivial line bundle over M, $(\cdot,\cdot)$ be the natural Hermitian structure on B, and let s_o be a unit section of B. Let β be a one-form on M that satisfies $d\beta = \omega$. Let ∇ be the connection on B defined by $\nabla_X s_o = -i(X \lrcorner \beta)s_o$ for all $X \in V_{\mathbb{C}}(M)$. Then let $(B,(\cdot,\cdot),\nabla)$ be the chosen prequantization bundle over (M,ω). Let $(B_1,(\cdot,\cdot),\nabla)$ and $(B_2,(\cdot,\cdot),\nabla)$ be the restrictions of the prequantization bundle $(B,(\cdot,\cdot),\nabla)$ to M_1 and M_2 respectively. We shall assume that the connection potential is given by $\beta_1 = \mathfrak{p}_1 d\mathfrak{q}_1 + df_1(\mathfrak{p}_1,\mathfrak{q}_1)$ on M_1, and $\beta_2 = \mathfrak{p}_2 d\mathfrak{q}_2 + df_2(\mathfrak{p}_2,\mathfrak{q}_2)$ on M_2.

Let $\mathcal{P}_1$ and $\mathcal{P}_{1c}$ be the canonically conjugate polarizations on M_1 spanned by $(\partial/\partial\mathfrak{p}_1)$ and $(\partial/\partial\mathfrak{q}_1)$ respectively. Let Q_1 and Q_{1c} be the effective configuration spaces with respect to $\mathcal{P}$ and $\mathcal{P}_c$ respectively. Let $pr_1 : M_1 \dashrightarrow Q_1$ and $pr_{c1} : M_1 \dashrightarrow Q_1$ be the corresponding projection maps. Note that Q_1 is identifiable with $R(\mathfrak{p}_1)$, and Q_{1c} is identifiable with $R(\mathfrak{q}_1)$. We shall replace the subscript 1 by 2 for the corresponding structures on M_2.

We shall start by quantizing the variables $\mathfrak{p}_j$ and $\mathfrak{q}_j$, j = 1, 2, in the polarization $\mathcal{P}_j$.

The quantization pre-Hilbert space $W_{\mathcal{P}_1}$ consists of square-integrable sections of the bundle $B \times \Delta_{-1/2}(\mathcal{P}_1)$ of the form

$$\Phi_1 = \varphi_1(\mathfrak{q}_1)\rho_1, \; \mathfrak{q}_1 \in R(\mathfrak{q}_1) \qquad (2.2.2.\text{Eq } 30a)$$

where

$$R(q_{\mathcal{f}_1}) = \mathbb{R},\ \rho_1 = \mathcal{s}_{10}|d\mathcal{f}_1|^{-1/2};\ \mathcal{s}_{10} = \{\exp \pm S_{\gamma_1}(\mathcal{f}_1, q_{\mathcal{f}_1})\}s_o. \qquad (2.2.2.\text{Eq } 30b)$$

The inner-product on W_{ρ_1} is given by

$$\langle \Phi_1, \Phi_1 \rangle_{\rho_1} = [2\pi\hbar]^{-1/2}\int_{\mathbb{R}} |\varphi_1(q_{\mathcal{f}_1})|^2 dq_{\mathcal{f}_1}. \qquad (2.2.2.\text{Eq } 31)$$

The quantization Hilbert space H_{ρ_1} is the completion of W_{ρ_1}.

The associated vector field generated by $\mathcal{f}_1$ is given by $pr_{1*}(X_{\mathcal{f}_1}) = (\partial/\partial q_{\mathcal{f}_1})$; for short, let $Y_{\mathcal{f}_1} = pr_{1*}(X_{\mathcal{f}_1})$. The variable $\mathcal{f}_1$ is quantizable in H_{ρ_1} because $Y_{\mathcal{f}_1}$ is complete on Q_1. The quantization operator $\widetilde{\mathcal{f}_1}$ in H_{ρ_1} is given by

$$\widetilde{\mathcal{f}_1}\,\Phi_1 = [\{-i\hbar(\partial/\partial q_{\mathcal{f}_1}) - (\partial f_1(\mathcal{f}_{1r}, q_{\mathcal{f}_1})/\partial q_{\mathcal{f}_1})\}\varphi_1(q_{\mathcal{f}_1})]\,\rho_1 \qquad (2.2.2.\text{Eq } 32)$$

[cf. Appendix 2.3, equation (A2.3.Eq 8)]. The spectrum of the operator $\widetilde{\mathcal{f}_1}$ is $(-\infty,\infty)$.

Similarly on M_2, the quantization pre-Hilbert space W_{ρ_2} consists of square-integrable sections of the bundle $B_2 \times \Delta_{-1/2}(\rho_2)$ of the form

$$\Phi_2 = \varphi_2(q_{\mathcal{f}_2})\rho_2,\ q_{\mathcal{f}_2} \in R(q_{\mathcal{f}_2}) \qquad (2.2.2.\text{Eq } 33a)$$

where

$$R(q_{\mathcal{f}_2}) = \mathbb{R},\ \rho_2 = \mathcal{s}_{20}|d\mathcal{f}_2|^{-1/2};\ \mathcal{s}_{20} = \{\exp \pm S_{\gamma_2}(\mathcal{f}_2, q_{\mathcal{f}_2})\}s_o. \qquad (2.2.2.\text{Eq } 33b)$$

The inner-product on W_{ρ_2} is given by

$$\langle \Phi_2, \Phi_2 \rangle_{\rho_2} = [2\pi\hbar]^{-1/2}\int_{\mathbb{R}} |\varphi_2(q_{\mathcal{f}_2})|^2 dq_{\mathcal{f}_2}. \qquad (2.2.2.\text{Eq } 34)$$

The quantization Hilbert space H_{ρ_2} is the completion of W_{ρ_2}.

The quantization operator $\widetilde{q}_{\mathcal{f}_2}$ in H_{ρ_2} is given by

$$\widetilde{\mathcal{f}_2}\,\Phi_2 = [\{-i\hbar(\partial/\partial q_{\mathcal{f}_2}) - (\partial f_2(\mathcal{f}_{2r}, q_{\mathcal{f}_2})/\partial q_{\mathcal{f}_2})\}\varphi_2(q_{\mathcal{f}_2})]\rho_2. \qquad (2.2.2.\text{Eq } 35)$$

The spectrum of the operator $\widetilde{\mathcal{f}_2}$ is $(-\infty,\infty)$.

Let $\mathcal{P}$ and $\mathcal{P}_c$ be the polarizations of the symplectic manifold $(M_1 \cup M_2, \omega)$ given by

$$\mathcal{P} = \mathcal{P}_1 \text{ and } \mathcal{P}_c = \mathcal{P}_{1c} \text{ on } M_1; \qquad (2.2.2.\text{Eq } 36a)$$

and

$$\mathcal{P} = \mathcal{P}_2 \text{ and } \mathcal{P}_c = \mathcal{P}_{2c} \text{ on } M_2. \qquad (2.2.2.\text{Eq } 36b)$$

One can interpret $\mathcal{P}$ and $\mathcal{P}$ as polarizations of (M,ω): strictly speaking, $\mathcal{P}$ and $\mathcal{P}_c$ are not polarizations of (M,ω) because they are not defined on the set $\{(p,q) \in M: p = 0\}$ in M; however, as this set is of measure zero we shall ignore this technicality.

We shall now quantize the Hamiltonian of the free particle p^2 in the polarization $\mathcal{P}$ as follows. We shall use the letter $\wp$ without the subscript to denote p^2 on M. Bearing the theorem on the canonical decomposition of global observables by Wan and McFarlane (1981) [cf. Appendix 2.8] in mind, we shall define $H_{\mathcal{P}}$ (the quantization Hilbert space associated with the polarization) by $H_{\mathcal{P}} = H_{\mathcal{P}_1} \oplus H_{\mathcal{P}_2}$, and we shall define the quantization operator $\widetilde{\wp}$ in $H_{\mathcal{P}}$ by $\widetilde{\wp} = \widetilde{\wp}_1 \oplus \wp_2$. Clearly the spectrum of $\widetilde{\wp}$ is $(-\infty,\infty)$ [cf. Naimark (1968), p209]. It is physically unacceptable to have $\widetilde{\wp}$ as the quantized observable corresponding to p because classically the free Hamiltonian p^2 is strictly positive. Therefore, $H_{\mathcal{P}}$ cannot be the quantum Hilbert space associated with the polarization $\mathcal{P}$, so we need to establish the quantum Hilbert space and quantum observable. This can be done as in the previous case by examining the link between generalized eigensections of $\widetilde{\wp}$ and the classical values of $\wp$.

(2.2.2.P2) Proposition

Each value p_{j0} of the classical variable p_j is a **generalized eigenvalue** of $\tilde{p}_j$ in H_{ρ_j} corresponding to the **generalized eigensection**

$$\Phi_{p_{j0}} = \varphi_{p_{j0}}(q_j)\,\rho_j \qquad (2.2.2.\text{Eq } 37a)$$

where

$$\varphi_{p_{j0}}(q_j) = [2\pi\hbar]^{-1/4}\{\exp \tfrac{i}{\hbar}S_{\sigma_{j0}}(p_{j0},q_j)\}\{\exp -\tfrac{i}{\hbar}S_{\gamma_j}(p_{j0},q_j)\}. \qquad (2.2.2.\text{Eq } 37b)$$

The generalized eigensections satisfy

$$\langle \Phi_{p_{j0}}, \Phi_{p'_{j0}} \rangle_{\rho_j} = \delta(p_{j0}-p'_{j0}). \qquad (2.2.2.\text{Eq } 38)$$

Proof

For the sake of tidiness, we shall drop the subscript j in some steps of the proof.

We have

$$S_{\sigma_0}(p_0,q) = p_0(q-q_r)+f(p_0,q)-f(p_0,q_r),$$
$$S_{\gamma}(p_0,q) = f(p_0,q)-f(p_r,q),$$
$$S_{\sigma_0}(p_0,q)-S_{\gamma}(p_0,q) = p_0(q-q_r)-f(p_0,q_r)+f(p_r,q),$$
$$[(\partial/\partial q)\{S_{\sigma_0}(p_0,q)-S_{\gamma}(p_0,q)\}] = p_0+(\partial f(p_r,q)/\partial q)$$

and

$$(\partial\varphi_{p_{j0}}(q_j)/\partial q_j) = \tfrac{i}{\hbar}[(\partial/\partial q_j)\{S_{\sigma_{j0}}(p_{j0},q_j)-S_{\gamma_j}(p_{j0},q_j)\}]\,\varphi_{p_{j0}}(q_j).$$

Therefore,

$$\tilde{p}_j\ \Phi_{p_{j0}} = [\{-i\hbar(\partial/\partial q_j)-(\partial f_j(p_{jr},q_j)/\partial q_j)\}\varphi_{p_{j0}}(q_j)]\rho_j = p_{j0}\Phi_{p_{j0}}. \qquad (2.2.2.\text{Eq } 39)$$

Thus $\Phi_{p_{j0}}$ is a generalized eigensection of $\tilde{p}_j$ corresponding to the generalized eigenvalue p_{j0}.

The crucial point is to observe that for each $p_{jo} \in R(p_j)$, the generalized eigensection $\Phi_{p_{jo}}$ can be constructed in three steps as follows: (i) Let σ_{jo} be the q_j-coordinate curve originating at the point (p_{jo}, q_{jr}); (note that σ_{jo} is determined by the value p_{jo}). Let $L_{\sigma_{jo}} : B_{(p_{jo}, q_{jr})} \dashrightarrow B_{(p_{jo}, q_j)}$ be the parallel transport along σ_{jo}.

Then $L_{\sigma_{jo}}([2\pi\hbar]^{-1/4} s_o(p_{jo}, q_{jr}))$ is a parallel section along σ_{jo} given explicitly by

$$L_{\sigma_{jo}}([2\pi\hbar]^{-1/4} s_o(p_{jo}, q_{jr})) = [2\pi\hbar]^{-1/4} \{\exp \tfrac{i}{\hbar}(\int_{q_{jr}}^{q_j} \beta_{q_j} dq_j)_{\sigma_{jo}}\} s_o(p_{jo}, q_j)$$
$$= [2\pi\hbar]^{-1/4} \{\exp \tfrac{i}{\hbar} S_{\sigma_{jo}}(p_{jo}, q_j)\} s_o(p_{jo}, q_j) \qquad (2.2.2.\text{Eq } 40)$$

(ii) Let γ_j be the p_j-coordinate curve originating at the point (p_{jo}, q_j); (note that q_j is constant along γ). Let $L_{\gamma_j} : B_{(p_{jo}, q_j)} \dashrightarrow B_{(p_j, q_j)}$ be the parallel transport along γ_j.

We shall extend the section along the curve σ_{jo} defined by equation (2.2.2.Eq 40) to the entire submanifold M as follows.

For each $q_j \in R(q_j)$, we define a parallel section along the curve γ_j (which is determined by the value of) by

$$L_{\gamma_j}[L_{\sigma_{jo}}\{(2\pi\hbar)^{-1/4} s_o(p_{jo}, q_j)\}]$$
$$= L_{\gamma_j}[(2\pi\hbar)^{-1/4} \{\exp \tfrac{i}{\hbar} S_{\sigma_{jo}}(p_{jo}, q_j)\} s_o(p_{jo}, q_j)]$$
$$= (2\pi\hbar)^{-1/4} \{\exp \tfrac{i}{\hbar} S_{\sigma_{jo}}(p_{jo}, q_j)\}\{\exp \tfrac{i}{\hbar}(\int_{p_{jo}}^{p_j} \beta_{p_j} dp_j)_{\gamma_j}\} s_o(p_j, q_j)$$
$$= (2\pi\hbar)^{-1/4} \{\exp \tfrac{i}{\hbar} S_{\sigma_{jo}}(p_{jo}, q_j) - \tfrac{i}{\hbar}(\int_{p_{jr}}^{p_{jo}} \beta_{p_j} dq_j)_{\gamma_j} + \tfrac{i}{\hbar}(\int_{p_{jr}}^{p_j} \beta_{p_j} dp_j)_{\gamma_j}\} s_o$$
$$= (2\pi\hbar)^{-1/4} \{\exp \tfrac{i}{\hbar} S_{\sigma_{jo}}(p_{jo}, q_j)\}\{\exp -\tfrac{i}{\hbar} S_{\gamma_j}(p_{jo}, q_j)\} \mathcal{S}_{jo}. \qquad (2.2.2.\text{Eq } 41)$$

Let $\varphi_{p_{jo}}(q_j)\mathcal{S}_{jo}$ be the global section of B_j that is given on each curve γ_j by equation (2.2.2.Eq 41). By construction $\varphi_{p_{jo}}(q_j)\mathcal{S}_{jo}$ is a polarized section of B_j (with respect to the polarization P_j), i.e. $\nabla_{X_{q_j}}(\varphi_{p_{jo}}(q_j)\mathcal{S}_{jo}) = 0$.

(iii) Let $\Phi_{p_{j0}} = \varphi_{p_{j0}}(q_j)\rho_j$ be the corresponding P_j-wave function.

Finally, the inner-product of the generalized eigensections $\Phi_{p_{j0}}$ and $\Phi_{p'_{j0}}$ is given by

$$
\begin{aligned}
&\langle \Phi_{p_{j0}}, \Phi_{p'_{j0}} \rangle_{P_j} \\
&= [2\pi\hbar]^{-1/2} \int_{\mathbb{R}} \varphi_{p_{j0}}(q_j)\,\varphi_{p'_{j0}}(q_j)\,dq_j \\
&= [2\pi\hbar]^{-1} \{\exp -\tfrac{i}{\hbar}(f_j(p_{j0}, q_{jr}) + f_j(p'_{j0}, q_{jr}))\} \int_{\mathbb{R}} \{\exp -\tfrac{i}{\hbar} q_j(p_{j0} - p'_{j0})\}\,dq_j \\
&= \delta(p_{j0} - p'_{j0}). \blacksquare
\end{aligned}
\qquad (2.2.2.\text{Eq } 42)
$$

We can construct generalized eigensections of the operators $\widehat{p}_j$ corresponding to generalized eigenvalues which lie outside $R(p_j)$ in three steps as follows.

(i) For each $p_{j0} \in R(p_j)$, the function $\varphi_{p_{j0}}(q_j)$ defined by equation (2.2.2.Eq 37b) can be rewritten as

$$
\varphi_{p_{j0}}(q_j) = [2\pi\hbar]^{-1/4} \{\exp \tfrac{i}{\hbar}[p_{j0}(q_j - q_{jr}) - f_j(p_{j0}, q_{jr}) + f_j(p_{jr}, q_j)]\}. \qquad (2.2.2.\text{Eq } 43)
$$

(ii) Let us formally extend the range of p_j from $R(p_j)$ to the whole of $\mathbb{R}$. Let $f_{j,\infty}(p_j, q_j)$ be a smooth function on $\mathbb{R}^2$ that satisfies the condition

$$
f_{j,\infty}(p_j, q_j) = f_j(p_j, q_j) \text{ on } R(p_j) \times R(q_j). \qquad (2.2.2.\text{Eq } 44)
$$

(iii) Then for each $p_{j0} \in \mathbb{R} - R(p_j)$, the generalized eigensection of $\widehat{p}_j$ corresponding to the generalized eigenvalue p_{j0} is given by

$$
\Phi_{p_{j0}} = \varphi_{p_{j0}}(q_j)\rho_j \qquad (2.2.2.\text{Eq } 45a)
$$

where

$$
\varphi_{p_{j0}}(q_j) = [2\pi\hbar]^{-1/4} \{\exp \tfrac{i}{\hbar}[p_{j0}(q_j - q_{jr}) - f_{j,\infty}(p_{j0}, q_{jr}) + f_{j,\infty}(p_{jr}, q_j)]\} \qquad (2.2.2.\text{Eq } 45b)
$$

Clearly the quantization Hilbert space H_{P_j} is spanned by the set $\{\Phi_{p_{j0}} : p_{j0} \in \mathbb{R}\}$.

It can easily be shown that

$$\int_{\mathbb{R}} d\mathfrak{f}_{j0}\ \varphi_{\mathfrak{f}_{j0}}(q_j)\,\varphi_{\mathfrak{f}_{j0}}(q'_j) = \delta(q_j - q'_j) \qquad (2.2.2.\text{Eq } 46)$$

(This property should not be confused with the inner-product of generalized eigensections of $\widetilde{\mathfrak{f}_j}$ given by equation (2.2.2.Eq 42).)

Let us digress for a moment to consider the quantizations of the variable $\mathfrak{f}$ in a general case (2) situation. In order to obtain physically acceptable results for the quantizations of the variable $\mathfrak{f}$ in canonically conjugate polarizations for a general case (2) situation we make the following assumptions.

Physical assumption PA2

The values assumed by the **quantum observable corresponding to the variable** $\mathfrak{f}$ should be contained in the range of values $R(\mathfrak{f})$ of the classical variable $\mathfrak{f}$.

Quantum assumption QA2

When quantizing $\mathfrak{f}$ in $\mathcal{P}$ the **quantum Hilbert space** $H(\mathcal{P})$ should be spanned by all generalized eigensections of the quantization operator $\widetilde{\mathfrak{f}}$ corresponding to generalized eigenvalues consistent with the physical assumption PA2 above. A corresponding statement also applies when quantizing $\mathfrak{f}$ in $\mathcal{P}_c$.

Let us return to the problem of establishing a physically acceptable quantization of the free Hamiltonian $\mathfrak{f} = p^2$ in the polarization $\mathcal{P}$. Firstly, we shall establish physically acceptable quantizations of $\mathfrak{f}_j$ in $\mathcal{P}_j$ as follows. According to the assumptions PA2 and QA2 physically acceptable quantizations of $\mathfrak{f}_j$ in $\mathcal{P}_j$ are achieved if the quantum Hilbert spaces $H(\mathcal{P}_j)$ are the subspaces of the quantization Hilbert spaces $H_{\mathcal{P}_j}$ spanned by the sets $\{\Phi_{\mathfrak{f}_{jo}}: \mathfrak{f}_{jo} \in R(\mathfrak{f}_j) = (0,\infty)\}$, and the corresponding quantum operators $\hat{\mathfrak{f}}_j$ are the restrictions $\tilde{\mathfrak{f}}_j$ in $H_{\mathcal{P}_j}$ to $H(\mathcal{P}_j)$. Secondly, we note that the operator $\hat{\mathfrak{f}}_1 \oplus \hat{\mathfrak{f}}_2$ in $H(\mathcal{P}_1) \oplus H(\mathcal{P}_2)$ is self-adjoint with positive spectrum because the operators $\hat{\mathfrak{f}}_j$ in $H(\mathcal{P}_j)$ are self-adjoint with positive spectra [cf. Naimark (1968), p209]. Therefore, a physically acceptable quantization of $\mathfrak{f}$ in the polarization $\mathcal{P}$ is established if the quantum Hilbert space $H(\mathcal{P})$ is $H(\mathcal{P}_1) \oplus H(\mathcal{P}_2)$, and the quantum operator $\hat{\mathfrak{f}}$ is $\hat{\mathfrak{f}}_1 \oplus \hat{\mathfrak{f}}_2$.

Let us now quantize $\mathfrak{f} = p^2$ in the polarization $\mathcal{P}_c$. As before we define the quantization Hilbert space associated with the polarization $\mathcal{P}_c$ by $H_{\mathcal{P}_c} = H_{\mathcal{P}_{1c}} \oplus H_{\mathcal{P}_{2c}}$.

The quantization pre-Hilbert spaces $W_{\mathcal{P}_{jc}}$ consist of square-integrable sections of the form

$$\Psi_{jc} = \psi_{jc}(\mathfrak{f}_j)\rho_{jc}, \ \mathfrak{f}_j \in R(\mathfrak{f}_j) \qquad (2.2.2.\text{Eq } 47a)$$

where

$$R(\mathfrak{f}_j) = (0,\infty), \ \rho_j = \mathcal{S}_{jc}|dq_j|^{-1/2}, \ \mathcal{S}_{jc} = \{\exp \pm S_{\sigma_j}(\mathfrak{f}_j, q_j)\}s_o. \qquad (2.2.2.\text{Eq } 47b)$$

The inner-products on $W_{\mathcal{P}_{jc}}$ are given by

$$\langle \Psi_{jc}, \Psi_{jc} \rangle_{\mathcal{P}_{jc}} = [2\pi\hbar]^{-1/2} \int_0^\infty |\psi_{jc}(\mathfrak{f}_j)|^2 d\mathfrak{f}_j. \qquad (2.2.2.\text{Eq } 48)$$

The quantization Hilbert spaces $H_{\mathcal{P}_{jc}}$ are the completion of $W_{\mathcal{P}_{jc}}$.

The quantization operators $\tilde{\mathfrak{f}}_{jc}$ in $H_{\mathcal{P}_{jc}}$ are self-adjoint multiplication operators given by

$$\tilde{\mathfrak{f}}_{jc}\Psi_{jc} = \mathfrak{f}_j \Psi_{jc}. \qquad (2.2.2.\text{Eq } 49)$$

Since $\widetilde{f}_{jc}$ in $H_{\mathcal{P}_c}$ satisfy the assumptions PA2 and QA2, it follows that the quantum Hilbert spaces $H(\mathcal{P}_{jc})$ and quantum operators $\widehat{f}_{jc}$ are given by $H(\mathcal{P}_{jc}) = H_{\mathcal{P}_{jc}}$ and $\widehat{f}_{jc} = \widetilde{f}_{jc}$ respectively. Therefore, a physically acceptable quantization of f in $\mathcal{P}_c$ is established if the quantum Hilbert space $H(\mathcal{P}_c)$ is $H(\mathcal{P}_{1c}) \oplus H(\mathcal{P}_{2c})$, and the quantum operator $\widehat{f}_c$ is $f_{1c} \oplus f_{2c}$.

The pairing maps between $H(\mathcal{P}_j)$ and $H(\mathcal{P}_{jc})$ are given by [cf. Appendix 2.6, equation (A2.6.Eq 1)]

$$\langle \Phi_j , \Psi_{jc} \rangle_{\mathcal{P}_j \mathcal{P}_{jc}} = [2\pi\hbar]^{-1} \int_{R_j} \int_{R_{jc}} \varphi_j(q_j) \overline{\psi_{jc}(f_j)} \{\exp \tfrac{i}{\hbar}[S_{\gamma_j}(f_j, q_j) - S_{\sigma_j}(f_j, q_j)]\} df_j dq_j \qquad (2.2.2.\text{Eq } 50)$$

where $R_j = R(q_j)$ and $R_{jc} = R(f_j)$. Let us formally extend the range of f_j to $\mathbb{R}$, and let us extend the domain of the function $\psi_{jc}(f_j)$ to $\mathbb{R}$ by putting $\psi_{jc}(f_j) = 0$ for $f_j \in \mathbb{R} - R(f_j)$. We recall that earlier we had smoothly extended the domains of the functions $f_j(f_j, q_j)$ from $R(f_j) \times R(q_j)$ to $\mathbb{R}^2$ by introducing the functions $f_{j,\infty}(f_j, q_j)$ [cf. equation (2.2.2.Eq 44)].

Therefore, the pairing maps between $H(\mathcal{P}_j)$ and $H(\mathcal{P}_{jc})$ can now be rewritten as

$$\langle \Phi_j, \Psi_{jc} \rangle_{\mathcal{P}_j \mathcal{P}_{jc}} = [2\pi\hbar]^{-3/4} \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi_j(q_j) \overline{\psi_{jc}(f_j)}\, \varphi_{f_j}(q_j) df_j dq_j \qquad (2.2.2.\text{Eq } 52)$$

where $\varphi_{f_j}(q_j)$ is given by equations (2.2.2.Eq 37b) and (2.2.2.Eq 45b).

The pairing maps defined above determine the unitary maps $V_j : H(\mathcal{P}_{jc}) \dashrightarrow H(\mathcal{P}_j)$ which are defined as follows. The map V_j map $\Psi_{jc} = \psi_{jc}(f_j)\rho_j$, where $\psi_{jc}(f_j) = 0$ for $f_j \in \mathbb{R} - R(f_j)$, to $\Phi_j = \varphi_j(q_j)\rho_j$ by

$$\varphi_j(q_j) = [2\pi\hbar]^{-1/4} \int_{\mathbb{R}} \psi_{jc}(f_j) \varphi_{f_j}(q_j) df_j . \qquad (2.2.2.\text{Eq } 53\text{a})$$

The inverse map $\mathbf{V}^{-1}$ is given by

$$\psi_{jc}(f_j) = [2\pi\hbar]^{-1/4} \int_{\mathbb{R}} \varphi_j(q_j) \overline{\varphi_{f_j}(q_j)} dq_j . \qquad (2.2.2.\text{Eq } 53\text{b})$$

The proof that the maps $\mathbf{V}_j$ are unitary is essentially the same as that given in Appendix 2.6 [cf. part(ii)]. Then let $\mathbf{V}:\mathbf{H}(\mathcal{P}_c) \dashrightarrow \mathbf{H}(\mathcal{P})$ be the unitary map defined by $\mathbf{V} = \mathbf{V}_1 \oplus \mathbf{V}_2$.

We shall show that $\hat{\mathfrak{p}}$ in $\mathbf{H}(\mathcal{P})$ and $\hat{\mathfrak{p}}_c$ in $\mathbf{H}(\mathcal{P}_c)$ are unitarily related as follows. It is sufficient to show that $\mathbf{V}_j\hat{\mathfrak{p}}_{jc}\mathbf{V}^{-1} = \hat{\mathfrak{p}}_j$. Let $\Phi_j = \varphi_j(\mathfrak{q}_j)\rho_j$ and $\Psi_{jc} = V_j^{-1}\Phi_j = \psi_{jc}(\mathfrak{p}_j)\rho_{jc}$ where $\psi_{jc}(\mathfrak{p}_j) = 0$ for $\mathfrak{p}_j \in \mathbb{R} - R(\mathfrak{p}_j)$. Then

$$
\begin{aligned}
&(\mathbf{V}_j\hat{\mathfrak{p}}_{jc}\mathbf{V}_j^{-1})\Phi_j \\
&= (V_j\hat{\mathfrak{p}}_{jc}\Psi_{jc}) \\
&= (\mathbf{V}_j\,\mathfrak{p}_j\,\Psi_{jc}) \\
&= \{(2\pi\hbar)^{-1/4}\int_{\mathbb{R}}\mathfrak{p}_j\psi_{jc}(\mathfrak{p}_j)\varphi_{\mathfrak{p}_j}(\mathfrak{q}_j)\,d\mathfrak{p}_j\}\rho_j \\
&= \{(2\pi\hbar)^{-1/4}\int_{\mathbb{R}}\psi_{jc}(\mathfrak{p}_j)[\{-i\hbar(\partial/\partial\mathfrak{q}_j)-(\partial f_j(\mathfrak{p}_{jr},\mathfrak{q}_j)/\partial\mathfrak{q}_j)\}\varphi_{\mathfrak{p}_j}(\mathfrak{q}_j)]\,d\mathfrak{p}_j\}\rho_j
\end{aligned}
$$

(by equation (2.2.2.Eq 39))

$$= \hat{\mathfrak{p}}_j\Phi_j. \tag{2.2.2.Eq 54}$$

Now let us consider the quantizations of the variable $\mathfrak{q}_j$. The variables $\mathfrak{q}_j$ are quantizable in the quantization Hilbert spaces $H_{\mathcal{P}_j}$, but not quantizable in the quantum Hilbert spaces $H(\mathcal{P}_j)$ by an argument similar to that given in proposition (A2.4.P1) of Appendix 2.4. The variables $\mathfrak{q}_j$ are not quantizable in $H_{\mathcal{P}_{jc}}$, since the associated vector fields $pr_{jc*}(X_{\mathfrak{q}_j})$ are not complete on Q_{jc}. Therefore, $\mathfrak{q}_j$ are not quantizable in $\mathbf{H}(\mathcal{P}_{jc})$. Hence we have established consistent quantizations of $\mathfrak{q}_j$ in the canonically conjugate polarizations $\mathcal{P}_j$ and $\mathcal{P}_{jc}$.

As a final check of our results we shall compare the quantization of $\mathfrak{p}$ in the polarization $\mathcal{P}$ with the quantization of the free Hamiltonian p^2 in the vertical polarization P. The symplectic manifold (M,ω) with the cartesian coordinates (p,q) is an example of a case (3) situation. When we study the case (3) situation we shall see that H(P) the quantum Hilbert space (associated with the vertical polarization P) is the quantization

Hilbert space H_P. In Appendix 2.7 we derive the quantum operator $\hat{p}^2$ in $H(P)$, and then we demonstrate that $\hat{\mathscr{p}}$ in $H(\mathscr{P})$ and $\hat{p}^2$ in $H(P)$ are unitarily related. This serves to show that our physical and quantization assumptions, and therefore our method of establishing of establishing unitarily equivalent quantizations in the canonically conjugate polarizations $\mathscr{P}$ and $\mathscr{P}_c$ for the case (2) situation is physically reasonable.

