A representation-independent form of Drinfel'd's map for the quantum group $U_{\mathrm{q}}(sl_2)$.

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by

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Abstract.

The quantum universal enveloping algebra $U_q(sl_2)$ is introduced. Methods for obtaining $U_q(sl_2)$ from an enveloping algebra $U(sl_2)$ over the Lie algebra sl_2 are discussed, with emphasis on the twist map $F: U(sl_2) \to U_q(sl_2)$, proposed by V. G. Drinfel'd. A form for the twist element $F \in U(sl_2) \otimes U(sl_2)$ which transposes with the substitution $q \to q^{-1}$ is derived. Next semirepresentations $\mathbf{l}^{\pm(j)}(u)$ are defined as (2*j*+1)-th order operator valued matrix representations for elements of the form $u \in U(sl_2) \otimes U(sl_2)$, with explicit construction of a set of second and third order semirepresentations $\mathbf{l}^{\pm}(F)$. A partial construction of Drinfel'd's twist element F is then effected from $\mathbf{l}^{\pm}(F)$. Finally, a general form for $F \in U(sl_2) \otimes U(sl_2)$ is induced from this construction.

Preface.

The genesis of the modern study of quantum groups, or quasitriangular Hopf algebras, is generally traced to a paper[1] delivered before the 1986 International Congress of Mathematicians by the Russian algebraicist, V. G. Drinfel'd. Abstract Hopf algebras had been studied by mathematicians[2] for their interesting symmetry properties for almost a century, but at that time had only just come under scrutiny by physicists as providing possible solutions for a range of physical problems. These solutions took the form of representations ('R-matrices') of elements satisfying the quantum Yang-Baxter equation (QYBE), $\overline{R}_{12}\overline{R}_{13}\overline{R}_{23} = \overline{R}_{23}\overline{R}_{13}\overline{R}_{12}$, developed by the theoretical physicist C.N. Yang[3] to describe order invariance in particle scattering.

The explicit (and by now well known) form for the quantum universal enveloping algebra $U_q(sl_2)$ as a q-deformed enveloping algebra over the complex Lie algebra sl_2 is due in large part to two other Russian mathematicians, P. P. Kulish and N. Yu. Reshetikhin.[4] In this algebra R-matrices play a central role in defining the quasicocommutativity–more specifically the quasitriangularity–of representations of the coalgebra structure of $U_q(sl_2)$, through the gauge transformation $\overline{\Delta}u_{21} = \overline{R}\overline{\Delta}u\overline{R}^{-1}$. Drinfel'd spent several years studying mechanisms for these transformations, publishing a series of papers in which he developed the hypothesis that a representation-independent form for the quasitriangular structure $\overline{R}$ can be obtained directly via a twist map $\overline{R} = F_{21}RF^{-1}$ from a corresponding symmetric structure R associated with an undeformed enveloping (tensor) algebra $U(sl_2)$ over the Lie algebra sl_2. The twist element F will then take the form of a counital 2–cocycle, an element of the algebra $U(sl_2) \otimes U(sl_2)$.

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[1] Drinfel'd (1986).
[2] see Sweedler (1969).
[3] Yang (1967).
[4] Kassel (1995), p. 138.

This thesis outlines and implements a method for constructing an explicit representation-independent form for Drinfel'd's twist map element $F \in U(sl_2) \otimes U(sl_2)$, treated as a q-dependent power series in $S_{\pm} \otimes S_{\mp}$. The first chapter derives several general properties of this counital 2–cocycle, first with respect to the substitution $q \to q^{-1}$, then as an expansion in the deforming parameter h. The second chapter presents the general mathematical framework for this paper in defining the $(2j+1)$-th order semirepresentations $\mathbf{l}^{\pm(j)}(u)$ as operator valued matrix representations of elements of the form $u \in U(sl_2) \otimes U(sl_2)$. It is demonstrated that these semirepresentations $\mathbf{l}^{\pm(j)}(u)$ uniquely determine the first $(2j+1)$ terms of the elements u as power series in $S_{\pm} \otimes S_{\mp}$, which will be shown to be the form taken by the twist element F. The two main chapters of this paper are then explicit constructions of second and third order semirepresentations $\mathbf{l}^{\pm}(F)$ leading directly to a partial construction of F to third order in $S_{\pm} \otimes S_{\mp}$. The final chapter presents a short, inductive argument for a general representation-independent form of Drinfel'd's twist element F.

Contents.

Chapter One. Introduction: Drinfel'd's construction of $U_q(sl_2)$

I.1. *The quantum universal enveloping algebra $U_q(sl_2)$.*

A logical starting point is the definition of the Hopf algebra $U_q(sl_2)$. Conditions for $U_q(sl_2)$ as a quantum group are set down. The main result of this section is then the construction of explicit forms for the quasitriangular structure $\overline{R} \in U_q(sl_2) \otimes U_q(sl_2)$, both as a series in $\overline{S}_+ \otimes \overline{S}_-$ and an expansion in the deforming parameter h. As this section is necessarily introductory, the forms of the definitions will closely follow those in the texts of Majid and Kassel.[1]

Definition I. The quantum universal enveloping algebra (QUEA) $U_q(sl_2)$ is generated by the elements $\overline{S}_\pm, \overline{S}_z$ satisfying a multiplication rule $U_q(sl_2) \otimes U_q(sl_2) \to U_q(sl_2)$ defined by the q-dependent commutation relations

$$[\overline{S}_z, \overline{S}_\pm] = \pm \overline{S}_\pm,$$

$$[\overline{S}_+, \overline{S}_-] = [2\overline{S}_z]_q. \tag{1.1}$$

The elements $\overline{S}_\pm, \overline{S}_z$ then generate $U_q(sl_2)$ over the field of complex numbers. The q–bracket $[u]_q$ is defined in terms of a complex-valued deforming parameter, $q \equiv e^h$, assumed to be non-zero and not to be a root of unity. Thus

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[1] Majid (1995), sections 2.1 and 3.2; Kassel (1995), chapters VI and VII.

$$[u]_q \equiv \frac{q^u - q^{-u}}{q - q^{-1}} = q^{u-1} + q^{u-1} + \ldots + q^{-(u-3)} + q^{-(u-1)}; \tag{1.2}$$

$$[-u]_q = -[u]_q,$$

$$[u' + u'']_q = q^{u''}[u']_q + q^{-u'}[u'']_q,$$

for all $u, u', u'' \in U_q(sl_2)$. From its definition, $[u]_q \to u$ as $q \to 1$, so that (1.1) reduce to the commutators for the complex Lie algebra sl_2 in the limit $q \to 1$, the bar notation indicating this q-dependence. $U_q(sl_2)$ is unital with identity element 1. The coproduct $\overline{\Delta}: U_q(sl_2) \to U_q(sl_2) \otimes U_q(sl_2)$ and counit then becomes

$$\overline{\Delta} q^{\pm \overline{S}_z} \equiv q^{\pm \overline{S}_z} \otimes q^{\pm \overline{S}_z},$$

$$\overline{\Delta}\overline{S}_{\pm} \equiv \overline{S}_{\pm} \otimes q^{\overline{S}_z} + q^{-\overline{S}_z} \otimes \overline{S}_{\pm}; \tag{1.3}$$

$$\varepsilon q^{\pm \overline{S}_z} \equiv 1, \ \varepsilon \overline{\mathrm{S}}_{\pm} \equiv 0.$$

Note that the coproduct structure twists with the substitution $q \to q^{-1}$. That is, defining $\overline{\Delta}u(q^{-1})$ such that $\overline{\Delta}u \to \overline{\Delta}u(q^{-1})$ with $q \to q^{-1}$,

$$\overline{\Delta}u(q^{-1}) = \overline{\Delta}u_{21} \tag{1.4}$$

for all $u \in U_q(sl_2)$. An antipode map $\overline{S}: U_q(sl_2) \to U_q(sl_2)$ then makes $U_q(sl_2)$ a Hopf algebra,

$$\overline{S}\overline{S}_{\pm} \equiv -q^{\pm 1}\overline{S}_{\pm}, \ \overline{S} q^{\pm \overline{S}_z} \equiv q^{\mp \overline{S}_z}. \tag{1.5}$$

Definition II. The Hopf algebra $U_q(sl_2)$ is quasicocommutative if there exists an invertible element $\overline{R} \in U_q(sl_2) \otimes U_q(sl_2)$ such that

$$\overline{\Delta}u_{21} = \overline{R}\,\overline{\Delta}u\overline{R}^{-1} \tag{1.6}$$

for all $u \in U_q(sl_2)$. The map (1.6) then defines $U_q(sl_2)$ as a quantum group provided $\overline{R}$ satisfies the quasitriangularity conditions,

$$(\overline{\Delta} \otimes id)\overline{R} = \overline{R}_{13}\overline{R}_{23}, \tag{1.7}$$

$$(id \otimes \overline{\Delta})\overline{R} = \overline{R}_{13}\overline{R}_{12}$$

with

$$(\varepsilon \otimes id)\overline{R} = (id \otimes \varepsilon)\overline{R} = 1; \tag{1.8}$$

$$(\overline{S} \otimes id)\overline{R} = \overline{R}^{-1}, \;\; (id \otimes \overline{S})\overline{R}^{-1} = \overline{R}.$$

After some algebra (1.7) gives $\overline{R}_{12}\overline{R}_{13}\overline{R}_{23} = \overline{R}_{23}\overline{R}_{13}\overline{R}_{12}$, the quantum Yang-Baxter equation referred to in the introduction. Majid derived an explicit form[2] for a quasitriangular structure $\overline{R} \in U_q(sl_2) \otimes U_q(sl_2)$ satisfying (1.6) which can be treated as a series in $\overline{S}_+ \otimes \overline{S}_-$,

$$\begin{aligned}\overline{R} &= q^{2\overline{S}_z \otimes \overline{S}_z} \sum_{n=0}^{\infty} \frac{q^{-n}p^n}{[n]_q!} q^{\frac{n(n-1)}{2}} (q^{\overline{S}_z}\overline{S}_+ \otimes q^{-\overline{S}_z}\overline{S}_-)^n \\ &= q^{2\overline{S}_z \otimes \overline{S}_z}\Big(1 \otimes 1 + q^{\pm 1}p(q^{\overline{S}_z} \otimes q^{-\overline{S}_z})(\overline{S}_+ \otimes \overline{S}_-) \\ &\qquad + \frac{qp^2}{[2]_q}(q^{2\overline{S}_z} \otimes q^{-2\overline{S}_z})(\overline{S}_+^2 \otimes \overline{S}_-^2) + \ldots\Big)\end{aligned} \tag{1.9}$$

where $p = q - q^{-1}$ and $[n]_q! \equiv [n]_q[n-1]_q \ldots [1]_q$. This form for the quasitriangular structure $\overline{R}$ can then be expanded in the deforming parameter *h*,

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[2]Majid (1995), p. 83.

$$\overline{R} = \sum_{n=0}^{\infty} r_{(n)} h^n = 1 \otimes 1 + rh + \mathrm{O}(h^2), \tag{1.10}$$

$$r \equiv 2S_z \otimes S_z + 2S_+ \otimes S_- .$$

Here r is a solution to the classical Yang-Baxter equation. Higher order terms can also be constructed, but are not necessary to the calculations which follow.

Proposition I. From its definition as a twist map (1.6), the quasitriangular structure $\overline{R} \in U_q(sl_2) \otimes U_q(sl_2)$ must invert with the substitution $q \to q^{-1}$. Defining $\overline{R}(q^{-1})$ such that $\overline{R} \to \overline{R}(q^{-1})$ with $q \to q^{-1}$,

$$\overline{R}(q^{-1}) = \overline{R}_{21}. \tag{1.11}$$

Proof. The proof follows from the definition of $\overline{R}$ and the observation that the coproduct (1.3) twists with the substitution $q \to q^{-1}$. Equating (1.4, 1.6) and substituting $q \to q^{-1}$,

$$\begin{aligned}\overline{\Delta} u_{21}(q^{-1}) &= \overline{R}(q^{-1}) \overline{\Delta} u(q^{-1}) \overline{R}^{-1}(q^{-1}) \\ &= \overline{R}(q^{-1}) \overline{\Delta} u_{21} \overline{R}^{-1}(q^{-1}) \\ &= \overline{R}(q^{-1}) \overline{R} \overline{\Delta} u \overline{R}^{-1} \overline{R}^{-1}(q^{-1}) \\ &= (\overline{R}(q^{-1}) \overline{R}) \overline{\Delta} u (\overline{R}(q^{-1}) \overline{R})^{-1},\end{aligned}$$

$$\therefore \qquad \overline{\Delta} u = (\overline{R}(q^{-1}) \overline{R}) \overline{\Delta} u (\overline{R}(q^{-1}) \overline{R})^{-1}$$

as $\overline{\Delta} u_{21}(q^{-1}) = \overline{\Delta} u$. The element $\overline{R}(q^{-1}) \overline{R} \in U_q(sl_2) \otimes U_q(sl_2)$ then commutes with $\overline{\Delta} u$ for all $u \in U_q(sl_2)$. In fact Majid's form for $\overline{R}$ fixes $\overline{R}(q^{-1}) \overline{R}$ as the identity, $1 \otimes 1$, whence it follows that $\overline{R}(q^{-1}) = \overline{R}^{-1}$. To show this, substitute $q \to q^{-1}$ in (1.9),

$$\overline{R}(q^{-1}) = q^{-2\overline{S}_z\otimes\overline{S}_z}\sum_{n=0}^{\infty}\frac{q^n(-p)^n}{[n]_q!}q^{-\frac{n(n-1)}{2}}(q^{-\overline{S}_z}\overline{S}_+\otimes q^{\overline{S}_z}\overline{S}_-)^n,$$

then construct $\overline{R}^{-1}$ from (1.9) and the map (1.8b). Using (1.5) and the fact that $\frac{1}{2}n(n-1) = (n-1)+(n-2)+\ldots+1$,

$$\overline{R}^{-1} = (\overline{S}\otimes id)\overline{R} = (\overline{S}\otimes id)q^{2\overline{S}_z\otimes\overline{S}_z}\sum_{n=0}^{\infty}\frac{q^{-n}p^n}{[n]_q!}q^{\frac{n(n-1)}{2}}(q^{\overline{S}_z}\overline{S}_+\otimes q^{-\overline{S}_z}\overline{S}_-)^n$$

$$= \sum_{n=0}^{\infty}\frac{q^{-n}p^n}{[n]_q!}q^{\frac{n(n-1)}{2}}(-q)^n(\overline{S}_+q^{-\overline{S}_z}\otimes 1)^n q^{-2\overline{S}_z\otimes\overline{S}_z}(1\otimes q^{-\overline{S}_z}\overline{S}_-)^n$$

$$= \sum_{n=0}^{\infty}\frac{q^{-n}p^n}{[n]_q!}q^{\frac{n(n-1)}{2}}(-q)^n q^n q^{-2(\overline{S}_z-n)\otimes\overline{S}_z}(q^{-\overline{S}_z}\overline{S}_+\otimes q^{-\overline{S}_z}\overline{S}_-)^n$$

$$= q^{-2\overline{S}_z\otimes\overline{S}_z}\sum_{n=0}^{\infty}\frac{q^n(-p)^n}{[n]_q!}q^{\frac{n(n-1)}{2}}(1\otimes q^{2\overline{S}_z})^n(q^{-\overline{S}_z}\overline{S}_+\otimes q^{-\overline{S}_z}\overline{S}_-)^n$$

$$= q^{-2\overline{S}_z\otimes\overline{S}_z}\sum_{n=0}^{\infty}\frac{q^n(-p)^n}{[n]_q!}q^{\frac{n(n-1)}{2}}q^{-2\{(n-1)+(n-2)+\ldots+1\}}(q^{-\overline{S}_z}\overline{S}_+\otimes q^{\overline{S}_z}\overline{S}_-)^n$$

$$= q^{-2\overline{S}_z\otimes\overline{S}_z}\sum_{n=0}^{\infty}\frac{q^n(-p)^n}{[n]_q!}q^{-\frac{n(n-1)}{2}}(q^{-\overline{S}_z}\overline{S}_+\otimes q^{\overline{S}_z}\overline{S}_-)^n = \overline{R}(q^{-1}),$$

which is the required result. ●

Definition III. The quantum inverse Killing form (QIKF) is an invertible element $\overline{Q}\in U_q(sl_2)\otimes U_q(sl_2)$ defined in terms of the quasitriangular structure $\overline{R}$ so as to commute with the coproduct structure of $U_q(sl_2)$.

Thus defining

$$\overline{Q} \equiv \overline{R}_{21}\overline{R}, \tag{1.13a}$$

it follows from (1.6) that

$$\overline{\Delta}u = \overline{Q}\Delta u\overline{Q}^{-1} \tag{1.13b}$$

for all $u \in U_q(sl_2)$.

I.2. *The map* $\tilde{\Phi}: U(sl_2) \to U_q(sl_2)$.

There exists a explicit form[3] for generators of the QUEA $U_q(sl_2)$ satisfying the commutation relations (1.1), as a function Φ of the elements of the complex Lie algebra sl_2. If an enveloping (tensor) algebra $U(sl_2)$ is constructed over sl_2, an invertible map $\tilde{\Phi}: U(sl_2) \to U_q(sl_2)$ is defined. The properties of this map, with the associated element $\Phi \in U(sl_2)$, are pivotal to the construction of an explicit form for Drinfel'd's map $F: U(sl_2) \to U_q(sl_2)$, and must be set out in detail.

Definition IV. The universal enveloping algebra $U(sl_2)$ is defined as the quotient algebra of the free associative algebra over the complex Lie algebra sl_2. The multiplication $U(sl_2) \otimes U(sl_2) \to U(sl_2)$ is unital with identity element 1, and is defined by the commutation relations for sl_2,

$$[S_\pm, S_z] = \pm S_\pm, \tag{1.14}$$

$$[S_+, S_-] = 2S_z,$$

modulo the relations $[u', u''] = u'u'' - u''u'$ for all $u', u'' \in U(sl_2)$. $U(sl_2)$ is then generated by the elements $1, S_z, S_\pm$. The bialgebra $U(sl_2)$ is counital with a primitive coproduct $\Delta: U(sl_2) \to U(sl_2) \otimes U(sl_2)$,

$$\Delta u \equiv u \otimes 1 + 1 \otimes u, \tag{1.15}$$

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[3] due to Curtright *et. al.* (1990).

$$\varepsilon u \equiv 0$$

for all $u \in U(sl_2)$. Finally, $U(sl_2)$ becomes a Hopf algebra with antipode map $S: U(sl_2) \to U(sl_2)$ given by

$$Su \equiv -u. \tag{1.16}$$

for all elements of sl_2.[4] Note that the QUEA $U_q(sl_2)$ is constructed so that $U_q(sl_2) \to U(sl_2)$ in the limit $q \to 1$, q being treated as a series in the deforming parameter h. Thus $U_q(sl_2)$ can be seen as a generalisation, a deformation of the enveloping algebra $U(sl_2)$ in the single complex-valued parameter q.

Definition V. Let $C \in U(sl_2)$ be the symmetric, second-order Casimir operator such that C is central in that it commutes with all elements u of the enveloping algebra $U(sl_2)$, and takes the form

$$C = 4S_z^2 + 2S_+S_- + \ 2S_-S_+ \equiv 4S(S+1). \tag{1.17a}$$

Thus

$$[C, u] = 0 \tag{1.17b}$$

for all $u \in U(sl_2)$. It can be shown that C actually generates the centre of $U(sl_2)$.[5] The central element S is defined so that for a given integer or half integer value of j, $S\phi_m^j = j\phi_m^j$, where ϕ_m^j, $m = -j, -j+1, \ldots, j$, are elements of the Hilbert space of functions chosen so that $S_z\phi_m^j = m\phi_m^j$. The ϕ_m^j then form a basis for the $2j+1$ dimensional carrier space of an irreducible representation of sl_2. Thus S is operator-valued but, unlike the Casimir operator C, $S \notin U(sl_2)$.[6]

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[4] Majid (1995), p.15.

[5] Kassel (1995), pp. 100, 103.

[6] The central element S should not be confused with the antipode map of (1.16). The latter will not appear in subsequent calculations, and is included purely for completeness.

Definition VI. Let $t \in U(sl_2) \otimes U(sl_2)$ be the symmetric, undeformed canonical two-tensor such that t commutes with the primitive coproduct defined on the enveloping algebra $U(sl_2)$, and takes the form

$$t \equiv 4S_z \otimes S_z + 2S_+ \otimes S_- + 2S_- \otimes S_+. \qquad (1.18a)$$

Thus

$$[t, \Delta u] = 0 \qquad (1.18b)$$

for all $u \in U(sl_2)$. [7] The map $\tilde{\phi}: U(sl_2) \otimes U(sl_2) \to U(sl_2)$ defined by the function $\phi(u' \otimes u'') = u''u'$; $u', u'' \in U(sl_2)$, then maps the canonical 2-tensor t to the (central) Casimir operator C.[8]

Proposition II. There exists an explicit form for the undeformed canonical two-tensor $t \in U(sl_2) \otimes U(sl_2)$ as a function of the Casimir operator C of (1.17a).[9] That is,

$$t = \frac{1}{2}(\Delta C - C \otimes 1 - 1 \otimes C). \qquad (1.19)$$

Proof. To show this, first construct the coproduct of the Casimir operator defined in (1.17a), then substitute for the primitive coproduct (1.15),

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[7] Kassel (1995), p. 405.

[8] Drinfel'd (1990a), p. 322.

[9] Kassel, *op. cit.*; Cornwell, unpublished notes.

$$\begin{aligned}
\Delta C &= \Delta(4S_z^2 + 2S_+S_- + 2S_-S_+) \\
&= 4(\Delta S_z)^2 + 2\Delta S_+\Delta S_- + 2\Delta S_-\Delta S_+ \\
&= 4(S_z \otimes 1 + 1 \otimes S_z)^2 + 2(S_+ \otimes 1 + 1 \otimes S_+)(S_- \otimes 1 + 1 \otimes S_-) \\
&\quad + 2(S_- \otimes 1 + 1 \otimes S_-)(S_+ \otimes 1 + 1 \otimes S_+) \\
&= 8S_z \otimes S_z + 4S_- \otimes S_+ + 4S_+ \otimes S_- + 4S_z^2 \otimes 1 + 41 \otimes S_z^2 \\
&\quad + 2S_+S_- \otimes 1 + 21 \otimes S_+S_- + 2S_-S_+ \otimes 1 + 21 \otimes S_-S_+ \\
&= 2t + C \otimes 1 + 1 \otimes C.
\end{aligned}$$

Rearranging then gives

$$t = \frac{1}{2}(\Delta C - C \otimes 1 - 1 \otimes C)$$

as required. ●

Definition VII. There exists a map $\tilde{\Phi}: U(sl_2) \to U_q(sl_2)$ from the generators of the enveloping algebra $U(sl_2)$ as elements of the complex Lie algebra sl_2, to the generators of the QUEA $U_q(sl_2)$ as satisfying the q-dependent commutation relations (1.1).

The element Φ is a functional of the central element S and the generators of $U(sl_2)$. Thus[10]

$$\Phi = \frac{[S - S_z]_q[S + S_z + 1]_q}{(S - S_z)(S + S_z + 1)}, \tag{1.20a}$$

so that

$$\bar{S}_z \equiv S_z,\ \bar{S}_+ \equiv \sqrt{\Phi}S_+,\ \bar{S}_- \equiv \sqrt{\Phi}S_-. \tag{1.20b}$$

[10]Curtright *et al.* (1991), p. 677.

The map $\widetilde{\Phi}: U(sl_2) \to U_q(sl_2)$ is then invertible provided the parameter q is not a root of unity. This form (1.20b) was generalised[11] in a single parameter λ, which enables an algebraically simplified variation to be defined which eliminates the radicals, but which still satisfies the deformed commutators (1.1),

$$\overline{S}_z \equiv S_z,\ \overline{S}_+ \equiv S_+,\ \overline{S}_- \equiv \Phi S_-. \tag{1.20c}$$

This is the form of the map $\Phi: U(sl_2) \to U_q(sl_2)$ that will be used throughout this thesis.

Notation. The form (1.20a) and subsequent calculations will be greatly simplified with the definition of the elements a, b, c, d as functionals of S and S_z,

$$\begin{aligned} a &\equiv S - S_z, \\ b &\equiv S + S_z + 1, \\ c &\equiv a + b = 2S + 1, \\ d &\equiv b - a = 2S_z + 1 \in U(sl_2), \end{aligned} \tag{1.21}$$

so that $\Phi \equiv (ab)^{-1}[a]_q[b]_q$. With the definition (1.17a) of the undeformed Casimir operator C, $c^2 = C + 1 \in U(sl_2)$, so that as the q-bracket $[c]_q$ involves an expansion in odd powers of c, $c^{-1}[c]_q \in U(sl_2)$. Finally,

$$4ab = 4S(S+1) - 4S_z(S_z+1) = c^2 - d^2 \in U(sl_2). \tag{1.22}$$

As $2S = a + b - 1$ and $2S_z = b - a - 1$, every functional of S, S_z can be rewritten as functionals of a, b, c, d. The undeformed commutation relations (1.14) are generalised in terms of the new variables,

$$S_\pm^n a^m = (a \mp n)^m S_\pm^n,$$

[11] *Ibid.*

$$S_{\pm}^{n} b^{m} = (b \pm n)^{m} S_{\pm}^{n},$$

$$[S_{\pm}, c] = 0; \tag{1.23}$$

$$S_{-}^{n} S_{+}^{m} = \prod_{k=0}^{n-1} (a-k)(b+k) S_{+}^{m-n},$$

with m, n positive integers, $m > n$. This last is a generalisation from the form (1.17a) of the Casimir operator as $S_- S_+ = 2S(S+1) - 2S_z^2 - S_+ S_- = 2S(S+1) - 2S_z^2 - S_- S_+ + 2S_z$, using (1.14b), so that $S_- S_+ = S(S+1) - S_z(S_z+1) = ab$. That all this will be advantageous will become clear as the calculations progress.

Proposition III. The form (1.20a) of Φ is an element of the enveloping algebra $U(sl_2)$. Then from (1.20b) every element of the QUEA $U_q(sl_2)$ is also an element of $U(sl_2)$. Thus Φ defines a homomorphism $\Phi: U(sl_2) \to U_q(sl_2)$.

Proof. Expanding the form (1.20b) of Φ with $p \equiv q - q^{-1}$ and using the functionals a, b, c, d, of (1.21),

$$\begin{aligned}
\Phi &\equiv (ab)^{-1} [a]_q [b]_q \\
&= (ab)^{-1} p^{-2} (q^{a} - q^{-a})(q^{b} - q^{-b}) \\
&= (ab)^{-1} p^{-2} (q^{c} - q^{-d} - q^{d} + q^{-c}) \\
&= (ab)^{-1} p^{-2} (q^{c} + q^{-c}) - (ab)^{-1} p^{-2} (q^{d} + q^{-d}).
\end{aligned}$$

Expanding the first term in the deforming parameter h while neglecting p^{-2} gives a series in even powers of c,

$$(q^{c} + q^{-c}) = \sum_{n} \frac{1}{n!} \{1 + (-1)^{n}\} c^{n} h^{n},$$

as the terms for which n is odd vanish. Hence as $ab, d, c^n \in U(sl_2)$, n even, and as Φ expands in positive powers of ab, $\Phi \in U(sl_2)$. The rest of the proof immediately follows.[12] ●

In what follows it will be important to note that the map $\breve{\Phi}: U(sl_2) \to U_q(sl_2)$ of (1.20c) also serves to map both irreducible and reducible representations of $U(sl_2)$ to the QUEA $U_q(sl_2)$. Further, the relations (1.20c) provide a map $\tilde{\Phi}: U(sl_2) \to U_q(sl_2)$ for the primitive coproduct structure of $U(sl_2)$. Thus both the deformed and undeformed coproducts define direct product representations of the same elements of $U_q(sl_2)$ as functionals of the elements of $U(sl_2)$. These representations will be equivalent, and can be related through the appropriate similarity transformations.[13]

I.3. *Drinfel'd's twist map* $\tilde{F}: U(sl_2) \to U_q(sl_2)$.

It has been shown that there exists a deforming map $\breve{\Phi}: U(sl_2) \to U_q(sl_2)$, due to Curtright *et al*, so that representations associated with coproducts so defined on the QUEA $U_q(sl_2)$ are equivalent. More specifically the map Φ acts to define a primitive coproduct structure $\Delta: U_q(sl_2) \to U_q(sl_2) \otimes U_q(sl_2)$ on $U_q(sl_2)$, as a limit of the q-deformed map $\overline{\Delta}: U_q(sl_2) \to U_q(sl_2) \otimes U_q(sl_2)$ of (1.3). This equivalence for undeformed and deformed coproducts on $U_q(sl_2)$ enabled Drinfel'd to develop a method for generating $U_q(sl_2)$ through a twist map $\tilde{F}: U(sl_2) \to U_q(sl_2)$, as a specific example of a more general map between quasi-Hopf algebras. This section outlines and expands the general properties of the twist element F thus defined as a basis for the explicit constructions of chapters III-V.

Definition VIII. Let $U_h(sl_2)$ be defined as an algebra of complex formal series in the deforming parameter h such that $U_h(sl_2) / hU_h(sl_2) = U(sl_2)$. $U_h(sl_2)$ is then a Hopf algebra with the (cocommutating) primitive coproduct $\Delta: U_h(sl_2) \to U_h(sl_2) \otimes U_h(sl_2)$ and antipode $S: U_h(sl_2) \to U_h(sl_2)$ of (1.15). The cocommutativity of this coproduct

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[12] see Cornwell, unpublished notes.

[13] Curtright *et al.* (1991).

allows $U_h(sl_2)$ to be equipped with the form of a quantum group with the definition of a quasitriangular structure $R \in U_h(sl_2) \otimes U_h(sl_2)$ as a series in the canonical two tensor t.[14] Explicitly, R is defined by

$$R \equiv q^{t/2} = \sum_n \frac{1}{2^n n!} t^n h^n, \tag{1.24}$$

$$\Delta u_{21} = \Delta u = R \Delta u R^{-1},$$

which follows from (1.18). From its general definition (1.13a) the QIKF $Q \in U_h(sl_2) \otimes U_h(sl_2)$ takes a natural form as an exponential series in t,

$$Q \equiv R_{21} R = q^t = \sum_n \frac{1}{n!} t^n h^n, \tag{1.25}$$

$$\Delta u = Q \Delta u Q^{-1}.$$

Definition IX. Let $F \in U(sl_2) \otimes U(sl_2)$ (strictly speaking, $F \in U_h(sl_2) \otimes U_h(sl_2)$) be an invertible element satisfying the conditions for a counital 2-cocycle, namely

$$(1 \otimes F)(id \otimes \Delta)F = (F \otimes 1)(\Delta \otimes id)F, \tag{1.26}$$

$$(\varepsilon \otimes id)F = (id \otimes \varepsilon)F = 1.$$

The set of elements of the form F provides a deforming (twist) map $\tilde{F}: U(sl_2) \to U_q(sl_2)$ from the primitive coproduct and antipode of the Hopf algebra $U(sl_2)$ to the q-dependent coproduct and antipode of the QUEA $U_q(sl_2)$. This map takes the form of a gauge transformation,

$$\bar{\Delta} u = F \Delta u F^{-1}, \tag{1.27}$$

$$\bar{S} u = f(Su) f^{-1},$$

[14]Drinfel'd (1990b), pp. 1419, 1440.

where $F \equiv F_{(1)} \otimes F_{(2)}, f \equiv F_{(1)} S F_{(2)}$, for all $u \in U(sl_2)$. This transformation then induces a twist map from the quasitriangular structure R of (1.24) to the deformed quasitriangular structure $\overline{R} \in U_q(sl_2) \otimes U_q(sl_2)$ of Majid (1.9),

$$\overline{R} = F_{21} R F^{-1} \tag{1.28}$$

so that

$$\overline{Q} = F Q F^{-1}.$$

In Drinfel'd's words, *$U_q(sl_2)$ is obtained from $U(sl_2)$ by twisting via the element F.*[15] ***It is the object of this thesis to construct an explicit form for an element F which satisfies these twist map conditions.***

Proposition IV. Let $F \in U(sl_2) \otimes U(sl_2)$ be a twist element satisfying (1.27, 1.28). Let $X \in U(sl_2) \otimes U(sl_2)$, $\overline{X} \in U_q(sl_2) \otimes U_q(sl_2)$ be arbitrary, invertible functionals of the canonical 2-tensor t and QIKF $\overline{Q}$ respectively. A more general twist element that provides a map $U(sl_2) \to U_q(sl_2)$ satisfying (1.27, 1.28) takes the form $\overline{X} F X$.

Proof. From (1.13b, 1.18b) and their definitions the elements $X, \overline{X}$ commute with the coproducts Δu and $\overline{\Delta} u, u \in U(sl_2)$, respectively. For (1.27) it is only necessary to show

$$\begin{aligned} \overline{\Delta} u &= (\overline{X} F X) \Delta u (\overline{X} F X)^{-1} \\ &= \overline{X} F (X \Delta u X^{-1}) F^{-1} \overline{X}^{-1} = \overline{X} F \Delta u F^{-1} \overline{X}^{-1} \end{aligned}$$

$$\therefore \qquad \overline{X}^{-1} \overline{\Delta} u \overline{X} = \overline{\Delta} u = F \Delta u F^{-1}.$$

which is (1.27) for F. The conditions (1.28) for $\overline{X} F X$ must then follow. ●

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[15] *Ibid.*, p.1422.

Proposition V. From its definition (VIII) as an exponential series in h, the undeformed quasitriangular structure R inverts with the substitution $q \rightarrow q^{-1}$. That the corresponding deformed quasitriangular structure $\overline{R}$ of Majid also inverts with $q \rightarrow q^{-1}$ has been demonstrated (Prop. I). It follows that the twist element F may preserve this property through the twist map $\overline{R} = F_{21}RF^{-1}$. The condition thus imposed on F is that it must twist with the substitution $q \rightarrow q^{-1}$. That is, we may assume that

$$F(q^{-1}) = F_{21}. \tag{1.29}$$

Proof. Substituting $q \rightarrow q^{-1}$ in the twist map (1.28) and using $R(q^{-1}) = R^{-1}$, $\overline{R}(q^{-1}) = \overline{R}^{-1}$,

$$\begin{aligned}\overline{R}(q^{-1}) &= F_{21}(q^{-1})R(q^{-1})F^{-1}(q^{-1}) \\ &= F_{21}(q^{-1})R^{-1}F^{-1}(q^{-1}).\end{aligned}$$

Inverting,

$$\begin{aligned}\overline{R}^{-1}(q^{-1}) &= (F_{21}(q^{-1})R^{-1}F^{-1}(q^{-1}))^{-1} \\ &= F(q^{-1})RF_{21}{}^{-1}(q^{-1}) = \overline{R}.\end{aligned}$$

$$\therefore \qquad \overline{R} = F(q^{-1})RF_{21}{}^{-1}(q^{-1}).$$

Identification with the twist map $\overline{R} = F_{21}RF^{-1}$ then gives the required result. This is actually a statement that if the element F satisfies the twist map condition (1.28a), then $F_{21}(q^{-1})$ must also satisfy (1.28a), but the restrictions (Prop. IV) placed on the general form of F then lead once again to (1.29). ●

Proposition VI. Let $F \in U(sl_2) \otimes U(sl_2)$ be an invertible element satisfying the conditions (1.28) for a twist map $\tilde{F}: U(sl_2) \rightarrow U_q(sl_2)$. Further let the element F provide a deforming map $\overline{\Delta}S_+ = F\Delta S_+F^{-1}$. The condition for F as a deforming map

$\overline{\Delta}\overline{S}_- = F\Delta\overline{S}_- F^{-1}$ reduces to a deforming map involving the Casimir operator C and canonical 2-tensor t,

$$\overline{\Delta}\overline{Y} = F\Delta\overline{Y}F^{-1}; \tag{1.30}$$

with

$$\overline{\Delta}\overline{Y} = q^{-1}\overline{Y}\otimes q^d + qq^{-d}\otimes\overline{Y} - [2]_q q^d\otimes q^{-d} + p^2(q^{-d/2}\otimes q^{d/2})(q^{-1}\overline{S}_-\otimes S_+ + qS_+\otimes\overline{S}_-),$$

$$\Delta\overline{Y} = (q^{\sqrt{2t+C\otimes 1+1\otimes C+1\otimes 1}} + q^{-\sqrt{2t+C\otimes 1+1\otimes C+1\otimes 1}}),$$

$$\overline{Y} \equiv q^c + q^{-c} = q^{\sqrt{C+1}} + q^{-\sqrt{C+1}}.$$

Proof. Rearranging $\overline{\Delta}\overline{S}_- = F\Delta\overline{S}_- F^{-1}$ and using $\overline{S}_- \equiv \Phi S_-$, $\Phi \equiv (ab)^{-1}[a]_q[b]_q$, $S_-S_+ = ab$ from (Def. VII),

$$\overline{\Delta}\Phi\overline{\Delta}S_- = F\Delta\Phi\Delta S_- F^{-1};$$

$$\Delta\Phi \equiv \Delta(ab)^{-1}\Delta([a]_q[b]_q) = \Delta([a]_q[b]_q)\Delta S_+^{-1}\Delta S_-^{-1},$$

$$\overline{\Delta}\Phi = \overline{\Delta}([a]_q[b]_q)\overline{\Delta}S_+^{-1}\overline{\Delta}S_-^{-1}.$$

Combining and using (1.27),

$$\overline{\Delta}([a]_q[b]_q)\overline{\Delta}S_+^{-1} = F\Delta([a]_q[b]_q)\Delta S_+^{-1}F^{-1}$$

$$\Rightarrow \quad \overline{\Delta}([a]_q[b]_q)\overline{\Delta}S_+^{-1} = F\Delta([a]_q[b]_q)(F^{-1}\overline{\Delta}S_+F)^{-1}F^{-1} = F\Delta([a]_q[b]_q)F^{-1}\overline{\Delta}S_+^{-1}$$

$$\Rightarrow \quad \overline{\Delta}([a]_q[b]_q) = F\Delta([a]_q[b]_q)F^{-1}.$$

Using the result from the proof of (Prop. III),

$$p^2[a]_q[b]_q = (q^c + q^{-c}) - (q^d + q^{-d}) = \overline{Y} - (q^d + q^{-d}),$$

$$\Rightarrow \qquad \overline{\Delta}\overline{Y} - \overline{\Delta}(q^d + q^{-d}) = F(\Delta\overline{Y} - \Delta(q^d + q^{-d}))F^{-1}$$

$$\Rightarrow \qquad \overline{\Delta}\overline{Y} = F\Delta\overline{Y}F^{-1}$$

as $\overline{\Delta}S_z = \Delta S_z$. The right side can be expanded using (1.19) and the definition of c in terms of the Casimir operator, $c^2 = C + 1 \in U(sl_2)$,

$$\begin{aligned}\Delta\overline{Y} &= \Delta(q^c + q^{-c}) = \Delta(q^{\sqrt{C+1}} + q^{-\sqrt{C+1}}) \\ &= (q^{\sqrt{\Delta C + 1\otimes 1}} + q^{-\sqrt{\Delta C + 1\otimes 1}}) \\ &= (q^{\sqrt{2t + C\otimes 1 + 1\otimes C + 1\otimes 1}} + q^{-\sqrt{2t + C\otimes 1 + 1\otimes C + 1\otimes 1}}).\end{aligned}$$

The left side can be evaluated using $\overline{S}_-S_+ = [a]_q[b]_q$ and the result from the proof of (Prop. III),

$$\begin{aligned}\overline{\Delta}\overline{Y} &= p^2\overline{\Delta}([a]_q[b]_q) + \overline{\Delta}(q^d + q^{-d}) \\ &= p^2\overline{\Delta}\overline{S}_-\overline{\Delta}S_+ + q^{1/2}\overline{\Delta}q^{S_z} + q^{-1/2}\overline{\Delta}q^{-S_z}.\end{aligned}$$

Substituting the coproducts (1.3) with $\overline{S}_+ \equiv S_+$ and recombining then gives (1.30b), as required. ●

Proposition VII. Even/odd terms in the expansion of the twist element F in the deforming parameter h are symmetric/antisymmetric. Thus

$$F = \sum_n f_{(n)}h^n, \tag{1.31}$$

$$(f_{(n)})_{21} = \begin{cases} +f_{(n)}, & n \text{ even.} \\ -f_{(n)}, & n \text{ odd.}\end{cases}$$

Expansion of the twist map (1.28a) then uniquely determines F to zero and first order in h, giving

$$f_{(0)} = 1 \otimes 1, \tag{1.32a}$$

$$f_{(1)} = \tfrac{1}{2}(S_- \otimes S_+ - S_+ \otimes S_-). \tag{1.32b}$$

Higher order terms in h can be expressed as commutators or anticommutators with the canonical 2-tensor t of (1.18a). Thus to second and third order in h,

$$[f_{(2)}, 2t] = r^s_{(3)} - \tfrac{1}{3}t^3 - \tfrac{1}{8}(r^a t + t r^a) r^a, \tag{1.33}$$

$$[f_{(3)}, 2t]_+ = \tfrac{1}{12} r^a t^3 - r^a_{(4)} + \tfrac{1}{4} r^s_{(3)} r^a - \tfrac{1}{2}(r^a t + t r^a) f_{(2)},$$

where $r^s_{(n)} \equiv r_{(n)} + (r_{(n)})_{21}$ and $r^a_{(n)} \equiv r_{(n)} - (r_{(n)})_{21}$ are symmetric and antisymmetric parts of the terms in the h-expansion (1.10) of the quasitriangular structure $\overline{R}$.

