AN AXIOMATIZATION OF 'IF'

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Introduction

If we translate the conditional connective 'if ... then' by the truth-functional connective '$\supset$', we accept arguments as valid that are obviously invalid in the untranslated version. An example is 'It is not the case that if today is sunny, then Fermat's conjecture is true. Therefore Fermat's conjecture is false.'[1] In translation as 'not $(A \supset B) \supset$ not B' the argument is valid, but the untranslated version is invalid.

Nor does the attempt to translate 'if ... then' by the strict implication connective '$\prec$' fare any better. Acceptance of this leads us to accept as valid, arguments like 'It is necessary that 2+2=4; therefore if today is sunny, then 2+2=4.' This is obviously invalid.

In examples of this kind we are mistranslating an argument involving a conditional into another, simpler argument and then pronouncing that argument valid. In these simpler arguments we have changed the relation between the antecedent and the consequent of the conditional from an intensional, non-truth-functional,

1. There are several excellent examples of such arguments, e.g. Stevenson (14), and Cooper (8) p.297.

relation to one that is merely truth-functional. We no longer require that the antecedent must be relevant to the consequent.

It is this lack of relevance between the antecedent and the consequent that makes the paradoxes of material and strict implication counter-intuitive. In our system C below we shall use '$\supset$' only to translate expressions containing 'and' and 'not', but not 'if ... then'. Since we want all the truths of propositional logic in C, we shall include $A \supset (B \supset A)$ and $\sim A \supset (A \supset B)$ in the system - these are not counter-intuitive when '$A \supset B$' is a translation of 'not (A and not B)'. We have an intensional connective '$\rightarrow$' which we reserve to translate 'if ... then'.

Since the truth of a conditional requires that the antecedent be relevant to the consequent, if '$\rightarrow$' is to be acceptable, then paradoxes of material implication using it must not be derivable in C. By matrices given at the end we shall prove that neither X2 $A \rightarrow (B \rightarrow A)$ nor X3 $\sim A \rightarrow (A \rightarrow B)$ is in C. (The Xi refers to the number of the formula in the list non-theorems given on page 43.). Although there are no modal operators in C, if we let $B v \sim B$ be a necessary proposition and $A \wedge \sim A$ be an impossible one, then paradoxes of strict implication are also shown not to be in C: X4 $A \rightarrow (B v \sim B)$, X5 $(A \wedge \sim A) \rightarrow B$, and X19 $(A \wedge \sim A) \rightarrow (B v \sim B)$.

If the truth values of the antecedent and consequent do not form a sufficient condition for the truth of a conditional, but they do form a necessary condition. A conditional is not true when its antecedent is true and its consequent is false. $(A \rightarrow B) \rightarrow (A \supset B)$

is a theorem of C and X1 $(A \supset B) \rightarrow (A \rightarrow B)$ is not in C.

Unfortunately, apart from the relevance requirement, there does not seem to be any general way of characterizing conditionals. We can only use this requirement to exclude certain formulas from our system; the paradoxes and all formulas known to yield them when used with other, intuitively valid formulas and rules of inference. But when do we know that we have included in our system all the valid principles involving conditionals? All the axioms and rules of inference of C are intuitively valid, but intuitions are at best unreliable, and it is impossible to know whether all valid principles are represented in C without some definite rule for involving valid conditionals.

The semantics given later are a Hintikka-set, and we shall prove the completeness of C relative to this. In this type of semantics we prove that C is complete relative to the intuitive principles reflected in our choice of axioms and rules of inference for C. If we decide to accept another principle as valid, we add it to the system as a new axiom, and add a new condition to our semantics. We then check to see that the paradoxes are still excluded from C. Since does not have, as yet, a decision procedure, we must do this by matrices.

The following are some formulas, some more some less intuitively valid that are known to lead to paradoxes in C and are excluded by the matrices: X6 $((A \vee B) \wedge \sim A) \rightarrow B$, X9 $((A \wedge B) \rightarrow C) =$

$((A \wedge \sim C) \rightarrow \sim B)$, X10 $((A \wedge B) \rightarrow C) \rightarrow (A \rightarrow (B \rightarrow C))$, X17 $(A \wedge (A \supset B)) \rightarrow B$, and X20 $A \rightarrow (B \rightarrow (A \wedge B))$. These produce paradoxes when used with $(A \wedge B) \rightarrow A$,

$(A \rightarrow (B \wedge C)) = ((A \rightarrow B) \wedge (A \rightarrow C))$,

$(A \rightarrow B) = (\sim B \rightarrow \sim A)$,

$((A \wedge (B \vee C)) = ((A \wedge B) \vee (A \wedge C))$, and

From A and $A \rightarrow B$ infer B.

All these seem more intuitive than the excluded formulas, so they are retained in C.

$(A \wedge B) \rightarrow A$ also brings us into conflict with a formula Cooper (8), Angell (3) and Stevenson (14) regard as valid: X14 $(A \rightarrow B) \rightarrow \sim(A \rightarrow \sim B)$. Angell bases his system largely on the intuitive validity of this formula. $(A \wedge B) \rightarrow A$ has two cases $(A \wedge \sim A) \rightarrow A$ and $(\sim A \wedge A) \rightarrow A$, which together with $(A \wedge B) \rightarrow (B \wedge A)$ and a rule of syllogism and X14 would make C inconsistent. We could follow Angell and eliminate $(A \wedge B) \rightarrow A$ or we could modify it as he suggests, but this is unnecessary since X14 is not unrestrictedly valid; X14 is only valid when A is possible. When A is allowed to be impossible then, 'If $\sqrt{2}$ is rational, then ~~it~~ a certain number <u>n</u> is even, and if $\sqrt{2}$ is rational, then <u>n</u> ~~it~~ is not even', is a counter example. Accordingly we retain $(A \wedge B) \rightarrow A$ in C, and a restricted form of X14, $(\sim(A \rightarrow \sim A) \rightarrow \sim((A \rightarrow B) \wedge (A \rightarrow \sim B))$, is derivable in C.

The rules of inference, adjunction - If A and B, then $A \wedge B$, and modus ponens in the form If A and $A \rightarrow B$, then B, are valid principles of inference which we want in C. The rule modus ponens for '$\supset$', If A and $A \supset B$, then B, is not valid, it violates our relevance requirement, and we want to exclude it. If we were to have his rule in C either as primitive or derivable, then by our deduction theorem we would have X17$(A \wedge (A \supset B)) \rightarrow B$. But we exclude X1 so modus ponens in this form is not in C. However, a rule for '$\supset$' in the form - If $\vdash_c A$ and $\vdash_c A \supset B$, then $\vdash_c B$ - is acceptable as a rule for generating theorems, it does not violate the relevance requirement, so we have it in C.

The system C.

The primitive symbols of C are the following:
the set of propositional symbols: p', p'', p''', ... the five symbols $\sim$

$\wedge$

$\rightarrow$

(

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Any symbol or sequence of symbols in the above list is a string. The formation rules specify which strings are wffs of C.

Formation rules

1. Any propositional symbol is a wff of C.
2. If A is a wff of C, then $\sim A$ is a wff of C.
3. If A and B are wffs of C, then $(A \wedge B)$ is a wff of C.
4. If A and B are wffs of C, then $(A \rightarrow B)$ is a wff of C.

In these rules A and B are metalinguistic variables, and in what follows upper case letters will be used as metalinguistic variables standing for wffs of C.

The following are defined connectives in C:

1. $A \vee B = \sim(\sim A \wedge \sim B)$

2. $A \supset B = \sim(A \wedge \sim B)$

3. $A \leftrightarrow B = (A \to B) \wedge (B \to A)$

4. $A \equiv B = (A \supset B) \wedge (B \supset A)$

(For convenience we omit the outermost brackets around a complete wff where no ambiguity is likely to arise).

The axiom schemata of C (It is not known whether the axion schemats are independent - one or more may be redundant).

A1. $(A \wedge B) \to (B \wedge A)$

A2. $(A \wedge B) \to A$

A3. $((A \wedge B) \wedge C) \to (A \wedge (B \wedge C))$

A4. $A \equiv \sim\sim A$

A5. $(A \wedge (B \vee C)) \to ((A \wedge B) \vee C)$

A6. $((A \to B) \wedge (B \to C)) \to (A \to C)$

A7. $(\sim A \to B) \to (\sim B \to A)$

A8. $(A \wedge (A \to B)) \to B$

A9. $(A \to B) \to \sim(A \wedge \sim B)$

A10. $((A \to B) \wedge (A \to C)) \equiv (A \to (B \wedge C))$

A11. $(A \to (B \to C)) \to ((A \wedge B) \to C)$

A12. $((A \equiv B) \wedge \alpha) \to \beta$, where α is a wff differing from β only in containing A as a subformula in one or more places where β has B.

A13. All instances of truth functionally tautologous schemata.

The rules of inference for C:

R1. Adjunction (adj.) From A and B to infer $A \wedge B$.

R2. Modus ponens for '$\to$' (mp$\to$) From A and $A \to B$ to infer B.

R3. Modus ponens for '$\supset$' (mp $\supset$) If $\vdash_c A$ and $\vdash_c A \supset B$, then

$\vdash_c B$. The symbol $\vdash_c$ is to mean '... is a theorem of C.'

