PRESENTATIONS FOR SEMIGROUP CONSTRUCTIONS AND RELATED COMPUTATIONAL METHODS

Isabel M. Araújo

Ph.D. Thesis

University of St Andrews

December 2000

Aos meus Pais

Contents

Declaration

I, Isabel Maria de Mendonça Machado de Araújo, declare that this thesis has been composed by myself, that it is a record of my own work, and that has not been accepted in any previous application for any degree.

Name: Isabel M.M.M. Araújo **Signature** **Date:** 21/12/2000

I was admitted to the Faculty of Science of the University of St Andrews as a candidate for the degree of Doctor of Philosophy in January 1998.

Name: Isabel M.M.M. Araújo **Signature:**. **Date:** 21/12/2000

I certify that Isabel Maria de Mendonça Machado de Araújo has satisfied the conditions of the Resolution and Regulations and is thus qualified to submit the accompanying thesis in application for the degree of Doctor of Philosophy.

Name: Nik Ruškuc **Signature** ... **Date:** 21/12/2000

I agree that access to my thesis in the University of St Andrews shall be unrestricted.

Name: Isabel M.M.M. Araújo **Signature** **Date:** 21/12/2000

Preface

The beginnings of combinatorial semigroup theory date from the middle of last century, motivated by the study of combinatorial group theory. Nevertheless, it was only in recent years that the subject saw large expansion, and became quite independent. In particular, the study of semigroup presentations and related decidability questions have been widely considered in recent publications. Also important has been the development of several computational tools.

In this thesis we consider presentations for semigroup constructions. This is motivated by the fact that a common approach when studying a semigroup is to decompose it in terms of some other semigroups from which it is easier to obtain information. Sometimes this is done by means of some construction (for example we may find that a given semigroup is the direct product of two of its subsemigroups). Thus it is natural to ask how properties of the semigroup relate to properties of its 'pieces' and, conversely, how a given property is preserved under a certain construction. We consider these questions with respect to the property of being finitely generated and finitely presented for some common constructions and decompositions. We also consider some related decidability questions.

The last part of this thesis is concerned with finitely generated commutative semigroups. We explore some decidability issues for this class of semigroups.

We start by introducing some basic definitions and results about semigroups in Chapter 1. Then, in Chapter 2 we introduce the notion of finite generation, finite presentability, and some combinatorial techniques used throughout the thesis. We also discuss some computational methods used. Chapters 3, 4, 5 and 6 deal with semigroup constructions. Thus, in Chapter 3 we consider the direct product of semigroups; in particular we prove that it is decidable whether the direct product of a finite semigroup with a finitely presented infinite semigroup is finitely presented. We also characterize finitely generated and finitely presented direct products of a finite number of semigroups. In Chapter 4 we consider the semidirect product of semigroups, and prove some necessary and some sufficient conditions for it to be finitely generated and finitely presented. In Chapter 5 we characterize finitely generated and finitely presented Bruck–Reilly extensions of groups, and prove some partial results for Bruck–Reilly extensions of monoids. In Chapter 6 we consider unions, bands, semilattices and strong semilattices of semigroups and ask when they preserve finite generation and finite presentability. Finally, Chapter 7 deals with commutative semigroups. In there we consider several decidability questions for this class of semigroups, and develop algorithms to answer them. Appendix A contains the implementation in GAP of the algorithms developed in Chapter 3, while Appendix B contains the code for the some of the algorithms from Chapter 7. Appendix C introduces some results on polynomial rings that are used in Chapter 7.

In this thesis we present results that have been proved during the author's study period at St Andrews University, under the supervision of Dr. Nik Ruškuc. Many of the results here presented are part of the papers Araújo and Ruškuc [5], Araújo and Ruškuc [6], Araújo, Branco, Fernandes, Gomes and Ruškuc [3] and Araújo, Kelarev and Solomon [4]. The first of these papers has already been published, while the second and the third one have been accepted for publication. A survey, by the author, on finite presentability of semigroup constructions has also been accepted for publication [2]. In this thesis though we have tried to go further and to give the reader a more complete picture of the theory we develop. This has been achieved by answering some pertinent questions that arise naturally in the context of each chapter, as well as through illustrating the main text with several examples. We also present some results that are part of work in progress.

Acknowledgements

This thesis is the product of my three years of study at the University of St Andrews. This work would not have been possible without the support of many people to whom I would like to thank.

First I would like to thank my supervisor, Dr. Nik Ruškuc. His guidance, help, support and ideas were essential at every stage. His enthusiasm and enjoyment in doing 'mathematics' encouraged and made me enjoy carrying out research. I would also like to thank Prof. Edmund Robertson for his helpful advice and support. I also have to mention everyone in the Algebra Group, who have created an enjoyable environment to work in.

My interest in semigroups dates from a few years back, when I took a course on Semigroup Theory for my undergraduate degree. Thus I want to thank Prof. Gracinda Gomes, who taught that course, and hence was the first person to introduce me to this subject. Furthermore, her support, guidance and friendship have been very important since then.

I have been fortunate enough to collaborate with several researchers during the last three years. Thus, I would like to thank them all for the pleasant moments we have spent working together, as well for letting me present in this thesis the results of our joint work. Namely, I would like to thank my co-authors: Prof. Mário Branco, Prof. Vitor Fernandes, Prof. Gracinda Gomes, Dr. Andrei Kelarev, Dr. Nik Ruškuc and Dr. Andrew Solomon. I would also like to mention Prof. Edmund Robertson and Dr. Alexander Hulpke. I also thank Dr. David Easdown for his hospitality at the University of Sydney for the time I spent there.

I acknowledge the financial support of the University of St Andrews Scholarship during the first five months of my studies, and from then on the financial support of the Sub-Programa Ciência e Tecnologia do 2° Quadro Comunitário de Apoio (grant number BD/15623/98). I would also like to thank the support of the Centro de Álgebra da Universidade de Lisboa, Fundação para a Ciência e a Tecnologia, and of the Programa Ciência, Tecnologia e Inovação do Quadro Comunitário de Apoio.

Finally a word for my family and friends. I would like to thank my parents, who have always supported me and encouraged me, even when that meant being very far away! I would also like to thank my sister and all my friends from Portugal, whose encouragement was always very important. Finally, I would like to mention all the friends that I have met during the last three years in St Andrews who supported me and made my time here fun and enjoyable.

Abstract

The main question that we consider in this thesis is how finite presentability of semigroups is preserved under a semigroup construction and decidability issues related to this problem. We also consider decidability questions for commutative semigroups and develop algorithms accordingly.

Thus, after introducing basic definitions in Chapter 1 and some techniques to deal with semigroup presentations and combinatorial semigroup theory in Chapter 2, we deal with semigroup constructions in Chapters 3, 4, 5 and 6. The constructions considered are direct products of semigroups, semidirect products of semigroups, Bruck–Reilly extensions of monoids and strong semilattices of semigroups, respectively. In Chapter 6 we also consider general unions of semigroups, as well as bands of semigroups and monoids, and semilattices of semigroups.

In Chapter 7 we study commutative semigroups from a computational point of view. Thus, given a commutative semigroup defined by a presentation we prove that several decidability questions have a positive answer and develop algorithms to compute those answers.

Chapter 1

Introduction to the study of semigroups

In this chapter we introduce the reader to the main concepts we will use throughout this thesis. Thus, in Section 1.1 we introduce basic definitions and state some basic results. Section 1.2 deals with fundamentals about Green's relations. In Section 1.3 we define some algebraic properties of semigroups and introduce some special classes of semigroups to which we shall refer later. Finally, in Section 1.4, we introduce the main concept that we will be dealing with: semigroup presentations.

The material here presented can be found in any major introductory text such as Grillet [27], Howie [28] or Lallement [32].

1.1 Basic definitions

A *semigroup* is a set S together with an associative binary operation (normally referred to as *multiplication*) on S. We say that S is *commutative* if the multiplication is commutative.

An element $e \in S$ is said to be an *idempotent* if $e = e^2$. The set of idempotent elements from S is denoted by $E(S)$. If all the elements of S are idempotents then S is a *band*; if furthermore S is commutative then S is a *semilattice*.

A semigroup may have no idempotent elements. On the other hand it may have some idempotents with special properties. Thus, if e_l (resp. e_r) is an element of S such that $e_l s = s$ (resp. $se_r = s$), for all $s \in S$, then e_l (resp. e_r) is called a *left identity* (resp. *right identity*) of S. If $e \in S$ is simultaneously a left and a right identity then we say that e is an *identity* of S. It is easy to see that a semigroup may have at most an identity. Whenever a semigroup has an identity then we call it a *monoid* and denote its identity by 1_S (or simply 1). If z_l (resp. z_r) is an element of S such that $z_l s = z_l$ (resp. $sz_r = z_r$), for all $s \in S$, then z_l (resp. z_r) is a *left zero* (resp. *right zero*). If $z \in S$ is both a left and a right zero, then z is a *zero* of S. A semigroup may contain at most one zero. If S contains a zero we say that S is a *semigroup with zero* and denote its zero by 0_S (or simply 0).

We may adjoin an identity to S to obtain a monoid. Thus we define the monoid S^1 to be equal to S if S has an identity, or to be equal to S with an identity adjoined otherwise and we define the monoid S^I to be S with an identity adjoined, regardless of whether S has already an identity. (This procedure of adjoining an identity is often used to prove results for semigroups from known results for monoids.) Also, we let S^0 be the semigroup obtained from S by adjoining a zero if S does not contain one, or be S itself otherwise.

For two subsets $A, B \subseteq S$ we write AB to denote the set

$$\{ab : a \in A, b \in B\}.$$

As usual, if $B = \{b\}$, we write Ab instead of $A\{b\}$.

A non-empty subset T of S is a *subsemigroup* of S if T is closed under multiplication. Examples of subsemigroups of S are S itself and $E(S)$, if S is commutative. The semigroup S is a *group* if for all $s \in S$

$$Sa = S \text{ and } aS = S$$

(this definition is equivalent to the usual definition of a group). We say that a subset T of S is a *subgroup* of S if T is a group with respect to the operation defined on S. Now, if S is a monoid and G is its largest subgroup containing 1_S then G is called the *group of units* of S, and the elements of G are called *units* (they are in fact the elements of S which have inverses in the group sense).

Now, if A is a non-empty subset of S, we define the *subsemigroup generated by A* to be the intersection of all subsemigroups of S which contain A; we denote it by $\langle A \rangle$. In fact we have the following

Proposition 1.1 *Let S be a semigroup and let $A \subseteq S$. The subsemigroup generated by A is the set of all products of one or more elements of A.*

A proof of the above stated fact can be found in, for example, [27].

If $A \subseteq S$ is such that the subsemigroup of S generated by A is S itself then we say that A *generates* S, or A is a generating set for S, and call the elements of A the *generators of* S. Notice that every semigroup has a generating set: the whole set of elements of the semigroup. We say that $s \in S$ is an *indecomposable element* if s cannot be written as a product of two elements of S. Any generating set for S must contain the set of all indecomposable elements of S.

A non-empty subset I of S is called a *left ideal* if $SI \subseteq I$, a *right ideal* if $IS \subseteq I$, and a (two-sided) *ideal* if I is both a left and a right ideal. It is easy to see that a non-empty intersection of left (resp. right, two-sided) ideals is again a left (resp. right, two-sided) ideal. Thus, we define the *left* (resp. *right, two-sided*) *ideal generated by a non-empty subset* X of S to be the intersection of all left (resp. right, two-sided) ideals of S which contain X. The left ideal generated by X is actually equal to S^1X, while the right ideal generated by X is XS^1 and the two-sided ideal generated by X is S^1XS^1. If $X = \{x\}$ is a singleton set then we call the left (resp. right, two-sided) ideal generated by X the *principal left* (resp. *right, two-sided*) *ideal of* x. Examples of ideals of S are S itself and $\{0\}$, if S contains a zero 0.

Let S and T be semigroups and let $\phi : S \to T$ be a mapping between them. We say that ϕ is a (*semigroup*) *morphism* (or *homomorphism*) if for all $s_1, s_2 \in S$

$$(s_1s_2)\phi = (s_1\phi)(s_2\phi).$$

If both S and T are monoids we say that the mapping ϕ is a *monoid morphism* if the following condition is also satisfied:

$$1_S\phi = 1_T.$$

We will say morphism for monoid morphism whenever no confusion arises. The *image* of ϕ is the set $S\phi = \{s\phi : s \in S\}$, and this is a subsemigroup of T; if $S\phi = T$ (i.e. ϕ is onto) then we say that ϕ is an *epimorphism*, and that T is a *homomorphic image of* S. If ϕ is one-one than we say that ϕ is a *monomorphism* or an *embedding*. In this case we also say that S *embeds in* T and we write $S \hookrightarrow T$. Finally, ϕ is an *isomorphism* if it is both a monomorphism and an epimorphism. This is equivalent to the existence of a morphism $\phi^{-1} : T \to S$ such that $\phi\phi^{-1}$ is the identity mapping on S and $\phi^{-1}\phi$ is the identity mapping on T. If ϕ is an isomorphism then we write $S \cong T$ and say that S and T are *isomorphic*.

If ϕ is a morphism from S into itself we call it an *endomorphism*. If, moreover, ϕ is an isomorphism, we call it an *automorphism*. The set of all endomorphisms of S is a monoid, under the operation of composition of mappings, and we denote it by $\mathrm{End}(S)$. The set of all automorphisms is a subgroup of $\mathrm{End}(S)$.

If $\phi : S \to T$ is a mapping between two semigroups S and T satisfying

$$(s_1 s_2)\phi = (s_2\phi)(s_1\phi)$$

for all $s_1, s_2 \in S$ then we say that ϕ is an *antihomomorphism*.

A *left action of a semigroup* S *on a set* X is a mapping

$$\tau : S \times X \to X$$

such that

$$(s_1, (s_2, x)\tau)\tau = (s_1 s_2, x)\tau \tag{1.1}$$

(for $s_1, s_2 \in S$, $x \in X$). When there is no danger of confusion we write ${}^{s}x$ (or sx) for $(s, x)\tau$. With this notation (1.1) becomes

$${}^{s_1}({}^{s_2}x) = {}^{s_1 s_2}x.$$

We then say that S *acts on* X. Any semigroup acts on itself by pre-multiplication. A *right action* is defined dually.

We now give some examples of well-known semigroups.

Example 1.2 The set $\mathbb{N}$ of natural numbers is a semigroup with respect to addition. Moreover, it is a commutative semigroup, which contains no identity, has no idempotents and is generated by the element 1 (which is, in fact, its only indecomposable element). By adjoining an identity to this semigroup we obtain $\mathbb{N}_0$, the semigroup of all non-negative integers.

Example 1.3 Let S be a semigroup which is generated by a single element $a \in S$. Then S is called a *monogenic semigroup*. If for all $m \neq n$, $a^m \neq a^n$ then S is clearly isomorphic to $\mathbb{N}$ (with respect to addition) and we call it an *infinite monogenic semigroup*. Otherwise, if there are repetitions among the powers of a, then S is finite. In particular, any cyclic group C_n of finite order n is a (finite) monogenic semigroup.

Example 1.4 Let X be a set and let T_X be the set of all mappings $X \to X$. Then T_X is a semigroup with respect to composition of mappings (which is an associative operation) which is called the *full transformation semigroup*. If $X = \{1, 2, \ldots, n\}$ then we write T_n instead of T_X. The full transformation semigroup plays a similar role in semigroup theory to the one of the symmetric group in group theory: every semigroup is a subsemigroup of some transformation semigroup.

1.2 Green's relations

On a semigroup S we consider five special equivalence relations, defined in terms of its ideal structure. Thus, for any $s_1, s_2 \in S$ we have $s_1 \mathcal{R} s_2$ if s_1 and s_2 generate the same principal right ideal (i.e. $s_1 S^1 = s_2 S^1$), $s_1 \mathcal{L} s_2$ if s_1 and s_2 generate the same principal left ideal (i.e. $S^1 s_1 = S^1 s_2$) and $s_1 \mathcal{J} s_2$ if s_1 and s_2 generate the same principal (two-sided) ideal (i.e. $S^1 s_1 S^1 = S^1 s_2 S^1$). We denote by $\mathcal{H}$ the intersection of $\mathcal{R}$ and $\mathcal{L}$, and by $\mathcal{D}$ their join. We write $\mathcal{R}^S$, $\mathcal{L}^S$, $\mathcal{H}^S$, $\mathcal{D}^S$, $\mathcal{J}^S$ whenever we need to make clear in which semigroup the relations are considered.

Now, these five equivalences are related as follows:

$$\mathcal{H} \subseteq \mathcal{R},\ \mathcal{L} \subseteq \mathcal{D} \subseteq \mathcal{J}.$$

In a group G all these equivalences are equal to the universal equivalence $G \times G$. In a finite semigroup $\mathcal{D} = \mathcal{J}$.

For a given $s \in S$ we denote the $\mathcal{R}$–class (resp. $\mathcal{L}$–class, $\mathcal{H}$–class, $\mathcal{D}$–class, $\mathcal{J}$–class) of s by R_a (resp. L_a, H_a, D_a, J_a). The relations $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{J}$ induce partial orders among equivalence classes, namely

$$\begin{aligned}
R_s \leq R_t &\quad \text{if} \quad sS^1 \subseteq tS^1, \\
L_s \leq L_t &\quad \text{if} \quad S^1 s \subseteq S^1 t, \text{ and} \\
J_s \leq J_t &\quad \text{if} \quad S^1 s S^1 \subseteq S^1 t S^1
\end{aligned}$$

$(s, t \in S)$. The sets $S/\mathcal{R}$, $S/\mathcal{L}$ and $S/\mathcal{J}$ are partially ordered sets. This also induces pre-orders $\leq_{\mathcal{R}}$, $\leq_{\mathcal{L}}$, $\leq_{\mathcal{J}}$ on the elements of S.

Now, relations $\mathcal{R}$, $\mathcal{L}$, $\mathcal{H}$ and $\mathcal{D}$ relate as follows: each $\mathcal{D}$–class is a union of $\mathcal{R}$–classes as well as a union of $\mathcal{L}$–classes and the intersection of an $\mathcal{R}$–class with an $\mathcal{L}$–class is either an $\mathcal{H}$–class or is empty. In fact, the intersection of an $\mathcal{R}$–class with an $\mathcal{L}$–class is non-empty if and only if they are contained in the same $\mathcal{D}$–class. Furthermore, if $s, t \in S$ are such that $s\mathcal{D}t$ then $|H_s| = |H_t|$. This is a consequence of the following important

Lemma 1.5 (Green's Lemma) *Let S be a semigroup and let $a, b \in S$.*

(i) If $a\mathcal{L}b$ and $u, v \in S^1$ are such that $ua = b$ and $vb = a$ then the mappings $\lambda_u : R_a \to R_b$, $x \mapsto ux$, and $\lambda_v : R_b \to R_a$, $y \mapsto vy$ are mutually inverse $\mathcal{L}$-class preserving bijections.

(ii) If $a\mathcal{R}b$ and $u, v \in S^1$ are such that $au = b$ and $bv = a$ then the mappings $\rho_u : L_a \to L_b$, $x \mapsto xu$, and $\rho_v : L_b \to L_a$, $y \mapsto yv$ are mutually inverse $\mathcal{R}$-class preserving bijections.

Also if H is an $\mathcal{H}$–class of S then either $H^2 \cap H = \emptyset$ or $H^2 = H$ and H is a subgroup of S. It immediately follows that H is a group if and only if H contains an idempotent, and that no $\mathcal{H}$–class of S can contain more then one idempotent. Furthermore, if an $\mathcal{H}$–class H of S is a group, then it is a maximal subgroup of S (with respect to inclusion).

We refer the reader to [28] or [32] to a more detailed introduction to the study of Green's relations on a semigroup.

1.3 Some algebraic properties

In this section we introduce some algebraic properties of semigroups. This will allow us to define some important classes of semigroups, to which we will refer throughout the text.

An element $s \in S$ is said to be *regular* if there exists $x \in S$ such that

$$sxs = s.$$

If all the elements of S are regular then S is a *regular* semigroup. Furthermore, if S is regular and for each $s \in S$ there exists a unique $s^{-1} \in S$ satisfying

$$ss^{-1}s = s \text{ and } s^{-1}ss^{-1} = s^{-1}$$

then S is said to be an *inverse semigroup* (and s^{-1} is called the *inverse* of s in S). Equivalently, S is an inverse semigroup if S is regular and $E(S)$ is a commutative subsemigroup of S. Clearly every group is an inverse semigroup. A semigroup S is said to be *completely regular* if every element is in a subgroup of S. A semigroup S is a *Clifford semigroup* if it is regular and all its idempotents are central (i.e. $ea = ae$ for every $s \in S$, $e \in E(S)$).