Proof. That even/odd terms in the expansion of the twist element F in the deforming parameter h are symmetric/antisymmetric follows immediately from (Prop. V). (1.32a) follows from the definition of F as reducing to the identity in the limit $q \to 1$. Rearranging the twist map (1.28a), then substituting the expansions (1.10, 1.24a, 1.31) of the quasitriangular structures $\overline{R}$, R and twist element F, multiplying and collecting terms to first order in h,

$$\{1 \otimes 1 + 2(S_z \otimes S_z + S_+ \otimes S_-)h + \ldots\}\{f_{(0)} + f_{(1)}h + \ldots\}$$

$$= \{f_{(0)} - f_{(1)}h + \ldots\}\{1 \otimes 1 + (2S_z \otimes S_z + S_+ \otimes S_- + S_- \otimes S_+)h + \ldots\},$$

$$\therefore \qquad f_{(1)} = \frac{1}{2}(S_- \otimes S_+ - S_+ \otimes S_-) f_{(0)}.$$

which is the required result. Higher order terms in h then take the form of (1.33). ●

Chapter Two. General semirepresentations of $U(sl_2)$.

The properties of Drinfel'd's twist element as a map $F\colon U(sl_2) \to U_q(sl_2)$ have now been set down. This chapter outlines a method for constructing nth order semirepresentations of F as elements of the algebra $M^{(n)}$ of matrices with operator valued entries. The form of these semirepresentations is to a great extent determined by their relationship to underlying (irreducible and direct product) representations defined for the complex Lie algebra sl_2. These relationships in turn fix an explicit representation-independent form for the twist element F. As will be shown, this representation-independent form can in part be recovered using these relationships. First, some properties of the algebra $M^{(n)}$ must be derived.

II.1. *The algebra $M^{(n)}$ of matrices with operator valued entries.*

Definition X. Let $\mathbf{M}^{(n)}$ be an nth order matrix of operator-valued entries, elements of the enveloping algebra $U(sl_2)$ with the central element S. The set of matrices $M^{(n)}$ then forms an algebra with the usual matrix multiplication and identity element given by $\mathbf{I}^{(n)} \equiv \mathit{diagonal}(1,1,\ldots,1)$.[1]

Definition XI. Let $\mathbf{G}^{(n)}$ be an nth order matrix of operator-valued, mutually commuting entries, functionals of S, S_z. That is,

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[1] It is possible to define a Hopf algebra on the q-deformed analogue of the algebra $M^{(2)}$ of matrices with algebra-valued entries. See Kassel (1995), chapter IV.

$$\mathbf{G}^{(n)} \equiv \begin{pmatrix} G_{11} & G_{12} & \cdots & G_{1n} \\ G_{21} & G_{22} & \cdots & G_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ G_{n1} & G_{n2} & \cdots & G_{nn} \end{pmatrix}, \tag{2.1}$$

where $G_{jk} \equiv G_{jk}(S, S_z)$. $\mathcal{G}^{(n)}$ is then an algebra with matrix multiplication and identity element $\mathbf{I}^{(n)}$. Clearly, as products of components of $\mathbf{G}^{(n)}$ are themselves functionals of S, S_z, $\mathcal{G}^{(n)}$ is a subalgebra of $\mathcal{M}^{(n)}$.

Proposition VIII. The elements $\mathbf{G}^{(n)} \in \mathcal{G}^{(n)}$ can be manipulated using standard matrix operations.[2] Specifically,

(i) A nonsingular element $\mathbf{G}^{(n)} \in \mathcal{G}^{(n)}$ will have a non-zero determinant. The inverse $(\mathbf{G}^{(n)})^{-1}$, $(\mathbf{G}^{(n)})^{-1}\mathbf{G}^{(n)} = \mathbf{G}^{(n)}(\mathbf{G}^{(n)})^{-1} = \mathbf{I}^{(n)}$, exists and can be constructed.

(ii) Suppose there exist n eigenvalues λ_k and n linearly independent eigenvectors $\mathbf{x}_k$, $k = 1,2,\ldots,n$, each operator-valued and a functional of S, S_z, as solutions to the eigenvector equation,

$$\mathbf{G}^{(n)}\mathbf{x}_k = \lambda_k \mathbf{x}_k. \tag{2.2}$$

The n eigenvalues λ_k of $\mathbf{G}^{(n)}$ are then solutions of the nth order characteristic polynomial,

$$(-1)^n(\lambda^n - \alpha_1\lambda^{n-1} - \alpha_2\lambda^{n-2} - \ldots - \alpha_n) = 0; \tag{2.3}$$

$$\alpha_k = k^{-1} tr\mathbf{A}_k,$$

$$\mathbf{A}_k = \mathbf{G}^{(n)}\mathbf{B}_{k-1},\ \mathbf{A}_1 = \mathbf{G}^{(n)},$$

$$\mathbf{B}_k = \mathbf{A}_k - \alpha_k\mathbf{I},$$

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[2] Cornwell (1984a), unpublished notes; Faddeeva (1959).

where $k = 1,2,\ldots,n$. The n corresponding eigenvectors $\mathbf{x}_k$ of $\mathbf{G}^{(n)}$ are the individual column vectors of each of the n matrices $\mathbf{D}_k$ defined by

$$\mathbf{D}_k \equiv \lambda_k^{n-1}\mathbf{I} + \lambda_k^{n-2}\mathbf{B}_1 + \ldots + \mathbf{B}_{n-1}, \tag{2.4}$$

$k = 1,2,\ldots,n$. A diagonalizing matrix $\mathbf{D} \in \mathcal{G}^{(n)}$ can then be constructed from the n eigenvectors as column vectors, so that

$$\lambda = \mathbf{D}^{-1}\mathbf{G}^{(n)}\mathbf{D}, \tag{2.5}$$

$$\lambda \equiv diagonal(\lambda_1, \lambda_2, \ldots, \lambda_n).$$

(iii) Equivalent elements of $\mathcal{G}^{(n)}$ have identical traces.

Proof. As functionals of S, S_z, the entries of $\mathbf{G}^{(n)} \in \mathcal{G}^{(n)}$ are elements of a unital Abelian algebra, or a unital commutative ring. Further, for the purposes of the constructions in this thesis, an inverse is assumed to exist for every non-zero functional of S, S_z, so the elements of this ring form a field. As this proposition holds for every algebra of matrices with entries that are elements of a field, it must hold for $\mathcal{G}^{(n)}$.[3] ●

Definition XII. Let $\mathbf{L}^{(n)} \in \mathcal{M}^{(n)}$ be an nth order matrix of operator-valued entries, elements of $U(sl_2)$ and the central element S, which takes the form

$$\mathbf{L}^{(n)} = \begin{pmatrix} Z_{11} & Z_{12}S_- & \cdots & Z_{1n}S_-^{n-1} \\ Z_{21}S_+ & Z_{22} & \cdots & Z_{2n}S_-^{n-2} \\ \vdots & \vdots & \ddots & \\ Z_{n1}S_+^{n-1} & Z_{n2}S_+^{n-2} & & Z_{nn} \end{pmatrix} \tag{2.6}$$

[3] Fraleigh(1982), pp. 206ff, 223ff.

where $Z_{jk} \equiv Z_{jk}(S,S_z)$. That the set of matrices $\mathcal{L}^{(n)}$ of the form (2.6) is an algebra with matrix multiplication and identity element $\mathbf{I}^{(n)} \in \mathcal{L}^{(n)}$ follows from the fact that if $\mathbf{L}^{(n)}, \mathbf{L}'^{(n)} \in \mathcal{L}^{(n)}$, the product $\mathbf{L}^{(n)}\mathbf{L}'^{(n)} \in \mathcal{L}^{(n)}$. Using the commutation relations (1.14),

$$\mathbf{L}^{(n)}\mathbf{L}'^{(n)} \equiv \begin{pmatrix} Z_{11} & Z_{12}S_- & \cdots & Z_{1n}S_-^{n-1} \\ Z_{21}S_+ & Z_{22} & \cdots & Z_{2n}S_-^{n-2} \\ \vdots & \vdots & \ddots & \\ Z_{n1}S_+^{n-1} & Z_{n2}S_+^{n-2} & & Z_{nn} \end{pmatrix}\begin{pmatrix} Z'_{11} & Z'_{12}S_- & \cdots & Z'_{1n}S_-^{n-1} \\ Z'_{21}S_+ & Z'_{22} & \cdots & Z'_{2n}S_-^{n-2} \\ \vdots & \vdots & \ddots & \\ Z'_{n1}S_+^{n-1} & Z'_{n2}S_+^{n-2} & & Z'_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} Z_{11}Z'_{11} + Z_{12}S_-Z'_{21}S_+ + \ldots + Z_{1n}S_-^{n-1}Z'_{n1}S_+^{n-1} & \cdots & Z_{11}Z'_{1n}S_-^{n-1} + Z_{12}S_-Z'_{2n}S_-^{n-2} + \ldots + Z_{1n}S_-^{n-1}Z'_{nn} \\ Z_{21}S_+Z'_{11} + Z_{22}Z'_{21}S_+ + \ldots + Z_{2n}S_-^{n-2}Z'_{n1}S_+^{n-1} & \cdots & Z_{21}S_+Z'_{1n}S_-^{n-1} + Z_{22}Z'_{2n}S_-^{n-2} + \ldots + Z_{2n}S_-^{n-2}Z'_{nn} \\ \vdots & \ddots & \vdots \\ Z_{n1}S_+^{n-1}Z'_{11} + Z_{n2}S_+^{n-2}Z'_{21}S_+ + \ldots + Z_{nn}Z'_{n1}S_+^{n-1} & \cdots & Z_{n1}Z'_{1n}S_-^{n-1} + Z_{n2}S_+^{n-2}Z'_{2n}S_-^{n-2} + \ldots + Z_{nn}Z'_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} Z_{11} & Z_{12}S_- & \cdots & Z_{1n}S_-^{n-1} \\ Z_{21}S_+ & Z_{22} & \cdots & Z_{2n}S_-^{n-2} \\ \vdots & \vdots & \ddots & \\ Z_{n1}S_+^{n-1} & Z_{n2}S_+^{n-2} & & Z_{nn} \end{pmatrix},$$

where the $S_\pm S_\mp, S_\pm^2 S_\mp^2, \ldots, S_\pm^n S_\mp^n$ have been factored through and combined using (1.23d), and entries $Z_{jk} \equiv Z_{jk}(S,S_z)$ defined in terms of the Z_{jk}, Z'_{jk}. Thus the product $\mathbf{L}^{(n)}\mathbf{L}'^{(n)}$ again takes the form of an element of $\mathcal{L}^{(n)}$. $\mathcal{L}^{(n)}$ is then a subalgebra of $\mathcal{M}^{(n)}$.

Proposition IX. A gauge transformation $\mathbf{X}_-: \mathcal{L}^{(n)} \to \mathcal{G}^{(n)}$ can be defined with an nth order diagonal matrix $\mathbf{X} \in \mathcal{M}^{(n)}$ of the form[4]

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[4]The operation of S_-^{-1} can be defined from the relations (1.23), so that $S_-^{-1} = (a-1)^{-1}(b+1)^{-1}S_+$. Strictly speaking $S_-^{-1} \notin U(sl_2)$, but will always appear in subsequent calculations in such a way that the $(a-1)^{-1}(b+1)^{-1}$ will factor out.

$$\mathbf{X}_- \equiv diagonal(1, S_-, \ldots, S_-^{n-1}), \tag{2.7a}$$

so that

$$\mathbf{G}^{(n)} = \mathbf{X}_- \mathbf{L}^{(n)} \mathbf{X}_-^{-1} \tag{2.7b}$$

where $\mathbf{L}^{(n)} \in \mathcal{L}^{(n)}$, $\mathbf{G}^{(n)} \in \mathcal{G}^{(n)}$. The map $\mathbf{X}_-: \mathcal{L}^{(n)} \to \mathcal{G}^{(n)}$ is then an algebra homomorphism. The map (2.7b) is clearly invertible, so that for $\mathbf{L}^{(n)} \in \mathcal{L}^{(n)}$, $\mathbf{G}^{(n)} \in \mathcal{G}^{(n)}$,

$$\mathbf{L}^{(n)} = \mathbf{X}_-^{-1} \mathbf{G}^{(n)} \mathbf{X}_-. \tag{2.7c}$$

Proof. Substituting the forms (2.6, 2.7a) for $\mathbf{L}^{(n)}, \mathbf{X}_-$ into (2.7b) and using the commutation relations (1.14, 1.23d),

$$\mathbf{G}^{(n)} = \mathbf{X}_- \mathbf{L}^{(n)} \mathbf{X}_-^{-1} = \begin{pmatrix} Z_{11} & Z_{12} S_- S_-^{-1} & \cdots & Z_{1n} S_-^{n-1} S_-^{-n+1} \\ S_- Z_{21} S_+ & S_- Z_{22} & \cdots & S_- Z_{2n} S_-^{n-1} S_-^{-n+1} \\ \vdots & \vdots & \ddots & \vdots \\ S_-^{n-1} Z_{n1} S_+^{n} & S_-^{n-1} Z_{n2} S_+^{n-1} S_-^{-1} & \cdots & S_-^{n-1} Z_{nn} S_-^{-n+1} \end{pmatrix}$$

$$= \begin{pmatrix} Z_{11} & Z_{12} & \cdots & Z_{1n} \\ abZ_{21}(S_z \pm 1) & Z_{22}(S_z \pm 1) & \cdots & Z_{2n}(S_z \pm 1) \\ \vdots & \vdots & \ddots & \vdots \\ \{\prod_{k=0}^{n-1} (a-k)(b+k)\} Z_{n1}(S_z \pm n) & \{\prod_{k=0}^{n-2} (a-k)(b+k)\} Z_{n2}(S_z \pm n) & \cdots & Z_{nn}(S_z \pm n) \end{pmatrix}$$

where $Z_{jk} \to Z_{jk}(S_z \pm m)$ with the substitution $S_z \to S_z \pm m$. Thus $\mathbf{G}^{(n)}$ is of the form (2.1), as required. The map (2.7b) is then an algebra homomorphism as $\mathbf{X}_- \mathbf{I}^{(n)} \mathbf{X}_-^{-1} = \mathbf{I}^{(n)}$. ●

Proposition X. Let $\mathbf{D} \in \mathcal{G}^{(n)}$ be the diagonalising matrix for the element $\mathbf{G}^{(n)} \in \mathcal{G}^{(n)}$ defined by (2.5a). Then the corresponding diagonalising matrix for the element

$\mathbf{L}^{(n)} \in \mathcal{L}^{(n)}$, where $\mathbf{G}^{(n)} = \mathbf{X}_{-}\mathbf{L}^{(n)}\mathbf{X}_{-}^{-1}$, exists and takes the form $\mathbf{X}_{-}^{-1}\mathbf{D}$. Further let λ be the matrix of eigenvalues for $\mathbf{G}^{(n)}$ defined by (2.5b). Then λ is also the matrix of eigenvalues for $\mathbf{L}^{(n)}$.[5]

Proof. Combining $\lambda = \mathbf{D}^{-1}\mathbf{G}^{(n)}\mathbf{D}$ with the gauge transformation $\mathbf{G}^{(n)} = \mathbf{X}_{-}\mathbf{L}^{(n)}\mathbf{X}_{-}^{-1}$,

$$\lambda = \mathbf{D}^{-1}\mathbf{G}^{(n)}\mathbf{D} = \mathbf{D}^{-1}\mathbf{X}_{-}\mathbf{L}^{(n)}\mathbf{X}_{-}^{-1}\mathbf{D},$$

$$\therefore \qquad \lambda = (\mathbf{X}_{-}^{-1}\mathbf{D})^{-1}\mathbf{L}^{(n)}\mathbf{X}_{-}^{-1}\mathbf{D},$$

which is the required result.●

II.2. *Representations of the algebra* $U(sl_2)$.

The positive and negative semirepresentations $\mathbf{l}^{\pm}(u)$ of elements u of the enveloping algebra $U(sl_2)$ are now defined as elements of the algebra $\mathcal{M}^{(n)}$ of matrices with operator-valued entries. It is shown that the condition for the existence of an element $u \in U(sl_2)$ corresponding to these semirepresentations can be expressed as a set of consistency conditions among direct product representations of u.

Definition XIII. For every non-negative half integer and integer value of j there exists an irreducible representation $\Gamma^{(j)}$ of order $2j+1$ generated by the action of the complex Lie algebra sl_2 on basis vectors ϕ_m^j of the Hilbert space of functions. Thus

$$S_z\phi_m^j = m\phi_m^j,\ S_+\phi_m^j = \phi_{m+1}^j, \tag{2.8}$$

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[5] Cornwell, unpublished notes.

$$S_-\phi^j_m = (j+m)(j-m+1)\phi^j_{m-1}$$

where the appropriate termination conditions hold at $j=\pm m$. The representation $\Gamma^{(j)}$ will then satisfy the commutation relations (1.14). The 2j+1 dimensional basis ϕ^j_m thus chosen for the carrier space of the irreducible representation $\Gamma^{(j)}$ is not orthonormal, but is obtained from the standard orthonormal basis by a rescaling which is chosen so as to both eliminate the appearance of square roots and to leave only a trivial factor for the action of S_+.[6]

Proposition XI. The nth order positive and negative semirepresentations $\mathbf{l}^{\pm(j)}(u)$, n=2j+1, of an element of the form $u \in U(sl_2)\otimes U(sl_2)$ can be defined from the nth order irreducible representation $\Gamma^{(j)}$ of (2.8). Let $u \equiv u_{(1)} \otimes u_{(2)}$.[7] Then

$$\mathbf{l}^{+(j)}(u) \equiv u_{(1)}\Gamma^{(j)}(u_{(2)}), \tag{2.9}$$

$$\mathbf{l}^{-(j)}(u) \equiv \Gamma^{(j)}(u_{(1)})u_{(2)}.$$

With the identity $\mathbf{l}^{\pm(j)}(1\otimes 1) \equiv \mathbf{I}^{(j)}$, the semirepresentations $\mathbf{l}^{\pm(j)}(u)$ are then algebra representations of $U(sl_2)$.[8] The positive and negative semirepresentations are related through the definition of the transpose element $u_{21} \in U(sl_2)\otimes U(sl_2)$, so that with $u_{21} \equiv u_{(2)} \otimes u_{(1)}$,

$$\mathbf{l}^{\pm(j)}(u_{21}) = \mathbf{l}^{\mp(j)}(u). \tag{2.10}$$

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[6] See Cornwell (1984b), p.441.

[7] In this and subsequent definitions a sum of terms of the form $u_{(1)}\otimes u_{(2)}$ is implied. For this notation see Majid (1995), p. 7.

[8] The form of the semirepresentation originated with the FRT method. See Faddeev *et al* (1987); Cornwell, unpublished notes.

Proof. To show this, let $u, u', u'' \in U(sl_2) \otimes U(sl_2)$ such that $u = u'u''$. With the definition of multiplication on $U(sl_2) \otimes U(sl_2)$,

$$
\begin{aligned}
\mathbf{I}^{-(j)}(u')\mathbf{I}^{-(j)}(u'') &= \Gamma^{(j)}(u'_{(1)})u'_{(2)}\Gamma^{(j)}(u''_{(1)})u''_{(2)} \\
&= \Gamma^{(j)}(u'_{(1)})\Gamma^{(j)}(u''_{(1)})u'_{(2)}u''_{(2)} \\
&= \Gamma^{(j)}(u'_{(1)}u''_{(1)})u'_{(2)}u''_{(2)} = \Gamma^{(j)}(u_{(1)})u_{(2)} \\
&= \mathbf{I}^{-(j)}(u),
\end{aligned}
$$

as required. The proof for $\mathbf{I}^{+(j)}(u)$ is analogous. That $\mathbf{I}^{+(j)}(u_{21}) = \mathbf{I}^{-(j)}(u)$ follows as

$$\mathbf{I}^{+(j)}(u_{21}) = u_{(2)}\Gamma^{(j)}(u_{(1)}) = \Gamma^{(j)}(u_{(1)})u_{(2)} = \mathbf{I}^{-(j)}(u).$$

The proof for $\mathbf{I}^{-(j)}(u_{21})$ is analogous.●

Definition XIV. Let $\Gamma^{(j)} \otimes \Gamma^{(j)}$ denote the direct product representations of elements of *U(sl$_2$)* constructed from the irreducible representations $\Gamma^{(j)}$ of (2.8). Set $u \equiv u_{(1)} \otimes u_{(2)}$. Then

$$\Gamma^{j_1} \otimes \Gamma^{j_2}(u)_{ik|pl} \equiv \Gamma^{j_1}(u_{(1)})_{ip} \otimes \Gamma^{j_2}(u_{(2)})_{kl} \tag{2.10}$$

defines the components for a direct product representation of an element of the form $u \in U(sl_2) \otimes U(sl_2)$. The primitive coproduct structure on *U(sl$_2$)* then defines a direct product representation for elements of *sl$_2$*. Thus

$$\Gamma^{j_1 \otimes j_2}(u) = (\Gamma^{j_1} \otimes \Gamma^{j_2})(\Delta u) \tag{2.11}$$

for all elements u of sl_2.[9]

Proposition XII. The semirepresentation matrices $\mathbf{I}^{\pm(j)}(u)$ are members of the algebra $M^{(n)}$ of nth order matrices with operator-valued entries with $n=2j+1$. They satisfy the relations

$$\Gamma^{(j)}(\mathbf{I}^{-(j)}(u)_{ip})_{kl} = \Gamma^{(j)}(\mathbf{I}^{+(j)}(u)_{kl})_{ip}, \tag{2.12}$$

with $i, p, k, l = 1, 2, \ldots, n$.

Proof. Condition (2.12) reduces to the statement that an element u exists if and only if its direct product representation exists. Elements u', u'' can always be constructed so that $\mathbf{I}^{-(j)}(u') = \mathbf{I}^{-(j)}(u)$ and $\mathbf{I}^{+(j)}(u'') = \mathbf{I}^{+(j)}(u)$. With $u' \equiv u'_{(1)} \otimes u'_{(2)}$, $u'' \equiv u''_{(1)} \otimes u''_{(2)}$,

$$\begin{aligned}
\Gamma^{(j)}(\mathbf{I}^{-(j)}(u)_{ip})_{kl} = \Gamma^{(j)}(\mathbf{I}^{-(j)}(u')_{ip})_{kl} &= \Gamma^{(j)}(u'_{(1)})_{ip} \otimes \Gamma^{(j)}(u'_{(2)})_{kl} \\
&= \Gamma^{(j)} \otimes \Gamma^{(j)}(u')_{ik|pl}, \\
\Gamma^{(j)}(\mathbf{I}^{+(j)}(u)_{kl})_{ip} = \Gamma^{(j)}(\mathbf{I}^{+(j)}(u'')_{kl})_{ip} &= \Gamma^{(j)}(u''_{(1)})_{ip} \otimes \Gamma^{(j)}(u''_{(2)})_{kl} \\
&= \Gamma^{(j)} \otimes \Gamma^{(j)}(u'')_{ik|pl},
\end{aligned}$$

$i, p, k, l = 1, 2, \ldots, n$. Substituting in (2.12) then gives

$$\Gamma^{(j)} \otimes \Gamma^{(j)}(u')_{ik|pl} = \Gamma^{(j)} \otimes \Gamma^{(j)}(u'')_{ik|pl}$$

which holds for all allowable j if and only if $u'=u''=u$, as required.●

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[9] Cornwell, unpublished notes.

Proposition XIII. The semirepresentation matrices $\mathbf{l}^{-(j_1)}(u),\mathbf{l}^{+(j_2)}(u)$ are members of the algebras $M^{(n_1)}, M^{(n_2)}$ of matrices of order n_1, n_2 respectively, with operator valued entries with $n_1 = 2j_1+1, n_2 = 2j_2+1$. They satisfy the relationships

$$\Gamma^{(j_1)}(\mathbf{l}^{-(j_2)}(u)_{ip})_{kl} = \Gamma^{(j_2)}(\mathbf{l}^{+(j_1)}(u)_{kl})_{ip}, \tag{2.13}$$

with $i,p = 1,2,\ldots,n_1$ and $k,l = 1,2,\ldots,n_2$.

Proof. This is simply a generalisation of (Prop. XII). Construct elements u', u'' so that $\mathbf{l}^{-(j_2)}(u') = \mathbf{l}^{-(j_2)}(u)$ and $\mathbf{l}^{+(j_1)}(u'') = \mathbf{l}^{+(j_1)}(u)$. With $u' \equiv u'_{(1)} \otimes u'_{(2)}$, $u'' \equiv u''_{(1)} \otimes u''_{(2)}$,

$$\begin{aligned}
\Gamma^{(j_1)}(\mathbf{l}^{-(j_2)}(u)_{ip})_{kl} &= \Gamma^{(j_1)}(\mathbf{l}^{-(j_2)}(u')_{ip})_{kl} = \Gamma^{(j_2)}(u'_{(1)})_{ip} \otimes \Gamma^{(j_1)}(u'_{(2)})_{kl} \\
&= \Gamma^{(j_2)} \otimes \Gamma^{(j_1)}(u')_{ik|pl}, \\
\Gamma^{(j_2)}(\mathbf{l}^{+(j_1)}(u)_{kl})_{ip} &= \Gamma^{(j_2)}(\mathbf{l}^{+(j_1)}(u'')_{kl})_{ip} = \Gamma^{(j_2)}(u''_{(1)})_{ip} \otimes \Gamma^{(j_1)}(u''_{(2)})_{kl} \\
&= \Gamma^{(j_2)} \otimes \Gamma^{(j_1)}(u'')_{ik|pl},
\end{aligned}$$

$i,p = 1,2,\ldots,n_1$ and $k,l = 1,2,\ldots,n_2$. Substituting in (2.13) then gives

$$\Gamma^{(j_2)} \otimes \Gamma^{(j_1)}(u')_{ik|pl} = \Gamma^{(j_2)} \otimes \Gamma^{(j_1)}(u'')_{ik|pl}$$

which holds for all allowable j_1, j_2 if and only if $u'=u''=u$, as required.●

II.3. *A general form for semirepresentations of the twist element F.*

Finally, general forms for the semirepresentations $\mathbf{l}^{\pm(j)}(F)$ of the twist element F are constructed. It is argued from properties of the twist map $\check{F}: U(sl_2) \to U_q(sl_2)$ that the

$\mathbf{l}^{\pm(j)}(F)$ are elements of the algebra $\mathcal{L}^{(n)}$ of matrices of the form (2.6). A method for partially recovering a representation-independent form for F as a series in $\mathrm{S}_{\pm}\otimes\mathrm{S}_{\mp}$ is presented.

Proposition XIV. Let $H\in U(sl_2)\otimes U(sl_2)$ take the form of a series in $\mathrm{S}_{\pm}\otimes\mathrm{S}_{\mp}$ as a subalgebra of the enveloping algebra $U(sl_2)$. That is,

$$H\equiv\alpha_0 Z_0+\sum_{k=1}^{\infty}\alpha'_k Z_k S_+^k\otimes S_-^k+\sum_{k=1}^{\infty}\alpha''_k Z''_k S_-^k\otimes S_+^k, \tag{2.14}$$

where $Z_0, Z_k, Z''_k\in U(sl_2)\otimes U(sl_2)$ are functionals of S, S_z, and $\alpha_0, \alpha'_k, \alpha''_k$ are real-valued coefficients that can be expanded in the deforming parameter h. The nth order semirepresentations $\mathbf{l}^{\pm(j)}(H)$, n=2j+1, are then members of the algebra $\mathcal{L}^{(n)}$ of matrices of the form (2.6). Further, a semirepresentation $\mathbf{g}^{\pm(j)}(H)\in G^{(n)}$ of elements of the form H is obtained through the gauge transformation defined by the diagonal matrix $\mathbf{X}_-$ (Prop IX). Thus

$$\mathbf{g}^{\pm(j)}(H)=\mathbf{X}_-\mathbf{l}^{\pm(j)}(H)\mathbf{X}_-^{-1}. \tag{2.15}$$

Proof. The proof comes directly from the construction of $\mathbf{l}^{-(j)}(H)$ using (2.9b) with the irreducible representations (2.8),

$$\begin{aligned}\mathbf{l}^{-(j)}(H)&=\alpha_0\mathbf{l}^{-(j)}(Z_0)+\sum_{k=1}^{\infty}\alpha'_k\mathbf{l}^{-(j)}(Z'_k)\mathbf{l}^{-(j)}(S_+^k\otimes S_-^k)+\sum_{k=1}^{\infty}\alpha''_k\mathbf{l}^{-(j)}(Z''_k)\mathbf{l}^{-(j)}(S_-^k\otimes S_+^k)\\&=\alpha_0\mathbf{l}^{-(j)}(Z_0)+\sum_{k=1}^{\infty}\alpha'_k\mathbf{l}^{-(j)}(Z'_k)\Gamma^{(j)}(S_+^k)S_-^k+\sum_{k=1}^{\infty}\alpha''_k\mathbf{l}^{-(j)}(Z''_k)\Gamma^{(j)}(S_-^k)S_+^k\end{aligned}$$

$$=\begin{pmatrix}\alpha_0\mathbf{l}^{-(j)}(Z_0)_{11} & \cdots & \alpha'_{n-1}\mathbf{l}^{-(j)}(Z'_{n-1})_{11}S_-^{n-1}\\ \vdots & \ddots & \vdots\\ \alpha''_{n-1}\mathbf{l}^{-(j)}(Z''_{n-1})_{(n-1)(n-1)}\Gamma^{(j)}(S_-^{n-1})_{(n-1)1}S_+^{n-1} & \cdots & \alpha_0\mathbf{l}^{-(j)}(Z_0)_{(n-1)(n-1)}\end{pmatrix}$$

which is of the form (2.6). That $\mathbf{g}^{-(j)}(H) \in G^{(n)}$ is a semirepresentation of elements $H \in U(sl_2) \otimes U(sl_2)$ of the form (2.14) then follows (Prop IX). The proof for $\mathbf{l}^{+(j)}(H), \mathbf{g}^{+(j)}(H)$ is analogous.●

Proposition XV. The nth order matrices $\mathbf{l}^{\pm(j)}(H) \in L^{(n)}$ are semirepresentations of an element $H \in U(sl_2) \otimes U(sl_2)$ of the form (2.14) provided $\mathbf{l}^{\pm(j)}(H)$ satisfy the consistency conditions of (Prop. XII). An explicit form for H can then be recovered from the semirepresentations $\mathbf{l}^{\pm(j)}(H)$ as solutions to a set of simultaneous equations defined by (2.11),

$$\Gamma^{(j)}(\mathbf{l}^{-(j)}(H)_{ip})_{kl} = \Gamma^{(j)}(\mathbf{l}^{+(j)}(H)_{kl})_{ip}. \tag{2.16}$$

Proof. The proof consists in substituting the form (Prop. XIV) of $\mathbf{l}^{\pm(j)}(H)$ in (2.16). The diagonal entries of $\mathbf{l}^{\pm(j)}(H)$ then give a set of n^2 simultaneous equations in α_0,

$$\alpha_0\Gamma^{(j)}(\mathbf{l}^{-(j)}(Z_0)_{ii})_{kk} = \alpha_0\Gamma^{(j)}(\mathbf{l}^{+(j)}(Z_0)_{kk})_{ii},$$

$i, k = 1, 2, \ldots, n$. Off diagonal entries of $\mathbf{l}^{\pm(j)}(H)$ corresponding to the second term of (2.14) give a set of $(n\text{-}1)^2$ simultaneous equations in α_1', α_1'',

$$\alpha_1'\Gamma^{(j)}(\mathbf{l}^{-(j)}(Z_1'S_+ \otimes S_-)_{(i-1)i})_{k(k-1)} = \alpha_1''\Gamma^{(j)}(\mathbf{l}^{+(j)}(Z_1''S_- \otimes S_+)_{k(k-1)})_{(i-1)i},$$

$i, k = 2, 3, \ldots, n$. Off diagonal entries of $\mathbf{l}^{\pm(j)}(H)$ corresponding to higher order terms of (2.14) give additional simultaneous equations in α_2', α_2'', etc.●

Proposition XVI. The twist map $\check{F}: U(sl_2) \to U_q(sl_2)$ leaves the undeformed coproduct $\Delta S_z = S_z \otimes 1 + 1 \otimes S_z$ unaltered. The general form (2.6) is then imposed on semirepresentations $\mathbf{l}^{\pm(j)}(F)$ of the twist element F.

Proof. That the twist map $\tilde{F}: U(sl_2) \to U_q(sl_2)$ leaves the coproduct $\Delta S_z = S_z \otimes 1 + 1 \otimes S_z$ unaltered follows from the expansion of (1.3a) in the deforming parameter h. The semirepresentations $\mathbf{l}^{\pm(j)}(\Delta S_z)$ are defined from (2.9b),

$$\mathbf{l}^{\pm(j)}(\Delta S_z) = \Gamma^{(j)}(S_z)1 + S_z\Gamma^{(j)}(1)$$

$$= diagonal(S_z + j, S_z + j - 1, \ldots, S_z - j).$$

It follows that $\mathbf{l}^{\pm(j)}(\Delta S_z) \in \mathcal{L}^{(n)}$. The equivalent semirepresentations $\mathbf{g}^{\pm(j)}(\Delta S_z) \in \mathcal{G}^{(n)}$ are obtained with the diagonal matrix $\mathbf{X}_-$ (2.7a, 2.15) giving

$$\mathbf{g}^{\pm(j)}(\Delta S_z) = \mathbf{X}_- \mathbf{l}^{\pm(j)}(\Delta S_z)\mathbf{X}_-^{-1} = (S_z + j)\mathbf{1}^{(j)}.$$

The semirepresentations $\mathbf{g}^{\pm(j)}(F)$ provide identity maps for $\mathbf{g}^{\pm(j)}(\Delta S_z)$. Hence $\mathbf{g}^{\pm(j)}(F), \mathbf{g}^{\pm(j)}(\Delta S_z)$ must commute. The entries of $\mathbf{g}^{\pm(j)}(F)$ are then functionals of S, S_z alone, and so $\mathbf{g}^{\pm(j)}(F) \in \mathcal{G}^{(n)}$. Finally from (Prop. IX), $\mathbf{l}^{\pm(j)}(F) \in \mathcal{L}^{(n)}$ as required.●

Proposition XVII. The positive (negative) semirepresentation $\mathbf{l}^{\pm(j)}(F)$ of the twist element F can be obtained from the negative (positive) semirepresentation $\mathbf{l}^{\mp(j)}(F)$ with the substitution $q \to q^{-1}$. That is, $\mathbf{l}^{\pm(j)}(F) \to \mathbf{l}^{\mp(j)}(F)$ with the substitution $q \to q^{-1}$.

Proof. The transpose of the twist element F is obtained with the substitution $q \to q^{-1}$. That is, $F \to F_{21}$ with $q \to q^{-1}$ (Prop. V). The positive (negative) semirepresentation of the twist element F is equal to the negative (positive) semirepresentation of the transpose of F (Prop. XI). The result then follows.●

Proposition XVIII. A general form for semirepresentations $\mathbf{g}^{\pm(j)}(F)$ of the twist element F can be constructed from nth order diagonalising matrices for the deformed

QIKF $\mathbf{g}^{\pm(j)}(\overline{Q})$ and (symmetric) canonical 2-tensor $\mathbf{g}^{\pm(j)}(t)$ (1.13a, 1.18). The $\mathbf{g}^{\pm(j)}(F)$ then satisfy the semirepresentation form of the gauge transformation (1.28) for the QIKF. Thus

$$\mathbf{g}^{\pm(j)}(F) \equiv \overline{D}^{\pm} D^{-1}, \tag{2.17}$$

$$\mathbf{g}^{\pm(j)}(\overline{Q}) = \mathbf{g}^{\pm(j)}(F)\mathbf{g}^{\pm(j)}(Q)\mathbf{g}^{\pm(j)}(F)^{-1},$$

where $\overline{\mathbf{D}}^{\pm}, \mathbf{D}$ are the diagonalising matrices for $\mathbf{g}^{\pm(j)}(\overline{Q})$ and $\mathbf{g}^{\pm(j)}(t)$, respectively. The forms (2.6) for $\mathbf{l}^{\pm(j)}(F)$ are then recovered through the gauge transformation (2.7),

$$\mathbf{l}^{\pm(j)}(F) = \mathbf{X}_{-}^{-1}\mathbf{g}^{\pm(j)}(F)\mathbf{X}_{-}. \tag{2.18}$$

Proof. From (1.9, 1.13, 1.20) the QIKF $\overline{Q}$ expands as a series in $S_{+} \otimes S_{-}$, so (Prop. XIV) the semirepresentation $\mathbf{l}^{-(j)}(\overline{Q}) \in L^{(n)}$, the algebra of matrices of the form (2.6). The gauge transformation (2.7) provides a map $\mathbf{l}^{-(j)}(\overline{Q}) \to \mathbf{g}^{-(j)}(\overline{Q}) \in G^{(n)}$. Thus (Prop X) a matrix $\overline{\mathbf{D}} \in G^{(n)}$ can be constructed so that

$$\overline{\Lambda} = \overline{\mathbf{D}}^{-1}\mathbf{g}^{-(j)}(\overline{Q})\overline{\mathbf{D}}, \tag{2.19}$$

$\overline{\Lambda}$ being the (diagonal) matrix of eigenvalues for $\mathbf{g}^{-(j)}(\overline{Q})$. The same argument holds for the form (1.18) of the canonical 2-tensor t, so that a matrix $\mathbf{D} \in G^{(n)}$ can be constructed for the semirepresentation $\mathbf{g}^{-(j)}(t)$,

$$\lambda = \mathbf{D}^{-1}\mathbf{g}^{-(j)}(t)\mathbf{D}. \tag{2.20}$$

The undeformed QIKF Q (Def. VIII) takes the form of an exponential series in the canonical 2-tensor t. It follows that the same matrix $\mathbf{D}$ diagonalises the semirepresentation $\mathbf{g}^{-(j)}(Q)$ of Q. Thus

$$\Lambda = \mathbf{D}^{-1}\mathbf{g}^{-(j)}(Q)\mathbf{D} \tag{2.21}$$

where $\Lambda \equiv q^{\lambda}$, the matrix of eigenvalues for $\mathbf{g}^{-(j)}(t)$. The deformed and undeformed QIKF's define (1.13, 1.27) equivalent deforming maps for the coproduct structure on $U(sl_2)$; it must follow that as representations $\mathbf{g}^{-(j)}(\overline{Q})$, $\mathbf{g}^{-(j)}(Q)$ diagonalise to the same matrix of eigenvalues (Def. VII). Thus $\overline{\Lambda} = \Lambda$. Equating $\overline{\Lambda}, \Lambda$ in (2.19, 2.21) and rearranging,

$$\overline{\mathbf{D}}^{-1}\mathbf{g}^{-(j)}(\overline{Q})\overline{\mathbf{D}} = \mathbf{D}^{-1}\mathbf{g}^{-(j)}(Q)\mathbf{D}$$

$$\Rightarrow \qquad \mathbf{g}^{-(j)}(\overline{Q}) = \overline{\mathbf{D}}\mathbf{D}^{-1}\mathbf{g}^{-(j)}(Q)\mathbf{D}\overline{\mathbf{D}}^{-1} = \overline{\mathbf{D}}\mathbf{D}^{-1}\mathbf{g}^{-(j)}(Q)(\overline{\mathbf{D}}\mathbf{D}^{-1})^{-1}.$$

Comparing with (2.17b) then gives the required result. The proof for $\mathbf{g}^{+(j)}(F)$ is analogous.●

Chapter Three. Partial construction of the twist element F from $\mathbf{l}^{\pm(1/2)}(F)$.

This section presents an explicit form for the first two terms of Drinfel'd's twist map element F, treated as an expansion in positive powers of $S_{\pm} \otimes S_{\mp}$,

$$F = F_0 + F_1 + F_2 + \ldots \in U(sl_2) \otimes U(sl_2), \tag{3.1}$$

F_n being nth order in $S_{\pm} \otimes S_{\mp}$. It begins with the construction of the j=1/2 semirepresentations of the coproduct structures, canonical 2-tensor and QIKF of the algebras $U(sl_2)$, $U_q(sl_2)$ starting from the action of the generators of the Lie algebra sl_2 and following the general principles laid out in previous sections.