Case (3)

In this case we have $R(\mathscr{p}) \times R(\mathscr{q}) = \mathbb{R}^2$. Clearly both the canonical variables $\mathscr{p}$ and $\mathscr{q}$ can be quantized in each of the quantization Hilbert spaces $H_{\mathscr{P}}$ and $H_{\mathscr{P}_c}$. The spectra of the quantization operators $\tilde{\mathscr{p}}$ and $\tilde{\mathscr{q}}$ in $H_{\mathscr{P}}$ are $R(\mathscr{p})$ and $R(\mathscr{q})$ respectively. Similarly, the spectra of the quantization operators $\tilde{\mathscr{p}}_c$ and $\tilde{\mathscr{q}}_c$ in $H_{\mathscr{P}_c}$ are $R(\mathscr{p})$ and $R(\mathscr{q})$ respectively. The physical assumptions PA1 and PA2, and the quantization assumptions QA1 and QA2 are applicable here. Then the quantum Hilbert spaces $H(\mathscr{P})$ and $H(\mathscr{P}_c)$, and the quantum operators $\hat{\mathscr{p}}$, $\hat{\mathscr{q}}$, $\hat{\mathscr{p}}_c$ and $\hat{\mathscr{q}}_c$ coincide with the quantization Hilbert spaces $H_{\mathscr{P}}$ and $H_{\mathscr{P}_c}$, and the quantization operators $\tilde{\mathscr{p}}$, $\tilde{\mathscr{q}}$, $\tilde{\mathscr{p}}_c$ and $\tilde{\mathscr{q}}_c$ respectively. Thus we have established unitarily equivalent quantizations of the variables $\mathscr{p}$ and $\mathscr{q}$ in the canonically conjugate polarizations $\mathscr{P}$ and $\mathscr{P}_c$; the link being given by the unitary map $\mathbf{U}:\mathbf{H}(\mathscr{P}) \dashrightarrow \mathbf{H}(\mathscr{P}_c)$ which maps $\Phi = \varphi(\mathscr{q})\rho$ to $\Psi_c = \psi_c(\mathscr{p})\rho_c$ by

$$\psi_c(\mathscr{p}) = [2\pi\hbar]^{-1/2}\int_{\mathbb{R}} \varphi(\mathscr{q})\{\exp \tfrac{i}{\hbar}S_\gamma(\mathscr{p},\mathscr{q})\}\{\exp -\tfrac{i}{\hbar}S_\sigma(\mathscr{p},\mathscr{q})\}d\mathscr{q} \qquad (2.2.2.\text{Eq } 55a)$$

with inverse map $\mathbf{U}$ given by

$$\varphi(\mathscr{q}) = [2\pi\hbar]^{-1/2}\int_{\mathbb{R}} \psi_c(\mathscr{p})\{\exp -\tfrac{i}{\hbar}S_\gamma(\mathscr{p},\mathscr{q})\}\{\exp\ iS_\sigma(\mathscr{p},\mathscr{q})\}d\mathscr{p} \qquad (2.2.2.\text{Eq } 55b)$$

Case (4)

In this case the canonical variable p can only be quantized in the quantization Hilbert space $H_{\mathcal{P}_c}$, and the variable q can only be quantized in the quantization Hilbert space $H_{\mathcal{P}}$. Therefore, it is not possible to set up unitarily equivalent quantizations of p and q in the polarizations $\mathcal{P}$ and $\mathcal{P}_c$ as we did in the previous cases. A way forward using local observables is possible. This will be discussed in Chapter 4.

2.3 HALF-DENSITY QUANTIZATIONS IN CANONICALLY CONJUGATE POLARIZATIONS AND THEIR UNITARY EQUIVALENCE IN TWO-DIMENSIONAL NONCONTRACTIBLE SYMPLECTIC MANIFOLDS

(2.3.1) Introduction

We shall start by giving a definition.

(2.3.1.D1) Definition [Prugovecki (1971), p315]

The self-adjoint operators $\widetilde{A}_1,\ldots,\widetilde{A}_k$ acting in a Hilbert space H constitute a **complete set of quantum observables** if the following two conditions are satisfied:

(CQO 1) There is a measure μ in the Borel sets of the k-dimensional Euclidean space $\mathbb{R}^k$ with support $R(\widetilde{\mathbf{A}}) = R(\widetilde{\mathbf{A}}_1) \times \ldots \times R(\widetilde{\mathbf{A}}_k)$ where $R(\widetilde{\mathbf{A}}_1),\ldots,R(\widetilde{\mathbf{A}}_k)$ are the spectra of $\widetilde{\mathbf{A}}_1,\ldots,\widetilde{\mathbf{A}}_k$ respectively.

(CQO 2) There is a unitary map $\mathbf{U}$ of H onto $L^2(\mathbb{R}^k,\mu)$ such that the operators

$$\widetilde{\mathbf{A}}'_i = \mathbf{U}\widetilde{\mathbf{A}}_i\mathbf{U}^{-1}, \quad i = 1,\ldots,k \qquad (2.3.1.\text{Eq 1a})$$

are the multiplication operators

$$\mathbf{A}'_i\Psi(x) = x_i\Psi(x) \qquad (2.3.1.\text{Eq 1b})$$

with domain

$$D_{A'_i} = \{\Psi(x): \int_{\mathbb{R}^k} x_i|\Psi(x)|\, d\mu(x)<\infty,\ \Psi\in L^2(\mathbb{R}^k,\mu)\}. \qquad (2.3.1.\text{Eq 1c})$$

If the above two requirements are fulfilled, the Hilbert space $L^2(\mathbb{R}^k,\mu)$ is called a **spectral representation space of the operators** $\widetilde{\mathbf{A}}_1,\ldots,\widetilde{\mathbf{A}}_k$, and the set of operators $\widetilde{\mathbf{A}}'_1,\ldots,\widetilde{\mathbf{A}}'_k$ is called the **spectral representation of the operators** $\widetilde{\mathbf{A}}_1,\ldots,\widetilde{\mathbf{A}}_k$.

In the previous section we proposed the following scheme for rendering the quantizations of canonical variables in canonically conjugate polarizations of 2-dimensional contractible symplectic manifolds unitarily equivalent. We replaced the quantization Hilbert spaces and quantization operators in the standard half-density quantization scheme by newly-defined structures which we called quantum Hilbert spaces and quantum operators respectively. In this section we shall study examples in which the canonically conjugate polarizations consists of a polarization with toroidal leaves and a polarization with non-compact leaves. The quantization in the polarization with toroidal leaves gives rise to BWS conditions and a quantization Hilbert space that consists of sections that are only defined on the isolated leaves of the polarization [cf. Chapter 1, section 1.2]. Hence the scheme proposed in the previous section is unsuitable for the examples in non-contractible symplectic manifolds that we shall consider.

Let us now consider the examples the we studied in the previous section to see if they can shed any light on how we could render quantizations of canonical variables in canonically conjugate polarizations of non-contractible symplectic manifolds unitarily equivalent. In the case (1) and case (3) situations we showed that the variable q is quantizable in the polarizations $\mathcal{P}$ and $\mathcal{P}_c$ with quantum operators $\hat{q}$ in $H(\mathcal{P})$ and $\hat{q}_c$ in $H(\mathcal{P}_c)$. Clearly $\hat{q}$ and $\hat{q}_c$ constitute a complete set of quantum observables in $H(\mathcal{P})$ and $H(\mathcal{P}_c)$ respectively with $H(\mathcal{P})$ the corresponding spectral representation space, and $\hat{q}$ the corresponding spectral representation. Similarly, in the case(2) or case (3) situations the variable p is quantizable in $H(\mathcal{P})$ and $H(\mathcal{P}_c)$ with $H(\mathcal{P}_c)$ the corresponding spectral representation space, and $\hat{p}_c$ the corresponding spectral representation. It seems natural to assume that such a unitary relationship between the quantum operators and their corresponding quantum Hilbert spaces should continue to exist when quantizing canonical variables in canonically conjugate polarizations of non-contractible 2-dimensional symplectic manifolds. We shall therefore use the notion of a complete set of quantum observables and its spectral representation when constructing quantum operators and quantum Hilbert spaces in the examples we shall consider.

(2.3.1) Notation

We shall restrict ourselves to the study of 2-dimensional noncontractible symplectic manifolds (M,ω) with canonical coordinates (I,θ) where (I,θ) are either action-angle variables, or in the case where $M = T^*S^1$ then θ is the polar angle (on S^1) and I is the canonical momentum.

Let $\mathcal{P}$ and $\mathcal{P}_c$ be the polarization spanned by $(\partial/\partial I)$ and $(\partial/\partial\theta)$ respectively. Let Q and Q_c be the effective configuration spaces with respect to the polarizations $\mathcal{P}$ and $\mathcal{P}_c$ respectively, and let $pr: M \dashrightarrow Q$ and $pr_c: M \dashrightarrow Q_c$ be respectively the corresponding projection maps.

Let $(I_r, \theta_r = 0)$ be the chosen reference point in M. Let γ and γ_o be the I-coordinate curves through the points (I_r,θ) and (I_r,θ_o) respectively. Let σ and σ_o be the θ-coordinate curves through the points $(I,\theta_r = 0)$ and $(I_o,\theta_r = 0)$ respectively. Let $R(I)$ denote the range of values of I along γ and let $R(\theta)$ denote the range of values of θ. We have $R(\theta) = \mathbb{R}$. Let β be a one-form on M given by

$$\beta = \beta_I dI+\beta_\theta d\theta = (I+c)d\theta+df(I,\theta) \qquad (2.3.2.\text{Eq } 1)$$

where $c \in \mathbb{R}$ and $f \in C^\infty(M)$. Clearly β satisfies $d\beta = \omega$.

Let $B = M \times \mathbb{C}$ be a trivial bundle on M, $(\cdot,\cdot)$ be the natural Hermitian structure on B, s_o be a unit section of B and let ∇ be a connection on B defined by

$$\nabla_X s_o = -i\hbar^{-1}(X \lrcorner \beta)s_o \text{ for all } X \in V_{\mathbb{C}}(M). \quad (2.3.2.\text{Eq } 2)$$

Let $(B,(\cdot,\cdot),\nabla)$ be the chosen prequantization bundle over (M,ω).

The various line integral of β along the coordinate curves are given by:

$$S_\sigma(I,\theta) = \int_\sigma \beta = \left(\int_0^\theta \beta_\theta d\theta\right)_{I=\text{const.}} = (I+c)\theta+f(I,\theta)-f(I,0), \quad (2.3.2.\text{Eq } 3a)$$

$$S_{\sigma_o}(I_o,\theta) = \int_{\sigma_o}\beta = \left(\int_0^\theta \beta_\theta d\theta\right)_{I=I_o} = (I_o+c)\theta+f(I_o,\theta)-f(I_o,0), \quad (2.3.2.\text{Eq } 3b)$$

$$S_\gamma(I,\theta) = \int_\gamma \beta = \left(\int_{I_r}^{I} \beta_I\, dI\right)_{\theta=\text{const.}} = f(I,\theta)-f(I_r,\theta) \quad (2.3.2.\text{Eq } 3c)$$

and

$$S_{\gamma_o}(I,\theta_o) = \int_{\gamma_o}\beta = \left(\int_{I_r}^{I} \beta_I\, dI\right)_{\theta=\theta_o} = f(I,\theta_o)-f(I_r,\theta_o). \quad (2.3.2.\text{Eq3d})$$

(2.3.3) A particle constrained to move on a circle

Consider the physical situation of a particle constrained to move on circle of radius 1 [cf. Martin (1981), pp46-47]. This is a simple version of the rigid rotor which is of some importance in physics [cf. Schiff (1968), p99]. The configuration space is $Q = S^1$; let θ be the polar angle on S^1. The phase space is $M = T^*Q = \mathbb{R} \times S^1$; let (I,θ) be the usual canonical coordinates on M such that I is the canonical momentum. Clearly M is noncontractible; in particular, it has the geometry of an infinitely long cylinder of radius 1. In quantum mechanics the spectrum of the quantum momentum observable is $\{n\hbar : n \in \mathbb{Z}\}$ [cf. Martin (1968), p99].

Let us quantize I in the polarization $\mathcal{P}$. The leaves of $\mathcal{P}$ are the I-coordinate curves γ which are noncompact since they are infinite lines. The quantization Hilbert space $H_{\mathcal{P}}$ consists of square-integrable sections of the bundle B x $\Delta_{-1/2}(\mathcal{P})$ of the form

$$\Phi = \varphi(\theta)\rho, \ \theta \in \mathbb{R}; \qquad (2.3.3.\text{Eq 1a})$$

where

$$\rho = s_0 |dI|^{-1/2}; \ s_0 = \{\exp \tfrac{i}{\hbar} S_\gamma(I,\theta)\} s_0. \qquad (2.3.3.\text{Eq 1b})$$

The inner-product on $H_{\mathcal{P}}$ is given by

$$\langle \Phi , \Phi \rangle_{\mathcal{P}} = (2\pi h)^{-1/2} \int_0^{2\pi} |\varphi(\theta)|^2 d\theta. \qquad (2.3.3.\text{Eq 2})$$

(Remark: (R1) Note that because s_0 is globally smooth it follows that the section Φ is globally smooth if

$$\varphi(\theta) = \varphi(\theta+2\pi). \qquad (2.3.3.\text{Eq 3}))$$

In this case the effective configuration space with respect to the polarization $\mathcal{P}$ coincides with the actual configuration space (of the classical system) S^1.

The associated vector field $pr_*(X_I)$ generated by I is given by $pr_*(X_I) = (\partial/\partial\theta)$. The canonical momentum I is quantizable in $H_{\mathcal{P}}$ because $pr_*(X_I)$ is complete on S^1. Then by equation (A2.3.Eq 5) [cf. Appendix 2.3], the quantization operator I in $H_{\mathcal{P}}$ is given by the expression

$$\tilde{I}\Phi = [\{-i\hbar(\partial/\partial\theta)-(\partial f(I_r,\theta)/\partial\theta)+c\}\varphi(\theta)]\rho; \qquad (2.3.3.\text{Eq 4})$$

(here we have made the following replacements in the operator expression given by equation (A2.3.Eq 5): $p_r \dashrightarrow I_r$, $f_1(p_{1r}, q_1) \dashrightarrow f(I_r,\theta)+c\theta$ and $(\partial/\partial q_1) \dashrightarrow (\partial/\partial\theta)$).

The classical range of values of I, R(I), is $\mathbb{R}$. From Proposition (2.2.2.P2) [cf. section (2.2)] and the condition given by equation (2.3.3.Eq 3), we conclude that $I_0 \in R(I)$ is an eigenvalue of $\tilde{I}$ if the corresponding section Φ_{I_0} which is given by

$$\Phi_{I_0} = (\hbar/2\pi)^{1/4} \{\exp \tfrac{i}{\hbar} S_{\sigma_0}(I_0,\theta)\}\{\exp -iS_\gamma(I_0,\theta)\}\rho$$
$$= (\hbar/2\pi)^{1/4} \{\exp \tfrac{i}{\hbar}[(I_0+c)\theta+f(I_r,\theta)-f(I_0,\theta)]\}\rho \qquad (2.3.3.\text{Eq 5})$$

is globally smooth. Since ρ is a globally smooth section of $B \times \Delta_{-1/2}(\mathcal{P})$ and $f \in C^{\infty}(M)$, it follows that Φ_{I_o} is globally smooth if

$$(I_o+c)2\pi = 2\pi n\hbar, \text{ where n is some integer.} \qquad (2.3.3.\text{Eq } 6)$$

For each $n \in \mathbb{Z}$, we shall write

$$I(n) = n\hbar - c. \qquad (2.3.3.\text{Eq } 7)$$

Thus the operator $\widetilde{I}$ in $H_{\mathcal{P}}$ has a discrete spectrum

$$R(\widetilde{I}) = \{I(n): n \in \mathbb{Z}\}. \qquad (2.3.3.\text{Eq } 8)$$

Then the normalized eigensection of $\widetilde{I}$ corresponding to the eigenvalue $I(n)$ is given by

$$\Phi_n = (\hbar/2\pi)^{1/4} \{\exp \tfrac{i}{\hbar}[n\hbar+f(I_r,\theta)-f(I(n),\theta)]\} . \qquad (2.3.3.\text{Eq } 9)$$

Note that by choosing $\beta = Id\theta + df$ the physically correct spectrum is obtained.

Let us now quantize I in the polarization $\mathcal{P}_c$. The leaves of $\mathcal{P}_c$ are the θ-coordinate curves. These leaves are toroidal; in particular, they are circles of radius 1. We shall follow the procedure for quantizing the action variable I given in section (1.2) (of Chapter 1).

A formal expression for a $\mathcal{P}_c$-wave function is given by

$$\Psi_c = \Psi(I)\rho_c, \ I \in R(I); \qquad (2.3.3.\text{Eq } 10a)$$

where

$$\rho_c = s_{co}|d\theta|^{-1/2} \text{ and } s_{co} = \{\exp \tfrac{i}{\hbar} S_{\sigma}(I,\theta)\}s . \qquad (2.3.3.\text{Eq } 10b)$$

The sections Ψ_c are only well defined on isolated θ-coordinate curves, and these curves are called the BWS leaves. The sections Ψ_c are only well defined on the curves σ_o (which are determined by the values I_o) if

$$S_{\sigma_o}(I,0) = S_{\sigma_o}(I,2\pi). \qquad (2.3.3.\text{Eq } 11a)$$

or equivalently if the BWS condition given by

$$\oint_{\sigma_o} \beta = 2\pi n\hbar, \text{ for some } n \in \mathbb{Z} \qquad (2.3.3.\text{Eq } 11b)$$

is satisfied. Hence the BWS conditions are satisfied if I_o takes the values

$$I(n) = n\hbar - c, \ n \in \mathbb{Z}. \qquad (2.3.3.\text{Eq } 12)$$

Let σ^n denote the θ-coordinate curve determined by the value $I(n)$. Let $B|_{\sigma^n}$ and $\Delta_{-1/2}(\mathcal{P}_c)|_{\sigma^n}$ denote the restrictions of the bundles B and $\Delta_{-1/2}(\mathcal{P}_c)$, respectively, to σ^n.

Then $\mathcal{P}_c$-wave functions on σ^n are sections of the bundle $B|_{\sigma^n} \times \Delta_{-1/2}(\mathcal{P}_c)|_{\sigma^n}$ of the form

$$\Psi_{cn} = a_n\{\exp \tfrac{i}{\hbar} S_{\sigma^n}(I(n), \theta)\} s_o(I(n), \theta)|d\theta|^{-1/2} \qquad (2.3.3.Eq\ 13)$$

where $a_n \in \mathbb{C}$.

The Hilbert space $\mathbf{H}^n_{\mathcal{P}}$ is defined to be the one-dimensional space consisting of square-integrable $\mathcal{P}$-wave functions on σ^n with respect to the inner-product

$$\langle \Psi_{cn}, \Psi_{cn} \rangle_{\mathcal{P}_c} = |a_n \overline{a_n}|. \qquad (2.3.3.Eq\ 14)$$

The quantization Hilbert space $\mathbf{H}_{\mathcal{P}}$ is defined by the direct sum

$$H_{\mathcal{P}_c} = \bigoplus_{n \in \mathbb{Z}} H^n_{\mathcal{P}_c}. \qquad (2.3.3.Eq\ 15)$$

The quantization operator $\widetilde{I}_c$ in $\mathbf{H}_{\mathcal{P}_c}$ is defined by

$$\widetilde{I}_c \Psi_{cn} = I(n)\, \Psi_{cn} \text{ where } \Psi_{cn} \in H^n_{\mathcal{P}_c}. \qquad (2.3.3.Eq\ 16)$$

The spectrum of $\widetilde{I}$, $R(\widetilde{I}_c)$, is given by

$$R(\widetilde{I}_c) = \{I(n) = n\hbar - c: n \in \mathbb{Z}\}. \qquad (2.3.3.Eq\ 17)$$

By theorem (A2.9.T1) [cf. Appendix 2.9], the operators $\widetilde{I}$ in $\mathbf{H}_{\mathcal{P}}$ and $\widetilde{I}_c$ in $H_{\mathcal{P}_c}$ are unitarily equivalent because they have a common spectrum. We can, in line with the assumptions PA1, QA1, PA2 and QA2 given in section (2.2), formally introduce the notions of quantum Hilbert spaces and quantum operators. It follows that the quantum Hilbert spaces $H(\mathcal{P})$ and $H(\mathcal{P}_c)$, and the quantum operators $\hat{I}$ and $\hat{I}_c$ should be given by $H(\mathcal{P}) = \mathbf{H}_{\mathcal{P}}$ and $H(\mathcal{P}_c) = \mathbf{H}_{\mathcal{P}_c}$, and $\hat{I} = \widetilde{I}$ and $\hat{I}_c = \widetilde{I}_c$ respectively.

In the next subsection we shall give a unified treatment of all this.

(2.3.4) The spectral representation and the spectral representation spaces

We can summarize the results of section (2.2) as follows. The variable $\mathfrak{p}$ was only quantizable in the canonically conjugate polarizations $\mathcal{P}$ and $\mathcal{P}_c$ in the case (2) and case (3) situations, and the corresponding quantum operators were denoted by $\hat{\mathfrak{p}}$ in $H(\mathcal{P})$ and $\hat{\mathfrak{p}}_c$ in $H(\mathcal{P}_c)$. Similarly, the variable $\mathfrak{q}$ was only quantizable in the canonically conjugate polarizations $\mathcal{P}$ and $\mathcal{P}_c$ in the case (1) and case (3) situations, and the corresponding quantum operators were denoted by $\hat{\mathfrak{q}}$ in $H(\mathcal{P})$ and $\hat{\mathfrak{q}}_c$ in $H(\mathcal{P}_c)$.

In all the examples on noncontractible symplectic manifolds that we shall discuss in this section the variable I is quantizable in the canonically conjugate polarizations $\mathcal{P}$ and $\mathcal{P}_c$. Let $\hat{I}$ and $\hat{I}_c$ be the corresponding quantum operators in the quantum Hilbert spaces $H(\mathcal{P})$ and $H(\mathcal{P}_c)$ respectively.

Remark: (R1) From now on we shall use the term **quantizable** to mean quantizable in the canonically conjugate polarizations $\mathcal{P}$ and $\mathcal{P}_c$.

We shall now present the ideas discussed earlier in the following postulates:

(2.3.4.PST1) Postulate PST1

In the situations where $\mathfrak{p}$ is quantizable the **quantum Hilbert space** $H(\mathcal{P}_c)$ is identifiable with the Hilbert space $L^2_c(\mathbb{R},\mu)$ which consists of functions of the classical variable $\mathfrak{p}$ that are square-integrable on $\mathbb{R}$ with respect to a measure μ whose support lies within $R(\mathfrak{p})$ (the range of values of the classical variable $\wp$). In the situations where q is quantizable an analogous statements applies to the quantum Hilbert space $H(\mathcal{P})$. In situations where I is quantizable an analogous statements also applies to the quantum Hilbert space $H(\mathcal{P}_c)$.

(2.3.4.PST2) Postulate PST2

In the situations where $\wp$ is quantizable the quantum operator $\hat{\wp}$ in $H(\mathcal{P})$ has $H(\mathcal{P}_c)$ as its spectral representation space, and $\hat{\wp}_c$ as its spectral representation. In situations where q is quantizable, the quantum operator $\hat{q}_c$ in $H(\mathcal{P}_c)$ has $H(\mathcal{P})$ as its spectral representation space, and $\hat{q}$ as its spectral representation. In situations where I is quantizable the quantum operator $\hat{I}$ in $H(\mathcal{P})$ has $H(\mathcal{P}_c)$ as its spectral representation space, and $\hat{I}_c$ as its spectral representation.

The essence of the above postulates lies in the extension of the definition of the quantum Hilbert space introduced in section (2.2). In particular, the quantum Hilbert space defined by postulate PST1 is basically the same as that given by the assumptions PA1, QA1, PA2 and QA2 except that the inner-product is now defined with respect a measure μ instead of the standard Lebesgue measure. This generalization enables us to deal with the quantizations of the canonical variables $\wp$ and q in the 4 cases studied in section (2.2) where the quantized observables have continuous spectra, and the quantizations of the action variable I where the quantized operator has a discrete spectra in a unified manner. In general, the measure μ is not uniquely defined by the requirement in postulate PST1 [cf. Prugovecki (1981), p324].

Postulate PST2 basically spells out the requirement that the quantizations in the polarizations $\mathcal{P}$ and $\mathcal{P}_c$ should be unitarily related.

We shall now apply our postulates to concrete examples.

(2.3.5) <u>Quantizations of canonical variables in canonically conjugate polarizations of contractible 2-dimensional symplectic manifolds revisited</u>

Clearly the results of section (2.2) are consistent with the postulates PST1 and PST2. We shall illustrate this point with the following example. Consider the situations when $\mathfrak{q}$ is quantizable. Clearly the quantum operator $\hat{\mathfrak{q}}_c$ in $H(\mathcal{P}_c)$ has $H(\mathcal{P})$ as its spectral representation space and $\hat{\mathfrak{q}}$ as its spectral representation. This unitary link is exactly what is required by postulate PST2. We based the postulate PST2 on this unitary link. The Hilbert space $H(\mathcal{P})$ (defined in section (2.2)) is identical with the Hilbert space $L^2(\mathbb{R},\mu)$ (defined by postulate PST1) when the measure μ is given by

$$\begin{aligned} d\mu(\mathfrak{q}) &= (2\pi\hbar)^{-1/2}\, d\mathfrak{q} \text{ for } \mathfrak{q} \in R(\mathfrak{q}) \\ &= 0 \text{ otherwise.} \end{aligned} \qquad (2.3.5.\text{Eq } 1)$$

Similarly, one can show that the quantizations of the variable $\mathfrak{p}$ given in section (2.2) is consistent with the postulates PST1 and PST2.

(2.3.6) Noncontractible symplectic manifolds

Let (M,ω) be a noncontractible 2-dimensional symplectic manifold with canonical coordinates (I,θ); we shall assume the notation given in subsection (2.3.2).

Then by postulate PST1 the quantum Hilbert space $H(\mathcal{P}_c)$ consists of elements of form

$$\Psi_c = \psi_c(I)\rho_c, \quad I \in R(I) \qquad (2.3.6.\text{Eq } 1b)$$

where

$$\psi_c \in L^2_c(\mathbb{R},\mu) \;;\; \rho_c = s_{co}|d\theta|^{-1/2} \;;\; s_{co} = \{\exp \tfrac{i}{\hbar} S_\sigma(I,\theta)\}s_o. \qquad (2.3.6.\text{Eq } 1b).$$

The measure μ can be determined in three steps as follows.

(μ1) Clearly μ cannot be a Lebesgue measure since μ is only well defined on the θ-coordinate curves σ^n (which are determined by the value $I(n)$) on which $I(n)$ takes the values [cf. equation (2.3.3.Eq 12)]

$$I(n) = n\hbar - c, \quad n \in \mathbb{Z}. \qquad (2.3.6.\text{Eq } 2)$$

(μ2) The quantization Hilbert space $H_\mathcal{P}$ and the quantization operator $\widetilde{I}$ are formally identical to those given explicitly in subsection (2.3.3) [cf. equations (2.3.3.Eq 1a), (2.3.3.Eq 1b), (2.3.3.Eq 3) and (2.3.3.Eq 4)]. We must now check to see if $H_\mathcal{P}$ and $\widetilde{I}$ can be regarded as the quantum Hilbert space and quantum operator respectively. One way of doing this is to check whether the spectral representation space of $\widetilde{I}$ in $H_\mathcal{P}$ can be identified with the Hilbert space $L^2_c(\mathbb{R},\mu)$. In practice this is done as follows:

(i) Suppose $R(\widetilde{I})$ (the spectrum of $\widetilde{I}$) is contained in $R(I)$ (the range of classical values of I); then the quantum operator $\hat{I}$ is taken to be the quantization operator $\widetilde{I}$, and the quantum Hilbert space $H(\mathcal{P})$ is taken to be the quantization Hilbert space $H_\mathcal{P}$

(ii) Let Φ_n be the globally smooth eigenfunctions of the operator $\widetilde{I}$ in

$H_{\wp}$ given formally by equation (2.3.3.Eq 9). Suppose $R(\tilde{I})$ is not contained in R(I); then we define the quantum Hilbert space $H(\wp)$ to be the subspace of $\mathbf{H}_{\wp}$ spanned by the set $\{\Phi_n : n \in \mathbb{Z},\ I(n) \in R(I)\}$. The quantum operator $\hat{I}$ is then defined to be the restriction of the quantization operator $\tilde{I}$ to $H(\wp)$. Let $R(\hat{I})$ be the spectrum of the quantum operator $\hat{I}$.

(μ3) It follows from postulate PST2 that the measure μ is a discrete measure with support R(I), and such that

$$\mu(\{I(n)\}) = 1 \text{ for every } I(n) \in R(\hat{I}). \qquad (2.3.6.\text{Eq } 3)$$

It is easy to check that the multiplication operator I on the Hilbert space $L^2_{\mathbb{C}}(\mathbb{R},\mu)$ possesses a discrete spectrum identical to $R(\hat{I})$ with normalized eigenfunctions $\Psi_{cn}(I)$ given by

$$I\Psi_{cn}(I) = I(n)\Psi_{cn}(I) \qquad (2.3.6.\text{Eq } 4a)$$

or explicitly

$$\Psi_{cn}(I) = 1 \text{ when } I = I(n).$$
$$= 0 \text{ when } I = I(n') \neq I(n),\quad n' \in \mathbb{Z}. \qquad (2.3.6.\text{Eq } 4b)$$

(Note that I(n) is an eigenvalue of the multiplication operator I only if I(n) belongs to R(I) the range of classical values of I.) Therefore, we shall define the quantum operator $\hat{I}_c$ in $H(\wp_c)$ by $\hat{I}_c = I$.

We shall now apply these ideas to two concrete examples.

(2.3.6.Ex 1) Example: The particle constrained to move on a circle revisited

In this case we have $R(I) = \mathbb{R}$ and $R(\tilde{I}) = \{I(n): n \in \mathbb{Z}\}$ [cf. equations (2.3.7.Eq 7) and (2.3.7.Eq 8)]. Hence the spectrum of $\tilde{I}$, $R(\tilde{I})$, is contained in R(I).

To summarize we have:

(i) $\mathbf{H}(\mathcal{P}) = \mathbf{H}_{\mathcal{P}}$, $\hat{I} = \tilde{I}$ and $R(\hat{I}) = R(\tilde{I})$. The measure μ is then given by equation (2.3.6.Eq 3).

(ii) The Hilbert space $\mathbf{H}(\mathcal{P}_c)$ consists of elements of the form $\Psi_c = \psi_c(I)\rho_c$ where $\psi_c(I)$ is a member of $L^2_c(\mathbb{R},\mu)$.

(iii) The operator $\hat{I}_c = I$ in $\mathbf{H}(\mathcal{P}_c)$ has the the same spectrum as $\hat{I}$ in $H(\mathcal{P})$. By theorem (A2.9.T1) [cf. Appendix 2.9], $\hat{I}$ in $H(\mathcal{P})$ and $\hat{I}_c$ in $H(\mathcal{P}_c)$ are unitarily related.

(2.3.6.Ex 2) Example: The one-dimensional harmonic oscillator [cf. Wan and McKenna (1984)]

Let $\mathbb{R}$ be the configuration space with cartesian coordinate q. The cotangent bundle $T^*\mathbb{R}$ is identifiable with $\mathbb{R}^2$. Let ω be the canonical two-form on $T^*\mathbb{R}$, and let (p,q) be the usual cartesian canonical coordinates on $T^*\mathbb{R}$. Let $M = T^*\mathbb{R}-\{(0,0)\}$; then (M,ω) is noncontractible 2-dimensional symplectic manifold. Let (I,θ) be action-angle variables on M given by

$$I = (p^2+q^2)/2, \ \theta = \tan^{-1}(q/p). \qquad (2.3.6.\text{Eq } 5)$$

The Hamiltonian of the harmonic oscillator is given by H = I.

Our problem is to establish unitarily equivalent quantizations of the action variable I in the canonically conjugate polarizations $\mathcal{P}$ and $\mathcal{P}_c$. However, unlike the previous example there is a constraint on I given by $I>0$, i.e. $R(I) = (0,\infty)$. This means we can no longer identify the quantum Hilbert space $H(\mathcal{P})$ and quantum operator $\hat{I}$ with the quantization Hilbert space $H_{\mathcal{P}}$ and quantization operator $\tilde{I}$ respectively.

Let us quantize I in the polarization $\mathcal{P}$. Then $W_{\mathcal{P}}$ consists of square-integrable sections of the bundle $B \times \Delta_{-1/2}(\mathcal{P})$ of the form $\Phi = \varphi(\theta)\rho$ where $\rho = \mathscr{S}_o|dI|^{-1/2}$ and $\mathscr{S}_o = \{\exp \frac{i}{\hbar} S_\gamma(I,\theta)\}s_o$. The quantization Hilbert space $H_{\mathcal{P}}$ is the completion of $W_{\mathcal{P}}$. By equation (2.3.3.Eq 4a), the quantization operator $\widetilde{I}$ in $H_{\mathcal{P}}$ is given by the expression $\widetilde{I}\Phi = [\{-i\hbar(\partial/\partial\theta)-(\partial f(I_r,\theta)/\partial\theta)+c\}\varphi(\theta)]\rho$. By equation (2.3.3.Eq 8), the spectrum of $\widetilde{I}$ is given by $R(\widetilde{I}) = \{I(n) = n\hbar - c\colon n \in \mathbb{Z}\}$.