Theorems and derived rules of inference of C needed later: DR Syll.

If $\vdash_c A \to B$ and $\vdash_c B \to C$, then $\vdash_c A \to C$. By A6 and mp $\to$ twice. DR Simp.

If $\vdash_c A \wedge B$, then $\vdash_c A$. By A2 and mp $\to$.

Theorem 1. $\vdash_c (A \wedge B) \to B$

1.	$(A \wedge B) \to (B \wedge A)$	A1
2.	$(B \wedge A) \to B$	A2
3.	Th 1.	1,2 DR Syll.

2. $\vdash_c A \to \sim\sim A$ — A4, df'=', DR Simp.

3. $\vdash_c \sim\sim A \to A$ — as Th 2

4. $\vdash_c A \to A$ — Ths 2,3 DR Syll.

5. $\vdash_c (A \to B) = (\sim B \to \sim A)$

1. $((A = \sim\sim A) \wedge ((\sim\sim A \to B) \to (\sim B \to \sim A))) \to ((A \to B) \to (\sim B \to \sim A))$ — A 12

2. $(A \to B) \to (\sim B \to \sim A)$ — 1, A4, A7 $\sim A/A$, adj. mp $\to$

3. $((A = \sim\sim A) \wedge ((\sim B \to \sim A) \to (\sim\sim A \to B))) \to ((\sim B \to \sim A) \to (A \to B))$ — A12

4. $(\sim B \to \sim A) \to (A \to B)$ — 3, A4, A7 B/A $\sim A/B$, adj., mp $\to$

5. $(A \to B) = (\sim B \to \sim A)$ — 2,4 adj., df'='

6. $\vdash_c \sim A \to \sim(A \wedge B)$ — Th 5 $A \wedge B/A$, A/B, df'=' DR Simp. A2, mp $\to$

7. $\vdash_c \sim(A \wedge A) \rightarrow \sim A$

1. $(A \rightarrow A) \wedge (A \rightarrow A)$ — Th. 4, ~~adj.~~

2. $A \rightarrow (A \wedge A)$ — A10, df '=', DR Simp, mp

3. $\sim(A \wedge A) \rightarrow \sim A$ — Th. 5, $(A \wedge A)/B$, df '=', DR Simp mp $\rightarrow$ twice.

DR Repl. If $\vdash_c A \equiv B$ and $\vdash_c \alpha$, then $\vdash_c \beta$. A12, adj and mp $\rightarrow$

8. $\vdash_c (A \wedge A) \equiv A$ — Th7.2, A2 A/B, adj, df '='

9. $\vdash_c A \rightarrow (A \vee B)$ — Th6, $\sim A/A$ $\sim B/B$, df 'v', Th2, DR Syll

10. $\vdash_c (A \equiv B) \equiv (B \equiv A)$

1. $(A \equiv B) \rightarrow (B \rightarrow A)$ — A2, df '='

2. $(A \equiv B) \rightarrow (A \rightarrow B)$ — Th1 df '='

3. $(A \equiv B) \rightarrow (B \equiv A)$ — A10, df '=', DR Simp, adj. 1,2, mp

4. $(B \equiv A) \rightarrow (A \equiv B)$ — as 3

5. Th10 — adj 3,4 df '='

11. $\vdash_c ((A \vee B) \rightarrow C) \equiv (A \rightarrow C) \wedge (B \rightarrow C))$

1. $(\sim C \rightarrow (\sim A \wedge \sim B)) \equiv ((\sim C \rightarrow \sim A) \wedge (\sim C \rightarrow \sim B))$ — DR Repl Th10, A10 $\sim C/A$ $\sim A/B$ $\sim B/C$

2. $(\sim(\sim A \wedge \sim B) \rightarrow \sim\sim C) \equiv ((\sim\sim A \rightarrow \sim\sim C) \wedge (\sim\sim B \rightarrow \sim\sim C))$ — A12, Th5, mp $\rightarrow$

3. Th11 — df 'v' DR. Repl. A4,

12. $\vdash_c ((B \wedge C) \vee (B \wedge D)) \rightarrow B$

1. $(B \wedge C) \rightarrow B$ — A2

2. $(B \wedge D) \rightarrow B$ — A2

3. Th12 — Th11, df =, DR Simp, adj 1,2 mp $\rightarrow$

13. $\vdash_c ((A \wedge B) \wedge C) \rightarrow (((A \wedge B) \wedge C) \wedge A)$

1.	$((A \wedge B) \wedge C) \rightarrow (A \wedge B)$	A2
2.	$(A \wedge B) \rightarrow A$	A2
3.	$((A \wedge B) \wedge C) \rightarrow A$	DR Syll 1,2
4.	$((A \wedge B) \wedge C) \rightarrow ((A \wedge B) \wedge C)$	Th4
5.	Th13	A10 df =, DR Simp, adj 3,4, mp $\rightarrow$

14. $\vdash_c ((A \wedge C) \wedge B) \rightarrow ((((A \wedge C) \wedge C) \wedge A) \wedge B)$

1.	$((A \wedge C) \wedge B) \rightarrow (A \wedge C)$	A2
2.	$(A \wedge C) \rightarrow C$	Th1
3.	$(A \wedge C) \rightarrow A$	A2
4.	$((A \wedge C) \wedge B) \rightarrow B$	Th1
5.	$((A \wedge C) \wedge B) \rightarrow C$	DR Syll 1,2
6.	$((A \wedge C) \wedge B) \rightarrow A$	" 1,3
7.	$((A \wedge C) \wedge B) \rightarrow ((A \wedge C) \wedge C)$	A10, df'=', DR Simp. adj 1,5, mp $\rightarrow$
8.	$((A \wedge C) \wedge B) \rightarrow (((A \wedge C) \wedge C) \wedge A)$	as 7, adj 7,4
9.	Th 14	as 8 adj 8,4

15. $_c (\sim\sim A \wedge B) \rightarrow ((\sim\sim A \wedge B) \wedge A)$

1.	$(\sim\sim A \wedge B) \rightarrow (\sim\sim A \wedge B)$	Th4
2.	$(\sim\sim A \wedge B) \rightarrow \sim\sim A$	A2
3.	$\sim\sim A \rightarrow A$	Th3
4.	$(\sim\sim A \wedge B) \rightarrow A$	Dr Syll 2,3
5.	Th15	A10, df'=', DR Simp, adj 1,4, mp $\rightarrow$

16. $\vdash_c (A \wedge B) \rightarrow (((A \vee C) \wedge A) \wedge B)$

1.	$(A \wedge B) \rightarrow A$	A2
2.	$A \rightarrow (A \vee C)$	Th10
3.	$(A \wedge B) \rightarrow (A \vee C)$	DR Syll 1,2
4.	$(A \wedge B) \rightarrow B$	Th1
5.	$(A \wedge B) \rightarrow ((A \vee C)$	A10, df'=', DR Simp, adj 3,1, mp $\rightarrow$
6.	Th16	as 5 adj 5,4

17. $\vdash_c (A \vee B) = (B \vee A)$

1.	$(\sim A \wedge \sim B) \rightarrow (\sim B \wedge \sim A)$	A1 $\sim A/A$, $\sim B/B$
2.	$(B \vee A) \rightarrow (A \vee B)$	DR Repl, Th5, df'v'
3.	$(A \vee B) = (B \vee A)$	as 2
4.	Th 17	adj 2,3 df'='

18. $\vdash_c (A \vee (B \wedge C)) = ((A \vee B) \wedge (A \vee C))$

1.	$A \rightarrow (A \vee B)$	Th 9
2.	$A \rightarrow (A \vee C)$	"
3.	$A \rightarrow ((A \vee B) \wedge (A \vee C)$	A10, df'=', DR Simp, adj 6,7 mp $\rightarrow$
4.	$(B \wedge C) \rightarrow B$	A2
5.	$B \rightarrow (A \vee B)$	Th9
6.	$(B \wedge C) \rightarrow (A \vee B)$	DR Syll 4,5
7.	$(B \wedge C) \rightarrow (A \vee C)$	as 4 adj 5,6
8.	$(A \vee (B \wedge C)) \rightarrow ((A \vee B) \wedge (A \vee C))$	Th11, df'=', DR Simp, adj 6,7 mp $\rightarrow$
9.	$((A \vee B) \wedge (A \vee C)) \rightarrow ((C \vee A) \wedge (B \vee A))$	DR Repl, Th17

10. $((C \vee A) \wedge (B \vee A)) \rightarrow (((C \vee A) \wedge B) \vee A)$ A5

11. $A \rightarrow (A \vee (B \wedge C))$ Th9

12 $(B \wedge (C \vee A)) \rightarrow ((B \wedge C) \vee A)$ A5

13 $((B \wedge (C \vee A)) \vee A) \rightarrow (A \vee (B \wedge C))$ Th11, df'=', DR Simp
adj 11,13 mp →

14 $(((C \vee A) \wedge B) \vee A) \rightarrow ((B \wedge (C \vee A)) \vee A)$ DR Repl, Th10

15. $((A \vee B) \wedge (A \vee C)) \rightarrow (A \vee (B \wedge C))$ DR Syll 10, 15, 14

16. Th18 adj 16,8 df'='

19. $\vdash_c ((B \vee C) \wedge A) \rightarrow ((B \wedge A) \vee (C \wedge A))$