In the main section most general forms for the twist map elements $\mathbf{l}^{\pm(1/2)}(F)$ are constructed in two arbitrary (operator-valued) variables $\nu'^{\pm}, \nu''^{\pm}$ from the diagonalising matrices of $\mathbf{l}^{\pm(1/2)}(t)$ and $\mathbf{l}^{\pm(1/2)}(\overline{Q})$, so that Drinfel'd's twist map conditions

$$\mathbf{l}^{\pm(1/2)}(\overline{R}) = \mathbf{l}^{\pm(1/2)}(F_{21})\mathbf{l}^{\pm(1/2)}(R)\mathbf{l}^{\pm(1/2)}(F)^{-1}, \tag{3.2a}$$

$$\mathbf{l}^{\pm(1/2)}(\overline{Q}) = \mathbf{l}^{\pm(1/2)}(F)\mathbf{l}^{\pm(1/2)}(Q)\mathbf{l}^{\pm(1/2)}(F)^{-1} \tag{3.2b}$$

are satisfied. These forms for $\mathbf{l}^{\pm(1/2)}(F)$ are as yet too general to act as deforming maps for the coproduct structure of $U(sl_2)$.

The remaining sections must then set conditions for the $\mathbf{l}^{\pm(1/2)}(F)$ as semirepresentations of the deforming map $\tilde{F}: \Delta u \to \overline{\Delta} u,\ u \in U(sl_2)$. It is shown that these conditions reduce to restrictions on the form of the variables $\nu'^{\pm}, \nu''^{\pm}$ in that they must be central, symmetric and invariant with respect to the substitution $q \to q^{-1}$.

Finally, explicit forms for the first two terms of F are recovered from the components of $\mathbf{l}^{\pm(1/2)}(F)$, as solutions to a set of simultaneous equations constructed from the direct product representations $\Gamma^{(1/2)}(\mathbf{l}^{\pm(1/2)}(F))$.

III.1. *Construction of j=1/2 semirepresentations of the algebra* $U(sl_2)$.

The j = 1/2 semirepresentations of the centre and coproduct structure of the algebra $U(sl_2)$ are constructed from the irreducible representation $\Gamma^{(1/2)}(g), g \in sl_2$. With $u \equiv \Sigma u_{(1)} \otimes u_{(2)},\ u_{(1)}, u_{(2)} \in U(sl_2)$,

$$\mathbf{l}^{-(1/2)}(u) \equiv \Gamma^{(1/2)}(u_{(1)})u_{(2)}, \tag{3.3a}$$

$$\mathbf{l}^{+(1/2)}(u) \equiv u_{(1)}\Gamma^{(1/2)}(u_{(2)}). \tag{3.3b}$$

This is a narrowing of the more general definition of the semirepresentations presented in (Prop. XI). Equivalent semirepresentations are obtained through a gauge transformation with the operator-valued diagonal matrix $\mathbf{X}_{-}$,

$$\mathbf{X}_{-} \equiv diagonal(1, S_{-}), \tag{3.4a}$$

so that

$$\mathbf{g}^{\pm(1/2)}(u) \equiv \mathbf{X}_{-}\mathbf{l}^{\pm(1/2)}(u)\mathbf{X}_{-}^{-1} \tag{3.4b}$$

the resulting semirepresentations $\mathbf{g}^{\pm(1/2)}(u)$ taking on much simplified forms. This transformation will greatly reduce the length of subsequent calculations, as will be seen.

$$\Gamma^{(1/2)}(S_+) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ \Gamma^{(1/2)}(S_-) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ \Gamma^{(1/2)}(S_z) = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}; \tag{3.5}$$

$$\Gamma^{(1/2)}(a) \equiv \Gamma^{(1/2)}(S - S_z) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\Gamma^{(1/2)}(b) \equiv \Gamma^{(1/2)}(S + S_z + 1) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.6}$$

$$\Gamma^{(1/2)}(c) \equiv \Gamma^{(1/2)}(a + b) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2\mathbf{1},\quad \Gamma^{(1/2)}(d) \equiv \Gamma^{(1/2)}(b - a) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Note that $\Gamma^{(1/2)}(S_\pm^n) = \mathbf{0}$ for $n \geq 2$. The variables *a, b, c, d* where $d \in U(sl_2)$ and $c^n \in U(sl_2)$, *n* even, were introduced in the first section and will be used to simplify calculations, as will the observation that $\Gamma^{(1/2)}(\overline{S}_-) = \Gamma^{(1/2)}(S_-)$, so that many of the structures involving q-brackets will reduce to the identity. It is this simplicity peculiar to the $j=1/2$ irreducible representation that reduces many of these calculations to manageable proportions.[1]

Definition XV. The $j = 1/2$ semirepresentations $\mathbf{l}^{\pm(1/2)}(\Delta u)$ of the elements $\Delta u \equiv 1 \otimes u + u \otimes 1$, $u \in U(sl_2)$ that define the (symmetric) primitive coproduct structure on $U(sl_2)$ are constructed with the irreducible representation (3.5). $\mathbf{l}^{\pm(1/2)}(\Delta u)$ are then algebra representations of $U(sl_2)$.

$$\mathbf{l}^{\pm(1/2)}(\Delta S_z) = \mathbf{l}^{\pm(1/2)}\left(1 \otimes S_z + S_z \otimes 1\right) = \Gamma^{(1/2)}(1) S_z + \Gamma^{(1/2)}(S_z) 1 = \begin{pmatrix} S_z + 1/2 & 0 \\ 0 & S_z - 1/2 \end{pmatrix},$$

$$\mathbf{l}^{\pm(1/2)}(\Delta S_+) = \mathbf{l}^{\pm(1/2)}\left(1 \otimes S_+ + S_+ \otimes 1\right) = \Gamma^{(1/2)}(1) S_+ + \Gamma^{(1/2)}(S_+) 1 = \begin{pmatrix} S_+ & 1 \\ 0 & S_+ \end{pmatrix}, \tag{3.7}$$

$$\mathbf{l}^{\pm(1/2)}(\Delta S_-) = \mathbf{l}^{\pm(1/2)}\left(1 \otimes S_- + S_- \otimes 1\right) = \Gamma^{(1/2)}(1) S_- + \Gamma^{(1/2)}(S_-) 1 = \begin{pmatrix} S_- & 0 \\ 1 & S_- \end{pmatrix}.$$

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[1] Note that $\Gamma^{(1/2)}(a)$ and $\Gamma^{(1/2)}(d)$ are singular. It is a measure of the balance of the constructions that follow that these matrices will only appear in non-singular q-expansions.

The equivalent semirepresentations $\mathbf{g}^{\pm(1/2)}(\Delta u)$ are then obtained from $\mathbf{l}^{\pm(1/2)}(\Delta u)$ with the gauge transform (3.4a), so that with $S_-^{-1} = S_+ S_+^{-1} S_-^{-1} = S_+ a^{-1} b^{-1}$ and recalling that $S_\pm S_z = (S_z \mp 1) S_\pm$,

$$\mathbf{g}^{\pm(1/2)}(\Delta S_z) = \mathbf{X}_- \mathbf{l}^{\pm(1/2)}(\Delta S_z)\mathbf{X}_-^{-1} = \begin{pmatrix} S_z + {}^1\!/_2 & 0 \\ 0 & S_-(S_z - {}^1\!/_2)S_-^{-1} \end{pmatrix} = (S_z + {}^1\!/_2\,)\mathbf{I},$$

$$\mathbf{g}^{\pm(1/2)}(\Delta S_+) \equiv \mathbf{X}_- \mathbf{l}^{\pm(1/2)}(\Delta S_+)\mathbf{X}_-^{-1} = \begin{pmatrix} S_+ & S_-^{-1} \\ 0 & S_- S_+ S_-^{-1} \end{pmatrix} = \begin{pmatrix} (a+1)(b-1) & 1 \\ 0 & ab \end{pmatrix} S_-^{-1}, \quad (3.8)$$

$$\mathbf{g}^{\pm(1/2)}(\Delta S_-) \equiv \mathbf{X}_- \mathbf{l}^{\pm(1/2)}(\Delta S_-)\mathbf{X}_-^{-1} = \begin{pmatrix} S_- & 0 \\ S_- & S_- \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} S_-.$$

Note that as Δu is symmetric, $\mathbf{l}^{-(1/2)}(\Delta u) = \mathbf{l}^{+(1/2)}(\Delta u)$ and $\mathbf{g}^{-(1/2)}(\Delta u) = \mathbf{g}^{+(1/2)}(\Delta u)$.

Definition XVI. The $j = 1/2$ semirepresentations $\mathbf{l}^{\pm(1/2)}(\overline{\Delta} u)$ of the elements $\overline{\Delta} u, u \in U(sl_2)$ that define the coproduct structure on $U_q(sl_2)$ can be constructed with the irreducible representations (3.5). $\mathbf{l}^{\pm(1/2)}(\overline{\Delta} u)$ are then algebra representations of $U_q(sl_2)$. Thus

$$\mathbf{l}^{\pm(1/2)}(\overline{\Delta} S_z) = \mathbf{l}^{\pm(1/2)}(\Delta S_z) = \begin{pmatrix} S_z + {}^1\!/_2 & 0 \\ 0 & S_z - {}^1\!/_2 \end{pmatrix},$$

$$\mathbf{l}^{\pm(1/2)}(\overline{\Delta} S_+) = \mathbf{l}^{\pm(1/2)}\left(q^{-S_z} \otimes S_+ + S_+ \otimes q^{S_z} \right) = \begin{pmatrix} q^{\pm 1/2} S_+ & q^{\pm 1/2} q^{\mp d/2} \\ 0 & q^{\mp 1/2} S_+ \end{pmatrix}, \quad (3.9)$$

$$\mathbf{l}^{\pm(1/2)}(\overline{\Delta} \overline{S}_-) = \mathbf{l}^{\pm(1/2)}\left(q^{-S_z} \otimes \overline{S}_- + \overline{S}_- \otimes q^{S_z} \right) = \begin{pmatrix} q^{\pm 1/2} \overline{S}_- & 0 \\ q^{\pm 1/2} q^{\mp d/2} & q^{\mp 1/2} \overline{S}_- \end{pmatrix}.$$

The equivalent semirepresentations $\mathbf{g}^{\pm(1/2)}(\overline{\Delta} u)$ are then obtained from $\mathbf{l}^{\pm(1/2)}(\overline{\Delta} u)$ with the gauge transform (3.4a), so that with $S_-^{-1} = S_+ S_+^{-1} S_-^{-1} = S_+ a^{-1} b^{-1}$, $\Phi \equiv a^{-1} b^{-1} [a]_q [b]_q$ and $S_\pm S_z = (S_z \mp 1) S_\pm$,

$$\mathbf{g}^{\pm(1/2)}(\overline{\Delta}S_z) = \mathbf{g}^{\pm(1/2)}(\Delta S_z) = (S_z + {}^1\!/_2\,)\mathbf{I},$$

$$\mathbf{g}^{\pm(1/2)}(\overline{\Delta}S_+) \equiv \mathbf{X}_-\mathbf{l}^{\pm(1/2)}(\overline{\Delta}S_+)\mathbf{X}_-^{-1} = \begin{pmatrix} q^{\pm 1/2}S_+ & q^{\pm 1/2}q^{\mp d/2}S_-^{-1} \\ 0 & q^{\mp 1/2}S_-S_+S_-^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} q^{\pm 1/2}(a+1)(b-1) & q^{\pm 1/2}q^{\mp d/2} \\ 0 & q^{\mp 1/2}ab \end{pmatrix} S_-^{-1}, \tag{3.10}$$

$$\mathbf{g}^{\pm(1/2)}(\overline{\Delta}\overline{S}_-) \equiv \mathbf{X}_-\mathbf{l}^{\pm(1/2)}(\overline{\Delta}\overline{S}_-)\mathbf{X}_-^{-1} = \begin{pmatrix} q^{\pm 1/2}\overline{S}_- & 0 \\ q^{\pm 1/2}S_-q^{\mp d/2} & q^{\mp 1/2}S_-\overline{S}_-S_-^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} q^{\pm 1/2}\Phi & 0 \\ q^{\mp 1/2}q^{\mp d/2} & q^{\mp 1/2}(a-1)^{-1}(b+1)^{-1}[a-1]_q[b+1]_q \end{pmatrix} S_-.$$

Note that as $\overline{\Delta}u \to \overline{\Delta}u_{21}$ with the substitution $q \to q^{-1}$ it must follow that $\mathbf{l}^{-(1/2)}(\overline{\Delta}u) \to \mathbf{l}^{+(1/2)}(\overline{\Delta}u)$ and $\mathbf{g}^{-(1/2)}(\overline{\Delta}u) \to \mathbf{g}^{+(1/2)}(\overline{\Delta}u)$ with $q \to q^{-1}$. This result and the results above will become important in section (III.4), where the conditions on the twist elements as deforming maps $\mathbf{l}^{\pm(1/2)}(F){:}\mathbf{l}^{\pm(1/2)}(\Delta u) \to \mathbf{l}^{\pm(1/2)}(\overline{\Delta}u)$ are derived.

Definition XVII. The undeformed canonical two tensor $t \in U(sl_2) \otimes U(sl_2)$ takes an especially simple form, being symmetric and linear in $S_\pm \otimes S_\mp$. From it's definition (VI) in section (I.2),

$$\mathbf{l}^{\pm(1/2)}(t) = \mathbf{l}^{\pm(1/2)}(4S_z \otimes S_z + 2S_+ \otimes S_- + 2S_- \otimes S_+)$$

$$= 4\Gamma^{(1/2)}(S_z)S_z + 2\Gamma^{(1/2)}(S_+)S_- + 2\Gamma^{(1/2)}(S_-)S_+,$$

$$\therefore \qquad \mathbf{l}^{\pm(1/2)}(t) = 2\begin{pmatrix} S_z & S_- \\ S_+ & -S_z \end{pmatrix}. \tag{3.11a}$$

The equivalent semirepresentations $\mathbf{g}^{\pm(1/2)}(t)$ then take forms whose components are independent of $S_\pm$, and hence mutually commuting. Recalling that $S_-S_+ = (S - S_z)(S + S_z + 1) = ab$, $d \equiv 2S_z + 1$ and $S_\pm S_z = (S_z \mp 1)S_\pm$,

$$\mathbf{g}^{\pm(1/2)}(t) \equiv \mathbf{X}_{-}\mathbf{l}^{\pm(1/2)}(t)\mathbf{X}_{-}^{-1} = 2\begin{pmatrix} S_z & S_{-}S_{-}^{-1} \\ S_{-}S_{+} & -S_{-}S_zS_{-}^{-1} \end{pmatrix},$$

$\therefore$
$$\mathbf{g}^{\pm(1/2)}(t) = \begin{pmatrix} d-1 & 2 \\ 2ab & -(d+1) \end{pmatrix}. \tag{3.11b}$$

Definition XVIII. Construction of the negative semirepresentation of the quasitriangular structure, $\mathbf{l}^{-(1/2)}(\overline{R})$, is more complicated in that it involves the manipulation of an operator valued infinite series. Treating Majid's form (Def. II) for $\overline{R}$ as an expansion in $S_{\pm} \otimes S_{\mp}$ and recalling that $\Gamma^{(1/2)}(S_{\pm}^{n}) = \mathbf{0}$ and hence $\mathbf{l}^{\pm(1/2)}(S_{\pm}^{n} \otimes S_{\mp}^{n}) = \mathbf{0}$ for $n \geq 2$, the form of the semirepresentation serves to truncate the series to terms that are at most quadratic in $S_{\pm} \otimes S_{\mp}$,

$$\begin{aligned}\mathbf{l}^{-(1/2)}(\overline{R}) &= \mathbf{l}^{-(1/2)}(q^{2S_z \otimes S_z})\mathbf{l}^{-(1/2)}\Big(1 \otimes 1 + q^{-1}p(q^{S_z} \otimes q^{-S_z})(S_{+} \otimes \Phi S_{-}) + \ldots\Big) \\ &= q^{2\Gamma(S_z)S_z}\Big(\Gamma^{(1/2)}(1)1 + q^{-1}p\Gamma^{(1/2)}(q^{S_z})q^{-S_z}\Gamma^{(1/2)}(S_{+})\Phi S_{-}\Big),\end{aligned}$$

$\therefore$
$$\mathbf{l}^{-(1/2)}(\overline{R}) = \begin{pmatrix} q^{S_z} & q^{-1/2}p\Phi S_{-} \\ 0 & q^{-S_z} \end{pmatrix}, \tag{3.12a}$$

with $\Phi \equiv a^{-1}b^{-1}[a]_q[b]_q$, $p \equiv q - q^{-1}$ and $d \equiv 2S_z + 1$. The equivalent semirepresentation, $\mathbf{g}^{-(1/2)}(\overline{R})$, is then constructed by means of the gauge transformation (3.4a),

$$\mathbf{g}^{-(1/2)}(\overline{R}) \equiv \mathbf{X}_{-}\mathbf{l}^{-(1/2)}(\overline{R})\mathbf{X}_{-}^{-1} = \begin{pmatrix} q^{S_z} & q^{-1/2}p\Phi S_{-}S_{-}^{-1} \\ 0 & S_{-}q^{-S_z}S_{-}^{-1} \end{pmatrix},$$

$\therefore$
$$\mathbf{g}^{-(1/2)}(\overline{R}) = q^{-1/2}\begin{pmatrix} q^{d/2} & p\Phi \\ 0 & q^{-d/2} \end{pmatrix}. \tag{3.12b}$$

The identical argument is used in the construction of the transpose element $\mathbf{l}^{-(1/2)}(\overline{R}_{21})$. Taking the transpose of Majid's form,

$$\mathbf{l}^{-(1/2)}(\overline{R}_{21}) = \mathbf{l}^{-(1/2)}(q^{2S_z \otimes S_z})\mathbf{l}^{-(1/2)}\Big(1 \otimes 1 + q^{-1}p(q^{-S_z} \otimes q^{S_z})(\Phi \mathrm{S}_- \otimes S_+) + \ldots\Big)$$

$$= q^{2\Gamma(S_z)S_z}\Big(\Gamma^{(1/2)}(1)1 + q^{-1}p\Gamma^{(1/2)}(q^{-S_z})q^{S_z}\Gamma^{(1/2)}(\Phi\mathrm{S}_-)S_+\Big),$$

$$\therefore \qquad \mathbf{l}^{-(1/2)}(\overline{R}_{21}) = \begin{pmatrix} q^{S_z} & 0 \\ q^{-1/2}pS_+ & q^{-S_z} \end{pmatrix}; \qquad (3.13a)$$

so that

$$\mathbf{g}^{-(1/2)}(\overline{R}_{21}) \equiv \mathbf{X}_-\mathbf{l}^{-(1/2)}(\overline{R}_{21})\mathbf{X}_-^{-1} = \begin{pmatrix} q^{S_z} & 0 \\ q^{-1/2}S_-pS_+ & S_-q^{-S_z}S_-^{-1} \end{pmatrix},$$

$$\therefore \qquad \mathbf{g}^{-(1/2)}(\overline{R}_{21}) = q^{-1/2}\begin{pmatrix} q^{d/2} & 0 \\ pab & q^{-d/2} \end{pmatrix}. \qquad (3.13b)$$

Definition XIX. Both the *positive* and *negative* semirepresentations $\mathbf{g}^{\pm(1/2)}(\overline{Q})$ of the QIKF follow directly from its definition (III) as the product of the quasitriangular structure with its transpose, namely

$$\mathbf{l}^{\pm(1/2)}(\overline{Q}) \equiv \mathbf{l}^{\pm(1/2)}(\overline{R}_{21})\mathbf{l}^{\pm(1/2)}(\overline{R}), \qquad (3.14a)$$

$$\mathbf{l}^{\pm(1/2)}(\overline{R}_{21}) = \mathbf{l}^{\mp(1/2)}(\overline{R}). \qquad (3.14b)$$

Transforming and combining these two equations and substituting for $\mathbf{g}^{\pm(1/2)}(\overline{R}_{21})$ and $\mathbf{g}^{\pm(1/2)}(\overline{R})$ then gives

$$\mathbf{g}^{-(1/2)}(\overline{Q}) \equiv \mathbf{X}_-\mathbf{l}^{-(1/2)}(\overline{Q})\mathbf{X}_-^{-1} = \mathbf{X}_-\mathbf{l}^{-(1/2)}(\overline{R}_{21})\mathbf{X}_-^{-1}\mathbf{X}_-\mathbf{l}^{-(1/2)}(\overline{R})\mathbf{X}_-^{-1}.$$

$$= \mathbf{g}^{-(1/2)}(\overline{R}_{21})\mathbf{g}^{-(1/2)}(\overline{R}) = q^{-1}\begin{pmatrix} q^{d/2} & 0 \\ pab & q^{-d/2} \end{pmatrix}\begin{pmatrix} q^{d/2} & p\Phi \\ 0 & q^{-d/2} \end{pmatrix},$$

$$\therefore \qquad \mathbf{g}^{-(1/2)}(\overline{Q}) = q^{-1}\begin{pmatrix} q^d & pq^{d/2}\Phi \\ pq^{d/2}ab & p^2[a]_q[b]_q + q^{-d} \end{pmatrix}, \tag{3.15a}$$

and

$$\mathbf{g}^{+(1/2)}(\overline{Q}) = \mathbf{g}^{+(1/2)}(\overline{R}_{21})\mathbf{g}^{+(1/2)}(\overline{R}) = \mathbf{g}^{-(1/2)}(\overline{R})\mathbf{g}^{-(1/2)}(\overline{R}_{21})$$

$$= q^{-1}\begin{pmatrix} q^{d/2} & p\Phi \\ 0 & q^{-d/2} \end{pmatrix}\begin{pmatrix} q^{d/2} & 0 \\ pab & q^{-d/2} \end{pmatrix}$$

$$\therefore \qquad \mathbf{g}^{+(1/2)}(\overline{Q}) = q^{-1}\begin{pmatrix} q^d + p^2[a]_q[b]_q & pq^{-d/2}\Phi \\ pq^{-d/2}ab & q^{-d} \end{pmatrix}. \tag{3.15b}$$

III.2 *Construction of general forms for* $\mathbf{l}^{\pm(1/2)}(F)$ *in two variables.*

This is the crucial section in that semirepresentations of Drinfel'd's twist map element are constructed from the diagonalising matrices for the canonical two-tensor and QIKF. Throughout it has been shown that the *negative* semirepresentations of the elements $u \in U(sl_2) \otimes U(sl_2)$ hitherto dealt with map to their corresponding *positive* semirepresentations with the substitution $q \to q^{-1}$, largely because the elements themselves possess the property that either $u = u_{21}$, or $u \to u_{21}$ with $q \to q^{-1}$. It will be demonstrated that the general forms for $\mathbf{l}^{\pm(1/2)}(F)$ thus obtained also possesses this important symmetry, so that $\mathbf{l}^{-(1/2)}(F) \to \mathbf{l}^{+(1/2)}(F)$ with the substitution $q \to q^{-1}$.

Proposition XIX. The canonical two-tensor $\mathbf{l}^{\pm(1/2)}(t)$ diagonalises to a matrix of eigenvalues whose components are central, in that they are functionals of the central element S. The diagonalising matrix takes the form $\mathbf{X}_-^{-1}\mathbf{D}\mathbf{V}$, $\mathbf{X}_-$ being the diagonal matrix (3.4a), and

$$\mathbf{D} = -\begin{pmatrix} 1 & 1 \\ a & -b \end{pmatrix} \tag{3.16a}$$

being the diagonalising matrix for the equivalent semirepresentations $\mathbf{g}^{\pm(1/2)}(t)$. **D** is non-singular as $|\mathbf{D}| = (a+b) = c \neq 0$, so that

$$\mathbf{D}^{-1} = -c^{-1}\begin{pmatrix} b & 1 \\ a & -1 \end{pmatrix}. \tag{3.16b}$$

V is a matrix of operator-valued components that are completely arbitrary save that they are functionals of S, S_z and independent of $S_{\pm}$,

$$\mathbf{V} \equiv diagonal(\nu', \nu''). \tag{3.17}$$

Then the matrix of eigenvalues

$$\lambda = \mathbf{D}^{-1}\mathbf{g}^{\pm(1/2)}(t)\mathbf{D} = diagonal(c-1, -c-1) \tag{3.18a}$$

so that

$$\mathbf{V}^{-1}\lambda\mathbf{V} = \mathbf{V}^{-1}\mathbf{D}^{-1}\mathbf{X}_{-}\mathbf{X}_{-}^{-1}\mathbf{g}^{\pm(1/2)}(t)\mathbf{X}_{-}\mathbf{X}_{-}^{-1}\mathbf{D}\mathbf{V}$$

$$= \mathbf{V}^{-1}\mathbf{D}^{-1}\mathbf{X}_{-}\mathbf{l}^{\pm(1/2)}(t)\mathbf{X}_{-}^{-1}\mathbf{D}\mathbf{V},$$

and therefore

$$\lambda = (\mathbf{X}_{-}^{-1}\mathbf{D}\mathbf{V})^{-1}\mathbf{l}^{\pm(1/2)}(t)\mathbf{X}_{-}^{-1}\mathbf{D}\mathbf{V}. \tag{3.18b}$$

Post-multiplication of **D** by the matrix **V** is equivalent to the multiplication of the individual (column) eigenvectors that compose **D** by the variables $\nu'^{\pm}, \nu''^{\pm}$, and is the direct expression of the arbitrariness of these eigenvectors. These variables will play a central role in the arguments of the remaining sections.

Proof. The proof consists in the construction of the sets of eigenvalues and eigenvectors for the matrix semirepresentations $\mathbf{g}^{\pm(1/2)}(t)$. Substitution into (3.18a) then leads to the required result. As discussed in Chapter Two, the components of $\mathbf{g}^{\pm(1/2)}(t)$ are mutually commuting, so these eigenvalues and eigenvectors can be obtained using standard numerical techniques. Specifically, the eigenvalues are solutions of the characteristic polynomial for $\mathbf{g}^{\pm(1/2)}(t)$ with n=2,

$$\begin{vmatrix} d-1-\lambda & 2 \\ 2ab & -(d+1)-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \qquad (d-1-\lambda)(d+1+\lambda)+4ab=0$$

$$\Rightarrow \qquad \lambda^2+2\lambda+(1-c^2)=0$$

with $c^2 - d^2 = 4ab$. The solutions to this quadratic equation are $\lambda = \pm c - 1$, as required. The eigenvectors of $\mathbf{g}^{\pm(1/2)}(t)$ are then the columns of the matrix $\mathbf{D}_\lambda$ defined by

$$\mathbf{D}_\lambda = \lambda \mathbf{1} + \mathbf{B}_1$$

with

$$\mathbf{B}_1 = \mathbf{g}^{\pm(1/2)}(t) - tr\mathbf{g}^{\pm(1/2)}(t)\mathbf{1},$$

$$\mathbf{D}_\lambda = \mathbf{g}^{\pm(1/2)}(t) + (\lambda + 2)\mathbf{1}.$$

For $\lambda = c-1$,

$$\mathbf{D}_\lambda = \begin{pmatrix} c+d & 2 \\ 2ab & c-d \end{pmatrix} = 2\begin{pmatrix} b & 1 \\ ab & a \end{pmatrix}.$$

For $\lambda = -c-1$,

$$\mathbf{D}_\lambda = \begin{pmatrix} -(c-d) & 2 \\ 2ab & -(c+d) \end{pmatrix} = 2\begin{pmatrix} -a & 1 \\ ab & -b \end{pmatrix}.$$

Neglecting arbitrary numerical factors, the diagonalising matrix $\mathbf{D}$ can then be constructed from the eigenvectors $\{1, a\}$ and $\{1, -b\}$, as required. ●

Proposition XX. The diagonalising matrix $\mathbf{D}$ for the canonical two-tensor $\mathbf{g}^{\pm(1/2)}(t)$ also diagonalises the undeformed Q-matrix, $\mathbf{g}^{\pm(1/2)}(Q)$, $Q \equiv q^t$ being introduced (Def. VIII) as the twist map for the (symmetric) primitive coproduct structure of $U(sl_2)$. The resulting matrix of eigenvalues Λ is expressed as a q-exponential of the (diagonal) matrix of eigenvalues λ. Thus

$$\mathbf{g}^{\pm(1/2)}(Q) \equiv q^{\mathbf{l}(t)}, \tag{3.19a}$$

$$\Lambda = \mathbf{D}^{-1}\mathbf{g}^{\pm(1/2)}(Q)\mathbf{D} = diagonal(q^{(c-1)}, q^{-(c+1)}). \tag{3.19b}$$

Proof. The result follows directly from the definition of the undeformed Q-Matrix $\mathbf{g}^{\pm(1/2)}(Q)$ as an exponential expansion in the canonical two-tensor $\mathbf{g}^{\pm(1/2)}(t)$.[2] ●

Proposition XXI. The QIKF $\mathbf{l}^{-(1/2)}(\overline{Q})$ also diagonalises to a matrix of eigenvalues whose components are functionals of the central element *S*. The diagonalising matrix takes the form $\mathbf{X}_-^{-1}\overline{\mathbf{D}}\overline{\mathbf{V}}$, where again $\mathbf{X}_-$ is the diagonal matrix (3.4a),

$$\overline{\mathbf{D}} = -\begin{pmatrix} 1 & 1 \\ abq^{c/2}[b]_q^{-1} & -abq^{c/2}[a]_q^{-1} \end{pmatrix}, \tag{3.20}$$

being the diagonalising matrix for the equivalent representation $\mathbf{g}^{-(1/2)}(\overline{Q})$. $\overline{\mathbf{D}}$ is non-singular as $|\overline{\mathbf{D}}| = abq^{c/2}([a]_q^{-1} + [b]_q^{-1}) \neq 0$.

$$\overline{\mathbf{V}} \equiv diagonal(\overline{v}', \overline{v}'') \tag{3.21}$$

is a matrix of (now *q*-dependent) operator-valued components that are completely arbitrary save that they are functionals of S, S_z. Then

$$\Lambda = \overline{\mathbf{D}}^{-1}\mathbf{g}^{-(1/2)}(\overline{Q})\overline{\mathbf{D}} = diagonal(q^{(c-1)}, q^{-(c+1)}) \tag{3.22a}$$

so that

$$\overline{\mathbf{V}}^{-1}\Lambda\overline{\mathbf{V}} = \overline{\mathbf{V}}^{-1}\overline{\mathbf{D}}^{-1}\mathbf{X}_-\mathbf{X}_-^{-1}\mathbf{g}^{-(1/2)}(\overline{Q})\mathbf{X}_-\mathbf{X}_-^{-1}\overline{\mathbf{D}}\overline{\mathbf{V}}.$$

$$= \overline{\mathbf{V}}^{-1}\overline{\mathbf{D}}^{-1}\mathbf{X}_-\mathbf{l}^{\pm(1/2)}(\overline{Q})\mathbf{X}_-^{-1}\overline{\mathbf{D}}\overline{\mathbf{V}},$$

and therefore

$$\Lambda = (\mathbf{X}_-^{-1}\overline{\mathbf{D}}\overline{\mathbf{V}})^{-1}\mathbf{l}^{-(1/2)}(\overline{Q})\mathbf{X}_-^{-1}\overline{\mathbf{D}}\overline{\mathbf{V}}. \tag{3.22b}$$

Note that from (3.19b) and (3.23b), *the undeformed Q-matrix* $\mathbf{g}^{\pm(1/2)}(Q)$ *and the QIKF* $\mathbf{g}^{-(1/2)}(\overline{Q})$ *diagonalise to the same matrix of eigenvalues*. Note also that $\mathbf{g}^{\pm(1/2)}(Q)$ and $\mathbf{g}^{-(1/2)}(\overline{Q})$ are equivalent as $tr\mathbf{g}(\overline{Q}) = q^{-1}\left(q^d + p^2[a]_q[b]_q + q^{-d}\right) = q^{c-1} + q^{-c-1}$.

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[2] See Cornwell (1984b), p. 408, and unpublished notes.

Proof. The proof consists in the construction of the sets of eigenvalues and eigenvectors for the matrix semirepresentation $\mathbf{g}^{-(1/2)}(\overline{Q})$. Substitution into (3.22b) then leads to the required result. As discussed in Chapter Two, the components of $\mathbf{g}^{-(1/2)}(\overline{Q})$ are mutually commuting, so these eigenvalues and eigenvectors can be obtained using standard numerical techniques. Specifically, the eigenvalues are solutions of the characterictic polynomial for $\mathbf{g}^{-(1/2)}(\overline{Q})$ with n=2,

$$(q^d - \mathrm{q}\lambda)(p^2[a]_q[b]_q + q^{-d} - \mathrm{q}\lambda) - p^2 q^d ab\Phi = 0$$

$$\Rightarrow \qquad \mathrm{q}\lambda^2 - (q^c + q^{-c})\lambda + q^{-1} = 0$$

with $q^d + q^{-d} + p^2[a]_q[b]_q = q^c + q^{-c}$. The solutions to this quadratic equation are $\Lambda = q^{\pm c-1}$, as required. The eigenvectors of $\mathbf{g}^{\pm(1/2)}(\overline{Q})$ are then the columns of the matrix $\overline{\mathbf{D}}_\lambda$ defined by

$$\overline{\mathbf{D}}_\lambda = \Lambda\mathbf{1} + \overline{\mathbf{B}}_1$$

with

$$\overline{\mathbf{B}}_1 = \mathbf{g}^{-(1/2)}(\overline{Q}) - tr\mathbf{g}^{-(1/2)}(\overline{Q})\mathbf{1},$$

$$\overline{\mathbf{D}}_\lambda = \mathbf{g}^{-(1/2)}(\overline{Q}) + (\Lambda - q^{-1}q^c - q^{-1}q^{-c})\mathbf{1}.$$

For $\Lambda = q^c q^{-1}$,

$$\overline{\mathbf{D}}_\lambda = q^{-1}\begin{pmatrix} q^d - q^{-c} & pq^{d/2}\Phi \\ pq^{d/2}ab & p^2[a]_q[b]_q + q^{-d} - q^{-c} \end{pmatrix},$$

$$= q^{-1}\begin{pmatrix} q^d - q^{-c} & pq^{d/2}\Phi \\ pq^{d/2}ab & q^d + q^c \end{pmatrix}.$$

For $\Lambda = q^{-c}q^{-1}$,

$$\overline{\mathbf{D}}_\lambda = q^{-1}\begin{pmatrix} q^d - q^{c} & pq^{d/2}\Phi \\ pq^{d/2}ab & p^2[a]_q[b]_q + q^{-d} - q^{c} \end{pmatrix},$$

$$= q^{-1}\begin{pmatrix} q^d - q^{c} & pq^{d/2}\Phi \\ pq^{d/2}ab & q^d + q^{-c} \end{pmatrix}.$$

Neglecting arbitrary (now operator valued) factors $q^{-1}(q^{d}-q^{-c})=q^{-1}pq^{-a}[b]_q$ and $q^{-1}(q^{d}-q^{c})=-q^{-1}pq^{b}[a]_q$, the diagonalising matrix $\overline{\mathbf{D}}$ can be constructed from the eigenvectors $\{1,\ abq^{c/2}[b]_q^{-1}\}$ and $\{1,\ -abq^{c/2}[a]_q^{-1}\}$, as required. ●

Proposition XXII. The diagonalising matrix for the *positive* semirepresentation $\mathbf{g}^{+(1/2)}(\overline{Q})$ of the QIKF can be obtained by the substitution $q\to q^{-1}$ in the matrix $\overline{\mathbf{D}}$ of (3.20),

$$\overline{\mathbf{D}}^{+}=-\begin{pmatrix}1 & 1\\ abq^{-c/2}[b]_q^{-1} & -abq^{-c/2}[a]_q^{-1}\end{pmatrix}. \tag{3.23}$$

$\mathbf{g}^{+(1/2)}(\overline{Q})$ then diagonalises to the same matrix of eigenvalues (3.18b) as $\mathbf{g}^{-(1/2)}(\overline{Q})$.

Proof. The proof follows from the action of the form (3.12b) of $\mathbf{g}^{+(1/2)}(\overline{Q})$ on the eigenvectors $\overline{\mathbf{x}}_1,\overline{\mathbf{x}}_2$ that compose (3.23). For $\overline{\mathbf{x}}_1=\{-1,-abq^{-c/2}[b]_q^{-1}\}$ and $\overline{\mathbf{x}}_2=\{-1,abq^{-c/2}[a]_q^{-1}\}$,

$$\mathbf{g}^{+(1/2)}(\overline{Q})\overline{\mathbf{x}}_1=q^{-1}\begin{pmatrix}q^{d}+p^{2}[a]_q[b]_q & pq^{-d/2}\Phi\\ pq^{-d/2}ab & q^{-d}\end{pmatrix}\begin{pmatrix}-1\\ -abq^{-c/2}[b]_q^{-1}\end{pmatrix}$$

$$=-q^{-1}\begin{pmatrix}q^{d}+p^{2}[a]_q[b]_q+pq^{-b}[a]_q\\ (pq^{a}[b]_q+q^{-d})abq^{-c/2}[b]_q^{-1}\end{pmatrix}=q^{-1}q^{c}\overline{\mathbf{x}}_1,$$

$$\mathbf{g}^{+(1/2)}(\overline{Q})\overline{\mathbf{x}}_2=q^{-1}\begin{pmatrix}q^{d}+p^{2}[a]_q[b]_q & pq^{-d/2}\Phi\\ pq^{-d/2}ab & q^{-d}\end{pmatrix}\begin{pmatrix}-1\\ abq^{-c/2}[a]_q^{-1}\end{pmatrix}$$

$$=-q^{-1}\begin{pmatrix}q^{d}+p^{2}[a]_q[b]_q-pq^{-b}[b]_q\\ (pq^{a}[a]_q-q^{-d})abq^{-c/2}[a]_q^{-1}\end{pmatrix}=q^{-1}q^{-c}\overline{\mathbf{x}}_2,$$

after substituting $p[a]_q = q^a - q^{-a}$, $p[b]_q = q^b - q^{-b}$ and multiplying through, as required. ●

Proposition XXIII. A general form for the j=1/2 negative semirepresentation $\mathbf{l}^{-(1/2)}(F)$ of Drinfel'd's twist map element can be constructed from the diagonalising matrices, $\overline{\mathbf{D}}$ and $\mathbf{D}$, of the QIKF and the canonical 2-tensor, giving

$$\mathbf{l}^{-(1/2)}(F) = \mathbf{X}_-^{-1}\overline{\mathbf{D}}\mathbf{V}\mathbf{D}^{-1}\mathbf{X}_- \tag{3.24}$$

where

$$\mathbf{V} \equiv diagonal(\nu'^-, \nu''^-)$$

is a matrix of arbitrary functionals ν'^-, ν''^- of S and S_z.[3] Allowing ν'^-, ν''^- to vary over the set of continuous functions of S, S_z then sweeps out a continuous set of elements $\mathbf{l}^{-(1/2)}(F)$. The explicit construction gives

$$\mathbf{l}^{-(1/2)}(F) \equiv \mathbf{X}_-^{-1}\mathbf{g}^{-(1/2)}(F)\mathbf{X}_- \tag{3.25}$$

with

$$\mathbf{g}^{-(1/2)}(F) = \nu'^-\mathbf{g}^{-(1/2)}(F') + \nu''^-\mathbf{g}^{-(1/2)}(F''),$$

$$\mathbf{g}^{-(1/2)}(F') = c^{-1}q^{-a/2}\begin{pmatrix} [b]_q & b^{-1}[b]_q \\ abq^{c/2} & aq^{c/2} \end{pmatrix},$$

$$\mathbf{g}^{-(1/2)}(F'') = c^{-1}q^{b/2}\begin{pmatrix} [a]_q & -a^{-1}[a]_q \\ -abq^{-c/2} & bq^{-c/2} \end{pmatrix}.$$

$\mathbf{l}^{-(1/2)}(F)$ reduces to the identity in the limit $q \to 1$ provided $\nu'^-, \nu''^- \to 1$ as $q \to 1$. Further, as the constituent matrices $\mathbf{X}_-, \overline{\mathbf{D}}$ and $\mathbf{D}$ are all non-singular, the sole condition

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[3] The notation is used to associate ν'^-, ν''^- with the negative semirepresentation of F. In practice ν'^-, ν''^- are products of the variables defined in (3.17) and (3.21), but their arbitrariness makes this distinction unnecessary.

for the non-singularity, and hence invertibility, of $\mathbf{l}^{-(1/2)}(F)$ is that the determinant of $\mathbf{V}$ be non-zero. Hence the variables ν'^{-}, ν''^{-} must always be chosen to be non-zero.