We shall now introduce a discrete measure ν as follows. Let ν be a measure with support $R(\widetilde{I})$ and such that

$$\nu(\{I(n)\}) = 1 \text{ for every } I(n) \in R(\widetilde{I}). \qquad (2.3.6.\text{Eq } 6)$$

Then in analogy with the definition of the quantum Hilbert space given in postulate PST1 we define the quantization Hilbert space $H_{\mathcal{P}_c}$ to be the space of square-integrable sections of $B \times \Delta_{-1/2}(\mathcal{P}_c)$ of the form $\Psi_c = \psi_c(I)\rho_c$ where $I \in \mathbb{R}$, $\psi_c(I) \in L^2_c(\mathbb{R},\mu)$ and $\rho_c = \mathscr{S}_{co}|d\theta|^{-1/2}$. The quantization operator $\widetilde{I}_c$ is then the multiplication operator I in $H_{\mathcal{P}_c}$. Clearly the the spectrum of $\widetilde{I}_c$ is given by $R(\widetilde{I}_c) = R(\widetilde{I})$. Then by theorem (A2.9.T1) the quantization operators $\widetilde{I}_c$ in $H_{\mathcal{P}_c}$ and $\widetilde{I}$ in $H_{\mathcal{P}}$ are unitarily equivalent. Since $R(\widetilde{I})$ contains negative eigenvalues, it follows that neither $\widetilde{I}$ nor I_c represent a physically acceptable quantized operator corresponding to the Hamiltonian H where H = I. Alternatively, one could say that $H_{\mathcal{P}_c}$ is not the quantum Hilbert space and $\widetilde{I}_c$ is not the quantum operator as $H_{\mathcal{P}_c}$ contradicts postulate PST1: ν does not vanish outside R(I) (the classical range of values of I). Similarly, by postulate PST2, $H_{\mathcal{P}}$ is not the quantum Hilbert space and $\widetilde{I}$ is not the quantum operator.

By equation (2.3.3.Eq 9), the eigensections of $\widetilde{I}$ corresponding to the eigenvalue $I(n) = n\hbar - c$, where $n \in \mathbb{Z}$, are given by $\Phi_n = (\hbar/2\pi)^{1/4} \{\exp \frac{i}{\hbar}[n\hbar + f(I_r,\theta) - f(I(n),\theta)]\}\rho$.

Since the spectrum $R(\widetilde{I})$ is not contained in $R(I)$, the quantum Hilbert space $H(\mathcal{P})$ is defined to be the subspace spanned by the set of eigensections $\{\Phi_n : I(n)>0\}$. The quantum operator $\hat{I}$ is then defined to be the restriction of the quantization operator $\widetilde{I}$ to $H(\mathcal{P})$. The spectrum of $\hat{I}$ is given by

$$R(\hat{I}) = \{I(n) : n \in \mathbb{Z}\} \cap R(I)$$
$$= \{I(n) : n \in \mathbb{Z} \text{ and } I(n)>0\}. \qquad (2.3.6.\text{Eq } 7)$$

The quantum Hilbert space $H(\mathcal{P}_c)$ and the quantum operator $\hat{I}_c$ are defined as follows. To conform with postulate PST2 we introduce a new discrete measure μ with support $R(\hat{I})$ and such that

$$\mu(\{I(n)\}) = 1 \text{ for every } I(n) \in R(\hat{I}). \qquad (2.3.6.\text{Eq } 8)$$

The quantum Hilbert space $H(\mathcal{P}_c)$ is defined to be the space of sections of $B \times \Delta_{-1/2}(\mathcal{P}_c)$ of the form $\Psi_c = \Psi_c(I)\rho_c$ where $\Psi_c(I)$ is a member of $L^2_c(\mathbb{R},\mu)$. The quantum operator $\hat{I}_c$ is then defined to be the multiplication operator in $H(\mathcal{P}_c)$.

Clearly the quantum operators $\hat{I}$ and $\hat{I}_c$ have a common spectrum; so by theorem (A2.9.T1), [cf.Appendix 2.9], the operators $\hat{I}$ and $\hat{I}_c$ are unitarily equivalent. Then by postulate PST2 $\hat{I}$ in $H(\mathcal{P})$ has $H(\mathcal{P}_c)$ as spectral representation space and $\hat{I}_c$ as its spectral representation.

Finally, by choosing the connection potential to be

$$\beta = (I+\hbar/2)d\theta + df. \qquad (2.3.6,\text{Eq } 9)$$

the eigenvalues of $\hat{I}$ become $I(n) = (n+1/2)\hbar$, where $n = 1,2,3,\ldots$. We have therefore established consistent quantizations of the variable I in the canonically conjugate polarizations $\mathcal{P}$ and $\mathcal{P}_c$ with physically correct spectrum.

(2.3.7) Concluding remark

We have established a unified scheme for dealing with the quantizations of canonical variables in suitably chosen canonically conjugate polarizations applicable whether or not the symplectic manifold is contractible.

APPENDICES A2.1-A2.9

APPENDIX 2.1

The pairing map and the linear map (induced by the pairing map)

Let (M,ω) be a 2k-dimensional symplectic manifold. Let $\mathcal{P}$ and $\mathcal{P}'$ be two reducible polarizations of (M,ω) that are transverse. Let $(B,(\cdot,\cdot),\nabla)$ be the chosen prequantization bundle over (M,ω). Let $H_{\mathcal{P}}$ and $H_{\mathcal{P}'}$ be the quantization Hilbert spaces associated with the polarizations $\mathcal{P}$ and $\mathcal{P}'$ respectively. Let $\langle\ ,\ \rangle_{\mathcal{P}}$ and $\langle\ ,\ \rangle_{\mathcal{P}'}$ be the inner-products on $H_{\mathcal{P}}$ and $H_{\mathcal{P}'}$ respectively. Then $H_{\mathcal{P}}$ consists of square-integrable sections of the bundle $B \times \Delta_{-1/2}(\mathcal{P})$ of the form $\Psi = s\nu$ that satisfy $\nabla_X s = 0$ and $\nabla_X \nu = 0$ for all $X \in V(M;\mathcal{P})$. Similarly, $H_{\mathcal{P}'}$ consists of square-integrable sections of the bundle $B \times \Delta_{-1/2}(\mathcal{P}')$ of the form $\Phi' = s'\nu'$ that that satisfy $\nabla_Y s' = 0$ and $\nabla_Y \nu' = 0$ for all $Y \in V(M;\mathcal{P}')$.

(A2.1.D1) Definition [cf. Blattner (1973); Woodhouse (1980), p160]

The **pairing map** between the quantization Hilbert spaces $H_{\mathcal{P}}$ and $H_{\mathcal{P}'}$ is the sesquilinear map

$$\langle\Psi, \Phi'\rangle_{\mathcal{P}\mathcal{P}'} : H_{\mathcal{P}} \times H_{\mathcal{P}'} \dashrightarrow \mathbb{C} \qquad \text{(A2.1.Eq 1a)}$$

given by

$$\langle\Psi, \Phi'\rangle_{\mathcal{P}\mathcal{P}'} = [2\pi\hbar]^{-k}\int_M (\Psi, \Phi')_{\mathcal{P}\mathcal{P}'} \qquad \text{(A2.1.Eq 1b)}$$

where $(\Psi, \Phi')_{\mathcal{P}\mathcal{P}'}$ is the one-TM-density defined by

$$(\Psi, \Phi')_{\mathcal{P}\mathcal{P}'} = (s,s')\,\nu\{X_i\}\overline{\nu'}\{Y_j\}\,|2k!\,\mathcal{E}_\omega(X_1,\ldots X_k, Y_1,\ldots,Y_k)|^{3/2} \qquad \text{(A2.1.Eq 1c)}$$

such that $\{X_i\}$ is a field of bases of $\mathcal{P}$, $\{Y_j\}$ is a field of bases of $\mathcal{P}'$ and $\mathcal{E}_\omega$ is the Liouville volume form.

(A2.1.D2) Definitions [cf. Blattner (1973); Woodhouse (1980), p160]

(D2.1) The **linear maps (induced by the pairing map)** are the maps $U_{\mathcal{P}\mathcal{P}'} : H_{\mathcal{P}} \dashrightarrow H_{\mathcal{P}'}$ and $U_{\mathcal{P}'\mathcal{P}} : H_{\mathcal{P}'} \dashrightarrow H_{\mathcal{P}}$ determined by the relations

$$\langle U_{\mathcal{P}\mathcal{P}'}\Psi, \Phi'\rangle_{\mathcal{P}'} = \langle \Psi, \Phi'\rangle_{\mathcal{P}\mathcal{P}'} \qquad \text{(A2.1.Eq 2a)}$$

and

$$\langle \Psi, U_{\mathcal{P}'\mathcal{P}}\Phi'\rangle_{\mathcal{P}} = \langle \Psi, \Phi'\rangle_{\mathcal{P}\mathcal{P}'} \qquad \text{(A2.1.Eq 2b)}$$

respectively.

(D2.2) The polarizations $\mathcal{P}$ and $\mathcal{P}'$ are said to be **unitarily related** if $U_{\mathcal{P}\mathcal{P}'}$ and $U_{\mathcal{P}'\mathcal{P}}$ are unitary maps.

Remark: (R1) In general the linear maps (induced by the pairing map) are not unitary maps. There are no general theorems for ascertaining whether $\mathcal{P}$ and $\mathcal{P}'$ are unitarily related; however, we do have the following theorem.

(A2.1.T1) Theorem [cf. Blattner (1974)]

Let $(\mathbb{R}^{2k},\omega)$ be a symplectic manifold where ω is any symplectic two-form on $\mathbb{R}^{2k}$. Then any two transverse, transalational invariant, reducible polarization of $(\mathbb{R}^{2k},\omega)$ are unitarily related.

Example: the pairing map between the quantization Hilbert spaces associated with the vertical and horizontal polarizations, and the linear maps (induced by the pairing map)

Let $Q \subseteq \mathbb{R}$ be the configuration space with cartesian coordinate q and let R(q) denote the range of q. Let $M = T^{*}Q$ be the phase space, ω be the canonical two-form on M and let (p,q) be the usual cartesian canonical coordinates on M. Let R(p) denote the range of p which is given by $R(p) = \mathbb{R}$. Let m be an arbitrary point in M. Let γ denote the p-coordinate curve and let σ denote the q-coordinate through the point m. Let P and P_c be respectively the vertical and horizontal polarizations of (M,ω), i.e. P is spanned by the vector field $(\partial/\partial p)$ and P_c is spanned by the vector field $(\partial/\partial q)$. Clearly the polarizations P and P_c are both reducible. The effective configuration space with respect to P is the actual configuration space Q. Let Q_c be the effective configuration space with respect to P_c; then Q_c is identifiable with R(p) and is coordinatized by p. Let $B = M \times \mathbb{C}$ be the trivial bundle over M, $(\cdot,\cdot)$ be the natural Hermitian structure of B, s_o be a unit section of B and let ∇ be a connection on B given by

$$\nabla_X s_o = -\tfrac{i}{\hbar}(X \lrcorner \beta)s_o, \text{ for all } X \in V(M) \qquad \text{(A2.1.Eq 3)}$$

where

$$\beta = pdq+df, \; f \in C^{\infty}(M). \qquad \text{(A2.1.Eq 4)}$$

Let $(B,(\cdot,\cdot),\nabla)$ be the chosen prequantization bundle over (M,ω).

Let (p_r,q_r) be an arbitrary point in M. Here is a list of integrals of the connection potential β along the coordinate curves that we shall use:

$$S_\gamma(p,q) = \int_\gamma \beta = \left(\int_{p_r}^{p} (\partial f/\partial p)dp\right)_{q=\text{const.}} = f(p,q)-f(p_r,q) \qquad \text{(A2.1.Eq 5a)}$$

and

$$S_\sigma(p,q) = \int_\sigma \beta = \left(\int_{q_r}^{q} p+(\partial f/\partial q)dq\right)_{p=\text{const.}} = pq-pq_r+f(p,q)-f(p,q_r). \qquad \text{(A2.1.Eq 5b)}$$

The quantization Hilbert space $\mathbf{H_P}$ consists of square-integrable sections of $B \times \Delta_{-1/2}(P)$ of the form

$$\Psi = \psi(q)\{\exp \tfrac{i}{\hbar}S_\gamma(p,q)\}s_o|dp|^{-1/2} \qquad \text{(A2.1.Eq 6a)}$$

with respect to the inner-product

$$\langle\Psi,\Psi\rangle_P = [2\pi\hbar]^{-1/2}\int_Q |\psi(q)|^2 dq. \qquad \text{(A2.1.Eq 6b)}$$

The quantization Hilbert space H_{P_c} consists of square-integrable sections of the bundle $B \times \Delta_{-1/2}(P_c)$ of the form

$$\Phi_c = \varphi_c(p)\{\exp \tfrac{i}{\hbar}S_\sigma(p,q)\}s_o\,|dq|^{-1/2} \qquad \text{(A2.1.Eq 7a)}$$

with respect to the inner-product

$$\langle \Phi_c, \Phi_c \rangle_{P_c} = [2\pi\hbar]^{-1/2}\int_{\mathbb{R}} |\varphi_c(p)|^2\,dp. \qquad \text{(A2.1.Eq 7b)}$$

Then the one-TM-density $(\Psi, \Phi_c)_{PP_c}$ defined by equation (A2.1.Eq 1c) is given explicitly

$$\begin{aligned}(\Psi, \Phi_c)_{PP_c} &= \Psi(q)\overline{\varphi_c}(p)\,|dp|^{-1/2}\{(\partial/\partial p)\}\,|dq|^{-1/2}\{(\partial/\partial q)\}\{\exp \tfrac{i}{\hbar}(S_\gamma(p,q)\} \times \\ &\qquad \{\exp -\tfrac{i}{\hbar}S_\sigma(p,q)\}\,|2!dp \wedge dq((\partial/\partial p),(\partial/\partial q))| \\ &= \Psi(q)\overline{\varphi_c}(p)\{\exp \tfrac{i}{\hbar}S_\gamma(p,q)\}\{\exp -\tfrac{i}{\hbar}S_\sigma(p,q)\} \qquad \text{(A2.1.Eq 8)}\end{aligned}$$

Therefore the pairing map between H_P and H_{P_c} is given by

$$\langle \Psi, \Phi_c \rangle_{PP_c} = [2\pi\hbar]^{-1}\int_Q\int_{\mathbb{R}} \Psi(q)\overline{\varphi_c}(p)\{\exp \tfrac{i}{\hbar}S_\gamma(p,q)\}\{\exp -\tfrac{i}{\hbar}S_\sigma(p,q)\}\,dp\,dq$$

(A2.1.Eq 9)

The linear map (induced by the pairing map) U_{PP_c} is given by

$$U_{PP_c}\Psi = [\,2\pi\hbar]^{-1/2}\,[\int_Q \Psi(q)\{\exp \tfrac{i}{\hbar}S_\gamma(p,q)\}\{\exp -iS_\sigma(p,q)\}\,dq]\,\rho_c$$

(A2.1.Eq 10a)

where

$$\rho_c = \{\exp \tfrac{i}{\hbar}S_\sigma(p,q)\}s_o\,|dq|^{-1/2}. \qquad \text{(A2.1.Eq 10b)}$$

The linear map (induced by the pairing map) U_{P_cP} is given by

$$U_{P_cP}\,\Phi_c = [2\pi\hbar]^{-1/2}\,[\int_{\mathbb{R}} \varphi_c(p)\{\exp -\tfrac{i}{\hbar}S_\gamma(p,q)\}\{\exp \tfrac{i}{\hbar}S_\sigma(p,q)\}\,dp]\,\rho$$

(A2.1.Eq 11a)

where

$$\rho = \{\exp \tfrac{i}{\hbar}S_\gamma(p,q)\}s_o\,|dp|^{-1/2}. \qquad \text{(A2.1.Eq 11b)}$$

Remark: (R2) In the case where $Q = \mathbb{R}$, we have $M = \mathbb{R}^2$. Then by Theorem (A2.1.T1) U_{PP_c} and U_{P_cP} are unitarily related. In particular when $\beta = p\,dq$, then U_{PP_c} and U_{P_cP} are given by

$$U_{PP_c}\Psi = [2\pi\hbar]^{-1/2}\,[\int_{\mathbb{R}} \Psi(q)\{\exp -\tfrac{i}{\hbar}pq\}dq]\,\rho_c \qquad \text{(A2.1.Eq 12a)}$$

and

$$U_{P_cP} \quad = [2\pi\hbar]^{-1/2}\,[\int_{\mathbb{R}} \varphi_c(p)\{\exp \tfrac{i}{\hbar}pq\}dq]\,\rho. \qquad \text{(A2.1.Eq 12b)}$$

Clearly U_{PP_c} and U_{P_cP} are identifiable with the the inverse Fourier transform and Fourier transform respectively.

APPENDIX 2.2

The parallel transport on a complex line bundle [cf. Kostant (1970), pp106-107; Simms and Woodhouse (1976), pp32-34; Woodhouse (1980), p116]

Let (M,ω) be a symplectic manifold such that there exists a global one-form β that satisfies $d\beta = \omega$. Let $B = M \times \mathbb{C}$ be the trivial bundle over M, $(\cdot,\cdot)$ be the natural Hermitian structure of B, s_o be a unit section of B and let ∇ be a connection on B given by

$$\nabla_X s_o = -i(X \lrcorner \beta)s_o, \text{ for all } X \in V(M).$$

Let $(0,a)$ be an open interval of $\mathbb{R}$ and let $\gamma : (0,a) \dashrightarrow M$ be a smooth curve in M originating at the point m_o. Let $m_o = \gamma(0)$. Let γ denote the set $\{m: \; t \in (0,a), \; \gamma(t) = m\}$.

(A2.2.D1) Definitions

(D1.1) The map $r:\gamma \dashrightarrow B$ is referred to as a **section along the curve** .

(D1.2) Let X_γ be a tangent vector along γ and let r be a section along γ. Then r is said to be a **parallel section along** γ if

$$(\nabla_{X_\gamma} r)(m) = 0 \text{ for all } m \in \gamma.$$

Remarks: (R1) Any section r along γ can be written in the form $r = \varphi s_o$ where $\varphi : \gamma \dashrightarrow \mathbb{C}$ is a complex valued function along γ.

(R2) Let r be a section along γ given by $r = \varphi s_o$. Clearly r is parallel along γ if

$$X_\gamma(\varphi) - i(X \lrcorner \beta \;)\varphi = 0 \text{ on } \gamma.$$

Hence r can be written in the form

$$r = z_o\{\exp i(\textstyle\int_\gamma \beta)\}s_o \text{ on}$$

where $z \in \mathbb{C}$ and $r(m_o) = z_o s_o(m_o) \in B_{m_o}$.

(A2.2.D2) Definition

The parallel transport along γ (from m_o to some point $m \in \gamma$) is the linear isomorphism

$$L_\gamma : B_{m_o} \dashrightarrow B_m$$

defined by

$$L_\gamma(b_o) = L_\gamma(z_o s_o(m_o)) = z_o\{\exp \pm(\int_{m_o}^{m} \beta_\gamma)\} s_o(m)$$

for all $b_o = z_o s_o(m_o) \in B_{m_o}$.

APPENDIX 2.3

Operator expressions

(A2.3.1) The expression for the operator $\widetilde{q}_c$ in $H_{\mathcal{P}_c}$

We have

$S_\sigma(p,q) = p(q-q_r)+f(p,q)-f(p,q_r)$ by equation (2.2.2.Eq 2d),

$s_{co} = \{\exp \frac{i}{\hbar} S_\sigma(p,q)\}s_o$ by equation (2.2.2.Eq 5b),

$\beta = p\,dq + df(p,q)$ by equation (2.2.2.Eq 1a),

and

$\nabla_X s_o = -\frac{i}{\hbar}(X \lrcorner \beta)s_o$ for all $X \in V_{\mathbb{C}}(M)$ by equation (2.2.2.Eq 3).

Then, for $X_q = -(\partial/\partial p)$,

$$\begin{aligned}\nabla_{X_q} s_{co} &= X_q(\{\exp \tfrac{i}{\hbar} S_\sigma(p,q)\})s_o + \{\exp \tfrac{i}{\hbar} S_\sigma(p,q)\}\nabla_{X_q} s_o \\ &= -\tfrac{i}{\hbar}\{(\partial S_\sigma(p,q)/\partial p) + (X_q \lrcorner \beta)\}s_{co} \\ &= \tfrac{i}{\hbar}\{(\partial f(p,q_r)/\partial q) - q + q_r\}s_{co}. \end{aligned} \tag{A2.3.Eq 1}$$

Hence for $s_c = \Psi_c(p)s_{co}$, we have

$$\nabla_{X_q} s_c = -(\partial \Psi_c(p)/\partial p)s_{co} + \Psi_c(p)\nabla_{X_q} s_{co}. \tag{A2.3.Eq 2}$$

We have $L_{X_q}|dq|^{-1/2} = 0$ by equation (1.1.6.Eq 16).

Let $\Psi_c = \Psi_c(p)s_{co}|dq|^{-1/2} \in H_{\mathcal{P}_c}$. Thus the expression for $\widetilde{q}_c$ in $H_{\mathcal{P}_c}$ is given by

$$\begin{aligned}\widetilde{q}_c \Psi &= \{(-i\hbar\nabla_{X_q} + q)\Psi_c(p)s_{co}\}|dq|^{-1/2} - i\hbar\Psi_c(p)s_{co}(L_{X_q}|dq|^{-1/2}) \\ &= [\{i\hbar(\partial/\partial p) + (\partial f(p,q_r)/\partial p) + q_r\}\Psi_c(p)]s_{co}|dq|^{-1/2}. \end{aligned} \tag{A2.3.Eq 3}$$

APPENDIX 2.4

Essential self-adjointness

(A2.4.D1) Definition [Hellwig (1964), p172]

An operator **A** in a Hilbert space **H** is said to be **essentially self-adjoint if**

(ESA1) **A** in **H** is symmetric; and

(ESA2) (**A**+i)**H** and (**A**-i)**H** are dense in **H**.

Remark: (R1) We shall denote the adjoint of an operator **A** by $\mathbf{A}^{+}$.

(A2.4.L1) Lemma

A symmetric operator A in a Hilbert space H is essentially self-adjoint if and only if its adjoint A^{+} has no eigenfunctions corresponding to eigenvalues $\pm i$, i.e. if and only if

$$(\mathbf{A}^{+} \pm i)\varphi = 0 \Rightarrow \varphi = 0. \qquad \text{(A2.4.Eq 1)}$$

Proof: This follows readily from the definition given above.■

Remark: (R2) Let **H** and **H'** be two arbitrary Hilbert spaces that are linked together by the unitary map **U** :**H** $\longrightarrow$ **H'**. Let **I**:**H** $\longrightarrow$**H** and **I'**:**H'** $\longrightarrow$ **H'** be the identity operators on **H** and **H'** respectively. Then $\mathbf{U}^{+}$ has the following properties [cf. Weidmann (1980), p85-86]:

(U1) $\mathbf{U}^{+} = \mathbf{U}^{-1}$; and

(U2) $\mathbf{U}^{+}\mathbf{U} = \mathbf{I}$ and $\mathbf{U}\mathbf{U}^{+} = \mathbf{I}'$.

(A2.4.L2) Lemma

Let H and H' be two Hilbert spaces linked together by the unitary operator $U:H \dashrightarrow H'$. Let A be a symmetric operator in H and let A' be the symmetric operator in H' given by $A' = UAU^+$. Then A is essentially self-adjoint if and only if A' is essentially self-adjoint.

Proof

Suppose φ is a vector in H such that

$$(A'^+ \pm i)\varphi = 0. \qquad \text{(A2.4.Eq 2)}$$

Then

$$(A'^+ \pm i)U^+U\varphi = (A'^+U^+ \pm iU^+)\varphi' = 0,\ \varphi' = U\varphi. \qquad \text{(A2.4.Eq 3)}$$

This implies $(UA'^+U^+ \pm i)\varphi' = 0$. Since $UA'^+U^+ \subseteq A^+$, we get

$$(A^+ \pm i)\varphi' = 0. \qquad \text{(A2.4.Eq 4)}$$

Now if A' is not essentially self-adjoint; then by Lemma (A2.1.L1) there exists $0 \neq \varphi \in H$ satisfying equation (A2.4.Eq 2). Therefore, there exists $\varphi' = U\varphi \neq 0$ satisfying equation (A2.4.Eq 4). Consequently A is not essentially self-adjoint. Similarly, we can argue that A being not essentially self-adjoint implies that A' being not essentially self-adjoint to establish the lemma. ∎

(A2.4.P1) Proposition

The restriction of the operator $\tilde{\wp}_c$ in $H_{\wp_c}$ to $H(\wp_c)$ is not essentially self-adjoint.

Proof:

Let $\tilde{\wp}_{cr}$ denote the restriction of $\tilde{\wp}_c$ to $H(\wp_c)$ and let $\hat{\wp} = U^{+}\tilde{\wp}_{cr}U$ where $U: H(\wp) \dashrightarrow H(\wp_c)$ is a unitary operator defined by equation (2.2.2.Eq 27a). Then $\hat{\wp}$ is an operator in $H(\wp)$. Explicitly we have

$$\hat{\wp}\ \Phi = [\{-i\hbar(\partial/\partial q)-(\partial f(\wp_r, q)/\partial q)\}\varphi(q)]\rho \qquad \text{(A2.4.Eq 5)}$$

where $\Phi = \varphi(q)\rho$ is given by equations (2.2.2.Eq 20a) and (2.2.2.Eq 20b). It is well known that the operator $\{-i\hbar(\partial/\partial q)-(\partial f(\wp_r, q)/\partial q)\}\varphi(q)$ is not essentially self-adjoint when operating on $\varphi(q)$ whose domain is $R(q) \neq \mathbb{R}$ [cf. Akhiezer and Glazman (1961), pp106-111; Wan and Viazminsky (1977)]. Thus $\tilde{\wp}_{cr}$ is not essentially self-adjoint by Lemma (A2.4.L2).■

APPENDIX 2.5

<u>Bochner's theorem</u> [cf. Riesz and Nagy (1955), pp291-292; Wan and McKenna (1984)]

Let (a,a') and (b,b') be open intervals in $\mathbb{R}$. We shall use the letters $\mathfrak{q}$ and $\mathfrak{p}$ for the usual cartesian coordinates in (a,a') and (b,b') respectively. Let $\mathbf{U}:L^2(a,a') \dashrightarrow L^2_c(b,b')$ be linear isomorphism which maps $\varphi(\mathfrak{q})$ to $\Psi_c(\mathfrak{p})$ by

$$\Psi_c(\mathfrak{p}) = \int_a^{a'} d\mathfrak{q}\,\varphi(\mathfrak{q})C(\mathfrak{q},\mathfrak{p}) \qquad \text{(A2.5.Eq 1a)}$$

where $C(\mathfrak{q},\mathfrak{p})$ is a smooth function on (a,a') x (b,b'). Let the inverse map U^{-1} be given by

$$\varphi(\mathfrak{q}) = \int_b^{b'} d\mathfrak{p}\,\Psi_c(\mathfrak{p})D(\mathfrak{q},\mathfrak{p}) \qquad \text{(A2.5.Eq 1b)}$$

where $D(\mathfrak{q},\mathfrak{p})$ is a smooth function (a,a') x (b,b'). The linear isomorphism $\mathbf{U}$ is unitary if the two functions $E(\mathfrak{p},\mathfrak{q})$ and $G(\mathfrak{q},\mathfrak{p})$ defined by

$$E(\mathfrak{p},\mathfrak{q}) = \int_{\mathfrak{p}_r}^{\mathfrak{p}} \overline{C(\mathfrak{q},x)}\,dx, \quad b<\mathfrak{p}_r<b'; \qquad \text{(A2.5.Eq 2a)}$$

$$G(\mathfrak{q},\mathfrak{p}) = \int_{\mathfrak{q}_r}^{\mathfrak{q}} \overline{D(x,\mathfrak{p})}\,dx, \quad a<\mathfrak{q}_r<a'; \qquad \text{(A2.5.Eq 2b)}$$

satisfy the following three conditions:

(BT1) Let

$$I(\mathfrak{p},\mathfrak{p}') = \int_a^{a'} E(\mathfrak{p},\mathfrak{q})\overline{E(\mathfrak{p}',\mathfrak{q})}\,d\mathfrak{q}, \qquad \text{(A2.5.Eq 3a)}$$

and let $y_1 = \min\{\mathfrak{p},\mathfrak{p}'\}$ and $y_2 = \max\{\mathfrak{p},\mathfrak{p}'\}$; then

$$\begin{aligned} I(\mathfrak{p},\mathfrak{p}') &= (y_1-\mathfrak{p}_r) \text{ if } (\mathfrak{p}-\mathfrak{p}_r)(\mathfrak{p}'-\mathfrak{p}_r)>0 \text{ and } (\mathfrak{p}-\mathfrak{p}_r)>0,\\ &= (\mathfrak{p}_r-y_2) \text{ if } (\mathfrak{p}-\mathfrak{p}_r)(\mathfrak{p}'-\mathfrak{p}_r)>0 \text{ and } (\mathfrak{p}-\mathfrak{p}_r)<0,\\ &= 0 \text{ if } (\mathfrak{p}-\mathfrak{p}_r)(\mathfrak{p}'-\mathfrak{p}_r)<0; \end{aligned} \qquad \text{(A2.5.Eq 3b)}$$

(BT2) Let

$$J(q,q') = \int_b^{b'} G(q,p)\overline{G(q',p)}d \ , \qquad \text{(A2.5.Eq 4a)}$$

and let $z_1 = \min\{q,q'\}$ and $z_2 = \max\{q,q'\}$; then

$$\begin{aligned} J(q,q') &= (z_1-q_r) \text{ if } (q-q_r)(q'-q_r)>0 \text{ and } (q-q_r)>0, \\ &= (q_r-z_2) \text{ if } (q-q_r)(q'-q_r)>0 \text{ and } (q-q_r)<0, \\ &= 0 \text{ if } (q-q_r)(q'-q_r)<0; \end{aligned} \qquad \text{(A2.5.Eq 4b)}$$

(BT3)

$$\int_{q_r}^{q} E(p,q)dq = \int_{p_r}^{p} G(q,p)dp. \qquad \text{(A2.5.Eq 5)}$$

APPENDIX 2.6

(A2.6.1) The pairing map between $\mathbf{H}_{\mathcal{P}}$ and $\mathbf{H}_{\mathcal{P}_c}$

The elements of $\mathbf{H}_{\mathcal{P}}$ are square-integrable and can be written in the form

$$\Phi = \varphi(q)\rho = \varphi(q)\{\exp \pm S_{\gamma}(p,q)\}s_0|dp|^{-1/2}$$

[cf. equations (2.2.2.Eq 20a) and (2.2.2.Eq 20b)]. The elements of $\mathbf{H}_{\mathcal{P}_c}$ are square-integrable and can be written in the form

$$\Psi_c = \psi_c(p)\rho_c = \psi_c(p)\{\exp \pm S_{\sigma}(p,q)\}s_0|dq|^{-1/2}$$

[cf. equations (2.2.2.Eq 5a) and (2.2.2.Eq 5b)]. The effective configuration spaces Q and Q_c are identifiable with $R(p)$ and $R(q)$ respectively. In the case (1) situation we have $R(p) = \mathbb{R}$ and $R(q) \neq \mathbb{R}$.