1. $((\sim A \vee \sim B) \wedge (\sim A \vee \sim C)) \rightarrow ((\sim A \vee (\sim B \wedge \sim C))$ Th16 df'=' DR Simp

2. $\sim ((\sim A \vee (\sim B \wedge \sim C)) \rightarrow \sim ((\sim A \vee \sim B) \wedge (\sim A \vee \sim C))$ DR Repl Th5

3. $\sim (\sim A \vee \sim\sim (\sim B \wedge \sim C)) \rightarrow \sim (\sim\sim (\sim A \vee \sim B) \wedge \sim\sim (\sim A \vee \sim C))$ DR Repl
A4

4. Th19 df'v'

20. $\vdash_c ((A \vee B) \wedge C) \rightarrow ((((A \vee B) \wedge C) \wedge A) \vee (((A \vee B) \wedge C) \wedge B))$

1. $((A \vee B) \wedge C) \rightarrow ((A \wedge C) \vee (B \wedge C))$ Th19

2. $((A \vee B) \wedge C) \rightarrow (A \vee B)$ A2

3. $((A \vee B) \wedge C) \rightarrow ((A \vee B) \wedge ((A \wedge C) \vee (B \wedge C)))$ A10, df'=', DR Simp,
adj 2,1, mp →

4. $((A \vee B) \wedge ((A \wedge C) \vee (B \wedge C))) \rightarrow (((A \wedge C) \vee (B \wedge C)) \wedge (A \vee B))$ Th10

5. $(((A \wedge C) \vee (B \wedge C)) \wedge (A \vee B)) \rightarrow (((A \wedge C) \wedge (A \vee B)) \vee ((B \wedge C) \wedge (A \vee B)))$
Th19

6. $(((A \wedge C) \wedge (A \vee B)) \vee ((B \wedge C) \wedge (A \vee B))) \rightarrow (((A \vee B) \wedge (A \wedge C)) \vee ((A \vee B) \wedge (B \wedge C$
Th10

7. Th 20 DR Syll 3,4,5,6

21. $\vdash_c A \rightarrow \sim(A \rightarrow \sim A)$

1.	$(A \rightarrow \sim A) \rightarrow \sim(A \wedge \sim\sim A)$	A9
2.	$\sim(A \wedge \sim\sim A) \rightarrow \sim(A \wedge A)$	DR Repl, A4
3.	$\sim(A \wedge A) \rightarrow \sim A$	Th7
4.	$(A \rightarrow \sim A) \rightarrow \sim A$	DR Syll 1,2,3
5.	Th 21	DR Repl Th5

Deduction Theorem

In the standard definition of a deduction of a wff A from a set of wffs Γ, axioms which are not members of Γ are allowed to be members of the string. Thus there is a deduction of a theorem from any set whatsoever. If we adopt this definition and then try to prove a deduction theorem for C, we shall be trying to prove that when A is an axiom of C then $\vdash_C B \rightarrow A$. We would fail since X2, $A \rightarrow (B \rightarrow A)$, is proved a non-theorem of C below.

C does have a restricted deduction theorem when the definition of a deduction is changed to: A deduction of a wff A from a set Γ of wffs of C is a finite string of wffs of C which has A as the last wff in the string and each wff in the string is either A) a member of Γ

or B) an immediate consequence of a preceeding wff or wffs in the string by a rule of inference.

Theorem 1. (Deduction theorem for C) If there is a deduction of A from a set Γ involving no use of mp $\supset$, then $\vdash_c \gamma \rightarrow A$, where γ is a conjunction of all members of Γ used in this deduction of A.

Proof Induction on the length of the deduction.

Basis n=1, then $A \in \Gamma$, so γ is A and by $\vdash_c A \rightarrow A$, $\vdash_c \gamma \rightarrow A$.

Induction step, assume the theorem for $n < k$, prove for n=k.

2 cases: 1. $A \in \Gamma$, then by A2 $(A \wedge B) \rightarrow A$, where B is the conjunction of the other members of Γ used in the deduction; $\vdash_c \gamma \rightarrow A$.

2. A is an immediate consequence of a wff or wffs preceding A in the deduction.

a) adj. A is $B \wedge C$, where B and C are wffs preceding A in the deduction. By the induction hypothesis $\vdash_c \gamma \to B$ and $\vdash_c \gamma \to C$. By A10, DR Simp, adj, mp $\to$, $\vdash_c \gamma \to B \wedge C$; that is $\vdash_c \gamma \to A$.

b) mp $\to$, A is an immediate consequence of two wffs C and $C \to A$. By the induction hypothesis $\vdash_c \gamma \to C$ and $\vdash_c \gamma \to (C \wedge A)$. A10, DR Simp, adj, and mp$\to$ give $\vdash_c \gamma \to (C \wedge (C \to A))$. Then by A8 and Dr Syll $\vdash_c \gamma \to A$. P If we were to allow the use of mp$\supset$ in a deduction, then from the set $\{A \vee \sim A, (A \vee \sim A) \supset (B \vee \sim B)\}$, since $\vdash_c A \vee \sim A$ and $\vdash_c (A \vee \sim A) \supset (B \vee \sim B)$, we could get $B \vee \sim B$, and then $\vdash_c (A \vee \sim A) \wedge ((A \vee \sim A) \supset (B \vee \sim B)) \to (B \vee \sim B)$. But this wff is proved a non-theorem of C, X21, by the matrices below (p. ~~45-47~~ 42).

Semantics for C

We want a formal semantics for C which will justify our assumed interpretation of the connective '$\to$' as the conditional 'if ... then'. It does not seem necessary to catch all the properties of ordinary language conditional statements, whatever they may be, in our formal semantics. E.g. conditionals can be devided into factual and counter-factual and further devided by verb mood and tense variation. One feature which all these conditional statements do have in common is that the antecedent is relevant to the consequent.[1] This feature our semantics must catch. It is the lack of any relevance between antecedent and consequent that makes the ''paradoxes'' of material and strict implication paradoxes. Anyone wishing to accept either material or strict implication as a formalization of

1. Anderson and Belnap seem to be the first to make relevance a foundation of their system. (1) (6)

conditionals must be willing to treat the connection between antecedent and consequent as truth-functional or quasi truth-functional, and it is just not so. It is a necessary condition for the truth of a conditional statement that it is not the case that the antecedent is true and the consequent false, but otherwise the truth values of the components do not determine the truth value of the conditional: neither '$A \supset B$' nor '$A \prec B$' is a sufficient condition for 'If A then B'.

That this is so can be seen by what we mean when we deny a conditional statement.[1] If someone were to say 'If I sell the heirlooms, then I will get out of debt.' and we, knowing something of their value, wished to deny this, we should say in abbreviation 'not (if A then B)'. If the truth of '$A \supset B$' were a sufficient condition for the truth if 'if A then B', we would be asserting '$A \wedge \sim B$'. But in denying his statement we are not predicting that the speaker will sell, in fact we might be about to try to dissuade him since we know they are of insufficient intrinsic worth. Nor are we predicting that he will remain in debt. We could consistently add to our denial that he will soon be out of debt since it was just announced that he had won the football pools. What we are denying is the 'because' in 'B because A'.

When as in the example, the consequent might be true, but not because the antecedent is, we can justify our denial of 'if A then B' by describing circumstances, different from the

1. Stevenson, (14) pp 38-39

actual ones, in which A is true and B is false. It is just this idea of circumstances different from the actual ones that we tacitly accept when we accept a counter-factual conditional as true. We will call these sets of circumstances alternative sets. The man who says 'If Hitler had invaded Britain, then he would have won the war', is presumably prepared to give a partial description of an alternative set in which 'Hitler invaded Britain' is true and then argue from political, economic, and social factors, which are the same in this alternative set as in the actual case, that 'he won the war' is true in this alternative.

It is this notion of a set of actual circumstances and a class of alternative sets which differ from it that will be the intuitive idea behind our formal semantics.

In these alternative sets we are accepting a truth-functional connection between antecedent and consequent. As Routley and Routley, (12)p,3, point out this does not mean that we are accepting a truth-functional connection for conditionals. The intensional (non-truth-functional) part of the connection is in the choice of the propositions that we put into the alternative sets.

Our remarks above about alternative sets also mean that if we are correct then the whole problem of counterfactual conditionals rests on the acceptance of a truth-functional relationship between antecedent and consequent. If this is removed and the connection is accepted as intensional, there no longer is any special problem about the verification or analysis of these.

They are to be treated exactly as factual conditionals.[1]

If we have a set of wffs, G, which is intuitively to represent the set of actual circumstances, we allow a statement of the form 'if A then B' to be in G only if in every alternative set for G in our class of alternative sets, when A is true in an alternative set, B is. We also require that G be consistent and complete - a wff $A \in G$ iff the negation of $A \notin G$.[2] The restriction on membership in G for wffs of the form 'if A then B' then is equivalent to: a wff of the form 'not (if A then B)' is a member of G only if there is at least one alternative set of G in which A is true, but B is not. This latter formulation is the one we shall use in the semantics.

Since we wish to have a normal interpretation for our conjunction '$\wedge$', negation '$\sim$', and disjunction 'v', G and the alternative sets for G must satisfy the following:

1. $A \wedge B$ is in a set iff A is in the set and B also is.
2. A is in a set iff $\sim\sim A$ is.
3. AvB is in a set iff A is or B is.