If S is a semigroup without zero then S is called a *simple semigroup* if it only has one ideal: S itself. If S has a zero 0, then S is a *0-simple* semigroup if S and $\{0\}$ are its only ideals and $S^2 \neq 0$. A simple semigroup S is a *completely simple* semigroup if S contains at least one minimal left ideal and one minimal right ideal (with respect to inclusion). A 0-simple semigroup S is a *completely 0-simple* semigroup if it contains at least one 0-minimal left ideal and at least one 0-minimal right ideal. (A 0-minimal left/right ideal is a minimal ideal, with respect to inclusion, within the set of non zero left/right ideals of S.) A semigroup S is said to be *bisimple* if it contains only one $\mathcal{D}$–class.

Since the idempotents of an inverse semigroup S commute, $E(S)$ is closed under multiplication and thus it is a semilattice. Moreover, on the set of idempotents of an arbitrary semigroup a natural partial order relation is defined by

$$e \leq f \iff ef = fe = e.$$

An inverse semigroup S is an *ω-semigroup* if its semilattice of idempotents is isomorphic to C_ω (the semilattice $\{e_i : i \in \mathbb{N}_0\}$ with multiplication given by $e_i e_j = e_{\max\{i,j\}}$ $(i, j \in \mathbb{N}_0)$).

A semigroup S is a *null semigroup* if it contains a zero 0, and all products $s_1 s_2$ are equal to 0 $(s_1, s_2 \in S)$. A semigroup S is a *left* (resp. *right*) *zero semigroup* if for all $s_1, s_2 \in S$

$$s_1 s_2 = s_1 \text{ (resp. } s_1 s_2 = s_2).$$

Notice that if S is a null semigroup then $S \backslash \{0\}$ is the smallest subset of S that generates S and if S is a left zero semigroup or a right zero semigroup then no proper subset of S generates S.

The definitions here introduced cover some classes of semigroups which have been widely studied. Most of those classes relate to regular semigroups or inverse semigroups. Petrich's [39] book and Meakin's survey [36] are basic texts dealing with the study of inverse semigroups. Much of Howie's introductory book [28] deals with properties of regular semigroups. More recently, Petrich and Reilly [40] give an exhaustive account of work done on completely regular semigroups (including completely simple semigroups).

1.4 Presentations

One of the fundamental concepts that we will be dealing with in this thesis is the one of presentation that we shall now introduce. In order to do that we start by defining a class of semigroups of major importance.

Thus, given a set A (which we call *alphabet*) we denote by A^+ the set of all finite non-empty words over A, and by A^* the set A^+ together with the empty word; we denote the empty word by 1 or ε. Both A^+ and A^* are semigroups with respect to the operation of concatenation of words. We call A^+ the *free semigroup on* A and A^* the *free monoid on* A. Clearly, A generates A^+ as a semigroup and A^* as a monoid. Any word $w \in A^+$ (or $w \in A^*$) can be written uniquely as a product of elements of A; we call the number of elements of A in such product the *length* of w, and we denote it by $|w|$. Also, for a word $w \in A^+$ (or A^*) and an element $a \in A$, we denote by $n(w, a)$ the number of occurrences of a in w.

The following Proposition and Corollary illustrate the universality of free semigroups.

Proposition 1.6 *Let A be an alphabet and let S be a semigroup. Then every mapping $\phi : A \to S$ extends uniquely to a homomorphism*

$$\overline{\phi} : A^+ \to S.$$

Moreover, the image of $\overline{\phi}$ is the subsemigroup of S generated by $A\phi$. In particular, S is generated by $A\phi$ if and only if $\overline{\phi}$ is surjective.

Corollary 1.7 *Every semigroup is a homomorphic image of a free semigroup.*

Proofs for these results can be found in, for example, [32]. By replacing 'semigroup' by 'monoid' in the above proposition and corollary, we obtain the similar results valid for A^*.

Now, let A be an alphabet. A *semigroup presentation* is a pair $\langle A \mid R\rangle$, where $R \subseteq A^+ \times A^+$. An element from A is called a *generating symbol* and an element $(u, v) \in R$ is called a *defining relation*, and is usually written as $u = v$. The semigroup presentation $\langle A \mid R\rangle$ defines the semigroup $S = A^+/\rho$, where ρ is the smallest congruence on A^+ containing R. More generally, we say that a semigroup S is defined by a presentation $\langle A \mid R\rangle$ if $S \cong A^+/\rho$. Very often we identify a word $w \in A^+$ and the element $w/\rho \in S$ it represents (and hence we also identify the generating symbols of the presentation with the generators of the semigroup). For any two words $w_1, w_2 \in A^+$ we write $w_1 \equiv w_2$ if they are identical words and $w_1 = w_2$ if they represent the same element of M (i.e. if $w_1/\rho = w_2/\rho$), in which case we say that S *satisfies the relation* $w_1 = w_2$.

For two words $w_1, w_2 \in A^+$, we say that w_2 *is obtained from* w_1 *by one application from one relation from* R if there exist $\alpha, \beta \in A^*$ and $(u, v) \in R$ such that $w_1 \equiv \alpha u \beta$ and $w_2 \equiv \alpha v \beta$. We say that the relation $w_1 = w_2$ is a *consequence of the set of relations* R, if there is a finite sequence

$$w_1 \equiv \gamma_1, \gamma_2, \ldots, \gamma_k \equiv w_2$$

of words from A^+, in which every γ_i $(1 < i \leq k)$ is obtained from γ_{i-1} by one application of one relation from R. Such a sequence is called *an elementary sequence with respect to* $\langle A \mid R\rangle$.

Proposition 1.8 [55, Proposition 2.3] *Let S be a semigroup, let A be a generating set for S and let $R \subseteq A^+ \times A^+$. Then S is defined by the presentation $\langle A \mid R\rangle$ if and only if S satisfies all the relations from R and for any two words $w_1, w_2 \in A^+$, such that $w_1 = w_2$ in S, $w_1 = w_2$ is a consequence from R.*

Presentations are universal means of defining semigroups, in the sense that every semigroup can be defined by a presentation. Indeed, an arbitrary semigroup S is defined by the presentation $\langle A \mid R\rangle$, where

$$A = \{a_s : s \in S\}$$

is an alphabet in one-one correspondence with S and

$$R = \{a_x a_y = a_{xy} : x, y \in S\} \subseteq A^+ \times A^+.$$

To see this let T be the semigroup defined by $\langle A \mid R \rangle$. By the definition of R, all those relations are satisfied by S, and thus there exists an epimorphism

$$\begin{aligned} \phi : T &\to S \\ a_s &\mapsto s \ (s \in S). \end{aligned}$$

Now, if $w_1, w_2 \in A^+$ are such that $w_1\phi = w_2\phi$ then there exists $x, y \in S$ such that $w_1 = a_x$ and $w_2 = a_y$. Hence $x = y$, and therefore ϕ is an isomorphism, which proves that $\langle A \mid R \rangle$ defines S. We call this presentation *the multiplication table presentation of* S, and denote it by $\mathcal{M}(S)$. In line with the convention of identifying words with the elements of the semigroup that they represent, we normally take S to be the set of generating symbols for $\mathcal{M}(S)$. Finally, notice that if S is a finite semigroup, both the set of generators and the set of relations of $\mathcal{M}(S)$ are finite.

By replacing A^+ by A^* in the above, we can define a *monoid presentation*. All the definitions and results stated are still true with the obvious adaptations. Also notice that if S is a monoid defined by a monoid presentation $\langle A \mid R \rangle$ then S is defined by the (semigroup) presentation

$$\langle A, e \mid R, ae = ea = a \ (a \in A) \rangle.$$

Also, if S is a monoid defined by a semigroup presentation $\langle A \mid R \rangle$ then there exists a word $w \in A^+$ representing the identity of S. Hence S is defined by the monoid presentation $\langle A \mid R, w = 1 \rangle$.

Whenever it is clear whether the presentation considered is a semigroup or a monoid presentation, we shall refer to it simply as a presentation.

Example 1.9 From Example 1.2 we know that $\mathbb{N}$ is a semigroup with respect to addition, and $\mathbb{N}_0$ is a monoid. Actually, $\mathbb{N}$ is the free semigroup on an alphabet consisting of a single element $\{a\}$, and it is defined by the presentation $\langle a \mid \rangle$ (with no relations). Similarly, $\mathbb{N}_0$ is the free monoid on the same alphabet, and it is defined by the monoid presentation $\langle a \mid \rangle$. More generally, the free semigroup (resp. free monoid) on an alphabet A is defined by the semigroup (resp. monoid) presentation with generators A and no relations.

Example 1.10 The cyclic group of order n is defined by the monoid presentation

$$\langle a \mid a^n = 1 \rangle.$$

On the other hand, if S is a monogenic finite semigroup on a, and $m, n \geq 1$ are the smallest integers such that $m \neq n$ and $a^m = a^n$, then S is defined by the presentation

$$\langle a \mid a^m = a^n \rangle.$$

Example 1.11 Consider the monoid B defined by the presentation

$$\langle b, c \mid bc = 1 \rangle.$$

It is an infinite monoid. Moreover, every word $w \in \{b, c\}^*$ is equal to an element $c^i b^j$ for some $i, j \geq 0$. Moreover, two words $c^i b^j, c^h b^k \in \{b, c\}^*$ are equal in B if and only if $i = j$ and $h = k$. This is called the *bicyclic monoid*.

We now define another two types of presentations that will appear later.

Let A be an alphabet. The *free group* on A is the monoid defined by the (monoid) presentation

$$\langle A, A^{-1} \mid aa^{-1} = a^{-1}a = 1 \ (a \in A) \rangle$$

where $A^{-1} = \{a^{-1} : a \in A\}$ is a set in one-one correspondence with A, disjoint from A. A *group presentation* is a pair $\langle B \mid Q \rangle$, where B is an alphabet, $B^{-1} = \{b^{-1} : b \in B\}$ is another alphabet disjoint from B and in one-one correspondence with it, and $Q \subseteq (B \cup B^{-1})^* \times (B \cup B^{-1})^*$. The group defined by the presentation $\langle B \mid Q \rangle$ is the monoid defined by the presentation

$$\langle B, B^{-1} \mid Q, bb^{-1} = b^{-1}b = 1 \ (b \in B) \rangle.$$

The *free inverse semigroup on A* is the semigroup I_A defined by the (semigroup) presentation

$$\langle A, A^{-1} \quad | \quad ww^{-1}w = w, \ ww^{-1}zz^{-1} = zz^{-1}ww^{-1} \ (w, z \in (A \cup A^{-1})^+) \rangle$$

where $A^{-1} = \{a^{-1} : a \in A\}$ is a set in one-one correspondence with A, disjoint from A and if $w \equiv a_1a_2 \cdots a_k \in A^+$ then $w^{-1} \equiv a_k^{-1} \cdots a_2^{-1}a_1^{-1} \in (A^{-1})^+$. An *inverse semigroup presentation* is a pair $\langle B \mid Q\rangle$, where B is an alphabet, $B^{-1} = \{b^{-1} : b \in B\}$ is another alphabet disjoint from B and in one-one correspondence with it, and $Q \subseteq (B \cup B^{-1})^+ \times (B \cup B^{-1})^+$. The inverse semigroup defined by the presentation $\langle B \mid Q\rangle$ is the semigroup defined by the presentation

$$\langle B, B^{-1} \mid Q,\ ww^{-1}w = w,\ ww^{-1}zz^{-1} = zz^{-1}ww^{-1}\ (w, z \in (B \cup B^{-1})^+)\rangle$$

(and if $w \equiv b_1b_2 \cdots b_k \in B^+$ then $w^{-1} \equiv b_k^{-1} \cdots b_2^{-1}b_1^{-1}$). Similarly we can define the *free inverse monoid* and an *inverse monoid presentation.*

Given two presentations defining the same semigroup, one may ask how they are related to each other. This problem leads to the introduction of Tietze transformations. These are operations that we may perform on a presentation without changing the semigroup defined by it. So, given a presentation $\langle A \mid R\rangle$ we obtain a new presentation $\langle A' \mid R'\rangle$ by one of the following transformations:

(T1) given $b \notin A$ and $w \in A^+$ we let $A' = A \cup \{b\}$ and $R' = R \cup \{b = w\}$;

(T2) if R contains a relation of the form $b = w$ where $b \in A$, and $w \in (A \setminus \{b\})^+$ we define a homomorphism $\phi : A^+ \to (A \setminus \{b\})^+$ by $a\phi = a$ for $a \neq b$, $b\phi = w$ and let $A' = a \setminus \{b\}$, $R' = (R \setminus \{b = w\})\phi$;

(T3) if $u = v$ is a relation which is a consequence of R we let $A' = A$, $R' = R \cup \{u = v\}$; or

(T4) if R contains a relation $u = v$ such that $u = v$ is a consequence of $R \setminus \{u = v\}$ then we let define $A' = A$, and $R' = R \setminus \{u = v\}$.

We then have the following theorem (see, for example, [55]).

Theorem 1.12 *Let $\langle A \mid R\rangle$ and $\langle B \mid Q\rangle$ be two presentations with A, B, R and Q finite. Then $\langle A \mid R\rangle$ and $\langle B \mid Q\rangle$ define the same semigroup if and only if one can be obtained from the other by a finite sequence of Tietze transformations.*

Chapter 2

Techniques in combinatorial semigroup theory

In this chapter we introduce some tools in combinatorial and computational semigroup theory that we shall be using throughout the text.

Although any semigroup can be defined by a presentation, we focus our attention on the ones that have finite presentations. Indeed, that is one of the reasons why semigroup presentations are useful: they are ways of dealing with infinite objects in a finite way.

In Section 2.1 we introduce the concept of finite generation and finite presentability and we prove some basic and some fundamental results. In Section 2.2 we consider the question of finding presentations for subsemigroups and subgroups of a semigroup defined by a presentation. In Section 2.3 we discuss some of the computational methods used throughout the thesis and the software used to produce some of the examples illustrating the text.

2.1 Finite presentability

Let S be a semigroup. We say that S is *finitely generated* if there exists a finite set $A \subseteq S$ such that A generates S. Examples of finitely generated semigroups are finite semigroups and a free semigroup over a finite alphabet. The following is a well known fact:

Proposition 2.1 *Let S be a finitely generated semigroup and let T be a homomorphic image of S. Then T is finitely generated as well.*

We say that S is *finitely presented* if S is defined by a presentation $\langle A \mid R \rangle$, with both A and R finite. The property of being finitely presented is invariant of generating set, that is if S is finitely presented with respect to a finite generating set A, then for any finite generating set B of S, there exists a finite set of relations Q such that S is defined by the presentation $\langle B \mid Q \rangle$. In fact, the following stronger result holds:

Proposition 2.2 *Let S be a semigroup defined by a finite presentation $\langle A \mid R \rangle$, and let B be another finite generating set for S. If S is defined by the presentation $\langle B \mid Q \rangle$, for a set of relations $Q \subseteq B^+ \times B^+$, then Q has a finite subset Q' such that S is defined by $\langle B \mid Q' \rangle$.*

Examples of finitely presented semigroups are free semigroups, monogenic semigroups and finite semigroups. Example 2.4 below exhibits a non finitely presented semigroup.

In Section 1.4 we defined different notions of presentations. Accordingly we can define different notions of finite presentability. Thus, if S is a monoid (resp. inverse semigroup, inverse monoid, group), we say that S is *finitely presented as a monoid* (resp. *finitely presented as an inverse semigroup*, *finitely presented as an inverse monoid*, *finitely presented as a group*) if it can be defined by a monoid presentation (resp. inverse semigroup presentation, inverse monoid presentation, group presentation) $\langle A \mid R \rangle$ with both A and R finite. Clearly we have the following

Proposition 2.3 *A group is finitely presented as a group if and only if it is finitely presented as a monoid. A monoid is finitely presented as a monoid if and only if it is finitely presented as a semigroup.*

Thus, when referring to finitely presented groups or monoids we shall simply say they are finitely presented.

A similar result is not true for inverse semigroups or inverse monoids. The most natural example is the following:

Example 2.4 Let I_x be the free inverse semigroup on $\{x\}$. Then I_x is defined by the inverse semigroup presentation $\langle x \mid \rangle$, and hence it is finitely presented as an inverse semigroup. Nevertheless, it is not finitely presented as a semigroup. (This was proved by Schein [59]. A proof of this can be found in [39].)

Similarly there are inverse monoids which are finitely presented as inverse monoids and are not finitely presented as monoids. The following proposition gives a sufficient condition for an inverse monoid to be finitely presented as a monoid:

Proposition 2.5 [57, Corollary 4.8] *An inverse monoid with finitely many $\mathcal{R}$– and $\mathcal{L}$– classes is finitely presented as an inverse monoid if and only if it is finitely presented as a monoid.*

Remark 2.6 The notion of finite generation has the obvious analogue for monoids, groups, and inverse semigroups or monoids. However, all these notions coincide in the sense that a monoid (resp. inverse semigroup, inverse monoid, group) is finitely generated as a monoid (resp. inverse semigroup, inverse monoid, group) if and only if it is finitely generated as a semigroup.

We end this section by presenting the following Proposition, which relates finite presentability of a semigroup and a 'large' subsemigroup of it.

Proposition 2.7 [56, Theorem 1.1 and 1.3] *Let T be a semigroup and let S be a subsemigroup of T such that $T \backslash S$ is finite. Then T is finitely generated (resp. finitely presented) if and only if S is finitely generated (resp. finitely presented). In particular, for a semigroup S, S^I is finitely presented if and only if S is finitely presented.*

2.2 Rewriting semigroup presentations

In this section we consider the problem of finding a presentation for a subsemigroup of a semigroup defined by a presentation. Finding presentations for subgroups of groups is a classical problem in combinatorial group theory, which was solved by Reidemeister and Schreier (see [63]). The semigroup case is solved by a modification of that method (see [13, 14, 15]).

So, let S be the semigroup defined by the presentation $\langle A \mid R\rangle$ and let T be the subsemigroup of S generated by the set

$$X = \{\zeta_i \mid i \in I\} \subseteq A^+.$$

The aim is to obtain a presentation for T in terms of the generators X. The first step is to introduce a new alphabet $B = \{b_i \mid i \in I\}$ in one-one correspondence $b_i \mapsto \zeta_i$ $(i \in I)$ with the set X (intuitively b_i $(i \in I)$ is an abstract image of the generator $\zeta_i \in X$) and consider the unique homomorphism

$$\pi : B^+ \to A^+$$

extending this correspondence. With this notation, given $u, v \in B^+$, to say that $u = v$ holds in S means that $u\pi = v\pi$ in S. We denote by $\mathfrak{L}(A,T)$ the set of all words in A^+ representing elements of T. Clearly, any word in $\mathfrak{L}(A,T)$ can be represented as an element of B^+, that is, there exists a mapping

$$\phi : \mathfrak{L}(A,T) \to B^+$$

with the property that, for all $w \in \mathfrak{L}(A,T)$, we have

$$(w\phi)\pi = w \text{ in } S.$$

Such a mapping is called a *rewriting mapping*. Once we know a rewriting mapping we can establish a presentation for T:

Theorem 2.8 [13, Theorem 2.1] *With the above notation, T is defined by the presentation with generators B and relations*

$$\begin{aligned} b_i &= \zeta_i\phi, \\ (w_1w_2)\phi &= (w_1\phi)(w_2\phi), \\ (w_3uw_4)\phi &= (w_3vw_4)\phi, \end{aligned}$$

where $i \in I$, $w_1, w_2 \in \mathfrak{L}(A,T)$, $(u = v) \in R$, $w_3, w_4 \in A^*$, and $w_3uw_4 \in \mathfrak{L}(A,T)$.

Notice that this presentation is in general infinite. However in many particular cases it is possible to find an equivalent finite presentation. Usually the crucial step is to find a suitable rewriting mapping.