Then by equation (A2.1.Eq 9) of Appendix 2.1 the pairing map between $\mathbf{H}$ and $\mathbf{H}$ is given by

$$\langle\Phi,\Psi_c\rangle_{\mathcal{P}\mathcal{P}_c} = [2\pi\hbar]^{-1/2}\int_{R(p)}\int_{R(q)}\varphi(q)\overline{\psi_c}(p)\{\exp \pm S_{\gamma}(p,q)\}\{\exp -\pm S_{\sigma}(p,q)\}dp\,dq$$

(A2.6.Eq 1)

(A2.6.2) The unitary map (induced by the pairing map) from $\mathbf{H}(\mathcal{P})$ to $\mathbf{H}(\mathcal{P}_c)$

Let us extend the range of q, from $R(q)$ to $\mathbb{R}$ and let $f_{\infty}(p,q)$ be a smooth function on $\mathbb{R}^2$ that satisfies the condition

$$f_{\infty}(p,q) = f(p,q) \text{ on } R(p) \times R(q)$$

[cf. equation (2.2.2.Eq 16)]. In the case (1) situation the quantum Hilbert space $H(\mathcal{P})$ is the quantization Hilbert space $H_{\mathcal{P}}$. The elements of $\mathbf{H}(\mathcal{P})$ are of the form

$$\Phi = \varphi(q)\rho,$$

where

$$\varphi(q) = 0,\ q \in \mathbb{R}-R(q);\ \rho = \{\exp \pm[-p(q-q_r)-f_{\infty}(p_r,q)+f_{\infty}(p,q_r)]\}s_0|dp|^{-1/2}.$$

The operator $\tilde{q}_c$ in $H_{\mathcal{P}_c}$ is given by equation (2.2.2.Eq 7a). By equations (2.2.2.Eq 9a), (2.2.2.Eq 9b), (2.2.2.Eq 17a) and (2.2.2.Eq 17b) the generalized eigensection (corresponding to the eigenvalue q) of the operator $\tilde{q}_c$ in $H_{\mathcal{P}_c}$ is given by

$$\Psi_{cq} = \psi_{cq}(p)\rho_c$$

where

$$\psi_{cq}(p) = [2\pi\hbar]^{-1/4}\{\exp \tfrac{i}{\hbar} S_\gamma(p,q)\}\{\exp -\tfrac{i}{\hbar} S_\sigma(p,q)\} \text{ for } q \in R(q),$$
$$\psi_{cq}(p) = [2\pi\hbar]^{-1/4}\{\exp \tfrac{i}{\hbar}[-p(q-q_r)-f_\infty(p_r,q)+f_\infty(p,q_r)]\} \text{ for } q \in \mathbb{R}-R(q).$$

Then the pairing map given by equation (A2.6.Eq 1) can be rewritten as

$$\langle \Phi, \Psi_c \rangle_{\mathcal{P}\mathcal{P}_c} = [2\pi\hbar]^{-3/4}\int_{\mathbb{R}}\int_{\mathbb{R}} \varphi(q)\overline{\psi_c}(p)\psi_{cq}(p)\,dp\,dq. \qquad \text{(A2.6.Eq 2)}$$

The pairing map determines a linear map $U:H(\mathcal{P}) \dashrightarrow H(\mathcal{P}_c)$ [cf. equation (A2.1.Eq 10a)] which maps $\Phi = \varphi(q)\rho$ (where $\varphi(q) = 0$ for $q \in \mathbb{R}-R(q)$) to $\Psi_c = \psi_c(p)\rho_c$ by

$$\psi_c(p) = [2\pi\hbar]^{-1/4}\int_{\mathbb{R}} \varphi(q)\psi_{cq}(p)\,dq. \qquad \text{(A 2.6.Eq 3)}$$

The inverse map $U^{-1}:H(\mathcal{P}_c) \dashrightarrow H(\mathcal{P})$ [cf. equation (A2.1.Eq 11a)] given by

$$\varphi(q) = [2\pi\hbar]^{-1/4}\int_{\mathbb{R}} \psi_c(p)\overline{\psi_{cq}}(p)\,dp. \qquad \text{(A2.6.Eq 4)}$$

To demonstrate that U^{-1} defined by equation (A2.6.Eq 4) is the inverse map we need to show that $U^{-1}U\Phi = \Phi$ and $UU^{-1}\Psi_c = \Psi_c$. This is checked as follows.

$$\begin{aligned}
\varphi(q) &= [2\pi\hbar]^{-1/2}\int_{\mathbb{R}} dp\,\overline{\psi_{cq}}(p)\int_{\mathbb{R}} dq'\varphi(q')\psi_{cq'}(p) \\
&= \int_{\mathbb{R}} dq'\varphi(q')\langle \Psi_{cq'}, \Psi_{cq}\rangle \quad \text{(by equation (2.2.2.Eq 6))} \\
&= \int_{\mathbb{R}} dq'\varphi(q')\delta(q-q') \quad \text{(by equation (2.2.2.Eq 10))} \\
&= \varphi(q).
\end{aligned}$$

and

$$\begin{aligned}
\psi_c(p) &= [2\pi\hbar]^{-1/2}\int_{\mathbb{R}} dq\,\psi_{cq}(p)\int_{\mathbb{R}} dp'\psi_c(p')\overline{\psi_{cq}}(p') \\
&= \int_{\mathbb{R}} dp'\psi_c(p')\{(2\pi\hbar)^{-1/2}\int_{\mathbb{R}} \psi_{cq}(p)\,\overline{\psi_{cq}}(p')\,dq\} \\
&= \int_{\mathbb{R}} dp'\psi_c(p')\delta(p-p') \quad \text{(by equation (2.2.2.Eq 18))} \\
&= \psi_c(p).
\end{aligned}$$

To show that **U** is a unitary map we use Bochner's theorem [cf. Appendix (2.5)] as follows. In this case the various sets and functions defined in Appendix 2.6 are given by:

$$(a,a') = (b,b') = \mathbb{R},$$
$$C(\mathfrak{q},\mathfrak{p}) = [2\pi\hbar]^{-1/4}\,\Psi_{c\mathfrak{q}}(\mathfrak{p}),$$
$$D(\mathfrak{q},\mathfrak{p}) = [2\pi\hbar]^{-1/4}\,\overline{\Psi_{c\mathfrak{q}}}(\mathfrak{p}),$$
$$E(\mathfrak{p},\mathfrak{q}) = \int_{\mathfrak{p}_r}^{\mathfrak{p}} \overline{C(\mathfrak{q},x)}\,dx = [2\pi\hbar]^{-1/4}\int_{\mathfrak{p}_r}^{\mathfrak{p}} \overline{\Psi}_{c\mathfrak{q}}(x)\,dx$$

and

$$G(\mathfrak{q},\mathfrak{p}) = \int_{\mathfrak{q}_r}^{\mathfrak{q}} \overline{D(x,\mathfrak{p})}\,dx = [2\pi\hbar]^{-1/4}\int_{\mathfrak{q}_r}^{\mathfrak{q}} \Psi_{cx}(\mathfrak{p})\,dx.$$

Then checking condition (BT1) of Bochner's theorem we get

$$\int_{\mathbb{R}} E(\mathfrak{p},\mathfrak{q})\overline{E(\mathfrak{p}',\mathfrak{q})}\,d\mathfrak{q} = \int_{\mathfrak{p}_r}^{\mathfrak{p}}\int_{\mathfrak{p}_r}^{\mathfrak{p}'} dx\,dx'\{[2\pi\hbar]^{-1/2}\int_{\mathbb{R}}\overline{\Psi}_{c\mathfrak{q}}(x)\,\Psi_{c\mathfrak{q}}(x')\,d\ \}$$
$$= \int_{\mathfrak{p}_r}^{\mathfrak{p}}\int_{\mathfrak{p}}^{\mathfrak{p}'} dx\,dx'\,\delta(x-x') \text{ (by equation (2.2.2.Eq 18))}$$

which clearly satisfies condition (BT1). Similarly, conditions (BT2) and (BT3) can be verified. Hence **U** is a unitary map.

(A2.6.3) <u>The operators $\hat{\mathfrak{q}}$ and $\hat{\mathfrak{q}}_c$ are unitarily related</u>

Finally, we need to show that the quantum operators $\hat{\mathfrak{q}}$ in $H(\mathcal{P})$ and $\hat{\mathfrak{q}}_c$ in $H(\mathcal{P}_c)$ are unitarily related: we need to check that $U\hat{\mathfrak{q}}U^{-1} = \hat{\mathfrak{q}}_c$. Let $\Phi = U^{-1}\Psi_c$ where $\Phi = \varphi(\mathfrak{q})\rho \in H(\mathcal{P})$ and $\Psi_c = \Psi_c(\mathfrak{p})\rho_c \in H(\mathcal{P}_c)$; (note that $\varphi(\mathfrak{q}) = 0$ for $\mathfrak{q} \in \mathbb{R} - R(\mathfrak{q})$.)

$$(U\hat{\mathfrak{q}}U^{-1})\Psi_c = U\hat{\mathfrak{q}}\Phi$$
$$= U\mathfrak{q}\varphi(\mathfrak{q})\rho \text{ (by equation (2.2.2.Eq 22a))}$$
$$= [(2\pi\hbar)^{-1/4}\int_{\mathbb{R}} \mathfrak{q}\,\varphi(\mathfrak{q})\,\Psi_{c\mathfrak{q}}(\mathfrak{p})\,d\mathfrak{q}] \quad \text{(by equation (A2.6.Eq 3))}$$
$$= [(2\pi\hbar)^{-1/4}\int_{\mathbb{R}} \varphi(\mathfrak{q})\{\,i\hbar(\partial/\partial\mathfrak{p}) + (\partial f(\mathfrak{p},\mathfrak{q}_r)/\partial\mathfrak{p}) + \mathfrak{q}_r\}\Psi_{c\mathfrak{q}}(\mathfrak{p})\,d\mathfrak{q}]$$

(by equation (2.2.2.Eq 11))

$$= \hat{\mathfrak{q}}_c\Psi_c$$

(by taking the curly-bracketed operator outside the integral).

APPENDIX 2.7

The quantization of p^2 in the polarizations P and $\mathcal{P}$ are unitarily related

(A2.7.1) Notation

Let $\mathbb{R}$ be the configuration space of a free particle and let q be the usual cartesian coordinate on $\mathbb{R}$. Let $M = T^*\mathbb{R} = \mathbb{R}^2$, ω be the canonical two-form on M and let (p,q) be the usual cartesian coordinates on M. Let P and P_c be the vertical and horizontal polarizations , respectively, of the cotangent bundle $T^*\mathbb{R}$.

Remark: (R1) We have not used Q to denote the configuration as we usually do because Q has already been reserved to denote the effective configuration space with respect to the polarization $\mathcal{P}$.

Let (p_r,q_r) be the chosen reference point in M. Let τ denote the p-coordinate curve in M through the point (p_r,q) and let λ denote the q-coordinate curve through the point (p,q_r). Let β be a one-form on M that satisfies $d\beta = \omega$. Since M is contractible, it follows from Poincare's lemma that there exists a $h(p,q) \in M$ such that

$$\beta = pdq+dh(p,q). \qquad \text{(A2.7.Eq 1a)}$$

We shall write

$$\beta = \beta_p dp+\beta_q dq \qquad \text{(A2.7.Eq 1b)}$$

where

$$\beta_p = (\partial h/\partial p) \text{ and } \beta_q = p+(\partial h/\partial q). \qquad \text{(A2.7.Eq 1c)}$$

The line integrals of β along the coordinate curves τ and λ are given by:

$$S_\tau(p,q) = \int_\tau \beta = \int_{p_r}^{p} \beta_p\, dp = h(p,q)-h(p_r,q) \qquad \text{(A2.7.Eq 2a)}$$

$$S_\lambda(p,q) = \int_\lambda \beta = \int_{q_r}^{q} \beta_q\, dq = p(q-q_r)+h(p,q)-h(p,q_r). \qquad \text{(A2.7.Eq 2b)}$$

Let $B = M \times \mathbb{C}$ be a trivial bundle over M, $(\cdot,\cdot)$ be the natural Hermitian structure on B, s_0 be a unit section of B and let ∇ be the connection on B defined by

$$\nabla_X s_0 = -\tfrac{i}{\hbar}(X \lrcorner \beta)s_0 \text{ for all } X \in V_{\mathbb{C}}(M). \qquad \text{(A2.7.Eq 3)}$$

Let $(B,(\cdot,\cdot),\nabla)$ be the chosen prequantization bundle over (M,ω).

(A2.7.2) <u>The quantization of p^2 in the vertical polarization P</u>

We shall now give a brief sketch of the method of quantizing the Hamiltonian p^2 in the vertical polarization P [cf. Wan and McKenna (1984)].

The quantization Hilbert space H_P consists of square-integrable sections of $B \times \Delta_{-1/2}(P)$ of the form

$$\mathbf{X} = \chi(q)\mu = \chi(q)\{\exp \tfrac{i}{\hbar}S_\tau(p,q)\}s_0\,|dp|^{-1/2}. \qquad \text{(A2.7.Eq 4)}$$

The variable p is quantizable in H_P , since the associated vector field $(\partial/\partial q)$ generated by p is complete in the configuration space $\mathbb{R}$. The quantization operator $\widetilde{p}$ in H_P is given by the expression [cf. equation (A2.3.Eq 6) in Appendix 2.3]

$$p\mathbf{X} = [\{-i\hbar(\partial/\partial q)-(\partial h(p_r,q)/\partial q)\}\chi(q)]\mu. \qquad \text{(A2.7.Eq 5)}$$

The Hilbert space H_{P_c} consists of square-integrable sections of $B \times \Delta_{-1/2}(P_c)$ of the form

$$K_c = K_c(p)\mu_c = K_c(p)\{\exp \tfrac{i}{\hbar}S_\lambda(p,q)\}s_0\,|dq|^{-1/2}. \qquad \text{(A2.7.Eq 6)}$$

The observables p and p^2 are both quantizable in H_{P_c} , and the quantization operator $\widetilde{p_c}$ and $\widehat{p_c^2}$ are given by the expressions

$$\widetilde{p_c}K_c = p\,K_c \text{ and } \widehat{p^2}\,K_c = p^2 K_c \qquad \text{(A2.7.Eq 7)}$$

respectively.

By equation (A2.1.Eq 11a) and theorem (A2.1.T1), we have the unitary map $T:H_{P_c} \dashrightarrow H_P$ defined by

$$TK_c = [(2\pi\hbar)^{-1/2}\int_{\mathbb{R}} K_c(p)\{\exp -\tfrac{i}{\hbar}S_\tau(p,q)\}\{\exp \tfrac{i}{\hbar}S_\lambda(p,q))\}dp]\mu. \qquad \text{(A2.7.Eq 8)}$$

We shall now show that $T\widetilde{p_c}T^{-1} = \widetilde{p}$ as follows. Let $T^{-1}X = K_c$. Then we have

$$(T\widetilde{p_c}T^{-1})X = T\widetilde{p_c}K_c$$

$$= TpK_c \text{ (by equation (A2.7.Eq 7))}$$

$$= [(2\pi h)^{-1/2}\int_{\mathbb{R}} pK_c(p)\{\exp -\tfrac{i}{\hbar}S_\tau(p,q)\}\{\exp \tfrac{i}{\hbar}S_\lambda(p,q)\}dp]\mu$$

$$= \widetilde{p}X \qquad \text{(A2.7.Eq 9)}$$

The last step was obtained using the fact [cf. equation(2.2.2.Eq 39)]

$$p\{\exp -\tfrac{i}{\hbar}S_\tau(p,q)\}\{\exp \tfrac{i}{\hbar}S_\lambda(p,q)\}$$

$$= [\{-i\hbar(\partial/\partial q)-(\partial h(p_r,q)/\partial q)\}\{\exp -\tfrac{i}{\hbar}S_\tau(p,q)\}\{\exp \tfrac{i}{\hbar}S_\lambda(p,q)\}]. \qquad \text{(A2.7.Eq 10)}$$

We shall now quantize p^2 in the polarization P as follows. Since p^2 is not an element of $C^\infty(M;P,1)$ we cannot quantize it directly in H_P, so we shall adopt the following method. Let $T^{-1}X = K_c$. We define the quantization operator $\widetilde{p}^2$ in H_P by

$$\widetilde{p}^2X = (T\widetilde{p_c^2}\,T^{-1})X. \qquad \text{(A2.7.Eq 11a)}$$

Explicitly,

$$\widetilde{p^2}X = (T\widetilde{p_c^2}\,K_c)$$

$$= Tp^2K_c \text{ (by equation (A2.7.Eq 7))}$$

$$= [\{-i\hbar(\partial/\partial q)-(\partial h(p_r,q)/\partial q)\}^2\chi(q)]\mu. \qquad \text{(A2.7.Eq 11b)}$$

The last step is obtained using equation (A2.7.Eq 10).

The symplectic manifold (M,ω) with cartesian coordinates (p,q) is an example of a case (3) situation, i.e. $R(p) = \mathbb{R}$, $R(q) = \mathbb{R}$. In the case (3) situation the quantization Hilbert space H_P and the quantum Hilbert space $H(P)$ coincide. Therefore, we shall define the quantum operator $\hat{p}^2$ in $H(P)$

by $\hat{p}^2 = \tilde{p}^2$.

(A2.7.3) <u>The quantum operators $\hat{p}^2$ in $H(P)$ and $\hat{\wp}$ in $H(\mathcal{P})$ are unitarily related.</u>

Let (p_j', q_j') denote the restriction of the cartesian canonical coordinates to the regions M_j'. Then in M_j', we can treat $\wp_j$ and $\mathcal{q}_j'$ as functions of (p_j', q_j'); so we shall write $\wp_j'(p_j', q_j') = p_j'^2$ and $\mathcal{q}(p_j', q_j') = (q_j'/2p_j')$.

Remark: (R2) The chosen reference point (p_r, q_r) should not be confused with the reference points $(\wp_{jr}, \mathcal{q}_{jr})$ in M_j'. Note that (p_r, q_r) is usually taken to be $(0,0)$; in which case, it does not belong to either to M_1 or M_2. Alternatively, (p_r, q_r) may lie in either M_1 or M_2. However, regardless of the choice of (p_r, q_r) the functions $S_\tau(p_j', q_j')$ and $S_\lambda(p_j', q_j')$ are well-defined in M_j' because the functions $h(p_r, q_j')$ and $h(p_j', q_r)$ are well defined in M_j'.

Let P_j' and P_{cj}' denote the restrictions of the polarizations P and P_c respectively to M_j'. Let $(B_j', (\cdot,\cdot), \nabla)$ denote the restriction of the prequantization bundle to M_j'. In M_j', we have:

$$(\partial/\partial \wp_j') = X_{-\mathcal{q}_j'} = (2p_j')^{-1}[q_j'(\partial/\partial q_j') + (\partial/\partial p_j')], \qquad \text{(A2.7.Eq 12a)}$$

$$(\partial/\partial \mathcal{q}_j') = X_{\wp_j'} = (2p_j')(\partial/\partial q_j'), \qquad \text{(A2.7.Eq 12b)}$$

$$\beta = p_j' dq_j' + dh(p_j', q_j')$$

$$= \wp_j' d\mathcal{q}_j' + df_j'(\wp_j', \mathcal{q}_j'); \qquad \text{(A2.7.Eq 12c)}$$

$$h(p_j', q_j') = -(1/2)p_j'q_j' + f_j'(\wp_j'(p_j', q_j'), \mathcal{q}_j'(p_j', q_j')), \qquad \text{(A2.7.Eq 12d)}$$

$$S_\tau(p_j', q_j') = h(p_j', q_j') - h(p_r, q_j'), \qquad \text{(A2.7.Eq 12e)}$$

$$S_{\sigma_j'}(\wp_j', \mathcal{q}_j') = \wp_j'(\mathcal{q}_j' - \mathcal{q}_{jr}) + f_j'(\wp_j', \mathcal{q}_j') - f_j'(\wp_j', \mathcal{q}_{jr}) \qquad \text{(A2.7.Eq 12f)}$$

and

$$S_{\gamma_j'}(\wp_j', \mathcal{q}_j') = f_j'(\wp_j', \mathcal{q}_j') - f_j'(\wp_{jr}, \mathcal{q}_j'). \qquad \text{(A2.7.Eq 12g)}$$

Before proceeding any further let us summarize our objective. We require to show that the quantum operators $\hat{p}^2$ in H(P) and $\hat{\wp}$ in $H(\mathcal{P})$ are unitarily related. To do this we need to construct a unitary map between H(P) and $H(\mathcal{P})$. However, the construction of the unitary map is complicated by the fact that for each j, P_j and $\mathcal{P}_j$ are not transverse because by equation (A2.7.Eq 12a) we have $P_{jm} + \mathcal{P}_{jm} \neq T_mM$ at $m \in \{m \in M_j: q = 0\}$, so we cannot construct a pairing map between the quantum Hilbert spaces H(P) and $H(\mathcal{P})$. On the other hand, the polarizations P_j and $\mathcal{P}_{c_j}$ are transverse in M_j by equation (A2.7.Eq 12b), so we can construct a pairing map between the quantum Hilbert spaces H(P) and $H(\mathcal{P}_c)$. We shall show that $\hat{p}^2$ in H(P) and $\hat{\wp}$ in $H(\mathcal{P})$ are unitarily related using the following five steps:

(i) Construct the pairing map between H(P) and $H(\mathcal{P}_c)$;

(ii) Construct the linear map (induced by the pairing map) from H(P) to $H(\mathcal{P}_c)$;

(iii) Use Bochner's theorem to show that the linear map constructed in step (ii) is unitary;

(iv) Show that the operators $\hat{p}^2$ in H(P) and $\hat{\wp}_c$ in $H(\mathcal{P}_c)$ are unitarily related.

<u>Remark</u>: (R3) By equation (2.2.2.Eq 54), $\hat{\wp}$ in $H(\mathcal{P})$ and $\hat{\wp}_c$ in $H(\mathcal{P}_c)$ are unitarily related; the link between $H(\mathcal{P})$ and $H(\mathcal{P}_c)$ is given by the unitary map $V: H(\mathcal{P}_c) \dashrightarrow H(\mathcal{P})$ which has been defined in equation (2.2.2.Eq 53a). Therefore, if $\hat{p}^2$ in H(P) and $\hat{\wp}_c$ in $H(\mathcal{P})$ are unitarily related, then $\hat{p}^2$ in H(P) and $\hat{\wp}$ in $H(\mathcal{P})$ are unitarily related.

(i) The pairing map between $\mathbf{H}(P)$ and $\mathbf{H}(\mathcal{P}_c)$

The elements of H(P) defined by equation (A2.7.Eq 4) can be written in the form

$$\mathbf{X} = \mathbf{X}_1 \oplus \mathbf{X}_2 \qquad \text{(A2.7.Eq 13a)}$$

where $\mathbf{X}_j$ denotes the restrictions of $\mathbf{X}$ to the regions M_j. Explicitly, $\mathbf{X}_j$ is given by

$$\mathbf{X}_j = \chi(q_j)\mu_j = \chi(q_j)\{\exp \tfrac{i}{\hbar} S_\tau(p_j, q_j)\} s_0 |dp_j|^{-1/2}. \qquad \text{(A2.7.Eq 13b)}$$

The elements of $H(\mathcal{P}_c)$ are of the form [cf. equations (2.2.2.Eq 47a) and (2.2.2.Eq 47b)]

$$\Psi_c = \Psi_{1c} \oplus \Psi_{2c} \qquad \text{(A2.7.Eq 14a)}$$

where

$$\Psi_{jc} = \psi_{jc}(\wp_j)\rho_{jc} = \psi_{jc}(\wp_j)\{\exp \tfrac{i}{\hbar} S_{\sigma_j}(\wp_j, q_j)\} s_0 |dq_j|^{-1/2}. \qquad \text{(A2.7.Eq 14b)}$$

Then the pairing map between H(P) and $\mathbf{H}(\mathcal{P}_c)$ is given by [cf. equations (A2.1.Eq 1b) and (A2.1.Eq 1c)]

$$\langle \mathbf{X}, \Psi_c \rangle_{P\mathcal{P}_c} = (2\pi\hbar)^{-1} \left[\int_{M_1} (\mathbf{X}_1, \Psi_{1c})_{P_1\mathcal{P}_{1c}} + \int_{M_2} (\mathbf{X}_2, \Psi_2)_{P_2\mathcal{P}_{2c}} \right]$$

$$\text{(A2.7.Eq 15a)}$$

where $(\mathbf{X}_j, \Psi_{jc})_{P_j\mathcal{P}_{jc}}$ are one-TM_j-densities given by

$$\begin{aligned}
&(\mathbf{X}_j, \Psi_{jc})_{P_j\mathcal{P}_{jc}} \\
&= \chi(q_j)\,\overline{\psi_{jc}}(\wp_j)\{\exp \tfrac{i}{\hbar} S_\tau(p_j, q_j)\}\{\exp -\tfrac{i}{\hbar} S_{\sigma_j}(\wp_j, q_j)\} \times \\
&\qquad |dp_j|^{-1/2}\,\{\partial/\partial p_j\}\,|dq_j|^{-1/2}\,\{\partial/\partial q_j\}\,|2|dp_j \wedge dq_j\,(\partial/\partial p_j, 2p_j(\partial/\partial q\ ))|^{3/2} \\
&= \chi(q_j)\,\overline{\psi_{jc}}(\wp_j)\{\exp \tfrac{i}{\hbar} S_\tau(p_j, q_j)\}\{\exp -\tfrac{i}{\hbar} S_{\sigma_j}(\wp_j, q_j)\}(4\wp_j)^{3/4}.
\end{aligned}$$

$$\text{(A2.7.Eq 15b)}$$

Let $L_j(q_j, \wp_j) = S_\tau(p_j, q_j) - S_{\sigma_j}(\wp_j, q_j)$; then $L_j(q_j, \wp_j)$ is given explicitly by

$$L_j(q_j, \wp_j) = (-1)^j q_j(\wp_j)^{1/2} + \wp_j q_{jr} - h(p_r, q_j) + f_j(\wp_j, q_{jr}). \qquad \text{(A2.7.Eq 16)}$$

The pairing map between $\mathbf{H}(P)$ and $\mathbf{H}(\mathcal{P}_c)$ is given explicitly by

$$\langle \mathbf{X}\ ,\Psi_c \rangle_{P\mathcal{P}_c}$$
$$= (2\pi\hbar)^{-1} \sum_{j=1}^{2} \quad [\int_{\mathbb{R}} dq_j \int_0^{\infty} d\wp_j\ \chi(q_j)\overline{\Psi_{jc}}(\wp_j)\{\exp \not{\imath} L_j(q_j,\wp_j)\}(4\wp_j)^{1/4}\ ]$$

(A2.7.Eq 17)

In the latter equation we have used the following facts:

$$R(q_j) = \mathbb{R},\ R(\wp_j) = (0,\infty) \text{ and } |\varepsilon_\omega| = (4\wp_j)^{-1/2}|d\wp_j \wedge dq_j|. \qquad \text{(A2.7.Eq 18)}$$

(ii) <u>The construction of the linear map between $\mathbf{H}(\mathcal{P}_c)$ and $\mathbf{H}(P)$</u>

Let $\mathbf{W}:\mathbf{H}(\mathcal{P}_c) \dashrightarrow H(P)$ denote the linear map (induced by the pairing map defined by equation (A2.1.Eq 11a). The linear map $\mathbf{W}$ maps $\Psi_c = \Psi_{1c} \oplus \Psi_{2c}$, $\Psi_{jc} = \psi_{jc}(\wp_j)\rho_{jc}$, to $\mathbf{X} = \chi(q)\mu$ by

$$\chi(q) = (2\pi\hbar)^{-1/2}\sum_{j=1}^{2} \ [\int_0^{\infty} \psi_{jc}(\wp_j)\{\exp -\not{\imath} L_j(q_j,\wp_j)\}(4\wp_j)^{-1/4}\ d\wp_j]$$

(A2.7.Eq 19a)

with inverse map $\mathbf{W}^{-1}$ given by [cf. equation (A2.1.Eq 10)]

$$\psi_{jc}(\wp_j) = (2\pi\hbar)^{-1/2} \ [\int_{\mathbb{R}} \chi(q\ )\{\exp \not{\imath} L_j(q_j,\wp_j)\}(4\wp_j)^{-1/4}\ dq_j]$$

(A2.7.Eq 19b)

To check that $\mathbf{W}^{-1}$ is the inverse map we need to show that $\mathbf{W}\mathbf{W}^{-1}\mathbf{X} = \mathbf{X}$.

$$\chi(q) = (2\pi\hbar)^{-1}\sum_{\substack{j=1\\ q_j=q}}^{2} \int_0^{\infty} d\wp_j (4\wp_j)^{-1/4}\ \{\exp -\not{\imath} L_j(q_j,\wp_j)\}[\int_{\mathbb{R}} dq'_j\ (q'_j)$$
$$\{\exp \not{\imath} L_j(q'_j,\wp_j)\}(4\wp_j)^{-1/4}\ ]$$
$$= \sum_{\substack{j=1\\ q_j=q}}^{2} \int_{\mathbb{R}} dq'_j\chi(q'_j)\{\exp \not{\imath}(h(p_r,q_j)-h(p_r,q'_j))\}[(2\pi\hbar)^{-1}\int_0^{\infty} d\wp_j (4\wp_j)^{-1/2}$$
$$\{\exp -\not{\imath}(\wp_j)^{-1/2}\ (q_j-q'_j)\}]$$
$$= \sum_{\substack{j=1\\ q_j=q}}^{2} \int_{\mathbb{R}} dq'_j\chi(q'_j)\{\exp -\not{\imath}(h(p_r,q_j)-h(p_r,q'_j))\}[(2\pi\hbar)^{-1}\int_{\mathbb{R}} dy_j$$
$$\{\exp -\not{\imath} y_j\ (q_j-q'_j)\}]$$
$$\text{(where } y_j = (-1)^j(\wp_j)^{1/2} \text{ and } dy_j = (-1)^j(4\wp_j)^{-1/2}\ d\wp_j)$$
$$= \sum_{\substack{j=1\\ q_j=q}}^{2} \int_{\mathbb{R}} dq'_j\chi(q'_j)\{\exp -\not{\imath}(h(p_r,q_j)-h(p_r,q'_j))\}]\delta(q_j-q'_j)$$
$$= \chi(q)$$

(A2.7.Eq 20)

Hence we have established that $\mathbf{W}^{-1}$ is the inverse of $\mathbf{W}$.

(iii) **W is a unitary map**

To show that W is unitary we shall use Bochner's theorem [cf. Appendix 2.5] as follows. The various functions defined in Appendix 2.5 are listed as follows:

$C(q,\not{p}) = (2\pi\hbar)^{-1/2}\{\exp \tfrac{i}{\hbar}L_j(q_j,\not{p}_j)\}(4\not{p}_j)^{-1/4}$ in M_j,

$D(q,\not{p}) = (2\pi\hbar)^{-1/2}\{\exp -\tfrac{i}{\hbar}L_j(q_j,\not{p}_j)\}(4\not{p}_j)^{-1/4}$ in M_j,

$E(\not{p},q) = (2\pi\hbar)^{-1/2}\sum_{\substack{j=1\\ q_j=q}}^{2}\int_{\not{p}_{jr}}^{\not{p}}\{\exp-\tfrac{i}{\hbar}L_j(q_j,x_j)\}(4x_j)^{-1/4}\,dx_j$ in M_j,

$G(q,\not{p}) = (2\pi\hbar)^{-1/2}\sum_{\substack{j=1\\ q_j=q}}^{2}\int_{q_r}^{q_j}\{\exp \tfrac{i}{\hbar}L_j(x_j,\not{p}_j)\}(4\not{p}_j)^{-1/4}\,dx_j$ in M_j.

Let us check condition (BT1) of Bochner's theorem.

$$
\begin{aligned}
I(\not{p},\not{p}') &= \int_{-\infty}^{\infty}E(\not{p},q)\overline{E(\not{p}',q)}\,dq\\
&= (2\pi\hbar)^{-1}\int_{-\infty}^{\infty}dq\Big[\sum_{\substack{j=1\\ q_j=q}}^{2}\int_{\not{p}_{jr}}^{\not{p}_j}\int_{\not{p}_{jr}}^{\not{p}_j'}dx_j'dx_j\{\exp -\tfrac{i}{\hbar}(L_j(q_j,x_j)-L_j(q_j,x_j'))\}\\
&\qquad (4x_j')^{-1/4}(4x_j)^{-1/4}\Big]\\
&= \sum_{j=1}^{2}\int_{\not{p}_{jr}}^{\not{p}_j}\int_{\not{p}_{jr}}^{\not{p}_j'}dx_j'dx_j(4x_j')^{-1/4}(4x_j)^{-1/4}\{\exp -\tfrac{i}{\hbar}(k_j(x_j,q_{jr})-k_j(x_j',q_{jr}))\}\\
&\qquad\Big[(2\pi\hbar)^{-1}\int_{-\infty}^{\infty}\{\exp-\tfrac{i}{\hbar}(-1)^j q_j(\sqrt{x_j}-\sqrt{x_j'})\}dq_j\Big]\\
&\qquad (\text{where } k_j(x_j,q_{jr}) = f_j(x_j,q_{jr})+x_j q_{jr})\\
&= \sum_{j=1}^{2}\int_{\not{p}_{jr}}^{\not{p}_j}\int_{\not{p}_j}^{\not{p}_j'}dx_j'dx_j(4x_j')^{-1/4}(4x_j)^{-1/4}\{\exp-\tfrac{i}{\hbar}(k_j(x_j,q_{jr})-k_j(x_j',q_{jr}))\}\\
&\qquad \delta(\sqrt{x}-\sqrt{x'})\\
&= \sum_{j=1}^{2}\int_{U_{jr}}^{U_j}\int_{U_{jr}}^{U_j'}du_j du_j'\delta(u_j-u_j')(4u_ju_j')^{-1/2}\{\exp -\tfrac{i}{\hbar}(k_j(x_j(u_j),q_{jr})-k_j(x_j'(u_j'),q_{jr}))\}\\
&\qquad (\text{where } u_j = (-1)^j\sqrt{x_j},\ u_{jr} = (-1)^j\sqrt{\not{p}_{jr}})
\end{aligned}
$$

Let $y_{j1} = \min\{\not{p}_j,\not{p}_j'\}$, $y_{j2} = \max\{\not{p}_j,\not{p}_j'\}$, $v_{j1} = \min\{u_j,u_j'\}$ and let $v_{j2} = \max\{u_j,u_j'\}$.