Proof. The proof follows directly from the substitution of the general form (3.24) into the definition (3.2b) for the deforming map $\mathbf{l}^{\pm(1/2)}(Q) \to \mathbf{l}^{-(1/2)}(\overline{Q})$, along with the observation that the matrices of eigenvectors for the undeformed Q-matrix and the QIKF are identical. Substituting,

$$\mathbf{l}^{-(1/2)}(\overline{Q}) = \mathbf{X}_{-}^{-1}\overline{\mathbf{D}}\mathbf{V}\mathbf{D}^{-1}\mathbf{X}_{-}\mathbf{l}^{\pm(1/2)}(Q)(\mathbf{X}_{-}^{-1}\overline{\mathbf{D}}\mathbf{V}\mathbf{D}^{-1}\mathbf{X}_{-})^{-1}$$

$$\Leftrightarrow \qquad \mathbf{X}_{-}\mathbf{l}^{-(1/2)}(\overline{Q})\mathbf{X}_{-}^{-1} = \overline{\mathbf{D}}\mathbf{V}\mathbf{D}^{-1}\mathbf{X}_{-}\mathbf{l}^{\pm(1/2)}(Q)\mathbf{X}_{-}^{-1}\mathbf{D}\mathbf{V}^{-1}\overline{\mathbf{D}}^{-1}$$

$$\Leftrightarrow \qquad \overline{\mathbf{D}}^{-1}\mathbf{g}^{-(1/2)}(\overline{Q})\overline{\mathbf{D}} = \mathbf{V}\mathbf{D}^{-1}\mathbf{g}^{\pm(1/2)}(Q)\mathbf{D}\mathbf{V}^{-1}$$

$$\Leftrightarrow \qquad \Lambda = \mathbf{V}\Lambda\mathbf{V}^{-1} = \Lambda,$$

as required. Substituting the diagonalising matrices (3.16b, 3.20) with (3.4a) and the appropriate substitutions for ν'^{-}, ν''^{-}, then multiplying through gives the explicit form (3.25) for $\mathbf{l}^{-(1/2)}(F)$. ●

Proposition XXIV. The *positive* semirepresentation of the general form of the twist map element $\mathbf{l}^{+(1/2)}(F)$ is obtained from the form (3.24, 3.25) of the *negative* semirepresentation $\mathbf{l}^{-(1/2)}(F)$ with the substitution $q \to q^{-1}$. Substituting $q \to q^{-1}$ in (3.24),

$$\mathbf{l}^{+(1/2)}(F) = \mathbf{X}_{-}^{-1}\overline{\mathbf{D}}^{+}\mathbf{V}^{+}\mathbf{D}^{-1}\mathbf{X}_{-}, \tag{3.26}$$

where

$$\mathbf{V}^{+} \equiv diagonal(\nu'^{+}, \nu''^{+})$$

is a matrix of arbitrary functionals ν'^{+}, ν''^{+} such that $\nu'^{-}, \nu''^{-} \to \nu'^{+}, \nu''^{+}$ with the $q \to q^{-1}$ substitution, $\mathbf{X}_{-}, \mathbf{D}$ are q-independent, and $\overline{\mathbf{D}}^{+}$ is the diagonalising matrix

(3.23) for the positive semirepresentation of the QIKF, $\mathbf{l}^{+(1/2)}(\overline{Q})$. Substituting $q \to q^{-1}$ in (3.25) gives the explicit construction for $\mathbf{l}^{+(1/2)}(F)$,

$$\mathbf{l}^{+(1/2)}(F) \equiv \mathbf{X}_-^{-1}\mathbf{g}^{+(1/2)}(F)\mathbf{X}_- \tag{3.27}$$

with

$$\mathbf{g}^{+(1/2)}(F) = \nu'^{+}\mathbf{g}^{+(1/2)}(F') + \nu''^{+}\mathbf{g}^{+(1/2)}(F''),$$

$$\mathbf{g}^{+(1/2)}(F') = c^{-1}q^{a/2}\begin{pmatrix} [b]_q & b^{-1}[b]_q \\ abq^{-c/2} & aq^{-c/2} \end{pmatrix},$$

$$\mathbf{g}^{+(1/2)}(F'') = c^{-1}q^{-b/2}\begin{pmatrix} [a]_q & -a^{-1}[a]_q \\ -abq^{c/2} & bq^{c/2} \end{pmatrix}.$$

As with $\mathbf{l}^{-(1/2)}(F)$, $\mathbf{l}^{+(1/2)}(F)$ reduces to the identity in the limit $q \to 1$ provided $\nu'^{+}, \nu''^{+} \to 1$ as $q \to 1$. The condition for the invertibility of $\mathbf{l}^{+(1/2)}(F)$ is that the determinant of $\mathbf{V}^{+}$ be non-zero, and hence the variables ν'^{+}, ν''^{+} must always be chosen to be non-zero.

Proof. The proof follows directly from the substitution of the general form (3.26) into the definition (3.2b) for the deforming map $\mathbf{l}^{\pm(1/2)}(Q) \to \mathbf{l}^{+(1/2)}(\overline{Q})$, with the observation that the eigenvalues for the undeformed Q-matrix and the QIKF are identical, and the further observation that the positive and negative semirepresentations of the QIKF also diagonalise to the identical set of eigenvalues. Substituting (3.26) in (3.2b),

$$\mathbf{l}^{+(1/2)}(\overline{Q}) = \mathbf{X}_-^{-1}\overline{\mathbf{D}}^{+}\mathbf{V}^{+}\mathbf{D}^{-1}\mathbf{X}_-\mathbf{l}^{\pm(1/2)}(Q)(\mathbf{X}_-^{-1}\overline{\mathbf{D}}^{+}\mathbf{V}^{+}\mathbf{D}^{-1}\mathbf{X}_-)^{-1}$$

$$\Leftrightarrow \qquad \mathbf{X}_-\mathbf{l}^{+(1/2)}(\overline{Q})\mathbf{X}_-^{-1} = \overline{\mathbf{D}}^{+}\mathbf{V}^{+}\mathbf{D}^{-1}\mathbf{X}_-\mathbf{l}^{\pm(1/2)}(Q)\mathbf{X}_-^{-1}\mathbf{D}(\mathbf{V}^{+})^{-1}(\overline{\mathbf{D}}^{+})^{-1}$$

$$\Leftrightarrow \qquad (\overline{\mathbf{D}}^{+})^{-1}\mathbf{g}^{+(1/2)}(\overline{Q})\overline{\mathbf{D}}^{+} = \mathbf{V}^{+}\mathbf{D}^{-1}\mathbf{g}^{\pm(1/2)}(Q)\mathbf{D}(\mathbf{V}^{+})^{-1}$$

$$\Leftrightarrow \qquad \Lambda = \mathbf{V}^{+}\Lambda(\mathbf{V}^{+})^{-1} = \Lambda,$$

as required. Substituting the diagonalising matrices (3.16b, 3.23) with (3.4a) and multiplying through then gives the explicit form (3.27) for $\mathbf{l}^{+(1/2)}(\mathcal{F})$. ●

III.3. *Properties of the general forms* $\mathbf{l}^{\pm(1/2)}(\mathcal{F})$ *as twist maps.*

This section lays out the conditions for the general forms (3.25, 3.27) for the semi-representations $\mathbf{l}^{\pm(1/2)}(\mathcal{F})$ as twist maps $\mathbf{l}^{\pm(1/2)}(\mathcal{F}):\mathbf{l}^{\pm(1/2)}(\mathcal{R}) \to \mathbf{l}^{\pm(1/2)}(\overline{\mathcal{R}})$. The arbitrary variables $\nu'^{\pm}, \nu''^{\pm}$ now take on a central role in that it will be shown that these conditions on the $\mathbf{l}^{\pm(1/2)}(\mathcal{F})$ will in each case reduce to a set of restrictions on the form of $\nu'^{\pm}, \nu''^{\pm}$.

Proposition XXV. The general forms (3.25, 3.27) for the $\mathbf{l}^{\pm(1/2)}(\mathcal{F})$ satisfy the twist map condition (3.2a), treated as an expansion in the deforming parameter *h*, provided $\nu'^{\pm}, \nu''^{\pm} \to 1$ in the limit $q \to 1$. To first order in *h* the expansion of the $\mathbf{l}^{\pm(1/2)}(\mathcal{F})$ takes the form

$$\mathbf{l}^{\pm(1/2)}(\mathcal{F}) \equiv \mathbf{l}^{\pm(1/2)}(\mathcal{f}_{(0)}) + \mathbf{l}^{\pm(1/2)}(\mathcal{f}_{(1)})h + \ldots \tag{3.28}$$

with

$$\mathbf{l}^{\pm(1/2)}(\mathcal{f}_{(0)}) = \mathbf{g}^{\pm(1/2)}(\mathcal{f}_{(0)}) = \mathbf{I};$$

$$\mathbf{l}^{\pm(1/2)}(\mathcal{f}_{(1)}) = \frac{1}{2}\begin{pmatrix} 0 & \pm\mathcal{S}_- \\ \mp\mathcal{S}_+ & 0 \end{pmatrix},$$

$$\mathbf{g}^{\pm(1/2)}(\mathcal{f}_{(1)}) \equiv \mathbf{X}_-\mathbf{l}^{\pm(1/2)}(\mathcal{f}_{(1)})\mathbf{X}_-^{-1} = \frac{1}{2}\begin{pmatrix} 0 & \pm 1 \\ \mp ab & 0 \end{pmatrix}.$$

Proof. An expansion of the twist map condition (3.2) has been carried out (Prop. VII). Constructing the semirepresentations of that expansion to zero and first order in *h* gives (3.28). Expansion of the forms (3.25, 3.27) for the $\mathbf{l}^{\pm(1/2)}(\mathcal{F})$ must now reduce to (3.28) to zero and first order in *h*. Defining

$$\nu'^{\pm} \equiv \nu'^{\pm}_{(0)} + \nu'^{\pm}_{(1)}h + \nu'^{\pm}_{(2)}h^2 + O(h^3),$$

$$\nu''^{\pm} \equiv \nu''^{\pm}_{(0)} + \nu''^{\pm}_{(1)}h + \nu''^{\pm}_{(2)}h^2 + O(h^3), \tag{3.29}$$

$$q^{u'}[u'']_q = u'' + u'u''h + \tfrac{1}{6}u''(3u'^2 + u''^2 - 1)h^2 + O(h^3),$$

with $u', u'' \in U(sl_2)$, the expansion of the q-bracket being calculated with *Mathematica*. Using these results and expanding $\mathbf{g}^{\pm(1/2)}(F')$, $\mathbf{g}^{\pm(1/2)}(F'')$ from (3.25, 3.27),

$$\mathbf{g}^{\pm(1/2)}(F') = c^{-1}\begin{pmatrix} (b \pm \frac{1}{2}abh + \ldots) & (1 \pm \frac{1}{2}ah + \ldots) \\ ab(1 \mp \frac{1}{2}bh + \ldots) & a(1 \mp \frac{1}{2}bh + \ldots) \end{pmatrix},$$

$$\mathbf{g}^{\pm(1/2)}(F'') = c^{-1}\begin{pmatrix} (a \mp \frac{1}{2}abh + \ldots) & -(1 \mp \frac{1}{2}bh + \ldots) \\ -ab(1 \pm \frac{1}{2}ah + \ldots) & b(1 \pm \frac{1}{2}ah + \ldots) \end{pmatrix};$$

so that

$$\mathbf{g}^{\pm(1/2)}(f'_{(0)}) = c^{-1}\begin{pmatrix} b & 1 \\ ab & a \end{pmatrix}, \quad \mathbf{g}^{\pm(1/2)}(f'_{(1)}) = c^{-1}\frac{1}{2}\begin{pmatrix} \pm b & \pm 1 \\ \mp b^2 & \mp b \end{pmatrix}a,$$

$$\mathbf{g}^{\pm(1/2)}(f''_{(0)}) = c^{-1}\begin{pmatrix} a & -1 \\ -ab & b \end{pmatrix}, \quad \mathbf{g}^{\pm(1/2)}(f''_{(1)}) = c^{-1}\frac{1}{2}\begin{pmatrix} \mp a & \pm 1 \\ \mp a^2 & \pm a \end{pmatrix}b. \tag{3.30}$$

From (3.25, 3.28), $\mathbf{g}^{\pm(1/2)}(F)$ can be treated as both an expansion in h and a functional of the variables $\nu'^{\pm}, \nu''^{\pm}$,

$$\mathbf{g}^{\pm(1/2)}(F) = \mathbf{g}^{\pm(1/2)}(f_{(0)}) + \mathbf{g}^{\pm(1/2)}(f_{(1)})h + \ldots,$$

$$\mathbf{g}^{\pm(1/2)}(F) = \nu'^{\pm}\mathbf{g}^{\pm(1/2)}(F') + \nu''^{\pm}\mathbf{g}^{\pm(1/2)}(F'').$$

Thus

$$\mathbf{g}^{\pm(1/2)}(F) = (\nu'^{\pm}_{(0)} + \nu'^{\pm}_{(1)}h + \ldots)\{\mathbf{g}^{\pm(1/2)}(f'_{(0)}) + \mathbf{g}^{\pm(1/2)}(f'_{(1)})h + \ldots\}$$

$$+(\nu''^{\pm}_{(0)} + \nu''^{\pm}_{(1)}h + \ldots)\{\mathbf{g}^{\pm(1/2)}(f''_{(0)}) + \mathbf{g}^{\pm(1/2)}(f''_{(1)})h + \ldots\},$$

after substituting the expansions (3.29) of $\nu'^{\pm}, \nu''^{\pm}$ and rearranging. Collecting terms in zero and first order in h and substituting from (3.30),

$$\mathbf{g}^{\pm(1/2)}(f_{(0)}) = \nu'^{\pm}_{(0)}\mathbf{g}^{\pm(1/2)}(f'_{(0)}) + \nu''^{\pm}_{(0)}\mathbf{g}^{\pm(1/2)}(f''_{(0)}),$$

$$\mathbf{g}^{\pm(1/2)}(f_{(1)}) = \nu'^{\pm}_{(0)}\mathbf{g}^{\pm(1/2)}(f'_{(1)}) + \nu'^{\pm}_{(1)}\mathbf{g}^{\pm(1/2)}(f'_{(0)}) + \nu''^{\pm}_{(0)}\mathbf{g}^{\pm(1/2)}(f''_{(1)}) + \nu''^{\pm}_{(1)}\mathbf{g}^{\pm(1/2)}(f''_{(0)})$$

so that

$$\mathbf{g}^{\pm(1/2)}(f_{(0)}) = c^{-1}\begin{pmatrix} (\nu'^{\pm}_{(0)}b + \nu''^{\pm}_{(0)}a) & (\nu'^{\pm}_{(0)} - \nu''^{\pm}_{(0)}) \\ ab(\nu'^{\pm}_{(0)} - \nu''^{\pm}_{(0)}) & (\nu'^{\pm}_{(0)}a + \nu''^{\pm}_{(0)}b) \end{pmatrix},$$

which comparing with (3.28b) immediately gives $\nu'^{\pm}_{(0)} = \nu''^{\pm}_{(0)} = 1$ as required, and then back-substituting $\nu'^{\pm}_{(0)} = \nu''^{\pm}_{(0)} = 1$,

$$\mathbf{g}^{\pm(1/2)}(f_{(1)}) = \frac{1}{2}\begin{pmatrix} 0 & \pm 1 \\ \mp ab & 0 \end{pmatrix} + c^{-1}\begin{pmatrix} (\nu'^{\pm}_{(1)}b - \nu''^{\pm}_{(1)}a) & (\nu'^{\pm}_{(1)} - \nu''^{\pm}_{(1)}) \\ ab(\nu'^{\pm}_{(1)} - \nu''^{\pm}_{(1)}) & a(\nu'^{\pm}_{(1)}a - \nu''^{\pm}_{(1)}b) \end{pmatrix}$$

which comparing with (3.28c) immediately gives $\nu'^{\pm}_{(1)} = \nu''^{\pm}_{(1)} = 0$, as required. ●

Proposition XXVI. The general forms (3.25, 3.27) for the semirepresentations $\mathbf{l}^{\pm(1/2)}(F)$ are invariant with respect to post-multiplication by the (symmetric) canonical 2-tensor $\mathbf{l}^{\pm(1/2)}(t)$. Post-multiplication by $\mathbf{l}^{\pm(1/2)}(t)$ is equivalent to the substitution $\nu'^{\pm}, \nu''^{\pm} \to (c-1)\nu'^{\pm}, -(c+1)\nu''^{\pm}$, $\pm(c-1)$ being the eigenvalues of $\mathbf{l}^{\pm(1/2)}(t)$, so that

$$\mathbf{l}^{\pm(1/2)}(F)\mathbf{l}^{\pm(1/2)}(t) = (c-1)\nu'^{\pm}\mathbf{l}^{\pm(1/2)}(F') - (c+1)\nu''^{\pm}\mathbf{l}^{\pm(1/2)}(F''). \qquad (3.31)$$

Proof. The proof comes out of evaluating $\mathbf{l}^{\pm(1/2)}(F')\mathbf{l}^{\pm(1/2)}(t)$ and $\mathbf{l}^{\pm(1/2)}(F'')\mathbf{l}^{\pm(1/2)}(t)$ individually. Transforming with $\mathbf{X}_-$,

$$\mathbf{X\_l}^{\pm(1/2)}(F')\mathbf{l}^{\pm(1/2)}(t)\mathbf{X}_-^{-1} = \mathbf{X\_l}^{\pm(1/2)}(F')\mathbf{X}_-^{-1}\mathbf{X\_l}^{\pm(1/2)}(t)\mathbf{X}_-^{-1} = \nu'^{\pm}\mathbf{g}^{\pm(1/2)}(F')\mathbf{g}^{\pm(1/2)}(t),$$

$$\mathbf{X\_l}^{\pm(1/2)}(F'')\mathbf{l}^{\pm(1/2)}(t)\mathbf{X}_-^{-1} = \mathbf{X\_l}^{\pm(1/2)}(F'')\mathbf{X}_-^{-1}\mathbf{X\_l}^{\pm(1/2)}(t)\mathbf{X}_-^{-1} = \nu''^{\pm}\mathbf{g}^{\pm(1/2)}(F'')\mathbf{g}^{\pm(1/2)}(t).$$

Substituting (3.11b, 3.25 and 3.27), and using $d-1+2a = 2b-d-1 = c-1$, $d-1-2b = -(d+1)-2a = -(c+1)$,

$$\nu'^{\pm}\mathbf{g}^{\pm(1/2)}(F')\mathbf{g}^{\pm(1/2)}(t) = c^{-1}q^{\pm a/2}\begin{pmatrix} [b]_q & b^{-1}[b]_q \\ abq^{\mp c/2} & bq^{\mp c/2} \end{pmatrix}\begin{pmatrix} d-1 & 2 \\ 2ab & -(d+1) \end{pmatrix}$$

$$= c^{-1}q^{\pm a/2}\begin{pmatrix} (d-1)[b]_q + 2a[b]_q & 2[b]_q - (d+1)b^{-1}[b]_q \\ (d-1)q^{\mp c/2}ab + 2q^{\mp c/2}a^2 b & 2q^{\mp c/2}ab - q^{\mp c/2}a(d+1) \end{pmatrix}$$

$$= c^{-1}q^{\pm a/2}(c-1)\begin{pmatrix} [b]_q & b^{-1}[b]_q \\ q^{\mp c/2}ab & q^{\mp c/2}a \end{pmatrix} = (c-1)\nu'^{\pm}\mathbf{g}^{\pm(1/2)}(F');$$

$$\nu''^{\pm}\mathbf{g}^{\pm(1/2)}(F'')\mathbf{g}^{\pm(1/2)}(t) = c^{-1}q^{\mp b/2}\begin{pmatrix} [a]_q & -a^{-1}[a]_q \\ -q^{\pm c/2}ab & q^{\pm c/2}b \end{pmatrix}\begin{pmatrix} d-1 & 2 \\ 2ab & -(d+1) \end{pmatrix}$$

$$= c^{-1}q^{\mp b/2}\begin{pmatrix} (d-1)[a]_q - 2b[a]_q & 2[a]_q + (d+1)a^{-1}[a]_q \\ -(d-1)q^{\pm c/2}ab + 2q^{\pm c/2}ab^2 & -2q^{\pm c/2}ab - q^{\pm c/2}b(d+1) \end{pmatrix}$$

$$= -c^{-1}q^{\mp b/2}(c+1)\begin{pmatrix} [a]_q & -a^{-1}[a]_q \\ -q^{\pm c/2}ab & q^{\pm c/2}b \end{pmatrix}$$

$$= -(c+1)\nu''^{\pm}\mathbf{g}^{\pm(1/2)}(F'')\mathbf{g}^{\pm(1/2)}(t).$$

The required result immediately follows. ●

Proposition XXVII. The general forms (3.25, 3.27) for the semirepresentations $\mathbf{l}^{\pm(1/2)}(F)$ are invariant with respect to pre-multiplication by the QIKF, $\mathbf{l}^{\pm(1/2)}(\overline{Q})$. Pre-multiplication by $\mathbf{l}^{\pm(1/2)}(\overline{Q})$ is equivalent to the substitution $\nu'^{\pm}, \nu''^{\pm} \to q^{(c-1)}\nu'^{\pm}, q^{-(c+1)}\nu''^{\pm}$, $q^{\pm(c-1)}$ being the eigenvalues of $\mathbf{l}^{\pm(1/2)}(\overline{Q})$, so that

$$\mathbf{l}^{\pm(1/2)}(\overline{Q})\mathbf{l}^{\pm(1/2)}(F) = q^{(c-1)}\nu'^{\pm}\mathbf{l}^{\pm(1/2)}(F') + q^{-(c+1)}\nu''^{\pm}\mathbf{l}^{\pm(1/2)}(F''). \quad (3.32)$$

Proof. The proof comes out of evaluating the much more complicated expressions $\mathbf{l}^{\pm(1/2)}(\overline{Q})\mathbf{l}^{\pm(1/2)}(F')$ and $\mathbf{l}^{\pm(1/2)}(\overline{Q})\mathbf{l}^{\pm(1/2)}(F'')$. Transforming (3.32) with $\mathbf{X}_-$,

$$\mathbf{X}_-\mathbf{l}^{\pm(1/2)}(\overline{Q})\mathbf{l}^{\pm(1/2)}(F')\mathbf{X}_-^{-1} = \mathbf{g}^{\pm(1/2)}(\overline{Q})\nu'^{\pm}\mathbf{g}^{\pm(1/2)}(F'),$$

$$\mathbf{X}_-\mathbf{l}^{\pm(1/2)}(\overline{Q})\mathbf{l}^{\pm(1/2)}(F'')\mathbf{X}_-^{-1} = \mathbf{g}^{\pm(1/2)}(\overline{Q})\nu''^{\pm}\mathbf{g}^{\pm(1/2)}(F'').$$

Substituting (3.15, 3.25, 3.27) and using $q^{\pm d} \pm q^b p[a]_q = q^{\mp d} + (p[a]_q \pm q^{\mp a})p[b]_q = q^{\pm c}$, $q^{\mp d} \pm q^{\pm a} p[b]_q = q^{\pm d} + (p[b]_q \pm q^{\mp b})p[a]_q = q^{\pm c}$,

$$\mathbf{g}^{-(1/2)}(\overline{Q})\nu'^{-}\mathbf{g}^{-(1/2)}(F') = q^{-1}\begin{pmatrix} q^d & pq^{d/2}\Phi \\ pq^{d/2}ab & q^{-d} + p^2[a]_q[b]_q \end{pmatrix}$$

$$\times \nu'^{-}c^{-1}q^{-a/2}\begin{pmatrix} [b]_q & b^{-1}[b]_q \\ abq^{c/2} & aq^{c/2} \end{pmatrix}$$

$$= \nu'^{-}c^{-1}q^{-a/2}q^{-1}\begin{pmatrix} (q^d + q^b p[a]_q)[b]_q & (q^d + q^b p[a]_q)b^{-1}[b]_q \\ (q^{-d} + (q^{-a} + p[a]_q)p[b]_q)abq^{c/2} & (q^{-d} + (q^{-a} + p[a]_q)p[b]_q)aq^{c/2} \end{pmatrix}$$

$$= \nu'^{-}c^{-1}q^{-a/2}q^{(c-1)}\begin{pmatrix} [b]_q & b^{-1}[b]_q \\ abq^{c/2} & aq^{c/2} \end{pmatrix} = q^{(c-1)}\nu'^{-}\mathbf{g}^{-(1/2)}(F'),$$

$$\mathbf{g}^{+(1/2)}(\overline{Q})\nu'^{+}\mathbf{g}^{+(1/2)}(F') = q^{-1}\begin{pmatrix} q^d + p^2[a]_q[b]_q & pq^{-d/2}\Phi \\ pq^{-d/2}ab & q^{-d} \end{pmatrix}$$

$$\times \nu'^{+}c^{-1}q^{a/2}\begin{pmatrix} [b]_q & b^{-1}[b]_q \\ abq^{-c/2} & aq^{-c/2} \end{pmatrix}$$

$$= \nu'^{+}c^{-1}q^{a/2}q^{-1}\begin{pmatrix} (q^d + (p[b]_q + q^{-b})p[a]_q)[b]_q & (q^d + (p[b]_q + q^{-b})p[a]_q)b^{-1}[b]_q \\ (q^{-d} + q^a p[b]_q)abq^{-c/2} & (q^{-d} + q^a p[b]_q)aq^{-c/2} \end{pmatrix}$$

$$= \nu'^{+}c^{-1}q^{a/2}q^{(c-1)}\begin{pmatrix} [b]_q & b^{-1}[b]_q \\ abq^{-c/2} & aq^{-c/2} \end{pmatrix} = q^{(c-1)}\nu'^{+}\mathbf{g}^{+(1/2)}(F');$$

$$\mathbf{g}^{-(1/2)}(\overline{Q})\nu''^{-}\mathbf{g}^{-(1/2)}(F'') = q^{-1}\begin{pmatrix} q^{d} & pq^{d/2}\Phi \\ pq^{d/2}ab & q^{-d}+p^{2}[a]_q[b]_q \end{pmatrix}$$

$$\times \nu''^{-}c^{-1}q^{-b/2}\begin{pmatrix} [a]_q & -a^{-1}[a]_q \\ -abq^{-c/2} & bq^{-c/2} \end{pmatrix}$$

$$= \nu''^{-}c^{-1}q^{-b/2}q^{-1}\begin{pmatrix} (q^{d}-q^{-a}p[b]_q)[a]_q & (q^{d}-q^{-a}p[b]_q)(-a^{-1}[a]_q) \\ -(q^{-d}+(p[b]_q-q^{b})p[a]_q)abq^{-c/2} & (q^{-d}+(p[b]_q-q^{b})p[a]_q)bq^{-c/2} \end{pmatrix}$$

$$= \nu''^{-}c^{-1}q^{-b/2}q^{-(c+1)}\begin{pmatrix} [a]_q & -a^{-1}[a]_q \\ -abq^{-c/2} & bq^{-c/2} \end{pmatrix} = q^{-(c+1)}\nu''^{-}\mathbf{g}^{-(1/2)}(F''),$$

$$\mathbf{g}^{+(1/2)}(\overline{Q})\nu''^{+}\mathbf{g}^{+(1/2)}(F'') = q^{-1}\begin{pmatrix} q^{d}+p^{2}[a]_q[b]_q & pq^{-d/2}\Phi \\ pq^{-d/2}ab & q^{-d} \end{pmatrix}$$

$$\times \nu''^{+}c^{-1}q^{b/2}\begin{pmatrix} [a]_q & -a^{-1}[a]_q \\ -abq^{c/2} & bq^{c/2} \end{pmatrix}$$

$$= \nu''^{+}c^{-1}q^{b/2}q^{-1}\begin{pmatrix} (q^{d}+(p[a]_q-q^{a})p[b]_q)[a]_q & (q^{d}+(p[a]_q-q^{a})p[b]_q)(-a^{-1}[a]_q) \\ (q^{-d}-q^{-b}p[a]_q)(-abq^{c/2}) & (q^{-d}-q^{-b}p[a]_q)bq^{c/2} \end{pmatrix}$$

$$= \nu''^{+}c^{-1}q^{b/2}q^{-(c+1)}\begin{pmatrix} [a]_q & -a^{-1}[a]_q \\ -abq^{c/2} & bq^{c/2} \end{pmatrix} = q^{-(c+1)}\nu''^{+}\mathbf{g}^{+(1/2)}(F'').$$

The required result immediately follows. ●

Proposition XXVIII. There exists a set of solutions such that $\nu'^{\pm} = \nu''^{\pm}$ for which the components of the semirepresentations $\mathbf{l}^{\pm(1/2)}(F)$ are elements of $U(sl_2)$. However, as $c \notin U(sl_2)$, the products $\mathbf{l}^{\pm(1/2)}(\overline{Q})\mathbf{l}^{\pm(1/2)}(F)$ and $\mathbf{l}^{\pm(1/2)}(F)\mathbf{l}^{\pm(1/2)}(t)$ define semirepresentations with components that are no longer elements of $U(sl_2)$.

Proof. The proof consists in setting $\nu'^{-} = \nu''^{-}$ in (3.25) and hinges on the observation that as $c^n \in U(sl_2), c \equiv a + b$ for n even,

$$q^{mc} + q^{-mc} = \sum_n \frac{1}{n!} h^n \{(mc)^n + (-mc)^n\} \in U(sl_2),$$

$$c^{-1}(q^{mc} - q^{-mc}) = \sum_n \frac{1}{n!} h^n c^{-1} \{(mc)^n - (-mc)^n\} \in U(sl_2)$$

for any rational number m, terms involving odd/even powers in c vanishing. Evaluating individual components of $\mathbf{g}^{-(1/2)}(F)$, and recalling that $d \equiv b - a, ab = c^2 - d^2 \in U(sl_2)$,

$$\mathbf{g}^{-(1/2)}(F)_{11} = c^{-1}\{q^{-a/2}[b]_q + q^{b/2}[a]_q\} = p^{-1}c^{-1}\{q^{-a/2}(q^b - q^{-b}) + q^{b/2}(q^a - q^{-a})\}$$

$$= q^{d/4} p^{-1}\{q^{d/2}c^{-1}(q^{c/4} - q^{-c/4}) + q^{-d/2}c^{-1}(q^{3c/4} - q^{-3c/4})\} \in U(sl_2),$$

$$\mathbf{g}^{-(1/2)}(F)_{22} = c^{-1}\{aq^{b/2} + bq^{-a/2}\} = c^{-1}\frac{1}{2}\{(c-d)q^{c/4}q^{d/4} + (c+d)q^{-c/4}q^{d/4}\}$$

$$= q^{d/4}\frac{1}{2}\{(q^{c/4} + q^{-c/4}) + dc^{-1}(q^{c/4} - q^{-c/4})\} \in U(sl_2),$$

$$\mathbf{g}^{-(1/2)}(F)_{21} = abc^{-1}\{q^{-a/2}q^{c/2} - q^{b/2}q^{-c/2}\} = abc^{-1}\{q^{-c/4}q^{d/4}q^{c/2} - q^{c/4}q^{d/4}q^{-c/2}\}$$

$$= q^{d/4}abc^{-1}\{q^{c/4} - q^{-c/4}\} \in U(sl_2),$$

$$\mathbf{g}^{-(1/2)}(F)_{12} = c^{-1}\{q^{-a/2}b^{-1}[b]_q - q^{b/2}a^{-1}[a]_q\}$$

$$= p^{-1}abc^{-1}\{q^{-a/2}a(q^b - q^{-b}) - q^{b/2}b(q^a - q^{-a})\}$$

$$= q^{d/4}p^{-1}abc^{-1}\frac{1}{2}\{(c-d)(q^{c/4}q^{d/2} - q^{-3c/4}q^{-d/2})$$

$$-(c+d)(q^{3c/4}q^{-d/2} - q^{-c/4}q^{d/2})\}$$

$$= q^{d/4}abp^{-1}\frac{1}{2}\{q^{d/2}(q^{c/4} + q^{-c/4}) - q^{-d/2}(q^{3c/4} + q^{-3c/4})$$

$$-dc^{-1}q^{d/2}(q^{c/4} - q^{-c/4}) - dc^{-1}q^{-d/2}(q^{3c/4} - q^{-3c/4})\} \in U(sl_2),$$

which is the required result. The proof that the components of $\mathbf{g}^{+(1/2)}(F)$ are elements of $U(sl_2)$ follows with the substitution $q \to q^{-1}$. That the products $\mathbf{l}^{\pm(1/2)}(\overline{Q})\mathbf{l}^{\pm(1/2)}(F)$ and $\mathbf{l}^{\pm(1/2)}(F)\mathbf{l}^{\pm(1/2)}(t)$ define semirepresentations with components that are no longer elements of $U(sl_2)$ follows directly from propositions VIII and IX. ●

III.4. *Conditions on* $\mathbf{l}^{\pm(1/2)}(F)$ *as coproduct deforming maps.*

This is a short section in which the conditions on the twist elements $\mathbf{l}^{\pm(1/2)}(F)$ as semirepresentations of the deforming map for the coproduct structure of $U(sl_2)$ are determined, which again will reduce to restrictions on the form of the arbitrary variables $\nu'^{\pm}, \nu''^{\pm}$. Note that as $u \in U(sl_2)$, $\Delta u_{21} = \Delta u$ and $\overline{\Delta} u \to \overline{\Delta} u_{21}$ with the substitution $q \to q^{-1}$,

$$\mathbf{l}^{-(1/2)}(\overline{\Delta} u)\mathbf{l}^{-(1/2)}(F) = \mathbf{l}^{-(1/2)}(F)\mathbf{l}^{-(1/2)}(\Delta u)$$

implies

$$\mathbf{l}^{+(1/2)}(\overline{\Delta} u)\mathbf{l}^{+(1/2)}(F) = \mathbf{l}^{+(1/2)}(F)\mathbf{l}^{+(1/2)}(\Delta u) \tag{3.33}$$

and the results derived for the negative, will also hold for the positive, semirepresentations $\mathbf{l}^{\pm(1/2)}(F)$.

Proposition XXIX. The semirepresentation $\mathbf{l}^{-(1/2)}(F)$ of Drinfel'd's twist element must commute with the coproduct $\mathbf{l}^{-(1/2)}(\Delta S_z)$, or

$$\mathbf{l}^{-(1/2)}(\Delta S_z)\mathbf{l}^{-(1/2)}(F) = \mathbf{l}^{-(1/2)}(F)\mathbf{l}^{-(1/2)}(\Delta S_z) \tag{3.34}$$

Proof. The ν'^{-}, ν''^{-} are arbitrary, so after transformation with $\mathbf{X}_{-}$ (3.34) must hold separately for the $\mathbf{g}^{-(1/2)}(F')$ and $\mathbf{g}^{-(1/2)}(F'')$ of (3.8),

$$\mathbf{g}^{-(1/2)}(\Delta S_z)\mathbf{g}^{-(1/2)}(F) = \mathbf{g}^{-(1/2)}(F)\mathbf{g}^{-(1/2)}(\Delta S_z),$$

so that

$$\mathbf{g}^{-(1/2)}(\Delta S_z)\mathbf{g}^{-(1/2)}(F') = \mathbf{g}^{-(1/2)}(F')\mathbf{g}^{-(1/2)}(\Delta S_z),$$

$$\mathbf{g}^{-(1/2)}(\Delta S_z)\mathbf{g}^{-(1/2)}(F'') = \mathbf{g}^{-(1/2)}(F'')\mathbf{g}^{-(1/2)}(\Delta S_z).$$

Substituting,

$$(S_z + {}^1\!/_2\,)\mathbf{I}\mathbf{g}^{-(1/2)}(F') = \mathbf{g}^{-(1/2)}(F')(S_z + {}^1\!/_2\,)\mathbf{I},$$

$$(S_z + {}^1\!/_2\,)\mathbf{I}\mathbf{g}^{-(1/2)}(F'') = \mathbf{g}^{-(1/2)}(F'')(S_z + {}^1\!/_2\,)\mathbf{I}$$

which is an identity as the components of $\mathbf{g}^{-(1/2)}(F')$, $\mathbf{g}^{-(1/2)}(F'')$ are independent of $S_{\pm}$. ●

Proposition XXX. The form (3.25) for $\mathbf{l}^{-(1/2)}(F)$ provides a deforming map $\mathbf{l}^{-(1/2)}(\Delta S_+) \to \mathbf{l}^{-(1/2)}(\overline{\Delta} S_+)$ if the arbitrary variables ν'^-, ν''^- are central,

$$\mathbf{l}^{-(1/2)}(\overline{\Delta} S_+)\mathbf{l}^{-(1/2)}(F) = \mathbf{l}^{-(1/2)}(F)\mathbf{l}^{-(1/2)}(\Delta S_+) \tag{3.35}$$

Proof. Again, the equivalent form for (3.35) must hold independently for $\mathbf{g}^{-(1/2)}(F')$, $\mathbf{g}^{-(1/2)}(F'')$ as ν'^-, ν''^- are arbitrary. Transforming with $\mathbf{X}_-$ and substituting from (3.8, 3.10),

$$\mathbf{g}^{-(1/2)}(\overline{\Delta} S_+)\mathbf{g}^{-(1/2)}(F) = \mathbf{g}^{-(1/2)}(F)\mathbf{g}^{-(1/2)}(\Delta S_+)$$

so that

$$\mathbf{g}^{-(1/2)}(\overline{\Delta} S_+)\mathbf{g}^{-(1/2)}(F') = \mathbf{g}^{-(1/2)}(F')\mathbf{g}^{-(1/2)}(\Delta S_+),$$

$$\mathbf{g}^{-(1/2)}(\overline{\Delta} S_+)\mathbf{g}^{-(1/2)}(F'') = \mathbf{g}^{-(1/2)}(F'')\mathbf{g}^{-(1/2)}(\Delta S_+).$$

Substituting,

$$\mathbf{g}^{-(1/2)}(\overline{\Delta} S_+)\nu'^-\mathbf{g}^{-(1/2)}(F') = q^{-1/2}S_+(ab)^{-1}\begin{pmatrix} ab & qq^{d/2} \\ 0 & q(a-1)(b+1) \end{pmatrix}$$

$$\times \nu'^{-} c^{-1} q^{-a/2} \begin{pmatrix} [b]_q & b^{-1}[b]_q \\ abq^{c/2} & aq^{c/2} \end{pmatrix}$$

$$= S_+ \nu'^{-} c^{-1} q^{1/2} q^{-a/2} \begin{pmatrix} q^{-1}[b]_q + q^b & q^{-1} b^{-1}[b]_q + b^{-1} q^b \\ (a-1)(b+1)q^{c/2} & (a-1)(b+1)b^{-1} q^{c/2} \end{pmatrix}$$

$$= S_+ \nu'^{-} c^{-1} q^{1/2} q^{-a/2} \begin{pmatrix} [b+1]_q & b^{-1}[b+1]_q \\ (a-1)(b+1)q^{c/2} & (a-1)(b+1)b^{-1} q^{c/2} \end{pmatrix},$$

$$\nu'^{-} \mathbf{g}^{-(1/2)}(F') \mathbf{g}^{-(1/2)}(\Delta S_+) = \nu'^{-} c^{-1} q^{-a/2} \begin{pmatrix} [b]_q & b^{-1}[b]_q \\ abq^{c/2} & aq^{c/2} \end{pmatrix} S_+ (ab)^{-1} \begin{pmatrix} ab & 1 \\ 0 & (a-1)(b+1) \end{pmatrix}$$

$$= \nu'^{-} S_+ c^{-1} q^{1/2} q^{-a/2} (ab)^{-1} \begin{pmatrix} ab[b+1]_q & [b+1]_q + (a-1)[b+1]_q \\ ab(a-1)(b+1)q^{c/2} & (a-1)(b+1)q^{c/2} + (a-1)^2 (b+1) q^{c/2} \end{pmatrix}$$

$$= \nu'^{-} S_+ c^{-1} q^{1/2} q^{-a/2} \begin{pmatrix} [b+1]_q & b^{-1}[b+1]_q \\ (a-1)(b+1)q^{c/2} & b^{-1}(a-1)(b+1) q^{c/2} \end{pmatrix};$$

$$\mathbf{g}^{-(1/2)}(\overline{\Delta} S_+) \nu''^{-} \mathbf{g}^{-(1/2)}(F'') = q^{-1/2} S_+ (ab)^{-1} \begin{pmatrix} ab & qq^{d/2} \\ 0 & q(a-1)(b+1) \end{pmatrix}$$

$$\times \nu''^{-} c^{-1} q^{b/2} \begin{pmatrix} [a]_q & -a^{-1}[a]_q \\ -abq^{-c/2} & bq^{-c/2} \end{pmatrix}$$

$$= S_+ \nu''^{-} c^{-1} q^{-1/2} q^{b/2} \begin{pmatrix} [a]_q - qq^{-a} & -a^{-1}[a]_q + a^{-1} qq^{-a} \\ -(a-1)(b+1)qq^{-c/2} & a^{-1}(a-1)(b+1)qq^{-c/2} \end{pmatrix}$$

$$= S_+ \nu''^{-} c^{-1} q^{1/2} q^{b/2} \begin{pmatrix} [a-1]_q & -a^{-1}[a-1]_q \\ -(a-1)(b+1)q^{-c/2} & a^{-1}(a-1)(b+1)q^{\ c/2} \end{pmatrix},$$

$$\nu''^{-} \mathbf{g}^{-(1/2)}(F'') \mathbf{g}^{-(1/2)}(\Delta S_+) = \nu''^{-} c^{-1} q^{b/2} \begin{pmatrix} [a]_q & -a^{-1}[a]_q \\ -abq^{-c/2} & bq^{-c/2} \end{pmatrix}$$

$$\times S_+ (ab)^{-1} \begin{pmatrix} ab & 1 \\ 0 & (a-1)(b+1) \end{pmatrix}$$

$$= \nu''^{-} S_{+} c^{-1} q^{1/2} q^{b/2} (ab)^{-1} \begin{pmatrix} [a-1]_q & -(a-1)^{-1}[a-1]_q \\ -(a-1)(b+1)q^{-c/2} & (b+1)q^{-c/2} \end{pmatrix} \begin{pmatrix} ab & 1 \\ 0 & (a-1)(b+1) \end{pmatrix}$$

$$= \nu''^{-} S_{+} c^{-1} q^{1/2} q^{b/2} \begin{pmatrix} [a-1]_q & -a^{-1}[a-1]_q \\ -(a-1)(b+1)q^{-c/2} & a^{-1}(a-1)(b+1)q^{-c/2} \end{pmatrix}.$$

Back-substitution and mass cancellation gives $S_{+}\nu'^{-} = \nu'^{-}S_{+}$, $S_{+}\nu''^{-} = \nu''^{-}S_{+}$, which is the required result. ●

Proposition XXXI. That the semirepresentation $\mathbf{l}^{-(1/2)}(F)$ provides a deforming map $\mathbf{l}^{-(1/2)}(\widetilde{F}){:}\mathbf{l}^{-(1/2)}(\Delta\overline{S}_{-}) \to \mathbf{l}^{-(1/2)}(\overline{\Delta}\overline{S}_{-})$ follows immediately from the definition of $\mathbf{l}^{-(1/2)}(F)$ as a map $\mathbf{l}^{-(1/2)}(\widetilde{F}){:}\mathbf{l}^{-(1/2)}(\Delta S_{+}) \to \mathbf{l}^{-(1/2)}(\overline{\Delta}S_{+})$. This has been shown (Prop. VI) to hold true provided the QIKF satisfies a very complicated expression, which in semirepresentation form becomes[4]

$$\mathbf{l}^{-(1/2)}(\overline{\Delta}\overline{Y}) = q^{2}\mathbf{l}^{-(1/2)}(\overline{Q}) + q^{-2}\mathbf{l}^{-(1/2)}(\overline{Q})^{-1} \tag{3.36a}$$

with

$$\overline{\Delta}\overline{Y} = q^{-1}\overline{Y} \otimes q^{d} + qq^{-d} \otimes \overline{Y} - [2]_q q^{d} \otimes q^{-d}$$
$$+ p^{2}(q^{-d/2} \otimes q^{d/2})(q^{-1}\overline{S}_{-} \otimes S_{+} + qS_{+} \otimes \overline{S}_{-})$$
$$\overline{Y} \equiv q^{c} + q^{-c}.$$

Proof. The proof consists simply in evaluating the equivalent form of (3.36a). Transforming with $\mathbf{X}_{-}$,

$$\mathbf{g}^{-(1/2)}(\overline{\Delta}\overline{Y}) = q^{2}\mathbf{g}^{-(1/2)}(\overline{Q}) + q^{-2}\mathbf{g}^{-(1/2)}(\overline{Q})^{-1}. \tag{3.36b}$$

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[4]The key step in deriving (3.36a) is the demonstration that $\mathbf{l}^{\pm(1/2)}(t) = -\mathbf{I} \pm \sqrt{(C+1)}\mathbf{I}$. See Cornwell, unpublished notes.