Thus, we now turn to the special case of finding a presentation for the subgroup of a monoid. (This was studied in [57], and generalizes to subgroups of semigroups by the usual technique of adjoining an identity.) So, let M be a monoid and let G be a subgroup of M (the identity of G is an idempotent e of M, not necessarily equal to 1_M). The monoid M acts on the set $\{Gs : s \in M\}$ by right multiplication. A set Gs ($s \in M$) is called a *right coset of G in M* if there exists $t \in M$ such that $Gst = G$. Thus let $\mathcal{C} = \{G_i : i \in I\}$ be the set consisting of all cosets of G in M. Notice that if $Gs \notin \mathcal{C}$ ($s \in M$) then $Gst \notin \mathcal{C}$ for every $t \in M$ (for otherwise there would exist $u \in M$ such that $G(st)u = G$ and hence also $Gs(tu) = G$). Therefore there is an action of S on $I \cup \{0\}$ (where we assume that $0 \notin I$) defined by

$$is = \begin{cases} j & \text{if } G_i s = G_j \\ 0 & \text{if } G_i s \notin \mathcal{C} \end{cases} \quad (i \in I),$$

$$0s = 0.$$

Lemma 2.9 [57, Proposition 2.4] *For each $i \in I$ there is $r_i, r_i' \in M$ such that $Gr_i = G_i$ and $gr_ir_i' = g$ for all $g \in G$. Furthermore r_i, r_i' can be chosen such that whenever $r_i \equiv \alpha\beta$ ($\alpha, \beta \in A^*$) then $\alpha \equiv r_j$ for some $j \in I$.*

We fix $r_i, r_i' \in M$ according to Lemma 2.9. We call the collection of such elements r_i, r_i' a *Schreier system of coset representatives.*

Theorem 2.10 [57, Theorem 2.7] *If M is generated by a set A then G is generated, as a monoid, by the set*

$$Y = \{er_iar_{ia}' : i \in I,\ a \in A,\ ia \neq 0\}.$$

Let M be defined by a (monoid) presentation $\langle A \mid R\rangle$. We introduce a new alphabet

$$B = \{[i, a] : i \in I,\ a \in A,\ ia \neq 0\}$$

representing the generating set Y of G, and define a mapping

$$\phi : \{(i, w) : i \in I,\ w \in A^*,\ iw \neq 0\} \to B^*$$

inductively by

$$(i, 1)\phi = 1,\ (i, aw)\phi = [i, a]\phi(ia, w)$$

$(i \in I, a \in A, w \in A^*, iaw \neq 0)$. Then we have

Theorem 2.11 [57, Corollary 2.15] *With the above notation, the presentation*

$$\begin{aligned}\langle B \mid\ & (i, u)\phi = (i, v)\phi\ (i \in I,\ (u = v) \in R,\ iu \neq 0),\\ & [i, a] = 1\ (i \in I,\ a \in A,\ r_i a \equiv r_j \textit{ for some } j \in I)\rangle\end{aligned}$$

defines G as a group.

The presentation given in Theorem 2.11 is finite whenever M is finitely presented (and is given by a finite presentation) and the set $\mathcal{C}$ of cosets of G in M is finite as well.

Remark 2.12 Notice that if G is a maximal subgroup of M, that is, G is a group $\mathcal{H}$–class of M, then $\mathcal{C}$ consists of all the $\mathcal{H}$–classes in the same $\mathcal{R}$–class of G. Indeed, it is easy to see that any coset is itself an $\mathcal{H}$–class and that the elements of a coset are $\mathcal{R}$–related to the elements of G. Moreover, by Green's Lemma (Lemma 1.5) it follows that any $\mathcal{H}$–class in the $\mathcal{R}$–class of G is in $\mathcal{C}$.

2.3 Computational methods

Computational semigroup theory started with the determination of semigroups of small orders (see Forsythe [18], Plemmons [41]). But it was with the development of techniques for transformation semigroups, and semigroups defined by presentations, that this discipline caught the attention of more researchers.

In this thesis we deal mainly with semigroups defined by presentations, and therefore we use and are interested in, computational methods to explore the structure of those. On one hand we use computational methods as a tool for our work: the formulation and proof of our results does not depend on computations, but the examples thus obtained certainly provided essential insight into the problems. They also illustrate the results obtained. One the other hand we consider and solve some decidability questions and develop some algorithms.

One natural question to ask is what we can say about a semigroup given in terms of a presentation. It is well known that most questions one may ask cannot be answered in general (not even for finitely presented semigroups). For example, there is no algorithm that takes a semigroup presentation, and two words on the alphabet of the presentation, and returns true if both words represent the same element of the semigroup defined by the presentation and returns false otherwise. This problem is called the *word problem* and it is well known that it is unsolvable. Thus, we cannot program a computer to perform this task. Moreover, the unsolvability of the word problem implies that, in general, we cannot find an answer to many other questions that one would like to ask about a semigroup – is it finite, is it commutative, is it a group – are some examples of this long list. One therefore may argue that the use of semigroup presentations is very limited, as one cannot use them to answer major questions about the objects they define. This seems, though, a very negative approach to the problem. In fact, while most questions do not have a general answer, most of them can be answered for 'nice' presentations. Or sometimes, given an extra piece of information, the questions become answerable (for example, a semigroup presentation defining a semigroup known to be commutative has a solvable word problem).

There are procedures that although do not give an answer in general might produce answers in certain situations: examples are the Todd–Coxeter and the Knuth–Bendix procedures. They are both described by Sims [63]; we describe a commutative version of Knuth–Bendix in Chapter 7. Given a presentation, Todd–Coxeter procedure will stop if and only if the semigroup defined by that presentation is finite. In that case it will return a transformation representation of the semigroup. Knuth–Bendix stops if and only if it finds a finite confluent rewriting system for the semigroup, and in that case it produces a solution of the word problem for the semigroup.

For our computations we started by using a version of Todd–Coxeter implemented by Trevor Walker in 1992, which is available locally at St Andrews. This is, however, a stand-alone implementation, with no possibility of expansion and therefore with many limitations. Motivated by the idea of creating a tool for her own work, which would be also useful for others, the author was one of the members of the team that implemented basic semigroup functionality in GAP (see [19]; [7] describes, in the form of a tutorial, the functionality available in GAP 4.1. for computations with semigroups). In particular, we have been responsible for the implementation of a version of Knuth–Bendix (based on Sims [63] description of the algorithm). This was available since the version 4.1 of GAP. An improved version will appear in the release 4.3, and is joint work with Alexander Hulpke.

Some of the algorithms described in Chapter 7 are part of the package commsemi of GAP (which is an alpha version and is deposited in the GAP archive – it can be retrieved from the GAP web pages). This package was joint work with Andrew Solomon.

Chapter 3

Direct products of semigroups

In this chapter we study finite generation and finite presentability of direct products of semigroups.

Given two monoids it is well known that their direct product is finitely generated if and only if both monoids are finitely generated, and it is finitely presented if and only if both monoids are finitely presented. This is a consequence of the fact that if S and T are monoids then they both are homomorphic images of $S \times T$ and that if $\langle A \mid R\rangle$ and $\langle B \mid Q\rangle$ are presentations for S and T, respectively, then their direct product is defined by the presentation

$$\langle A,\, B \mid R,\, Q,\, ab = ba \ (a \in A,\, b \in B)\rangle. \tag{3.1}$$

This result does not generalize for direct products of semigroups. Indeed, the additive semigroup of natural numbers $\mathbb{N}$ is finitely generated by a single element, but the direct product $\mathbb{N} \times \mathbb{N}$ is not finitely generated: it is easy to see that any generating set must contain the infinite set $\{(1, n) : n \in \mathbb{N}\}$.

It is thus natural to ask the following

Question 3.1 When is the direct product of two semigroups finitely generated and when is it finitely presented?

Notice that there are three distinct cases to be considered: the direct product of two finite semigroups, the direct product of one finite with one infinite semigroup and the direct product of two infinite semigroups. In the first case the direct product is a finite semigroup and hence it is always finitely generated and finitely presented. The other two cases have different answers. The question was answered in both cases by Robertson, Ruškuc and Wiegold [47]. The purpose of this chapter is to take this study further. Section 3.1 describes finitely generated direct products and gives some new results on the size of minimal generating sets. Section 3.2 introduces the results proved by Robertson, Ruškuc and Wiegold in [47] with respect to finite presentability. The main notion there is the one of a stable semigroup. In Section 3.3 we give a constructive characterization of finite stable semigroups. In Section 3.4 we give some examples of stable and non-stable semigroups which will lead to examples of finitely presented and non-finitely presented direct products. In Section 3.5, we present the algorithms arising from the decidability questions answered in the previous sections. Finally, in Section 3.6, we consider direct products of finitely many semigroups, and generalize the theorems from Sections 3.1 and 3.2 characterizing finitely generated and finitely presented direct products.

The results from Sections 3.1 and 3.6 appear here for the first time while the results from Sections 3.3 and 3.4 have appeared in [5].

3.1 Finite generation: known results and some considerations on the rank

As mentioned before, the direct product of two finitely generated semigroups need not be finitely generated. The reason why $\mathbb{N} \times \mathbb{N}$ is not finitely generated is that it contains an infinite set of indecomposable elements. Notice that S does not have indecomposable elements if and only if $S^2 = S$, where $S^2 = SS = \{s_1 s_2 : s_1, s_2 \in S\}$. We thus have

Theorem 3.2 [47, Theorem 2.1] *Let S and T be two infinite semigroups. Then $S \times T$ is finitely generated if and only if both S and T are finitely generated and $S^2 = S$ and $T^2 = T$.*

and

Theorem 3.3 [47, Theorem 8.2] *Let S be a finite semigroup and let T be an infinite semigroup. Then the direct product $S \times T$ is finitely generated if and only* *if $S^2 = S$ and T is finitely generated.*

The *rank* of a semigroup S, $\operatorname{rank}(S)$, is the minimum possible cardinality of a generating set of S. We now look at the rank of the direct product of two semigroups. If S and T are monoids then

$$\operatorname{rank}(S \times T) \leq \operatorname{rank}(S) + \operatorname{rank}(T).$$

(We consider the rank of a monoid S to be the minimum size of a semigroup generating set for it.) In fact, this formula is valid in a more general setting:

Proposition 3.4 *Let S be a semigroup with a right identity d, and T be a semigroup with a left identity e (or vice-versa). Then*

$$\operatorname{rank}(S \times T) \leq \operatorname{rank}(S) + \operatorname{rank}(T).$$

PROOF Let A and B be generating sets for S and T, respectively, and let $(s,t) \in S \times T$. Then there exist $a_1, a_2, \ldots, a_n \in A$ and $b_1, b_2, \ldots, b_m \in B$ such that $s = a_1 a_2 \cdots a_n$ and $t = b_1 b_2 \cdots b_m$. Then we have

$$\begin{aligned}
(s,t) &= (a_1 a_2 \cdots a_n, b_1 b_2 \cdots b_m) \\
&= (a_1 a_2 \cdots a_n d^m, e^n b_1 b_2 \cdots b_m) \\
&= (a_1, e)(a_2, e) \cdots (a_n, e)(d, b_1)(d, b_2) \cdots (d, b_m)
\end{aligned}$$

and thus $S \times T$ is generated by the set

$$\{(a, e) : a \in A\} \cup \{(d, b) : b \in B\}.$$

We therefore conclude that

$$\operatorname{rank}(S \times T) \leq \operatorname{rank}(S) + \operatorname{rank}(T).$$

□

The proof of Theorem 3.2, in [47], gives an upper bound for the rank of direct product of two infinite semigroups, in general. Namely, for two infinite semigroups S and T,

$$\operatorname{rank}(S \times T) \leq 4 \operatorname{rank}(S) \operatorname{rank}(T).$$

An example in the same paper ([47, Example 2.8]) exhibits a direct product of two semigroups S and T satisfying

$$\operatorname{rank}(S \times T) = 2\operatorname{rank}(S)\operatorname{rank}(T).$$

We now prove that $2\operatorname{rank}(S)\operatorname{rank}(T)$ is in fact an upper bound for $\operatorname{rank}(S \times T)$. It then follows that this is the possible best upper bound in general, and that it is actually attained. Our proof holds for both the finite and the infinite cases.

Proposition 3.5 *Let S and T be two semigroups satisfying $S^2 = S$ and $T^2 = T$ and let $A = \{a_i : i \in I\}$ and $B = \{b_j : j \in J\}$ be generating sets for S and T respectively. Fix elements $s_i \in S$ $(i \in I)$ and $t_j \in T$ $(j \in J)$ and functions $\xi : I \to I$ and $\theta : J \to J$ such that $a_i = a_{i\xi} s_i$ for all $i \in I$ and $b_j = t_j b_{j\theta}$ for all $j \in J$. Then the set*

$$(A \times \{t_j : j \in J\}) \cup (\{s_i \; i \in I\} \times B).$$

generates $S \times T$.

Proof Let $s \in S$ and $t \in T$ be arbitrary. Suppose s can be decomposed as a product $a_{i_1} a_{i_2} \cdots a_{i_n}$ of n generators from A, and that t can be decomposed as a product $b_{j_1} b_{j_2} \cdots b_{j_m}$ of m generators from B. By induction on m we prove that s can be written as a product

$$a_{i_1} \cdots a_{i_n \xi^m} s_{i_n \xi^{m-1}} \cdots s_{i_n \xi} s_{i_n}$$

of n elements of A, followed by m elements of $\{s_i : i \in I\}$. Indeed, if $m = 1$, by replacing a_{i_n} by $a_{i_n \xi} s_n$ the result follows. Now, if we have s written as

$$a_{i_1} \cdots a_{i_n \xi^{m-1}} s_{i_n \xi^{m-2}} \cdots s_{i_n \xi} s_{i_n}$$

then by replacing $a_{i_n \xi^{m-1}}$ by $a_{i_n \xi^m} s_{i_n \xi^{m-1}}$ we get the desired result. Similarly, t can be written as a product of n elements from $\{t_j : j \in J\}$ followed by m elements from B. Therefore, (s,t) can be written as a product of n elements from $A \times \{t_j : j \in J\}$ followed by m elements from $\{s_i : i \in I\} \times B$. Thus $S \times T$ is generated by the set

$$(A \times \{t_j : j \in J\}) \cup (\{s_i \; i \in I\} \times B).$$

□

Corollary 3.6 *Let S and T be two semigroups satisfying $S^2 = S$ and $T^2 = T$. Then*

$$\text{rank}(S \times T) \leq 2\text{rank}(S)\text{rank}(T).$$

We now examine some direct products of some special kinds of semigroups.

Proposition 3.7 *Let S be a semigroup satisfying $S^2 = S$. Suppose there exists a generating set A of S such that $A \subseteq A^2$, of size* $\text{rank}(S)$. *Then, for any semigroup T satisfying $T^2 = T$,*

$$\text{rank}(S \times T) \leq \text{rank}(S)\text{rank}(T).$$

PROOF We start by noticing that if A is a generating set of minimal size for a semigroup S, then A satisfies $A \subseteq A^2$ if and only if any element $s \in S$ that can be written as a product of n ($n \geq 1$) elements of A can also be written as a product of m elements of A, for any $m \geq n$.

Now let B be a generating set for T and let $(s,t) \in S \times T$. Then there exist $a_1, a_2, \ldots, a_n \in A$ and $b_1, b_2, \ldots, b_m \in B$ such that $s = a_1 a_2 \cdots a_n$, $t = b_1 b_2 \cdots b_m$. If $n = m$ then

$$(s,t) = (a_1, b_1)(a_2, b_2) \cdots (a_n, b_m).$$

If $n < m$ then there are elements $a_{k_1}, a_{k_2}, \ldots, a_{k_{m-n}} \in A$ such that we can write $s = a_1 a_2 \cdots a_n a_{k_1} a_{k_2} \cdots a_{k_{m-n}}$ as a product of m elements of A. We now have both s and t written as a product of the same number of elements, and the result follows as before. Otherwise, using the fact that $T^2 = T$, we can write t as a product of k elements from B, for some $k \geq n$. We are now in the second case (if $k > n$) or in the first one (if $k = n$). Thus it follows that $S \times T$ is generated by the set $A \times B$. Notice that A is fixed, but it is of minimal size, and hence

$$\text{rank}(S \times T) \leq \text{rank}(S)\text{rank}(T).$$

□

The following are examples of finite and infinite semigroups satisfying Proposition 3.7.

Example 3.8 If S is a rectangular band, a left zero semigroup, or a semilattice it satisfies the conditions of Proposition 3.7. Then for any T satisfying $T^2 = T$ we have that

$$\mathrm{rank}(S \times T) \leq \mathrm{rank}(S)\mathrm{rank}(T).$$

Example 3.9 Let S be a semigroup defined by a presentation

$$\langle A \mid a^2 = a \ (a \in A)\rangle$$

for some finite set A. Then S satisfies the conditions of Proposition 3.7 and hence, for any semigroup T satisfying $T^2 = T$ we have that $S \times T$ is finitely generated and

$$\mathrm{rank}(S \times T) \leq \mathrm{rank}(S)\mathrm{rank}(T).$$

Proposition 3.7 is gives a best possible upper bound , in general, for direct products satisfying those conditions. Indeed, the following example shows that this is actually attained.

Example 3.10 Let SL_A and SL_B be free semilattices on finite sets A and B, respectively. As noted in Example 3.8,

$$\mathrm{rank}(\mathrm{SL}_A \times \mathrm{SL}_B) \leq \mathrm{rank}(\mathrm{SL}_A)\mathrm{rank}(\mathrm{SL}_B).$$

We are going to prove that in fact, the equality holds. Recall that A and B are minimum generating sets for SL_A and SL_B, respectively. Let $Z = \{(x_i, y_i) : i \in I\}$ be a generating set for $\mathrm{SL}_A \times \mathrm{SL}_B$ and let $(a, b) \in A \times B$. Then there exist $(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k) \in Z$ such that

$$(a, b) = (x_1, y_1)(x_1, y_2) \cdots (x_k, y_k)$$

and thus $a = x_1 x_2 \cdots x_k$, $b = y_1 y_2 \cdots y_k$. But then it follows that $x_i = a$, $y_i = b$ $(1 \leq i \leq k)$. Therefore $A \times B \subseteq Z$. But since $A \times B$ generate $\mathrm{SL}_A \times \mathrm{SL}_B$ we can conclude that it is indeed a minimum generating set for it and hence the equality follows.

The following is another example where such an upper bound also holds. However, the semigroups do not satisfy the conditions of Proposition 3.7.

Example 3.11 Let I_X and I_Y be the free inverse semigroup on sets X and Y, respectively. Recall that I_X is generated (as a semigroup) by $X \cup X^{-1} = \{x, x^{-1} : x \in X\}$ and that this is a generating set of minimal size for it. Similarly, $Y \cup Y^{-1} = \{y, y^{-1} : y \in Y\}$ is a generating set of minimal size for I_Y. From Proposition 3.5 it follows that $\mathrm{I}_X \times \mathrm{I}_Y$ is generated by the set

$$\begin{gathered}
(\,(X \cup X^{-1}) \times \{yy^{-1}, y^{-1}y : y \in Y\}\,) \;\cup \\
(\,\{xx^{-1}, x^{-1}x : x \in X\} \times (Y \cup Y^{-1})\,) = \\
\{(x, yy^{-1}),\, (x, y^{-1}y),\, (x^{-1}, yy^{-1}), (x^{-1}, y^{-1}y),\, (xx^{-1}, y), \\
(xx^{-1}, y^{-1}),\, (x^{-1}x, y),\, (x^{-1}x, y^{-1}) \;:\; x \in X,\, y \in Y\}
\end{gathered}$$

and this is a generating set of size

$$8|X||Y| = 2\mathrm{rank}(\mathrm{I}_X)\mathrm{rank}(\mathrm{I}_Y).$$

But notice that

$$\begin{aligned}
(x^{-1}, y^{-1}y) &= (x^{-1}x, y^{-1})(x^{-1}, yy^{-1})(xx^{-1}, y) \\
(xx^{-1}, y^{-1}) &= (x, y^{-1}y)(x^{-1}x, y^{-1})(x^{-1}, yy^{-1}) \\
(x, yy^{-1}) &= (xx^{-1}, y)(x, y^{-1}y)(x^{-1}x, y^{-1}) \\
(x^{-1}x, y) &= (x^{-1}, yy^{-1})(xx^{-1}, y)(x, y^{-1}y)
\end{aligned}$$

and hence it follows that $\mathrm{I}_X \times \mathrm{I}_Y$ is generated by the set

$$\{(x, y^{-1}y),\, (x^{-1}, yy^{-1}),\, (xx^{-1}, y),\, (x^{-1}x, y^{-1}) : x \in X, y \in Y\}.$$

Therefore

$$\mathrm{rank}(\mathrm{I}_X \times \mathrm{I}_Y) \leq 4|X||Y| = \mathrm{rank}(\mathrm{I}_X)\mathrm{rank}(\mathrm{I}_Y).$$

We will prove that the equality holds. So let Z be a generating set for $\mathrm{I}_X \times \mathrm{I}_Y$,

$$Z = \{(w_i, z_i) : i \in I\}.$$

Notice that, if a word w represents the same element as x (resp. x^{-1}) in I_X ($x \in X$) then its first letter has to be x (resp. x^{-1}). Similarly, if a word z represents the same element than y (resp. y^{-1}) in I_Y ($y \in Y$) then its first letter is y (resp. y^{-1}). Hence, to be able to generate $(x, y) \in X \times Y$ the set Z has to contain an element (w_j, z_j) with the first letter of w_j being an x and the first letter of z_j being an y. Similar arguments for (x, y^{-1}), (x^{-1}, y) and (x^{-1}, y^{-1}) $(x \in X,\ y \in Y)$ imply that $Z \geq 4|X||Y|$ and hence

$$\operatorname{rank}(\mathrm{I}_X \times \mathrm{I}_Y) = 4|X||Y| = \operatorname{rank}(\mathrm{I}_X)\operatorname{rank}(\mathrm{I}_Y).$$

3.1.1 Rees matrix semigroups

We now study the rank of direct products of finite Rees matrix semigroups over groups.