If in addition to 1-3 we also make the alternative sets consistent and complete as we did G, then we have the paradoxes of strict implication in G. Assume all alternative sets satisfy

1. Ayers (4).
2. Routley and Routley (12).

the condition A is a member of a set iff $\sim$A is not. Now consider the tautology, Av$\sim$A. If A is a member of any alternative set then Av$\sim$A is (3). If A is not a member then $\sim$A is, so Av$\sim$A is (3). Av$\sim$A is a member of any alternative set. In the same way any tautology T is a member of any alternative set, and any contradiction C is in no alternative set. Thus 'if A then T' and 'if C then A' will be members of G where A is any wff whatsoever. This breaks our intuitive relevance requirement.

In addition to the effects such a condition has there are other reasons for not requiring that the alternative sets be consistent and complete. Our alternative sets have features in common with arguments. As we shall see below we can use the conditions C1-C10 in our formal semantics to generate alternative sets starting from the antecedent of a conditional. If we can show that in every alternative set formed from the antecedent. A, the consequent, B, must also be a member, then we can use the alternative sets to justify the conditional 'if A then B'. Arguments start from a premise or premises and proceed by licensed inferences to a conclusion. They justify a conditional statement with the premise or premises as antecedent and conclusion as consequent. Our alternative sets however, are infinite and arguments are presumably not, and arguments are directed towards a conclusion. Still we shall allow the same freedoms to our alternative sets that we allow to arguments. We use necessarily false premises in reductio arguments so we shall allow our alternative sets to have as members necessarily false propositions - contradictions. Nor shall we require that the alternative sets be complete. This last means that every tautology is not necessarily a member of every alternative set.[1]

In addition to these conditions we wish to add some conditions for the '$\rightarrow$' connective which will correspond to the axioms of C. These are C4-C10 listed below in the formal semantics; each one corresponds to one of the axioms A6-A12.

The formal semantics for C[2]

A C-model is an ordered pair $\langle K, G \rangle$, where K is a class of sets of wffs of C, each set k in K satisfying the conditions C1-C10 below, and G is a set in K which satisfies in addition the two conditions G1 and G2.

G1. $A \in G$ iff $\sim A \notin G$.

G2. A wff of the form $\sim(A \rightarrow B)$ is a member of G only if there is at least one set in K, k, such that $A \in k$ and $B \notin k$.

C1. $A \wedge B \in k$ iff $A \in k$ and $B \in k$.

C2. $A \in k$ iff $\sim\sim A \in k$.

C3. $(A \vee B) \in k$ iff $A \in k$ or $B \in k$.

C4. If $(A \rightarrow B) \wedge (B \rightarrow C) \in k$, then $A \rightarrow C \in k$.

~~If A B k, then (B C) (A C) k.~~

C5. If $\sim A \rightarrow B \in k$, then $\sim B \rightarrow A \in k$.

1. R. and V. Routley, (12). Our alternative sets are now their normal set-ups.

2. A Hintikka-type semantics, Hintikka, (10), (11).

C6. If $A \wedge (A \rightarrow B) \in k$, then $B \in k$.

C7. If $A \rightarrow B \in k$, then $\sim(A \wedge \sim B) \in k$.

C8. $(A \rightarrow B) \wedge (A \rightarrow C) \in k$ iff $A \rightarrow (B \wedge C) \in k$.

C9. If $A \rightarrow (B \rightarrow C) \in k$, then $(A \wedge B) \rightarrow C \in k$.

C10. If $(A \equiv B) \wedge \alpha \in k$, then $\beta \in k$, where α is a wff differing from β only in containing A as a subformula in one of more places where β has B.

A wff A is said to be satisfiable in a C-model $M = \langle K,G \rangle$ iff $A \in G$.

A wff A is said to be valid iff A is satisfied in every C-model M - or equivalently iff $\sim A$ is not satisfied in any C-model M.

A C-model M in which A is not satisfied is said to be a counter-model for A.

Theorem 2. All the theorems of C are valid.

Proof. The axioms are valid.

The condition G1 on G in a C-model M can be used to show that any substitution instance of a tautology is valid -A13. We do this by using the reductio method for the classical propositional calculus, letting 0 and 1 stand for non-membership and membership respectively in a set G. Assume the wff is not a member of G, where A is a substitution instance of a tautology, and write 0 under the main connective. Assign 0 or 1 to the well formed parts of A using truth table considerations for the truth functional connectives '$\wedge$', '$\sim$', and the connectives defined in terms of these. If we find that we cannot reach a consistent membership assignment, then A must be a member of G in any C-model M; A is valid.

The following example is a tautology with the membership assignment marked ~~directly~~ below the wff, the order of assignment below that, and the inconsistencies underlined.

$$((A \to B) \wedge C) \supset (((A \to B) \wedge \sim C) \supset \sim (A \to B))$$

$$\begin{array}{cccccccccccc} 1 & 1 & \underline{1} & 0 & \quad & 1 & \underline{11} & \underline{0} & \quad & 0 & 0 & 1 \\ 3 & 2 & 4 & 1 & \quad & 7 & 68 & 9 & \quad & 5 & 10 & 11 \end{array}$$

To show that any of the other axioms of C, A1-A12, are valid it is only necessary to show that the negations of these are not satisfiable in any C-model M. If the negation were a member of G for some C-model M, then a wff of the form $\sim (A \to B)$ is. For all but A4 and A10 this is just the negation, but for these it need be only one conjunct of the axiom. By G2 there is in K at least one set such that A is a member and B is not. An examination of the axioms and the conditions defining a C-model shows that this is the case only if some wff both is and is not a member of this set. Therefore no such set is specifiable and $(A \to B) \in G$ for any C-model M; the axioms are valid.

For example, suppose A1 is not valid, then $\sim ((A \wedge B) \to (B \wedge A)) \in G$ for some C-model M. In K of this C-model then there must be at least one set k, such that $A \wedge B \in k$ and $B \wedge A \notin k$, by G2. By C1 $A \in k$ and $B \in k$ and either $B \notin k$ or $A \notin k$. In either case some wff both is and is not a member of k. This procedure is easily seen to apply to the remaining axioms.

It remains to prove that the primitive rules of inference preserve validity.

1. adj. If A is valid and B is valid, then by C1 $A \wedge B$ is a member of G for any C-model; $A \wedge B$ is valid.

2. mp $\rightarrow$. If A is valid and $A \rightarrow B$ is valid, by 1 $A \wedge (A \rightarrow B)$ is valid. By C6 B is valid.

3. mp $\supset$. If A is valid and $A \supset B$ is valid, by 1 $A \wedge (A \supset B)$ is valid. By dfs '$\supset$' and 'v' $A \wedge (\sim A v B)$ is valid. A5 is valid,
so by 2 $(A \wedge \sim A) v B$ is valid. By C3 $A \wedge \sim A \in G$ or $B \in G$ for any C-model M. G1 and C1 together forbid $A \wedge \sim A$ to be a member of any G, therefore B must be a member of G for any C-model M; B is valid.

From theorem 2 and G1 it follows that C is consistent. If A is a theorem of G, then A is valid so $\sim A \notin G$ for any C-model M, $\sim A$ is not valid and hence not a theorem of C.

Before we prove the completeness of C with respect to the semantics we need definitions of a consistent wff, a consistent set, a maximal consistent set, and we need to prove a property of maximal consistent sets.

A wff A of C is said to be consistent with respect to C iff $\sim A$ is not a theorem of C, $\nvdash_C \sim A$.

A finite set of wffs of C is said to be consistent with respect to C iff the negation of the wff formed by a conjunction of all the members of the set is not a theorem of C.

An infinite set of wffs of C is said to consistent with respect to C iff no finite subset of it is inconsistent with respect to C.

A set of wffs of C, Γ, is a maximal consistent set with respect to C iff it is consistent and there is no wff A that can be added to Γ without producing a set that is inconsistent. (From here on by 'consistent' we shall understand 'consistent with respect to C', and by 'wff' 'wff of C'.)

For any maximal consistent set Γ and any wff A, exactly one of A and $\sim$A is a member of Γ.

Proof. For no wff A can both A and $\sim$ A be members of Γ. Since $\vdash_c \sim(A \wedge \sim A)$, by A13, the finite set $\{A, \sim A\}$ is inconsistent. But Γ is consistent ex hypothesi.

Suppose neither A nor A are members of Γ. Ex hypothesi Γ is maximal so there is a finite subset Γ_i of Γ such that $\Gamma_i \cup \{A\}$ is inconsistent and a finite subset Γ_j of Γ such that $\Gamma_j \cup \{\sim A\}$ is inconsistent. Let Δ be $\Gamma_i \cup \Gamma_j$, then Δ is finite and $\Delta \cup \{A\}$ and $\Delta \cup \{\sim A\}$ are both inconsistent sets. Write δ for a conjunction of all the wffs that are members of Δ. $\vdash_c \sim(\delta \wedge A)$ and $\vdash_c \sim(\delta \wedge \sim A)$, and by A13 $\vdash_c (\sim(A \wedge B) \wedge \sim(A \wedge \sim B)) \supset \sim A$. Then by adj and mp $\rightarrow \vdash_c \sim \delta$. This means Δ is inconsistent and Δ is a finite subset of Γ, but Γ is consistent ex hypothesi.