The QIKF $\mathbf{g}^{-(1/2)}(\overline{Q})$ is obtained from (3.15a). The inverse form $\mathbf{g}^{-(1/2)}(\overline{Q})^{-1}$ is constructed with the result, demonstrated in the first section, that $\overline{R},\overline{R}_{21} \to \overline{R}^{-1},\overline{R}_{21}^{-1}$ with the substitution $q \to q^{-1}$. $\mathbf{g}^{-(1/2)}(\overline{R})^{-1}$ and $\mathbf{g}^{-(1/2)}(\overline{R}_{21})^{-1}$ are thus obtained by substituting $q \to q^{-1}$ in (3.12b, 3.13b),

$$\mathbf{g}^{-(1/2)}(\overline{Q})^{-1} = \mathbf{g}^{-(1/2)}(\overline{R})^{-1}\mathbf{g}^{-(1/2)}(\overline{R}_{21})^{-1} = q\begin{pmatrix} q^{-d/2} & -p\Phi \\ 0 & q^{d/2} \end{pmatrix}\begin{pmatrix} q^{-d/2} & 0 \\ -pab & q^{d/2} \end{pmatrix}$$

$$= q\begin{pmatrix} q^{-d} + p^2[a]_q[b]_q & -p\Phi q^{d/2} \\ -pabq^{d/2} & q^d \end{pmatrix}.$$

Then

$$q^2\mathbf{g}^{-(1/2)}(\overline{Q}) + q^{-2}\mathbf{g}^{-(1/2)}(\overline{Q})^{-1} = q\begin{pmatrix} q^d & p\Phi q^{d/2} \\ pabq^{d/2} & q^{-d} + p^2[a]_q[b]_q \end{pmatrix}$$

$$+q^{-1}\begin{pmatrix} q^{-d} + p^2[a]_q[b]_q & -p\Phi q^{d/2} \\ -pabq^{d/2} & q^d \end{pmatrix}$$

$$= \begin{pmatrix} pq^d + q^{-1}(q^c + q^{-c}) & p^2\Phi q^{d/2} \\ p^2abq^{d/2} & -pq^d + q(q^c + q^{-c}) \end{pmatrix}.$$

The right hand side of (3.36b) is now evaluated using $\Gamma^{(1/2)}(\overline{Y}) = \Gamma^{(1/2)}(q^c) + \Gamma^{(1/2)}(q^{-c}) = (q^{1/2} + q^{-1/2})\mathbf{I}$. Thus

$$\mathbf{l}^{-(1/2)}(\overline{\Delta}\overline{Y}) = q^{-1}\Gamma^{(1/2)}(\overline{Y})q^d + q\Gamma^{(1/2)}(q^{-d})\overline{Y} - [2]_q\Gamma^{(1/2)}(q^d)q^{-d}$$

$$+p^2\Gamma^{(1/2)}(q^{-d/2})q^{d/2}\{\Gamma^{(1/2)}(q^{-1}\overline{S}_-)S_+ + q\Gamma^{(1/2)}(S_+)\overline{S}_-)\}$$

$$= \begin{pmatrix} q^{-1}(q^c + q^{-c}) + pq^d & p^2q^{d/2}\overline{S}_- \\ q^{-1}p^2q^{d/2}S_+ & q(q^c + q^{-c}) - q^{-2}pq^d \end{pmatrix},$$

$$\mathbf{g}^{-(1/2)}(\overline{\Delta}\overline{Y}) \equiv \mathbf{X}_-\mathbf{l}^{-(1/2)}(\overline{\Delta}\overline{Y})\mathbf{X}_-^{-1} = \begin{pmatrix} q^{-1}(q^c + q^{-c}) + pq^d & p^2q^{d/2}\Phi \\ p^2abq^{d/2} & q(q^c + q^{-c}) - pq^d \end{pmatrix}.$$

Which proves that both sides of the equation are equal. ●

III.5 *Partial construction of Drinfel'd's twist element F from* $\mathbf{l}^{\pm(1/2)}(F)$.

A construction of the first two terms of the form (3.1) of Drinfel'd's twist map F is now effected. The starting point will be the assumption that the individual terms of F are tensor products of the components of the semirepresentations $\mathbf{l}^{\pm(1/2)}(F)$. As the variables $\nu'^{\pm}, \nu''^{\pm}$ have been shown to be central, the general forms (3.25, 3.27) for the $\mathbf{l}^{\pm(1/2)}(F)$ reduce to

$$\mathbf{l}^{\pm(1/2)}(F) = \nu'^{\pm}\mathbf{l}^{\pm(1/2)}(F') + \nu''^{\pm}\mathbf{l}^{\pm(1/2)}(F'') \tag{3.37}$$

$$\mathbf{l}^{\pm(1/2)}(F') = c^{-1}q^{\pm a/2}\begin{pmatrix} [b]_q & b^{-1}[b]_q S_- \\ q^{\pm 1/2}q^{\mp c/2}S_+ & (a+1)q^{\pm 1/2}q^{\mp c/2} \end{pmatrix},$$

$$\mathbf{l}^{\pm(1/2)}(F'') = c^{-1}q^{\mp b/2}\begin{pmatrix} [a]_q & -a^{-1}[a]_q S_- \\ -q^{\pm 1/2}q^{\pm c/2}S_+ & (b-1)q^{\pm 1/2}q^{\pm c/2} \end{pmatrix}.$$

Proposition XXXII. The positive semirepresentation $\mathbf{l}^{+(1/2)}(F)$ is obtained from $\mathbf{l}^{-(1/2)}(F)$ with the substitution $q \to q^{-1}$. The $\mathbf{l}^{\pm(1/2)}(F)$ are then semirepresentations of an element $F \in U(sl_2) \otimes U(sl_2)$ if and only if the variables $\nu'^{\pm}, \nu''^{\pm}$ are invariant with respect to the $q \to q^{-1}$ substitution.[5]

Proof. That $\mathbf{l}^{-(1/2)}(F) \to \mathbf{l}^{+(1/2)}(F)$ with the substitution $q \to q^{-1}$ has been demonstrated. The condition that the $\mathbf{l}^{+(1/2)}(F)$ of (3.37) are semirepresentations of an element $F \in U(sl_2) \otimes U(sl_2)$ is equivalent to the condition (Prop. XII) that the direct product representations $\Gamma^{(1/2)}(\mathbf{l}^{\pm(1/2)}(F))$ for F satisfy

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[5] As $\mathbf{l}^{\pm(1/2)}(\overline{Q})\mathbf{l}^{\pm(1/2)}(F)$ is also a valid twist map, the product being equivalent to the substitution $\nu'^{\pm}, \nu''^{\pm} \to q^{c-1}\nu'^{\pm}, q^{-c-1}\nu''^{\pm}$ (Prop. IX), it follows that not all elements $\mathbf{l}^{\pm(1/2)}(F)$ satisfying the conditions for a twist map are semirepresentations of an element F.

$$\Gamma^{(1/2)}(\mathbf{l}^{\pm(1/2)}(F)_{jk})_{lm} = \Gamma^{(1/2)}(\mathbf{l}^{\mp(1/2)}(F)_{lm})_{jk} \tag{3.38}$$

so that

$$\begin{aligned}
\Gamma^{(1/2)}(\mathbf{l}^{\pm(1/2)}(F)_{11})_{11} &= \Gamma^{(1/2)}(\mathbf{l}^{\mp(1/2)}(F)_{11})_{11}, \\
\Gamma^{(1/2)}(\mathbf{l}^{\pm(1/2)}(F)_{11})_{22} &= \Gamma^{(1/2)}(\mathbf{l}^{\mp(1/2)}(F)_{22})_{11}, \\
\Gamma^{(1/2)}(\mathbf{l}^{\pm(1/2)}(F)_{22})_{22} &= \Gamma^{(1/2)}(\mathbf{l}^{\mp(1/2)}(F)_{22})_{22}, \\
\Gamma^{(1/2)}(\mathbf{l}^{\pm(1/2)}(F)_{12})_{21} &= \Gamma^{(1/2)}(\mathbf{l}^{\mp(1/2)}(F)_{21})_{12}.
\end{aligned} \tag{3.39}$$

Substituting the components of (3.37),[6]

$$\begin{aligned}
\Gamma^{(1/2)}(\mathbf{l}^{-(1/2)}(F)_{11}) &= \nu'^{-}\Gamma^{(1/2)}(q^{-a/2}[b]_q) + \nu''^{-}\Gamma^{(1/2)}(q^{b/2}[a]_q) \\
&= \nu'^{-}\begin{pmatrix} [2]_q & 0 \\ 0 & q^{-1/2} \end{pmatrix} + \nu''^{-}\begin{pmatrix} 0 & 0 \\ 0 & q^{1/2} \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
\Gamma^{(1/2)}(\mathbf{l}^{-(1/2)}(F)_{22}) &= \nu'^{-}\Gamma^{(1/2)}(q^{-1/2}q^{b/2}(a+1)) + \nu''^{-}\Gamma^{(1/2)}(q^{-1/2}q^{-a/2}(b-1)) \\
&= \nu'^{-}\begin{pmatrix} q^{1/2} & 0 \\ 0 & 2 \end{pmatrix} + \nu''^{-}\begin{pmatrix} q^{-1/2} & 0 \\ 0 & 0 \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
\Gamma^{(1/2)}(\mathbf{l}^{-(1/2)}(F)_{12}) &= \nu'^{-}\Gamma^{(1/2)}(q^{-a/2}b^{-1}[b]_q S_-) - \nu''^{-}\Gamma^{(1/2)}(q^{b/2}a^{-1}[a]_q S_-) \\
&= \nu'^{-}\begin{pmatrix} 0 & 0 \\ q^{-1/2} & 0 \end{pmatrix} - \nu''^{-}\begin{pmatrix} 0 & 0 \\ q^{1/2} & 0 \end{pmatrix}, \\
\Gamma^{(1/2)}(\mathbf{l}^{-(1/2)}(F)_{21}) &= \nu'^{-}\Gamma^{(1/2)}(q^{-1/2}q^{b/2}S_+) - \nu''^{-}\Gamma^{(1/2)}(q^{-1/2}q^{-a/2}S_+) \\
&= \nu'^{-}\begin{pmatrix} 0 & q^{1/2} \\ 0 & 0 \end{pmatrix} - \nu''^{-}\begin{pmatrix} 0 & q^{-1/2} \\ 0 & 0 \end{pmatrix};
\end{aligned}$$

and

[6] As $\nu'^{\pm}, \nu''^{\pm}$ are functionals of S, the identification $\Gamma(\nu'^{\pm}), \Gamma(\nu''^{\pm}) \to \nu'^{\pm}\mathbf{I}, \nu''^{\pm}\mathbf{I}$ can be made here.

$$\Gamma^{(1/2)}(\mathbf{l}^{+(1/2)}(F)_{11}) = \nu'^{+}\begin{pmatrix} [2]_q & 0 \\ 0 & q^{1/2} \end{pmatrix} + \nu''^{+}\begin{pmatrix} 0 & 0 \\ 0 & q^{-1/2} \end{pmatrix},$$

$$\Gamma^{(1/2)}(\mathbf{l}^{+(1/2)}(F)_{22}) = \nu'^{+}\begin{pmatrix} q^{-1/2} & 0 \\ 0 & 2 \end{pmatrix} + \nu''^{+}\begin{pmatrix} q^{1/2} & 0 \\ 0 & 0 \end{pmatrix},$$

$$\Gamma^{(1/2)}(\mathbf{l}^{+(1/2)}(F)_{12}) = \nu'^{+}\begin{pmatrix} 0 & 0 \\ q^{1/2} & 0 \end{pmatrix} - \nu''^{+}\begin{pmatrix} 0 & 0 \\ q^{-1/2} & 0 \end{pmatrix},$$

$$\Gamma^{(1/2)}(\mathbf{l}^{+(1/2)}(F)_{21}) = \nu'^{+}\begin{pmatrix} 0 & q^{-1/2} \\ 0 & 0 \end{pmatrix} - \nu''^{+}\begin{pmatrix} 0 & q^{1/2} \\ 0 & 0 \end{pmatrix}.$$

Back substituting into (3.39),

$$\nu'^{-}[2]_q = \nu'^{+}[2]_q,$$

$$\nu'^{-}q^{1/2} + \nu''^{-}q^{-1/2} = \nu'^{+}q^{1/2} + \nu''^{+}q^{-1/2},$$

$$\nu'^{-}q^{-1/2} + \nu''^{-}q^{1/2} = \nu'^{+}q^{-1/2} + \nu''^{+}q^{1/2},$$

$$\nu'^{-} = \nu'^{+},$$

$$\nu'^{-}q^{-1/2} - \nu''^{-}q^{1/2} = \nu'^{+}q^{-1/2} - \nu''^{+}q^{1/2},$$

$$\nu'^{-}q^{1/2} - \nu''^{-}q^{-1/2} = \nu'^{+}q^{1/2} - \nu''^{+}q^{-1/2}.$$

From the first and fourth equations $\nu'^{-} = \nu'^{+}$, and so the remaining equations give $\nu''^{-} = \nu''^{+}$, as required. ●

Proposition XXXIII. The zero order term of Drinfel'd's 2-cocycle F as an expansion in $S_{\pm} \otimes S_{\mp}$, F_0, can be constructed from the diagonal elements of the $\mathbf{l}^{\pm(1/2)}(F)$, and takes the form

$$F_0 = V'F_0' + V''F_0'' \tag{3.40}$$

with

$$F_0' = N(2[2]_q - 1)^{-1}(2q^{a/2}[b]_q \otimes q^{-a/2}[b]_q - q^{-1}q^{a/2}[b]_q \otimes q^{b/2}(a+1)$$

$$-qq^{-b/2}(a+1)\otimes q^{-a/2}[b]_q+[2]_q q^{-b/2}(a+1)\otimes q^{b/2}(a+1)),$$

$$F_0''=N(q^{-b/2}[a]_q\otimes q^{-a/2}(b-1)+q^{a/2}(b-1)\otimes q^{b/2}[a]_q).$$

Note that $F_0\to(F_0)_{21}$ with the substitution $q\to q^{-1}$, as required. The V',V'' are arbitrary, symmetric and constructed from the center of $U(sl_2)\otimes U(sl_2)$ so that $\mathbf{l}^{\pm(1/2)}(V')=v'$, $\mathbf{l}^{\pm(1/2)}(V'')=v''$ and $V',V''\to 1\otimes 1$ in the limit $q\to 1$. The factor $N=2c^{-1}\otimes c^{-1}$ with the q–valued constant $(2[2]_q-1)^{-1}$ comes out of the calculations and ensures that F_0 also reduces to the identity as $q\to 1$.

Proof. First construct an arbitrary element $F_0'\in U(sl_2)\otimes U(sl_2)$ as a linear combination of tensor products of the components of $\mathbf{l}^{\pm(1/2)}(F')$,

$$F_0'=N\alpha_0(\alpha_1 q^{a/2}[b]_q\otimes q^{-a/2}[b]_q+\alpha_2 q^{a/2}[b]_q\otimes q^{b/2}(a+1)$$

$$+\alpha_3 q^{-b/2}(a+1)\otimes q^{-a/2}[b]_q+\alpha_4 q^{-b/2}(a+1)\otimes q^{b/2}(a+1))$$

with $N=2c^{-1}\otimes c^{-1}$ a normalising factor so that $\mathbf{l}^{\pm(1/2)}(N)=c^{-1}$, and coefficients α_j assumed to be real-numbered functions of q. Constructing $\mathbf{l}^{\pm(1/2)}(F_0')$ with (3.6),

$$\mathbf{l}^{-(1/2)}(F_0')=c^{-1}\alpha_0\begin{pmatrix}(\alpha_1[2]_q+\alpha_3 q^{-1})q^{-a/2}[b]_q+(\alpha_2[2]_q+\alpha_4 q^{-1})q^{b/2}(a+1) & 0\\ 0 & (\alpha_1 q^{1/2}+\alpha_3 2q^{-1/2})q^{-a/2}[b]_q+(\alpha_2 q^{1/2}+\alpha_4 2q^{-1/2})q^{b/2}(a+1)\end{pmatrix},$$

$$\mathbf{l}^{+(1/2)}(F_\circ')=c^{-1}\alpha_0\begin{pmatrix}(\alpha_1[2]_q+\alpha_2 q)q^{a/2}[b]_q+(\alpha_3[2]_q+\alpha_4 q)q^{-b/2}(a+1) & 0\\ 0 & (\alpha_1 q^{-1/2}+\alpha_2 2q^{1/2})q^{a/2}[b]_q+(\alpha_3 q^{-1/2}+\alpha_4 2q^{1/2})q^{-b/2}(a+1)\end{pmatrix}.$$

Equating these diagonal components with (3.37) then gives a set of simultaneous equations in the α_j,

$$\alpha_1[2]_q + \alpha_2 q = \alpha_0^{-1} \qquad \alpha_1 q^{-1/2} + \alpha_2 2q^{1/2} = 0$$

$$\alpha_1[2]_q + \alpha_3 q^{-1} = \alpha_0^{-1} \qquad \alpha_1 q^{1/2} + \alpha_3 2q^{-1/2} = 0$$

$$\alpha_2 q^{1/2} + \alpha_4 2q^{-1/2} = \alpha_0^{-1} q^{-1/2} \qquad \alpha_2[2]_q + \alpha_4 q^{-1} = 0$$

$$\alpha_3 q^{-1/2} + \alpha_4 2q^{1/2} = \alpha_0^{-1} q^{1/2} \qquad \alpha_3[2]_q + \alpha_4 q = 0.$$

Solving gives $\alpha_0 = (2[2]_q - 1)^{-1}$ and $\alpha_1 = 2$, $\alpha_2 = -q^{-1}$, $\alpha_3 = -q, \alpha_4 = [2]_q$, which substituting back into the general form F_0' gives the required result. Now set

$$F_0'' = N\beta_0(\beta_1 q^{-b/2}[a]_q \otimes q^{b/2}[a]_q + \beta_2 q^{-b/2}[a]_q \otimes q^{-a/2}(b-1)$$
$$+\beta_3 q^{a/2}(b-1) \otimes q^{b/2}[a]_q + \beta_4 q^{a/2}(b-1) \otimes q^{-a/2}(b-1))$$

with $N = 2c^{-1} \otimes c^{-1}$, and the β_j again assumed to be real-numbered functions of q. Constructing $\mathbf{l}^{\pm(1/2)}(F_0'')$ with (3.6),

$$\mathbf{l}^{-(1/2)}(F_0'') = c^{-1}\beta_0 \begin{pmatrix} \beta_3 q^{b/2}[a]_q + \beta_4 q^{-a/2}(b-1) & 0 \\ 0 & \beta_1 q^{-1/2} q^{b/2}[a]_q + \beta_4 q^{-1/2} q^{-a/2}(b-1) \end{pmatrix},$$

$$\mathbf{l}^{+(1/2)}(F_0'') = c^{-1}\beta_0 \begin{pmatrix} \beta_2 q^{-b/2}[a]_q + \beta_4 q^{a/2}(b-1) & 0 \\ 0 & \beta_1 q^{1/2} q^{-b/2}[a]_q + \beta_3 q^{1/2} q^{a/2}(b-1) \end{pmatrix}.$$

Equating diagonal components with (3.13) immediately gives $\beta_1 = \beta_4 = 0$ and $\beta_0 = \beta_2 = \beta_3 = 1$. Substituting back into the general form F_0'' then gives the required result. ●

Proposition XXXIV. The first order term of Drinfel'd's 2-cocycle F as an expansion in $S_\pm \otimes S_\mp$, F_1, can be constructed from the off–diagonal elements of $\mathbf{l}^{\pm(1/2)}(F)$, and takes the form

$$F_1 = V'F_1' + V''F_1'' \tag{3.41}$$

with

$$F_1' = N(qq^{-b/2}S_+ \otimes q^{-a/2}b^{-1}[b]_q S_- + q^{-1}q^{a/2}b^{-1}[b]_q S_- \otimes q^{b/2}S_+),$$

$$F_1'' = -N(q^{a/2}S_+ \otimes q^{b/2}a^{-1}[a]_q S_- + q^{-b/2}a^{-1}[a]_q S_- \otimes q^{-a/2}S_+).$$

Again note that $F_1 \to (F_1)_{21}$ with the substitution $q \to q^{-1}$. The normalising factor $N = 2c^{-1} \otimes c^{-1}$ comes out of the calculations, but F_1 now vanishes in the limit $q \to 1$, as required.

Proof. The steps in the construction of the first order term of the 2-cocycle F are identical to those for the zero order term of F, but involve the off-diagonal elements of $\mathbf{l}^{\pm(1/2)}(F)$. Set

$$F_1' = N(\alpha_1 q^{-b/2}S_+ \otimes q^{-a/2}b^{-1}[b]_q S_- + \alpha_2 q^{a/2}b^{-1}[b]_q S_- \otimes q^{b/2}S_+)$$

with $N = 2c^{-1} \otimes c^{-1}$, and α_1, α_2 assumed to be real-numbered functions of q. Constructing $\mathbf{l}^{\pm(1/2)}(F_1')$ with (3.6),

$$\mathbf{l}^{-(1/2)}(F_1') = \begin{pmatrix} 0 & \alpha_1 q^{-1}q^{-a/2}b^{-1}[b]_q S_- \\ \alpha_2 q^{1/2}q^{b/2}S_+ & 0 \end{pmatrix},$$

$$\mathbf{l}^{+(1/2)}(F_1') = \begin{pmatrix} 0 & \alpha_2 q q^{a/2}b^{-1}[b]_q S_- \\ \alpha_1 q^{-1/2}q^{-b/2}S_+ & 0 \end{pmatrix}.$$

Equating off-diagonal components with (3.13) immediately gives $\alpha_1 = q$ and $\alpha_2 = q^{-1}$, which substituting back into the general form F_1' gives the required result. Now set

$$F_1'' = N(\beta_1 q^{a/2} S_+ \otimes q^{b/2} a^{-1}[a]_q S_- + \beta_2 q^{-b/2} a^{-1}[a]_q S_- \otimes q^{-a/2} S_+)$$

with $N = 2c^{-1} \otimes c^{-1}$, and β_1, β_2 again assumed to be real-numbered functions of q. Constructing $\mathbf{l}^{\pm(1/2)}(F_1'')$ with (3.6),

$$\mathbf{l}^{-(1/2)}(F_1'') = \begin{pmatrix} 0 & \beta_1 q^{b/2} a^{-1}[a]_q S_- \\ \beta_2 q^{-1/2} q^{-a/2} S_+ & 0 \end{pmatrix},$$

$$\mathbf{l}^{+(1/2)}(F_1'') = \begin{pmatrix} 0 & \beta_2 q^{-b/2} a^{-1}[a]_q S_- \\ \beta_1 q^{1/2} q^{a/2} S_+ & 0 \end{pmatrix}.$$

Finally, equating off-diagonal components with (3.37) gives $\beta_1 = \beta_2 = -1$, which substituting back into the general form for F_1'' then gives the required result. ●

Chapter Four. Partial construction of the twist element F from $\mathbf{I}^{\pm(1)}(F)$.

The work of the previous chapter is now expanded upon with an explicit construction of the third term of Drinfeld's twist map element F, treated as an expansion in positive powers of $S_{\pm} \otimes S_{\mp}$,

$$F = F_0 + F_1 + F_2 + \ldots \in U(sl_2) \otimes U(sl_2), \tag{4.1}$$

F_n being nth order in $S_{\pm} \otimes S_{\mp}$. The arguments of Chapter Three are followed almost verbatim, but now begin with the construction of the j=1 (three by three) semirepresentations of the coproduct structures, canonical 2-tensor and QIKF of the algebras.

The main section will be a construction of the most general forms for the twist elements $\mathbf{I}^{\pm(1)}(F)$ now in *three* arbitrary (operator-valued) variables $w'^{\pm}, w''^{\pm}, w'''^{\pm}$, so that

$$\mathbf{I}^{\pm(1)}(\overline{R}) = \mathbf{I}^{\pm(1)}(F_{21})\mathbf{I}^{\pm(1)}(R)\mathbf{I}^{\pm(1)}(F)^{-1}, \tag{4.2a}$$

$$\mathbf{I}^{\pm(1)}(\overline{Q}) = \mathbf{I}^{\pm(1)}(F)\mathbf{I}^{\pm(1)}(Q)\mathbf{I}^{\pm(1)}(F)^{-1} \tag{4.2b}$$

are satisfied. These forms for $\mathbf{I}^{\pm(1)}(F)$ are again too general to act as deforming maps for the coproduct stucture $\Delta: U(sl_2) \to U(sl_2) \otimes U(sl_2)$.

The remaining sections demonstrate that the conditions for the semirepresentations $\mathbf{I}^{\pm(1)}(F)$ as deforming maps $\mathbf{I}^{\pm(1)}(F): \mathbf{I}^{\pm(1)}(\Delta u) \to \mathbf{I}^{\pm(1)}(\overline{\Delta} u)$, $u \in U(sl_2)$ are identical in form to those derived for the $\mathbf{l}^{\pm(1/2)}(F)$ in that the variables $w'^{\pm}, w''^{\pm}, w'''^{\pm}$ must be central, symmetric and invariant with respect to the substitution $q \to q^{-1}$.

Using (Prop. XIII) the direct product representations $\Gamma^{(1/2)}(\mathbf{l}^{\pm(1)}(F)), \Gamma^{(1)}(\mathbf{l}^{\pm(1/2)}(F))$ it is shown that these $\mathbf{l}^{\pm(1)}(F)$ are consistent with the results of Chapter Three in that $\mathbf{l}^{\pm(1)}(F)$ and $\mathbf{l}^{\pm(1/2)}(F)$ are semirepresentations of the same element *F*. Finally, an explicit form for the third term (4.1) for *F* is recovered from the components of the $\mathbf{l}^{\pm(1)}(F)$, as solutions to a set of simultaneous equations constructed from the direct product relations $\Gamma^{(1)}(\mathbf{l}^{\pm(1)}(F))$.

IV.1. *Construction of $j=1$ semirepresentations of the algebra $U(sl_2)$.*

The $j=1$ semirepresentations of the centre and coproduct structure of the algebra $U(sl_2)$ are constructed from the irreducible representation $\Gamma^{(1)}$ of the Lie algebra sl_2. With $u \equiv \Sigma u_{(1)} \otimes u_{(2)}$, $u_{(1)}, u_{(2)} \in U(sl_2)$,

$$\mathbf{l}^{-(1)}(u) \equiv \Gamma^{(1)}(u_{(1)})u_{(2)}, \tag{4.3a}$$

$$\mathbf{l}^{+(1)}(u) \equiv u_{(1)}\Gamma^{(1)}(u_{(2)}). \tag{4.3b}$$

Equivalent semirepresentations are obtained through a gauge transformation with the operator-valued diagonal matrix $\mathbf{X}_-$,

$$\mathbf{X}_- \equiv diagonal(1, S_-, S_-^2), \tag{4.4a}$$

so that

$$\mathbf{g}^{\pm(1)}(u) \equiv \mathbf{X}_- \mathbf{l}^{\pm(1)}(u) \mathbf{X}_-^{-1} \text{4b)}$$

the resulting semirepresentations $\mathbf{g}^{\pm(1)}(u)$ taking on a much simplified form.

$$\Gamma^{(1)}(S_+)=\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix},\ \Gamma^{(1)}(S_-)=\begin{pmatrix}0&0&0\\2&0&0\\0&2&0\end{pmatrix},\ \Gamma^{(1)}(S_z)=\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}; \tag{4.5}$$

$$\Gamma^{(1)}(a)=\Gamma^{(1)}(S-S_z)=\begin{pmatrix}0&0&0\\0&1&0\\0&0&2\end{pmatrix},$$

$$\Gamma^{(1)}(b)=\Gamma^{(1)}(S+S_z+1)=\begin{pmatrix}3&0&0\\0&2&0\\0&0&1\end{pmatrix}, \tag{4.6}$$

$$\Gamma^{(1)}(c)=\Gamma^{(1)}(a+b)=\begin{pmatrix}3&0&0\\0&3&0\\0&0&3\end{pmatrix}=3\mathbf{1},\quad \Gamma^{(1)}(d)=\Gamma^{(1)}(b-a)=\begin{pmatrix}3&0&0\\0&1&0\\0&0&-1\end{pmatrix}.$$

Note that now $\Gamma^{(1)}(S_\pm^n)=\mathbf{0}$ for $n\geq 3$. The variables *a, b, c, d* where $d\in U(sl_2)$ and $c^n\in U(sl_2)$, *n* even, are again introduced to simplify the form of the elements $\mathbf{l}^{\pm(1)}(F)$ in subsequent sections.

Definition XX. The $j=1$ semirepresentations $\mathbf{l}^{\pm(1)}(\Delta u)$ of the symmetric coproduct structure $\Delta u\equiv 1\otimes u+u\otimes 1,\ u\in U(sl_2)$, are defined with the irreducible representation $\Gamma^{(1)}$. $\mathbf{l}^{\pm(1)}(\Delta u)$ are then algebra representations of *U*(sl_2). Thus

$$\mathbf{l}^{\pm(1)}(\Delta S_z)=\mathbf{l}^{\pm(1)}\big(1\otimes S_z+S_z\otimes 1\big)=\Gamma^{(1)}(1)S_z+\Gamma^{(1)}(S_z)1=\begin{pmatrix}S_z+1&0&0\\0&S_z&0\\0&0&S_z-1\end{pmatrix},$$

$$\mathbf{l}^{\pm(1)}(\Delta S_+)=\mathbf{l}^{\pm(1)}\big(1\otimes S_++S_+\otimes 1\big)=\Gamma^{(1)}(1)S_++\Gamma^{(1)}(S_+)1=\begin{pmatrix}S_+&1&0\\0&S_+&1\\0&0&S_+\end{pmatrix}, \tag{4.7}$$

$$\mathbf{l}^{\pm(1)}(\Delta S_-)=\mathbf{l}^{\pm(1)}\big(1\otimes S_-+S_-\otimes 1\big)=\Gamma^{(1)}(1)S_-+\Gamma^{(1)}(S_-)1=\begin{pmatrix}S_-&0&0\\2&S_-&0\\0&2&S_-\end{pmatrix}.$$

The equivalent semirepresentations $\mathbf{g}^{\pm(1)}(\Delta u)$ are then obtained from $\mathbf{l}^{\pm(1)}(\Delta u)$ with the gauge transform (4.4a), so that with $S_-^{-1} = S_+ S_+^{-1} S_-^{-1} = S_+ a^{-1} b^{-1}$ and recalling that $S_\pm^n S_z = (S_z \mp n) S_\pm^n$,

$$\mathbf{g}^{\pm(1)}(\Delta S_z) \equiv \mathbf{X}_- \mathbf{l}^{\pm(1)}(\Delta S_z) \mathbf{X}_-^{-1} = \begin{pmatrix} S_z + 1 & 0 & 0 \\ 0 & S_- S_z S_-^{-1} & 0 \\ 0 & 0 & S_-^2 (S_z - 1) S_-^{-2} \end{pmatrix} = (S_z + 1)\mathbf{1},$$

$$\mathbf{g}^{\pm(1)}(\Delta S_+) \equiv \mathbf{X}_- \mathbf{l}^{\pm(1)}(\Delta S_+) \mathbf{X}_-^{-1} = \begin{pmatrix} S_+ & S_-^{-1} & 0 \\ 0 & S_- S_+ S_-^{-1} & S_- S_-^{-2} \\ 0 & 0 & S_-^2 S_+ S_-^{-2} \end{pmatrix}$$

$$= \begin{pmatrix} (a+1)(b-1) & 1 & 0 \\ 0 & ab & 1 \\ 0 & 0 & (a-1)(b+1) \end{pmatrix} S_-^{-1}, \tag{4.8}$$

$$\mathbf{g}^{\pm(1)}(\Delta S_-) \equiv \mathbf{X}_- \mathbf{l}^{\pm(1)}(\Delta S_-) \mathbf{X}_-^{-1} = \begin{pmatrix} S_- & 0 & 0 \\ 2S_- & S_-^2 S_-^{-1} & 0 \\ 0 & 2S_-^2 S_-^{-1} & S_-^3 S_-^{-2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} S_-.$$

Note that as Δu is symmetric, $\mathbf{l}^{-(1)}(\Delta u) = \mathbf{l}^{+(1)}(\Delta u)$ and $\mathbf{g}^{-(1)}(\Delta u) = \mathbf{g}^{+(1)}(\Delta u)$.

Definition XXI. Further, the $j = 1$ semi-representations $\mathbf{l}^{-(1)}(\overline{\Delta} u)$ of the deformed coproduct $\overline{\Delta} u \in U(sl_2) \otimes U(sl_2)$ are defined with the irreducible representation $\Gamma^{(1)}$. $\mathbf{l}^{\pm(1)}(\overline{\Delta} u)$ are then algebra representations of $U_q(sl_2)$. Thus

$$\mathbf{l}^{\pm(1)}(\overline{\Delta} S_z) = \mathbf{l}^{\pm(1)}(\Delta S_z) = \begin{pmatrix} S_z + 1 & 0 & 0 \\ 0 & S_z & 0 \\ 0 & 0 & S_z - 1 \end{pmatrix},$$

$$\mathbf{l}^{\pm(1)}(\overline{\Delta}S_+) = \mathbf{l}^{\pm(1)}\left(q^{-S_z}\otimes S_+ + S_+\otimes q^{S_z}\right) = \begin{pmatrix} q^{\pm1}S_+ & q^{\pm1/2}q^{\mp d/2} & 0 \\ 0 & S_+ & q^{\pm1/2}q^{\mp d/2} \\ 0 & 0 & q^{\mp1}S_+ \end{pmatrix}, \tag{4.9}$$

$$\mathbf{l}^{\pm(1)}(\overline{\Delta}\overline{S}_-) = \mathbf{l}^{\pm(1)}\left(q^{-S_z}\otimes \overline{S}_- + \overline{S}_-\otimes q^{S_z}\right) = \begin{pmatrix} q^{\pm1}\overline{S}_- & 0 & 0 \\ [2]_q q^{\pm1/2}q^{\mp d/2} & \overline{S}_- & 0 \\ 0 & [2]_q q^{\pm1/2}q^{\mp d/2} & q^{\mp1}\overline{S}_- \end{pmatrix}.$$

The equivalent semirepresentations $\mathbf{g}^{\pm(1)}(\overline{\Delta}u)$ are then obtained from $\mathbf{l}^{\pm(1)}(\overline{\Delta}u)$ with the gauge transform (4.4a), so that with $S_-^{-1} = S_+S_+^{-1}S_-^{-1} = S_+a^{-1}b^{-1}$, $\Phi \equiv a^{-1}b^{-1}[a]_q[b]_q$ and $S_\pm^n S_z = (S_z \mp n)S_\pm^n$,

$$\mathbf{g}^{\pm(1)}(\overline{\Delta}S_z) = \mathbf{g}^{\pm(1)}(\Delta S_z) = (S_z+1)\mathbf{I},$$

$$\mathbf{g}^{\pm(1)}(\overline{\Delta}S_+) \equiv \mathbf{X}_-\mathbf{l}^{\pm(1)}(\overline{\Delta}S_+)\mathbf{X}_-^{-1} = \begin{pmatrix} q^{\pm1}S_+ & q^{\pm1/2}q^{\mp d/2}S_-^{-1} & 0 \\ 0 & S_-S_+S_-^{-1} & S_-q^{\pm1/2}q^{\mp d/2}S_-^{-2} \\ 0 & 0 & q^{\mp1}S_-^2S_+S_-^{-2} \end{pmatrix}$$

$$= \begin{pmatrix} q^{\pm1}(a+1)(b-1) & q^{\pm1/2}q^{\mp d/2} & 0 \\ 0 & ab & q^{\pm1/2}q^{\mp d/2} \\ 0 & 0 & q^{\mp1}(a-1)(b+1) \end{pmatrix} S_-^{-1} \tag{4.10}$$

$$\mathbf{g}^{\pm(1)}(\overline{\Delta}\overline{S}_-) \equiv \mathbf{X}_-\mathbf{l}^{\pm(1)}(\overline{\Delta}\overline{S}_-)\mathbf{X}_-^{-1} = \begin{pmatrix} q^{\pm1}\overline{S}_- & 0 & 0 \\ [2]_q q^{\pm1/2}S_-q^{\mp d/2} & S_-\overline{S}_-S_-^{-1} & 0 \\ 0 & [2]_q q^{\pm1/2}S_-^2q^{\mp d/2}S_-^{-1} & q^{\mp1}S_-^2\overline{S}_-S_-^{-2} \end{pmatrix}$$

$$
= \begin{pmatrix} q^{\pm 1}\Phi & 0 & 0 \\ [2]_q q^{\mp 1/2} q^{\mp d/2} & (a-1)^{-1}(b+1)^{-1}[a-1]_q[b+1]_q & 0 \\ 0 & [2]_q q^{\mp 3/2} q^{\mp d/2} & q^{\mp 1}(a-2)^{-1}(b+2)^{-1}[a-2]_q[b+2]_q \end{pmatrix} S_-
$$

Note that as $\overline{\Delta}u \to \overline{\Delta}u_{21}$ with the substitution $q \to q^{-1}$ it must follow that $\mathbf{l}^{-(1)}(\overline{\Delta}u) \to \mathbf{l}^{+(1)}(\overline{\Delta}u)$ and $\mathbf{g}^{-(1)}(\overline{\Delta}u) \to \mathbf{g}^{+(1)}(\overline{\Delta}u)$ with $q \to q^{-1}$. This result and the results above will become important in section (IV.4), where the conditions on the twist elements as deforming maps $\mathbf{l}^{\pm(1)}(F)$:$\mathbf{l}^{\pm(1)}(\Delta u) \to \mathbf{l}^{\pm(1)}(\overline{\Delta}u)$ are derived.