So let G be a group with identity 1_G, and let I and Λ be non-empty sets. Let $P = (p_{\lambda i})$ be a $\Lambda \times I$ matrix over G. On the set $S = I \times G \times \Lambda$ multiplication is defined by

$$(i, a, \lambda)(j, b, \mu) = (i, ap_{\lambda j}b, \mu).$$

The operation thus defined is associative and the semigroup obtained is called the $I \times \Lambda$ *Rees matrix semigroup over the group* G, *with sandwich matrix* P, and is denoted by $\mathcal{M}(G; I, \Lambda; P)$. (For more details see [28].) Also, as noted in [54], P can be chosen to be in the normal form, that is,

$$p_{\lambda 1} = p_{1i} = 1 \ (i \in I,\ \lambda \in \Lambda).$$

Rees matrix semigroups were introduced by Rees [43] and form the class of completely simple semigroups. A presentation for a Rees matrix semigroup (in the general case of Rees matrix semigroups over monoids) was given in [29]. In [8], Ayik and Ruškuc consider finite generation and presentability of those semigroups.

The rank of finite Rees matrix semigroups was investigated by Ruškuc in [54], where he proves that if $S = \mathcal{M}(G; I, \Lambda; P)$ is finite, and $\overline{G}$ is the subgroup of G generated by $\{p_{\lambda i} : i \neq 1, \lambda \neq 1\}$, then

$$\operatorname{rank}(S) = \max\{|I|, |\Lambda|, \operatorname{rank}(G : \overline{G})\} \tag{3.2}$$

where $\operatorname{rank}(G : \overline{G}) = \min\{|Z| : \langle \overline{G} \cup Z \rangle = G\}$ (and is called the *rank of* G *modulo* $\overline{G}$).

We start by showing that the direct product of two Rees matrix semigroups is also a Rees matrix semigroup. This is a consequence of the fact that the class of Rees matrix semigroup over groups forms a variety, that is, it is closed under homomorphic images, subalgebras and direct products (details can be found in [28] or [40]). We give an alternative proof of this fact, which will give us the actual Rees matrix semigroup obtained as the direct product.

Proposition 3.12 *Let $S = \mathcal{M}[G;\ I,\ \Lambda;\ P]$ and $T = \mathcal{M}[H;\ I',\ \Lambda';\ Q]$ be two Rees matrix semigroups with P and Q in the normal form. Then $S \times T$ is isomorphic to $\mathcal{M} = [G \times H;\ I \times I',\ \Lambda \times \Lambda';\ R]$ where R is a $(|\Lambda||\Lambda'| \times |I||I'|)$ matrix given by $R = (r_{(\lambda,\mu)(i,j)})$ and*

$$r_{(\lambda,\mu)(i,j)} = (p_{\lambda i}, q_{\mu j}).$$

Furthermore, R is also in the normal form.

Proof Define a mapping

$$\phi : S \times T \to \mathcal{M}[G \times H;\ I \times I',\ \Lambda \times \Lambda';\ R]$$

by

$$((i, a, \lambda), (j, b, \mu)) \mapsto ((i, j), (a, b), (\lambda, \mu)).$$

This is clearly a well-defined bijection. It remains to prove that it is a homomorphism. So let $((i, a, \lambda), (j, b, \mu)), ((i', a', \lambda'), (j', b', \mu')) \in S \times T$. Then we have

$$\begin{aligned}
&[((i, a, \lambda), (j, b, \mu))((i', a', \lambda'), (j', b', \mu'))]\phi = \\
&= ((i, ap_{\lambda i'}a', \lambda'), (j, bq_{\mu j'}b', \mu'))\phi \\
&= ((i, j), (ap_{\lambda i'}a', bq_{\mu j'}b'), (\lambda', \mu')) \\
&= ((i, j), (a, b)(p_{\lambda i'}, q_{\mu j'})(a', b'), (\lambda', \mu')) \\
&= ((i, j), (a, b), (\lambda, \mu))((i', j'), (a', b'), (\lambda', \mu')) \\
&= ((i, a, \lambda), (j, b, \mu))\phi\, ((i', a', \lambda'), (j', b', \mu'))\phi.
\end{aligned}$$

Now, to see that R is in the normal form if P and Q are, consider $(1, 1) \in I \times I'$ to be the first column of R and $(1, 1) \in \Lambda \times \Lambda'$ to be its first row. Then we have

$$r_{(1,1)(i,j)} = (p_{1i}, q_{1j}) = (1_G, 1_H)$$

and

$$r_{(\lambda,\mu)(1,1)} = (p_{\lambda 1}, q_{\mu 1}) = (1_G, 1_H)$$

and thus R is in the normal form. □

Let $S = \mathcal{M}[G; I, \Lambda; P]$ and $T = \mathcal{M}[H; I', \Lambda'; Q]$ be two finite Rees matrix semigroups with P and Q in the normal form. From (3.2) and Proposition 3.12 it follows that

$$\operatorname{rank}(S \times T) = \max\{\, |I||I'|,\, |\Lambda||\Lambda'|,\, \operatorname{rank}(G \times H : L)\,\}$$

where L is the subgroup of $G \times H$ generated by the set

$$\{r_{(\lambda,\mu)(i,j)} \,:\, (1,1) \neq (\lambda,\mu) \in \Lambda \times \Lambda',\, (1,1) \neq (i,j) \in I \times I'\}.$$

The following Lemma describes L in terms of some subgroups of G and H.

Lemma 3.13 *Let $S = \mathcal{M}[G; I, \Lambda; P]$ and $T = \mathcal{M}[H; I', \Lambda'; Q]$ be two finite Rees matrix semigroups with P and Q in the normal form. Then if L is the subgroup of $G \times H$ described above, then $L = \overline{G} \times \overline{H}$, where $\overline{G}$ is the subgroup of G generated by $\{p_{\lambda i} \,:\, 1 \neq \lambda \in \Lambda,\, 1 \neq i \in I\}$ and $\overline{H}$ is the subgroup of H generated by $\{q_{\mu j} \,:\, 1 \neq \mu \in \Lambda',\, 1 \neq j \in I'\}$.*

PROOF First notice that $L \subseteq \overline{G} \times \overline{H}$ since for any $(1,1) \neq (\lambda,\mu) \in \Lambda \times \Lambda'$ and $(1,1) \neq (i,j) \in I \times I'$

$$r_{(\lambda,\mu)(i,j)} = (p_{\lambda i}, q_{\mu j}) \in \overline{G} \times \overline{H}.$$

Now let $(a,b) \in \overline{G} \times \overline{H}$. Then there exist $1 \neq \lambda_1, \ldots, \lambda_n \in \Lambda$, $1 \neq i_1, \ldots, i_n \in I$, $1 \neq \mu_1, \ldots, \mu_m \in \Lambda'$, $1 \neq j_1, \ldots, j_m \in I'$ (without loss of generality suppose that $n < m$) such that

$$\begin{aligned}
(a,b) &= (p_{\lambda_1 i_1} \cdots p_{\lambda_n i_n}, q_{\mu_1 j_1} \cdots q_{\mu_m j_m}) \\
&= (p_{\lambda_1 i_1}, q_{\mu_1 j_1}) \cdots (p_{\lambda_n i_n}, q_{\mu_n j_n})(1, q_{\mu_{n+1} j_{n+1}}) \cdots (1, q_{\mu_m j_m}) \\
&= r_{(\lambda_1,\mu_1)(i_1,j_1)} \cdots r_{(\lambda_n,\mu_n)(i_n,j_n)} r_{(1,\mu_{n+1})(1,j_{n+1})} \cdots r_{(1,\mu_m)(1,j_m)}
\end{aligned}$$

and thus $(a,b) \in L$. $\square$

We also have

Lemma 3.14 *With notation as above we have*

$$\operatorname{rank}(G \times H : \overline{G} \times \overline{H}) \leq \operatorname{rank}(G : \overline{G}) + \operatorname{rank}(H : \overline{H}).$$

PROOF Let X be such that $\overline{G} \cup X$ generates G, and Y be a set such that $\overline{H} \cup Y$ generates H. Let $(g, h) \in G \times H$. Then there exists $x \in G \backslash \overline{G} \cup \{1_G\}$ and $y \in H \backslash \overline{H} \cup \{1_H\}$ such that $g \in x\overline{G}$ and $h \in y\overline{H}$, so that $g = xg'$ and $h = yh'$ for some $g' \in \overline{G}$ and some $h' \in \overline{H}$. So

$$(g, h) = (xg', yh') = (x, 1_H)(1_G, y)(g', h').$$

Now, $(x, 1_H)$ can certainly be written as a product of elements from $(\overline{G} \cup X) \times \{1_H\} \subseteq (\overline{G} \times \overline{H}) \cup (X \times \{1_H\})$ and $(1_G, y)$ can be written as a product of elements from $\{1_G\} \times (\overline{H} \cup Y) \subseteq (\overline{G} \times \overline{H}) \cup (\{1_G\} \times Y)$. Hence we have proved that $G \times H$ is generated by the set

$$(\overline{G} \times \overline{H}) \cup (X \times \{1_H\}) \cup (\{1_G\} \times Y).$$

But if X and Y are chosen to be minimal then $|X| = \text{rank}(G : \overline{G})$ and $|Y| = \text{rank}(H : \overline{H})$. Therefore we have

$$\begin{aligned} \text{rank}(G \times H : \overline{G} \times \overline{H}) &= \min\{|Z| : \langle \overline{G} \times \overline{H} \cup Z \rangle = G \times H\} \\ &\leq |X| + |Y| = \text{rank}(G : \overline{G}) + \text{rank}(H : \overline{H}) \end{aligned}$$

as required. □

We can now prove the following

Theorem 3.15 *Let S and T be two finite Rees matrix semigroup as above. Then*

$$\text{rank}(S \times T) \leq \text{rank}(S)\text{rank}(T)$$

except if one of S or T is a cyclic group, in which case $\text{rank}(S \times T) = \text{rank}(S) + \text{rank}(T)$.

PROOF If follows from Lemma 3.14 that

$$\text{rank}(S \times T) \leq \max\{|I||I'|, |\Lambda||\Lambda'|, \text{rank}(G : \overline{G}) + \text{rank}(H : \overline{H})\}.$$

Now,

$$|I||I'| \leq \max\{|I|, |\Lambda|, \text{rank}(G : \overline{G})\} \max\{|I'|, |\Lambda'|, \text{rank}(H : \overline{H})\},$$

$$|\Lambda||\Lambda'| \leq \max\{|I|, |\Lambda|, \operatorname{rank}(G : \overline{G})\} \max\{|I'|, |\Lambda'|, \operatorname{rank}(H : \overline{H})\}$$

and

$$\begin{aligned}\operatorname{rank}(G : \overline{G}) + \operatorname{rank}(H : \overline{H}) &\leq \operatorname{rank}(G : \overline{G}) \operatorname{rank}(H : \overline{H}) \\ \leq \max\{|I|, |\Lambda|, \operatorname{rank}(G : \overline{G})\} &\max\{|I'|, |\Lambda'|, \operatorname{rank}(H : \overline{H})\}\end{aligned}$$

except when $\operatorname{rank}(G : \overline{G}) = 0$, $\operatorname{rank}(H : \overline{H}) = 0$, $\operatorname{rank}(G : \overline{G}) = 1$ or $\operatorname{rank}(H : \overline{H}) = 1$. It thus follows that

$$\operatorname{rank}(S \times T) \leq \operatorname{rank}(S)\operatorname{rank}(T) \tag{3.3}$$

provided $\operatorname{rank}(G : \overline{G}) \neq 0$, $\operatorname{rank}(H : \overline{H}) \neq 0$, $\operatorname{rank}(G : \overline{G}) \neq 1$ and $\operatorname{rank}(H : \overline{H}) \neq 1$. Expression (3.3) is also true if

$$\max\{|I||I'|, |\Lambda||\Lambda'|, \operatorname{rank}(G : \overline{G}) + \operatorname{rank}(H : \overline{H})\} = |I||I'|$$

or

$$\max\{|I||I'|, |\Lambda||\Lambda'|, \operatorname{rank}(G : \overline{G}) + \operatorname{rank}(H : \overline{H})\} = |\Lambda||\Lambda'|.$$

So suppose that

$$\begin{gathered}\max\{|I||I'|, |\Lambda||\Lambda'|, \operatorname{rank}(G : \overline{G}) + \operatorname{rank}(H : \overline{H})\} = \\ \operatorname{rank}(G : \overline{G}) + \operatorname{rank}(H : \overline{H}).\end{gathered}$$

If $\operatorname{rank}(G : \overline{G}) = 0$ then it follows that

$$\operatorname{rank}(H : \overline{H}) \geq |I||I'|$$

and hence $\operatorname{rank}(H : \overline{H}) \geq |I'|$ and also

$$\operatorname{rank}(H : \overline{H}) \geq |\Lambda||\Lambda'|$$

and hence $\operatorname{rank}(H : \overline{H}) \geq |\Lambda'|$. It follows that $\operatorname{rank}(T) = \operatorname{rank}(H : \overline{H})$ and hence

$$\begin{aligned}\operatorname{rank}(S)\operatorname{rank}(T) &= \operatorname{rank}(S)\operatorname{rank}(H : \overline{H}) \\ &\geq \operatorname{rank}(H : \overline{H}).\end{aligned}$$

So in this case we have

$$\operatorname{rank}(S \times T) \leq \operatorname{rank}(S) \operatorname{rank}(T).$$

The case when $\mathrm{rank}(H : \overline{H}) = 0$ is similar.

Now suppose that $\mathrm{rank}(G : \overline{G}) = 1$. Then

$$\mathrm{rank}(S \times T) = 1 + \mathrm{rank}(H : \overline{H}) \leq \mathrm{rank}(S) + \mathrm{rank}(T).$$

Therefore we also have

$$\mathrm{rank}(S \times T) \leq \mathrm{rank}(S)\mathrm{rank}(T)$$

unless $\mathrm{rank}(S)$=1 or $\mathrm{rank}(T) = 1$, which clearly would imply that S is a cyclic group, or T is a cyclic group, respectively. The case when $\mathrm{rank}(H : \overline{H}) = 1$ is similar. □

The next example shows that the bound given in Theorem 3.15 is actually attained.

Example 3.16 Let $S = \mathcal{M}[C_2; \{1,2,3,4\}, \{1,2\}; P]$ and $T = \mathcal{M}[C_3; \{1,2,3,4\}, \{1,2\}; Q]$ be two Rees matrix semigroups, where C_2 is the cyclic group of order 2 and C_3 is the cyclic group of order 3, P and Q are 2×4 matrices with all entries equal to the identity of C_2 and to the identity of C_3, respectively. Then

$$S \times T = \mathcal{M}[C_2 \times C_3; \{1, 2, \ldots, 16\}, \{1,2,3,4\}; R]$$

where R is a 16×4 matrix with all entries being the identity of $C_2 \times C_3$. Now, by (3.2), $\mathrm{rank}(S) = 4$, $\mathrm{rank}(T) = 4$ and

$$\mathrm{rank}(S \times T) = 16 = \mathrm{rank}(S)\mathrm{rank}(T).$$

The proof of Theorem 3.15 also suggests the following

Corollary 3.17 *Let $S = \mathcal{M}[G; I, \Lambda; P]$ and $T = \mathcal{M}[H; I', \Lambda'; Q]$ be two Rees matrix semigroup over finite groups G and H, respectively. Suppose*

$$\max\{|I||I'|,\ |\Lambda||\Lambda'|, \mathrm{rank}(G : \overline{G}) + \mathrm{rank}(H : \overline{H})\} = \mathrm{rank}(G : \overline{G}) + \mathrm{rank}(H : \overline{H})\}$$

(where $\overline{G}$ and $\overline{H}$ are defined as before). Then

$$\mathrm{rank}(S \times T) \leq \mathrm{rank}(S) + \mathrm{rank}(T).$$

PROOF The result follows immediately from

$$\begin{aligned}\operatorname{rank}(S \times T) &\leq \operatorname{rank}(G : \overline{G}) + \operatorname{rank}(H : \overline{H}) \\ &\leq \operatorname{rank}(S) + \operatorname{rank}(T).\end{aligned}$$

□

Notice that this result is not a consequence of Proposition 3.4, since, in general, a Rees matrix semigroup does not have left or right identities.

Once again the bound given in Corollary 3.17 is attained, as the following example shows.

Example 3.18 Let $S = \mathcal{M}[G; \{1,2\}, \{1,2\}; P]$ and $T = \mathcal{M}[H; \{1,2\}, \{1,2\}; Q]$ be Rees matrix semigroups, with $G = C_2 \times C_2 \times C_2$, $H = C_2 \times C_2$ (where C_2 is the cyclic cyclic group of order 2) and with P and Q being 2×2 matrices with all entries equal to the identity of G and the identity of H, respectively. Then

$$S \times T = \mathcal{M}[G \times H; \{1,2,3,4\}, \{1,2,3,4\}; R]$$

where R is a 4×4 matrix with all entries equal to the identity of $G \times H = C_2 \times C_2 \times C_2 \times C_2 \times C_2$. Notice that $\operatorname{rank}(G : \overline{G}) = \operatorname{rank}(G)$, $\operatorname{rank}(H : \overline{H}) = \operatorname{rank}(H)$ and $\operatorname{rank}(G \times H : \overline{G} \times \overline{H}) = \operatorname{rank}(G \times H) = \operatorname{rank}(G) + \operatorname{rank}(H)$. Thus, by (3.2), $\operatorname{rank}(S) = 3$, $\operatorname{rank}(T) = 2$ and

$$\operatorname{rank}(S \times T) = 5 = \operatorname{rank}(S) + \operatorname{rank}(T).$$

It is clear that given two semigroups S and T the rank of their direct product has to be always greater than or equal to the maximum between $\operatorname{rank}(S)$ and $\operatorname{rank}(T)$. We have also seen that in general the rank of the direct product is much bigger than that. We now present an example of a direct product of two semigroups (which are not monoids) where this minimum number of generators is enough to generate the direct product.

Example 3.19 Let $S = \mathcal{M}[G; \{1,2\}, \{1,2\}; P]$ and $T = \mathcal{M}[H; \{1,2\}, \{1,2\}, Q]$ be Rees matrix semigroups, with $G = C_2 \times C_2 \times C_2 \times C_2 \times C_2$ and $H = C_3 \times C_3$ (where C_2 is the cyclic group of order 2, C_3 is the cyclic group of order 3) and with P and Q being 2×2 matrices with all entries equal to identity of G and H, respectively. Then

$$S \times T = \mathcal{M}[G \times H; \{1,2,3,4\}, \{1,2,3,4\}; R],$$

where R is a 4×4 matrix with all entries equal to the identity of $G \times H = C_2 \times C_2 \times C_2 \times C_2 \times C_2 \times C_3 \times C_3$. Notice that $\operatorname{rank}(G : \overline{G}) = \operatorname{rank}(G)$, $\operatorname{rank}(H : \overline{H}) = \operatorname{rank}(H)$ and $\operatorname{rank}(G \times H : \overline{G} \times \overline{H}) = \operatorname{rank}(G \times H)$. Furthermore $\operatorname{rank}(G \times H) = \operatorname{rank}(G)$. This follows from the fact they G and H have coprime orders. Thus, by (3.2), $\operatorname{rank}(S) = 5$, $\operatorname{rank}(T) = 2$ and

$$\operatorname{rank}(S \times T) = 5 = \operatorname{rank}(S) = \max\{\operatorname{rank}(S), \operatorname{rank}(T)\}.$$

3.2 Finite presentability: known results

One of the crucial steps in characterizing finitely presented direct products of semigroups is the definition of a stable semigroup that we shall now introduce. This was originally introduced in [47].

Let $\mathcal{P} = \langle A \mid R \rangle$ be a presentation. A pair of words $(w_1, w_2) \in A^+ \times A^+$ is called a *critical pair* (with respect to $\mathcal{P}$) if $w_1 = w_2$ in S and for every elementary sequence $w_1 \equiv \gamma_1, \gamma_2, \ldots, \gamma_m \equiv w_2$ there exists i $(1 \leq i \leq m)$ such that $|\gamma_i| < \min\{|w_1|, |w_2|\}$. A semigroup S is said to be *stable* if $S^2 = S$ and there exists a finite presentation $\langle A \mid R \rangle$ which defines S and has no critical pairs. (The original definition in [47] does exclude semigroups with indecomposable elements. However, all the results have this necessary condition.) In fact, if a semigroup is stable then it is possible to find such a presentation for any generating set as the following proposition states.

Proposition 3.20 [47, Proposition 3.4] *Let S be a stable semigroup. Then for any generating set A of S there exists a stable presentation $\langle A \mid R \rangle$ of S, with respect to A, which has no critical pairs.*

Examples of stable semigroups are given in Section 3.4.

The second part of Question 3.1 is then answered by the following theorems. The first one concerns the direct product of two infinite semigroups.