We shall prove that C is complete with respect to the semantics - a wff A is valid only if $\vdash_c$A. First we need 4 lemmas.

Lemma 1 There is an effective ennumeration of the wffs of C.

Proof We use the proof in Hunter, G. Metalogic, 1971, pp 108-109. Change the numeral assignment keeping p, ', $\sim$, as they are, and replacing '$\supset$' by '$\wedge$'.

To '$\rightarrow$' we assign the numeral 100000.

To '(' we assign the numeral 1000000.

To ')' we assign the numeral 10000000.

The rest of the proof proceeds as given.

<u>Lemma 2</u> If Γ is any consistent set of wffs then Γ can be increased to a maximal consistent set.

<u>Proof</u> First let A_n be the n-th wff in the enumeration of lemma 1, then we define an infinite sequence of sets of wffs as follows:
Γ_0 is Γ, and Γ_n is $\Gamma_{n-1} \cup \{A_n\}$ if this is consistent. If not then Γ_n is Γ_{n-1}.

Let Γ' be the infinite union of $\Gamma_0, \ldots, \Gamma_n, \ldots$.

Γ' is consistent. If not then there is a finite subset Δ of Γ' such that the negation of a conjunction of the members of Δ is a theorem of C. Each wff in Δ has a position in the enumeration of lemma 1. Since Δ is a finite set there is a highest numbered wff in Δ; let A_n be this wff. Γ_n then, formed by $\Gamma_{n-1} \cup \{A_n\}$, must be inconsistent since it contains all the wffs in Δ. But Γ_n is consistent by construction. Therefore Γ' is consistent.

Γ' is maximal. If not then there is some wff A_n, where n is again its number in the enumeration, which can be added to Γ' without producing an inconsistent set. But in that case A_n was added to Γ_{n-1} to form Γ_n. So $A_n \in \Gamma'$.

For use in lemma 3 we shall need an enumeration of the wffs in Γ' of the form $\sim(A \rightarrow B)$, where Γ' is as in lemma 2. Since by lemma 1 there is an effective enumeration of the wffs of C, there is an effective enumeration of the wffs of C of the form $\sim(A \rightarrow B)$. This is formed simply by deleting from $A_1, \ldots, A_n, \ldots$ all wffs not of this form Let the resulting enumeration, $A_i, \ldots, A_j, \ldots$ be E. Each of these still has its number in the original enumeration. Let Γ_i and Γ be as in lemma 2. In the formation of Γ_i, if A_i was not added to Γ_{i-1} and A_i is in E, then A_i is deleted from E. This defines an enumeration of the wffs of C of the form $\sim(A \rightarrow B)$ that are in Γ'. Let $B_1, \ldots, B_n, \ldots$ be this enumeration which need not be effective.

Lemma 3. Every consistent set of wffs is satisfiable in some C-model M.

Proof From lemma 2 if Γ is any consistent set of wffs, it has a maximal consistent extension. Let this extension be Γ'. The form of the proof will be to show that we can define a class K' of sets of wffs, and that K' and ' are the K and G respectively of a C-model $M = \langle K, G \rangle$. Then since all the members of Γ' are satisfiable in this C-model, all the members of Γ are.

We define K': For each wff in the enumeration $B_1, \ldots, B_n, \ldots$ we define a finite series of sets of wffs, $k'_1, \ldots, k'_n$. Where A is the antecedent in $\sim(A \rightarrow B)$, there are eliminated from A all occurrences of connectives other than the primitives, '$\wedge$', '$\sim$', '$\rightarrow$', and the defined connective 'v'. Negations over conjunctions and disjunctions are to be removed by A4, D.R.Repl, and df'v'. We wish to reduce A to constitutent wffs of A, where each wff is a unnegated or singly negated propositional variable or a wff of the form $(C \rightarrow D)$ or $\sim(C \rightarrow D)$. The definition of the sets is in two stages.

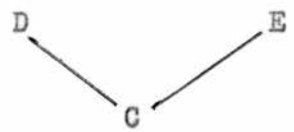
Stage 1. In this stage we will be using C1, C2 and C3 in one direction only. C1.1. If $A \wedge B \in k$, then $A \in k$ and $B \in k$.

C2.1 If $\sim\sim A \in k$, then $A \in k$.

C3.1. If $(AvB) \in k$, then $A \in k$ or $B \in k$.

These will be used as 1,2 and 3 below to reduce A to its constituents.

First we define a tree for A. Each formula in the tree is immediately connected to at most two formulas above it.



Each branch is formed by one of the following:

1) If C is of the form $F_1 \wedge F_2$, then D (the formula immediately above it to the left) is F_1 and E (the formula immediately above it to the right) is F_2.

2) If C is of the form $\sim\sim C$, then D is C. In this case there is no E.

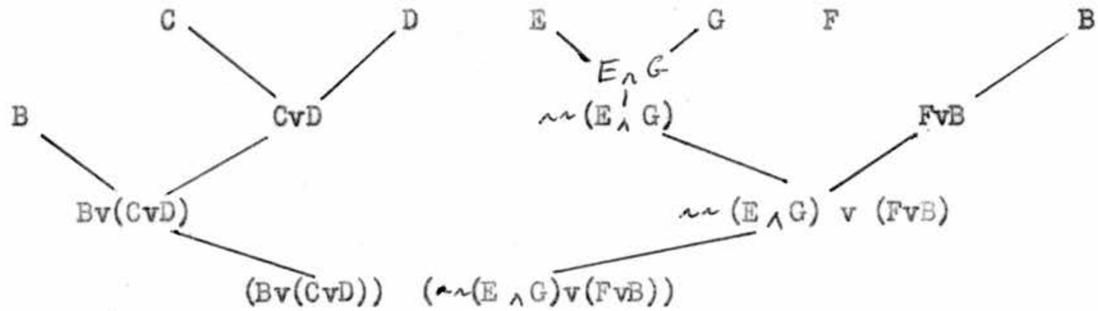
3) If C is of the form $F_1 v F_2$, then D is F_1 and E is F_2.

Start with A at the bottom. Construct a tree by repeated application of 1, 2, 3 until no further applications are possible. (If none are possible then the tree consists of A itself.) A unique order may be got by requiring that the application of 1, 2, and 3 always be made first to the leftmost unfinished branch. A finished branch is one to which none of 1, 2, or 3 is applicable.

We also define a choice - at any application of 3 to a formula of the form F_1vF_2 this formula is said to be a choice and the branches to F_1 and to F_2 are said to pass through a choice.

The resulting tree will have finitely many long branches, and each branch will end at the top in either an at most singly negated propositional variable or in a wff of the form $(C \rightarrow D)$ or $\sim(C \rightarrow D)$. The tree will end at the bottom in A itself.

Example Let A be of the form $(Bv(CvD)) \wedge (\sim\sim(E \wedge G)v(FvB))$, where B, C,D,E,F, and G are formulas not further reducible by C1-C3. Then the tree for A is:



There four choices: $Bv(CvD)$, CvD, $\sim\sim(E \wedge G)v(FvB)$, FvB.

We define a finite series of sets $k'_1, \ldots, k'_n$, corresponding to the choices we have in the tree. Intuitively we want a different set for each of the branches through a choice. We want a different set of end points of the tree for every combination of endpoints from which A can be reached as a member of the set by applications of C1-C3 in the other direction. In our example (B, E, G) is a set from which A can be reached as a member of the set. From B by C3 $Bv(CvD)$ is a member of the set. From E and G, $E \wedge G$ is a member of C1, then by C2 $\sim\sim(E \wedge G)$ is a member. By C3 $\sim\sim(E \wedge G)v(FvB)$ is a member, and finally by C1 $(Bv(CvD)) \wedge (\sim\sim(E \wedge G)v(FvB))$ is a member of the set; A is a member of the set.

In forming the sets, every branch that does not go through a choice has its end point in every set. If there are no choices in the tree then there is only one set. Since there are only finitely many finitely long branches it is possible to check each branch in the tree for A. To form the sets, first check the leftmost branch in the tree. If it does not go through a choice put its end point in the set, and go on to the next branch, going from left to right. If some branches go through only one choice, the end points of all the branches going to the left through the choice will be in one set, or series of sets, and the end points of all the branches going to the right through the choice will be in another set or sets. If some branches go through more than one choice, as branches 2,3, 6 and 7 do in our example, the end points of the branches going to the left through the topmost choice will be in one set or series of sets. The end points of the branches to the right will be in another set or sets.

In our example, we form the first set by putting the end point of the leftmost branch, B, in set 1. The branch ending in B passes through one choice, $Bv(CvD)$, so we would put into set 1 with B the end points of any other branches passing through this choice to the left and through no other choice to the left, if there were any. We omit the end points of all the branches to the right through this choice. The next branch, 4, also goes through one choice, $\sim\sim(E \wedge G)v(FvB)$, so we put E and G into set one, and omit all the end points of branches passing to the right through this choice. Our set 1 is (B,E,G). The second set is formed by putting into a set with B end points from the rightmost group of omitted end points, in this case the end points of the branches going to the right through the choice $\sim\sim(E\wedge G)v(FvB)$. Since these branches go through a further choice, we put the end points of the branches passing to the left through the topmost choice in our second set, F. Set 2 then is (B,F). In this way we form 9 sets for A. in our example.