Then for $(\not{p}_j-\not{p}_{jr})(\not{p}_j'-\not{p}_{jr})>0$ and $(\not{p}_j-\not{p}_{jr})>0$, we have

$$I(\not{p},\not{p}') = \sum_{j=1}^{2}\int_{U_{jr}}^{V_{j1}}dv_{j1}\,2v_{j1} = \sum_{j=1}^{2}(y_{j1}-\not{p}_{jr}).$$

For $(\not{p}_j-\not{p}_{jr})(\not{p}_j'-\not{p}_{jr})>0$ and $(\not{p}_j-\not{p}_{jr})<0$, we have

$$I(\not{p},\not{p}') = \sum_{j=1}^{2}\int_{V_{j2}}^{U_{jr}}dv_{j2}\,2v_{j2} = \sum_{j=1}^{2}(\not{p}_{jr}-y_{j2}).$$

For $(\wp_j - \wp_{jr})(\wp_j' - \wp_{jr}) < 0$, we have

$$I(\wp, \wp') = 0.$$

The values assumed by $I(\wp, \wp')$ clearly satisfy condition (BT1); similarly, conditions (BT2) and (BT3) can be checked. Hence **W** is a unitary map.

(iv) <u>The quantization operators $\hat{p}^2$ and $\hat{\wp}$ are unitarily related</u>

We shall show that $\hat{p}^2$ in H(P) and $\hat{\wp}_c$ in $H(\mathcal{P}_c)$ are unitarily related as follows. Let $W^{-1}X = \Psi_c$ where $\Psi_c = \Psi_{1c} \oplus \Psi_{2c}$, $\Psi_{jc} = \psi_{jc}(\wp_j)\rho_{jc}$, and $X = \chi(q)\mu$.

$(W\hat{\wp}_c W^{-1})X$

$= W\hat{\wp}_c \Psi_c$

$= W(\wp_1 \Psi_{1c} \oplus \wp_2 \Psi_{2c})$

$= (2\pi\hbar)^{-1/2} \left[\sum_{J=1}^{2} \int_0^\infty d\wp_j \; \wp_j \psi_{jc}(\wp_j)\{\exp -iL_j(q_j, \wp_j)\}(4\wp_j)^{-1/4}\right]\mu$

(where $q_j = q$)

$= (2\pi\hbar)^{-1/2} \left[\sum_{J=1}^{2} \int_0^\infty d\wp_j \; x \right.$

$\left. \{(-i\hbar(\partial/\partial q_j) - (\partial h(p_r, q_j)/\partial q_j)\}^2 \{\exp -iL_j(q_j, \wp_j)\}(4\wp_j)^{-1/4}\right]$

(by $\{-i\hbar(\partial/\partial q_j) - (\partial h(p_r, q_j)/\partial q_j)\}\{\exp -iL_j(q_j, \wp_j)\}$

$= (-1)^{J+1}(\wp_j)^{1/2}\{\exp -iL_j(q_j, \wp_j)\}$)

$= \hat{p}^2 X.$

The last step is obtained by taking the operator $\{-i\hbar(\partial/\partial q_j) - (\partial h(p_r, q_j)/\partial q_j)\}^2$ outside the integral sign and comparing the result with expression for $\hat{p}^2$ given by equation (A2.7.Eq 11b). Hence we have established that $\hat{p}^2$ in H(P) and $\hat{\wp}_c$ in $H(\mathcal{P}_c)$ are unitarily related. Therefore $\hat{p}^2$ in H(P) and $\hat{\wp}$ in $H(\mathcal{P})$ are unitarily related.

APPENDIX 2.8

Theorem on the canonical decomposition of global observables [cf. Wan and McFarlane (1981)]

We shall restate the theorem in terms of the half-density quantization scheme.

Let $\mathcal{P}$ be a reducible polarization of a 2k-dimensional symplectic manifold (M,ω). Let Q be the effective configuration space with respect to and let $pr:M \dashrightarrow Q$ be the corresponding projection map. Let $(B,(\cdot,\cdot),\nabla)$ be the chosen prequantization bundle over (M,ω). Let ζ be an observable that satisfies the following conditions: (i) $\zeta \in C^{\infty}(M;\mathcal{P},1)$; and (ii) the associated vector field $pr_{*}(X_{\zeta})$ is complete almost everywhere on Q. Let $\{Q_{\alpha}\}$ be a family of submanifolds of M that satisfy the following conditions: (i) the associated vector field $pr_{*}(X_{\zeta})$ is complete on each Q_{α}, and (ii) $\{Q_{\alpha}\}$ partitions M. Let $M_{\alpha} = \{m \in M: pr\ (m) \in Q_{\alpha}\}$ and let ω_{α} be the restrictions of ω to M_{α}. Then $(M_{\alpha},\omega_{\alpha})$ are symplectic manifolds. Let ζ_{α} be the restrictions of ζ to $(M_{\alpha},\omega_{\alpha})$. Let $\mathcal{P}_{\alpha}$ be the polarizations of $(M_{\alpha},\omega_{\alpha})$ defined by $\mathcal{P}_{\alpha} = \mathcal{P}$ on M_{α}. Let $(B_{\alpha},(\cdot,\cdot),\nabla)$ be the restrictions of $(B,(\cdot,\cdot),\nabla)$ to $(M_{\alpha},\omega_{\alpha})$. Then $(B_{\alpha},(\cdot,\cdot),\nabla)$ are prequantization bundles on $(M_{\alpha},\omega_{\alpha})$.

Let $H_{\mathcal{P}}$ be the quantization Hilbert space associated with the polarization $\mathcal{P}$ and let $H_{\mathcal{P}_{\alpha}}$ be the quantization Hilbert spaces associated with the polarizations $\mathcal{P}_{\alpha}$. By definition, ζ is quantizable in the polarization $\mathcal{P}$, and ζ_{α} are quantizable in the polarizations $\mathcal{P}_{\alpha}$. Let $\widetilde{\zeta}$ in $H_{\mathcal{P}}$ be the quantization operator corresponding to ζ, and let $\widetilde{\zeta}_{\alpha}$ in $H_{\mathcal{P}_{\alpha}}$ be the quantization operators corresponding to ζ_α.

(A2.8.T1) <u>Theorem</u>

The quantization Hilbert space $H_\wp$ can be decomposed in terms of the quantization Hilbert spaces $H_{\wp_\alpha}$ to give

$$H_\wp = \bigoplus_\alpha H_{\wp_\alpha}$$

where $\bigoplus_\alpha$ is the direct sum over the index α. The quantization operator $\tilde{\zeta}$ can be decomposed in terms of the quantization operators $\tilde{\zeta}_\alpha$ to give

$$\tilde{\zeta} = \bigoplus_\alpha \tilde{\zeta}_\alpha.$$

APPENDIX 2.9

Theorem on unitary equivalence

(A2.9.T1) Theorem

Two self-adjoint operators **A** and **A'** in Hilbert spaces H and H' respectively are unitarily equivalent if they possess identical spectrum which is purely discrete and nondegenerate.

Proof

Let the common spectrum be $\{a_n : n = 1,2,...\}$, and let the corresponding sets of normalized eigenvectors be $C = \{\varphi_n : n = 1,2,...\}$ in **H** for **A** and $C' = \{\psi_n' : n = 1,2,...\}$ in H' for A'. Then C constitutes an orthonormal basis in **H** because the projectors $|\varphi_n\rangle\langle\varphi_n|$ associated with the eigenvectors form a resolution of the identity by the spectral theorem. Similarly, C' is an orthonormal basis in **H'**. There exists a unitary operator **U** which maps C to C' by $U\varphi_n = \psi_n'$ for all n [cf. Prugovecki (1981), p215]. It is then straightforward to verify that $UAU^{-1} = A'$. This is done by checking that UAU^{-1} has $\{a_n : n = 1,2,...\}$ as spectrum, and C' as the corresponding eigenfunctions. ■

CHAPTER 3

THE MASLOV-WKB METHOD, THE BWS CONDITIONS AND THE MODIFIED MASLOV-WKB METHOD

MASLOV-WKB METHOD, THE BWS CONDITIONS AND THE MODIFIED MASLOV-WKB METHOD

(3.1) Notation

Let (a,b) be an open interval in $\mathbb{R}$. Let $Q = (a,b)$ be the configuration space of a Hamiltonian system and let q be the usual cartesian coordinate on Q. Let $R(q)$ denote the range of q. Let $M = T^*Q = \mathbb{R} \times Q$, ω be the canonical two-form on M and let (p,q) be the usual cartesian coordinates on (M,ω).

Let q_o be a reference point in Q. Let $V(q)$ be a potential well in Q that satisfies the following conditions:

$$V(q) = \sum_{r=0}^{\infty} A_r q^r, \; A_r \text{ are real constants;} \qquad (3.1.\text{Eq } 1a)$$

$$0 \leqslant \lim_{q\to-\infty} V(q) = \lim_{q\to+\infty} V(q) = E_o \leqslant \infty \text{ and } (q-q_o)(\partial V/\partial q) \geqslant 0; \qquad (3.1.\text{Eq } 1b)$$

i.e. $V(q)$ has a single minimum at $q = q_o$.

Let $H(p,q)$ be the Hamiltonian of a particle in the potential well $V(q)$ given by

$$H(p,q) = (p^2/2) + V(q). \qquad (3.1.\text{Eq } 2)$$

We shall adopt the notation given in section (1.3) (of Chapter 1) with a few minor modifications. For the sake of completeness we shall now give the notation that we shall require as follows.

Let $M_o = \{\text{closed integral curves of } X_H\} - \{(0,q_o)\}$. Let ω_o be the restriction of the canonical two-form ω to M_o.

Let $\gamma^E(t)$ be the integral curve of X_H that originates at the point $m_o = (p = (2E)^{1/2}, q = q_o)$. Let $T(E)$ be the period of the integral curve $\gamma^E(t)$.

Let (I,θ) be the action variables on M_o given by

$$I = \oint_{\gamma^\epsilon} pdq, \ \theta = (2\pi t/T(E)). \qquad (3.1.Eq\ 3)$$

Let $R(I)$ be the range of I. Let $H(I)$ be the Hamiltonian expressed as a function of I.

Let $\gamma^\epsilon(\theta) = (p(\theta),q(\theta))$ be the integral curve of X_H that originates at the point ($p = (2E)^{1/2}$, $q = q_o$) with θ instead of t as parameter. Then $p(\theta)$ and $q(\theta)$ satisfy the following differential equations:

$$(\partial q(\theta)/\partial\theta) = [T(E)p(\theta)/2\pi] \text{ and } (\partial p(\theta)/\partial\theta) = [T(E)/2\pi](\sum_{r=1}^{\infty} rA_r q^{r-1}). \qquad (3.1.Eq\ 4)$$

We shall assume $(\partial q(\theta)/\partial\theta)$ has exactly two stationary points in the range $[0,2\pi)$. Similarly, we shall assume that $(\partial p(\theta)/\partial\theta)$ has exactly two stationary points in the range $[0,2\pi)$. Then let $\theta_o,\theta_1,\theta_2,\theta_3 \in [0,2\pi)$ such that they satisfy the following four conditions:

(i) $\theta_o = 0$;

(ii) $\theta_o < \theta_1 < \theta_2 < \theta_3 < 2\pi$;

(iii) $(\partial q(\theta)/\partial\theta) = 0$ at $\theta = \theta_o,\theta_2$;

(iv) $(\partial p(\theta)/\partial\theta) = 0$ at $\theta = \theta_1,\theta_3$.

Remark: (R1) Let P and P_c be the vertical and horizontal polarizations (of the cotangent bundle T^*Q) respectively. In the next subsection we shall identify the position and momentum representations with the quantum Hilbert spaces $H(P)$ and $H(P_c)$ respectively. Therefore, for the present it is sufficient to know that the position and momentum representations are suitably chosen subspaces of $L^2(Q)$ and $L^2_c(\mathbb{R})$ respectively. (As before the subscript c in $L^2_c(\mathbb{R})$ is used to indicate that it is the space of square-integrable functions of p.)

The Schrodinger equation in the position representation is given by

$$(\hat{H}-E)\Psi(q) = 0 \qquad (3.1.\text{Eq } 5a)$$

where

$$\hat{H} = [-i\hbar(\partial^2/\partial q^2)+V(q)]. \qquad (3.1.\text{Eq } 5b)$$

The Schrodinger equation in the momentum representation is given by

$$(\hat{H}_c-E)\varphi_c(p) = 0 \qquad (3.1.\text{Eq } 6a)$$

where

$$\hat{H}_c = [(p^2/2)+\sum_{r=0}^{\infty}(i\hbar)^r A_r(\partial/\partial p)^r]. \qquad (3.1.\text{Eq } 6b)$$

Let

$$\Delta_j = [-2j\pi-(2\pi-\theta_2),2j\pi+\theta_2]. \qquad (3.1.\text{Eq } 7)$$

Let $\Psi(\theta)$ be a function on γ^E; then

$$(\overline{\Psi}^j)(q) = \sum_{\substack{\theta\in\Delta_j \\ q(\theta)=q}} \Psi(\theta) \text{ if } q\in\{q: V(q)\leqslant E\}$$
$$= 0 \text{ otherwise}, \qquad (3.1.\text{Eq } 8)$$

and

$$(\overline{\Psi}^{c,j})(p) = \sum_{\substack{\theta\in\Delta_j \\ p(\theta)=p}} \Psi(\theta) \text{ if } p\in\{p: p^2\leqslant[2E-\min(V)]\}$$
$$= 0 \text{ otherwise}. \qquad (3.1.\text{Eq } 9)$$

Let $J(\theta)$ and $J_c(\theta)$ be functions of θ defined by

$$J(\theta) = +A_K = -k\pi \text{ for } \theta\in(2k\pi-(2\pi-\theta_3),2k\pi+\theta_1],$$

$$J(\theta) = -A_K = -k\pi-\pi/2 \text{ for } \theta\in(2k\pi+\theta_1,2k\pi+\theta_3],$$

$$J_c(\theta) = B^+_{k'} = -k'\pi-\pi/4 \text{ for } \theta\in(2k'\pi,2k'\pi+\theta_2],$$

$$J_c(\theta) = B^-_{k'} = -k'\pi-3\pi/4 \text{ for } \theta\in(2k'\pi+\theta_2,2k'\pi+2\pi],$$

$$(3.1.\text{Eq } 10)$$

where $k,\ k'\in\mathbb{Z}$.

Let $\phi(\theta)$ and $\phi_c(\theta)$ be the two WKB-like functions of θ defined by

$$\phi(\theta) = |(\partial q/\partial\theta)|^{-1/2}\{\exp \pm\int_0^\theta p(\theta)[\partial q(\theta)/\partial\theta]d\theta\}\{\exp iJ(\theta)\}; \qquad (3.1.\text{Eq } 11a)$$

$$\phi_c(\theta) = |(\partial p/\partial\theta)|^{-1/2}\{\exp -\pm\int_0^\theta q(\theta)[\partial p(\theta)/\partial\theta]d\theta\}\{\exp iJ_c(\theta)\}. \qquad (3.1.\text{Eq } 11b)$$

Let $e(\theta)$ and $e_c(\theta)$ be two smooth real-valued functions of θ that satisfy the following three conditions:

(i) $e(\theta)+e_c(\theta) = 1$ for all $\theta \in \mathbb{R}$;

(ii) For all $\theta \in \mathbb{R}$ and $k \in \mathbb{Z}$, we have

$e(\theta) = e(\theta+2k\pi)$ and $e_c(\theta) = e_c(\theta+2k\pi)$ (periodic conditions).

(iii) $e(\theta) = 0$ in the neighbourhood of the points belonging to the set Π . Similarly, $e_c(\theta) = 0$ in the neighbourhood of points belonging to the set Π_c .

(Here

$$\Pi = \{\theta \in \mathbb{R} : \exists k \in \mathbb{Z} \text{ with either } \theta = \theta_1+2k\pi, \text{ or } \theta = \theta_3+2k\pi]\},$$

and

$$\Pi_c = \{\theta \in \mathbb{R} : \exists\, k \in \mathbb{Z} \text{ with either } \theta = 2k\pi, \text{ or } \theta = \theta_2+2k\pi\}.)$$

3.2 THE MASLOV-WKB METHOD FOR $Q \neq \mathbb{R}$

In section (1.3) (of Chapter 1) we outlined the Maslov-WKB method for the Hamiltonian system of a particle in a potential well with configuration space $Q = \mathbb{R}$. In this section we shall seek to answer the following question: what modifications to the Maslov-WKB method are necessary if the configuration space is not the entire $\mathbb{R}$?.

We shall assume that $Q \neq \mathbb{R}$ throughout this section. Let us now formally extend the range of q from R(q) to $\mathbb{R}$; however, we shall continue to use R(q) to denote the range of q in Q.

Let $L^2(\mathbb{R})$ be the space of square integrable functions of q, and let $L^2_c(\mathbb{R})$ be the space of square-integrable functions of p. Then $L^2(\mathbb{R})$ and $L^2_c(\mathbb{R})$ are unitarily related by the Fourier transform $F: L^2_c(\mathbb{R}) \dashrightarrow L^2(\mathbb{R})$ which is given by

$$(F\,\Psi_c)(q) = (2\pi\hbar)^{-1/2}\int_{\mathbb{R}}\{\exp \tfrac{i}{\hbar}pq\}\Psi_c(p)dp,\ \Psi_c(p) \in L^2_c(\mathbb{R}); \qquad (3.2.\text{Eq } 1a)$$

and the inverse map F^{-1} is given by

$$(F^{-1}\varphi)(p) = (2\pi\hbar)^{-1/2}\int_{\mathbb{R}}\{\exp -\tfrac{i}{\hbar}pq\}\varphi(q)dq,\ \varphi(q) \in L^2(\mathbb{R}). \qquad (3.2.\text{Eq } 1b)$$

Clearly the phase space (M,ω) with cartesian canonical coordinates (p,q) is an example of the case (1) situation studied in Chapter 2: $R(p) = \mathbb{R}$ and $R(q) \neq \mathbb{R}$. Let P and P_c be the vertical and horizontal polarizations, respectively, of the cotangent bundle T^*Q. We shall identify the position and momentum representations with the quantum Hilbert spaces $H(P)$ and $H(P_c)$ respectively, as follows. Let the position representation be $L^2(Q)$ and let the momentum representation be $F^{-1}L^2(Q)$. Clearly the momentum representation $F^{-1}L^2(Q)$ is a closed subspace of $L^2_c(\mathbb{R})$. The momentum representation and position representation are related by the unitary map

$\mathbf{U}: \mathbf{F}^{-1} L^2(Q) \subset L^2_c(\mathbb{R}) \longrightarrow L^2(Q)$ which maps $\psi_c(p)$ to $\varphi(q)$ by

$$\varphi(q) = (2\pi\hbar)^{-1/2} \int_{\mathbb{R}} dp\{\exp \tfrac{i}{\hbar} pq\}\, \psi_c(p); \qquad (3.2.\text{Eq } 2a)$$

and the inverse map $\mathbf{U}^{-1}$ is given by

$$\psi_c(p) = (2\pi\hbar)^{-1/2} \int_{\mathbb{R}} dq\{\exp -\tfrac{i}{\hbar} pq\}\, \varphi(q). \qquad (3.2.\text{Eq } 2b)$$

Then the Hamiltonian operators in $\hat{H}$ in $L^2(Q)$ and $\hat{H}_c$ in $\mathbf{F}^{-1} L^2(Q)$ which are given by equations (3.1.Eq 5b) and (3.1.Eq 6b) are unitarily related by $\mathbf{U}$:

$$\mathbf{U}^{-1} \hat{H} \mathbf{U} = \hat{H}_c. \qquad (3.2.\text{Eq } 3)$$

Our objective is to construct approximate solutions of the Schrodinger equation (in the position representation) given by equation (3.1.Eq 5a). For $Q = \mathbb{R}$, we defined $\Phi^E(q)$, the Maslov-WKB wave function (corresponding to the energy E) of the Schrodinger equation, by

$$\Phi^E(q) = \lim_{j \to \infty} \{(1/j)[(\mathbf{F}\, \overline{\phi_c e_c}^{c,j})(q) + (\overline{\phi e}^{j})(q)]\} \qquad (3.2.\text{Eq } 4)$$

[cf. equation (1.3.4.Eq 10)]. Clearly some modifications to the definition of $\Phi^E(q)$ will be necessary.

If for the moment, we ignore the fact that $\overline{\phi_c e_c}^{c,j}$ may not be in the entire domain of $\mathbf{U}$; then formally we have

$$(\mathbf{F}\, \overline{\phi_c e_c}^{c,j})(q) = (\mathbf{U}\, \overline{\phi_c e_c}^{c,j})(q), \text{ for } q \in \{q: V(q)<E\}. \qquad (3.2.\text{Eq } 5)$$

Therefore, Theorem (1.3.4.T1) and corollary (1.3.4.C1) are formally unchanged if we replace $(\mathbf{F}\, \overline{\phi_c e_c}^{c,j})(q)$ by $(\mathbf{U}\, \overline{\phi_c e_c}^{c,j})(q)$. An obvious definition of the Maslov-WKB wave function (corresponding to the energy E) would be

$$\Phi^E(q) = \lim_{j \to \infty} \{(1/j)[(\mathbf{U}\, \overline{\phi_c e_c}^{c,j})(q) + (\overline{\phi e}^{j})(q)]\}. \qquad (3.2.\text{Eq } 6)$$

However, this definition will only makes sense if $(\mathbf{U}\, \overline{\phi_c e_c}^{c,j})(q)$ is well defined: $(\overline{\phi_c e_c}^{c,j})(p)$ must belong to the domain of $\mathbf{U}$ in the limit $\hbar \longrightarrow 0$. This requirement can be restated as follows: $(\overline{\phi_c e_c}^{c,j})(p)$ must approximate an element of the momentum representation $F^{-1}L^2(Q)$ as $\hbar \dashrightarrow 0$.

(3.2.T1) Theorem

(i) We have

$$\int_{\mathbb{R}-R(q)} |(F\overline{\phi_c e_c}^{c,j})(q)|^2 dq = O(\hbar^{2k}) \text{ for some } k = 1,2,3,\ldots.$$

(3.2.Eq 7)

(ii) $(\overline{\phi_c e_c}^{c,j})(p)$ approximates an element of the momentum representation space $F^{-1}L^2(Q)$ as $\hbar \dashrightarrow 0$.

Proof:

(i) See Appendix 3.1.

(ii) This assertion follows from (i), since by definition $F^{-1}L^2(Q)$ consists of elements $\Psi_c'(p)$ whose Fourier transform $(F\Psi_c')(q)$ vanishes on the set $\mathbb{R}-R(q)$.■

It follows from assertion (ii) of the above theorem that the function $\Phi^E(q)$ given by equation (3.2.Eq 6) is well defined in the limit $\hbar \dashrightarrow 0$; so we shall give the following definition.

(3.2.D1) Definition

The Maslov-WKB wave function (corresponding to the energy E) of the Schrodinger equation (in the position representation) is defined by

$$\Phi^E(q) = \lim_{j\to\infty} \{(1/j)[(U\overline{\phi_c e_c}^{c,j})(q)+(\overline{\phi e}^{j})(q)]\}. \quad (3.2.\text{Eq } 8)$$

(Here the Schrodinger equation referred to is given by equation (3.1.Eq 5a), and $E \in (\min(V),E_o)$ where $\min(V) = V(q_o)$.)

Clearly Theorem (1.3.4.T2) (of section (1.3)) remains unaltered.

Therefore, we have the **Maslov-WKB conditions** given by

$$[2\pi]^{-1} \oint_{\gamma\in} p\,dq = (n+1/2)\hbar, \text{ for some integer } n. \qquad (3.2.\text{Eq } 9)$$

Remark: (R1) We shall use the superscript W for the values and spectra predicted by the Maslov-WKB conditions.

The allowed values of I predicted by the Maslov-WKB conditions are:

$$I^{W}(n) = (n+1/2)\hbar, \text{ where } n \in \mathbb{Z} \text{ and } I^{W}(n) \in R(I). \qquad (3.2.\text{Eq } 10)$$

Thus, the discrete part of the spectrum of the Hamiltonian operator $\hat{H}$ predicted by the Maslov-WKB conditions is

$$R_{D}^{W}(\hat{H}) = \{E^{W}(n) = H(I^{W}(n)) : n \in \mathbb{Z} \text{ and } E^{W}(n) \in (\min(V), E_{o})\}. \qquad (3.2.\text{Eq } 11)$$

(Here the subscript D is used to indicate the fact that $R_{D}^{W}(\hat{H})$ is the discrete part of the spectrum of $\hat{H}$.)

3.3 A COMPARISON OF THE BWS CONDITIONS (IN THE HALF-DENSITY QUANTIZATION SCHEME) WITH THE MASLOV-WKB CONDITIONS

We shall use three examples of Hamiltonian systems consisting of a particle in a potential well to compare the BWS conditions (in the half-density quantization scheme) with the Maslov-WKB conditions.

In addition to the notation given in section (3.1) we shall use the following notation. In section (3.1) we defined the submanifold M_o by M_o = {closed integral curves of X_H}--{$(0,q_o)$} where the q_o is the point in Q where the potential well V(q) has its minima. Let ω_o be the restriction of the canonical two-form ω on M to M . The action variables (I,θ) on (M_o,ω_o) are defined by equation (3.1.Eq 3). Let $\mathcal{P}_c$ be the polarization on (M ,) spanned by the vector field $(\partial/\partial\theta)$. Let $B = M_o \times \mathbb{C}$ be the trivial bundle on M_o, $(\cdot,\cdot)$ be the natural Hermitian structure on B, s_o be a unit section of B and let ∇ be the connection on (M_o,ω_o) defined by

$$\nabla_X s_o = -i\!\!\!/(X \lrcorner \beta)s_o, \text{ for all } X \in V_{\mathbb{C}}(M) \qquad (3.3.\text{Eq 1a})$$

where the connection potential β is given by

$$\beta = pdq+cd\theta, \ c \in \mathbb{R}. \qquad (3.3.\text{Eq 1b})$$

Let $(B,(\cdot,\cdot),\nabla)$ be the chosen prequantization bundle over (M_o,ω_o).

The half-density quantization of the Hamiltonian H and the action variable I in the polarization $\mathcal{P}_c$ gives rise to the following BWS conditions [cf. equation (1.2.4.Eq 6)]:

$$[2\pi]^{-1}\oint_{\gamma\in}\beta = n\hbar, \text{ for some integer } n. \qquad (3.3.\text{Eq 2})$$

The allowed values of I predicted by the BWS conditions are:

$$I(n) = n\hbar - c, \ n \in \mathbb{Z} \text{ and } I \in R(I). \qquad (3.3.\text{Eq 3})$$

Therefore, the discrete part of the spectrum of $\hat{H}$ predicted by the BWS conditions is

$$R_{\mathcal{D}}(\hat{H}) = \{E(n) = H(I(n)) : n \in \mathbb{Z} \text{ and } E(n) \in (\min(V), E_0)\}. \qquad (3.3.\text{Eq } 4)$$

(3.3.Ex1) Example: The one-dimensional simple harmonic oscillator [cf. Wan and McKenna (1984); example (2.3.6.Ex 2) of Chapter 2]

We have $Q = \mathbb{R}$, $M = T^*Q = \mathbb{R}^2$, $V(q) = q^2$, $H = (p^2+q^2)/2$, $X_H = p(\partial/\partial q) - q(\partial/\partial p)$, $M_0 = \mathbb{R}^2 - \{(0,0)\}$, $I = H$ and $\theta = \tan^{-1}(q/p)$.

The discrete part of the spectrum of $\hat{H}$ predicted by the BWS conditions is

$$R_{\mathcal{D}}(\hat{H}) = \{E(n) = I(n) : I(n) = n\hbar - c,\ n \in \mathbb{Z} \text{ and } E(n) > 0\}. \qquad (3.3.\text{Eq } 5)$$

The physically correct spectrum is obtained if $c = -\hbar/2$.

The discrete part of the spectrum of $\hat{H}$ predicted by the Maslov-WKB conditions is

$$R^W_{\mathcal{D}}(\hat{H}) = \{E^W(n) = I^W(n) : I^W(n) = (n+1/2)\hbar,\ n = 0,1,2,\ldots\}. \qquad (3.3.\text{Eq } 6)$$

This is the physically correct spectrum of the operator $\hat{H}$.

(3.3.Ex 2) The modified Posch-Teller potential [cf. McKenna and Wan (1984); Flugge (1974), pp94-100]

We have $Q = \mathbb{R}$; $M = T^*Q = \mathbb{R}^2$; $V_0 = -[\hbar^2\alpha^2\lambda(\lambda-1)]/2$ where α, λ are real constants; $V(q) = -V_0/\mathrm{ch}^2(\alpha q)$; $V(q)$ has a single minima at $q = 0$; $M_0 = \{(p,q) \in M : V_0 < H < 0\} - \{(0,0)\}$; $I = [(-2V_0)^{1/2} - (-2H)^{1/2}]/\alpha$; $\theta = \sin^{-1}((\{H/(V_0-H)\}^{1/2}\,\mathrm{sh}(\alpha q))$; $R(I) = (0, [\lambda(\lambda-1)]^{1/2}\hbar)$ and $H(I) = -[(-2V_0)^{1/2} - \alpha I]^2/2$.

The physically correct eigenvalues belonging to the discrete part of the spectrum of $\hat{H}$ are known to be [cf. Flugge (1974)]

$$E(n) = -[\hbar^2\alpha^2(\lambda-1-n)^2]/2, \ n = 1,2,\ldots \text{ and } n\leqslant\lambda-1. \qquad (3.3.\text{Eq } 7)$$

The BWS conditions give physically correct results for the discrete part of the spectrum of $\hat{H}$ if the connection potential is chosen to be

$$\beta = pdq+\hbar(\lambda-1)^{1/2}[(\lambda-1)^{1/2} - \lambda^{1/2}]d\theta. \qquad (3.3.\text{Eq } 8)$$

In this case the allowed values of I predicted by the BWS conditions are

$$I(n) = \{n-(\lambda-1)^{1/2}[(\lambda-1)^{1/2} - \lambda^{1/2}]\}\hbar, \ n = 0,1,\ldots\leqslant\lambda-1. \qquad (3.3.\text{Eq } 9)$$

The allowed values of I predicted by the Maslov-WKB conditions are

$$I^{W}(n) = (n+1/2)\hbar, \ n = 0,1,2,\ldots\leqslant\lambda^{1/2}(\lambda-1)^{1/2} -1/2. \qquad (3.3.\text{Eq } 10)$$

Therefore, $R^{W}_{\mathcal{D}}(\hat{H})$ the discrete part of the spectrum of $\hat{H}$ predicted by the Maslov-WKB conditions consists of the following set of values:

$$E^{W}(n) = -(\alpha^2\hbar^2/2)[\lambda^{1/2}(\lambda-1)^{1/2} -(n+1/2)]^2, \ n = 0,1,2,\ldots\leqslant\lambda^{1/2}(\lambda-1)^{1/2} -1/2.$$

$$(3.3.\text{Eq } 11)$$

Let us compare the physically correct eigenvalues E(n) given by equation (3.3.Eq 7) with the values $E^{W}(n)$. Let $\Delta(n) = |[100(E(n)-E^{W}(n))]/E(n)|$; $\Delta(n)$ is the percentage error between the values E(n) and $E^{W}(n)$. For the sake of convenience, we shall put $\lambda = 4.5$. Then:

(i) The physically correct spectrum of $\hat{H}$ consists of the following eigenvalues:

$E(0) = -6.125\alpha^2\hbar$, $E(1) = -3.125\alpha^2\hbar^2$, $E(2) = -1.125\alpha^2\hbar^2$ and $E(4) = -0.125\alpha^2\hbar^2$.