Theorem 3.21 [47, Theorem 3.5] *Let S and T be infinite semigroups. The direct product $S \times T$ is finitely presented if and only if S and T are (finitely presented and) stable.*

For the direct product of a finite and an infinite semigroup we have:

Theorem 3.22 [47, Theorem 8.2] *Let S be a finite semigroup and T be an infinite semigroup. Then the direct product $S \times T$ is finitely presented if and only the following conditions are satisfied:*

(i) *S is stable; and*

(ii) *T is finitely presented.*

This last theorem suggests some further definitions. We say that a finite semigroup S *preserves finite generation* (resp. *finite presentability*) in direct products if it satisfies the following property: for every infinite semigroup T the direct product $S \times T$ is finitely generated (resp. finitely presented) if and only if T is finitely generated (resp. finitely presented). Also, we say that S *destroys finite generation* (resp. *finite presentability*) in direct products if $S \times T$ is not finitely generated (resp. finitely presented) for any infinite semigroup T. In this terminology, Theorems 3.3 and 3.22 combined give

Theorem 3.23 *For a finite semigroup S exactly one of the following conditions is true:*

(I) *S destroys finite generation (and hence finite presentability as well) in direct products; or*

(II) *S preserves finite generation but destroys finite presentability in direct products; or*

(III) *S preserves finite presentability (and also finite generation) in direct products.*

It also follows that a finite semigroup S satisfies condition (i) of the theorem if and only if $S^2 \neq S$; it satisfies condition (ii) if and only if $S^2 = S$ and it is not stable; and it satisfies condition (iii) if and only if ($S^2 = S$ and) it is stable.

3.3 Stability

As seen in the previous section, the notion of stability plays a fundamental role in the characterization of finitely presented direct products. In fact, if we are able to decide whether a finitely presented semigroup is stable, then we can also decide when is the direct product of two finitely presented semigroups finitely presented. Hence the question arises as to whether we can decide if a finitely presented semigroup is stable. It is easy to see that a positive answer does not follow straight from the definition. In this chapter we give a constructive characterization of finite stable semigroups. The corresponding question for infinite semigroups is still open.

Notice that if a semigroup S is finite then an easy algorithm will decide whether $S^2 = S$.

So let S be a finite semigroup satisfying $S^2 = S$. We start by introducing some notation and definitions. Denote by $\mathfrak{R}(S)$, $\mathfrak{L}(S)$ the set of all $\mathcal{R}$-maximal and $\mathcal{L}$-maximal elements of S, respectively, that is

$$\mathfrak{R}(S) = \{s \in S : (\forall t \in S)(s \leq_{\mathcal{R}} t \Rightarrow t\mathcal{R}s)\},$$

$$\mathfrak{L}(S) = \{s \in S : (\forall t \in S)(s \leq_{\mathcal{L}} t \Rightarrow t\mathcal{L}s)\}.$$

Now, for an arbitrary $s \in S$ we define a graph $\Gamma(S, s)$ as follows. The set of vertices is

$$\{(\alpha, \mu, \omega) : \alpha \in \mathfrak{R}(S), \mu \in S, \omega \in \mathfrak{L}(S), \alpha\mu\omega = s\}. \tag{3.4}$$

Two vertices $(\alpha_1, \mu_1, \omega_1)$ and $(\alpha_2, \mu_2, \omega_2)$ are joined by an edge if and only if

$$(\alpha_1 = \alpha_2 \ \&\ \mu_1\omega_1 = \mu_2\omega_2) \text{ or } (\alpha_1\mu_1 = \alpha_2\mu_2 \ \&\ \omega_1 = \omega_2). \tag{3.5}$$

The graph $\Gamma(S, s)$ is called the *stability graph of* s. When no confusion arises this graph is simply denoted by $\Gamma(s)$. Examples can be found in Section 3.4. With this notation we have the following

Theorem 3.24 *The following two conditions are equivalent for a finite semigroup S, satisfying $S^2 = S$:*

(i) *S is stable; and*

(ii) *all graphs $\Gamma(s)$ $(s \in S)$ are connected.*

Equivalently, a semigroup S preserves finite presentability in direct products if and only if $S^2 = S$ and all graphs $\Gamma(s)$ $(s \in S)$ are connected. Clearly, the graphs $\Gamma(s)$ $(s \in S)$ can be effectively constructed from the multiplication table of S. Therefore it is decidable whether a finite semigroup is stable and thus whether it preserves finite presentability in direct products. In Section 3.5 we describe such a decision algorithm.

In order to prove Theorem 3.24 we start by proving the following

Lemma 3.25 *Let S be a finite semigroup satisfying $S^2 = S$.*

(i) *For every $s \in S$ there exist $\lambda(s) \in \mathfrak{R}(S)$ and $s' \in S$ such that $s = \lambda(s)s'$, and, dually, there exist $\rho(s) \in \mathfrak{L}(S)$ and $s'' \in S$ such that $s = s''\rho(s)$.*

(ii) *For every $s \in \mathfrak{R}(S)$ there exists $\eta(s) \in S$ such that $s = s\eta(s)$; dually, for every $s \in \mathfrak{L}(S)$ there exists $\zeta(s) \in S$ such that $s = \zeta(s)s$.*

PROOF Let $s \in S$ be arbitrary. If $s \in \mathfrak{R}(S)$ then use $S^2 = S$ to write $s = s_1s_2$, for some $s_1, s_2 \in S$. From $s \prec_{\mathcal{R}} s_1$ it follows that $s_1 \in \mathfrak{R}(S)$, and so we may let $\lambda(s) = s_1$, $s' = s_2$. Also, if $s_3 \in S$ is such that $s_1 = ss_3$, then $s = ss_3s_2$, and we may let $\eta(s) = s_3s_2$. If $s \notin \mathfrak{R}(S)$, then we can choose $s_1 \in \mathfrak{R}(S)$ such that $s \prec_{\mathcal{R}} s_1$. Hence there exists $s_2 \in S$ such that $s = s_1s_2$ so we let $\lambda(s) = s_1$, $s' = s_2$.

Now if $s \in \mathfrak{L}(S)$ then use $S^2 = S$ to write $s = s_1s_2$ for some $s_1, s_2 \in S$. From $s \prec_{\mathcal{L}} s_2$ it follows that $s_2 \in \mathfrak{L}(S)$, and so we may let $\rho(s) = s_2$, $s'' = s_1$. Also if $s_3 \in S$ is such that $s_2 = s_3s$, then $s = s_1s_3s$, and we may let $\zeta(s) = s_1s_3$. If $s \notin \mathfrak{L}(S)$, then we can choose $s_2 \in \mathfrak{L}(S)$ such that $s \prec_{\mathcal{L}} s_2$. Hence there exists $s_1 \in S$ such that $s = s_1s_2$ so we let $\rho(s) = s_2$, $s'' = s_1$. □

In what follows we assume that $\lambda(s), s', \rho(s), s'', \eta(s), \zeta(s)$ are fixed (whenever defined) in accord with Lemma 3.25.

In both propositions below we consider S as a generating set for itself. In this context, for $w \in S^+$ we write $\overline{w}$ for the element of S represented by w. We also consider the multiplication table presentation: recall that it has the form $\mathcal{M}(S) = \langle S | R \rangle$ where R consists of the relations of the form $s_1s_2 = s_3$ $(s_1, s_2, s_3 \in S)$ where $\overline{s_1s_2} = s_3$.

Proposition 3.26 *Let S be a finite semigroup satisfying $S^2 = S$. If all the graphs $\Gamma(s)$ $(s \in S)$ are connected then S has no critical pairs with respect to its multiplication table, and hence S is stable.*

PROOF Let $s \in S$ be arbitrary, and let $w_1, w_2 \in S^+$ be any two words such that $\overline{w_1} = \overline{w_2} = s$. We are going to prove that there exists an elementary sequence from w_1 to w_2, with respect to the multiplication table $\mathcal{M}(S)$, in which every term has length at least $M = \min\{|w_1|, |w_2|\}$. If $M = 1$ there is nothing to prove. So assume that

$$w_1 \equiv a_1 a_2 \ldots a_m, \; w_2 \equiv b_1 b_2 \ldots b_n$$

with $m, n \geq 2$, $a_i, b_i \in S$.

We claim that there exists an elementary sequence

$$w_1 \equiv \beta_1, \beta_2, \ldots, \beta_k \equiv \lambda(a_1)(\eta\lambda(a_1))^M p\rho(a_m), \tag{3.6}$$

where $p = \overline{a_1' a_2 \ldots a_{m-1} a_m''}$. Indeed, we can use the relations $a_1 = \lambda(a_1)a_1'$, $a_m = a_m''\rho(a_m)$ and $\lambda(a_1) = \lambda(a_1)(\eta\lambda(a_1))$ (which hold in S by Lemma 3.25, and hence are in $\mathcal{M}(S)$) to transform w_1 into the word

$$\lambda(a_1)(\eta\lambda(a_1))^M a_1' a_2 \ldots a_{m-1} a_m'' \rho(a_m),$$

without ever decreasing the length of a word. Now we can use *any* elementary sequence from

$$a_1' a_2 \ldots a_{m-1} a_m''$$

to p and transform the word

$$\lambda(a_1)(\eta\lambda(a_1))^M a_1' a_2 \ldots a_{m-1} a_m'' \rho(a_m)$$

into

$$\lambda(a_1)(\eta\lambda(a_1))^M p\rho(a_m).$$

The subword $(\eta\lambda(a_1))^M$ guarantees that every intermediate word has length at least M.

Dually, we have an elementary sequence

$$\lambda(b_1)(\eta\lambda(b_1))^M q\rho(b_n) \equiv \gamma_1, \gamma_2, \ldots, \gamma_l \equiv w_2 \tag{3.7}$$

where $q = \overline{b_1' b_2 \dots b_{n-1} b_n''}$, in which every term has length at least M.

Now we are going to find an elementary sequence from $\lambda(a_1)(\eta\lambda(a_1))^M p\rho(a_m)$ to $\lambda(b_1)(\eta\lambda(b_1))^M q\rho(b_n)$ by using the assumption that the graph $\Gamma(s)$ is connected. Note that $(\lambda(a_1), p, \rho(a_m))$ and $(\lambda(b_1), q, \rho(b_n))$ are vertices in $\Gamma(s)$. So there is a path

$$(\lambda(a_1), p, \rho(a_m)) = (\alpha_1, \mu_1, \omega_1), (\alpha_2, \mu_2, \omega_2), \dots, (\alpha_h, \mu_h, \omega_h) = (\lambda(b_1), q, \rho(b_n))$$

in $\Gamma(s)$. We claim that for each i $(1 \leq i \leq h-1)$ there exists an elementary sequence

$$\alpha_i(\eta(\alpha_i))^M \mu_i\omega_i \equiv \delta_{i1}, \delta_{i2}, \dots, \delta_{ir_i} \equiv \alpha_{i+1}(\eta(\alpha_{i+1}))^M \mu_{i+1}\omega_{i+1} \tag{3.8}$$

in which every term has length at least M. According to (3.5), there are the following two cases to be considered:

Case 1: $\alpha_i = \alpha_{i+1}, \mu_i\omega_i = \mu_{i+1}\omega_{i+1}$. In this case a suitable sequence is

$$\begin{aligned}
&\alpha_i(\eta(\alpha_i))^M \mu_i\omega_i,\\
&\alpha_i(\eta(\alpha_i))^M \overline{\mu_i\omega_i} \equiv \alpha_{i+1}(\eta(\alpha_{i+1}))^M \overline{\mu_{i+1}\omega_{i+1}},\\
&\alpha_{i+1}(\eta(\alpha_{i+1}))^M \mu_{i+1}\omega_{i+1}.
\end{aligned}$$

Case 2: $\alpha_i\mu_i = \alpha_{i+1}\mu_{i+1}, \omega_i = \omega_{i+1}$. In this case we first use the relation $\omega_i = (\zeta(\omega_i))\omega_i$ to transform $\alpha_i(\eta(\alpha_i))^M \mu_i\omega_i$ into $\alpha_i(\eta(\alpha_i))^M \mu_i(\zeta(\omega_i))^M\omega_i$. Then we use the relation $\alpha_i = \alpha_i(\eta(\alpha_i))$ to transform this last word into $\alpha_i\mu_i(\zeta(\omega_i))^M\omega_i$; the subword $(\zeta(\omega_i))^M$ guarantees that every intermediate word has length at least M. We now proceeding dually to Case 1, first obtaining the sequence

$$\begin{aligned}
&\alpha_i\mu_i(\zeta(\omega_i))^M\omega_i,\\
&\overline{\alpha_i\mu_i}(\zeta(\omega_i))^M\omega_i \equiv \overline{\alpha_{i+1}\mu_{i+1}}(\zeta(\omega_{i+1}))^M\omega_{i+1},\\
&\alpha_{i+1}\mu_{i+1}(\zeta(\omega_{i+1}))^M\omega_{i+1}
\end{aligned}$$

and then use the relation $\alpha_i = \alpha_i(\eta(\alpha_i))$ to transform this last word into

$$\alpha_{i+1}(\eta(\alpha_{i+1}))^M \mu_{i+1}(\zeta(\omega_{i+1}))^M\omega_{i+1}$$

and then use the relation $\omega_{i+1} = (\zeta(\omega_{i+1}))\omega_{i+1}$ to transform it into

$$\alpha_{i+1}(\eta(\alpha_{i+1}))^M \mu_{i+1}\omega_{i+1}$$

as desired. Once again any intermediate word has length at least M.

By concatenating sequence (3.6) with sequences (3.8) for $i = 1, \ldots, h-1$ and sequence (3.7) we obtain an elementary sequence from w_1 to w_2 in which every term has length at least M. Hence the pair (w_1, w_2) is not a critical pair with respect to $\mathcal{M}(S)$, and therefore S is stable as required. □

Proposition 3.27 *Let S be a finite semigroup satisfying $S^2 = S$, and let $s \in S$ be arbitrary. If $\Gamma(s)$ is not connected then, for any finite presentation $\langle S|R\rangle$ of S (with respect to the generating set S), there is a critical pair (w_1, w_2) such that $\overline{w_1} = \overline{w_2} = s$. Consequently, S is not stable.*

Proof Let $(\alpha_1, \mu_1, \omega_1)$ and $(\alpha_2, \mu_2, \omega_2)$ be two vertices of $\Gamma(s)$ such that there is no path connecting them. Consider an arbitrary finite presentation $\langle S|R\rangle$ for S, and let

$$N = \max\{|u|, |v| : (u = v) \in R\}$$

be the maximum length of a left hand side or a right hand side of a relation from R. Next let

$$w_1 \equiv \alpha_1\mu_1(\zeta(\omega_1))^N\omega_1 \text{ and } w_2 \equiv \alpha_2\mu_2(\zeta(\omega_2))^N\omega_2.$$

By Lemma 3.25 (ii) and the definition of $\Gamma(s)$ it follows that $\overline{w_1} = \overline{w_2} = s$. We are now going to prove that (w_1, w_2) is a critical pair. Assume that it is not. Then there is an elementary sequence $w_1 \equiv \beta_1, \beta_2, \ldots, \beta_m \equiv w_2$ such that $|\beta_i| \geq N + 3$ $(1 \leq i \leq m)$. Write $\beta_i \equiv a_i\gamma_i b_i$ with $a_i, b_i \in S$, $\gamma_i \in S^+$ $(1 \leq i \leq m)$.

Claim 1 *For each i $(1 \leq i \leq m-1)$ the vertices $(\lambda(a_i), a_i'\gamma_i b_i'', \rho(b_i))$ and $(\lambda(a_{i+1}), a_{i+1}'\gamma_{i+1}b_{i+1}'', \rho(b_{i+1}))$ are connected by an edge in $\Gamma(s)$.*

Proof Since β_{i+1} is obtained from β_i by one application of a relation from R, we can write $\beta_i \equiv \delta u\xi$, $\beta_{i+1} \equiv \delta v\xi$, where $\delta, \xi \in S^*$ and $u = v$ is a relation from R. Moreover, since $|\beta_i| \geq N + 3$ and $|u| \leq N$, at least one of δ or ξ is non-empty. Assume that δ is non-empty. Then we clearly have $a_i = a_{i+1}$, and hence $\lambda(a_i) = \lambda(a_{i+1})$ and $a_i' = a_{i+1}'$. Also since u is a subword of $\gamma_i b_i$, we have $\gamma_i b_i = \gamma_{i+1}b_{i+1}$, and hence

$$a_i'\gamma_i b_i''\rho(b_i) = a_i'\gamma_i b_i = a_{i+1}'\gamma_{i+1}b_{i+1} = a_{i+1}'\gamma_{i+1}b_{i+1}''\rho(b_{i+1})$$

by Lemma 3.25 (i). Therefore, in this case, the given vertices are indeed joined by an edge by (3.5). The case where ξ is non-empty is considered in an analogous way. □

Claim 2 *In $\Gamma(s)$ there is a path from $(\alpha_1, \mu_1, \omega_1)$ to $(\lambda(a_1), a_1'\gamma_1 b_1'', \rho(b_1))$ and also a path from $(\alpha_2, \mu_2, \omega_2)$ to $(\lambda(a_m), a_m'\gamma_m b_m'', \rho(b_m))$.*

PROOF We prove the first statement; the second is proved analogously. By Lemma 3.25 (ii) we have $\mu_1(\zeta(\omega_1))^N\omega_1 = \mu_1\omega_1$, and hence there is an edge between $(\alpha_1, \mu_1, \omega_1)$ and $(\alpha_1, \mu_1(\zeta(\omega_1))^N, \omega_1)$. Since $\alpha_1\mu_1(\zeta(\omega_1))^N\omega_1 \equiv w_1 \equiv \beta_1 \equiv a_1\gamma_1 b_1$ we have $\alpha_1 = a_1$, $\mu_1(\zeta(\omega_1))^N \equiv \gamma_1$ and $\omega_1 = b_1$. Therefore we have $\alpha_1\mu_1(\zeta(\omega_1))^N = \lambda(a_1)a_1'\gamma_1$ by Lemma 3.25 (i), and so there is an edge between $(\alpha_1, \mu_1(\zeta(\omega_1))^N, \omega_1)$ and $(\lambda(a_1), a_1'\gamma_1, \omega_1)$. Similarly, from $a_1'\gamma_1\omega_1 = a_1'\gamma_1 b_1''\rho(b_1)$ it follows that there is an edge between $(\lambda(a_1), a_1'\gamma_1, \omega_1)$ and $(\lambda(a_1), a_1'\gamma_1 b_1'', \rho(b_1))$, completing the proof of the claim. □

By combining Claims 1 and 2 we conclude that there is a path connecting $(\alpha_1, \mu_1, \omega_1)$ and $(\alpha_2, \mu_2, \omega_2)$, contradicting our assumption that they are in different connected components of $\Gamma(s)$. Therefore (w_1, w_2) is indeed a critical pair. We have shown that S has critical pairs with respect to any finite presentation having S as the set of generators. It now follows that S is not stable by Proposition 3.20, which asserts that non-existence of critical pairs is invariant under change of generators. □

Theorem 3.24 is an immediate consequence of Proposition 3.26 and 3.27.

3.4 Examples and applications

It was shown in [47] that the question of whether a semigroup S preserves or destroys finite presentability in direct products is related to existence of certain kinds of weak identity elements. We say that an element $e \in S$ is a *relative left* (resp. *right*) *identity* for an element $s \in S$ if $es = s$ (resp. $se = s$). Recall that if e is a relative left (resp. right) identity for every $s \in S$ we say that e is a left (resp. right) identity of S. With this terminology [47, Theorem 7.2 and Remark 7.5] state that either of the following conditions is sufficient for S to be stable:

(I1) every element of S has a relative left identity and a relative right identity; or

(I2) S has a left identity or it has a right identity.

It immediately follows that finitely presented monoids and finitely presented semigroups, which are regular, are examples of stable semigroups. Also, if S is a finitely generated commutative semigroup, satisfying $S^2 = S$, then S is stable. Indeed, by Rédei's Theorem (see Theorem 7.2), it follows that S is finitely presented. Now, if A is a minimal generating set for S (that is, no generator of A is redundant) then for any $a \in A$, since $S^2 = S$, there exists a non-empty word w_a such that $a = w_a a = a w_a$, and therefore S every element of S has a relative left and a relative right identity.