Definition XXII. The undeformed canonical two tensor t then takes on a simple form that is symmetric and linear in $S_\pm \otimes S_\mp$. From its definition,

$$
\begin{aligned}
\mathbf{l}^{\pm(1)}(t) &\equiv \mathbf{l}^{\pm(1)}(4S_z \otimes S_z + 2S_+ \otimes S_- + 2S_- \otimes S_+) \\
&= 4\Gamma^{(1)}(S_z)S_z + 2\Gamma^{(1)}(S_+)S_- + 2\Gamma^{(1)}(S_-)S_+,
\end{aligned}
$$

$$
\therefore \qquad \mathbf{l}^{\pm(1)}(t) = \begin{pmatrix} 4S_z & 2S_- & 0 \\ 4S_+ & 0 & 2S_- \\ 0 & 4S_+ & -4S_z \end{pmatrix}. \tag{4.11a}
$$

The equivalent semirepresentations $\mathbf{g}^{\pm(1/2)}(t)$ then take a form whose components are independent of $S_\pm$, and hence mutually commuting. Recalling that $S_-S_+ = (S-S_z)(S+S_z+1) = ab$, $d \equiv 2S_z+1$ and $S_\pm^n S_z = (S_z \mp n)S_\pm^n$,

$$
\mathbf{g}^{\pm(1)}(t) \equiv \mathbf{X}_- \mathbf{l}^{\pm(1)}(t)\mathbf{X}_-^{-1} = \begin{pmatrix} 4S_z & 2S_-S_-^{-1} & 0 \\ 4S_-S_+ & 0 & 2S_-^2S_-^{-2} \\ 0 & 4S_-^2S_+S_-^{-1} & -4S_-^2S_zS_-^{-2} \end{pmatrix}
$$

$$\therefore \qquad \mathbf{g}^{\pm(1)}(t) = \begin{pmatrix} 2(d-1) & 2 & 0 \\ 4ab & 0 & 2 \\ 0 & 4(\mathrm{a}-1)(\mathrm{b}+1) & -2(d+3) \end{pmatrix}. \tag{4.11b}$$

Definition XXIII. Definition of the negative semirepresentation of the quasitriangular structure, $\mathbf{l}^{-(1)}(\overline{R})$, is more complicated in that it involves the manipulation of an operator valued infinite series. Treating Majid's form (Def. II) for $\overline{R}$ as an expansion in $S_{\pm} \otimes S_{\mp}$ and recalling that $\Gamma^{(1)}(S_{\pm}^{n}) = \mathbf{0}$ and hence $\mathbf{l}^{\pm(1)}(S_{\pm}^{n} \otimes S_{\mp}^{n}) = \mathbf{0}$ for $n \geq 3$,

$$\begin{aligned} \mathbf{l}^{-(1)}(\overline{R}) &= \mathbf{l}^{-(1)}(q^{2S_z \otimes S_z})\mathbf{l}^{-(1)}\Big(1 \otimes 1 + q^{-1}p(q^{S_z} \otimes q^{-S_z})(S_+ \otimes \overline{S}_-) \\ &\qquad + qp^2[2]_q^{-1}(q^{2S_z} \otimes q^{-2S_z})(S_+^2 \otimes \overline{S}_-^2) + \dots\Big) \\ &= q^{2\Gamma(S_z)S_z}\Big(\Gamma^{(1)}(1)1 + q^{-1}p\Gamma^{(1)}(q^{S_z})q^{-S_z}\Gamma^{(1)}(S_+)\overline{S}_- \\ &\qquad + qp^2[2]_q^{-1}\Gamma^{(1)}(q^{2S_z})q^{-2S_z}\Gamma^{(1)}(S_+^2)\overline{S}_-^2 + \dots\Big), \end{aligned}$$

$$\therefore \qquad \mathbf{l}^{-(1)}(\overline{R}) = \begin{pmatrix} q^{2S_z} & pq^{S_z}\overline{S}_- & [2]_q^{-1}p^2q^{-1}\overline{S}_-^2 \\ 0 & 1 & pq^{-1}q^{-S_z}\overline{S}_- \\ 0 & 0 & q^{-2S_z} \end{pmatrix}, \tag{4.12a}$$

with $p \equiv q - q^{-1}$, $\overline{S}_- \equiv \Phi S_-$ and $\Phi \equiv a^{-1}b^{-1}[a]_q[b]_q$. The equivalent semirepresention is then constructed by means of the gauge transformation (4.4a),

$$\mathbf{g}^{-(1)}(\overline{R}) \equiv \mathbf{X}_-\mathbf{l}^{-(1)}(\overline{R})\mathbf{X}_-^{-1} = \begin{pmatrix} q^{2S_z} & pq^{S_z}\overline{S}_-S_-^{-1} & [2]_q^{-1}p^2q^{-1}\overline{S}_-^2S_-^{-2} \\ 0 & S_-S_-^{-1} & pq^{-1}S_-q^{-S_z}\overline{S}_-S_-^{-2} \\ 0 & 0 & S_-^2q^{-2S_z}S_-^{-2} \end{pmatrix},$$

$$\therefore\ \mathbf{g}^{-(1)}(\overline{R}) = \begin{pmatrix} q^{2S_z} & pq^{S_z}\Phi & [2]_q^{-1}p^2q^{-1}\Phi\,(a-1)^{-1}(b+1)^{-1}[a-1]_q[b+1]_q \\ 0 & 1 & pq^{-2}q^{-S_z}(a-1)^{-1}(b+1)^{-1}[a-1]_q[b+1]_q \\ 0 & 0 & q^{-4}q^{-2S_z} \end{pmatrix}. \quad (4.12b)$$

The identical argument is used in the construction of the transpose element $\mathbf{l}^{-(1)}(\overline{R}_{21})$. Taking the transpose of Majid's form,

$$\mathbf{l}^{-(1)}(\overline{R}_{21}) = \mathbf{l}^{-(1)}(q^{2S_z\otimes S_z})\mathbf{l}^{-(1)}\Big(1\otimes 1 + q^{-1}p(q^{-S_z}\otimes q^{S_z})(\overline{S}_-\otimes S_+)$$

$$+qp^2[2]_q^{-1}(q^{-2S_z}\otimes q^{2S_z})(\overline{S}_-^2\otimes S_+^2)+\ldots\Big)$$

$$= q^{2\Gamma(S_z)S_z}\Big(\Gamma^{(1)}(1)1 + q^{-1}p\Gamma^{(1)}(q^{-S_z})q^{S_z}\Gamma^{(1)}(\overline{S}_-)S_+$$

$$+qp^2[2]_q^{-1}\Gamma^{(1)}(q^{-2S_z})q^{2S_z}\Gamma^{(1)}(\overline{S}_-^2)S_+^2+\ldots\Big),$$

$$\therefore \qquad \mathbf{l}^{-(1)}(\overline{R}_{21}) = \begin{pmatrix} q^{2S_z} & 0 & 0 \\ [2]_q pq^{-1}q^{S_z}S_+ & 1 & 0 \\ [2]_q p^2q^{-1}S_+^2 & [2]_q pq^{-S_z}S_+ & q^{-2S_z} \end{pmatrix}; \qquad (4.13a)$$

$$\mathbf{g}^{-(1)}(\overline{R}_{21}) \equiv \mathbf{X}_-\mathbf{l}^{-(1)}(\overline{R}_{21})\mathbf{X}_-^{-1} = \begin{pmatrix} q^{2S_z} & 0 & 0 \\ [2]_q pq^{-1}S_-q^{S_z}S_+ & S_-S_-^{-1} & 0 \\ [2]_q p^2q^{-1}S_-^2S_+^2 & [2]_q pS_-^2q^{-S_z}S_+S_-^{-1} & S_-^2q^{-2S_z}S_-^{-2} \end{pmatrix},$$

$$\therefore \mathbf{g}^{-(1)}(\overline{R}_{21}) = \begin{pmatrix} q^{2S_z} & 0 & 0 \\ [2]_q pq^{-1}q^{S_z}ab & 1 & 0 \\ [2]_q p^2q^{-1}ab(a-1)(b+1) & [2]_q pq^{-2}q^{-S_z}(a-1)(b+1) & q^{-4}q^{-2S_z} \end{pmatrix}. \quad (4.13b)$$

Definition XXIV. Both the *positive* and *negative* semirepresentations $\mathbf{g}^{\pm(1)}(\overline{Q})$ of the QIKF follow directly from its definition (III) as the product of the quasitriangular structure with its transpose, namely

$$\mathbf{l}^{\pm(1)}(\overline{Q}) \equiv \mathbf{l}^{\pm(1)}(\overline{R}_{21})\mathbf{l}^{\pm(1)}(\overline{R}),$$

$$\mathbf{l}^{\pm(1)}(\overline{R}_{21}) = \mathbf{l}^{\mp(1)}(\overline{R}). \tag{4.14}$$

Transforming and combining these two equations and substituting the forms (4.12b, 4.13b) for $\mathbf{g}^{\pm(1)}(\overline{R}_{21})$, $\mathbf{g}^{\pm(1)}(\overline{R})$ then gives

$$\mathbf{g}^{-(1)}(\overline{Q}) \equiv \mathbf{X}_{-}\mathbf{l}^{-(1)}(\overline{Q})\mathbf{X}_{-}^{-1} = \mathbf{X}_{-}\mathbf{l}^{-(1)}(\overline{R}_{21})\mathbf{X}_{-}^{-1}\mathbf{X}_{-}\mathbf{l}^{-(1)}(\overline{R})\mathbf{X}_{-}^{-1} = \mathbf{g}^{-(1)}(\overline{R}_{21})\mathbf{g}^{-(1)}(\overline{R})$$

$$= \begin{pmatrix} q^{2S_z} & 0 & 0 \\ [2]_q pq^{-1}q^{S_z}ab & 1 & 0 \\ [2]_q p^2q^{-1}ab(a-1)(b+1) & [2]_q pq^{-2}q^{-S_z}(a-1)(b+1) & q^{-4}q^{-2S_z} \end{pmatrix}$$

$$\times \begin{pmatrix} q^{2S_z} & pq^{S_z}\Phi & [2]_q^{-1}p^2q^{-1}\Phi(a-1)^{-1}(b+1)^{-1}[a-1]_q[b+1]_q \\ 0 & 1 & pq^{-2}q^{-S_z}(a-1)^{-1}(b+1)^{-1}[a-1]_q[b+1]_q \\ 0 & 0 & q^{-4}q^{-2S_z} \end{pmatrix},$$

$$\therefore \qquad \mathbf{g}^{-(1)}(\overline{Q}) = \begin{pmatrix} q^{4S_z} & pq^{3S_z} & [2]_q^{-1}p^2q^{-1}q^{2S_z} \\ [2]_q pq^{3S_z}[a]_q[b]_q & [2]_q p^2q^{2S_z}[a]_q[b]_q+1 & p^3q^{-1}q^{S_z}[a]_q[b]_q + pq^{-2}q^{-S_z} \\ [2]_q p^2q^{-1}q^{2S_z}[a]_q[a-1]_q[b]_q[b+1]_q & [2]_q pq^{-1}q^{S_z}(p^2[a]_q[b]_q + q^{-1}q^{S_z})[a-1]_q[b+1]_q & p^2q^{-2}(p^2[a]_q[b]_q + [2]_q q^{-2}q^{-2S_z})[a-1]_q[b+1]_q + q^{-8}q^{-4S_z} \end{pmatrix}, \tag{4.15a}$$

and

$$\mathbf{g}^{+(1)}(\overline{Q}) = \mathbf{g}^{+(1)}(\overline{R}_{21})\mathbf{g}^{+(1)}(\overline{R}) = \mathbf{g}^{-(1)}(\overline{R})\mathbf{g}^{-(1)}(\overline{R}_{21})$$

$$= \begin{pmatrix} q^{2S_z} & pq^{S_z}\Phi & [2]_q^{-1}p^2q^{-1}\Phi(a-1)^{-1}(b+1)^{-1}[a-1]_q[b+1]_q \\ 0 & 1 & pq^{-2}q^{-S_z}(a-1)^{-1}(b+1)^{-1}[a-1]_q[b+1]_q \\ 0 & 0 & q^{-4}q^{-2S_z} \end{pmatrix}$$

$$\times \begin{pmatrix} q^{2S_z} & 0 & 0 \\ [2]_q pq^{-1}q^{S_z}ab & 1 & 0 \\ [2]_q p^2q^{-1}ab(a-1)(b+1) & [2]_q pq^{-2}q^{-S_z}(a-1)(b+1) & q^{-4}q^{-2S_z} \end{pmatrix},$$

$$\therefore \qquad \mathbf{g}^{+(1)}(\overline{Q}) = \begin{pmatrix} (p^2q^{-2}[a-1]_q[b+1]_q + [2]_q q^{2S_z})p^2[a]_q[b]_q + q^{4S_z} & p^3q^{-3}q^{-S_z}[a-1]_q[b+1]_q + pq^{S_z} & [2]_q^{-1}p^2q^{-5}q^{-2S_z} \\ [2]_q(p^3q^{-3}q^{-S_z}[a-1]_q[b+1]_q + pq^{S_z})[a]_q[b]_q & [2]_q p^2q^{-4}q^{-2S_z}[a-1]_q[b+1]_q + 1 & pq^{-6}q^{-3S_z} \\ [2]_q p^2q^{-5}q^{-2S_z}[a]_q[a-1]_q[b]_q[b+1]_q & [2]_q pq^{-6}q^{-3S_z}[a-1]_q[b+1]_q & q^{-8}q^{-4S_z} \end{pmatrix}. \tag{4.15b}$$

IV.2. *Construction of general forms for* $\mathbf{l}^{\pm(1)}(F)$ *in three variables.*

The negative $j=1$ semirepresentation of Drinfel'd's twist map element is now constructed from the diagonalising matrices for the canonical two-tensor and QIKF. It will then be demonstrated that the general form for the negative semirepresentation $\mathbf{l}^{-(1)}(F)$ thus obtained maps to its corresponding positive semirepresentation, $\mathbf{l}^{-(1)}(F) \to \mathbf{l}^{+(1)}(F)$, with the substitution $q \to q^{-1}$.

Proposition XXXV. The canonical two-tensor $\mathbf{l}^{\pm(1)}(t)$ diagonalises to a matrix of eigenvalues whose components are central, in that they are functionals of the central element S. The diagonalising matrix takes the form $\mathbf{X}_-^{-1}\mathbf{DV}$, where $\mathbf{X}_-$ is the diagonal matrix (4.4a),

$$\mathbf{D} = -\begin{pmatrix} 1 & 1 & 1 \\ 2a & -(d+1) & -2b \\ 2a(a-1) & -2(a-1)(b+1) & 2b(b+1) \end{pmatrix} \tag{4.16a}$$

being the diagonalising matrix for the equivalent representation $\mathbf{g}^{\pm(1)}(t)$. $\mathbf{D}$ is then non-singular as $|\mathbf{D}| = 2c(c-1)(c+1) \neq 0$, so that

$$\mathbf{D}^{-1} = -(2c)^{-1}(c-1)^{-1}(c+1)^{-1}\begin{pmatrix} (c-1)2b(b+1) & 2(c-1)(b+1) & (c-1) \\ 4cab & -2c(d+1) & -2c \\ (c+1)2a(a-1) & -2(c+1)(a-1) & (c+1) \end{pmatrix}. \tag{4.16b}$$

Further

$$\mathbf{V} \equiv diagonal(w'^{\pm}, w''^{\pm}, w'''^{\pm}) \tag{4.17}$$

is a matrix of operator-valued components that are completely arbitrary save that they are functionals of S, S_z and independent of $S_{\pm}$. Then the matrix of eigenvalues

$$\lambda = \mathbf{D}^{-1}\mathbf{g}^{\pm(1)}(t)\mathbf{D} = diagonal(-2(c-1), 4, 2(c+1)) \tag{4.18a}$$

so that

$$\mathbf{V}^{-1}\lambda\mathbf{V} = \mathbf{V}^{-1}\mathbf{D}^{-1}\mathbf{X}_-\mathbf{X}_-^{-1}\mathbf{g}^{\pm(1)}(t)\mathbf{X}_-\mathbf{X}_-^{-1}\mathbf{DV}$$

$$= \mathbf{V}^{-1}\mathbf{D}^{-1}\mathbf{X}_-\mathbf{l}^{\pm(1)}(t)\mathbf{X}_-^{-1}\mathbf{DV},$$

$$\therefore \qquad \lambda = (\mathbf{X}_-^{-1}\mathbf{DV})^{-1}\mathbf{l}^{\pm(1)}(t)\mathbf{X}_-^{-1}\mathbf{DV}. \tag{4.18b}$$

Post-multiplication of $\mathbf{D}$ by the matrix $\mathbf{V}$ is equivalent to the multiplication of the individual (column) eigenvectors that compose $\mathbf{D}$ by the variables $w'^{\pm}, w''^{\pm}, w'''^{\pm}$, and is the direct expression of the arbitrariness of these eigenvectors. These variables will play a central role in the arguments of the remaining sections.

Proof. The technique whereby the sets of eigenvalues and eigenvectors are obtained for a matrix of the form (4.11b) has been demonstrated (Prop. XIX). Yielding to the increased complexity of the j=1 semirepresentations, it would be simpler to demonstrate that the column vectors of **D** are eigenvectors of $\mathbf{g}^{\pm(1)}(t)$ with corresponding eigenvalues λ by direct substitution of (4.11b, 4.16a) in the eigenvector equation,

$$\mathbf{g}^{\pm(1)}(t)\mathbf{x}_k = \lambda_k \mathbf{x}_k$$

with

$$\lambda_1 = -2(c-1), \lambda_2 = 4, \lambda_3 = 2(c+1);$$

$$\mathbf{x}_1 = -\{1, 2a, 2a(a-1)\},$$

$$\mathbf{x}_2 = -\{1, -(d+1), -2(a-1)(b+1)\},$$

$$\mathbf{x}_3 = -\{1, -2b, -2b(b+1)\}.$$

Substituting,

$$\mathbf{g}^{\pm(1)}(t)\mathbf{x}_1 = -\begin{pmatrix} 2(d-1) & 2 & 0 \\ 4ab & 0 & 2 \\ 0 & 4(\mathrm{a}-1)(\mathrm{b}+1) & -2(d+3) \end{pmatrix}\begin{pmatrix} 1 \\ 2a \\ 2a(a-1) \end{pmatrix}$$

$$= -\begin{pmatrix} 2(d-1)+4a \\ (2b+2(a-1))2a \\ (4(b+1)-2(d+3))2a(a-1) \end{pmatrix} = -2(c-1)\mathbf{x}_1,$$

$$\mathbf{g}^{\pm(1)}(t)\mathbf{x}_2 = -\begin{pmatrix} 2(d-1) & 2 & 0 \\ 4ab & 0 & 2 \\ 0 & 4(\mathrm{a}-1)(\mathrm{b}+1) & -2(d+3) \end{pmatrix}\begin{pmatrix} 1 \\ -(d+1) \\ -2(a-1)(b+1) \end{pmatrix}$$

$$= -\begin{pmatrix} 2(d-1)-2(d+1) \\ 4(ab-(a-1)(b+1)) \\ (-2(d+1)+2(d+3))2(a-1)(b+1) \end{pmatrix} = 4\mathbf{x}_2,$$

$$\mathbf{g}^{\pm(1)}(t)\mathbf{x}_3 = -\begin{pmatrix} 2(d-1) & 2 & 0 \\ 4ab & 0 & 2 \\ 0 & 4(a-1)(b+1) & -2(d+3) \end{pmatrix}\begin{pmatrix} 1 \\ -2b \\ 2b(b+1) \end{pmatrix}$$

$$= -\begin{pmatrix} 2(d-1)-4b \\ (2a+2(b+1))2b \\ -(4(a-1)+2(d+3))2b(b+1) \end{pmatrix} = 2(c+1)\mathbf{x}_3.$$

Neglecting arbitrary numerical factors, the diagonalising matrix $\mathbf{D}$ can be constructed from the eigenvectors $\mathbf{x}_k$, as required. ●

Proposition XXXVI. The diagonalising matrix $\mathbf{D}$ for the canonical two-tensor $\mathbf{g}^{\pm(1)}(t)$ also diagonalises the undeformed Q-matrix, $\mathbf{g}^{\pm(1)}(Q)$, $Q = q^t$ being introduced (Def. VIII) as the twist map for the (symmetric) primitive coproduct structure of $U(sl_2)$. The resulting matrix of eigenvalues Λ is expressed as a q-exponential of the (diagonal) matrix of eigenvalues λ. Thus,

$$\mathbf{g}^{\pm(1)}(Q) \equiv q^{\mathbf{l}(t)}, \tag{4.19a}$$

$$\Lambda = \mathbf{D}^{-1}\mathbf{g}^{\pm(1)}(Q)\mathbf{D} = diagonal(q^{-2(c-1)}, q^4, q^{2(c+1)}). \tag{4.19b}$$

Proof. The result follows directly from the definition of the undeformed Q-Matrix $\mathbf{g}^{\pm(1)}(Q)$ as an exponential expansion in the canonical two-tensor $\mathbf{g}^{\pm(1)}(t)$.[1] ●

Proposition XXXVII. The QIKF $\mathbf{l}^{-(1)}(\overline{Q})$ also diagonalises to a matrix of eigenvalues whose components are functionals of the central element S. The diagonalising matrix takes the form $\mathbf{X}_{-}^{-1}\overline{\mathbf{D}\mathbf{V}}$, where again $\mathbf{X}_{-}$ is the diagonal matrix (4.4a),

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[1] See Cornwell (1984b), p. 408, and unpublished notes.

$$\overline{\mathbf{D}} = -\begin{pmatrix} 1 & 1 & 1 \\ q^{-1/2}q^{-c/2}[2]_q[a]_q & q^{1/2}(q^{-c/2}[a]_q - q^{-1}q^{c/2}[b+1]_q) & -q^{-1/2}q^{c/2}[2]_q[b]_q \\ q^{-1}q^{-c}[2]_q[a]_q[a-1]_q & -[2]_q[a-1]_q[b+1]_q & q^{-1}q^{c}[2]_q[b]_q[b+1]_q \end{pmatrix}, \tag{4.20}$$

is the diagonalising matrix for the equivalent representation $\mathbf{g}^{-(1)}(\overline{Q})$, $\overline{\mathbf{D}}$ being non-singular as $|\overline{\mathbf{D}}| \neq 0$. and

$$\overline{\mathbf{V}} \equiv diagonal(\overline{w}'^{\pm}, \overline{w}''^{\pm}, \overline{w}'''^{\pm}) \tag{4.21}$$

is a matrix of (now q-dependent) operator-valued components that are completely arbitrary save that they are functionals of S, S_z. Then

$$\Lambda = \overline{\mathbf{D}}^{-1}\mathbf{g}^{-(1)}(\overline{Q})\overline{\mathbf{D}} = diagonal(q^{-2(c-1)}, q^4, q^{2(c+1)}), \tag{4.22a}$$

so that

$$\begin{aligned} \overline{\mathbf{V}}^{-1}\Lambda\overline{\mathbf{V}} &= \overline{\mathbf{V}}^{-1}\overline{\mathbf{D}}^{-1}\mathbf{X}_{-}\mathbf{X}_{-}^{-1}\mathbf{g}^{-(1)}(\overline{Q})\mathbf{X}_{-}\mathbf{X}_{-}^{-1}\overline{\mathbf{D}}\overline{\mathbf{V}} \\ &= \overline{\mathbf{V}}^{-1}\overline{\mathbf{D}}^{-1}\mathbf{X}_{-}\mathbf{l}^{\pm(1)}(\overline{Q})\mathbf{X}_{-}^{-1}\overline{\mathbf{D}}\overline{\mathbf{V}}, \end{aligned}$$

$$\therefore \qquad \Lambda = (\mathbf{X}_{-}^{-1}\overline{\mathbf{D}}\overline{\mathbf{V}})^{-1}\mathbf{l}^{-(1)}(\overline{Q})\mathbf{X}_{-}^{-1}\overline{\mathbf{D}}\overline{\mathbf{V}}. \tag{4.22b}$$

From (4.19b) and (4.22b), the undeformed Q-matrix $\mathbf{g}^{\pm(1)}(Q)$ and the QIKF $\mathbf{g}^{-(1)}(\overline{Q})$ diagonalise to the same matrix of eigenvalues.

Proof. As with the diagonalising matrix (4.16a) for the canonical 2-tensor, the proof consists in the demonstration that the column vectors of $\overline{\mathbf{D}}$ are eigenvectors of $\mathbf{g}^{-(1)}(\overline{Q})$ with corresponding eigenvalues Λ,

$$\mathbf{g}^{-(1)}(\overline{Q})\overline{\mathbf{x}}_k = \Lambda_k\overline{\mathbf{x}}_k \tag{4.23a}$$

with

$$\Lambda_1 = q^{-2(c-1)}, \Lambda_2 = q^4, \Lambda_3 = q^{2(c+1)}; \tag{4.23b}$$

$$\begin{aligned}
\overline{\mathbf{x}}_1 &= \{1, q^{-1/2}q^{-c/2}[2]_q[a]_q, q^{-1}q^{-c}[2]_q[a]_q[a-1]_q\}, \\
\overline{\mathbf{x}}_2 &= \{1, q^{1/2}q^{-c/2}[a]_q - q^{-1/2}q^{c/2}[b+1]_q, -[2]_q[a-1]_q[b+1]_q\}, \\
\overline{\mathbf{x}}_3 &= \{1, -q^{-1/2}q^{c/2}[2]_q[b]_q, q^{-1}q^{c}[2]_q[b]_q[b+1]_q\}.
\end{aligned} \tag{4.23c}$$

Neglecting arbitrary (operator valued) factors, direct substitution of (4.15a, 4.23c) into (4.23a) gives the required result. ●

Proposition XXXVIII. The diagonalising matrix for the *positive* semirepresentation $\mathbf{g}^{+(1)}(\overline{Q})$ of the deformed QIKF can be obtained by the substitution $q \to q^{-1}$ in the matrix $\overline{\mathbf{D}}$ of (4.20). Thus

$$\overline{\mathbf{D}}^{+} = \begin{pmatrix} 1 & 1 & 1 \\ [2]_q q^{1/2} q^{c/2}[a]_q & q^{-1/2}(q^{c/2}[a]_q - qq^{-c/2}[b+1]_q) & -[2]_q q^{1/2} q^{-c/2}[b]_q \\ [2]_q qq^{c}[a]_q[a-1]_q & -[2]_q[a-1]_q[b+1]_q & [2]_q qq^{-c}[b]_q[b+1]_q \end{pmatrix}. \tag{4.24}$$

$\mathbf{g}^{+(1)}(\overline{Q})$ then diagonalises to the same matrix of eigenvalues (4.19b) as $\mathbf{g}^{-(1)}(\overline{Q})$.

Proof. As with the diagonalising matrix (4.16a) for the canonical 2-tensor, the proof consists in the demonstration that the column vectors of $\overline{\mathbf{D}}$ are eigenvectors of $\mathbf{g}^{-(1)}(\overline{Q})$ with corresponding eigenvalues Λ,

$$\mathbf{g}^{+(1)}(\overline{Q})\overline{\mathbf{x}}_k = \Lambda_k \overline{\mathbf{x}}_k \tag{4.25a}$$

with

$$\Lambda_1 = q^{-2(c-1)}, \Lambda_2 = q^4, \Lambda_3 = q^{2(c+1)}; \tag{4.25b}$$

$$\bar{\mathbf{x}}_1 = \{1, [2]_q q^{1/2} q^{c/2}[a]_q, [2]_q q q^c [a]_q [a-1]_q\},$$

$$\bar{\mathbf{x}}_2 = \{1, q^{-1/2} q^{c/2}[a]_q - q^{1/2} q^{-c/2}[b+1]_q, -[2]_q [a-1]_q [b+1]_q\}, \tag{4.25c}$$

$$\bar{\mathbf{x}}_3 = \{1, -[2]_q q^{1/2} q^{-c/2}[b]_q, [2]_q q q^{-c}[b]_q [b+1]_q\}.$$

Neglecting arbitrary (operator valued) factors, direct substitution of (4.15b, 4.25c) into (4.25a) gives the required result. ●

Proposition XXXIX. A general form for the j=1 negative semirepresentation $\mathbf{l}^{-(1)}(F)$ of Drinfel'd's twist map element can be constructed from the diagonalising matrices, $\overline{\mathbf{D}}$ and $\mathbf{D}$, of the QIKF and the canonical 2-tensor, giving

$$\mathbf{l}^{-(1)}(F) = \mathbf{X}_-^{-1}\overline{\mathbf{D}}\mathbf{V}\mathbf{D}^{-1}\mathbf{X}_-, \tag{4.26}$$

$$\mathbf{V} \equiv diagonal(w'^-, w''^-, w'''^-)$$

being a matrix of arbitrary functionals w'^-, w''^-, w'''^- of S and S_z. Allowing w'^-, w''^-, w'''^- to vary over the set of continuous functions of S, S_z then sweeps out a continuous set of elements $\mathbf{l}^{-(1)}(F)$. The explicit construction gives

$$\mathbf{l}^{-(1)}(F) \equiv \mathbf{X}_-^{-1}\mathbf{g}^{-(1)}(F)\mathbf{X}_- \tag{4.27}$$

with

$$\mathbf{g}^{-(1)}(F) = w'^-\mathbf{g}^{-(1)}(F') + w''^-\mathbf{g}^{-(1)}(F'') + w'''^-\mathbf{g}^{-(1)}(F''');$$

$$\mathbf{g}^{-(1)}(F') = n'^{-1} q^{-a} \left(\begin{array}{cc} 2[b]_q[b+1]_q & 2b^{-1}[b]_q[b+1]_q \\ 2[2]_q q^{1/2} q^{c/2} ab[b+1]_q & 2[2]_q q^{1/2} q^{c/2} a[b+1]_q \\ 2[2]_q q q^c ab(a-1)(b+1) & 2[2]_q q q^c a(a-1)(b+1) \end{array} \right.$$

$$\left. \begin{matrix} b^{-1}(b+1)^{-1}[b]_q[b+1]_q \\ [2]_q q^{1/2} q^{c/2} a(b+1)^{-1}[b+1]_q \\ [2]_q q q^c a(a-1) \end{matrix} \right),$$

$$\mathbf{g}^{-(1)}(F'') = n''^{-1} q^{1/2} q^{d/2} \left(\begin{matrix} 2[a]_q[b]_q & -(ab)^{-1}[a]_q[b]_q(d+1) \\ 2ab\Phi'' & -\Phi''(d+1) \\ -2[2]_q ab(a-1)(b+1) & [2]_q(a-1)(b+1)(d+1) \end{matrix} \right.$$

$$\left. \begin{matrix} -(ab)^{-1}[a]_q[b]_q \\ -\Phi'' \\ [2]_q(a-1)(b+1) \end{matrix} \right),$$

$$\mathbf{g}^{-(1)}(F''') = n'''^{-1} q^{b} \left(\begin{matrix} 2[a]_q[a-1]_q & -2a^{-1}[a]_q[a-1]_q \\ -2[2]_q q^{1/2} q^{-c/2} ab[a-1]_q & 2[2]_q q^{1/2} q^{-c/2} b[a-1]_q \\ 2[2]_q q q^{-c} ab(a-1)(b+1) & -2[2]_q q q^{-c} b(a-1)(b+1) \end{matrix} \right.$$

$$\left. \begin{matrix} a^{-1}(a-1)^{-1}[a]_q[a-1]_q \\ -[2]_q q^{1/2} q^{-c/2} b(a-1)^{-1}[a-1]_q \\ [2]_q q q^{-c} b(b+1) \end{matrix} \right);$$

$$n' \equiv 2c(c+1),\, n'' \equiv (c+1)(c-1),\, n''' \equiv 2c(c-1),$$

$$\Phi'' \equiv q^{-1/2} q^{c/2}[a]_q - q^{1/2} q^{-c/2}[b+1]_q.$$

$\mathbf{l}^{-(1)}(F)$ reduces to the identity in the limit $q \to 1$ provided $w'^-, w''^-, w'''^- \to 1$ as $q \to 1$. Further, as the constituent matrices $\mathbf{X}_-, \overline{\mathbf{D}}$ and $\mathbf{D}$ are all non-singular, the sole condition for the non-singularity, and hence invertibility, of $\mathbf{l}^{-(1)}(F)$ is that the determinant of $\mathbf{V}$ be non-zero. Hence the variables w'^-, w''^-, w'''^- must always be chosen to be non-zero.

Proof. The proof follows directly from the substitution of the general form (4.26) into the definition (4.2b) for the deforming map $\mathbf{l}^{-(1)}(Q) \to \mathbf{l}^{-(1)}(\overline{Q})$, along with the observation that the matrices of eigenvectors for the undeformed Q-matrix and the QIKF are identical. Substituting,

$$\mathbf{l}^{-(1)}(\overline{Q}) = \mathbf{X}_-^{-1}\overline{\mathbf{D}}\mathbf{V}\mathbf{D}^{-1}\mathbf{X}_-\mathbf{l}^{\pm(1)}(Q)(\mathbf{X}_-^{-1}\overline{\mathbf{D}}\mathbf{V}\mathbf{D}^{-1}\mathbf{X}_-)^{-1}$$

$$\Leftrightarrow \qquad \mathbf{X}_-\mathbf{l}^{-(1)}(\overline{Q})\mathbf{X}_-^{-1} = \overline{\mathbf{D}}\mathbf{V}\mathbf{D}^{-1}\mathbf{X}_-\mathbf{l}^{-(1)}(Q)\mathbf{X}_-^{-1}\mathbf{D}\mathbf{V}^{-1}\overline{\mathbf{D}}^{-1}$$

$$\Leftrightarrow \qquad \overline{\mathbf{D}}^{-1}\mathbf{g}^{-(1)}(\overline{Q})\overline{\mathbf{D}} = \mathbf{V}\mathbf{D}^{-1}\mathbf{g}^{-(1)}(Q)\mathbf{D}\mathbf{V}^{-1}$$

$$\Leftrightarrow \qquad \Lambda = \mathbf{V}\Lambda\mathbf{V}^{-1} = \Lambda,$$

as required. Substituting the diagonalising matrices (4.16b, 4.20) with (4.4a) and multiplying through then gives the explicit form (4.27) for $\mathbf{l}^{-(1)}(F)$. ●

Proposition XL. The *positive* semirepresentation of the general form of the twist map element $\mathbf{l}^{+(1)}(F)$ is obtained from the form (4.26, 4.27) of the *negative* semirepresentation $\mathbf{l}^{-(1)}(F)$ with the substitution $q \to q^{-1}$. Substituting $q \to q^{-1}$ in (4.26),

$$\mathbf{l}^{+(1)}(F) = \mathbf{X}_-^{-1}\overline{\mathbf{D}}^{+}\mathbf{V}^{+}\mathbf{D}^{-1}\mathbf{X}_-, \tag{4.28}$$

$$\mathbf{V}^{+} \equiv diagonal(w'^{+}, w''^{+}, w'''^{+}).$$

$\mathbf{V}^{+}$ is a matrix of arbitrary functionals $w'^{+}, w''^{+}, w'''^{+}$ such that $w'^{-}, w''^{-}, w'''^{-} \to w'^{+}, w''^{+}, w'''^{+}$ with the $q \to q^{-1}$ substitution, $\mathbf{X}_-, \mathbf{D}$ are q-independent, and $\overline{\mathbf{D}}^{+}$ is the diagonalising matrix (4.24) for the positive semirepresentation of the QIKF, $\mathbf{l}^{+(1)}(\overline{Q})$. Substituting $q \to q^{-1}$ in (4.27) gives the explicit construction for $\mathbf{l}^{+(1)}(F)$,

$$\mathbf{l}^{+(1)}(F) = \mathbf{X}_-^{-1}\mathbf{g}^{+(1)}(F)\mathbf{X}_- \tag{4.29}$$

with

$$\mathbf{g}^{+(1)}(F) = w'^{+}\mathbf{g}^{+(1)}(F') + w''^{+}\mathbf{g}^{+(1)}(F'') + w'''^{+}\mathbf{g}^{+(1)}(F''');$$

$$\mathbf{g}^{+(1)}(F') = n'^{-1}q^{a}\begin{pmatrix} 2[b]_q[b+1]_q & 2b^{-1}[b]_q[b+1]_q & b^{-1}(b+1)^{-1}[b]_q[b+1]_q \\ 2[2]_q q^{-1/2}q^{-c/2}ab[b+1]_q & 2[2]_q q^{-1/2}q^{-c/2}a[b+1]_q & [2]_q q^{-1/2}q^{-c/2}a(b+1)^{-1}[b+1]_q \\ 2[2]_q q^{-1}q^{-c}ab(a-1)(b+1) & 2[2]_q q^{-1}q^{-c}a(a-1)(b+1) & [2]_q q^{-1}q^{-c}a(a-1) \end{pmatrix},$$

$$\mathbf{g}^{+(1)}(F'') = n''^{-1}q^{-1/2}q^{-d/2}\begin{pmatrix} 2[a]_q[b]_q & -(ab)^{-1}[a]_q[b]_q(d+1) & -(ab)^{-1}[a]_q[b]_q \\ 2ab\Phi''^{+} & -\Phi''^{+}(d+1) & -\Phi''^{+} \\ -2[2]_q ab(a-1)(b+1) & [2]_q(a-1)(b+1)(d+1) & [2]_q(a-1)(b+1) \end{pmatrix},$$

$$\mathbf{g}^{+(1)}(F''') = n'''^{-1}q^{-b}\begin{pmatrix} 2[a]_q[a-1]_q & -2a^{-1}[a]_q[a-1]_q & a^{-1}(a-1)^{-1}[a]_q[a-1]_q \\ -2[2]_q q^{-1/2}q^{c/2}ab[a-1]_q & 2[2]_q q^{-1/2}q^{c/2}b[a-1]_q & -[2]_q q^{-1/2}q^{c/2}b(a-1)^{-1}[a-1]_q \\ 2[2]_q q^{-1}q^{c}ab(a-1)(b+1) & -2[2]_q q^{-1}q^{c}b(a-1)(b+1) & [2]_q q^{-1}q^{c}b(b+1) \end{pmatrix};$$

$$n' \equiv 2c(c+1),\, n'' \equiv (c+1)(c-1),\, n''' \equiv 2c(c-1),$$

$$\Phi''^{+} \equiv q^{1/2}q^{-c/2}[a]_q - q^{-1/2}q^{c/2}[b+1]_q.$$

As with $\mathbf{l}^{-(1)}(F)$, $\mathbf{l}^{+(1)}(F)$ reduces to the identity in the limit $q\to 1$ provided $w'^{+}, w''^{+}, w'''^{+} \to 1$ as $q\to 1$. The condition for the invertibility of $\mathbf{l}^{+(1)}(F)$ is that the determinant of $\mathbf{V}^{+}$ be non-zero, and hence the variables $w'^{+}, w''^{+}, w'''^{+}$ must always be chosen to be non-zero.

Proof. The proof follows directly from the substitution of the general form (4.28) into the definition (4.2b) for the deforming map $\mathbf{l}^{+(1)}(Q) \to \mathbf{l}^{+(1)}(\overline{Q})$, with the observation that the eigenvalues for the undeformed Q-matrix and the QIKF are identical, and the further observation that the positive and negative semirepresentations of the QIKF also diagonalise to the same set of eigenvalues. Substituting (4.28) in (4.2b),

$$\mathbf{l}^{+(1)}(\overline{Q}) = \mathbf{X}_-^{-1}\overline{\mathbf{D}}^+\mathbf{V}^+\mathbf{D}^{-1}\mathbf{X}_-\mathbf{l}^{+(1)}(Q)(\mathbf{X}_-^{-1}\overline{\mathbf{D}}^+\mathbf{V}^+\mathbf{D}^{-1}\mathbf{X}_-)^{-1}$$

$$\Leftrightarrow \qquad \mathbf{X}_-\mathbf{l}^{+(1)}(\overline{Q})\mathbf{X}_-^{-1} = \overline{\mathbf{D}}^+\mathbf{V}^+\mathbf{D}^{-1}\mathbf{X}_-\mathbf{l}^{+(1)}(Q)\mathbf{X}_-^{-1}\mathbf{D}(\mathbf{V}^+)^{-1}(\overline{\mathbf{D}}^+)^{-1}$$

$$\Leftrightarrow \qquad (\overline{\mathbf{D}}^+)^{-1}\mathbf{g}^{+(1)}(\overline{Q})\overline{\mathbf{D}}^+ = \mathbf{V}^+\mathbf{D}^{-1}\mathbf{g}^{+(1)}(Q)\mathbf{D}(\mathbf{V}^+)^{-1}$$

$$\Leftrightarrow \qquad \Lambda = \mathbf{V}^+\Lambda(\mathbf{V}^+)^{-1} = \Lambda,$$

as required. Substituting the diagonalising matrices (4.16b, 4.24) with (4.4a) and multiplying through then gives the explicit form (4.29) for $\mathbf{l}^{+(1)}(F)$. ●

IV.3. *Properties of the general forms* $\mathbf{l}^{\pm(1)}(F)$ *as twist maps.*

This section lays out the conditions for the general forms (4.27, 4.29) of the semi-representations $\mathbf{l}^{\pm(1)}(F)$ as twist maps $\mathbf{l}^{\pm(1)}(F){:}\mathbf{l}^{\pm(1)}(R) \to \mathbf{l}^{\pm(1)}(\overline{R})$. The arbitrary variables $w'^{\pm}, w''^{\pm}, w'''^{\pm}$ now take on a central role in that it will be shown that these conditions on the $\mathbf{l}^{\pm(1)}(F)$ will in each case reduce to a set of restrictions on the form of $w'^{\pm}, w''^{\perp}, w'''^{\perp}$.