(ii) $R^{W}_{\mathcal{D}}(H)$ consists of the following values:

$E^{W}(0) = -6.0157\alpha^2\hbar^2$, $E^{W}(1) = -3.047\alpha^2\hbar^2$, $E^{W}(2) = -1.0783\alpha^2\hbar^2$ and $E^{W}(4) = -0.1100\alpha^2\hbar^2$.

(iii) The percentage errors are:

$$\Delta(0) = 11\%,\ \Delta(1) = 8\%,\ \Delta(2) = 5\% \text{ and } \Delta(3) = 2\%.$$

(3.3.Ex2) The isotonic oscillator [example (1.2.4.Ex1)]

We have $Q = \mathbb{R}^{+} = (0,\infty)$, $M = T^{*}Q = \mathbb{R} \times \mathbb{R}^{+}$, $V(q) = (q-1/q)^{2}$, $M_{0} = \mathbb{R} \times \mathbb{R}^{+} - \{(1,0)\}$, $I = H/8^{1/2}$ and $\theta = -\cos^{-1}(\{2q^{2}-H-2\}/\{H^{2}+4H\}^{1/2})$.

Note that unlike the previous examples the configuration space Q does not coincide with $\mathbb{R}$.

The physically correct spectrum of H is known to be

$$\{E(n) = 8^{1/2}[n+1/2+(1/4)(8/\hbar^{2}+1)^{1/2}-(1/4)(8/\hbar^{2})^{1/2}]\hbar:\ n = 0,1,2,\ldots\}. \quad (3.3.\text{Eq } 12)$$

In example (1.2.4.Ex1) of section (1.2) we showed that the BWS conditions give the physically correct result for the discrete part of the spectrum of $\hat{H}$ if the connection potential is chosen to be

$$\beta = pdq-[(1/2)+(1/4)(8/\hbar^{2}+1)^{1/2}-(1/4)(8/\hbar^{2})^{1/2}]hd\theta. \quad (3.3.\text{Eq } 13)$$

The spectrum $R^{W}_{\mathcal{D}}(\hat{H})$ predicted by the Maslov-WKB conditions is given by

$$R^{W}_{\mathcal{D}}(\hat{H}) = \{E^{W}(n) = 8^{1/2}(n+1/2)\hbar:\ n = 0,1,2,\ldots\}. \quad (3.3.\text{Eq } 14)$$

Then the percentage error between the exact eigenvalues of $\hat{H}$ and the corresponding values $E^{W}(n)$ are given by

$$\Delta(n) = [(E(n)-E^{W}(n))]/E(n) = 6.25\hbar^{2}\%,\ n = 0,1,2,\ldots \quad (3.3.\text{Eq } 15)$$

The following points emerge from the three examples considered:

(i) In general, the Maslov-WKB conditions predict physically incorrect results for the discrete part of the spectrum of $\widehat{H}$. However, they do give approximate eigenvalues; the percentage error between these approximate eigenvalues and the exact results depends on the problem under consideration.

(ii) The BWS conditions give physically correct results for the discrete part of the spectrum of $\widehat{H}$ if the following two conditions are satisfied:

(a) The physically correct discrete part of the spectrum of $\widehat{H}$ is given by $\{H(I(n)): I(n) = n\hbar - c,\ n \in \mathbb{Z},\ c \in \mathbb{Z}$ is fixed and $\min V(q) \leqslant H(I(n)) \leqslant \max V(q)\}$. (This condition is satisfied by a large number of examples that are of interest in physics.)

(b) The connection potential β is chosen to be $\beta = pdq + cd\theta$ where c is determined by physically correct spectrum given above.

3.4 A MODIFIED MASLOV-WKB METHOD

We have demonstrated that the Maslov-WKB method is unable to produce exact discrete eigenvalues of $\hat{H}$ (the Hamiltonian operator of a particle in potential well) in many cases. The question then arises as to whether we can modify the Maslov-WKB method to incorporate the BWS conditions, so as to give the exact eigenvalues. The modification must involve the flexibility arising from the choice of connection potential made in the half-density quantization scheme.

We shall assume the results and notation given in the previous sections of this chapter. In particular, we have the $R_{\mathcal{D}}(\hat{H})$ the discrete part of the spectrum of $\hat{H}$ predicted by the BWS conditions [cf. equation (3.3.Eq 4)] and $R^{W}_{\mathcal{D}}(H)$ the discrete part of the spectrum of $\hat{H}$ predicted by the Maslov-WKB conditions [cf. equation (3.2.Eq 11)] given by

$$R_{\mathcal{D}}(\hat{H}) = \{E(n) = H(I(n)):\ n \in \mathbb{Z} \text{ and } E(n) \in (\min(V), E_0)\} \qquad (3.4.\text{Eq } 1a)$$

and

$$R^{W}_{\mathcal{D}}(H) = \{E^{W}(n) = H(I^{W}(n)):\ n \in \mathbb{Z} \text{ and } E^{W}(n) \in (\min(V), E_0)\}. \qquad (3.4.\text{Eq } 1b)$$

respectively.

Clearly $R_{\mathcal{D}}(\hat{H})$ and $R^{W}_{\mathcal{D}}(\hat{H})$ coincide if the connection ∇ of the prequantization bundle $(B, (\cdot,\cdot), \nabla)$ (over (M_0, ω_0)) is determined by the connection potential $\beta = pdq + (k+1/2)\hbar d\theta$, where k is some integer. Therefore, we shall study the following two cases separately:

(CP1) $\beta = pdq + (k+1/2)\hbar d\theta$, where $k \in \mathbb{Z}$;

(CP2) $\beta = pdq + cd\theta$, where $c \neq (k+1/2)\hbar$ for all $k \in \mathbb{Z}$.

<u>Case</u> <u>CP1:</u> $\beta = pdq+(k+1/2)\hbar d\theta,\ k \in \mathbb{Z}$

In this case as pointed out earlier we have $R_{\mathcal{D}}(\hat{H}) = R^{W}_{\mathcal{D}}(\hat{H})$. Then for each $E(n) \in R_{\mathcal{D}}(\hat{H})$, we have the Maslov-WKB wave function $\Phi_n(q)$ (corresponding to the energy $E(n)$) defined in remark (R2) of section (3.2).

<u>Remark</u>: (R1) By equations (3.1.Eq 11a) and (3.1.Eq 11b), we have:

$$\phi(\theta+2j\pi)e(\theta+2j\pi) = \phi(\theta)\{\exp ij[(2\pi I/\hbar)-\pi]\}e(\theta),\ j \in \mathbb{Z}; \qquad (3.4.\text{Eq } 2a)$$

$$\phi_c(\theta+2j\pi)e_c(\theta+2j\pi) = \phi_c(\theta)\{\exp ij[(2\pi I/\hbar)-\pi]\}e_c(\theta),\ j \in \mathbb{Z}. \qquad (3.4.\text{Eq } 2b)$$

(Here we have used the fact that I is given by $I = \oint_{\gamma E} pdq = -\oint_{\gamma E} qdp$.) Now, in the case (CP1) situation, we have:

$$I(n) = (n+1/2)\hbar; \qquad (3.4.\text{Eq } 3a)$$

$$\phi(\theta+2j\pi)e(\theta+2j\pi) = \phi(\theta)e(\theta) \text{ on } \gamma^{E(n)}; \qquad (3.4.\text{Eq } 3b)$$

$$\phi_c(\theta+2j\pi)e_c(\theta+2j\pi) = \phi_c(\theta)e_c(\theta) \text{ on } \gamma^{E(n)}; \qquad (3.4.\text{Eq } 3c)$$

$(\overline{\phi e}^{j})(q) = j(\overline{\phi e}^{0})(q)$, where $\phi(\theta)e(\theta)$ is defined on $\gamma^{E(n)}$;

$(\overline{\phi_c e_c}^{c,j})(p) = j(\overline{\phi_c e_c}^{c,0})(p)$, where $\phi_c(\theta)e_c(\theta)$ is defined on $\gamma^{E(n)}$;

Therefore, $\Phi_n(q)$ is also given by

$$\Phi_n(q) = (U\,\overline{\phi_c e_c}^{c,0})(q)+(\overline{\phi e}^{0})(q). \qquad (3.4.\text{Eq } 4)$$

where it is understood that $\phi_c(\theta)e_c(\theta)$ and $\phi(\theta)e(\theta)$ are defined on $\gamma^{E(n)}$.

<u>Case</u> <u>CP2:</u> $\beta = pdq+cd\theta$, where $c \neq (k+1/2)\hbar$ for all $k \in \mathbb{Z}$

We have a new situation here in this case: for each $E(n) \in R_{\mathcal{D}}(\hat{H})$, we have $\Phi^{E(n)}(q) = 0$ by Theorem (1.3.4.T2) (of section (1.3)). Hence we cannot use the standard Maslov-WKB method to construct approximate solutions of the Schrodinger equation. However, equation (3.4.Eq 4) points to a way to circumvent Theorem (1.3.4.T2). We simply construct a new function on the closed integral curve $\gamma^{E(n)}$ determined by $I(n) = n\hbar - c$ rather than $I(n) = n\hbar$ as follows.

(3.4.D1) Definition

Let

$$\Psi_n(q) = (U\overline{\phi_c e_c}^{c,0})(q) + (\overline{\phi e}^{0})(q). \tag{3.4.Eq 5}$$

Here $\phi_c(\theta)e_c(\theta)$ and $\phi(\theta)e(\theta)$ are functions on the curve $\gamma^{E(n)}$ which is determined by the value $I(n) = n\hbar - c$, i.e., $E(n) = H(I(n))$. We shall refer to $\Psi_n(q)$ as the **modified Maslov-WKB wave function** (corresponding to the **energy** $E(n)$).

By definition $\Psi_n(q)$ is non-zero on the set $\{q: V(q) < E(n) = H(I(n))\}$. The validity of the above construction is justified by the following theorem whose proof is given in Appendix 3.2.

(3.4.T1) Theorem

We have

$$\|(\hat{H} - E(n))\Psi_n(q)\| = O(\hbar^2). \tag{3.4.Eq 6}$$

Therefore, $\Psi_n(q)$ is an approximate eigenfunction (corresponding to the predicted eigenvalue $E(n)$) of the Hamiltonian operator $\hat{H}$. In particular, the degree of approximation is as good as the original Maslov's approximation [cf. Theorem (1.3.4.T2), assertion (ii)].

Remarks: (R2) Clearly in the case (CP1) situation we have $\Phi_n(q) = \Psi_n(q)$ [cf. equations (3.4.Eq 4) and (3.4.Eq 5)]. Therefore, from now on we shall write $\Psi_n(q)$ for the approximate eigenfunction (corresponding to the energy $E(n)$) for both case (CP1) and case (CP2) situations.

(R3) Our choice of $\Psi_n(q)$ is the first term in the sum for the original Maslov-WKB wave functions defined by equation (3.2.Eq 8). We could just as easily have chosen $(1/j)[(U\,\overline{\phi_c e_c}^{\,c,j})(q)+(\overline{\phi\, e}^{\,j})(q)]$, where j is a positive integer, to be modified Maslov-WKB wave function. It follows from equations (3.4.Eq 2a) and (3.4.Eq 2b) that this sum differs from $\Psi_n(q)$ by a constant factor.

3.5 APPLICATIONS OF THE MODIFIED MASLOV-WKB METHOD

Let us consider the pairing problem in the quantization of a Hamiltonian H with closed integral curves. We shall assume the notation given in section (3.1). We can quantize H in the polarization $\mathcal{P}_c$ spanned by $(\partial/\partial\theta)$; so let $\hat{H}_{oc}$ be the quantum operator in the quantum Hilbert space $H(\mathcal{P}_c)$, and let Φ^n_{oc} be the eigenfunctions of $\hat{H}_{oc}$ and let E(n) be the eigenvalues of $\hat{H}_{oc}$. The question now is: what is the corresponding operator $\hat{H}$ in $H(P) = L^2(Q)$?. In other words, what is the unitary operator U which maps $H(\mathcal{P}_c)$ to H(P) and $\hat{H}_{oc}$ to $\hat{H}$?. We shall present an approximate solution to this problem based on the following simple observation: U is determined if we know the eigenfunctions φ_n and eigenvalues of the operator $\hat{H}$ in $L^2(Q)$. Now the eigenvalues of $\hat{H}$ are known as they can be taken to coincide with that of $\hat{H}_{oc}$. What we do not know are φ_n. Here the modified Maslov-WKB method comes into play. This enables us to construct approximate eigenfunctions Ψ_n of $\hat{H}$ consistent with the eigenvalues E(n). Using Ψ_n and E(n) we can construct an approximate unitary map U. Thus we have established an approximate pairing between the polarizations $\mathcal{P}$ and P. In the next chapter we shall discuss some examples in which this modified Maslov-WKB method is applicable.

APPENDICES A3.1-A3.2

APPENDIX 3.1

(A3.1.L1) Lemma [cf. Eckmann and Seneor (1976)]

Let $C(p)$ be a compactly supported smooth function of p with support Γ. Let $D(p)$ be a real smooth function of p and let $D' = (\partial D/\partial p)$.

(i) If the following integral exists for some j = 1,2,3,...

$$[-\hbar/i]^j \int_\Gamma dp\{\exp \pm D(p)\}[(\partial/\partial p)\{[D']^{-1}(\partial/\partial p)([D']^{-1}\ldots(\partial/\partial p)[C/D']\}]$$

then,

$$\int_\Gamma dp\{\exp \pm D(p)\}C(p)$$
$$= [-\hbar/i]^j \int_\Gamma dp\{\exp \pm D(p)\}[(\partial/\partial p)\{[D']^{-1}(\partial/\partial p)([D']^{-1}\ldots(\partial/\partial p)[C/D']\}] \qquad (3.1.\text{Eq } 1)$$

(ii) In particular, if $D'(p) \neq 0$ on Γ, then

$$\int_\Gamma dp\{\exp \pm D(p)\}C(p) = O(\hbar^j) \text{ for some } j = 1,2,3,\ldots. \qquad (\text{A}3.1.\text{Eq } 2)$$

Proof:

(i) The left hand side of equation (A3.1.Eq 1) is obtained by integrating the right hand side j-times.

(ii) This assertion follows from the fact that

$$[(\partial/\partial p)\{[D']^{-1}\ldots.(\partial/\partial p)[C/D']\}$$

is a compactly supported function. ■

For each positive integer r, we have

$$\Delta_r = [2r\pi-(2\pi-\theta_2),2r\pi+\theta_2].$$

Let us subdivide Δ_r into the following subintervals:

$$[2k\pi,2k\pi+\theta_2],\ k \in \mathbb{Z} \text{ and } -r\leqslant k\leqslant r;$$

$$[2k'\pi-(2\pi-\theta_2),2k'\pi],\ k' \in \mathbb{Z} \text{ and } -r\leqslant k'\leqslant r.$$

Let us fix k; then on the subinterval $[2k\pi,2k\pi+\theta_2]$ the map $\theta \dashrightarrow p(\theta)$ has a unique inverse which we shall denote by $\theta_k^+(p)$. Similarly, let us fix k'; then on the subinterval $[2k'\pi-(2\pi-\theta_2),2k'\pi]$ the map $\theta \dashrightarrow p(\theta)$ has a unique inverse which we shall denote by $\theta_{k'}^-(p)$.

Note that for $p \in \{p: \exists\, q \in Q \text{ s.t. } (p,q) \in \gamma^{\epsilon}\}$, we have

$$(\widetilde{\phi_c e_c}^{c,r})(p) = \sum_{\substack{\theta\in\Delta r\\ p(\theta)=p}} \phi_c(\theta)e_c(\theta)$$
$$= \sum_{\pm}\sum_{k=-r}^{r} \phi_c(\theta_k^{\pm}(p))e_c(\theta_k^{\pm}(p)). \qquad \text{(A3.1.Eq 3)}$$

(A3.1.T1) Theorem

We have

$$\int_{\mathbb{R}-R(q)} |(F\widetilde{\phi_c e_c}^{c,r})(q)|^2 dq = O(\hbar^{2j+1}) \text{ for some } j = 1,2,3,\ldots.$$

(A3.1.Eq 4)

Proof:

Let Γ_c be the support of $(\widetilde{\phi_c e_c}^{c,r})(p)$. Then for each $q \in \mathbb{R}-R(q)$, we have

$$(F\widetilde{\phi_c e_c}^{c,r})(q)$$
$$= (2\pi\hbar)^{-1/2}\sum_{\pm}\sum_{k=-r}^{k=r} \Big[\int_{\Gamma_c} dp\{\exp \tfrac{i}{\hbar}(pq-\int_0^{\theta_k^{\pm}(p)} q(\theta)(\partial p/\partial\theta)d\theta)\}\,|(\partial p/\partial\theta)|^{-1/2}(\theta_k^{\pm}(p))$$
$$\times\, e_c(\theta_k^{\pm}(p))\{\exp iJ_c(\theta_k^{\pm}(p))\}\Big] \text{ (by equation (A3.1.Eq 3))}$$
$$= [2\pi\hbar]^{-1/2}\sum_{\pm}\sum_{k=-r}^{k=r}\Big[\int_{\Gamma_c} dp\{\exp iD_k^{\pm}(p)\}C_k^{\pm}(p)\Big] \qquad \text{(A3.1.Eq 5)}$$

where

$$C_k^{\pm}(p) = |(\partial p/\partial\theta)|^{-1/2}(\theta_k^{\pm}(p))\{\exp \tfrac{i}{\hbar}J_c(\theta_k^{\pm}(p))\}e_c(\theta_k^{\pm}(p)) \text{ if } p \in \Gamma_c$$
$$= 0 \text{ otherwise,}$$

and

$$D_k^{\pm}(p) = pq-\int_0^{\theta_k^{\pm}} q(\theta)(\partial p/\partial\theta)d\theta \text{ if } p \in \Gamma_c$$
$$= 0 \text{ otherwise.}$$

Then we have

$$(\partial D_K^{\pm}/\partial p) = q - q(\theta_K^{\pm}(p)) \text{ if } p \in \Gamma_c;$$

$$|(\partial D_K^{\pm}/\partial p)|_{p\in\Gamma_c} \geqslant \min\{|q-q(\theta_K^{\pm}(p))| : p \in \Gamma_c\};$$

and

$$|(\partial D_K^{\pm}/\partial p)|_{p\in\Gamma_c} > 0 \text{ if } q \in \mathbb{R} - \{q: \exists\, p \text{ s.t } (p,q) \in \gamma^{\epsilon}\}.$$

Therefore, there exists $\varepsilon > 0$ such that

$$|(\partial D_K^{\pm}/\partial p)|_{p\in\Gamma_c} > q(\theta_3)+\varepsilon-q \text{ for } q \leqslant q(\theta_3) \qquad \text{(A3.1.Eq 6a)}$$

and

$$|(\partial D_K^{\pm}/\partial p)|_{p\in\Gamma_c} > q-q(\theta_1)-\varepsilon \text{ for } q \geqslant q(\theta_3). \qquad \text{(A3.1.Eq 6b)}$$

Since the function $\theta_K^{\pm}(p)$ is a compactly supported smooth function, $(\partial^j C_K^{\pm}/\partial p^j)$ is bounded and $|(\partial^j C_K^{\pm}/\partial p^j)|$ is finite, for each j.

Then for each $q \in \{q: \text{Either } q \leqslant q(\theta_3) \text{ or } q \geqslant q(\theta_1)\}$, we have

$$|(F\overline{\phi_c e_c}^{c,r})(q)|$$
$$= \hbar^j [2\pi\hbar]^{-1/2} \left|\sum_{\pm}\sum_{K=-r}^{r} \int_{\Gamma_c} dp\{\exp \pm D_K^{\pm}(p)\}(\partial D_K^{\pm}/\partial p)^{-j}(\partial^j C_K^{\pm}/\partial p^j)\right|$$

for some j = 1,2,3,... (by Lemma (A3.1.L1))

$$\leqslant \hbar^{j-1/2}(2\pi)[\max(p\in\Gamma_c)-\min(p\in\Gamma_c)][\min|(\partial D_K^{\pm}/\partial p)|]^{-j}[\max(|(\partial^j C_K^{\pm}/\partial p^j)|]$$
$$< \hbar^{j-1/2}[\min|(\partial D_K^{\pm}/\partial p)|]^{-j} L, \text{ (where L is a positive number).}$$

Let

$$\square = \{q: \text{either } q \geqslant q(\theta_1) \text{ or } q \leqslant q(\theta_3)\}.$$

Thus

$$\int_{\square} |(F\overline{\phi_c e_c}^{c,r})(q)|^2 dq \leqslant \hbar^{2j+1} L^2 \int_{\square} [\min(|(\partial D_K^{\pm}/\partial p)]^{-2j} dq$$
$$= O(\hbar^{2j+1}).$$

The last step follows from the fact that the integral $\int_{\Box}[\min(|(\partial D_K^{\pm}/\partial p)|]^{-2j}\,dp$ is finite; since by equations (A3.1.Eq 6a) and (A3.1.Eq 6b) we have

$$\int_{\Box}[\min|(\partial D_K^{\pm}/\partial p)|]^{-2j}\,dq \leqslant 2[-2j+1]|\varepsilon|^{-2j+1}.$$

Since $\mathbb{R}-R(q)\subseteq\Box$, we have

$$\int_{\mathbb{R}-R(q)} |(F\overline{\phi_c\, e_c}^{c,r})(q)|^2 dq = O(h^{2j+1}) \text{ for some } j = 1,2,3,\ldots.$$ ■

APPENDIX 3.2

Here is a short list of results that we shall use:

$$(\partial q(\theta)/\partial\theta) = [T(E)p(\theta)/2\pi] \text{ and } (\partial p(\theta)/\partial\theta) = [T(E)/2\pi](\sum_{r=1}^{\infty} rA_r q^{r-1});$$

(A3.2.Eq 1)

$$\hat{H} = [-i\hbar(\partial^2/\partial q^2)+V(q)]; \qquad \text{(A3.2.Eq 2)}$$

$$\hat{H}_c = [(p^2/2)+\sum_{r=0}^{\infty} (i\hbar)^r A_r(\partial/\partial p)^r] \qquad \text{(A3.2.Eq 3)}$$

$$(U\Psi_c)(q) = (2\pi\hbar)^{-1/2}\int_{\mathbb{R}} dp\{\exp i\hbar pq\}\Psi_c(p); \qquad \text{(A3.2.Eq 4)}$$

$$(U^{-1}\varphi)(p) = (2\pi\hbar)^{-1/2}\int_{\mathbb{R}} dq\{\exp -i\hbar pq\}\varphi(q); \qquad \text{(A3.2.Eq 5)}$$

$$U\hat{H}_c U^{-1} = \hat{H}; \qquad \text{(A3.2.Eq 6)}$$

Let $\phi_W^E(q) = f(q)\{\exp i\hbar S(q)\}$ and $\phi_{c,W}^E(p) = g(p)\{\exp i\hbar W(p)\}$.

$$(\hat{H}-E)\phi_W^E(q) = \hbar^0\{(1/2)(\partial S/\partial q)^2+V(q)-E\}\phi_W^E(q)$$

$$+ \hbar(-i)\{(1/2)(\partial^2 S/\partial q^2)f+(\partial f/\partial q)(\partial S/\partial q)\}\{\exp i\hbar S\}$$

$$+ \text{ higher order terms of } \hbar \qquad \text{(A3.2.Eq 7)}$$

$$(\hat{H}_c-E)\phi_{c,W}^E(p) = \hbar^0[(p^2/2)+\{\sum_{K=0}^{\infty} A_K(-\partial W/\partial p)^K\}-E]\phi_{c,W}^E(p)$$

$$- i\hbar[\sum_{r=1}^{\infty}(-1)^r A_r r\{[(r-1)/2](\partial W/\partial p)^{r-2}(\partial^2 W/\partial p^2)g+(\partial W/\partial p)^{r-1}(\partial g/\partial p)\}]\{\exp i\hbar W\}$$

$$+ \text{ higher order terms in } \hbar \qquad \text{(A3.2.Eq 8)}$$

(A3.2.L1) Lemma

Let $\boxtimes = R(q)-\{q: \exists\, p \text{ s.t } (p,q)\in\gamma^E\}$. Then we have

$$\int_{\boxtimes} |(U\overline{\phi_c(\partial e_c/\partial\theta)}^0)(q)|^2 dq = O(\hbar^{2j+1}) \text{ for some } j = 1,2,3,\ldots.$$

(A3.2.Eq 9)

Proof:

The function $(\partial e_c/\partial\theta)$, like $e_c(\theta)$, satisfies the following condition: $(\partial e_c/\partial\theta) = 0$ in the neighbourhood of points belonging to the set $\{\theta\in\mathbb{R}: \theta = 2k\pi,\ \theta = 2k\pi+\theta_2,\ k\in\mathbb{Z}\}$. Therefore, this proof is formally the same as the proof of Theorem (A3.1.T1) of Appendix 3.1. after we have replaced the Fourier transform **F** by the unitary map **U**, and $e_c(\theta)$ by

$(\partial e_c/\partial\theta)$. ■

(A3.2.T1) <u>Theorem</u>

Let

$$\Psi^{\epsilon}(q) = (U\,\overline{\phi_c e_c}^{c,0})(q)+(\overline{\phi e}^{c})(q) \qquad \text{(A3.2.Eq 10)}$$

where $\phi_c(\theta)e_c(\theta)$ and $\phi(\theta)e(\theta)$ are functions on the closed integral curve γ^{ϵ}.
Then we have

$$\|(\hat{H}-E)\Psi^{\epsilon}(q)\| = O(\hbar^2). \qquad \text{(A3.2.Eq 11)}$$

(Note that this theorem is more general than Theorem (3.4.T1).)

<u>Proof</u>:

For $q \in \{q \in R(q): \exists p \text{ s.t } (p,q) \in \gamma^{\epsilon}\}$, we have

$$\begin{aligned}
(\hat{H}-E)\Psi^{\epsilon}(q) &= \\
&= (\hat{H}-E)(U\,\overline{\phi_c e_c}^{c,0})(q)+(\hat{H}-E)(\overline{\phi e}^{0})(q) \\
&= UU^{-1}(\hat{H}-E)U(\overline{\phi_c e_c}^{c,0})(q)+(\hat{H}-E)(\overline{\phi e}^{0})(q) \\
&= U[(\hat{H}_c-E)(\overline{\phi_c e_c}^{c,0})(p)]+(\hat{H}-E)(\overline{\phi e}^{0})(q)
\end{aligned}$$

(by equation (A3.2.Eq 6))

$$= -i\hbar[2\pi/T(E)][\{U\,\overline{\phi_c(\partial e_c/\partial\theta)}^{c,0}\}(q)+\{\overline{\phi(\partial e/\partial\theta)}^{0}\}(q)]$$

(by equations (A3.2.Eq 7) and (A3.2.Eq 8))

$$= -i\hbar[2\pi/T(E)][\overline{\phi\{\partial(e+e_c)/\partial\theta)\}}^{0}]+O(\hbar^2)$$

(by Theorem (1.3.4.T1) of section (1.3))

$$= O(\hbar^2). \qquad \text{(A3.2.Eq 12)}$$

The last step is obtained using the fact that $(e+e_c) = 1$ [cf. section (3.1) of Chapter 3].

Let $\oplus = \{q : \exists\, p \text{ s.t } (p,q) \in \gamma^{E}\}$. Thus

$$\int_{\oplus} |(H-E)\Psi^{E}(q)|^2 dq = O(\hbar^2). \qquad \text{(A3.2.Eq 13)}$$

Let $\boxdot = R(q) - \oplus$. Then for each $q \in \boxdot$, we have

$$(H-E)\Psi^{E}(q) = [U(\hat{H}_c - E)(\overline{\phi_c e_c}^{c,0})](q)$$

$$= -i\hbar[2\pi/T(E)][U\,\overline{\phi_c(\partial e_c/\partial\theta)}^{c,0}](q) + O(\hbar^2)$$

(A3.2.Eq 14)

(by Theorem (1.3.4.T1) of section (1.3))

Thus

$$\int_{\boxdot} |(\hat{H}-E)\Psi^{E}(q)|^2 dq = \hbar \int_{\boxdot} |U\,\overline{\phi_c(\partial e_c/\partial\theta)}^{c,0}|^2 dq$$

(by equation (A3.2.Eq 13))

$$= O(\hbar^{2j+2}) \text{ for some } j = 1,2,3,\ldots \qquad \text{(A3.2.Eq 15)}$$

(by Lemma (A3.2.L1))

This proves our assertion. ■

CHAPTER 4

LOCALIZATION OF OBSERVABLES IN AN EFFECTIVE CONFIGURATION SPACE AND IN THE PHASE SPACE, AND THE MODIFIED MASLOV-WKB METHOD FOR CERTAIN MULTILINEAR MOMENTUM OBSERVABLES WITH CLOSED INTEGRAL CURVES

LOCALIZATION OF OBSERVABLES IN AN EFFECTIVE CONFIGURATION SPACE AND IN THE PHASE SPACE, AND THE MODIFIED MASLOV-WKB METHOD FOR MULTILINEAR MOMENTUM OBSERVABLES WITH CLOSED INTEGRAL CURVES

(4.1) Introduction

In section (4.2) we shall attempt to establish unitarily equivalent quantizations of a general observable ζ in suitably chosen canonically conjugate polarizations. In general, we shall see that it is not always possible to establish such quantizations, so we shall attempt to see whether we can establish unitarily equivalent quantizations for a local observable that possesses the same properties as ζ locally. Motivated by the physical limitations of measuring devices Wan et al have systematically studied the notion of local quantum observables in quantum mechanics [cf. McFarlane and Wan (1981a); Wan and Jackson (1984); Wan, Jackson and McKenna (1984); Wan and McLean (1985)]. The work of Wan et al includes quantizing classical momentum observables localized in the configuration space. In section (4.2) we shall show that by transforming to an effective configuration space we are able to effect the localization of ζ such that this local observable can be quantized as a complete momentum observable in a unitarily equivalent manner in suitably chosen canonically conjugate polarizations.

In section (4.3) we shall localize the Hamiltonian H (of a particle in a potential well) in the phase space and effect the quantization of the localized Hamiltonian in a suitably chosen polarization. We shall then use the approximate eigenfunctions of the Hamiltonian operator $\hat{H}$ in $L^2(Q)$ constructed by the modified Maslov-WKB method to determine the quantum operator in $L^2(Q)$ corresponding to the localized Hamiltonian.

In section (4.4) we shall extend the modified Maslov-WKB method to multilinear momenta ,i.e. observables that are polynomials of the momentum observable p with functions of q as coefficients.

4.2 THE LOCALIZATION OF OBSERVABLES IN A EFFECTIVE CONFIGURATION SPACE

Let (M,ω) be a 2-dimensional symplectic manifold, ζ be an observable on (M,ω), $R(\zeta)$ be the classical range of values of ζ and let $\gamma_m(t)$ be the integral curve of the Hamiltonian vector field X_ζ originating at the point m.

Then according to the **Hamilton box theorem** [cf. Abraham and Marsden (1980), pp391-392] we have the following results. Suppose $X_\zeta(m) \neq 0$ for some $m \in M$, then there is a neighbourhood U of m with canonical coordinates $(\mathfrak{p},\mathfrak{q})$ given by

$$\mathfrak{p} = \zeta \text{ with range denoted by } R(\mathfrak{p}), \qquad (4.2.\text{Eq 1a})$$

$$\mathfrak{q} = t \text{ with range denoted by } R(\mathfrak{q}), \qquad (4.2.\text{Eq 1b})$$

such that the map $F:U \dashrightarrow R(\mathfrak{p}) \times R(\mathfrak{q})$ given by

$$F(m) = (\mathfrak{p},\mathfrak{q} = t) \qquad (4.2.\text{Eq 1c})$$

is bijective with

$$F\gamma_{F^{-1}(\mathfrak{p},0)}(t) = (\mathfrak{p},\mathfrak{q} = t). \qquad (4.2.\text{Eq 1d})$$

Let ω_U denote the restriction of the symplectic two-form ω to U; then (U,ω_U) is a 2-dimensional symplectic manifold with global coordinates $(\mathfrak{p},\mathfrak{q})$. We shall consider the following four case situations:

(1) $R(\mathfrak{p}) = \mathbb{R}$; $R(\mathfrak{q}) \neq \mathbb{R}$;

(2) $R(\mathfrak{p}) \neq \mathbb{R}$; $R(\mathfrak{q}) = \mathbb{R}$;

(3) $R(\mathfrak{p}) = \mathbb{R}$; $R(\mathfrak{q}) = \mathbb{R}$;

(4) $R(\mathfrak{p}) \neq \mathbb{R}$; $R(\mathfrak{q}) \neq \mathbb{R}$;

Let $\mathcal{P}$ and $\mathcal{P}_c$ be the canonically conjugate polarizations of (U,ω_U) spanned by the vector fields $(\partial/\partial\mathfrak{p})$ and $(\partial/\partial\mathfrak{q})$ respectively. Then $R(\mathfrak{q})$ is the effective configuration space with respect to $\mathcal{P}$ and $R(\mathfrak{p})$ is the effective configuration space with respect to $\mathcal{P}_c$. Let β_U a one-form on U that satisfies $d\beta_U = \omega_U$.