If S is a finite semigroup it is easy to prove, using Theorem 3.24, that (I1) and (I2) are sufficient conditions for S to be stable. Indeed, if (I1) holds then, for any $s \in S$, one may take s' and s'' to be s, and $\lambda(s)$ and $\rho(s)$ to be the relative left and right identities for s respectively. Now for any two vertices $(\alpha_1, \mu_1, \omega_1)$ and $(\alpha_2, \mu_2, \omega_2)$ of $\Gamma(s)$ the sequence

$$\begin{aligned}&(\alpha_1, \mu_1, \omega_1), (\lambda(\alpha_1), \alpha_1\mu_1, \omega_1),\\ &(\lambda(\alpha_1), \alpha_1\mu_1\omega_1, \rho(\omega_1)) = (\lambda(\alpha_1), s, \rho(\omega_1)), (\lambda(\alpha_2), s, \rho(\omega_1)),\\ &(\lambda(\alpha_2), s, \rho(\omega_2)) = (\lambda(\alpha_2), \alpha_2\mu_2\omega_2, \rho(\omega_2)), (\lambda(\alpha_2), \alpha_2\mu_2, \omega_2), (\alpha_2, \mu_2, \omega_2)\end{aligned}$$

is a path, and so $\Gamma(s)$ is connected. Similarly, if S has, say, a left identity e (condition (I2)) then for any two vertices $(\alpha_1, \mu_1, \omega_1)$, $(\alpha_2, \mu_2, \omega_2)$ of a graph $\Gamma(s)$ the sequence

$$(\alpha_1, \mu_1, \omega_1), (e, \alpha_1\mu_1, \omega_1), (e, \alpha_2\mu_2, \omega_2), (\alpha_2, \mu_2, \omega_2)$$

is a path. It is also easy to see that neither (I1) nor (I2) implies the other. Indeed, in the 3-element semigroup $\langle a, b | a^2 = a, ab = b, b^3 = b^2 = ba\rangle$ the element a is a left identity, but the element b has no relative right identity. Also, in the 3-element semilattice $\langle a, b | a^2 = a, b^2 = b, ab = ba\rangle$ every element is its own relative left and relative right identity, but there is no left or right identity. Therefore neither of the above two conditions is necessary for S to preserve finite presentability in direct products. A question then arises as to whether their disjunction is necessary for S to preserve finite presentability. The following example shows that this is not the case.

Example 3.28 Let S be the semigroup with multiplication table

$$\begin{array}{c|cccccc} & a & b & c & d & e & 0 \\ \hline a & a & a & a & d & 0 & 0 \\ b & b & b & b & d & 0 & 0 \\ c & a & b & c & d & 0 & 0 \\ d & 0 & 0 & 0 & 0 & 0 & 0 \\ e & 0 & 0 & 0 & 0 & e & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}.$$

Clearly $S^2 = S$. We have $\mathfrak{R}(S) = \{c, e\}$ and $\mathfrak{L}(S) = \{c, d, e\}$. By (3.4) the graphs $\Gamma(a)$, $\Gamma(b)$, $\Gamma(c)$ and $\Gamma(e)$ all have a single vertex, namely (c, a, c), (c, b, c), (c, c, c) and (e, e, e) respectively, and, in particular, they are connected. The graph $\Gamma(d)$ has three vertices, (c, a, d), (c, b, d) and (c, c, d) and, since $ad = bd = cd (= d)$, there are edges between any two of these vertices; hence $\Gamma(d)$ is connected (see Figure 3.1).

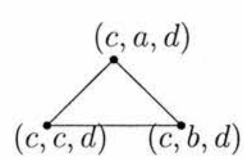

Figure 3.1: The graph $\Gamma(S, d)$ from Example 3.28

Finally, we consider the graph $\Gamma(0)$ (which has 29 vertices and it is represented on Figure 3.2) and we show that it is also connected. Let $(\alpha_1, \mu_1, \omega_1)$ be any vertex of $\Gamma(0)$. From $\alpha_1\mu_1\omega_1 = 0$ it follows that $\alpha_1\mu_1 = 0$ or $\mu_1\omega_1 = 0$; in any case there is an edge between $(\alpha_1, \mu_1, \omega_1)$ and $(\alpha_1, 0, \omega_1)$ by (3.5). Similarly, if $(\alpha_2, \mu_2, \omega_2)$ is any other vertex of $\Gamma(0)$, it is adjacent to $(\alpha_2, 0, \omega_2)$. Finally, the sequence $(\alpha_1, 0, \omega_1)$, $(\alpha_2, 0, \omega_1)$, $(\alpha_2, 0, \omega_2)$ is a path. Therefore $\Gamma(0)$ is indeed connected. By Proposition 3.26 we conclude that S preserves finite presentability, although S has no left or right identity and the element d has no relative right identity.

An example of a finite semigroup which preserves finite generation but destroys finite presentability is given in [47, Example 8.4]; it has 11 elements. We

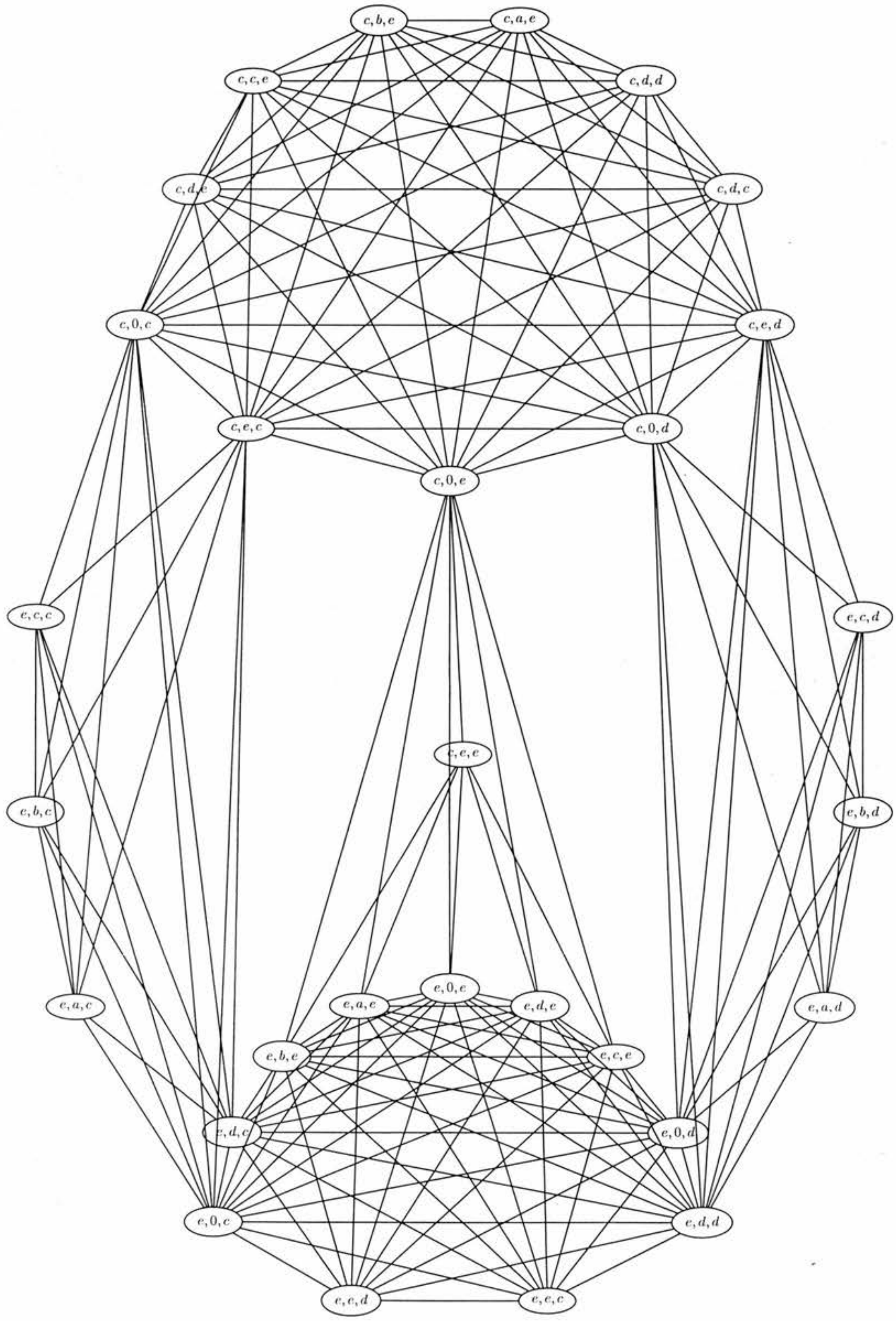

Figure 3.2: The graph $\Gamma(S, 0)$ from Example 3.28

now exhibit two such semigroups with only 4 elements.

Example 3.29 Let S be the semigroup with multiplication table

$$\begin{array}{c|cccc} & a & b & c & 0 \\ \hline a & 0 & 0 & 0 & 0 \\ b & a & b & b & 0 \\ c & a & c & c & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}.$$

Clearly $S^2 = S$. We have $\mathfrak{R}(S) = \{b, c\}$ and $\mathfrak{L}(S) = \{a, b, c\}$. By (3.4), the graph $\Gamma(a)$ has vertices (b, b, a), (b, c, a), (c, b, a) and (c, c, a) (see Figure 3.3). Indeed,

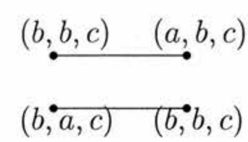

Figure 3.3: The graph $\Gamma(S, a)$ from Example 3.29

since $ba = ca$, there is an edge joining (b, b, a) and (b, c, a), and there is an edge joining (c, b, a) and (c, c, a), but there are no other edges in the graph. Therefore $\Gamma(a)$ is not connected and so S destroys finite presentability though it preserves finite generation.

Example 3.30 Let S be the semigroup with multiplication table

$$\begin{array}{c|cccc} & a & b & c & 0 \\ \hline a & 0 & 0 & 0 & 0 \\ b & 0 & 0 & b & 0 \\ c & a & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}.$$

Clearly $S^2 = S$. Now we have $\mathfrak{R}(S) = \{b, c\}$ and $\mathfrak{L}(S) = \{a, c\}$. The graphs $\Gamma(a)$, $\Gamma(b)$ and $\Gamma(c)$ all have a single vertex, namely (c, c, a), (b, c, c) and (c, c, c), respectively, and hence they are connected. However, we claim that the graph $\Gamma(0)$ is not connected (although every element of S is both a relative left and a relative right identity for 0). (See Figure 3.4.) Indeed, there is no edge connecting

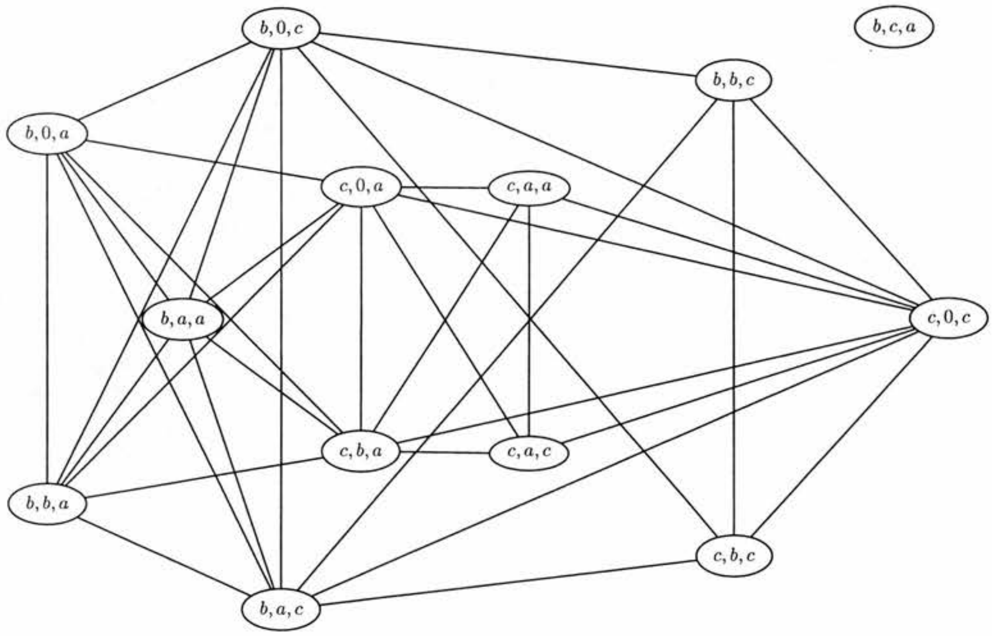

Figure 3.4: The graph $\Gamma(S, 0)$ from Example 3.30

the vertex (b, c, a) to any of the other 12 vertices of $\Gamma(0)$. This is because a vertex $(\alpha, \mu, \omega) \neq (b, c, a)$ satisfies $\alpha\mu = 0$ or $\mu\omega = 0$, whereas $bc = b \neq 0$ and $ca = a \neq 0$. Therefore we conclude that S preserves finite generation but destroys finite presentability. Notice that in this example the non connected graph arises from an element with both relative left and relative right identities.

We will denote the semigroup from Example 3.29 by T_1 and the semigroup from Example 3.30 by T_2.

Remark 3.31 The above examples are minimal, in the sense that any semigroup of order less than 4 either destroys finite generation or preserves finite presentability. Indeed, the unique semigroup of order one is actually a monoid and hence is preserves finite presentability. Now, there are four semigroups of order two and eighteen of order three:

S_1	0	a
0	0	0
a	0	0

,

S_2	0	a
0	0	0
a	0	a

,

S_3	a	b
a	a	a
b	b	b

,

S_4	0	a
0	a	b
a	b	a

,

S_5	0	a	b
0	0	0	0
a	0	0	0
b	0	0	0

S_6	0	a	b
0	0	0	0
a	0	0	0
b	0	0	a

S_7	0	a	b
0	0	0	0
a	0	0	0
b	0	0	b

S_8	0	a	b
0	0	0	0
a	0	0	0
b	0	a	b

S_9	0	a	b
0	0	0	0
a	0	a	a
b	0	b	b

S_{10}	0	a	b
0	0	0	0
a	0	a	a
b	0	a	a

S_{11}	0	a	b
0	0	0	0
a	0	0	a
b	0	a	b

S_{12}	0	a	b
0	0	0	0
a	0	a	0
b	0	0	b

S_{13}	0	a	b
0	0	0	0
a	0	a	a
b	0	a	b

S_{14}	a	b	c
a	a	a	a
b	a	a	a
c	c	c	c

S_{15}	0	a	b
0	0	0	0
a	0	a	b
b	0	b	a

S_{16}	a	b	c
a	a	a	a
b	a	b	a
c	c	c	c

S_{17}	a	b	c
a	a	a	c
b	a	a	c
c	c	c	a

S_{18}	a	b	c
a	a	b	b
b	b	a	a
c	b	a	a

S_{19}	a	b	c
a	a	a	a
b	a	b	c
c	c	c	c

S_{20}	a	b	c
a	a	a	a
b	b	b	b
c	c	c	c

S_{21}	a	b	c
a	a	a	c
b	a	b	c
c	c	c	a

S_{22}	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

(This list is from [18], and can also be found in [41]. We use zero as an element of the semigroup, whenever the semigroup has a zero element, to emphasize that fact.) Clearly semigroups S_1, S_5, S_6, S_7, S_{10}, S_{14}, S_{17} and S_{18} have indecomposable elements, since not every element appears in the multiplication table. Hence those semigroups destroy finitely generation in direct products. Now, S_4, S_{15}, S_{19}, S_{21} and S_{22} are monoids, and all the other semigroups satisfy at least one of (I1) or (I2) above. Thus these semigroups preserve finite presentability in direct products.

It is also true that T_1 and T_2 are the only two semigroups of order 4, which are non-stable and do not have indecomposable elements. To see this we have used the implementation, in GAP, of the algorithm described in Section 3.5 (see Appendix A), and the list of the multiplication tables for all non-isomorphic and non-anti-isomorphic semigroups of order 4 given in [18].

We also have

Proposition 3.32 *For all $n \geq 4$ there exists a non-stable semigroup S of order n satisfying $S^2 = S$.*

PROOF We use induction on n. Both T_1 and T_2 satisfy the case $n = 4$. So suppose that S is a non-stable semigroup of order $n-1$, satisfying $S^2 = S$. Let S^0 denote the semigroup obtained from S by adjoining a zero. Notice that for any $s \in S$, the graphs $\Gamma(S, s)$ and $\Gamma(S^0, s)$ are equal. It therefore follows that S^0 is not stable. Clearly $(S^0)^2 = S^0$ and S^0 has order n. □

It follows from the proof of Proposition 3.32, that there is a non-stable semigroup of any order $n \geq 4$, with no indecomposable elements, which has T_1 as a subsemigroup, and the same is true for T_2. The following example exhibits two semigroups of order 5, which are not obtained as in the proof of Proposition 3.32: one of them admits T_1 as a subsemigroup and the other one admits T_2 as a subsemigroup.

Example 3.33 Let S_1, S_2 be the semigroups defined by the multiplication tables

$$\begin{array}{c|ccccc} & a & b & c & d & 0 \\ \hline a & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 & 0 & 0 \\ c & 0 & 0 & 0 & c & 0 \\ d & a & b & 0 & d & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}, \qquad \begin{array}{c|ccccc} & a & b & c & d & 0 \\ \hline a & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 & 0 & 0 \\ c & a & b & c & c & 0 \\ d & a & b & d & d & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array},$$

respectively. Notice that neither of them can be obtained by adjoining a zero to a semigroup of order 4. They are both non-stable semigroups and contain no indecomposable elements. For S_1, $\mathfrak{R}(S_1) = \{c, d\}$ and $\mathfrak{L}(S_1) = \{a, b, d\}$ and the graph $\Gamma(S_1, 0)$ is not connected. Indeed, both vertices (c, d, a) and (c, d, b) are not connected to any other vertex. Clearly there is no edge between them. Also, any other vertex $(\alpha, \mu, \omega) \neq (c, d, a), (c, d, b)$ satisfies $\alpha\mu = 0$ or $\mu\omega = 0$, whereas $cd = c \neq 0$, $da = a \neq 0$ and $db = b$. Now, for S_2, $\mathfrak{R}(S_2) = \{c, d\}$ and $\mathfrak{L}(S_2) = \{a, b, c, d\}$. The graphs $\Gamma(S_2, a)$ and $\Gamma(S_2, b)$ are not connected. As for $\Gamma(S_2, a)$ it has 4 vertices, namely (c, c, a), (c, d, a), (d, c, a) and (d, d, a) and there is an edge joining (c, c, a) and (c, d, a) and another edge joining (d, c, a) and (d, d, a). However there are no other edges in the graph and hence it is not connected. The graph $\Gamma(S_2, b)$ is similar. Now, $\{a, c, d, 0\}$ forms a subsemigroup of S_1 which is isomorphic to T_2 and it it forms a subsemigroup of S_2 isomorphic to T_1.

However, there are non-stable semigroups of higher orders (without indecomposable elements) that do not have T_1 or T_2 as subsemigroups. Thus, although T_1 and T_2 are of minimal size, they are not minimal in the sense that they do not have to be subsemigroups of any other non-stable semigroup without indecomposable elements. The following is such an example.

Example 3.34 Let S be the semigroup defined by the multiplication table

	a	b	c	d	0
a	0	0	0	0	0
b	0	0	0	b	0
c	0	0	0	c	0
d	a	b	b	d	0
0	0	0	0	0	0

Now we have $\mathfrak{R}(S) = \{c, d\}$ and $\mathfrak{L}(S) = \{a, d\}$. The graphs $\Gamma(a)$, $\Gamma(c)$ and $\Gamma(d)$ all have a single vertex, namely (d, d, a), (c, d, d) and (d, d, d), respectively, and hence they are connected. The graph $\Gamma(b)$ has two vertices, namely (d, b, d) and (d, c, d), which are clearly connected by an edge. However, the graph $\Gamma(0)$ is not connected, Indeed, there is no edge connecting the vertex (c, d, a) to any of the other 14 vertices of $\Gamma(0)$. This is again because a vertex $(\alpha, \mu, \omega) \neq (c, d, a)$ satisfies $\alpha\mu = 0$ or $\mu\omega = 0$, whereas $bc = b \neq 0$ and $ca = a \neq 0$. Therefore we conclude that S preserves finite generation but destroys finite presentability.

However, we claim that neither T_1 nor T_2 is a subsemigroup of S. Suppose to the contrary that T_1 was a subsemigroup of S. Then, by looking at the multiplication table of T_1 we know that there would exist $x, y \in S$ with $x \neq y$ and such that $x^2 = x$ and $y^2 = y$. This is clearly not the case in S. If T_2 was a subsemigroup of S then since d is a unique idempotent of S then there would exist $z \in S$ with $z \neq 0$ such that $dz = 0$. Again this is not the case. The claim is hence proved.

The examples of semigroups of order 5 were found by inspection on the list of semigroups of order 5 given in [41], and using the implementation in GAP of the algorithms described in Section 3.5.

Also, notice that having T_1 or T_2 as a subsemigroup, does not imply that the semigroup is non-stable. Indeed, if we add an identity to a non-stable semigroup we obtain a stable one, which has the original one as a subsemigroup.

3.5 Algorithms

As mentioned before, the results of Section 3.3 give rise to a decision algorithm. Thus, given a finite semigroup S it is algorithmically decidable whether S preserves finite generation or presentability in direct products or whether it destroys them. Furthermore, given a direct product of a finite semigroup S and an infinite finitely presented semigroup T, we will have an answer to the question of whether $S \times T$ is finitely generated and finitely presented.

In this section we present these algorithms. They rely mostly on the fact that the semigroup S is finite, and hence we are able to iterate over all its elements. These algorithms have been implemented in GAP (see Appendix A), and used to produce some of the examples that appear in Section 3.4.