1. (B, E, G)
2. (B, F)
3. (B)
4. (C, E, G)
5. (C, F)
6. (C, B)
7. (D, E, G)
8. (D, F)
9. (D, B)

I. We shall need a property of these sets later: $\vdash_c A \rightarrow C_0$, where C_0 is a disjunction each disjunct of which is a conjunction of all the members of one set in the series formed in stage 1, and there is one disjunct for each set. When there is only one set in the series this is clear, since each step in the tree corresponds to a theorem of C. If A includes a choice, say A is $(C \vee D) \wedge E$, then since $\vdash_c ((C \vee D) \wedge E) \rightarrow ((C \wedge E) \vee (D \wedge E))$, $\vdash_c A \rightarrow C_0$.

Stage 2. To each of k_1', ..., k_n' defined in stage 1 we add new wffs as members so that each of the resulting enlarged sets satisfies C1-C10.

C1. First apply C1 in the direction C1.2 - If $A \in k$ and $B \in k$, then $A \wedge B \in k$, to each wff in a set in order. The conjunction of each wff and each one of the wffs occurring before it in the set or itself is a new member of the set. Both the conjunction with a wff as the left conjunct and with it as the right conjunct are added to the set as members. We apply C1.1 in the other direction - If $A \wedge B \in k$, then $A \in k$ and $B \in k$, only to wffs not added as members by applications of C1.2 to the set. Each time it is applied to a wff each conjunct is a new member of the set.

Example. Take set 1 formed in our example in stage 1, $\{B, E, G\}$. After the first application of C1 set 1 is $\{B, E, G, B \wedge B, E \wedge B, B \wedge E, E \wedge E, G \wedge B, B \wedge G, G \wedge E, E \wedge G, G \wedge G\}$.

C2. We apply C2 to each wff in the set defined by C1, first in the direction C2.2 - If $A \in k$, then $\sim\sim A \in k$. C2.1 - If $\sim\sim A \in k$, then $A \in k$, is applied then only to wffs not added as members by C2.2. After the application of C2 set 1 is $\{B, E, G, \ldots, G \wedge G, \sim\sim B, \sim\sim E, \sim\sim G, \sim\sim(B \wedge B), \sim\sim(E \wedge B), \sim\sim(B \wedge E), \sim\sim(E \wedge E), \sim\sim(E \wedge E) \sim\sim(G \wedge B), \sim\sim(B \wedge G), \sim\sim(G \wedge E), \sim\sim(E \wedge G), \sim\sim(G \wedge G)\}$

C3. We apply C3 to the set defined by the application of C2, again first in the direction C3.2 - If $A \in k$ or $B \in k$, then AvB k. To do this for each wff in the set we add as the other disjunct, first as the right and then as the left disjunct, one wff from the enumeration of wffs of C in lemma 1, $A_1, \ldots, A_n, \ldots$. The first time C3.2 is applied to a particular wff we use A_1 as the other disjunct, the second time A_2, etc. It is clear that in this way we define each disjunction one of whose disjuncts is a wff already in the set.

Example. If D is a wff in k' and C3.2 has already been applied to D n-1 times, then at the nth time we define two new wffs DvA_n and A_nvD.

We apply C3 in the direction C3.1 - If $AvB \in k$, then $A \in k$ or $B \in k$, only to wffs not added as members to the set by C3.2. Each time C3.1 is applied in a set k_i' one disjunct, the left one is a new member of k_i'. The right disjunct is a new member of a new set k_{n+1}', which is the same as k_1' up to the wff which was the left disjunct of the disjunction to which C3.1 was applied. Add this new set to the end of the series of sets $k_1', \ldots, k_n'$. In none of the further series of applications of C3.1 is C3.1 applied to any wff it was applied to in an earlier application, since the set will already satisfy C3.1. In the new set k'_{n+1} the application of conditions is started exactly where it was in k'_1. If the application of C3.1 which defined the new set was the nth application of a condition to a wff in k'_1, the next application of a condition in either set is the n+1th. After the application of C3 set 1 is $\{B, E, G, \ldots, G \wedge G, \sim\sim B, \sim\sim E, \sim\sim G, \sim\sim(B \wedge B), \ldots, \sim\sim(G \wedge G),$ $B \vee A_1, A_1 \vee B, E \vee A_1, A_1 \vee E, G \vee A_1, A_1 \vee G, \ldots, \sim\sim(G \wedge G) \vee A_1, A_1 \vee \sim\sim(G \wedge G)\}$.

C4-C10. We apply each of these conditions in turn to the entire set defined by the earlier applications of conditions. After C10 start again with C1, etc.

Since C8 is also an "iff" condition, we divide it into two conditions, C8.1 and C8.2, one for each direction, as in C1-C3. We apply one of these first, say C8.1, to the set defined by the series of applications of C7, and then C8.2 to the set defined by applications of C8.1. Again we do not apply C8.2 to the wffs added as members by C8.1.

It is clear that A itself is eventually added to each set in the series as a member. By the definition of the sets in stage 1, repeated applications of C1.2, C2.2 and C3.2 will result in adding A to the set in a finite number of steps.

We have defined a series of sets $k'_1, \ldots, k'_n, \ldots$ each of which satisfies C1-C10 and has A as a member. We have defined one series for each wff of the form $\sim(A \rightarrow B)$ in Γ'. Let K' be the class of these series of sets and Γ' itself. It still remains only to prove that Γ' satisfies G1, G2, and C1-C10 in order to prove that K' and Γ' are the K and G respectively of some C-model $M=\langle K,G\rangle$.

1. ' satisfies G1. Since Γ' is a maximal consistent set and we proved earlier that for any maximal consistent set a wff A is a member iff $\sim A$ is not a member.

It follows from 1 that every theorem of G is a member of Γ'; for if $\vdash_c A$ then $\{\sim A\}$ is an inconsistent set and therefore not a subset of Γ', a consistent set. Therefore $A \in \Gamma'$.

2. If $A \in \Gamma'$ and $A \rightarrow B \in \Gamma'$, then $B \in \Gamma'$. Suppose not, then by 1 $\sim B \in \Gamma'$. Since Γ' is consistent, $\{A, A \rightarrow B, \sim B\}$ is a consistent set, and therefore $\vdash_c ((A \wedge (A \rightarrow B)) \wedge \sim B)$. But $\vdash_c \sim((A \wedge (A \rightarrow B)) \wedge \sim B)$ from A9 $((A \wedge (A \rightarrow B)) \rightarrow B) \rightarrow \sim(A \wedge (A \rightarrow B) \wedge \sim B)$, A8 $(A \wedge (A \rightarrow B)) \rightarrow B$ and mp $\rightarrow$. So $B \in \Gamma'$.

3. Since A2 $(A \wedge B) \rightarrow A$ and $\vdash_c A \rightarrow (A \vee B)$ are members of Γ' by the considerations in 1, C1 and C3 are satisfied in one direction by the use of these and 2. In the other direction obviously no consistent set containing A and B as members is made inconsistent by the introduction of $A \wedge B$ as a member. Γ' is a maximal consistent set so $A \wedge B \in \Gamma'$ if $A, B \in \Gamma'$. If $A \vee B \in \Gamma'$, then by 1 $\sim(A \vee B) \notin \Gamma'$. By df'v', th 3, and 2 $\sim A \wedge \sim B \in \Gamma'$. By C1 either $\sim A \notin \Gamma'$ or $\sim B \notin \Gamma'$. By 1 either $A \in \Gamma'$ or $B \in \Gamma'$.

4. The remaining conditions C2 and C4-C10 are shown to be

satisfied by the axioms, which are members of Γ' by 1, and 2.

Before we prove that Γ' satisfies G2 we shall prove a property of the series of sets defined in stage 2.

Lemma 4 $\vdash_c A \rightarrow C_k$ is a disjunction, one disjunct for each set in the series of sets defined at the kth application of a condition to a set, and each disjunct is a conjunction of all the wffs in that set. Since at the kth application of a condition to a set the number of sets in the series $k_2', \ldots, k_n'$ is finite, and each set has a finite number of members, there is such a disjunction.

Proof. Induction on the number, n, of times in each set a condition is applied to a wff.

Basis n=0. $\vdash_c A \rightarrow C_0$, where C_0 is the disjunction at the end of stage 1, by I. stage 1.

Induction step - Assume for $n = k-1$ applications, and prove for $n=k$ applications. D_i is a conjunction of all the members of a set at the ith application of a condition to a wff in the set.

First we shall prove that if D_k is formed from D_{k-1} by adding as a conjunct to D_{k-1} a new wff defined by the application of a condition to a conjunct of D_{k-1}, $\vdash_c D_{k-1} \rightarrow D_k$. 10 cases, one for each condition.

1. Cl.1 If $A \wedge B$ is a conjunct in D_{k-1} and A is defined by Cl.1 and made a conjunct of D_k, then $\vdash_c D_{k-1} \rightarrow D_k$ since $\vdash_c ((A \wedge B) \wedge C) \rightarrow (((A \wedge B) \wedge C) \wedge A)$.