Let $B_U = U \times \mathbb{C}$ be the trivial bundle over U, $(\cdot,\cdot)$ be the natural Hermitian structure on B_U, s_o be a unit section of B_U and let ∇ be the connection over B_U defined by

$$\nabla_X \, s_o = -i(X \lrcorner \beta_U)s_o \text{ for all } X \in V_{\mathbb{C}}(U). \qquad \text{(A4.2.Eq 2)}$$

Let $(B_U,(\cdot,\cdot),\nabla)$ be the chosen prequantization bundle over (U,ω_U).

Our problem now is to establish unitarily equivalent quantizations of $\mathfrak{p} = \zeta$ in the canonically conjugate polarizations $\mathcal{P}$ and $\mathcal{P}_c$. In chapter 2 we spelled out the conditions under which $\mathfrak{p}$ is quantizable, i.e. $R(\mathfrak{q})$ should be $\mathbb{R}$. This condition is only satisfied in case (1) and case (4) situations, so in these cases we quantize ζ as we did in chapter 2. So the range of $\mathfrak{q}$, $R(\mathfrak{q})$, in the case (2) and case (3) situations poses a major obstacle to quantizing $\mathfrak{p}$ in these two cases. To circumvent this difficulty we propose a procedure which will enable us to quantize $\mathfrak{p}$ locally in the case (2) and case (3) situations. This is a generalization of the idea of localization of (cartesian) momentum observables put forward by Wan, Jackson and McKenna (1984).

The procedure for quantizing $\mathfrak{p}$ locally in the case (2) and case (3) situations is given as follows. Let Λ be an open interval in $R(\mathfrak{q})$ such that the closure of Λ is compact and let Λ_o be an open interval contained in Λ. Let $\xi(\mathfrak{q})$ be a C^∞ function of $\mathfrak{q}$ that satisfies the following two conditions:

(i) $\xi(\mathfrak{q}) = 1$ on Λ_o;

(ii) $\xi(\mathfrak{q}) = 0$ on $U-\Lambda$.

Therefore, $\xi(\mathfrak{q})$ is a function of compact support on $R(\mathfrak{q})$.

Let $V = F^{-1}(R(\mathcal{p}) \times \Lambda)$ and $V_o = F^{-1}(R(\mathcal{p}) \times \Lambda_o)$. Then let ω_V be the restriction of ω_U to V. Then (V,ω_V) is a symplectic submanifold of (U,ω_U).

Then we can choose canonical coordinates $(\mathcal{p}',\mathcal{q}')$ on (V,ω_V) given by

$$\mathcal{p}' = \xi(\mathcal{q})\mathcal{p}, \quad \mathcal{q}' = \int_{\mathcal{q}_o}^{\mathcal{q}} [\xi(\mathcal{q})]^{-1} d\mathcal{q}. \qquad \text{(A4.2.Eq 3)}$$

where $\mathcal{q}_o$ is a chosen reference point in Λ_o. Let $R(\mathcal{p}')$ denote the classical range of $\mathcal{p}'$ and let $R(\mathcal{q}')$ denote the range of $\mathcal{q}'$. Clearly $R(\mathcal{q}') = \mathbb{R}$ [cf. Wan, Jackson and McKenna (1984)].

Let $\mathcal{P}'$ and $\mathcal{P}'_c$ be the canonically conjugate polarizations of (V,ω_V) spanned by the vector fields $(\partial/\partial\mathcal{p}')$ and $(\partial/\partial\mathcal{q}')$ respectively.

<u>Remark</u>: (R1) Note that $\mathcal{p}' = \mathcal{p} = \zeta$ on V_o. In particular, $R(\mathcal{q})$ is identifiable with the effective configuration space with respect to the polarization $\mathcal{P}$; so $\mathcal{p}'$ is the localization of ζ in the effective configuration space $R(\mathcal{q})$.
(R2) If Λ_o is chosen to arbitrarily close to Λ, then V_o is arbitrarily close to V [cf. Abraham and Marsden (1980), p81]. One can also choose V to be arbitrarily close to U.

Since $R(\mathcal{q}') = \mathbb{R}$, $\mathcal{p}'$ is quantizable in the canonically conjugate polarizations $\mathcal{P}'$ and $\mathcal{P}'_c$. Since $\mathcal{p}' = \mathcal{p} = \zeta$ in V_o and $\mathcal{p}' = 0$ outside V, we would argue that the quantization of $\mathcal{p}'$ amounts to a local quantization of $\mathcal{p} = \zeta$. Therefore, we have established unitarily equivalent quantizations of the observable ζ at least locally, in canonically conjugate polarizations.

To sum up:
(i) In case (1) and case (4) situations we quantize ζ in the same way as we did in chapter 2;
(ii) In the case (2) and case (3) situations we first localize ζ in the

effective configuration space with respect to $\mathcal{P}$, and then we quantize the localized observable in suitably chosen canonically conjugate polarizations.

4.3 THE LOCALIZATION OF THE HAMILTONIAN (OF A PARTICLE IN POTENTIAL WELL) IN PHASE SPACE

(4.3.1) Notation

(4.3.1.D1) Definition

Let (M,ω) be the phase space of a classical system and let ζ be an observable of (M,ω). The observable is said to be **localized in the phase space** if the support of ζ is compact in M.

Remarks: (R1) If ζ is an observable that is localized in the phase space then the Hamiltonian vector field X_ζ is complete [cf. Abraham and Marsden (1980), p70].

(R2) Observables that are localized in phase space are of interest in physics because of the physical limitations of the measuring devices: a measuring device has a finite size, and usually measures a finite range. Clearly a measuring device with a finite range would be incapable of measuring an observable whose values go up to infinity.

Consider the example of a Hamiltonian of a particle in the potential well whose values go up to infinity. What we want to construct is a modified Hamiltonian with only a finite range of values. We can achieve this by localizing the Hamiltonian in phase space as follows.

We shall assume the notation and results given in chapter 3.

Let (E',E'') be an open interval in $(\min\{V(q)\},\max\{V(q)\})$ and let $[E'_0,E'_0]$ be a closed interval in (E',E''). Let $\xi(H)$ be a C^∞ function of H which vanishes outside (E',E''), and which equals 1 inside $[E'_0,E''_0]$; then according to Abraham and Marsden (1980) [cf. p81] the function $\xi(H)$ exists

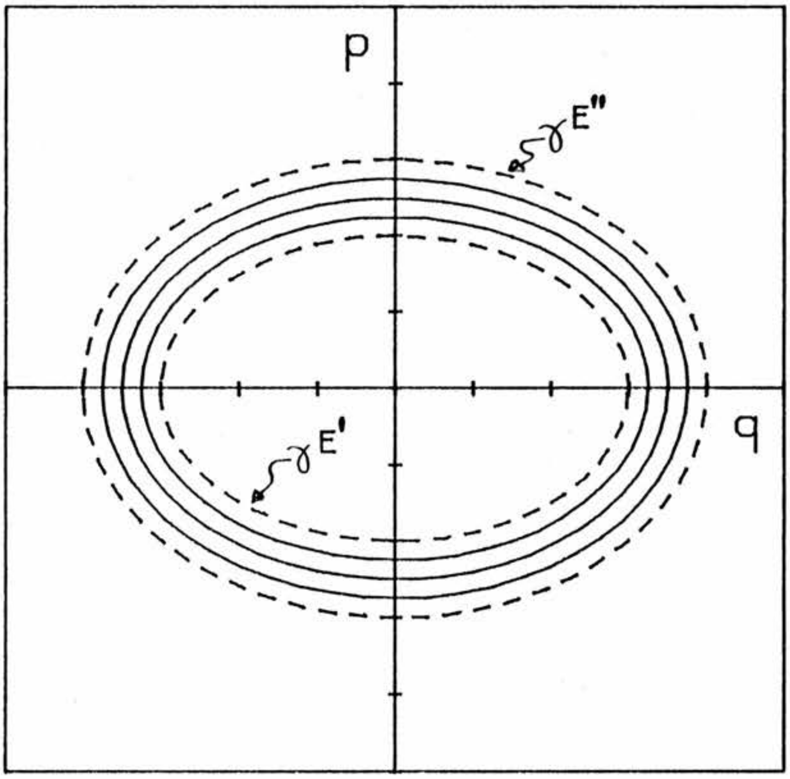

Fig 4-1: The loops are integral curves of X_H. The Hamiltonian H is localized in the region bounded by the integral curves $\gamma^{E'}$ and $\gamma^{E''}$.

even when $[E_o', E_o'']$ is arbitrarily close to (E', E''). The function $\xi(H)$ is referred to as a "bump function".

Let $\zeta = \xi(H)H$. Then ζ is localized in phase space as depicted by Fig 4-1. We therefore call ζ a **localization of** H **in the phase space.** The region

$$\Lambda_o = \{m \in M: E_o' \leqslant H(m) \leqslant E_o''\} \qquad (4.3.1.\text{Eq } 1)$$

is referred to as the **centre of localization.**

We recall that M_o is the region in M given by $M_o = \{\text{closed integral curves of } X_H\} - \{(0,q_o)\}$ where q_o is the point in Q at which V(q) has a minima. Explicitly, we have $M_o = \{m \in M: \min(V) \leqslant H(m) \leqslant \max(V)\} - \{(0,q_o)\}$. We also defined ω_o to be the restriction of the canonical two-form ω to M_o. We introduced action-angle variables (I,θ) on (M_o,ω_o) given by equation (3.1.Eq 3).

<u>Remarks</u>: (R1) By definition the support of ζ, supp(ζ), is contained in M_o: except in the case where $E_o' = \min(V)$, then supp(ζ) is contained in $M_o \cup \{(0,q_o)\}$.

(R2) Let $\chi_{\Lambda_o}(m)$ be a function on M_o which equals 1 inside Λ_o, and vanishes outside Λ_o. Note that $\chi_{\Lambda_o}(m)$ is not a smooth function. Therefore, when $[E_o', E_o'']$ is arbitrarily close to (E', E''); then ζ is a smooth approximation of the function $\chi_{\Lambda_o}(m)H(m)$.

Let ζ_o be the restriction of ζ to M_o. Since H is a function of the action variable I, it follows that ζ_o is also a function of I; so we shall write $\zeta_o = \zeta_o(I)$.

(4.3.2) The quantization of ζ_o

Let us quantize ζ_o in the polarization $\mathcal{P}_c$ spanned by the vector field $(\partial/\partial\theta)$. We recall that quantizations in the polarization $\mathcal{P}_c$ gives rise to BWS conditions [section (1.2) of chapter 1; section (3.3) of chapter 3]. In order to quantize ζ_o we need to quantize the action variable I first. Let $\hat{I}_c$ be the quantum operator corresponding to I in $H(\mathcal{P}_c)$. According to the BWS conditions the set of allowed values of I is

$$R(\hat{I}_c) = \{I(n): I(n) = n\hbar - c,\ n \in \mathbb{Z} \text{ and } \min(V) \leqslant H(I(n)) \leqslant \max(V)\}. \tag{4.3.2.Eq 1}$$

(Here $R(\hat{I}_c)$ is the spectrum of $\hat{I}_c$ given by equation (3.3.Eq 3).) According to section (2.3) of chapter 2 the quantum Hilbert space $H(\mathcal{P}_c)$ is identifiable with $L^2_c(\mathbb{R},\mu)$ where μ is the discrete measure with support $R(\hat{I}_c)$ defined by

$$\mu(\{I(n)\}) = 1 \text{ for each } I(n) \in R(\hat{I}_c). \tag{4.3.2.Eq 2}$$

Let $\hat{\zeta}_{oc}$ be the quantum operator (corresponding to ζ_o) in $H(\mathcal{P}_c)$; then $\hat{\zeta}_{oc}$ is the multiplication operator ζ_o in $H(\mathcal{P}_c)$, and the spectrum $R(\hat{\zeta}_{oc})$ of $\hat{\zeta}_{oc}$ is given by

$$R(\hat{\zeta}_{oc}) = \{\zeta_o^n = \zeta_o(I(n)): I(n) = n\hbar - c,\ n \in \mathbb{Z} \text{ and } E' \leqslant H(I(n)) \leqslant E''\}. \tag{4.3.2.Eq 3}$$

We can also write down the normalized eigenfunctions $\phi_{cn}(I)$ of $\hat{\zeta}_{oc}$ in exactly the same way as we did for $\hat{I}_c$[cf. equations (2.3.6.Eq 4a) and (2.3.6.Eq 4b), section (2.3)] as follows. For each $n \in \{n \in \mathbb{Z}: E' \leqslant H(I(n)) \leqslant E''\}$, the normalized eigenfunctions $\phi_{cn}(I)$ (corresponding to the eigenvalue ζ_o^n) of $\hat{\zeta}_{oc}$ are given by

$$\begin{aligned} \phi_{cn}(I) &= 1 \text{ when } I = I(n). \\ &= 0 \text{ when } I = I(n') \neq I(n),\ n' \in \mathbb{Z}. \end{aligned} \tag{4.3.2.Eq 4}$$

<u>Remark</u>: (R1) Note that $\zeta_o^n = \zeta_o(I(n))$ is only a eigenvalue of $\hat{\zeta}_{oc}$ if $n \in \{n \in \mathbb{Z}: E' \leqslant H(I(n)) \leqslant E''\}$.

(R2) $H(\mathcal{P}_c)$ is the Hilbert space spanned by the eigenfunctions $\psi_{cn}(I)$ of the quantum operator $\hat{I}_c$ which are given by

$$\psi_{cn}(I) = 1 \text{ when } I = I(n).$$

$$= 0 \text{ when } I = I(n') \neq I(n), \quad n' \in \mathbb{Z}. \qquad (4.3.2.\text{Eq } 5)$$

Clearly the eigenfunctions $\psi_{cn}(I)$ and $\phi_{cn}(I)$ are identical for $n \in \{n \in \mathbb{Z}: E' \leqslant H(I(n) \leqslant E''\}$.

Since ζ is not a simple observable we are unable to construct a unique quantum operator corresponding to ζ in $L^2(Q)$ using the canonical quantization procedure. However, we can employ the modified Maslov-WKB method outlined in section (3.4) of chapter 3 to obtain an approximate pairing between $H(\mathcal{P}_c)$ and $L^2(Q)$ as follows. By equation (3.3.Eq 4) the discrete part of the spectrum of $\hat{H}$ predicted by the BWS conditions is

$$R_{\mathcal{D}}(\hat{H}) = \{E(n) = H(I(n)): n \in \mathbb{Z} \text{ and } E(n) \in (\min(V), \max(V))\}. \qquad (4.3.\text{Eq } 6)$$

For each $E(n) \in R_{\mathcal{D}}(H)$, we constructed the modified Maslov-WKB wave function $\Psi_n(q)$ given by equation (3.4.Eq 5); $\Psi_n(q)$ is an approximate eigenfunction of $\hat{H}$ corresponding to the eigenvalue $E(n)$ which has been determined by the BWS conditions. We can use these eigenfunctions to construct an approximate pairing in accordance with the method spelled out in section (3.5) of chapter 3. In other words, the operator $\hat{\zeta}_{oc}$ in $H(\mathcal{P}_c)$ is mapped to an operator $\hat{\zeta}$ in $L^2(Q)$ determined, albeit approximately, by requiring $\hat{\zeta}$ to possess eigenvalues ζ_o^n with eigenfunctions $\Psi_n(q)$. (Here n belongs to the set $\{n \in \mathbb{Z}: E' \leqslant H(I(n) \leqslant E''\}$.) Then $R(\hat{\zeta})$ the spectrum of $\hat{\zeta}$ is given by $R(\hat{\zeta}) = R(\hat{\zeta}_{oc})$.

<u>Remark</u>: (R2) According to the BWS conditions the set of eigenvalues that the Hamiltonian operator $\hat{H}$ and the quantum operator $\hat{\zeta}$ have in common is given by

$$R(\hat{\zeta}) \cap R_{D}(\hat{H}) = \{\zeta_{o}^{n} = E(n) : E(n) = H(I(n)) \text{ and } E_{o}' \leqslant H(I(n)) \leqslant E_{o}''\}.$$

(4.3.Eq 7)

4.4 THE MODIFIED MASLOV-WKB METHOD FOR CERTAIN MULTILINEAR MOMENTUM OBSERVABLES WITH CLOSED INTEGRAL CURVES

(4.4.1) Notation

Let Q be an open interval in $\mathbb{R}$ with cartesian coordinate q and let $M = T^{*}Q$. Let ω be the canonical two-form on M and let (p,q) be the usual cartesian canonical coordinates on M.

Let ζ be an observable on (M,ω) that satisfies the following conditions:

(MMO1) ζ is given by

$$\zeta = \sum_{k=1}^{\infty} \xi_k(q)p^k \qquad (4.4.1.\text{Eq 1a})$$

where $\xi_k(q)$ are analytic functions of q. The function ζ is referred to as a **multilinear momentum observable** [cf. McFarlane and Wan (1981b)]. In addition, we shall assume that ζ can also be written in the form

$$\zeta = \sum_{k=1}^{\infty} \eta_k(p)q^k \qquad (4.4.1.\text{Eq 1b})$$

where $\eta_k(p)$ are analytic functions of p.

(MMO2) X_ζ is a complete Hamiltonian vector field with a single critical point at (0,0) and closed integral curves.

(MMO3) The observable ζ never takes the same value on two different integral curves of X_ζ.

(MMO4) Each integral curve γ^E, where E is the value of ζ on any point on the curve, has exactly two stationary points with respect to q and exactly two stationary points with respect to p. Then let m_0, m_1, m_2 and m_3 be points on γ^E such that [cf. Fig 4-2]

$$(\partial\zeta/\partial q) = 0 \text{ at } m_0 = (p_0,q_0) \text{ and } m_2 = (p_2,q_2), \qquad (4.4.1.\text{Eq 2b})$$

$$(\partial\zeta/\partial p) = 0 \text{ at } m_1 = (p_1,q_1) \text{ and } m_3 = (p_3,q_3). \qquad (4.4.1.\text{Eq 2b})$$

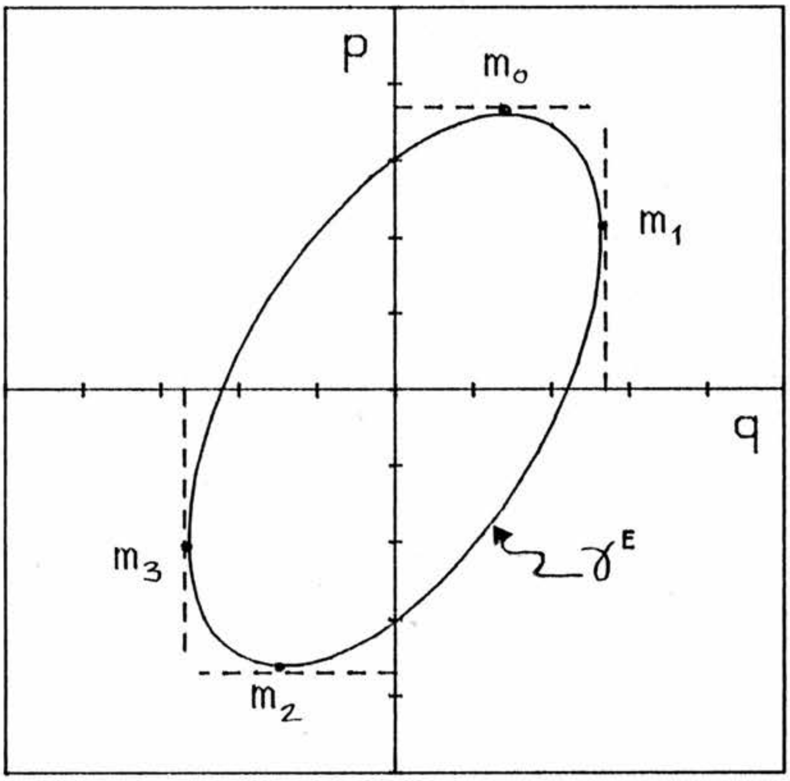

Fig 4-2: The loop is the integral curve of X_ζ corresponding to the value E of ζ. ζ has exactly two stationary points with respect to q on γ^E. ζ has exactly two stationary with respect to p on γ^E. We have

$$\partial\zeta/\partial q = 0 \text{ at } m_0 \text{ and } m_2;$$

$$\partial\zeta/\partial p = 0 \text{ at } m_1 \text{ and } m_3.$$

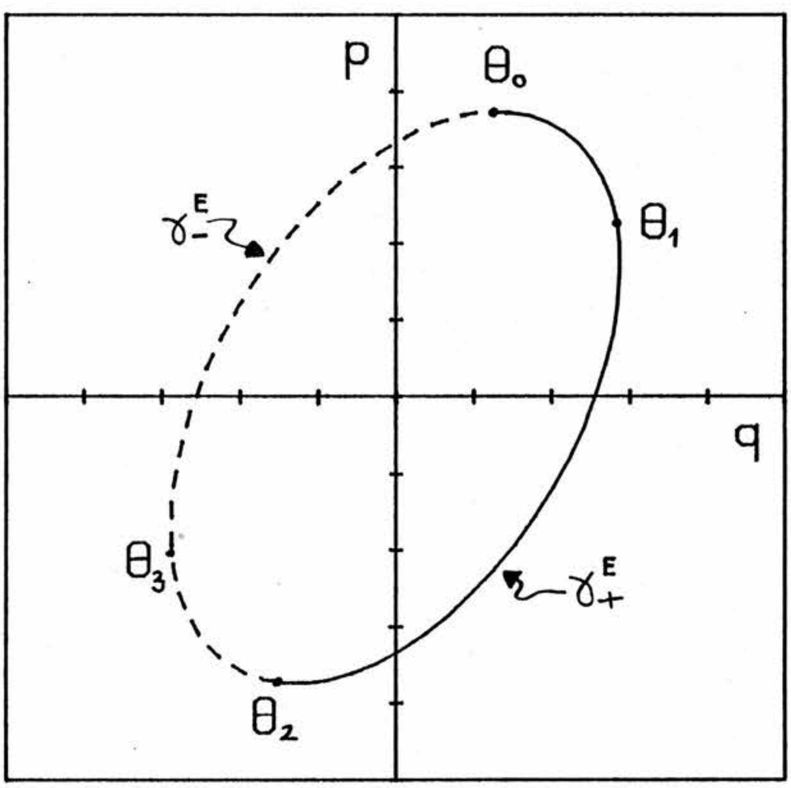

Fig 4-3a: The arcs γ_+^E and γ_-^E are depicted.

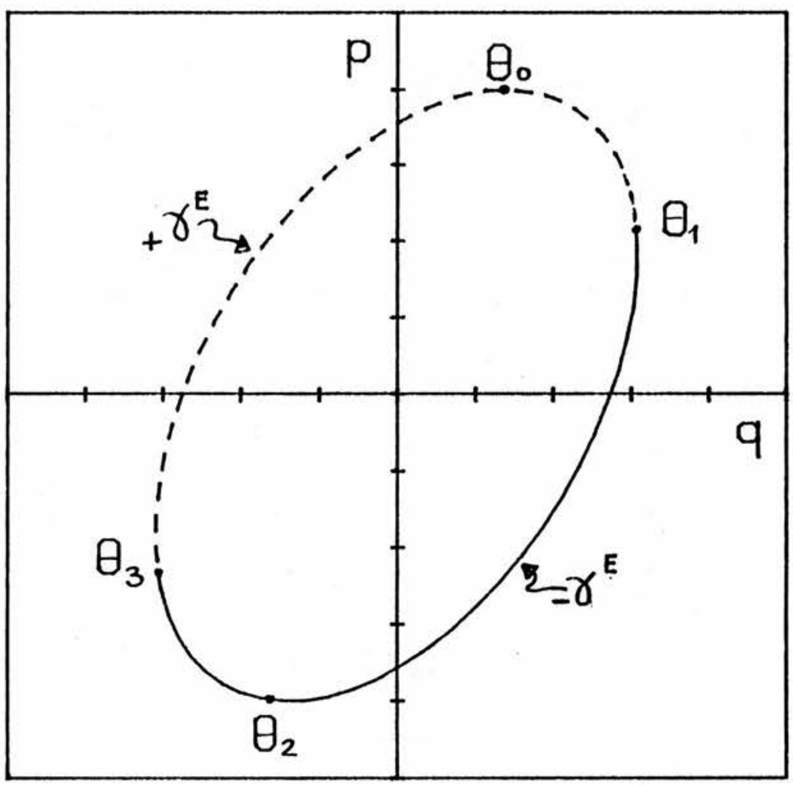

Fig 4-3b: The arcs and are depicted.

As before we let $M_o = M - \{(0,0)\}$ and let ζ_o be the restriction of ζ to M_o. Let $\gamma^{\epsilon}(t)$ be the integral curve of X_ζ that originates at the point m_o and let $T(E)$ be the period of $\gamma^{\epsilon}(t)$. Let (I,θ) be action-angle variables on (M_o,ω_o) defined by

$$I = \oint_{\gamma^\epsilon} pdq, \ \theta = (2\pi t/T(E)). \qquad (4.4.1.\text{Eq } 3)$$

Since ζ_o is only dependent on the action variable I we shall write $\zeta_o = \zeta_o(I)$.

Let $\gamma^{\epsilon}(\theta) = (p(\theta),q(\theta))$ be the integral curve of X_{ζ_o} that originates at m_o and parameterized by θ instead of t. Then $p(\theta)$ and $q(\theta)$ satisfy the following differential equations:

$$(\partial p(\theta)/\partial\theta) = -[T(E)/2\pi](\partial\zeta(p(\theta),q(\theta))/\partial q) \qquad (4.4.1.\text{Eq } 4\text{a})$$

$$(\partial q(\theta)/\partial\theta) = [T(E)/2\pi](\partial\zeta(p(\theta),q(\theta))/\partial p) \qquad (4.4.1.\text{Eq } 4\text{b})$$

with constant of motion $\zeta(p(\theta),q(\theta)) = E$.

Let $\theta_o,\theta_1,\theta_2,\theta_3 \in [0,2\pi)$ such that θ_o, θ_1, θ_2 and θ_3 correspond to m_o, m_1, m_2 and m_3 respectively, i.e.

$$(\partial p(\theta)/\partial\theta) = 0 \text{ at } \theta = \theta_o,\theta_2; \qquad (4.4.1.\text{Eq } 5\text{a})$$

$$(\partial q(\theta)/\partial\theta) = 0 \text{ at } \theta = \theta_1,\theta_3. \qquad (4.4.1.\text{Eq } 5\text{b})$$

Remark: (R1) The equations (4.4.1.Eq 5a) and (4.4.1.Eq 5b) are formally equivalent to equations (1.3.1.Eq 9a) and (1.3.1.Eq 9b). This formal equivalence will prove important when we consider the modified Maslov-WKB method for ζ.

Here is a list of additional notation we shall require when we consider the modified Maslov-WKB method: Let ${}_{+}\gamma^{E}$, ${}_{-}\gamma^{E}$, γ_{+}^{E} and γ_{-}^{E} be the arcs on the integral curve γ^{E} given by [cf. Fig 4-3a and Fig 4-3b]

$${}_{+}\gamma^{E} = \{(p(\theta),q(\theta)) \in \gamma^{E} : \theta \in [2\pi-\theta_3,\theta_1]\}; \qquad (4.4.1.\text{Eq } 6a)$$

$${}_{-}\gamma^{E} = \{(p(\theta),q(\theta)) \in \gamma^{E} : \theta \in [\theta_1,\theta_3]\}; \qquad (4.4.1.\text{Eq } 6b)$$

$$\gamma_{+}^{E} = \{(p(\theta),q(\theta)) \in \gamma^{E} : \theta \in [\theta_0,\theta_2]\}; \qquad (4.4.1.\text{Eq } 6c)$$

$$\gamma_{-}^{E} = \{(p(\theta),q(\theta)) \in \gamma^{E} : 0 \in [\theta_2,2\pi]\}; \qquad (4.4.1.\text{Eq } 6d)$$

Let

$${}_{+}S^{E}(q) = \int_{m_0}^{m} pdq \text{ on } {}_{+}\gamma^{E} \text{ with } m \in {}_{+}\gamma^{E}; \qquad (4.4.1.\text{Eq } 7a)$$

$${}_{-}S^{E}(q) = \int_{m_0}^{m} pdq \text{ on } {}_{-}\gamma^{E} \text{ with } m \in {}_{-}\gamma^{E}; \qquad (4.4.1.\text{Eq } 7b)$$

$$W_{+}^{E}(p) = -\int_{m_0}^{m} qdp \text{ on } \gamma_{+}^{E} \text{ with } m \in \gamma_{+}^{E}; \qquad (4.4.1.\text{Eq } 7c)$$

$$W_{-}^{E}(p) = -\int_{m_0}^{m} qdp \text{ on } \gamma_{-}^{E} \text{ with } m \in \gamma_{-}^{E}; \qquad (4.4.1.\text{Eq } 7d)$$

(here all the integrals are along γ^{E}).

Let $\beta = pdq + cd\theta$ be a one-form on M_0 where $c \in \mathbb{R}$. Let $(B,(\cdot,\cdot),\nabla)$ be the chosen prequantization bundle on M_0 defined in the usual way such that the connection ∇ is determined by the connection potential β.

(4.4.2) Quantization of ζ

We shall split this subsection in to two parts which we shall denote by (a) and (b) respectively: in part (a) we shall quantize ζ in the polarization $\mathcal{P}_c$ spanned by the vector field $(\partial/\partial\theta)$, in part (b) we shall give the modified Maslov-WKB method for multilinear momentum observables like ζ.

(a) Quantization of ζ_o in the polarization $\mathcal{P}_c$

Let us quantize ζ_o in the polarization $\mathcal{P}_c$ spanned by $(\partial/\partial\theta)$. Let $\hat{I}_c$ be the quantum operator corresponding to I in $H(\mathcal{P}_c)$ and let R(I) be the classical range of I on (M_o,ω_o). According to the BWS conditions the spectrum of $\hat{I}$, $R(\hat{I}_c)$, is given by

$$R(\hat{I}_c) = \{I(n) = n\hbar - c : n \in \mathbb{Z},\ \ I(n) \in R(I)\}. \qquad (4.4.2.\text{Eq } 1)$$

The Hilbert space $\mathbf{H}(\mathcal{P}_c)$ is identifiable with $L^2_c(\mathbb{R},\mu)$ where μ is a discrete measure with support $R(\hat{I}_c)$ and such that

$$\mu(\{I(n)\} = 1 \text{ for each } I(n) \in R(\hat{I}_c). \qquad (4.4.2.\text{Eq } 2)$$

Let $\hat{\zeta}_{oc}$ be the quantum operator (corresponding to ζ_o) in $\mathbf{H}(\mathcal{P}_c)$; then $\hat{\zeta}_{oc}$ is the multiplication operator ζ_o in $H(\mathcal{P}_c)$. The spectrum $R(\hat{\zeta}_{oc})$ of is given by

$$R(\hat{\zeta}_{oc}) = \{E(n) = \zeta_o(I(n)) : n \in \mathbb{Z} \text{ and } E(n) \in R(\zeta_o)\} \qquad (4.4.2.\text{Eq } 3)$$

where $R(\zeta_o)$ is the classical range of ζ_o.