Proposition XLI. The general forms (4.27, 4.29) for the $\mathbf{l}^{\pm(1)}(F)$ satisfy the twist map conditions (4.2), treated as expansions in the deforming parameter h, provided $w'^{\pm}, w''^{\pm}, w'''^{\pm} \to 1$ in the limit $q \to 1$. To first order in h expansion of the $\mathbf{l}^{\pm(1)}(F)$ takes the form

$$\mathbf{l}^{\pm(1)}(F) \equiv \mathbf{l}^{\pm(1)}(f_{(0)}) + \mathbf{l}^{\pm(1)}(f_{(1)})h + \ldots \tag{4.30}$$

with

$$\mathbf{l}^{\pm(1)}(f_{(0)}) = \mathbf{g}^{\pm(1)}(f_{(0)}) = \mathbf{I};$$

$$\mathbf{l}^{\pm(1)}(f_{(1)}) = \frac{1}{2}\begin{pmatrix} 0 & \pm S_- & 0 \\ \mp 2S_+ & 0 & \pm S_- \\ 0 & \mp 2S_+ & 0 \end{pmatrix},$$

$$\mathbf{g}^{\pm(1)}(f_{(1)}) \equiv \mathbf{X}_- \mathbf{l}^{\pm(1)}(f_{(1)}) \mathbf{X}_-^{-1} = \frac{1}{2}\begin{pmatrix} 0 & \pm 1 & 0 \\ \mp 2ab & 0 & \pm 1 \\ 0 & \mp 2(a-1)(b+1) & 0 \end{pmatrix}.$$

Proof. An expansion of the twist map condition (4.2) has been carried out (Prop. VII). Construction of the semirepresentations of that expansion to zero and first order in h gives (4.30). Expansion of the forms (4.27, 4.29) for the $\mathbf{l}^{\pm(1)}(F)$ must now reduce to (4.30) to zero and first order in h. Defining

$$\begin{aligned} w'^{\pm} &\equiv w'^{\pm}_{(0)} + w'^{\pm}_{(1)} h + w'^{\pm}_{(2)} h^2 + O(h^3), \\ w''^{\pm} &\equiv w''^{\pm}_{(0)} + w''^{\pm}_{(1)} h + w''^{\pm}_{(2)} h^2 + O(h^3), \\ w'''^{\pm} &\equiv w'''^{\pm}_{(0)} + w'''^{\pm}_{(1)} h + w'''^{\pm}_{(2)} h^2 + O(h^3); \end{aligned} \tag{4.31}$$

$$q^{u'}[u'']_q[u''']_q = u''u''' + u'u''u'''h + \tfrac{1}{6}u''u'''(3u'^2 + u''^2 + u'''^2 - 2)h^2 + O(h^3),$$

with $u', u'', u''' \in U(sl_2)$, the expansion of the product involving q-brackets being calculated with *Mathematica*. Using these results and expanding $\mathbf{g}^{\pm(1)}(F')$, $\mathbf{g}^{\pm(1)}(F'')$, $\mathbf{g}^{\pm(1)}(F''')$ from (4.27, 4.29) to zero and first order in h,

$$\begin{aligned} \mathbf{g}^{\pm(1)}(F') &= \mathbf{g}^{\pm(1)}(f'_{(0)}) + \mathbf{g}^{\pm(1)}(f'_{(1)})h + \ldots \\ \mathbf{g}^{\pm(1)}(F'') &= \mathbf{g}^{\pm(1)}(f''_{(0)}) + \mathbf{g}^{\pm(1)}(f''_{(1)})h + \ldots \end{aligned} \tag{4.32}$$

$$\mathbf{g}^{\pm(1)}(F''') = \mathbf{g}^{\pm(1)}(f'''_{(0)}) + \mathbf{g}^{\pm(1)}(f'''_{(1)})h + \ldots$$

so that

$$\mathbf{g}^{\pm(1)}(f'_{(0)}) = n'^{-1}\begin{pmatrix} 2b(b+1) & 2(b+1) & 1 \\ 4ab(b+1) & 4a(b+1) & 2a \\ 4ab(a-1)(b+1) & 4a(a-1)(b+1) & 2a(a-1) \end{pmatrix},$$

$$\mathbf{g}^{\pm(1)}(f'_{(1)}) = n'^{-1}\begin{pmatrix} \pm 2ab(b+1) & \pm 2a(b+1) & \pm a \\ \mp 2ab(b+1)(d+1) & \mp 2a(b+1)(d+1) & \mp a(d+1) \\ \mp 4ab(a-1)(b+1)^2 & \mp 4a(a-1)(b+1)^2 & \mp 2a(a-1)(b+1) \end{pmatrix};$$

$$\mathbf{g}^{\pm(1)}(f''_{(0)}) = n''^{-1}\begin{pmatrix} 2ab & -(d+1) & -1 \\ -2ab(d+1) & (d+1)^2 & (d+1) \\ -4ab(a-1)(b+1) & 2(a-1)(b+1)(d+1) & 2(a-1)(b+1) \end{pmatrix},$$

$$\mathbf{g}^{\pm(1)}(f''_{(1)}) = n''^{-1}\begin{pmatrix} \mp ab(d+1) & \pm\frac{1}{2}(d+1)^2 & \pm\frac{1}{2}(d+1) \\ \pm 2ab(d+1) \mp 4(ab)^2 & \mp(d+1)^2 \pm 2ab(d+1) & \mp(d+1) \pm 2ab \\ \pm 2ab(a-1)(b+1)(d+1) & \mp(a-1)(b+1)(d+1)^2 & \mp(a-1)(b+1)(d+1) \end{pmatrix};$$

$$\mathbf{g}^{\pm(1)}(f'''_{(0)}) = n'''^{-1}\begin{pmatrix} 2a(a-1) & -2(a-1) & 1 \\ -4ab(a-1) & 4b(a-1) & -2b \\ 4ab(a-1)(b+1) & -4b(a-1)(b+1) & 2b(b+1) \end{pmatrix},$$

$$\mathbf{g}^{\pm(1)}(f'''_{(1)}) = n'''^{-1}\begin{pmatrix} \mp 2ab(a-1) & \pm 2b(a-1) & \mp b \\ \pm 2ab(a-1)(d+1) & \mp 2b(a-1)(d+1) & \pm b(d+1) \\ \pm 4ab(a-1)^2(b+1) & \mp 4b(a-1)^2(b+1) & \pm 2b(a-1)(b+1) \end{pmatrix}$$

with n', n'', n''' defined in (4.27). From (4.27, 4.30), $\mathbf{g}^{\pm(1)}(F)$ can be treated as both an expansion in h and a functional of the variables $w'^{\pm}, w''^{\pm}, w'''^{\pm}$,

$$\mathbf{g}^{\pm(1)}(F) = \mathbf{g}^{\pm(1)}(f_{(0)}) + \mathbf{g}^{\pm(1)}(f_{(1)})h + \ldots,$$

$$\mathbf{g}^{\pm(1)}(F) = w'^{\pm}\mathbf{g}^{\pm(1)}(F') + w''^{\pm}\mathbf{g}^{\pm(1)}(F'') + w'''^{\pm}\mathbf{g}^{\pm(1)}(F''').$$

Thus

$$\mathbf{g}^{\pm(1)}(F) = (w'^{\pm}_{(0)} + w'^{\pm}_{(1)}h+\ldots)(\mathbf{g}^{\pm(1)}(f'_{(0)}) + \mathbf{g}^{\pm(1)}(f'_{(1)})h+\ldots)$$

$$+(w''^{\pm}_{(0)} + w''^{\pm}_{(1)}h+\ldots)(\mathbf{g}^{\pm(1)}(f''_{(0)}) + \mathbf{g}^{\pm(1)}(f''_{(1)})h+\ldots)$$

$$+(w'''^{\pm}_{(0)} + w'''^{\pm}_{(1)}h+\ldots)(\mathbf{g}^{\pm(1)}(f'''_{(0)}) + \mathbf{g}^{\pm(1)}(f'''_{(1)})h+\ldots),$$

after substituting the expansions (4.31) of $w'^{\pm}, w''^{\pm}, w'''^{\pm}$ and rearranging. Collecting terms in zero and first order in h and substituting from (4.32),

$$\mathbf{g}^{\pm(1)}(f_{(0)}) = w'^{\pm}_{(0)}\mathbf{g}^{\pm(1)}(f'_{(0)}) + w''^{\pm}_{(0)}\mathbf{g}^{\pm(1)}(f''_{(0)}) + w'''^{\pm}_{(0)}\mathbf{g}^{\pm(1)}(f'''_{(0)}),$$

$$\mathbf{g}^{\pm(1)}(f_{(1)}) = w'^{\pm}_{(0)}\mathbf{g}^{\pm(1)}(f'_{(1)}) + w'^{\pm}_{(1)}\mathbf{g}^{\pm(1)}(f'_{(0)}) + w''^{\pm}_{(0)}\mathbf{g}^{\pm(1)}(f''_{(1)}) + w''^{\pm}_{(1)}\mathbf{g}^{\pm(1)}(f''_{(0)})$$

$$+ w'''^{\pm}_{(0)}\mathbf{g}^{\pm(1)}(f'''_{(1)}) + w'''^{\pm}_{(1)}\mathbf{g}^{\pm(1)}(f'''_{(0)}).$$

Defining $n \equiv 2c(c+1)(c-1)$, recalling that $4ab = c^2 - d^2$ and-arguing from the results of the previous section-setting $w'^{\pm}_{(0)} = w''^{\pm}_{(0)} = w'''^{\pm}_{(0)} = 1,\ w'^{\pm}_{(1)} = w''^{\pm}_{(1)} = w'''^{\pm}_{(1)} = 0$,

$$\mathbf{g}^{\pm(1)}(f_{(0)}) = n^{-1}\begin{pmatrix} (c-1)2b(b+1)+4abc+(c+1)2a(a-1) & (c-1)2(b+1)-2c(d+1)-(c+1)2(a-1) & (c-1)-2c+(c+1) \\ (c-1)4ab(b+1)-4abc(d+1)-(c+1)4ab(a-1) & (c-1)a(b+1)+4c(d+1)^2+(c+1)b(a-1) & (c-1)2a+2c(d+1)-(c+1)2b \\ (c-1)4ab(a-1)(b+1)-8abc(a-1)(b+1)+(c+1)4ab(a-1)(b+1) & (c-1)4a(a-1)(b+1)+4c(a-1)(b+1)(d+1)-(c+1)4b(a-1)(b+1) & (c-1)2a(a-1)+4c(a-1)(b+1)+(c+1)2b(b+1) \end{pmatrix}$$

$$= n^{-1}\begin{pmatrix} c(2c^2+2d+2ab)-2cd-2c & 2c(d-d+1)-2c & (c-1)-2c+(c+1) \\ 4abc(d-d+1-1+1)-4ab(c+1-1) & 2c(4ab+d^2+1)-4c & 2c-2c \\ 4ab(a-1)(b+1)(c-1-2c+c+1) & 4(a-1)(b+1)(cd-cd) & 2c^3-2cd-4c+2c-2d^2 \end{pmatrix} = \mathbf{I},$$

$\mathbf{g}^{\pm(1)}(f_{(1)})$

$$= n^{-1}\begin{pmatrix} \pm(c-1)2ab(b+1)\mp 2abc(d+1)\mp(c+1)2ab(a-1) & \pm(c-1)2a(b+1)\pm c(d+1)^2\pm(c+1)2b(a-1) & \pm(c-1)a\pm c(d+1)\mp(c+1)b \\ \mp(c-1)2ab(b+1)(d+1)\mp 8c(ab)^2\pm 4abc(d+1)\pm(c+1)2ab(a-1)(d+1) & \mp(c-1)2a(b+1)(d+1)\pm 4abc(d+1)\mp 2c(d+1)^2\mp(c+1)2b(a-1)(d+1) & \mp(c-1)a(d+1)\pm 4cab\mp 2c(d+1)\pm(c+1)b(d+1) \\ \mp(c-1)4ab(a-1)(b+1)^2\pm 4abc(a-1)(b+1)(d+1)\pm(c+1)4ab(a-1)^2(b+1) & \mp(c-1)4a(a-1)(b+1)^2\mp 2c(a-1)(b+1)(d+1)^2\mp(c+1)4b(a-1)^2(b+1) & \mp(c-1)2a(a-1)(b+1)\mp 2c(a-1)(b+1)(d+1)\pm(c+1)2b(a-1)(b+1) \end{pmatrix}$$

$$= n^{-1}\begin{pmatrix} \pm 2abc(d-d+1-1)\mp 2ab(c+1-1) & \pm c(4ab-2d+d^2+2d+1)\mp 2c & \mp c(d-d-1)\mp c \\ \mp 2abc(4ab+d^2+d)\pm 2abc(d+1) & \mp 2c(d+1)(d-d+1)\pm 2(d+1)c & \mp c((d+1)(2-d)-4ab)\pm c(d+1) \\ \mp 4abc(a-1)(b+1)(d-d+2-1)\pm 4ab(a-1)(b+1)(c+1-1) & \mp 2c(a-1)(b+1)(4ab+d^2+1)\pm 4(a-1)(b+1)c & \mp 2c(a-1)(b+1)\pm 2(a-1)(b+1)c \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & \pm 1 & 0 \\ \mp 2ab & 0 & \pm 1 \\ 0 & \mp 2(a-1)(b+1) & 0 \end{pmatrix}$$

which are the required results. ●

Proposition XLII. The general forms (4.27, 4.29) for the semirepresentations $\mathbf{I}^{\pm(1)}(F)$ are invariant with respect to post-multiplication by the (symmetric) canonical 2-tensor $\mathbf{I}^{\pm(1)}(t)$. Post-multiplication by $\mathbf{I}^{\pm(1)}(t)$ is equivalent to the substitution $w'^{\pm}, w''^{\pm}, w'''^{\pm} \rightarrow -2(c-1)w'^{\pm}, 4w''^{\pm}, 2(c+1)w'''^{\pm}$, $-2(c-1), 4, 2(c+1)$ being the eigenvalues of $\mathbf{I}^{\pm(1)}(t)$, so that

$$\mathbf{I}^{\pm(1)}(F)\mathbf{I}^{\pm(1)}(t) = -2(c-1)w'^{\pm}\mathbf{I}^{\pm(1)}(F') + 4w''^{\pm}\mathbf{I}^{\pm(1)}(F'') + 2(c+1)w'''^{\pm}\mathbf{I}^{\pm(1)}(F'''). \quad (4.33)$$

Proof. The proof comes out of evaluating $\mathbf{I}^{\pm(1)}(F')\mathbf{I}^{\pm(1)}(t), \mathbf{I}^{\pm(1)}(F'')\mathbf{I}^{\pm(1)}(t)$ and $\mathbf{I}^{\pm(1)}(F''')\mathbf{I}^{\pm(1)}(t)$ individually using (4.11, 4.27, 4.29). Substituting into (4.33) then gives the required result. ●

Proposition XLIII. The general forms (4.27, 4.29) for the semirepresentations $\mathbf{I}^{\pm(1)}(F)$ are invariant with respect to pre-multiplication by the QIKF, $\mathbf{I}^{\pm(1)}(\overline{Q})$. Pre-multiplication by $\mathbf{I}^{\pm(1)}(\overline{Q})$ is equivalent to the substitution $w'^{\pm}, w''^{\pm}, w'''^{\pm} \rightarrow q^{-2(c-1)}w'^{\pm}, q^{4}w''^{\pm}, q^{2(c+1)}w'''^{\pm}$, $q^{-2(c-1)}, q^{4}, q^{2(c+1)}$ being the eigenvalues of $\mathbf{I}^{\pm(1)}(\overline{Q})$, so that

$$\mathbf{I}^{\pm(1)}(\overline{Q})\mathbf{I}^{\pm(1)}(F) = q^{-2(c-1)}w'^{\pm}\mathbf{I}^{\pm(1)}(F') + q^{4}w''^{\pm}\mathbf{I}^{\pm(1)}(F'') + q^{2(c+1)}w'''^{\pm}\mathbf{I}^{\pm(1)}(F'''). \quad (4.34)$$

Proof. The proof comes out of evaluating the individual products $\mathbf{I}^{\pm(1)}(\overline{Q})\mathbf{I}^{\pm(1)}(F')$, $\mathbf{I}^{\pm(1)}(\overline{Q})\mathbf{I}^{\pm(1)}(F'')$ and $\mathbf{I}^{\pm(1)}(\overline{Q})\mathbf{I}^{\pm(1)}(F''')$ using (4.15, 4.27, 4.29). Substituting into (4.34) then gives the required result. ●

IV.4. *Conditions on the* $\mathbf{I}^{\pm(1)}(F)$ *as coproduct deforming maps.*

This is a short section in which the conditions on the twist elements $\mathbf{I}^{\pm(1)}(F)$ as semirepresentations of the deforming map for the coproduct structure of $U(sl_2)$ are determined, which again will reduce to restrictions on the form of the arbitrary variables

$w'^{\pm}, w''^{\pm}, w'''^{\pm}$. Note that as $\Delta u_{21} = \Delta u$ and $\overline{\Delta} u \to \overline{\Delta} u_{21}$, $u \in U(sl_2)$, with the substitution $q \to q^{-1}$,

$$\mathbf{l}^{-(1)}(\overline{\Delta} u)\mathbf{l}^{-(1)}(F) = \mathbf{l}^{-(1)}(F)\mathbf{l}^{-(1)}(\Delta u)$$

implies

$$\mathbf{l}^{+(1)}(\overline{\Delta} u)\mathbf{l}^{+(1)}(F) = \mathbf{l}^{+(1)}(F)\mathbf{l}^{+(1)}(\Delta u) \tag{4.35}$$

and the results derived for the negative, will also hold for the positive, semirepresentations $\mathbf{l}^{\pm(1)}(F)$.

Proposition XLIV. The semirepresentation $\mathbf{l}^{-(1)}(F)$ of Drinfel'd's twist element must commute with the coproduct $\mathbf{l}^{-(1)}(\Delta S_z)$, or

$$\mathbf{l}^{-(1)}(\Delta S_z)\mathbf{l}^{-(1)}(F) = \mathbf{l}^{-(1)}(F)\mathbf{l}^{-(1)}(\Delta S_z) \tag{4.36}$$

Proof. The $w'^{\pm}, w''^{\pm}, w'''^{\pm}$ are arbitrary, so after transformation with $\mathbf{X}_{-}$ (4.36) must hold separately for the $\mathbf{g}^{-(1)}(F')$, $\mathbf{g}^{-(1)}(F'')$ and $\mathbf{g}^{-(1)}(F''')$ of (4.27),

$$\mathbf{g}^{-(1)}(\Delta S_z)\mathbf{g}^{-(1)}(F) = \mathbf{g}^{-(1)}(F)\mathbf{g}^{-(1)}(\Delta S_z),$$

so that

$$\mathbf{g}^{-(1)}(\Delta S_z)\mathbf{g}^{-(1)}(F') = \mathbf{g}^{-(1)}(F')\mathbf{g}^{-(1)}(\Delta S_z),$$

$$\mathbf{g}^{-(1)}(\Delta S_z)\mathbf{g}^{-(1)}(F'') = \mathbf{g}^{-(1)}(F'')\mathbf{g}^{-(1)}(\Delta S_z),$$

$$\mathbf{g}^{-(1)}(\Delta S_z)\mathbf{g}^{-(1)}(F''') = \mathbf{g}^{-(1)}(F''')\mathbf{g}^{-(1)}(\Delta S_z).$$

Substituting from (4.8),

$$(S_z + 1)\mathbf{I}\mathbf{g}^{-(1)}(F') = \mathbf{g}^{-(1)}(F')(S_z + 1)\mathbf{I},$$

$$(S_z + 1)\mathbf{I}\mathbf{g}^{-(1)}(F'') = \mathbf{g}^{-(1)}(F'')(S_z + 1)\mathbf{I},$$

$$(S_z+1)\mathbf{I}\mathbf{g}^{-(1)}(F''')=\mathbf{g}^{-(1)}(F''')(S_z+1)\mathbf{I},$$

which is an identity as the components of $\mathbf{g}^{-(1)}(F')$, $\mathbf{g}^{-(1)}(F'')$ and $\mathbf{g}^{-(1)}(F''')$ are independent of $S_{\pm}$. ●

Proposition XLV. The form (4.27) for $\mathbf{l}^{-(1)}(F)$ provides a deforming map $\mathbf{l}^{-(1)}(\Delta S_+)\to\mathbf{l}^{-(1)}(\overline{\Delta}S_+)$ if the arbitrary variables w'^-, w''^-, w'''^- are central, that is

$$\mathbf{l}^{-(1)}(\overline{\Delta}S_+)\mathbf{l}^{-(1)}(F)=\mathbf{l}^{-(1)}(F)\mathbf{l}^{-(1)}(\Delta S_+) \tag{4.37}$$

provided

$$S_+w'=w'S_+,\ S_+w''=w''S_+,$$
$$S_+w'''=w'''S_+.$$

Proof. Again, the equivalent form for (4.37) must hold independently for $\mathbf{g}^{-(1)}(F')$, $\mathbf{g}^{-(1)}(F'')$ and $\mathbf{g}^{-(1)}(F''')$ as w'^-, w''^-, w'''^- are arbitrary. Transforming with $\mathbf{X}_-$ gives

$$\mathbf{g}^{-(1)}(\overline{\Delta}S_+)\mathbf{g}^{-(1)}(F)=\mathbf{g}^{-(1)}(F)\mathbf{g}^{-(1)}(\Delta S_+)$$

so that

$$\mathbf{g}^{-(1)}(\overline{\Delta}S_+)\mathbf{g}^{-(1)}(F)=\mathbf{g}^{-(1)}(F)\mathbf{g}^{-(1)}(\Delta S_+),$$

$$\mathbf{g}^{-(1)}(\overline{\Delta}S_+)\mathbf{g}^{-(1)}(F'')=\mathbf{g}^{-(1)}(F'')\mathbf{g}^{-(1)}(\Delta S_+),$$

$$\mathbf{g}^{-(1)}(\overline{\Delta}S_+)\mathbf{g}^{-(1)}(F''')=\mathbf{g}^{-(1)}(F''')\mathbf{g}^{-(1)}(\Delta S_+).$$

Substituting from (4.8, 4.9, 4.27) then eventually gives $S_+w'=w'S_+$, $S_+w''=w''S_+$ and $S_+w'''=w'''S_+$, which is the required result. ●

Proposition XLVI. That the semirepresentation $\mathbf{l}^{-(1)}(F)$ provides a deforming map $\mathbf{l}^{-(1)}(F){:}\mathbf{l}^{-(1)}(\Delta\overline{S}_-)\to\mathbf{l}^{-(1)}(\overline{\Delta}\overline{S}_-)$ follows immediately from the definition of $\mathbf{l}^{-(1)}(F)$ as a map $\mathbf{I}^{-(1)}(F):\mathbf{I}^{-(1)}(\Delta S_+)\rightarrow\mathbf{I}^{-(1)}(\overline{\Delta}S_+)$. The deforming map for $\mathbf{I}^{-(1)}(\Delta\overline{S}_-)$ then reduces (Prop. VI) to a gauge transformation for a q-functional $\mathbf{I}^{-(1)}(\Delta\overline{Y})$ of S, S_z. Thus

$$\mathbf{I}^{-(1)}(\overline{\Delta}\overline{Y})\mathbf{I}^{-(1)}(F)=\mathbf{I}^{-(1)}(F)\mathbf{I}^{-(1)}(\Delta\overline{Y}), \tag{4.38}$$

with

$$\mathbf{I}^{-(1)}(\overline{\Delta}\overline{Y})=\begin{pmatrix} (q^2+q^{-4})q^d-[2]_q q^3 q^{-d}+q^{-2}\overline{Y} & p^2 q^{-1/2}q^{d/2}\overline{S}_- & 0 \\ p^2 q^{-3/2}q^{d/2}[2]_q S_+ & (q^2+q^{-4})q^d-[2]_q q q^{-d}+\overline{Y} & p^2 q^{1/2}q^{d/2}\overline{S}_- \\ 0 & p^2 q^{-1/2}q^{d/2}[2]_q S_+ & (q^2+q^{-4})q^d-[2]_q q^{-1}q^{-d}+q^2\overline{Y} \end{pmatrix},$$

$$\mathbf{I}^{-(1)}(\Delta\overline{Y})=q^{\sqrt{2\mathbf{I}^{-(1)}(t)+(C+9)\mathbf{I}}}+q^{-\sqrt{2\mathbf{I}^{-(1)}(t)+(C+9)\mathbf{I}}},$$

$$\overline{Y}\equiv q^c+q^{-c}.$$

Proof. The negative j=1 semirepresentation of the map $\overline{\Delta}\overline{Y}=F\Delta\overline{Y}F^{-1}$ is constructed using (1.30) with the irreducible representation (4.5, 4.6). That (4.38) holds can at least be partially demonstrated though expansion of $\mathbf{I}^{-(1)}(\overline{\Delta}\overline{Y}),\mathbf{I}^{-(1)}(\Delta\overline{Y})$ in the deforming parameter h, thus

$$\mathbf{I}^{-(1)}(\Delta\overline{Y})=2\mathbf{I}+\{2\mathbf{I}^{-(1)}(t)+(C+9)\mathbf{I}\}h^2+O(h^3).$$

Substituing into (4.38a) with the expansion (4.30) of the twist element $\mathbf{I}^{-(1)}(F)$ and rearranging gives an identity up to second order in h. That is,

$$\mathbf{I}^{-(1)}(\overline{\Delta}\overline{Y})=2\mathbf{I}+\{2\mathbf{I}^{-(1)}(t)+(C+9)\mathbf{I}\}h^2+O(h^3)$$

which follows from the form of (Prop VI). ●

IV.5. *Partial construction of Drinfel'd's twist element F from $\mathbf{l}^{\pm(1)}(F)$.*

Finally, a construction of the third term of the form (4.1) of Drinfel'd's twist map F is effected. As in Chapter Three, the starting point will be the assumption that the individual terms of F are tensor products of the components of the semirepresentations $\mathbf{l}^{\pm(1)}(F)$. The variables $w'^{\pm}, w''^{\pm}, w'''^{\pm}$ have been shown (Prop. XLV) to be central, so the general forms (4.27, 4.29) for the $\mathbf{l}^{\pm(1)}(F)$ reduce to

$$\mathbf{l}^{\pm(1)}(F) = w'^{\pm}\mathbf{l}^{\pm(1)}(F') + w''^{\pm}\mathbf{l}^{\pm(1)}(F'') + w'''^{\pm}\mathbf{l}^{\pm(1)}(F'''); \tag{4.39}$$

$$\mathbf{g}^{\pm(1)}(F') = n'^{-1}q^{\pm a}\begin{pmatrix} 2[b]_q[b+1]_q & 2b^{-1}[b]_q[b+1]_qS_- & b^{-1}(b+1)^{-1}[b]_q[b+1]_qS_-^2 \\ 2[2]_q q^{\pm 1/2}q^{\mp c/2}[b]_qS_+ & 2[2]_q q^{\pm 1/2}q^{\mp c/2}(a+1)[b]_q & [2]_q q^{\pm 1/2}q^{\mp c/2}(a+1)b^{-1}[b]_qS_- \\ 2[2]_q q^{\pm 1}q^{\mp c}S_+^2 & 2[2]_q q^{\pm 1}q^{\mp c}(a+2)S_+ & [2]_q q^{\pm 1}q^{\mp c}(a+1)(a+2) \end{pmatrix},$$

$$\mathbf{g}^{\pm(1)}(F'') = n''^{-1}q^{\mp 1/2}q^{\mp d/2}\begin{pmatrix} 2[a]_q[b]_q & -(ab)^{-1}[a]_q[b]_q(d+1)S_- & -(ab)^{-1}[a]_q[b]_qS_-^2 \\ 2\Phi''^{\pm}S_+ & -\Phi''^{\pm}(d-1) & -\Phi'''^{\pm}S_- \\ -2q^{\pm 2}[2]_qS_+^2 & q^{\pm 2}[2]_q(d-3)S_+ & q^{\pm 2}[2]_q(a+1)(b-1) \end{pmatrix},$$

$$\mathbf{g}^{\pm(1)}(F''') = n'''^{-1}q^{\mp b}\begin{pmatrix} 2[a]_q[a-1]_q & -2a^{-1}[a]_q[a-1]_qS_- \\ -2[2]_q q^{\pm 1/2}q^{\pm c/2}[a]_qS_+ & 2[2]_q q^{\pm 1/2}q^{\pm c/2}(b-1)[a]_q \\ 2[2]_q q^{\pm 1}q^{\pm c}S_+^2 & -2[2]_q q^{\pm 1}q^{\pm c}(b-2)S_+ \end{pmatrix.}$$

$$\left.\begin{matrix} a^{-1}(a-1)^{-1}[a]_q[a-1]_q S_-^2 \\ -[2]_q q^{\pm 1/2} q^{\pm c/2} a^{-1}(b-1)[a]_q S_- \\ [2]_q q^{\pm 1} q^{\pm c}(b-1)(b-2) \end{matrix}\right);$$

with

$$n' \equiv 2c(c+1), n'' \equiv (c+1)(c-1), n''' \equiv 2c(c-1),$$

$$\Phi'''^{\pm} \equiv q^{\pm 3/2} q^{\mp c/2}[a-1]_q - q^{\pm 1/2} q^{\pm c/2}[b]_q.$$

Proposition XLVII. The positive semirepresentation $\mathbf{l}^{+(1)}(F)$ is obtained from $\mathbf{l}^{-(1)}(F)$ with the substitution $q \to q^{-1}$. The $\mathbf{l}^{\pm(1)}(F)$ are then semirepresentations of an element $F \in U(sl_2) \otimes U(sl_2)$ if and only if the variables $w'^{\pm}, w''^{\pm}, w'''^{\pm}$ are invariant with respect to the $q \to q^{-1}$ substitution.

Proof. That $\mathbf{l}^{-(1)}(F) \to \mathbf{l}^{+(1)}(F)$ with the substitution $q \to q^{-1}$ has been demonstrated. The condition that the $\mathbf{l}^{\pm(1)}(F)$ of (4.39) are semirepresentations of an element $F \in U(sl_2) \otimes U(sl_2)$ is equivalent to the condition (Prop. XII) that the direct product representation $\Gamma^{(1)}(\mathbf{l}^{\pm(1)}(F))$ for F satisfies

$$\Gamma^{(1)}(\mathbf{l}^{\pm(1)}(F)_{jk})_{lm} = \Gamma^{(1)}(\mathbf{l}^{\mp(1)}(F)_{lm})_{jk}, \tag{4.40}$$

so that

$$\begin{aligned} &\Gamma^{(1)}(\mathbf{l}^{\pm(1)}(F)_{11})_{11} = \Gamma^{(1)}(\mathbf{l}^{\mp(1)}(F)_{11})_{11}, \\ &\Gamma^{(1)}(\mathbf{l}^{\pm(1)}(F)_{11})_{22} = \Gamma^{(1)}(\mathbf{l}^{\mp(1)}(F)_{22})_{11}, \\ &\Gamma^{(1)}(\mathbf{l}^{\pm(1)}(F)_{11})_{33} = \Gamma^{(1)}(\mathbf{l}^{\mp(1)}(F)_{33})_{11}, \\ &\Gamma^{(1)}(\mathbf{l}^{\pm(1)}(F)_{22})_{22} = \Gamma^{(1)}(\mathbf{l}^{\mp(1)}(F)_{22})_{22}, \\ &\Gamma^{(1)}(\mathbf{l}^{\pm(1)}(F)_{22})_{33} = \Gamma^{(1)}(\mathbf{l}^{\mp(1)}(F)_{33})_{22}, \\ &\Gamma^{(1)}(\mathbf{l}^{\pm(1)}(F)_{33})_{33} = \Gamma^{(1)}(\mathbf{l}^{\mp(1)}(F)_{33})_{33}, \end{aligned} \tag{4.41}$$

$$\Gamma^{(1)}(\mathbf{I}^{\pm(1)}(F)_{12})_{21} = \Gamma^{(1)}(\mathbf{I}^{\mp(1)}(F)_{21})_{12},$$

$$\Gamma^{(1)}(\mathbf{I}^{\pm(1)}(F)_{12})_{32} = \Gamma^{(1)}(\mathbf{I}^{\mp(1)}(F)_{32})_{12},$$

$$\Gamma^{(1)}(\mathbf{I}^{\pm(1)}(F)_{23})_{21} = \Gamma^{(1)}(\mathbf{I}^{\mp(1)}(F)_{21})_{23},$$

$$\Gamma^{(1)}(\mathbf{I}^{\pm(1)}(F)_{23})_{32} = \Gamma^{(1)}(\mathbf{I}^{\mp(1)}(F)_{32})_{23},$$

$$\Gamma^{(1)}(\mathbf{I}^{\pm(1)}(F)_{13})_{31} = \Gamma^{(1)}(\mathbf{I}^{\mp(1)}(F)_{31})_{13}.$$

Constructing the direct preduct representations $\Gamma^{(1)}(\mathbf{I}^{\pm(1)}(F')_{jk})$, $\Gamma^{(1)}(\mathbf{I}^{\pm(1)}(F'')_{jk})$, and $\Gamma^{(1)}(\mathbf{I}^{\pm(1)}(F''')_{jk})$ using (4.39) and the irreducible representations (4.5, 4.6),

$$\Gamma^{(1)}(\mathbf{I}^{\pm(1)}(F')_{11}) = \Gamma^{(1)}(2n'q^{\pm a}[b]_q[b+1]_q) = \frac{1}{12}\begin{pmatrix} [3]_q[4]_q & 0 & 0 \\ 0 & q^{\pm 1}[2]_q[3]_q & 0 \\ 0 & 0 & q^{\pm 2}[2]_q \end{pmatrix},$$

$$\Gamma^{(1)}(\mathbf{I}^{\pm(1)}(F'')_{11}) = \Gamma^{(1)}(2n''q^{\mp 1/2}q^{\mp d/2}[a]_q[b]_q) = \frac{1}{4}[2]_q\begin{pmatrix} 0 & 0 & 0 \\ 0 & q^{\mp 1} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\Gamma^{(1)}(\mathbf{I}^{\pm(1)}(F''')_{11}) = \Gamma^{(1)}(2n'''q^{\mp b}[a]_q[a-1]_q) = \frac{1}{6}[2]_q\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & q^{\mp 1} \end{pmatrix};$$

$$\Gamma^{(1)}(\mathbf{I}^{\pm(1)}(F')_{22}) = \Gamma^{(1)}(2n'[2]_q q^{\pm 1/2}q^{\pm a}q^{\mp c/2}[b]_q(a+1)) = \frac{1}{12}[2]_q\begin{pmatrix} q^{\mp 1}[3]_q & 0 & 0 \\ 0 & 2[2]_q & 0 \\ 0 & 0 & 3q^{\pm 1} \end{pmatrix},$$

$$\Gamma^{(1)}(\mathbf{I}^{\pm(1)}(F'')_{22}) = \Gamma^{(1)}(-n''q^{\mp 1/2}q^{\mp d/2}\Phi'''^{\pm}(d-1)) = -\frac{1}{4}\begin{pmatrix} q^{\mp 2}-[3]_q & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & q^{\pm 2}-[3]_q \end{pmatrix},$$

$$\Gamma^{(1)}(\mathbf{I}^{\pm(1)}(F''')_{22}) = \Gamma^{(1)}(2n'''[2]_q q^{\mp b}q^{\pm 1/2}q^{\pm c/2}[a]_q(b-1)) = \frac{1}{6}[2]_q\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$$\Gamma^{(1)}(\mathbf{l}^{\pm(1)}(F')_{33}) = \Gamma^{(1)}(n' q^{\pm a} q^{\pm 1} q^{\mp c}[2]_q (a+1)(a+2)) = \frac{1}{12} q^{\mp 2}[2]_q \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3q^{\pm 1} & 0 \\ 0 & 0 & 6q^{\pm 2} \end{pmatrix},$$

$$\Gamma^{(1)}(\mathbf{l}^{\pm(1)}(F'')_{33}) = \Gamma^{(1)}(n'' q^{\mp 1/2} q^{\mp d/2} q^{\pm 2}[2]_q (a+1)(b-1)) = \frac{1}{4}[2]_q \begin{pmatrix} 1 & 0 & 0 \\ 0 & q^{\pm 1} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Gamma^{(1)}(\mathbf{l}^{\pm(1)}(F''')_{33}) = \Gamma^{(1)}(n''' q^{\mp b} q^{\pm 1} q^{\pm c}[2]_q (b-1)(b-2)) = \frac{1}{6}[2]_q \begin{pmatrix} q^{\pm 1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$$\Gamma^{(1)}(\mathbf{l}^{\pm(1)}(F')_{12}) = \Gamma^{(1)}(2n' q^{\pm a} b^{-1}[b]_q [b+1]_q S_-) = \frac{1}{12}[2]_q \begin{pmatrix} 0 & 0 & 0 \\ 0 & q^{\pm 1}[3]_q & 0 \\ 0 & 0 & 2q^{\pm 2} \end{pmatrix},$$

$$\Gamma^{(1)}(\mathbf{l}^{\pm(1)}(F'')_{12}) = \Gamma^{(1)}(-n'' q^{\mp 1/2} q^{\mp d/2} (d+1)(ab)^{-1}[a]_q [b]_q S_-) = -\frac{1}{2}[2]_q \begin{pmatrix} 0 & 0 & 0 \\ q^{\mp 1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Gamma^{(1)}(\mathbf{l}^{\pm(1)}(F''')_{12}) = \Gamma^{(1)}(-2n''' q^{\mp b} a^{-1}[a]_q [a-1]_q S_-) = -\frac{1}{6}[2]_q \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & q^{\mp 1} & 0 \end{pmatrix};$$

$$\Gamma^{(1)}(\mathbf{l}^{\pm(1)}(F')_{21}) = \Gamma^{(1)}(2n'[2]_q q^{\pm 1/2} q^{\pm a} q^{\mp c/2}[b]_q S_+) = \frac{1}{12}[2]_q \begin{pmatrix} 0 & q^{\mp 1}[3]_q & 0 \\ 0 & 0 & [2]_q \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Gamma^{(1)}(\mathbf{l}^{\pm(1)}(F'')_{21}) = \Gamma^{(1)}(2n'' q^{\mp 1/2} q^{\mp d/2} \Phi'''^{\pm} S_+) = \frac{1}{4} \begin{pmatrix} 0 & q^{\mp 2} - [3]_q & 0 \\ 0 & 0 & \mp p[2]_q \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Gamma^{(1)}(\mathbf{l}^{\pm(1)}(F''')_{21}) = \Gamma^{(1)}(-2n'''[2]_q q^{\mp b} q^{\pm 1/2} q^{\pm c/2}[a]_q S_+) = -\frac{1}{6}[2]_q \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix};$$

$$\Gamma^{(1)}(\mathbf{l}^{\pm(1)}(F')_{23}) = \Gamma^{(1)}(n'[2]_q q^{\pm 1/2} q^{\pm a} q^{\mp c/2}[b]_q b^{-1}(a+1)S_-) = \frac{1}{12}[2]_q \begin{pmatrix} 0 & 0 & 0 \\ [2]_q & 0 & 0 \\ 0 & 3q^{\pm 1} & 0 \end{pmatrix},$$

$$\Gamma^{(1)}(\mathbf{l}^{\pm(1)}(F'')_{23}) = \Gamma^{(1)}(-n'' q^{\mp 1/2} q^{\mp d/2} \Phi'''^{\pm} S_-) = -\frac{1}{4} \begin{pmatrix} 0 & 0 & 0 \\ \mp p[2]_q & 0 & 0 \\ 0 & [3]_q - q^{\pm 2} & 0 \end{pmatrix},$$

$$\Gamma^{(1)}(\mathbf{l}^{\pm(1)}(F''')_{23}) = \Gamma^{(1)}(-n'''[2]_q q^{\mp b} q^{\pm 1/2} q^{\pm c/2}[a]_q a^{-1}(b-1)S_-) = -\frac{1}{6}[2]_q \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$$\Gamma^{(1)}(\mathbf{l}^{\pm(1)}(F')_{32}) = \Gamma^{(1)}(2n' q^{\pm a} q^{\pm 1} q^{\mp c}[2]_q(a+2)S_+) = \frac{1}{12}[2]_q \begin{pmatrix} 0 & 2q^{\mp 2} & 0 \\ 0 & 0 & 3q^{\pm 1} \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Gamma^{(1)}(\mathbf{l}^{\pm(1)}(F'')_{32}) = \Gamma^{(1)}(n'' q^{\mp 1/2} q^{\mp d/2} q^{\pm 2}[2]_q(d-3)S_+) = -\frac{1}{4}[2]_q \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & q^{\pm 1} \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Gamma^{(1)}(\mathbf{l}^{\pm(1)}(F''')_{32}) = \Gamma^{(1)}(-2n''' q^{\mp b} q^{\pm 1} q^{\pm c}[2]_q(b-2)S_+) = -\frac{1}{6}[2]_q \begin{pmatrix} 0 & q^{\pm 1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$$\Gamma^{(1)}(\mathbf{l}^{\pm(1)}(F')_{13}) = \Gamma^{(1)}(n' q^{\pm a} b^{-1}(b+1)^{-1}[b]_q[b+1]_q S_-^2) = \frac{1}{12}[2]_q \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ q^{\pm 2} & 0 & 0 \end{pmatrix},$$