Some basic algorithms are assumed, as for example a function `Elements`(S) (which returns the list of elements of a finite semigroup S) and functionality related with Greens' relations. In particular, `GreensRMaximalElementsOfSemigroup` and also `GreensLMaximalElementsOfSemigroup` which for a finite semigroup S, return the set of all maximal $\mathcal{R}$ and $\mathcal{L}$ elements of S, respectively. Finally we assume that we have a function `Graph` (that given a finite set of vertices and a finite set of edges between the vertices, creates the graph) and a function `IsConnectedGraph`. Clearly, since we are only concerned with finite semigroups, all these exist. Furthermore, in a system like GAP, all these algorithms are either implemented there (or can be found in a share package) or else can be very easily implemented based on the available functionality.

Now, to check if a finite semigroup has indecomposable elements we can, for each element $x \in S$ (or for each generator of S), try to find two elements $s, t \in S$ such that $st = x$. If we cannot find them by looping through all elements of S then x is indecomposable. So we have `IsSemigroupWithIndecomposableElements` that given a finite semigroup S, returns *true* if the semigroup has indecomposable elements (i.e. $S^2 \neq S$) and *false* otherwise. We therefore have the following

Algorithm 3.35 `PreservesFiniteGenerationInDirectProducts`

INPUT: a finite semigroup S

OUTPUT: *true* or *false*

if `IsSemigroupWithIndecomposableElements`(S) **then**
  return *false*
else
  return *true*
end if

Now, in order to produce an algorithm `IsStableSemigroup` we start by creating the stability graphs of the elements of S.

Algorithm 3.36 `StabilityGraphOfElement`

INPUT: a finite semigroup S and an element $x \in S$

OUTPUT: the graph $\Gamma(S, x)$, as defined in Section 3.3

{We start by computing the set of vertices, using (3.4)}
$V \leftarrow \emptyset$
for all $a \in$ `GreensRMaximalElementsOfSemigroup`(S) **do**
  for all $c \in$ `GreensLMaximalElementsOfSemigroup`(S) **do**
    for all $b \in$ `Elements`(S) **do**
      if $abc = x$ **then**
        $V \leftarrow V \cup \{(a, b, c)\}$

      end if
    end for
  end for
end for
{Now we compute the set of edges, using (3.5)}
$E \leftarrow \emptyset$
for all $(a_1, b_1, c_1) \in V$ **do**
  for all $(a_2, b_2, c_2) \in V$ **do**
    if $(a_1, b_1, c_1) \neq (a_2, b_2, c_2)$ and $[(a_1 = a_2$ and $b_1c_1 = b_2c_2)$ or $(a_1b_1 = a_2b_2$ and $c_1 = c_2)]$ **then**
      $E \leftarrow E \cup \{[(a_1, b_1, c_1), (a_2, b_2, c_2)]\}$
    end if
  end for
end for
{We are now ready to build the graph}
return `Graph`(V, E)

Algorithm 3.37 `IsStableSemigroup`

INPUT: a finite semigroup S

OUTPUT: *true* or *false*

for all $x \in$ `Elements`(S) **do**
  $\Gamma \leftarrow$`StabilityGraphOfElement`(S, x)
  if not `IsConnectedGraph`(Γ) **then**
    {S is not stable by Theorem 3.24}
    return *false*
  end if
end for
{If all the graphs were connected then, again by Theorem 3.24, S is stable}
return *true*

and hence we have

Algorithm 3.38 `PreservesFinitePresentabilityInDirectProducts`

INPUT: a finite semigroup S

OUTPUT: *true* or *false*

```
if IsStableSemigroup(S) then
    return true
else
    return false
end if
```

3.6 A generalization: direct products of finitely many semigroups

Given a finite family of semigroups S_i ($i \in \{1, 2, \ldots, n\}$) we consider the direct product of all these semigroups

$$Z = \prod_{i=1}^{n} S_i = S_1 \times S_2 \times \cdots \times S_n.$$

Notice that Z is isomorphic to the direct product of the semigroup $S_1 \times S_2 \times \cdots S_{n-1}$ with S_n and to any direct product of those semigroups in any order. It is natural to ask when such a direct product is finitely generated and when it is finitely presented.

For finite generation we have the following consequence of Theorems 3.2 and 3.3.

Theorem 3.39 *Let S_i ($i \in I$) be a finite family of infinite semigroups and let T_j ($j \in J$) be a finite family of finite semigroups (with $I \neq \emptyset$ and $|I| + |J| \geq 2$). Then the direct product $Z = (\prod_{i \in I} S_i) \times (\prod_{j \in J} T_j)$ is finitely generated if and only if*

(i) for each $i \in I$, S_i is finitely generated;

(ii) if $|I| > 1$, $S_i^2 = S_i$; and

(iii) for each $j \in J$, $T_j = T_j^2$.

Proof If $J = \emptyset$ then

$$Z = \prod_{i \in I} S_i.$$

Since $|J| = 0$ then $|I| \geq 2$. We use induction on $|I|$. The result for $|I| = 2$ is Theorem 3.2. Now, suppose that the result holds for a direct product of k infinite semigroups and that $I = \{1, 2, \ldots, k, k+1\}$. Using Theorem 3.2 once more we have that Z is finitely generated if and only if $(\prod_{i=1}^{k} S_i)$ and S_{k+1} are both finitely generated and

$$(\prod_{i=1}^{k} S_i)^2 = \prod_{i=1}^{k} S_i, \ S_{k+1}^2 = S_{k+1}.$$

Clearly, $(\prod_{i=1}^{k} S_i)^2 = \prod_{i=1}^{k} S_i$ if and only if for each $i \in \{1, 2, \ldots, k\}$, $S_i^2 = S_i$. By the induction hypothesis $(\prod_{i=1}^{k} S_i)$ is finitely generated if and only if each S_i is finitely generated and $S_i^2 = S_i$ $(i \in \{1, 2, \ldots, k\})$ which concludes the induction.

Now, if $J \neq \emptyset$, and since $T = \prod_{j \in J} T_j$ is a finite semigroup, it follows from Theorem 3.3 that Z is finitely generated if and only if S is finitely generated and $T^2 = T$. It is clear that $T^2 = T$ if and only if $T_j^2 = T_j$ for all $j \in J$. Now, if $|I| \geq 2$ then, by the first case, S is finitely generated if and only if S_i is finitely generated and $S_i^2 = S_i$ $(i \in I)$, as required. □

For finite presentability we have:

Theorem 3.40 *Let S_i $(i \in I)$ be a finite family of infinite semigroups and let T_j $(j \in J)$ be a finite family of finite semigroups (with $I \neq \emptyset$ and $|I| + |J| \geq 2$). Then the direct product $Z = (\prod_{i \in I} S_i) \times (\prod_{j \in J} T_j)$ is finitely presented if and only if*

(i) for each $i \in I$, S_i is finitely presented;

(ii) if $|I| \geq 2$, for each $i \in I$, S_i is stable; and

(iii) for each $j \in J$, T_j is stable.

The main step for the proof of this theorem is the following

Proposition 3.41 *Let S and T be stable semigroups. Then the direct product $S \times T$ is also stable.*

We start by introducing some new definitions. Let $A = \{a_i : i \in I\}$ be an alphabet. We say that a presentation over A is *uniform* if it has the form

$$\langle A \mid a_i = a_{i\zeta} a_{i\eta}, \ R \ (i \in I) \rangle$$

where $\zeta, \eta : I \to I$ are mappings, and each $(u = v) \in R$ satisfies $|u| = |v|$. Notice that if S is defined by a uniform presentation, in terms of a generating set A, then $A \subseteq A^2$ (i.e. every generator of A can be expressed as a product of two generators of A). This implies that any word in A^+ of length m is equal in S to a word of length n, for any $n \geq m$.

The following proposition characterizes semigroups that can be defined by uniform presentations, and establishes a connection between uniform presentations and stability.

Proposition 3.42 [47, Proposition 5.5] *A semigroup S can be defined by a uniform presentation if and only if $S^2 = S$. Furthermore, if S is finitely presented then it can be defined by a finite uniform presentation, and if S is stable then it has no critical pairs with respect to a finite uniform presentation.*

Now, given two alphabets $A = \{a_i : i \in I\}$ and $B = \{b_j : j \in J\}$, we define the *decomposition mapping*

$$\nu : \{(w_1, w_2) \in A^+ \times B^+ : |w_1| = |w_2|\} \to (A \times B)^+$$

by

$$(a_{i_1} a_{i_2} \cdots a_{i_n}, b_{j_1} b_{j_2} \cdots b_{j_n})\nu \equiv (a_{i_1}, b_{j_1})(a_{i_2}, b_{j_2}) \cdots (a_{i_n}, b_{j_n}).$$

Notice that if $\alpha, \gamma \in A^+$, $\beta, \gamma \in B^+$ are such that $|\alpha| = |\beta|$ and $|\gamma| = |\delta|$ then

$$(\alpha\gamma, \beta\delta)\nu \equiv (\alpha, \beta)\nu\,(\gamma, \delta)\nu. \tag{3.9}$$

A presentation for $S \times T$ is then given in the following

Proposition 3.43 [47, Proposition 5.7] *Let S and T be two stable semigroups. Let $A = \{a_i : i \in I\}$, $B = \{b_j : j \in J\}$ be alphabets such that*

$$\langle A \mid a_i = a_{i\zeta} a_{i\eta},\ R\ (i \in I)\rangle \quad \textit{and} \tag{3.10}$$

$$\langle B \mid b_j = b_{j\theta} b_{j\iota},\ Q\ (j \in J)\rangle \tag{3.11}$$

are finite uniform presentations for S and T, respectively. If ν denotes the decomposition mapping, then the direct product $S \times T$ is defined by the presentation

$$\mathcal{P} = \langle A \times B \quad | \quad (u_1, \alpha)\nu = (v_1, \alpha)\nu, \tag{3.12}$$

$$(\beta, u_2)\nu = (\beta, v_2)\nu, \tag{3.13}$$

$$(a_{i_1} a_{i_2\zeta} a_{i_2\eta}, \gamma)\nu = (a_{i_1\zeta} a_{i_1\eta} a_{i_2}, \gamma)\nu, \tag{3.14}$$

$$(\delta, b_{j_1} b_{j_2\theta} b_{j_2\iota})\nu = (\delta, b_{j_1\theta} b_{j_1\iota} b_{j_2})\nu, \tag{3.15}$$

$$(a_i, b_j) = (a_{i\zeta}, b_{j\theta})(a_{i\eta}, b_{j\iota}) \tag{3.16}$$

$$((u_1 = v_1) \in R;\ \alpha \in B^+;\ |u_1| = |\alpha|;\ (u_2 = v_2) \in Q;$$
$$\beta \in A^+;\ |u_2| = |\beta|; i_1, i_2 \in I;\ \gamma \in B^+;\ |\gamma| = 3;$$
$$j_1, j_2 \in J;\ \delta \in A^+;\ |\delta| = 3;\ i \in I;\ j \in J)\rangle.$$

We now have all the necessary ingredients to prove Proposition 3.41.

PROOF[of Proposition 3.41] We use the same notation as in Proposition 3.43. First notice that if both S and T are stable then $S \times T$ is finitely presented (by Theorem 3.21 in the case S and T are both infinite, by Theorem 3.22 if one is infinite and the other one is finite, and because $S \times T$ is finite if both S and T are finite). Also, it is clear, that if $S = S^2$ and $T = T^2$ then also $S \times T = (S \times T)^2$. It remains to be proved that $S \times T$ has no critical pairs with respect to the presentation $\mathcal{P}$.

We start by defining a mapping $\lambda_A : A^+ \times \mathbb{N}_0 \to A^+$ by

$$\begin{aligned} (w'a_i, n)\lambda_A &\equiv w' a_{i\zeta} a_{i\eta\zeta} a_{i\eta^2\zeta} \dots a_{i\eta^{n-1}\zeta} a_{i\eta^n} \\ (w, 0)\lambda_A &\equiv w, \end{aligned}$$

for $w \equiv w'a_i \in A^+$ and $n > 0$. We also define a mapping $\lambda_B : B^+ \times \mathbb{N}_0 \to B^+$ by

$$\begin{aligned} (w'b_j, n)\lambda_B &\equiv w' b_{j\theta} b_{j\iota\theta} b_{j\iota^2\theta} \dots b_{j\iota^{n-1}\theta} b_{j\iota^n} \\ (w, 0)\lambda_B &\equiv w, \end{aligned}$$

for $w \equiv w'b_j \in B^+$ and $n > 0$. Then, for any $w \in A^+$ (resp. $w \in B^+$) and $n \in \mathbb{N}_0$, the relation $(w, n)\lambda_A = w$ holds in S (resp. $(w, n)\lambda_B = w$ holds in T) and $|(w, n)\lambda_A| = |w| + n$ (resp. $|(w, n)\lambda_B| = |w| + n$). These are immediate consequences of the fact that $(w, n)\lambda_A$ (resp. $(w, n)\lambda_B$) is obtained from w by n applications of relations $a_i = a_{i\zeta} a_{i\eta}$ $(i \in I)$ (resp. $b_j = b_{j\theta} b_{j\iota}$ $(j \in J)$). The mapping λ_A (resp. λ_B) formalizes the fact that any word in A^+ (resp. B^+) of length m is equal in S (resp. in T) to a word of length n, for any $n \geq m$. We write λ instead of λ_A or λ_B, whenever it is clear which mapping we are referring to.

Now, let $w_1, w_2 \in (A \times B)^+$ be two words such that $w_1 = w_2$ holds in $S \times T$. We have to show that there exists an elementary sequence from w_1 to w_2, with respect to $\mathcal{P}$, with no term having length less than $\min\{|w_1|, |w_2|\}$. In order to do this we will follow the proof from [47] that $w_1 = w_2$ is a consequence of $\mathcal{P}$: to prove that $w_1 = w_2$ follows from $\mathcal{P}$, the authors build an elementary sequence from w_1 to w_2 with respect to $\mathcal{P}$. That sequence will be suitable for our proof.

We may assume that $|w_1| = |w_2|$. Indeed, if $|w_1| < |w_2|$, by applying relations (3.16) to w_1, we can obtain a word w_1' with the same length as w_2 (and each time we apply such a relation we are increasing the length of the word). So we obtain an elementary sequence from w_1 to w_1' in which no term has length less than $\min\{|w_1'|, |w_1|\}$. Such a sequence, composed with a sequence from w_1' to w_2 with no term having length less than $\min\{|w_1'|, |w_2|\}$, will produce an elementary sequence with respect to $\mathcal{P}$, each term of which will have length greater than or equal to $\min\{|w_1|, |w_2|\}$. If $|w_1| > |w_2|$ we can swap w_1 and w_2 and proceed as described above.

Now let $(a_{i_1}, b_{j_1}), (a_{i_2}, b_{j_2}), \ldots, (a_{i_p}, b_{j_p}), (a_{k_1}, b_{l_1}), (a_{k_2}, b_{l_2}), \ldots, (a_{k_p}, b_{l_p}) \in (A \times B)^+$ be such that

$$w_1 \equiv (a_{i_1}, b_{j_1})(a_{i_2}, b_{j_2}) \cdots (a_{i_p}, b_{j_p}),$$
$$w_2 \equiv (a_{k_1}, b_{l_1})(a_{k_2}, b_{l_2}) \cdots (a_{k_p}, b_{l_p}).$$

Since the relation $w_1 = w_2$ holds in $S \times T$, the relation

$$a_{i_1} a_{i_2} \ldots a_{i_p} = a_{k_1} a_{k_2} \ldots a_{k_p}$$

must hold in S and the relation

$$b_{j_1} b_{j_2} \cdots b_{j_p} = b_{l_1} b_{l_2} \cdots b_{l_p}$$

must hold in T. Recall that, by assumption, S and T have no critical pairs with respect to (3.10) and (3.11), respectively. Thus there exists an elementary sequence

$$a_{i_1} a_{i_2} \ldots a_{i_p} \equiv \xi_1, \xi_2, \ldots, \xi_q \equiv a_{k_1} a_{k_2} \ldots a_{k_p}$$

with respect to (3.10) in which $|\xi_m| \geq p$ for all m $(1 \leq m \leq q)$. Also, there exists an elementary sequence

$$b_{j_1}b_{j_2}\cdots b_{j_p} \equiv \chi_1, \chi_2, \ldots, \chi_r \equiv b_{l_1}b_{l_2}\cdots b_{l_p}$$

with respect to (3.11) in which $|\chi_m| \geq p$ for all m $(1 \leq m \leq r)$.

Denote by α the word $a_{k_1}a_{k_2}\ldots a_{k_p}$ and by by β the word $b_{j_1}b_{j_2}\ldots b_{j_p}$. Notice that $w_1 \equiv (\xi_1, (\beta, |\xi_1| - p)\lambda)\nu$ and $w_2 \equiv ((\alpha, |\chi_r| - p)\lambda, \chi_r)\nu$.

Claim 1 *For every m $(1 \leq m \leq q)$ there is an elementary sequence from*

$$(\xi_m, (\beta, |\xi_m| - p)\lambda)\nu \text{ to } (\xi_{m+1}, (\beta, |\xi_{m+1}| - p)\lambda)\nu,$$

with respect to $\mathcal{P}$, in which no term has length less than the minimum length of these two elements.

PROOF We distinguish four cases, depending on how ξ_{m+1} is obtained from ξ_m.

Case 1. $\xi_m \equiv \xi_{m+1}$. In this case

$$(\xi_m, (\beta, |\xi_m| - p)\lambda)\nu \equiv (\xi_{m+1}, (\beta, |\xi_{m+1}| - p)\lambda)\nu,$$

and the claim is trivially true.

Case 2. $\xi_m \equiv \gamma a_i \delta$, $\xi_{m+1} \equiv \gamma a_{i\zeta} a_{i\eta} \delta$, *with $i \in I$ and $\gamma, \delta \in A^*$.* If we let $|\xi_m| - p = s$ then clearly $|\xi_{m+1}| - p = s + 1$. Thus we have

$$(\beta, |\xi_m| - p)\lambda \equiv b_{j_1}b_{j_2}\ldots b_{j_{p-1}}b_{j_p\theta}b_{j_p\iota\theta}\ldots b_{j_p\iota^{s-1}\theta}b_{j_p\iota^s}, \tag{3.17}$$

and

$$(\beta, |\xi_{m+1}| - p)\lambda \equiv b_{j_1}b_{j_2}\ldots b_{j_{p-1}}b_{j_p\theta}b_{j_p\iota\theta}\ldots b_{j_p\iota^{s-1}\theta}b_{j_p\iota^s\theta}b_{j_p\iota^{s+1}}. \tag{3.18}$$

Let us now decompose the word $(\beta, |\xi_m| - p)\lambda$ as

$$(\beta, |\xi_m| - p)\lambda \equiv \chi_1 b_{j_l} \chi_2, \tag{3.19}$$

where, $\chi_1, \chi_2 \in B^*$, $|\chi_1| = |\gamma|$ and $|\chi_2| = |\delta|$.

Now we have

$$\begin{aligned}
&(\xi_m, (\beta, |\xi_m| - p)\lambda)\nu \equiv && \\
&\quad \equiv (\gamma a_i \delta, \chi_1 b_{j_l} \chi_2)\nu && \text{(by (3.19))} \\
&\quad \equiv (\gamma, \chi_1)\nu\, (a_i, b_{j_l})\nu\, (\delta, \chi_2)\nu && \text{(by (3.9))} \\
&\quad = (\gamma, \chi_1)\nu\, (a_{i\zeta}, b_{j_l\theta})(a_{i\eta}, b_{j_l\iota})\, (\delta, \chi_2)\nu && \text{(relation (3.16))} \\
&\quad \equiv (\gamma a_{i\zeta} a_{i\eta} \delta, \chi_1 b_{j_l\theta} b_{j_l\iota} \chi_2)\nu && \text{(by (3.9))} \\
&\quad \equiv (\xi_{m+1}, \chi_1 b_{j_l\theta} b_{j_l\iota} \chi_2)\nu && (\xi_{m+1} \equiv \gamma a_{i\zeta} a_{i\eta} \delta) \\
&\quad \equiv (\xi_{m+1}, b_{j_1} b_{j_2} \ldots b_{j_{l-1}} b_{j_l\theta} b_{j_l\iota} b_{j_{l+1}} \ldots b_{j_{p\iota^s - 1}\theta} b_{j_{p\iota^s}})\nu && \text{(by (3.17) and (3.19))} \\
&\quad = (\xi_{m+1}, b_{j_1} b_{j_2} \ldots b_{j_{l-1}} b_{j_l} b_{j_{l+1}} \ldots b_{j_{p\iota^s - 1}\theta} b_{j_{p\iota^s}\theta} b_{j_{p\iota^s + 1}})\nu && \text{(relations (3.15))} \\
&\quad \equiv (\xi_{m+1}, (\beta, |\xi_{m+1}| - p)\lambda)\nu && \text{(by (3.18))}.
\end{aligned}$$

The application of relations (3.16) increases the length of the word by one and the application of relations (3.15) preserves the length. Hence the claim is true in this case.