(b) The modified Maslov-WKB method

Let $L^2(Q)$ and $F^{-1}L^2(Q)$ be the position and momentum representations given in section (3.2) of chapter 3. Let $U : F^{-1}L^2(Q) \dashrightarrow L\ (Q)$ be the unitary map given by equation (3.2.Eq 2a). By equations (4.4.1.Eq 1a) and (4.4.1.Eq 2a) we can write ζ in the forms

$$\zeta = \sum_{k=0}^{\infty} \xi_k(q)p^k \qquad (4.4.2.\text{Eq } 4a)$$

or

$$\zeta = \sum_{k=0}^{\infty} \eta_k(p)q^k. \qquad (4.4.2.\text{Eq } 4b)$$

According to the canonical quantization scheme a general expression for the quantum operator $\hat{\zeta}$ (corresponding to the classical observable ζ) in $L^2(Q)$ is given by

$$\hat{\zeta} = \sum_{k=1}^{\infty} (-i\hbar)^k \sum_{j=0}^{k} D^j{}_k(q)(\partial^j/\partial q^j) + \xi_0(q) \qquad (4.4.2.\text{Eq } 5)$$

where $D^j{}_k(q)$ are smooth functions of q not involving $\hbar$ [cf. McFarlane (1980)]. Since $\hat{\zeta}$ should be formally self-adjoint, we require $D^k{}_k$ and $D^k{}_{k-1}$ to be given by

$$D^k{}_k = \xi_k(q) \text{ and } D^k{}_{k-1} = (1/2)k(\partial\xi_k/\partial q) \qquad (4.4.2.\text{Eq } 6)$$

respectively [cf. Woodhouse (1980), p80]. Hence the operator $\hat{\zeta}$ in $L^2(Q)$ is partially determined by the expression

$$\hat{\zeta}\varphi(q) = [\sum_{k=1}^{\infty} (-i\hbar)^k \{\xi_k(q)(\partial^k/\partial q^k)+(k/2)(\partial\xi_k/\partial q)(\partial^{k-1}/\partial q^{k-1})\}]\varphi(q)+\xi_0(q)\varphi(q)$$

$$(4.4.2.\text{Eq } 7)$$

where the terms omitted are undetermined.

Similarly, let $\hat{\zeta}_c$ be the quantum operator (corresponding to ζ) in the momentum representation $F^{-1}L^2(Q)$; then $\hat{\zeta}_c$ is partially determined by the expression

$$\hat{\zeta}_c\psi_c(p) = [\sum_{k=1}^{\infty} (i\hbar)^k \{\eta_k(p)(\partial^k/\partial p^k)+(k/2)(\partial\eta_k/\partial p)(\partial^{k-1}/\partial p^{k-1})\}]\psi_c(p)+\eta_k(p)\psi_c(p).$$

$$(4.4.2.\text{Eq } 8)$$

Let $E \in R(\zeta)$ where $R(\zeta)$ is the classical range of ζ. Then according to Appendix 4.2 the WKB solutions of $\hat{\zeta}$ corresponding to $\zeta = E$ is given by

$$\phi^E_W(q) = {}_+K|(\partial q/\partial\theta)|_+^{-1/2}\{\exp \tfrac{i}{\hbar}{}_+S^E(q)\} + {}_-K|(\partial q/\partial\theta)|_-^{-1/2}\{\exp \tfrac{i}{\hbar}{}_-S^E(q)\}$$

$$(4.4.2.\text{Eq } 9\text{a})$$

where ${}_+K$ and ${}_-K$ are constants, and ${}_+S^E(q)$ and ${}_-S^E(q)$ are given by equations (4.4.1.Eq 7a) and (4.4.1.Eq 7b) respectively, and

$$|(\partial q/\partial\theta)|_\pm^{-1/2} = [T(E)/2\pi]\,|\sum_{k=1}^{\infty} k\xi_k(\partial_\pm S^E/\partial q)^{k-1}|^{-1/2}. \qquad (4.4.2.\text{Eq } 9\text{b})$$

In Appendix 4.2 we also show that the WKB-solutions of $\hat{\zeta}_c$ corresponding to $\zeta = E$ is given by

$$\phi_{c,w}(p) = K_+ |(\partial p/\partial\theta)|_+^{-1/2} \{\exp \tfrac{i}{\hbar} W_+^{\epsilon}(p)\} + K_- |(\partial p/\partial\theta)|_-^{-1/2} \{\exp \tfrac{i}{\hbar} W_-^{\epsilon}(p)\}$$

(4.4.2.Eq 10a)

where K_+ and K_- are constants, $W_+^{\epsilon}(p)$ and $W_-^{\epsilon}(p)$ are given by equations (4.4.1.Eq 7c) and (4.4.1.Eq 7d), and

$$|(\partial p/\partial\theta)|^{-1/2} = [T(E)/2\pi] \left| \sum_{K=1}^{\infty} k\eta_K (-\partial W/\partial p)^{K-1} \right|^{-1/2}. \qquad (4.4.2.\text{Eq } 10b)$$

If we compare the WKB solutions $\phi_W^{\epsilon}(q)$ and $\phi_{c,W}^{\epsilon}(p)$ with the corresponding expressions for the WKB-solutions of $\hat{H}$ and $\hat{H}_c$, respectively, (the Hamiltonian operators of a particle in a potential well) given by equations (1.3.1.Eq 6) and (1.3.1.Eq 12) respectively we see that they are formally identical. In particular, the conditions given by (MM04) and equations (4.4.1.Eq 5a) and (4.4.1.Eq 5b) are equivalent to the conditions given by equations (1.3.1.Eq 9a) and (1.3.1.Eq 9b). This means that the results obtained by the Maslov-WKB method in section (1.3) (of chapter 1) and the results obtained by the modified Maslov-WKB method in chapter 3 remain applicable to the multilinear momentum observable ζ. So we shall assume the results in the above mentioned sections. Having constructed the modified Maslov-WKB wave functions $\Psi_n(q)$ corresponding to values in $R(\hat{\zeta}_{oc})$ we can proceed to establish a pairing between $H(\mathcal{P}_c)$ and $H(P)$ (the quantization Hilbert space associated with the vertical polarization P on T^*Q) as we did in section (4.3). The validity of the construction is justified by the following theorem whose proof is given in Appendix 4.3.

(4.4.2.T1) <u>Theorem</u>

We have

$$\|(\hat{\zeta}-E(n))\,\Psi_n(q)\| = O(\hbar^2). \qquad (4.4.2.\text{Eq } 11)$$

where $\hat{\zeta}$ is the operator given by equation (4.4.1.Eq 7), $E(n)$ belongs to $R(\hat{\zeta}_{oc})$ and $\Psi_n(q)$ is the modified Maslov-WKB wave function (corresponding to the value E(n)) defined by equation (3.4.Eq 5).

Remark (R1) The multilinear momentum observable ζ could be localized in phase space using the procedure outlined in section (4.3).

APPENDICES A4.1-A4.3

APPENDIX 4.1

We have

$$\hat{\zeta}\varphi(q) = [\sum_{k=1}^{\infty}(-i\hbar)^k\{\xi_k(q)(\partial^k/\partial q^k)+(k/2)(\partial\xi_k/\partial q)(\partial^{k-1}/\partial q^{k-1})\}]\varphi(q)+\xi_0(q)\varphi(q)$$

(A4.1.Eq 1)

$$\hat{\zeta}_c\psi_c(p) = [\sum_{k=1}^{\infty}(i\hbar)^k\{\eta_k(p)(\partial^k/\partial p^k)+(k/2)(\partial\eta_k/\partial p)(\partial^{k-1}/\partial p^{k-1})\}]\psi_c(p)+\eta_0(p)\psi_c(p).$$

(A4.1.Eq 2)

$$(\partial^k\{\exp \pm G(y)\}/\partial y^k) = [\sum\{k!/a!b!\ldots c!\}(\pm)^r\{\exp \pm G(y)\}\{\partial G/\partial y\}^a \times \{(\partial^2G/\partial y^2)/2!\}^b\ldots\{(\partial^t G/\partial y^t)/t!\}^c. \qquad \text{(A4.1.Eq 3)}$$

(Here the symbol $\sum$ indicates the summation over all solutions in non-negative integers of the equations

$$a+2b+\ldots+tc = k \text{ and } a+b+\ldots+c = r.)$$

Equation (A4.1.Eq 3) is obtained by using the formula for the k-th derivative of a composite function which is given by equation (A1.5.Eq 2) [cf. Appendix 1.5].

(A4.1.T1) Theorem

We have:

$$\text{(i) } \hat{\zeta}[f(q)\{\exp \pm S(q)\}] = [\sum_{k=0}^{\infty}\xi_k(\partial S/\partial q)^k][f\{\exp \pm S\}]$$
$$+(-i\hbar)[\sum_{k=1}^{\infty} k\{(1/2)(k-1)\xi_k(\partial S/\partial q)^{k-2}(\partial^2 S/\partial q^2)f+\xi_k(\partial S/\partial q)^{k-1}(\partial f/\partial q)$$
$$+(1/2)(\partial\xi_k/\partial q)(\partial S/\partial q)^{k-1}f\}][\exp \pm S\} + \text{H.O.T in } \hbar. \qquad \text{(A4.1.Eq 4)}$$

$$\text{(ii) } \hat{\zeta}_c[g(p)\{\exp \pm W\}] = [\sum_{k=0}^{\infty}\eta_k(-\partial W/\partial p)^k][g\{\exp \pm W\}]$$
$$+(-i\hbar)[\sum_{k=1}^{\infty} k\{(1/2)(k-1)\eta_k(-\partial W/\partial p)^{k-2}(\partial^2 W/\partial p^2)g-\eta_k(-\partial W/\partial p)^{k-1}(\partial g/\partial p)$$
$$-(1/2)(\partial\eta_k/\partial p)(-\partial W/\partial p)^{k-1}g][\exp \pm W\} + \text{H.O.T in } \hbar. \qquad \text{(A4.1.Eq 5)}$$

Proof:

(i) We have

$(-i\hbar)^K[(\partial^K/\partial q^K)f(q)\{\exp \tfrac{i}{\hbar}S(q)\}]$

$= (-i\hbar)^K[f(\partial^K\{\exp iS\}/\partial q^K)+k(\partial f/\partial q)(\partial^{K-1}\{\exp \tfrac{i}{\hbar}S\}/\partial q^{K-1})+\ldots$

$+(\partial^K f/\partial q)\{\exp \tfrac{i}{\hbar}S\}]$ (by equation (A1.5.Eq 1) of Appendix 1.5)

$= [(\partial S/\partial q)^K][f\{\exp \tfrac{i}{\hbar}S\}]-i\hbar[(1/2)k(k-1)(\partial S/\partial q)^{K-2}(\partial^2 S/\partial q^2)f+\ldots$

$+k(\partial S/\partial q)^{K-1}(\partial f/\partial q)]\{\exp iS\}$ + H.O.T in $\hbar$ (A4.1.Eq 6)

(by equation (A4.1.Eq 3)).

Similarly, we have

$(-i\hbar)^K(\partial^{K-1}/\partial q^{K-1})[f(q)\{\exp iS\}]$

$= -i\hbar(\partial S/\partial q)^{K-1}[f\{\exp \tfrac{i}{\hbar}S\}]$ + H.O.T in $\hbar$ (A4.1.Eq 7)

Then by equations (A4.1.Eq 1), (A4.1.Eq 6) and (A4.1.Eq 7) we get

$\hat{\xi}[f(q)\{\exp iS(q)\}] = [\sum_{K=0}^{\infty}\xi_K (\partial S/\partial q)^K][f\{\exp \tfrac{i}{\hbar}S\}]$

$+(-i\hbar)[\sum_{K=1}^{\infty} k\{(1/2)(k-1)\xi_K(\partial S/\partial q)^{K-1}(\partial^2 S/\partial q^2)f+\xi_K(\partial S/\partial q)^{K-1}(\partial f/\partial q)$

$+(1/2)(\partial\xi_K/\partial q)(\partial S/\partial q)^{K-1} f\}]\{\exp \tfrac{i}{\hbar}S\}$ + H.O.T in $\hbar$.

(ii) We have

$(i\hbar)^K[(\partial^K/\partial p^K)g(p)\{\exp \tfrac{i}{\hbar}W(p)\}]$

$= (i\hbar)^K[g(\partial^K\{\exp \tfrac{i}{\hbar}W\}/\partial p^K)+k(\partial g/\partial p)(\partial^{K-1}\{\exp \tfrac{i}{\hbar}W\}/\partial p^{K-1})+\ldots$

$+(\partial^K g/\partial p^K)\{\exp \tfrac{i}{\hbar}W\}]$ (by equation (A1.5.Eq 1) of Appendix 1.5)

$= [(-\partial W/\partial p)\ ][g\{\exp \tfrac{i}{\hbar}W\}]-i\hbar[(1/2)k(k-1)(-\partial W/\partial p)^{K-2}(\partial^2 W/\partial p^2)g+\ldots$

$-k(-\partial W/\partial p)^{K-1}(\partial g/\partial p)]\{\exp \tfrac{i}{\hbar}W\}$ + H.O.T in $\hbar$ (A4.1.Eq 8)

Similarly, we have

$(i\hbar)^K(\partial^{K-1}/\partial p^{K-1})[g(p)\{\exp \tfrac{i}{\hbar}W(p)\}]$

$= \quad i\hbar \quad (-\partial W/\partial p)^{K-1}[g\{\exp \tfrac{i}{\hbar}W\}]$ + H.O.T in $\hbar$. (A4.1.Eq 9)

Then by equations (A4.1.Eq 2), (A4.1.Eq 8) and (A4.1.Eq 9), we get

$$\hat{\xi}_c[g(p)\{\exp \pm W\}]$$
$$= [\sum_{k=0}^{\infty} \eta_k(-\partial W/\partial p)^k][g\{\exp \pm W\}]$$
$$+(-i\hbar)[\sum_{k=1}^{\infty} k\{(1/2)(k-1)\eta_k(-\partial W/\partial p)^{k-2}(\partial^2 W/\partial p^2)g-\eta_k(-\partial W/\partial p)^{k-1}(\partial g/\partial p)$$
$$-(1/2)(\partial\eta_k/\partial p)(-\partial W/\partial p)^{k-1}g]\{\exp \pm W\} + \text{H.O.T in } \hbar. \blacksquare$$

(A4.1.T2) <u>Theorem</u>

The equation

$$[\sum_{k=1}^{\infty}\{(1/2)k\ (k-1)\xi_k(\partial S/\partial q)^{k-2}(\partial^2 S/\partial q^2)f+k\xi_k(\partial S/\partial q)^{k-1}(\partial f/\partial q)$$
$$+(1/2)k(\partial\xi_k/\partial q)(\partial S/\partial q)^{k-1}f\} = 0 \qquad \text{(A4.1.Eq 10a)}$$

has as solution

$$f(q) = K|\sum_{k=1}^{\infty} k\xi_k(q)(\partial S/\partial q)^{k-1}|^{-1/2}$$
$$= K'|(\partial q/\partial\theta)|^{-1/2} \qquad \text{(A4.1.Eq 10b)}$$

where K and K' are constants, and

$$(\partial q/\partial\theta) = [T(E)/2\pi][\sum_{k=1}^{\infty} k\xi_k(\partial S/\partial q)^{k-1}] \qquad \text{(A4.1.Eq 10c)}$$

(In the equation (A4.1.Eq 10c) we have replaced p by $(\partial S/\partial q)$ [cf. equation (4.4.1.Eq 4b).)

<u>Proof</u>:

We shall split the proof into two cases according to whether $[\sum_{r=1}^{\infty} r\xi_r(\partial S/\partial q)^{r-1}]$ is either positive or negative.

<u>Case 1:</u> $[\sum_{r=1}^{\infty} r\xi_r(\partial S/\partial q)^{r-1}]>0$

In this case we have

$$f(q) = K[\sum_{r=1}^{\infty} r\xi_r(\partial S/\partial q)^{r-1}]^{-1/2} \qquad \text{(A4.1.Eq 11a)}$$

$$(\partial f/\partial q) = K^{-2}(-1/2)f^3[\sum_{r=1}^{\infty} r(\partial\xi_r/\partial q)(\partial S/\partial q)^{r-1}]$$
$$-K^{-2}f^3[\sum_{j=1}^{\infty}(1/2)j(j-1)\xi_j(\partial S/\partial q)^{j-2}(\partial^2 S/\partial q^2)]$$
$$\text{(A4.1.Eq 11b)}$$

and

$$[\sum_{k=1}^{\infty} k\xi_k(\partial S/\partial q)^{k-1}(\partial f/\partial q)]$$
$$= K^2 f^{-2}(\partial f/\partial q)$$
$$= -(1/2)f\ [\sum_{r=1}^{\infty} r(\partial\xi_r/\partial q)(\partial S/\partial q)^{r-1}]$$
$$-f\ [\sum_{j=1}^{\infty}(1/2)j(j-1)\xi_j(\partial S/\partial q)^{j-2}(\partial^2 S/\partial q^2)]$$

(A4.1.Eq 11c)

Then evaluating the left hand side of equation (A4.1.Eq 10a) we get

$$(1/2)f[\sum_{j=1}^{\infty} j(j-1)\xi_j(\partial S/\partial q)^{j-2}(\partial^2 S/\partial q^2)]+[\sum_{k=1}^{\infty} k\xi_k(\partial S/\partial q)^{k-1}(\partial f/\partial q)]$$
$$+(1/2)f[\sum_{r=1}^{\infty} r(\partial\xi_r/\partial q)(\partial S/\partial q)^{r-1}]$$

= 0 (by equation (A4.1.Eq 11c)).

<u>Case 2:</u> $[\sum_{r=1}^{\infty} r\xi_r(\partial S/\partial q)^{r-1}]<0$

The proof for this case is similar to that in case 1. ■

(A4.1.T3) <u>Theorem</u>

The equation

$$[\sum_{k=1}^{\infty}\{(1/2)k(k-1)\eta_k(-\partial W/\partial p)^{k-2}(\partial^2 W/\partial p^2)g-k\eta_k(-\partial W/\partial p)^{k-1}(\partial g/\partial p)$$
$$-(1/2)k(\partial\eta_k/\partial p)(-\partial W/\partial p)^{k-1}g\}] = 0 \quad \text{(A4.1.Eq 12a)}$$

has as solution

$$g(p) = K|\sum_{k=}^{\infty} k\eta_k(p)(-\partial W/\partial p)^{k-1}|^{-1/2}$$
$$= K'|(\partial p/\partial\theta)|^{-1/2} \quad \text{(A4.1.Eq 12b)}$$

where K and K' are constants, and

$$(\partial p/\partial\theta) = -[T(E)/2\pi][\sum_{k=1}^{\infty} k\eta_k(-\partial W/\partial p)^{k-1}] \quad \text{(A4.1.Eq 12c)}$$

(In the equation (A4.1.Eq 12c) we have replaced q by (- W/ p) [cf. equation (4.4.1.Eq 4a).)

Proof:

We shall split the proof into two cases according to whether $[\sum_{r=1}^{\infty} r\eta_r(-\partial W/\partial p)^{r-1}]$ is either positive or negative.

Case 1: $[\sum_{r=1}^{\infty} r\eta_r(-\partial W/\partial p)^{r-1}]>0$

In this case we have

$$g(p) = K[\sum_{r=1}^{\infty} r\eta_r(-\partial W/\partial p)^{r-1}]^{-1/2} \qquad \text{(A4.1.Eq 13a)}$$

$$(\partial g/\partial p) = K^{-2}(-1/2)g^3[\sum_{r=1}^{\infty} r(\partial\eta_r/\partial p)(-\partial W/\partial p)^{r-1}]$$
$$+K^{-2}g^3[\sum_{j=1}^{\infty} (1/2)j(j-1)\eta_j(-\partial W/\partial p)^{j-2}(\partial^2 W/\partial p^2)] \qquad \text{(A4.1.Eq 13b)}$$

and

$$-[\sum_{k=1}^{\infty} k\eta_k(-\partial W/\partial p)^{k-1}(\partial g/\partial p)]$$
$$= -K^2 g^{-2}(\partial g/\partial p)$$
$$= (1/2)g[\sum_{r=1}^{\infty} r(\partial\eta_r/\partial p)(-\partial W/\partial p)^{r-1}]$$
$$-g[\sum_{j=1}^{\infty} (1/2)j(j-1)\eta_j(-\partial W/\partial p)^{j-2}(\partial^2 W/\partial p^2)] \qquad \text{(A4.1.Eq 13c)}$$

Then evaluating the left hand side of equation (A4.1.Eq 12a), we get

$$(1/2)g[\sum_{j=1}^{\infty} j(j-1)\eta_j(-\partial W/\partial p)^{j-2}(\partial^2 W/\partial p^2)]-[\sum_{k=1}^{\infty} k\eta_k(-\partial W/\partial p)^{k-1}(\partial g/\partial p)]$$
$$-(1/2)g[\sum_{r=1}^{\infty} r(\partial\eta_r/\partial p)(-\partial W/\partial p)^{r-1}]$$

= 0 (by equation (A4.1.Eq 13c)).

Case 2: $[\sum_{r=1}^{\infty} r\eta_r(-\partial W/\partial p)^{r-1}]<0$

The proof for this case is similar to that in case 1. ■

APPENDIX 4.2

The WKB-method for a multilinear momentum with closed integral curves

(A4.2.1) The Hamilton-Jacobi equations

The Hamilton-Jacobi equations are:

$$\zeta((\partial S/\partial q),q) = E, \text{ where p is replaced by } (\partial S/\partial q); \qquad \text{(A4.2.1.Eq 1a)}$$

and

$$\zeta(p,(-\partial W/\partial p)) = E, \text{ where q is replaced by } (-\partial W/\ p). \qquad \text{(A4.2.1.Eq 1b)}$$

The Hamilton-Jacobi equation have as solutions ${}_{+}S^{\varepsilon}(q)$, ${}_{-}S^{\varepsilon}(q)$, $W^{+}_{\varepsilon}(p)$ and $W^{\varepsilon}_{-}(p)$ which are given by equations (4.4.1.Eq 7a)-(4.4.1.Eq 7d).

Remark: (R1) The solutions ${}_{+}S^{\varepsilon}(q)$, ${}_{-}S^{\varepsilon}(q)$, $W^{\varepsilon}_{+}(p)$ and $W^{\varepsilon}_{-}(p)$ are formally identical to the corresponding Hamilton-Jacobi solutions of the Hamiltonian system of a particle in a potential well which are given by equations (1.3.2.Eq 3a)-(1.3.1.Eq 3d).

(A4.2.2) The WKB solutions

We shall construct WKB solutions of the following equations:

$$(\hat{\zeta}-E)\varphi(q) = 0, \quad \varphi \in L\ (Q); \qquad \text{(A4.2.2.Eq 1a)}$$

$$(\hat{\zeta}_c-E)\psi_c(p) = 0, \quad \psi_c \in \mathbf{F}^{-1}L^2(Q). \qquad \text{(A4.2.2.Eq 1b)}$$

Let us put $\phi^{\varepsilon}_{w}(q) = f(q)\{\exp \pm S(q)\}$ in equation (A4.2.2.Eq 1a), and let us put $\phi^{\varepsilon}_{c,w}(p) = g(p)\{\exp \pm W(p)\}$ in equation (A4.2.2.Eq 1b). Then in terms of an expansion in $\hbar$, we get [cf. Theorem (A4.1.T1), Appendix 4.1]

$$(\hat{\zeta}-E)\varphi(q) = [\zeta((\partial S/\partial q),q)-E][f\{\exp \pm S\}]$$
$$+(-i\hbar)[\sum_{k=1}^{\infty} k\{(1/2)(k-1)\xi_k(\partial S/\partial q)^{k-2}(\partial^2 S/\partial q^2)f+\xi_k(\partial S/\partial q)^{k-1}(\partial f/\partial q)$$
$$+(1/2)(\partial\xi_k/\partial q)(\partial S/\partial q)^{k-1}f\}]\{\exp \pm S\} + \text{H.O.T in } \hbar. \qquad \text{(A4.2.2.Eq 2)}$$

$$(\hat{\zeta}_c - E)\phi^{\varepsilon}_{c,w}(p) = [\zeta(p,(-\partial W/\partial p))-E][g\{\exp \pm W\}]$$
$$+(-i\hbar)[\sum_{k=1}^{\infty} k\{(1/2)(k-1)\eta_k(-\partial W/\partial p)^{k-2}(\partial^2 W/\partial p^2)g-\eta_k(-\partial W/\partial p)^{k-1}(\partial g/\partial p)$$
$$-(1/2)(\partial\eta_k/\partial p)(-\partial W/\partial p)^{k-1} g]\{\exp \pm W\} + \text{H.O.T in } \hbar. \qquad \text{(A4.2.2.Eq 3)}$$

It follows from theorems (A4.1.T2) and (A4.1.T3) and the Hamilton-Jacobi equations (A4.2.1.Eq 1a) and (A4.2.1.Eq 1b) that the WKB solutions are given by

$$\phi^{\varepsilon}_{w}(q) = {}_{+}K|(\partial q/\partial\theta)|_{+}^{-1/2}\{\exp \pm_{+}S^{\varepsilon}(q)\} + {}_{-}K|(\partial q/\partial\theta)|_{-}^{-1/2}\{\exp \pm_{-}S^{\varepsilon}(q)\}$$

where ${}_{+}K$ and ${}_{-}K$ are constants, and ${}_{+}S^{\varepsilon}(q)$ and ${}_{-}S^{\varepsilon}(q)$ are given by equations (4.4.1.Eq 7a) and (4.4.1.Eq 7b) respectively, and

$$|(\partial q/\partial\theta)|_{\pm}^{-1/2} = [T(E)/2\pi]|\sum_{k=1}^{\infty} k\,\xi_k(\partial_{\pm}S^{\varepsilon}/\partial q)^{k-1}|^{-1/2};$$

and

$$\phi^{\varepsilon}_{c,w}(p) = K_{+}|(\partial p/\partial\theta)|_{+}^{-1/2}\{\exp \pm W^{\varepsilon}_{+}(p)\} + K_{-}|(\partial p/\partial\theta)|_{-}^{-1/2}\{\exp \pm W^{\varepsilon}_{-}(p)\}$$

where K_{+} and K_{-} are constants, $W^{\varepsilon}_{+}(p)$ and $W^{\varepsilon}_{-}(p)$ are given by equations (4.4.1.Eq 7c) and (4.4.1.Eq 7d), and

$$|(\partial p/\partial\theta)|^{-1/2} = [T(E)/2\pi]|\sum_{k=1}^{\infty} k\,\eta_k(-\partial W^{\varepsilon}_{\pm}/\partial p)^{k-1}|^{-1/2}.$$

APPENDIX 4.3

Let $\phi(\theta)$ and $\phi_c(\theta)$ be the WKB-like wave functions of θ given by equations (3.1.Eq 11a) and (3.1.Eq 11b) respectively. Let $e(\theta)$ and $e_c(\theta)$ be the real valued functions of θ defined in section (3.1) of Chapter 3. Let $U: F^{-1}L^2(Q) \dashrightarrow L^2(Q)$ be the unitary map defined by equation (3.2.Eq 2a). Let $\hat{\xi}$ in $L^2(Q)$ be the operator given by equation (4.4.2.Eq 7), and let $\hat{\xi}_c$ in $F^{-1}L^2(Q)$ be the operator given by equation (4.4.2.Eq 8); we shall assume that $U\hat{\xi}_c U^{-1} = \hat{\xi}$.

Then by equations (4.4.1.Eq 1a), (4.4.1.Eq 1b), (4.4.1.Eq 4a) and (4.4.1.Eq 4b), we have

$$(\partial q(\theta)/\partial\theta) = [T(E)/2\pi][\sum_{k=1}^{\infty} k\xi_k p^{k-1}] = [T(E)/2\pi][\sum_{k=1}^{\infty} k\,\xi_k(\partial S/\partial p)^{k-1}];$$

(A4.3.Eq 1a)

and

$$(\partial p(\theta)/\partial\theta) = -[T(E)/2\pi][\sum_{k=1}^{\infty} k\eta_k q^{k-1}] = -[T(E)/2\pi][\sum_{k=1}^{\infty} k\,\eta_k(-\partial W/\partial p)^{k-1}];$$

(A4.3.Eq 1a)

(here S and W are respectively the solutions of the Hamilton-Jacobi equations given by equations (A4.2.Eq 1a) and (A4.2.Eq 1b)).

Then we have

$$(\hat{\xi}-E)(\overline{\phi e}^{0})(q) = -i\hbar[\overline{\sum_{k=1}^{\infty} k\,\xi_k\{p(\theta)\}^{k-1}(\partial e/\partial q)}^{0}] + O(\hbar^2)$$

(by equation (A4.1.Eq 4))

$$= -i\hbar[2\pi/T(E)][\overline{\phi(\partial e/\partial\theta)}^{0}](q) + O(\hbar^2) \qquad \text{(A4.1.Eq 2)}$$

(by equation (A4.3.Eq 1a)).

Similarly, by equations (A4.1.Eq 5) and (A4.3.Eq 1b) we have

$$(\hat{\xi}_c-E)(\overline{\phi_c e_c}^{c,0})(p) = -i\hbar[2\pi/T(E)][\overline{\phi_c(\partial e_c/\partial\theta)}^{c,0}] + O(\hbar^2). \qquad \text{(A4.3.Eq 3)}$$

(A4.3..T1) Theorem

We have

$$\|[\hat{\zeta}-E(n)]\Psi_n(q)\| = O(\hbar^2) \qquad \text{(A4.3.Eq 4a)}$$

where E(n) belongs to $R(\hat{\zeta}_{oc})$ and $\Psi_n(q)$ is the modified Maslov-WKB wave function (corresponding to the value E(n)) defined by

$$\Psi_n(q) = (U\,\overline{\phi_c e_c}^{c,o})(q)+(\overline{\phi e}^{o})(q). \qquad \text{(A4.3.Eq 4b)}$$

Proof:

It is clear from equations (A4.1.Eq 3) and (A4.1.Eq 4) that the proof is formally identical to the proof for Theorem (A3.2.T1) [cf. Appendix 3.2]. ∎

CHAPTER 5

CONCLUSIONS AND PROSPECTS

CONCLUSIONS AND PROSPECTS

In Chapter 2 we proposed a scheme to establish unitarily equivalent quantizations in certain canonically conjugate polarizations of 2-dimensional symplectic manifolds. In this scheme we dealt with examples on contractible and noncontractible symplectic manifolds in a unified manner. The scheme we proposed was based on the following physical reasoning. Let ζ be an observable, with classical range $R(\zeta)$, that is quantizable in the canonically conjugate polarizations $\mathcal{P}$ and $\mathcal{P}_c$. Then the spectra of the quantized operators corresponding to ζ in the polarizations $\mathcal{P}$ and $\mathcal{P}_c$ should: (1) lie in the classical range $R(\zeta)$, and (2) be identical to each other.

In Chapter 3 we modified the Maslov-WKB method (for the one-dimensional Hamiltonian system of a particle in a potential well) to incorporate the BWS conditions, so as to enable us to construct an approximate pairing between the polarization $\mathcal{P}_c$ (which has toroidal leaves and is spanned by X_H on a submanifold of the phase space) and the vertical polarization P.

We began Chapter 4 with an attempt to construct unitarily equivalent quantizations of a general observable ζ (of an arbitrary 2-dimensional symplectic manifold) in canonically conjugate polarizations $\mathcal{P}$ and $\mathcal{P}_c$ which are spanned by the vector fields $(\partial/\partial\zeta)$ and $(\partial/\partial t)$, respectively, where t is the is the flow parameter of the integral curves of the Hamiltonian vector field X_H. We saw that, in general, one could not establish unitarily equivalent quantizations of ζ in $\mathcal{P}$ and $\mathcal{P}_c$. So we localized ζ in the effective configuration space with respect to the polarization $\mathcal{P}$, and then attempted to set up unitarily equivalent quantizations of this localized observable in suitably chosen canonically conjugate polarizations. The latter attempt was successful.

In the next two parts of Chapter 4, we showed how the modified Maslov-WKB method could be used to quantize the following observables in the vertical polarization P:

(1) The Hamiltonian (of a particle in a potential well) localized in the phase space, and

(2) certain multlinear observables with closed integral curves.

It is hoped that with the knowledge gained from an increasing number of explicit examples we can progressively enlarge the class of observables quantizable in a unitarily equivalent manner independent of the choice of polarization. This would also help towards a solution of the pairing problem in geometric quantization.

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We shall only list the important symbols defined in the text here.