$$\Gamma^{(1)}(\mathbf{l}^{\pm(1)}(F'')_{13}) = \Gamma^{(1)}(-n'' q^{\mp 1/2} q^{\mp d/2}(ab)^{-1}[a]_q[b]_q S_-^2) = -\frac{1}{4}[2]_q \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\Gamma^{(1)}(\mathbf{l}^{\pm(1)}(F''')_{13}) = \Gamma^{(1)}(n''' q^{\mp b} a^{-1}(a-1)^{-1}[a]_q[a-1]_q S_-^2) = -\frac{1}{6}[2]_q \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ q^{\mp 1} & 0 & 0 \end{pmatrix};$$

$$\Gamma^{(1)}(\mathbf{l}^{\pm(1)}(F')_{31}) = \Gamma^{(1)}(2n' q^{\pm a} q^{\pm 1} q^{\mp c}[2]_q S_+^2) = \tfrac{1}{12}[2]_q \begin{pmatrix} 0 & 0 & q^{\mp 2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Gamma^{(1)}(\mathbf{l}^{\pm(1)}(F'')_{31}) = \Gamma^{(1)}(-2n'' q^{\mp 1/2} q^{\mp d/2} q^{\pm 2}[2]_q S_+^2) = -\tfrac{1}{4}[2]_q \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Gamma^{(1)}(\mathbf{l}^{\pm(1)}(F''')_{31}) = \Gamma^{(1)}(2n''' q^{\mp b} q^{\pm 1} q^{\pm c}[2]_q S_+^2) = \tfrac{1}{6}[2]_q \begin{pmatrix} 0 & 0 & q^{\pm 1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Back-substituting in (4.41) by way of (4.39) and making the replacement $\Gamma^{(1)}(w'^{\pm}), \Gamma^{(1)}(w''^{\pm}), \Gamma^{(1)}(w'''^{\pm}) \to w'^{\pm}, w''^{\pm}, w'''^{\pm}$ gives a set of simultaneous equations in the $w'^{\pm}, w''^{\pm}, w'''^{\pm}$, or

$$w'^{\pm}\tfrac{1}{12}[3]_q[4]_q = w'^{\mp}\tfrac{1}{12}[3]_q[4]_q,$$

$$w'^{\pm}\tfrac{1}{12}q^{\pm 1}[2]_q[3]_q + w''^{\pm}\tfrac{1}{4}q^{\mp 1}[2]_q = w'^{\mp}\tfrac{1}{12}q^{\pm 1}[2]_q[3]_q - w''^{\mp}\tfrac{1}{4}(q^{\pm 2} - [3]_q),$$

$$w'^{\pm}\tfrac{1}{12}q^{\pm 2}[2]_q + w''^{\pm}\tfrac{1}{4}[2]_q + w'''^{\pm}\tfrac{1}{6}q^{\mp 1}[2]_q = w'^{\mp}\tfrac{1}{12}q^{\pm 2}[2]_q + w''^{\mp}\tfrac{1}{4}[2]_q + w'''^{\mp}\tfrac{1}{6}q^{\mp 1}[2]_q,$$

$$w'^{\pm}\tfrac{1}{6}[2]_q^2 + w'''^{\pm}\tfrac{1}{6}[2]_q = w'^{\mp}\tfrac{1}{6}[2]_q^2 + w'''^{\mp}\tfrac{1}{6}[2]_q,$$

$$w'^{\pm}\tfrac{1}{4}q^{\pm 1}[2]_q + w''^{\pm}\tfrac{1}{4}([3]_q - q^{\pm 2}) = w'^{\mp}\tfrac{1}{4}q^{\pm 1}[2]_q + w''^{\mp}\tfrac{1}{4}q^{\mp 1}[2]_q,$$

$$w'^{\pm}\tfrac{1}{2}[2]_q = w'^{\mp}\tfrac{1}{2}[2]_q,$$

$$w'^{\pm}\tfrac{1}{12}q^{\pm 1}[2]_q[3]_q - w''^{\pm}\tfrac{1}{4}q^{\mp 1}[2]_q = w'^{\mp}\tfrac{1}{12}q^{\pm 1}[2]_q[3]_q + w''^{\mp}\tfrac{1}{4}(q^{\pm 2} - [3]_q),$$

$$w'^{\pm}\tfrac{1}{6}q^{\pm 2}[2]_q - w'''^{\pm}\tfrac{1}{6}q^{\mp 1}[2]_q = w'^{\mp}\tfrac{1}{6}q^{\pm 2}[2]_q - w'''^{\mp}\tfrac{1}{6}q^{\mp 1}[2]_q,$$

$$w'^{\pm}\tfrac{1}{12}[2]_q^2 - w''^{\pm}\tfrac{1}{4}p[2]_q - w'''^{\pm}\tfrac{1}{6}[2]_q = w'^{\mp}\tfrac{1}{12}[2]_q^2 - w''^{\mp}\tfrac{1}{4}p[2]_q - w'''^{\mp}\tfrac{1}{6}[2]_q,$$

$$w'^{\pm}\tfrac{1}{4}q^{\pm 1}[2]_q - w''^{\pm}\tfrac{1}{4}([3]_q - q^{\pm 2}) = w'^{\mp}\tfrac{1}{4}q^{\pm 1}[2]_q - w''^{\mp}\tfrac{1}{4}q^{\mp 1}[2]_q,$$

$$w'^{\pm}\tfrac{1}{12}q^{\pm 2}[2]_q - w''^{\pm}\tfrac{1}{4}[2]_q + w'''^{\pm}\tfrac{1}{6}q^{\mp 1}[2]_q = w'^{\mp}\tfrac{1}{12}q^{\pm 2}[2]_q - w''^{\mp}\tfrac{1}{4}[2]_q + w'''^{\mp}\tfrac{1}{6}q^{\mp 1}[2]_q.$$

The first and sixth equations immediately give $w'^{\pm} = w'^{\mp}$. Substituting in the fourth and eighth equations gives $w'''^{\pm} = w'''^{\mp}$. Then after some algebra the remaining equations will give $w''^{\pm} = w''^{\mp}$, as required. ●

Proposition XLVIII. The $\mathbf{I}^{\pm(1)}(F)$ are semirepresentations of the same element $F \in U(sl_2) \otimes U(sl_2)$ as the semirepresentations $\mathbf{I}^{\pm(1/2)}(F)$ obtained in Chapter Three if and only if the arbitrary variables w', w'', w''' and v', v'' satisfy certain internal consistency conditions. These conditions are that

$$\Gamma^{(1)}(v') = \tfrac{1}{2}[2]_q \Gamma^{(1/2)}(w'), \tag{4.42}$$

and

$$\Gamma^{(1)}(v'') = \Gamma^{(1/2)}(w'').$$

No condition is set on w'''. The variables w', w'', w''' and v', v'' are completely arbitrary, save for the conditions obtained earlier, namely that they be non-zero, central, invariant with respect to the substitution $q \to q^{-1}$ and reduce to the identity in the limit $q \to 1$. Thus in practice (4.42) is a statement that the zero and first order term in $S_{\pm} \otimes S_{\mp}$ of Drinfel'd's 2-cocycle F obtained in section (III.5) can be constructed from $\mathbf{I}^{\pm(1)}(F)$ with the appropriate choice of w', w'', w'''.

Proof. The condition that the $\mathbf{I}^{\pm(1)}(F)$ of (4.39) and the $\mathbf{I}^{\pm(1/2)}(F)$ of (3.37) are semirepresentations of the same element $F \in U(sl_2) \otimes U(sl_2)$ is equivalent to the condition (Prop XIII) that the direct product representations $\Gamma^{(1)}(\mathbf{I}^{-(1/2)}(F))$ and $\Gamma^{(1/2)}(\mathbf{I}^{+(1)}(F))$ satisfy

$$\Gamma^{(1)}(\mathbf{I}^{-(1/2)}(F)_{jk})_{lm} = \Gamma^{(1/2)}(\mathbf{I}^{+(1)}(F)_{lm})_{jk}, \tag{4.43}$$

or equivalently

$$\Gamma^{(1)}(\mathbf{I}^{-(1/2)}(F)_{11})_{11} = \Gamma^{(1/2)}(\mathbf{I}^{+(1)}(F)_{11})_{11},$$

$$
\begin{aligned}
&\Gamma^{(1)}(\mathbf{l}^{-(1/2)}(F)_{11})_{22} = \Gamma^{(1/2)}(\mathbf{l}^{+(1)}(F)_{22})_{11},\\
&\Gamma^{(1)}(\mathbf{l}^{-(1/2)}(F)_{11})_{33} = \Gamma^{(1/2)}(\mathbf{l}^{+(1)}(F)_{33})_{11},\\
&\Gamma^{(1)}(\mathbf{l}^{-(1/2)}(F)_{22})_{11} = \Gamma^{(1/2)}(\mathbf{l}^{+(1)}(F)_{11})_{22},\\
&\Gamma^{(1)}(\mathbf{l}^{-(1/2)}(F)_{22})_{22} = \Gamma^{(1/2)}(\mathbf{l}^{+(1)}(F)_{22})_{22},\\
&\Gamma^{(1)}(\mathbf{l}^{-(1/2)}(F)_{22})_{33} = \Gamma^{(1/2)}(\mathbf{l}^{+(1)}(F)_{33})_{22},\\
&\Gamma^{(1)}(\mathbf{l}^{-(1/2)}(F)_{12})_{21} = \Gamma^{(1/2)}(\mathbf{l}^{+(1)}(F)_{21})_{12},\\
&\Gamma^{(1)}(\mathbf{l}^{-(1/2)}(F)_{12})_{32} = \Gamma^{(1/2)}(\mathbf{l}^{+(1)}(F)_{32})_{12},\\
&\Gamma^{(1)}(\mathbf{l}^{-(1/2)}(F)_{21})_{12} = \Gamma^{(1/2)}(\mathbf{l}^{+(1)}(F)_{12})_{21},\\
&\Gamma^{(1)}(\mathbf{l}^{-(1/2)}(F)_{21})_{23} = \Gamma^{(1/2)}(\mathbf{l}^{+(1)}(F)_{23})_{21}.
\end{aligned}
\tag{4.44}
$$

Constructing $\Gamma^{(1/2)}(\mathbf{l}^{+(1)}(F')_{jk})$, $\Gamma^{(1/2)}(\mathbf{l}^{+(1)}(F'')_{jk})$, and $\Gamma^{(1/2)}(\mathbf{l}^{+(1)}(F''')_{jk})$ using (4.39) and the j=1/2 irreducible representations (3.5, 3.6) from the previous section,

$$\Gamma^{(1/2)}(\mathbf{l}^{+(1)}(F')_{11}) = \Gamma^{(1/2)}(2n'q^{a}[b]_q[b+1]_q) = \tfrac{1}{6}[2]_q\begin{pmatrix} [3]_q & 0 \\ 0 & q \end{pmatrix},$$

$$\Gamma^{(1/2)}(\mathbf{l}^{+(1)}(F'')_{11}) = \Gamma^{(1/2)}(2n''q^{-1/2}q^{-d/2}[a]_q[b]_q) = \tfrac{2}{3}q^{-1/2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\Gamma^{(1/2)}(\mathbf{l}^{+(1)}(F''')_{11}) = \Gamma^{(1/2)}(2n'''q^{-b}[a]_q[a-1]_q) = \mathbf{0};$$

$$\Gamma^{(1/2)}(\mathbf{l}^{+(1)}(F')_{22}) = \Gamma^{(1/2)}(2n'[2]_q q^{1/2}q^{a}q^{-c/2}[b]_q(a+1)) = \tfrac{1}{6}[2]_q\begin{pmatrix} q^{-1/2}[2]_q & 0 \\ 0 & 2q^{1/2} \end{pmatrix},$$

$$\Gamma^{(1/2)}(\mathbf{l}^{+(1)}(F'')_{22}) = \Gamma^{(1/2)}(-n''q^{-1/2}q^{-d/2}\Phi'''^{+}(d-1)) = -\tfrac{1}{3}\begin{pmatrix} q^{-1}-[2]_q & 0 \\ 0 & q-[2]_q \end{pmatrix},$$

$$\Gamma^{(1/2)}(\mathbf{l}^{+(1)}(F''')_{22}) = \Gamma^{(1/2)}(2n'''[2]_q q^{-b}q^{1/2}q^{c/2}[a]_q(b-1)) = \mathbf{0};$$

$$\Gamma^{(1/2)}(\mathbf{l}^{+(1)}(F')_{33})=\Gamma^{(1)}(n'q^{a}qq^{-c}[2]_q(a+1)(a+2))=\tfrac{1}{12}[2]_q\begin{pmatrix}2q^{-1}&0\\0&6\end{pmatrix},$$

$$\Gamma^{(1/2)}(\mathbf{l}^{+(1)}(F'')_{33})=\Gamma^{(1/2)}(n''q^{-1/2}q^{-d/2}q^{2}[2]_q(a+1)(b-1))=\tfrac{1}{3}[2]_q\begin{pmatrix}q^{1/2}&0\\0&0\end{pmatrix},$$

$$\Gamma^{(1/2)}(\mathbf{l}^{+(1)}(F''')_{33})=\Gamma^{(1/2)}(n'''q^{-b}qq^{c}[2]_q(b-1)(b-2))=\mathbf{0};$$

$$\Gamma^{(1/2)}(\mathbf{l}^{+(1)}(F')_{12})=\Gamma^{(1/2)}(2n'q^{a}b^{-1}[b]_q[b+1]_qS_{-})=\tfrac{1}{6}[2]_q\begin{pmatrix}0&0\\q&0\end{pmatrix},$$

$$\Gamma^{(1/2)}(\mathbf{l}^{+(1)}(F'')_{12})=\Gamma^{(1/2)}(-n''q^{-1/2}q^{-d/2}(d+1)(ab)^{-1}[a]_q[b]_qS_{-})=-\tfrac{1}{3}\begin{pmatrix}0&0\\q^{-1/2}&0\end{pmatrix},$$

$$\Gamma^{(1/2)}(\mathbf{l}^{+(1)}(F''')_{12})=\Gamma^{(1/2)}(-n'''q^{-b}a^{-1}[a]_q[a-1]_qS_{-})=\mathbf{0};$$

$$\Gamma^{(1/2)}(\mathbf{l}^{+(1)}(F')_{21})=\Gamma^{(1/2)}(2n'[2]_qq^{1/2}q^{a}q^{-c/2}[b]_qS_{+})=\tfrac{1}{6}[2]_q^{2}\begin{pmatrix}0&q^{-1/2}\\0&0\end{pmatrix},$$

$$\Gamma^{(1/2)}(\mathbf{l}^{+(1)}(F'')_{21})=\Gamma^{(1/2)}(2n''q^{-1/2}q^{-d/2}\Phi'''^{+}S_{+})=\tfrac{2}{3}\begin{pmatrix}0&q^{-1}-[2]_q\\0&0\end{pmatrix},$$

$$\Gamma^{(1/2)}(\mathbf{l}^{+(1)}(F''')_{21})=\Gamma^{(1/2)}(-2n'''[2]_qq^{-b}q^{1/2}q^{c/2}[a]_qS_{+})=\mathbf{0};$$

$$\Gamma^{(1/2)}(\mathbf{l}^{+(1)}(F')_{23})=\Gamma^{(1/2)}(n'[2]_qq^{1/2}q^{a}q^{-c/2}[b]_qb^{-1}(a+1)S_{-})=\tfrac{1}{6}[2]_q\begin{pmatrix}0&0\\q^{1/2}&0\end{pmatrix},$$

$$\Gamma^{(1/2)}(\mathbf{l}^{+(1)}(F'')_{23})=\Gamma^{(1/2)}(-n''q^{-1/2}q^{-d/2}\Phi'''^{\pm}S_{-})=-\tfrac{1}{3}\begin{pmatrix}0&0\\ [2]_q-q&0\end{pmatrix},$$

$$\Gamma^{(1/2)}(\mathbf{l}^{+(1)}(F''')_{23})=\Gamma^{(1/2)}(-n'''[2]_qq^{-b}q^{1/2}q^{c/2}[a]_qa^{-1}(b-1)S_{-})=\mathbf{0};$$

$$\Gamma^{(1/2)}(\mathbf{l}^{+(1)}(F')_{32})=\Gamma^{(1/2)}(2n'q^{a}qq^{-c}[2]_q(a+2)S_{+})=\tfrac{1}{3}[2]_q\begin{pmatrix}0&q^{-1}\\0&0\end{pmatrix},$$

$$\Gamma^{(1/2)}(\mathbf{l}^{+(1)}(F'')_{32}) = \Gamma^{(1/2)}(n''q^{-1/2}q^{-d/2}q^{2}[2]_q(d-3)S_+) = -\tfrac{1}{3}[2]_q\begin{pmatrix} 0 & q^{1/2} \\ 0 & 0 \end{pmatrix},$$

$$\Gamma^{(1/2)}(\mathbf{l}^{+(1)}(F''')_{32}) = \Gamma^{(1/2)}(-2n'''q^{-b}qq^{c}[2]_q(b-2)S_+) = \mathbf{0};$$

$$\Gamma^{(1/2)}(\mathbf{l}^{+(1)}(F')_{13}) = \Gamma^{(1/2)}(n'q^{a}b^{-1}(b+1)^{-1}[b]_q[b+1]_q S_-^2) = \mathbf{0},$$

$$\Gamma^{(1/2)}(\mathbf{l}^{+(1)}(F'')_{13}) = \Gamma^{(1/2)}(-n''q^{-1/2}q^{-d/2}(ab)^{-1}[a]_q[b]_q S_-^2) = \mathbf{0},$$

$$\Gamma^{(1/2)}(\mathbf{l}^{+(1)}(F''')_{13}) = \Gamma^{(1/2)}(n'''q^{-b}a^{-1}(a-1)^{-1}[a]_q[a-1]_q S_-^2) = \mathbf{0};$$

$$\Gamma^{(1/2)}(\mathbf{l}^{+(1)}(F')_{31}) = \Gamma^{(1/2)}(2n'q^{a}qq^{-c}[2]_q S_+^2) = \mathbf{0},$$

$$\Gamma^{(1/2)}(\mathbf{l}^{+(1)}(F'')_{31}) = \Gamma^{(1/2)}(-2n''q^{-1/2}q^{-d/2}q^{2}[2]_q S_+^2) = \mathbf{0},$$

$$\Gamma^{(1/2)}(\mathbf{l}^{+(1)}(F''')_{31}) = \Gamma^{(1/2)}(2n'''q^{-b}qq^{c}[2]_q S_+^2) = \mathbf{0},$$

which last follows as $\Gamma^{(1/2)}(S_\pm^2) = \mathbf{0}$. Now constructing $\Gamma^{(1)}(\mathbf{l}^{-(1/2)}(F')_{jk})$ and $\Gamma^{(1)}(\mathbf{l}^{-(1/2)}(F'')_{jk})$ with (3.37) of section (III.5) and the j=1 irreducible representations (4.5, 4.6),

$$\Gamma^{(1)}(\mathbf{l}^{-(1/2)}(F')_{11}) = \Gamma^{(1)}(c^{-1}q^{-a/2}[b]_q) = \frac{1}{3}\begin{pmatrix} [3]_q & 0 & 0 \\ 0 & q^{-1/2}[2]_q & 0 \\ 0 & 0 & q^{-1} \end{pmatrix},$$

$$\Gamma^{(1)}(\mathbf{l}^{-(1/2)}(F'')_{11}) = \Gamma^{(1)}(c^{-1}q^{b/2}[a]_q) = \frac{1}{3}\begin{pmatrix} 0 & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & q^{1/2}[2]_q \end{pmatrix};$$

$$\Gamma^{(1)}(\mathbf{l}^{-(1/2)}(F')_{22}) = \Gamma^{(1)}(c^{-1}q^{-1/2}q^{b/2}(a+1)) = \frac{1}{3}\begin{pmatrix} q & 0 & 0 \\ 0 & 2q^{1/2} & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

$$\Gamma^{(1)}(\mathbf{l}^{-(1/2)}(F'')_{22}) = \Gamma^{(1)}(c^{-1}q^{-1/2}q^{-a/2}(b-1)) = \frac{1}{3}\begin{pmatrix} 2q^{-1/2} & 0 & 0 \\ 0 & q^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$$\Gamma^{(1)}(\mathbf{l}^{-(1/2)}(F')_{12}) = \Gamma^{(1)}(c^{-1}q^{-a/2}b^{-1}[b]_q S_-) = \frac{1}{3}\begin{pmatrix} 0 & 0 & 0 \\ q^{-1/2}[2]_q & 0 & 0 \\ 0 & 2q^{-1} & 0 \end{pmatrix},$$

$$\Gamma^{(1)}(\mathbf{l}^{-(1/2)}(F'')_{12}) = \Gamma^{(1)}(-c^{-1}q^{b/2}a^{-1}[a]_q S_-) = -\frac{1}{3}\begin{pmatrix} 0 & 0 & 0 \\ 2q & 0 & 0 \\ 0 & q^{1/2}[2]_q & 0 \end{pmatrix};$$

$$\Gamma^{(1)}(\mathbf{l}^{-(1/2)}(F')_{21}) = \Gamma^{(1)}(c^{-1}q^{-1/2}q^{b/2}S_+) = \frac{1}{3}\begin{pmatrix} 0 & q & 0 \\ 0 & 0 & q^{1/2} \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Gamma^{(1)}(\mathbf{l}^{-(1/2)}(F'')_{21}) = \Gamma^{(1)}(-c^{-1}q^{-1/2}q^{-a/2}S_+) = -\frac{1}{3}\begin{pmatrix} 0 & q^{-1/2} & 0 \\ 0 & 0 & q^{-1} \\ 0 & 0 & 0 \end{pmatrix}.$$

Back-substituting in (4.44) by way of (3.37, 4.39) gives a set of simultaneous equations in $\Gamma^{(1/2)}(w'), \Gamma^{(1/2)}(w'')$ and $\Gamma^{(1)}(v'), \Gamma^{(1)}(v'')$. Note that as $\Gamma^{(1/2)}(\mathbf{l}^{+(1)}(F''')) = \mathbf{0}$, all the coefficients of $\Gamma^{(1/2)}(w''')$ vanish and no additional conditions can be placed on the variable w'''. Thus

$$\Gamma^{(1)}(v')\tfrac{1}{3}[3]_q = \Gamma^{(1/2)}(w')\tfrac{1}{6}[2]_q[3]_q,$$

$$\Gamma^{(1)}(v')\tfrac{1}{3}q^{-1/2}[2]_q + \Gamma^{(1)}(v'')\tfrac{1}{3}q = \Gamma^{(1/2)}(w')\tfrac{1}{6}q^{-1/2}[2]_q^2 - \Gamma^{(1/2)}(w'')\tfrac{1}{3}(q^{-1} - [2]_q),$$

$$\Gamma^{(1)}(v')\tfrac{1}{3}q^{-1} + \Gamma^{(1)}(v'')\tfrac{1}{3}q^{1/2}[2]_q = \Gamma^{(1/2)}(w')\tfrac{1}{6}q^{-1}[2]_q + \Gamma^{(1/2)}(w'')\tfrac{1}{3}q^{1/2}[2]_q,$$

$$\Gamma^{(1)}(v')\tfrac{1}{3}q + \Gamma^{(1)}(v'')\tfrac{2}{3}q^{-1/2} = \Gamma^{(1/2)}(w')\tfrac{1}{6}q[2]_q + \Gamma^{(1/2)}(w'')\tfrac{2}{3}q^{-1/2},$$

$$\Gamma^{(1)}(v')\tfrac{2}{3}q^{1/2} + \Gamma^{(1)}(v'')\tfrac{1}{3}q^{-1} = \Gamma^{(1/2)}(w')\tfrac{1}{3}q^{1/2}[2]_q - \Gamma^{(1/2)}(w'')\tfrac{1}{3}(q - [2]_q),$$

$$\Gamma^{(1)}(v')\tfrac{1}{3}q^{-1/2}[2]_q - \Gamma^{(1)}(v'')\tfrac{2}{3}q = \Gamma^{(1/2)}(w')\tfrac{1}{6}q^{-1/2}[2]_q^2 + \Gamma^{(1/2)}(w'')\tfrac{2}{3}(q^{-1} - [2]_q),$$

$$\Gamma^{(1)}(v')\tfrac{2}{3}q^{-1} - \Gamma^{(1)}(v'')\tfrac{1}{3}q^{1/2}[2]_q = \Gamma^{(1/2)}(w')\tfrac{1}{3}q^{-1}[2]_q - \Gamma^{(1/2)}(w'')\tfrac{1}{3}q^{1/2}[2]_q,$$

$$\Gamma^{(1)}(v')\tfrac{1}{3}q - \Gamma^{(1)}(v'')\tfrac{1}{3}q^{-1/2} = \Gamma^{(1/2)}(w')\tfrac{1}{6}q[2]_q - \Gamma^{(1/2)}(w'')\tfrac{1}{3}q^{-1/2},$$

$$\Gamma^{(1)}(v')\tfrac{1}{3}q^{1/2} - \Gamma^{(1)}(v'')\tfrac{1}{3}q^{-1} = \Gamma^{(1/2)}(w')\tfrac{1}{6}q^{1/2}[2]_q + \Gamma^{(1/2)}(w'')\tfrac{1}{3}(q - [2]_q),$$

$$\Gamma^{(1)}(v') = \Gamma^{(1/2)}(w')\tfrac{1}{2}[2]_q.$$

The first and sixth equations immediately give $\Gamma^{(1)}(v') = \Gamma^{(1/2)}(w')\tfrac{1}{2}[2]_q$, which substituting into the remaining equations gives $\Gamma^{(1)}(v'') = \Gamma^{(1/2)}(w'')$, as required. •

Proposition XLIX. The second order term of Drinfel'd's twist element F as an expansion in $S_\pm \otimes S_\mp$, F_2, can be constructed from the extreme off–diagonal elements of $\mathbf{l}^{\pm(1)}(F)$, and takes the form

$$F_2 = W'F_2' + W''F_2'' + W'''F_2''' \tag{4.45}$$

with

$$F_2' = N'(q^3 q^{-b} S_+^2 \otimes q^{-a} b^{-1}[b]_q (b+1)^{-1}[b+1]_q S_-^2 + q^{-3} q^a b^{-1}[b]_q (b+1)^{-1}[b+1]_q S_-^2 \otimes q^b S_+^2),$$

$$F_2'' = -N''(q^2 q^{-d/2} S_+^2 \otimes q^{d/2}(ab)^{-1}[a]_q[b]_q S_-^2 + q^{-2} q^{-d/2}(ab)^{-1}[a]_q[b]_q S_-^2 \otimes q^{d/2} S_+^2),$$

$$F_2''' = N'''(q^a S_+^2 \otimes q^b a^{-1}[a]_q (a-1)^{-1}[a-1]_q S_-^2 + q^{-b} a^{-1}[a]_q (a-1)^{-1}[a-1]_q S_-^2 \otimes q^{-a} S_+^2);$$

$$\mathbf{l}^{\pm(1)}(N') = n', \mathbf{l}^{\pm(1)}(N'') = n'', \mathbf{l}^{\pm(1)}(N''') = n''',$$

$$\mathbf{l}^{\pm(1)}(W') = w', \mathbf{l}^{\pm(1)}(W'') = w'', \mathbf{l}^{\pm(1)}(W''') = w'''.$$

Note that $F_2 \to (F_2)_{21}$ with the substitution $q \to q^{-1}$. The normalising factors N', N'', N''' come out of the calculations, but F_2 vanishes in the limit $q \to 1$, as required.

Proof. Construct arbitrary elements $F_2', F_2'', F_2''' \in U(sl_2) \otimes U(sl_2)$ from linear combinations of tensor products of the extreme off-diagonal components of $\mathbf{l}^{\pm(1)}(F'), \mathbf{l}^{\pm(1)}(F'')$ and $\mathbf{l}^{\pm(1)}(F''')$. In other words set

$$F_2' = N'(\alpha_1 q^{-b} S_+^2 \otimes q^{-a} b^{-1} [b]_q (b+1)^{-1} [b+1]_q S_-^2$$
$$+ \alpha_2 q^a b^{-1} [b]_q (b+1)^{-1} [b+1]_q S_-^2 \otimes q^b S_+^2)$$

with the coefficents α_1, α_2 assumed to be real-numbered functions of q. Constructing the semirepresentations $\mathbf{l}^{\pm(1)}(F_2')$ using the irreducible representations (4.5, 4.6),

$$\mathbf{l}^{-(1)}(F_2') = n'^{-1} \begin{pmatrix} 0 & 0 & \alpha_1 q^{-3} q^{-a} b^{-1} [b]_q (b+1)^{-1} [b+1]_q S_-^2 \\ 0 & 0 & 0 \\ \alpha_2 2[2]_q q^b S_+^2 & 0 & 0 \end{pmatrix},$$

$$\mathbf{l}^{+(1)}(F_2') = n'^{-1} \begin{pmatrix} 0 & 0 & \alpha_2 q^3 q^a b^{-1} [b]_q (b+1)^{-1} [b+1]_q S_-^2 \\ 0 & 0 & 0 \\ \alpha_1 2[2]_q q^{-2} q^{-b} S_+^2 & 0 & 0 \end{pmatrix}.$$

Comparing $\mathbf{l}^{\pm(1)}(F_2')$ with the extreme off-diagonal components of (4.39) immediately gives $\alpha_1 = q^3$ and $\alpha_2 = q^{-3}$, which substituting back into the general form F_2' gives the required result. Now set

$$F_2'' = N''(\beta_1 q^{-d/2} S_+^2 \otimes q^{d/2} (ab)^{-1} [a]_q [b]_q S_-^2$$
$$+ \beta_2 q^{-d/2} (ab)^{-1} [a]_q [b]_q S_-^2 \otimes q^{d/2} S_+^2)$$

with the coefficents β_1, β_2 assumed to be real-numbered functions of q. Constructing the semirepresentations $\mathbf{l}^{\pm(1)}(F_2'')$ using (4.5, 4.6),

$$\mathbf{l}^{-(1)}(F_2'') = n''^{-1}\begin{pmatrix} 0 & 0 & \beta_1 q^{-3/2} q^{d/2} (ab)^{-1}[a]_q[b]_q S_-^2 \\ 0 & 0 & 0 \\ \beta_2 2[2]_q q^{1/2} q^{d/2} S_+^2 & 0 & 0 \end{pmatrix},$$

$$\mathbf{l}^{+(1)}(F_2'') = n''^{-1}\begin{pmatrix} 0 & 0 & \beta_2 q^{3/2} q^{-d/2} (ab)^{-1}[a]_q[b]_q S_-^2 \\ 0 & 0 & 0 \\ \beta_1 2[2]_q q^{-1/2} q^{-d/2} S_+^2 & 0 & 0 \end{pmatrix}.$$

Comparing $\mathbf{l}^{\pm(1)}(F_2'')$ with the extreme off-diagonal components of (4.39) immediately gives $\beta_1 = -q^2$ and $\beta_2 = -q^{-2}$, which substituting back into the general form F_2'' again gives the required result. Finally set

$$F_2''' = N'''(\gamma_1 q^a S_+^2 \otimes q^b a^{-1}[a]_q (a-1)^{-1}[a-1]_q S_-^2$$
$$+ \gamma_2 q^{-b} a^{-1}[a]_q (a-1)^{-1}[a-1]_q S_-^2 \otimes q^{-a} S_+^2)$$

with the coefficents γ_1, γ_2 assumed to be real-numbered functions of q. Constructing the semirepresentations $\mathbf{l}^{\pm(1)}(F_2''')$ using (4.5, 4.6),

$$\mathbf{l}^{-(1)}(F_2''') = n'''^{-1}\begin{pmatrix} 0 & 0 & \gamma_1 q^b a^{-1}[a]_q (a-1)^{-1}[a-1]_q S_-^2 \\ 0 & 0 & 0 \\ \gamma_2 2[2]_q q^{-1} q^{-a} S_+^2 & 0 & 0 \end{pmatrix},$$

$$\mathbf{l}^{+(1)}(F_2''') = n'''^{-1}\begin{pmatrix} 0 & 0 & \gamma_2 q^{-b} a^{-1}[a]_q (a-1)^{-1}[a-1]_q S_-^2 \\ 0 & 0 & 0 \\ \gamma_1 2[2]_q q q^a S_+^2 & 0 & 0 \end{pmatrix}.$$

Comparing $\mathbf{l}^{\pm(1)}(F_2''')$ with the extreme off-diagonal components of (4.39) immediately gives $\gamma_1 = \gamma_2 = 1$, which substituting back into the general form F_2''' once again gives the required result. ●

Chapter Five. Conclusions: higher order terms for the twist element F.

Proposition L. There does exist a representation-independent form for Drinfel'd's twist map $F: U(sl_2) \to U_q(sl_2)$. Specifically, there exists an explicit form for a twist element $F \in U(sl_2) \otimes U(sl_2)$ satisfying conditions (Def. IX) as a map $F: U(sl_2) \to U_q(sl_2)$. The form of the element, F, is a counital 2-cocycle which reduces to the identity as $q \to 1$. The form of the map, $\overline{R} = F_{21} R F$, preserves the property (Prop. I, Def. VIII) that the undeformed and deformed quasitriangular structures $R, \overline{R}$ invert with the substitution $q \to q^{-1}$, so that F must twist with the substitution $q \to q^{-1}$ (Prop. V).

A general form exists for the twist element F as a series in $S_\pm \otimes S_\mp$, in two arbitrary operator valued variables V', V''. That is,

$$F \equiv V' \sum_{n=0}^{\infty} F'_n + V'' \sum_{n=0}^{\infty} F''_n . \tag{5.1}$$

where the F'_n, F''_n are nth order in $S_\pm \otimes S_\mp$. The variables V', V'' take the form of q-functionals of the central element S such that V', V'' reduce to the identity in the limit $q \to 1$, and are invariant with respect to the substitution $q \to q^{-1}$(Props. XXX, XXXII). Further it has at least been partially shown (Prop. XXVIII) that the form of the V', V'' can be restricted ($V' = V''$) so that $F \in U(sl_2) \otimes U(sl_2)$.

Explicit forms for the zero order terms F'_0, F''_0 have been determined (Prop. XXXIII). Higher order terms F'_n, F''_n can be defined in terms of functionals Z_n, Z'_n of S, S_z. Thus

$$F'_n \equiv Z_n S_+^n \otimes S_-^n + Z_{n21}(q^{-1}) S_-^n \otimes S_+^n , \tag{5.2a}$$

$$F''_n \equiv Z'_n S^n_+ \otimes S^n_- + Z''_{n21}(q^{-1}) S^n_- \otimes S^n_+, \tag{5.2b}$$

where $Z'_{n21}(q^{-1}), Z''_{n21}(q^{-1})$ are obtained from Z'_n, Z''_n by twisting and the substitution $q \to q^{-1}$. The form (5.2) then ensures that the element F twists with the substitution $q \to q^{-1}$, as required. The only condition placed on the Z'_n, Z''_n is that $Z'_n \to -Z''_n$ in the limit $q \to 1$.

Finally, forms for Z'_n, Z''_n can be generalised from F'_1, F''_1, F'_2, F''_2 (Props. XXXIV, XLIX) giving

$$Z'_n \equiv \prod_{k=0}^{n-1} N_k q^{k+1} q^{-b/2} \otimes q^{-a/2} (b+k)^{-1} [b+k]_q, \tag{5.3a}$$

$$Z''_n \equiv -\prod_{k=0}^{n-1} N_k q^{a/2} \otimes q^{b/2} (a-k)^{-1} [a-k]_q, \tag{5.3b}$$

where $N_k^{-1} = (2+k)^{-1}(c+k) \otimes (c+k)$.

Proof. The proof is actually more of an inductive argument. Generalising (Props. XXIII, XXXIX), to an arbitrary $n = 2j+1$ with integer or half-integer j, semirepresentations $\mathbf{l}^{\pm(j)}(F)$ can be constructed from the nth order diagonalising matrices for the canonical 2-tensor $\mathbf{l}^{\pm(j)}(t)$ and QIKF $\mathbf{l}^{\pm(j)}(\overline{Q})$,

$$\mathbf{l}^{\pm(j)}(F) = \overline{\mathbf{D}} \mathbf{V} \mathbf{D}^{-1}, \tag{5.4}$$

the diagonal matrix $\mathbf{V}$ being composed of the n arbitrary operator valued elements $\nu^{(k)}; k = 1, \ldots, n$, functionals of the central element S. Multiplying through then gives the form

$$\mathbf{l}^{\pm(j)}(F) = \sum_{k=1}^{n} \nu^{(k)} \mathbf{l}^{\pm(j)}(F^{(k)}). \tag{5.5a}$$

From (Props. XXXIV, XLIX) a general form for F_{n-1}, that is the (n-1) order term in $S_\pm \otimes S_\mp$ in the expansion of the twist element F, can be constructed from the extreme off-diagonal entries of $\mathbf{l}^{\pm(j)}(F)$ and will also take the form of a series in the n arbitrary variables $V^{(k)}$. That is

$$F_{n-1} \equiv \sum_{k=1}^{n} V^{(k)} F_{n-1}^{(k)}, \tag{5.5b}$$

where $v^{(k)} = \mathbf{l}^{\pm(j)}(V^{(k)})$, and each $F_{n-1}^{(k)}$ is expressed as a tensor product of the extreme off-diagonal entries of the corresponding matrix $\mathbf{l}^{\pm(j)}(F^{(k)})$.

Now let l be an arbitrary integer or half-integer with $p \equiv 2(j+l)+1$ so that $p > n$. The same element F_{n-1} can be constructed in a similar manner from higher order semirepresentations $\mathbf{l}^{\pm(j+l)}(F)$, and results in a series of p arbitrary variables $W^{(i)}; i = 1,\ldots,p$,

$$\mathbf{l}^{\pm(j+l)}(F) = \sum_{i=1}^{p} w^{(k)} \mathbf{l}^{\pm(j+l)}(F^{(k)}). \tag{5.6a}$$

It follows that

$$F_{n-1} \equiv \sum_{i=1}^{p} W^{(i)} F_{n-1}^{(i)}. \tag{5.6b}$$

Although the F_{n-1} will in this case be composed of tensor products of entries of $\mathbf{l}^{\pm(j+l)}(F)$ that are off-diagonal according to the magnitude $|p-n|$, the F_{n-1} will not be composed of exteme off-diagonal entries of $\mathbf{l}^{\pm(j+l)}(F)$.

The $n+p$, variables $V^{(k)}, W^{(i)}$ thus defined are not completely arbitrary. The requirement (Props. XIII, XLVIII) for $\mathbf{l}^{\pm(j)}(F), \mathbf{l}^{\pm(j+l)}(F)$ as semirepresentations of the same element F results in a narrowing of the arbitrariness of $V^{(k)}, W^{(i)}$ in that substitution of (5.5a, 5,6a) into the consistency conditions (2.13) and identification of $v^{(k)}, w^{(i)}$ with $V^{(k)}, W^{(i)}$ gives a set of sumultaneous equations which define the p

variables $W^{(i)}$ as linear combinations of the $n < p$ variables $V^{(k)}$. Specifically for *n*=2, the *p* variables $W^{(i)}; i = 1,\ldots, p$ of (5.6b) are defined in terms of only *two* arbitrary variables, say V', V''. Thus

$$W^{(i)} = W^{(i)}(V', V''), \tag{5.7}$$

$i = 1,\ldots, p$. Substituting in (5.6b) and rearranging then gives the individual terms in the form (5.1) for *F*.

Finally, for arbitrarily large *j*, it is not in general possible to give an explicit construction for each of the $n = 2j+1$ terms $F_{n-1}^{(k)}$ in (5.5b). However, the first ($F_{n-1}^{(1)}$) and *n*th ($F_{n-1}^{(n)}$) terms can be induced from F_1', F_1'', F_2', F_2'' (Props. XXXIV, XLIX), which leads to the form (5.2, 5.3) for F_n. The one complication is that the corresponding variables $V^{(1)}, V^{(n)}$ in (5.5b) cannot be immediately identified with the V', V'' of (5.1) unless the remaining $V^{(k)}$ are made to vanish, which is not allowed. But it is argued from the arbitrariness of these $V^{(k)}$ that they will always provide solutions to equations of the form (5.7) which will enable the constuction of a form for F_n as the *n*th order term of the twist element (5.1).●

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