Cl.2 If A,B are two conjuncts in D_{k-1} and $A \wedge B$ is added to D_{k-1} as a conjunct to form D_k, then $\vdash_c D_{k-1} \rightarrow D_k$ since $\vdash_c ((A \wedge C) \wedge B) \rightarrow ((((A \wedge C) \wedge C) \wedge A \wedge B)$.

2. C2.1 If $\sim\sim A$ is a conjunct in D_{k-1}, by $\vdash_c (\sim\sim A \wedge B) \rightarrow ((\sim\sim A \wedge B) \wedge A)$, $\vdash_c D_{k-1} \rightarrow D_k$.

C2.2 Use $\vdash_c (A \wedge B) \rightarrow ((A \wedge B) \wedge \sim\sim A)$.

3. C3.1 If AvB is a conjunct in D_{k-1}, then $\vdash_c D_{k-1} \rightarrow (D_k v D_{k.1})$, where $D_{k.1}$ is the new set defined in stage 2, C3 above. This is shown by $\vdash_c ((AvB) \wedge C) \rightarrow ((((AvB) \wedge C) \wedge A) v (((AvB) \wedge C) \wedge B))$.

C3.2 If A is a conjunct in D_{k-1}, then $\vdash_c D_{k-1} \rightarrow D_k$ by $\vdash_c (A \wedge B) \rightarrow (A \wedge (B \wedge (AvC)))$.

4-10 In each case of C4-C10 we prove the theorem of C used to show that $\vdash_c D_{k-1} \rightarrow D_k$ by A2, DR Syll, Th 4 $(A \rightarrow A)$, adj, A9, df '=', DR Simp, and mp $\rightarrow$. E.g. C6.

1.	$((A \wedge (A \rightarrow B)) \wedge C) \rightarrow (A \wedge (A \rightarrow B))$	A2
2.	$(A \wedge (A \rightarrow B)) \rightarrow B$	A8
3.	$((A \wedge (A \rightarrow B)) \wedge C) \rightarrow B$	1,2 DR Syll
4.	$((A \wedge (A \rightarrow B)) \wedge C) \rightarrow ((A \wedge (A \rightarrow B)) \rightarrow C)$	Th 4
5.	$(((A \wedge (A \rightarrow B)) \wedge C) \rightarrow ((A \wedge (A \rightarrow B)) \wedge C)) \wedge (((A \wedge (A \rightarrow B)) \wedge C) \rightarrow B)$	3,4 adj
6.	$((A \wedge (A \rightarrow B)) \wedge C) \rightarrow (((A \wedge (A \rightarrow B)) \wedge C) \wedge B)$	A9, Df '=' DR Simp, 6, mp $\rightarrow$

This completes the proof that $\vdash_c D_{k-1} \rightarrow D_k$.

Since $\vdash_c D_{k-1} \rightarrow D_k$ for each disjunct in C_{k-1}, then where iD_{k-1} is the i-th disjunct in C_{k-1}, then

1	$iD_{k-1} \rightarrow iD_k$	by the above
2	$iD_k \rightarrow (iD_k v\ i{+}1D_k)$	Th9
3	$iD_{k-1} \rightarrow (iD_k v\ i{+}1D_k)$	1,2 DR Syll

4. $i{+}1D_{k-1} \rightarrow (iD_k \vee i{+}1\ D_k)$ as 3

5. $(iD_{k-1} \vee i{+}1D_{k-1}) \rightarrow (iD_k \vee i{+}1D_k)$ Th10, DR Simp, adj 3,4 mp$\rightarrow$.

That is $\vdash_c C_{k-1} \rightarrow C_k$. By the induction hypothesis $\vdash_c A \rightarrow C_{k-1}$. Therefore by Dr Syll $\vdash_c A \rightarrow C_k$. This completes the proof of lemma 4.

We still need to prove that Γ' satisfies G2 in order to prove that K' and Γ' are the K and G of some C-model $M = \langle K, G \rangle$.

5. Γ' satisfies G2.

Proof. For every wff of the form $\sim(A \rightarrow B)$ in Γ' we have defined a series of sets in stages 1 and 2 above. In this series there is at least one set $k_i^!$ in which $A \in k_i^!$ and $B \notin k_i^!$. If not, then B is a member of every set at some point in the definition of the sets, say by the k-th application of a condition to a wff in every set and A will also be a member of every set by some point, again say the Kth application. By lemma 4 $\vdash_c A \rightarrow C_k$, where C_k is as in lemma 4, and if B is a conjunct in every disjunct in C_k, then by $\vdash_c ((B \wedge C) \vee (B \wedge D)) \rightarrow B$, $\vdash_c C_k \rightarrow B$. By DR Syll $\vdash_c A \rightarrow B$. Since all theorem of C are in Γ', $A \rightarrow B \in \Gamma'$. Γ' satisfies G1 by 1 above so $\sim(A \rightarrow B) \notin \Gamma'$. But $\sim(A \rightarrow B) \in \Gamma'$, therefore there is at least one set $k_i^!$ in K' such that $A \in k_i^!$ and $B \notin k_i^!$. Γ' satisfies G2.

This completes the proof of lemma 3 - that any consistent set is satisfiable in some C-model M.

<u>Theorem 3.</u> If A is valid, then $\vdash_c$ A.

<u>Proof</u> Suppose $\nvdash_c$A, then $\{\sim A\}$ is a consistent set. By lemma 3 there is some C-model M in which it is satisfiable. By G1 A is not satisfiable in this model. Therefore A is not valid. If a is valid, then $\vdash_c$A.

<u>Theorem 4</u> $\vdash_c$A iff A is valid. Directly from theorems 2 and 3.

<u>On a decision procedure for C.</u>

The formal semantics for C has a necessary condition, G2, but not a sufficient condition for the membership of $\sim(A \rightarrow B)$ in G in a C-model M. We can use the reductio method in Theorem 2 to decide whether a wff is an instance of a tautologous schema, and therefore to decide for wffs not involving a '$\rightarrow$' connective whether they are theorems. But the existence of some set $k \in K$ with A as a member and which can be shown not to have B as a member is not a sufficient condition for $\sim(A \rightarrow B) \in G$ in some C-model M $= \langle K, G \rangle$. Example: Take $(p' \rightarrow \sim(p' \rightarrow \sim p'))$, here p' is the antecedent and $\sim(p' \rightarrow \sim p')$ the consequent. We use stage 1 in lemma 3 to define our initial series of sets from the antecedent. There is only one set, k, and k has only one member, p' itself. We use stage 2 to increase k to a series of sets each of which will satisfy C1-C10. Again there will be only one set, since all disjunctions introduced into k are defined by applications of C3.2, and C3.1, the condition
USED TO DEFINE

new sets in the series, is not applied to wffs defined by C3.2 Since the only condition that can introduce wffs into k that contain a '$\rightarrow$' connective is C3.2, all wffs introduced by this condition are disjunctions. This means that there can be no wffs in k which have a '$\rightarrow$' as their main connective or the negations of such wffs - so the consequent $\sim(p' \rightarrow \sim p')$ is not a member of k. k satisfies C1-C10. But $\sim(p' \rightarrow \sim(p' \rightarrow \sim p') \notin G$ in any C-model M since $\vdash_C p' \rightarrow \sim(p' \rightarrow \sim p')$ and therefore a member of G in every C-model M.

We need a set of conditions strong enough to make G2 a sufficient and necessary condition. Then, hopefully, we would be able to set out a decision procedure using them. The following set of conditions C1' - C7' has not been proved to make G2 both sufficient and necessary, but it has worked for all the wffs known to be theorems or non-theorems tried on it.

C1'. $A \wedge B \in k$ iff $A \in k$ and $B \in k$.

C2'. $A \vee B \in k$ iff $A \in k$ or $B \in k$.

If $(A \rightarrow B) \wedge (B \rightarrow C) \in k$, then $A \rightarrow C \in k$.

C3'. ~~If A B k, then (B C) (A C) k.~~

~~If ((B C) (A C)) k, then (A B) k.~~

If $\sim(A \rightarrow C) \in k$, then $\sim((A \rightarrow B) \wedge (B \rightarrow C)) \in k$.

C4'. If $A \wedge (A \rightarrow B) \in k$, then $B \in k$.

If $\sim B \in k$, then $\sim(A \wedge (A \rightarrow B)) \in k$.

C5'. If $A \rightarrow B \in k$, then $\sim(A \wedge \sim B) \in k$.

If $A \wedge \sim B \in k$, then $\sim(A \rightarrow B) \in k$.

C6'. If $A \rightarrow (B \rightarrow C) \in k$, then $(A \wedge B) \rightarrow C \in k$.

If $\sim((A \wedge B) \rightarrow C) \in k$, then $\sim(A \rightarrow (B \rightarrow C)) \in k$.

C7'. If $A \equiv B \in G$ and $\alpha \in k$, then $\beta \in k$.
If $\sim\beta \in k$, then $\sim(A \equiv B) \in G$ or $\sim\alpha \in k$.

We can omit any conditions similar to C2, C5 and C8, since these will be provided for by the new condition C7'.
If A is a member of some set k satisfying C1'-C7', then $\sim\sim A$ will be a member of k, since $\vdash_C A \equiv \sim\sim A$ hence in G in every C-model; and by C7' $\sim\sim A \in k$.

This semantics easily covers our previous example. When $p' \in k$, by C7' $\sim\sim p' \in k$ since $p' \equiv \sim\sim p' \in G$. By C1' $p' \wedge \sim\sim p' \in k$, and by C5' $\sim(p' \rightarrow \sim p') \in k$, which is our consequent. Therefore $\sim(p' \rightarrow \sim(p' \rightarrow \sim p'))$ cannot be in G in any C-model, so $\vdash_C p' \rightarrow \sim(p' \rightarrow \sim p')$.

However, it is now known whether it is possible or not to give a decision procedure even with this semantics. C7' in particular will be troublesome, but there does not seem to be any way to weaken it. We need to have in k wffs like $A \equiv \sim\sim A$, $(A \wedge B) \equiv (B \wedge A)$, etc. which are theorems of C and so in G in every C-model M. Without the guarantee that, say $A \equiv \sim\sim A$, is in every set we could define a set $k \in K$ in which $A \rightarrow B$ is a member and $\sim\sim A \rightarrow B$ is not. But $\vdash_C (A \rightarrow B) \rightarrow (\sim\sim A \rightarrow B)$, so $\sim((A \rightarrow B) \rightarrow (\sim\sim A \rightarrow B) \notin G$ in any C-model M, thus defeating our efforts to make G2 a sufficient as well as a necessary condition.

Comparison of C with other systems designed to avoid paradoxes

C contains Hallden's system (9) and the matrices he gives show that it does not contain C. C also contains Sugihara's system SA (15). SA is the same as Hallden's system with our A7 added to it and two other defined connectives, (our notation) $\Diamond A = \sim(A \rightarrow \sim A)$ and $A \dashv B = \sim \Diamond (A \wedge \sim B)$. SA does not contain C. The following 4-valued matrices (non-characteristic) assign a designated value to all the axioms of SA, and the rules of inference of SA preserve this property. A8 does not take a designated value for the assignment A=2 and B=1. 2 and 3 are designated.

$A \rightarrow B$	0	1	2	3
0	3	3	3	3
1	0	3	3	3
2	0	1	3	3
3	0	0	0	3

A	$\sim$A
0	3
1	2
2	1
3	0

$A \wedge B$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	2	2
3	0	1	2	3

Sugihara suggests adding other axioms to SA, one of which is A8 but he does not show that any of them are independent, only that the resulting system is free of implicational paradoxes.

C seems neither to contain nor be contained in the propositional, non-model portion of Anderson and Belnap's system E (1), (6). E lacks A13 and R3 (mp $\supset$), and I do not know whether E has A12 as a theorem. E has as axioms E1 $((A \rightarrow A) \rightarrow B) \rightarrow B$ and E3 $((A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$ neither of which seem to be theorems of C.

The most interesting system for comparison with C is the propositional part of Barkers (5) system SIR. SIR contains the three systems mentioned above and is highly similar to C, but does not seem to contain nor be contained by C. The rules of inference of SIR are the same as those of C or can be derived in C. The axioms are all theorems of C except 1.08 $(\sim(\sim A \to B) \to C) = (\sim A \to (\overset{\sim B}{\cancel{B}} \to C))$. From this can be derived permutation, $(A \to (B \to C)) = (B \to (A \to C))$. It is not known whether this wff is a theorem of C. It was not made an axiom in accordance with a possible counter example offered by V. Routley.[1] Substitute Arthur knows that if A then A/ A, A/B, A/C. The formula then is in ordinary language, 'If Arthur knows that if A then A, then if A then A iff if A, then if Arthur knows that if A then A, then A.' The left side is true so if we detach it, and then if A is true and we detach it, we are left with 'If Arthur knows that if A then A, then A', which is not true.

If we decide to reject this counter example and that we want permutation in the system C, we can add it as A14 and add a new condition to the semantics, C11 If $A \to (B \to C) \in k$, then $B \to (A \to C) \in k$. We do not need to make C11 an 'iff' condition. This new system C then contains SIR.

SIR contains as theorems and rules of inference all of C except A11 and A12. If A11 is derivable in SIR, Barker does not list it in his list of theorems, and A12 does not seem to be derivable.

1. (13) p20-21.

SIR does have a rule of inference corresponding to both of these; if $\vdash_{SIR} A \to (B \to C)$, then $\vdash_{SIR} (A \wedge B) \to C$ and if $\vdash_{SIR} A = B$, then A and B may replace each other wherever they occur.

Barker gives (p 82) two sets of non-characteristic matrices which he uses to prove that some wffs are non-theorems of SIR.

Set 1 (Designated values 2,3)[1]

A	~A
0	3
1	2
2	1
3	0

A → B	0	1	2	3
0	3	3	3	3
1	0	2	2	3
2	0	1	2	3
3	0	0	0	3

A ∧ B	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	2	2
3	0	1	2	3

Set 2 (Designated values 4, 5, 6, 7)

A	~A
0	7
1	6
2	5
3	4
4	3
5	2
6	1
7	0

A → B	0	1	2	3	4	5	6	7
0	7	7	7	7	7	7	7	7
1	0	6	0	6	0	0	6	7
2	0	0	5	5	0	5	0	7
3	0	0	0	4	6	0	0	7
4	0	1	2	3	4	5	6	7
5	0	0	2	2	0	5	0	7
6	0	1	0	1	0	0	6	7
7	0	0	0	0	0	0	0	7

A ∧ B	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	0	1	0	0	1	1
2	0	0	2	2	0	2	0	2
3	0	1	2	3	0	2	1	3
4	0	0	0	0	4	4	4	4
5	0	0	2	2	4	5	4	5
6	0	1	0	1	4	4	6	6
7	0	1	2	3	4	5	6	7

In 6 A1-A11 and A13 are always assigned designated values in these matrices and the rules of inference preserve this property. Due to the form of A12 it is not clear whether A12 always takes a designated value or not. If A=B takes a designated value then by

1. The set 1 matrices are a finite adaptation of matrices in Sugihara(15). The matrix for '→' was used in Anderson and Belnap (1). The set 2 matrices were used in Belnap (7)

examining the matrices for '$\wedge$' and '$\rightarrow$', we see that in both sets this can only be if A and B take the same values. In that case α and β must take the same values. When $A \equiv B$ is not designated it is not clear what values α and β may take. However, none of the wffs listed below, excluded by the matrices from SIR, could be theorems of C, since the matrices satisfy all axioms and rules of inference that could be used in their proof. If any of these wffs had a proof in C which used A12 then they must also use mp , or they would be in the form of A12, which they are not. In order to use mp $\rightarrow$ the antecedents must be theorems, so any use of A12 could be replaced by DR Repl. If $\vdash_C A \equiv B$ and $\vdash_C \alpha$, then $\vdash_C \beta$. This rule is in SIR and preserves designated values for these matrices. Therefore the following non-theorems of SIR are also non-theorems of C.

An assignment for which the wffs take a non-designated value is given after them. Not all of Barker's non-theorems are listed. Set 1.

X1 $(A \supset B) \rightarrow (A \rightarrow B)$ (1,0)

X2 $A \rightarrow (B \rightarrow A)$ (1,3)

X3 $\sim A \rightarrow (A \rightarrow B)$ (1,0)

X4 $A \rightarrow (B \vee \sim B)$ (3, 1)

X5 $(A \wedge \sim A) \rightarrow B$ (1, 0)

X6 $((A \vee B) \wedge \sim A) \rightarrow B$ (1, 0)

X7 $A \rightarrow ((A \wedge \sim B) \vee (A \wedge B))$ (3, 2)

to show that if '$A \rightarrow B$' is provable in E, then A and B share a variable. (Barker's note p 82-83)

X8 $A \rightarrow (A \rightarrow B)$ (1, 0)

X9 $((A \wedge B) \rightarrow C) = ((A \wedge \sim C) \rightarrow \sim B)$ (1, 1, 0)

X10 $((A \wedge B) \rightarrow C) = (A \rightarrow (B \rightarrow C))$ (3, 1, 1)

X11 $(A \rightarrow B) \vee (A \rightarrow \sim B)$ (3, 1)

X12 $(A \rightarrow B) \rightarrow ((C \wedge A) \rightarrow (B \wedge C)$ (1,3,1)

X13 $(A \rightarrow \sim (B \wedge C)) \rightarrow ((B \wedge A) \rightarrow \sim C)$ (1, 1, 3)

X14 $(A \rightarrow B) \rightarrow \sim (A \rightarrow \sim B)$ (0, 0)

X15 $(A \rightarrow \sim A) \rightarrow (A \rightarrow B)$ (1,0)

X16 $(\sim A \rightarrow A) \rightarrow (B \rightarrow A)$ (1, 3)

X17 $(A \wedge (A \supset B)) \rightarrow B$ (2, 0)

Set 2

X18 $((A \wedge B) \rightarrow C) \rightarrow ((A \rightarrow C) \vee (B \rightarrow C))$ (1, 2, 6)

X19 $(A \wedge \sim A) \rightarrow (B \vee \sim B)$ (5, 6)

X20 $A \rightarrow (B \rightarrow (A \wedge B))$ (2, 1)

X21 $((A \vee \sim A) \wedge ((A \vee \sim A) \supset (B \vee \sim B))) \rightarrow (B \vee \sim B)$ (5, 6)

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