Case 3. $\xi_m \equiv \gamma a_{i\zeta} a_{i\eta} \delta$, $\xi_{m+1} \equiv \gamma a_i \delta$, *with* $i \in I$ *and* $\gamma, \delta \in A^*$. This case is the converse of Case 2 and thus the elementary sequence is the same as in that case (in reverse order).

Case 4. $\xi_m \equiv \gamma u \delta$, $\xi_{m+1} \equiv \gamma v \delta$, *with* $(u = v) \in R$ *and* $\gamma, \delta \in A^*$. Recall that we have $|u| = |v|$ since (3.10) is a uniform presentation. Thus we have

$$|\xi_m| - p = |\xi_{m+1}| - p,$$

and hence

$$(\beta, |\xi_m| - p)\lambda \equiv (\beta, |\xi_{m+1}| - p)\lambda;$$

we denote this last word by β'. Decompose β' as $\beta' \equiv \beta'_1 \beta'_2 \beta'_3$, where $\beta'_1, \beta'_2, \beta'_3 \in B^*$, $|\beta'_1| = |\gamma|$, $|\beta'_2| = |u|$ and $|\beta'_3| = |\delta|$. Now we have

$$\begin{aligned}
(\xi_m, (\beta, |\xi_m| - p)\lambda)\nu &\equiv (\gamma u \delta, \beta'_1 \beta'_2 \beta'_3)\nu \\
&\equiv (\gamma, \beta'_1)\nu\, (u, \beta'_2)\nu\, (\delta, \beta'_3)\nu \\
&= (\gamma, \beta'_1)\nu\, (v, \beta'_2)\nu\, (\delta, \beta'_3)\nu \qquad \text{(relation (3.12))} \\
&\equiv (\gamma v \delta, \beta'_1 \beta'_2 \beta'_3)\nu \\
&\equiv (\xi_{m+1}, (\beta, |\xi_{m+1}| - p)\lambda)\nu,
\end{aligned}$$

and the application of relation (3.12) preserves the length of the word. $\square$

Claim 2 *For every m $(1 \leq m \leq r)$ there is an elementary sequence from*

$$((\alpha, |\chi_m| - p)\lambda, \chi_m)\nu \text{ to } ((\alpha, |\chi_{m+1}| - p)\lambda, \chi_{m+1})\nu,$$

with respect to $\mathcal{P}$, in which no term has length less than the minimum length of these two elements.

PROOF We distinguish four cases, depending on how χ_{m+1} is obtained from χ_m.

Case 1. $\chi_m \equiv \chi_{m+1}$. In this case

$$((\alpha, |\chi_m| - p)\lambda, \chi_m)\nu \equiv ((\alpha, |\chi_{m+1}| - p)\lambda, \chi_{m+1},)\nu,$$

and the claim is trivially true.

Case 2. $\chi_m \equiv \gamma b_j \delta$, $\chi_{m+1} \equiv \gamma b_{j\theta} b_{j\iota} \delta$, *with $j \in J$ and $\gamma, \delta \in B^*$.* If we let $|\chi_m| - p = s$ then clearly $|\chi_{m+1}| - p = s + 1$. Thus we have

$$(\alpha, |\chi_m| - p)\lambda \equiv a_{i_1} a_{i_2} \dots a_{i_{p-1}} a_{i_p\zeta} b_{i_p\eta\zeta} \dots a_{i_p\eta^{s-1}\iota} a_{i_p\eta^s}, \tag{3.20}$$

and

$$(\alpha, |\chi_{m+1}| - p)\lambda \equiv a_{i_1} a_{i_2} \dots a_{i_{p-1}} a_{i_p\zeta} a_{i_p\eta\zeta} \dots a_{i_p\eta^{s-1}\zeta} a_{i_p\eta^s\zeta} a_{i_p\eta^{s+1}}. \tag{3.21}$$

Let us now decompose the word $(\alpha, |\chi_m| - p)\lambda$ as

$$(\alpha, |\chi_m| - p)\lambda \equiv \alpha_1 a_{i_l} \alpha_2, \tag{3.22}$$

where $\alpha_1, \alpha_2 \in A^*$, $|\alpha_1| = |\gamma|$ and $|\alpha_2| = |\delta|$. Now we have

$$\begin{aligned}
&((\alpha, |\chi_m| - p)\lambda, \chi_m)\nu \equiv \\
&\quad \equiv (\alpha_1 a_{i_l} \chi_2, \gamma b_j \delta)\nu && \text{(by (3.22))} \\
&\quad \equiv (\alpha_1, \gamma)\nu\, (a_{i_l}, b_j)\nu\, (\alpha_2, \delta)\nu && \text{(by (3.9))} \\
&\quad = (\alpha_1, \gamma)\nu\, (a_{i_l\zeta}, b_{j\theta})(a_{i_l\eta}, b_{j\iota})\, (\alpha_2, \delta)\nu && \text{(relation (3.16))} \\
&\quad \equiv (\alpha_1 a_{i_l\theta} a_{i_l\iota} \alpha_2, \gamma b_{j\theta} b_{j\iota} \delta)\nu && \text{(by (3.9))} \\
&\quad \equiv (\alpha_1 a_{i_l\zeta} a_{i_l\eta} \alpha_2, \chi_{m+1})\nu && (\beta_{m+1} \equiv \gamma b_{j\theta} b_{j\iota} \delta) \\
&\quad \equiv (a_{i_1} a_{i_2} \dots a_{i_{l-1}} a_{i_l\zeta} a_{i_l\eta} a_{i_{l+1}} \dots a_{i_p\eta^{s-1}\zeta} a_{i_p\eta^s}, \chi_{m+1})\nu && \text{(by (3.20))} \\
&\quad = (a_{i_1} a_{i_2} \dots a_{i_{l-1}} a_{i_l} a_{i_{l+1}} \dots a_{i_p\eta^{r-1}\zeta} a_{i_p\eta^s\zeta} a_{i_p\eta^{s+1}}, \chi_{m+1})\nu && \text{(relations (3.14))} \\
&\quad \equiv ((\alpha, |\chi_{m+1}| - p)\lambda, \chi_{m+1})\nu && \text{(by (3.21)).}
\end{aligned}$$

The application of relations (3.16) increases the length of the word by one and the application of relations (3.14) preserve the length. Hence the claim is true in this case.

Case 3. $\chi_m \equiv \gamma b_{j\theta} b_{j\iota} \delta$, $\chi_{m+1} \equiv \gamma b_j \delta$, *with* $j \in J$ *and* $\gamma, \delta \in B^*$. This case is the converse of Case 2 and thus the elementary sequence is the same as in that case (in reverse order).

Case 4. $\chi_m \equiv \gamma u \delta$, $\chi_{m+1} \equiv \gamma v \delta$, *with* $(u = v) \in R$ *and* $\gamma, \delta \in B^*$. Recall that we have $|u| = |v|$ since (3.11) is a uniform presentation. Thus we have

$$|\chi_m| - p = |\chi_{m+1}| - p,$$

and hence

$$(\alpha, |\chi_m| - p)\lambda \equiv (\alpha, |\chi_{m+1}| - p)\lambda;$$

we denote this last word by α'. Decompose α' as $\alpha' \equiv \alpha'_1 \alpha'_2 \alpha'_3$, where $\alpha'_1, \alpha'_2, \alpha'_3 \in A^*$, $|\alpha'_1| = |\gamma|$, $|\alpha'_2| = |u|$ and $|\alpha'_3| = |\delta|$. Now we have

$$\begin{aligned}
((\alpha, |\chi_m| - p)\lambda, \chi_m)\nu &\equiv (\alpha'_1 \alpha'_2 \alpha'_3, \gamma u \delta)\nu \\
&\equiv (\alpha'_1, \gamma)\nu\, (\alpha'_2, u)\nu\, (\alpha'_3, \delta)\nu \\
&= (\alpha'_1, \gamma)\nu\, (\alpha'_2, v)\nu\, (\alpha'_3, \delta)\nu \qquad \text{(relation (3.13))} \\
&\equiv (\alpha'_1 \alpha'_2 \alpha'_3, \gamma v \delta)\nu \\
&\equiv ((\alpha, |\chi_{m+1}| - p)\lambda, \chi_{m+1})\nu,
\end{aligned}$$

and the application of relation (3.13) preserves the length of the word. □

Now, notice that

$$|(\xi_m, (\beta, |\xi_m| - p)\lambda)\nu| = |\xi_m| \geq p$$

$(1 \leq m \leq q)$ and that

$$|((\alpha, |\chi_1| - p)\lambda, \chi_m)\nu| = |\chi_m| \geq p$$

$(1 \leq m \leq r)$. Then, using Claim 1, we then have that there is an elementary sequence from

$$(\xi_1, (\beta, |\xi_1| - p)\lambda)\nu \text{ to } (\xi_q, (\beta, |\xi_q| - p)\lambda)\nu$$

with respect to $\mathcal{P}$ in which no term has length less than $\min\{|\xi_1|, |\xi_q|\} = p$. And using Claim 2, we have that there is an elementary sequence from

$$((\alpha, |\chi_1| - p)\lambda, \chi_1)\nu \text{ to } ((\alpha, |\chi_r| - p)\lambda, \chi_r)\nu$$

with respect to $\mathcal{P}$ in which no term has length less than $\min\{|\chi_1|, |\chi_q|\} = p$. Recall that $w_1 \equiv (\xi_1, (\beta, |\xi_1| - p)\lambda)\nu$, $(\xi_q, (\beta, |\xi_q| - p)\lambda)\nu \equiv ((\alpha, |\chi_1| - p)\lambda, \chi_1)\nu$ and $w_2 \equiv ((\alpha, |\chi_r| - p)\lambda, \chi_r)$.

Therefore we conclude that there is an elementary sequence from w_1 to w_2 with respect to $\mathcal{P}$ in which no term has length less than $\min w_1, w_2$. Thus $S \times T$ is stable. □

Corollary 3.44 *Let S_i $(i \in I)$ be a finite family of stable of semigroups. Then the direct product $Z = \prod_{i\in I} S_i$ is stable.*

Proof We use induction on the size of I. If $|I| = 1$ the result is trivially true, and if $|I| = 2$ then the result is Proposition 3.41. Now, let $I = \{1, 2, \ldots, k, k+1\}$ and assume that the direct product of k stable semigroups is stable. Thus,

$$\prod_{i=1}^{k} S_i$$

is stable semigroup. But then, by the case $|I| = 2$ it follows that Z is also stable. □

Proof[of Theorem 3.40] If $J = \emptyset$ then

$$Z = \prod_{i\in I} S_i$$

and $|I| \geq 2$. We use induction on $|I|$. The result for $|I| = 2$ is Theorem 3.21. Now, suppose that the result holds for a direct product of k infinite semigroups and that $I = \{1, 2, \ldots, k, k+1\}$. Using Theorem 3.21 once more we have that Z is finitely presented if and only if $\prod_{i=1}^{k} S_i$ and S_{k+1} are both stable. Now, if $\prod_{i=1}^{k} S_i$ is stable then it is, in particular, finitely presented, and thus, by the induction hypothesis, each S_i is stable. Conversely, by Corollary 3.44, if every S_i $(1 \leq i \leq k)$ is stable then $\prod_{i=1}^{k} S_i$ is also stable.

Now, if $J \neq \emptyset$, and since $T = \prod_{j\in J} T_j$ is a finite semigroup, it follows from Theorem 3.22 that Z is finitely presented if and only if $S = \prod_{i\in I} S_i$ is finitely presented and T is stable. Moreover, if $|I| \geq 2$ then, by the first case, S is finitely presented if and only if each S_i is stable. Now, if each T_j $(j \in J)$ is stable then

T is stable. It still remains to prove that if Z is finitely presented then each T_j ($j \in J$) is stable. In order to see this notice that

$$Z \cong (\prod_{i\in I} S_i \times \prod_{\substack{j \in J \\ j \neq k}} T_j) \times T_k$$

for any $k \in J$. Thus, if Z is finitely presented then, by applying Theorem 3.22 once again, T_k must be stable, for any $k \in J$. $\square$

Chapter 4

Semidirect products of semigroups and monoids

In this chapter we study semidirect products of semigroups. Our aim is to find necessary and sufficient conditions for the semidirect product of two semigroups to be finitely generated and finitely presented.

Let S and T be semigroups, and let $\phi : T \to \mathrm{End}(S)$ be an antihomomorphism. We denote $s(t\phi)$ by ${}^{t}s$. Since ϕ is an antihomomorphism we have

$$ {}^{t}(s_1 s_2) = {}^{t}s_1\, {}^{t}s_2 \text{ and } {}^{t_1 t_2}s = {}^{t_1}({}^{t_2}s) $$

$(s, s_1, s_2 \in S,\ t, t_1, t_2 \in T)$. That is, ϕ determines a left action of T on S. On the set $S \times T$ we define a binary operation

$$ (s,t)(u,v) = (s\, {}^{t}u, tv) $$

obtaining a semigroup, which we denote by $S \rtimes_\phi T$, and which is called the *semidirect product of S by T with respect to ϕ*. If both S and T are monoids, with identities 1_S and 1_T, respectively, and

$$ {}^{1_T}s = s \text{ and } {}^{t}1_S = 1_S, \tag{4.1} $$

then we say that $S \rtimes_\phi T$ is a *monoid semidirect product*; we denote it by $S \rtimes^{M}_\phi T$.

Remark 4.1 Notice that, in general, the semidirect product of two monoids need not be a monoid semidirect product. Indeed, if T contains a zero, and ϕ maps every $t \in T$ to the endomorphism of S that takes all the elements to zero, then conditions (4.1) are not fulfilled. Clearly the semidirect product of two monoids is a monoid semidirect product if and only if ϕ is a monoid morphism, and the image of ϕ is a subsemigroup of $\mathrm{End}(S)$ consisting only of monoid endomorphisms.

Well known examples of semidirect products are the direct product (where ϕ maps every element onto the identity mapping of S) and the wreath product of semigroups.

In Section 4.1 we present some known results about finite generation and finite presentability of wreath products of monoids. Then, in Section 4.2, we consider the general case, stating basic results, raising questions and exhibiting some examples. Finally, in Section 4.3, we consider the semidirect product of two semigroups with respect to an action by an idempotent morphism.

Proposition 4.15 of Section 4.2 appears here for the first time and was proved jointly with Ruškuc and Thomson. The results from Section 4.3 also appear here for the first time, and are joint work with Branco, Fernandes, Gomes and Ruškuc. Many interesting questions remain unanswered, calling for further investigation on this topic.

4.1 Wreath product: known results

In this section we review some known results about finite generation and finite presentability of wreath products. We will use some of the results presented here to obtain some examples in the following sections.

We start by defining the wreath product of monoids. So, let S and T be monoids. The *cartesian product* of $|T|$ copies of S is denoted by S^T and can be thought of as the set of all functions $f : T \to S$. In this sense, S^T is a monoid, with multiplication given by

$$x(fg) = (xf)(xg) \ (x \in T,\ f, g \in S^T)$$

and identity $\overline{1}$ (where $\overline{1}$ is the function mapping all elements of T to the identity of S). The *unrestricted wreath product* $S\mathrm{Wr}T$ of S by T is the set $S^T \times T$ with multiplication

$$(f, b)(g, b') = (f\,{}^{b}g, bb')$$

where ${}^{b}g$ is given by the action $(b, f) \mapsto {}^{b}f$ of T on S^T defined by

$$x\,{}^{b}f = (xb)f \ (x \in T). \tag{4.2}$$

In fact, $S\mathrm{Wr}T$ is the (monoid) semidirect product of S^T by T with respect to this action, and thus $S\mathrm{Wr}T$ is a monoid with identity $(\overline{1}, 1_T)$. The *direct power* of $|T|$ copies of S is denoted by $S^{(T)}$ and consists of all elements $f \in S^T$ with finite support. In other words, each $f \in S^{(T)}$ satisfies $xf = 1_S$ for all but finitely many $x \in T$. The *(restricted) wreath product* $S\mathrm{wr}T$ of S by T is then defined to be the submonoid of $S\mathrm{Wr}T$ generated by $S^{(T)} \times T$. If all the sets $bc^{-1} = \{x \in T : b = xc\}$ $(b, c \in T)$ are finite then $S\mathrm{wr}T = S^{(T)} \times T$ (as sets). In particular, this holds if T is a group. We refer the reader to [37] for a more detailed introduction.

If both S and T are groups then we have the following well known result for finite generation.

Theorem 4.2 *Let S and T be groups. Then the wreath product $S\mathrm{wr}T$ is finitely generated if and only if both S and T are finitely generated.*

This result does not generalize for the monoid case: for example the wreath product $\mathbb{N}_0\mathrm{wr}\mathbb{N}_0$ (where $\mathbb{N}_0$ denotes the additive monoid of natural numbers, including zero) is not finitely generated. Indeed we have:

Theorem 4.3 [46, Theorem 1.1] *Let S and T be monoids and let G be the group of units of T. Then the wreath product $S\mathrm{wr}T$ of S by T is finitely generated if and only if both S and T are finitely generated, and either S is trivial, or $T = VG$ for some finite subset V of T.*

As for finite presentability, it turns out that the result for the wreath product of two groups (proved by Baumslag [9]) generalizes for the monoid case (with a different proof however):

Theorem 4.4 [46, Theorem 1.2] *Let S and T be monoids. Then the wreath product $S\mathrm{wr}T$ is finitely presented if and only if either S is trivial and T is finitely presented or T is finite and S is finitely presented.*

Remark 4.5 Notice that the definition of unrestricted wreath product $S\mathrm{Wr}T$ is not left-right symmetric. By considering the mapping $f \in S^T$ acting on the left, the action (4.2) becomes a right action $(f, b) \mapsto f^b$ of T on S^T. The semidirect product of S^T by T with respect to this right action is called the *left unrestricted wreath product* of S by T and is denoted $S\mathrm{Wr}_l T$. Similarly we can define the *left (restricted) wreath product* of S by T by taking the submonoid of $S\mathrm{Wr}_l T$ generated by the functions of finite support. In general $S\mathrm{Wr}T \ncong S\mathrm{Wr}_l T$ and $S\mathrm{wr}T \ncong S\mathrm{wr}_l T$. It is clear from Theorem 4.4 that $S\mathrm{wr}T$ is finitely presented if and only if $S\mathrm{wr}_l T$ is finitely presented. The same is not true for finite generation.

Remark 4.6 Although the definition of unrestricted wreath product can be extended for two arbitrary semigroups, the same is not true for the (restricted) wreath product. In fact, in order to define the support of an element we made use of the fact that S was a monoid. One common approach here is to allow the definition of support to be with respect to a fixed idempotent of S. In this way we can define the wreath product of two semigroups S and T, provided S contains an idempotent e, which is denoted by $S_e\mathrm{wr}T$ and is called the *wreath product of S by T with respect to the idempotent e*. All finite generation results and finite presentability results in this case are much more technical; such a theory was developed in [45] and [65].

4.2 The general case

On dealing with the general case we start by looking at finite generation.

The semidirect product of two finitely generated semigroups is not finitely generated in general. Indeed, we have seen in Chapter 3 that even in the special case of the direct product this does not have to be the case. The following shows that, for two arbitrary semigroups S and T, T finitely generated does not imply $S \rtimes_\phi T$ finitely generated, even when S is finite and $S^2 = S$. (Recall that in such a case the direct product $S \times T$ would be finitely generated; see Theorem 3.3.)

Example 4.7 Let $S = \{0, 1\}$ be the two–element semilattice, and let $T = \langle a, b \mid \rangle$ be the free semigroup of rank 2. Define

$$^a0 = {}^a1 = 0, \quad {}^b0 = {}^b1 = 1.$$

Then $S \rtimes_\phi T$ is not finitely generated. Indeed, it is clear that each of the infinitely many elements of the form $(1, a^n) \in S \rtimes_\phi T$ has to belong to any generating set. Nevertheless S is finite and it satisfies $S^2 = S$.

However, for the monoid semidirect product we have following

Proposition 4.8 [65] *Let S and T be finitely generated monoids and suppose that a monoid semidirect product $S \underset{M}{\rtimes}_\phi T$ can be defined. Then $S \underset{M}{\rtimes}_\phi T$ is finitely generated.*

Proof Suppose that S is generated by a set A and T is generated by a set B. Then let

$$C = \{(a, 1_T) : a \in A\} \cup \{(1_S, b) : b \in B\}.$$

We will prove that the set C generates $S \underset{M}{\rtimes}_\phi T$. So let $(s,t) \in S \underset{M}{\rtimes}_\phi T$. Since $s \in S$, there are $a_1, a_2, \ldots, a_m \in A$ such that $s = a_1 a_2 \cdots a_m$ and similarly there are $b_1, b_2, \ldots, b_n \in B$ such that $t = b_1 b_2 \cdots b_n$. Then

$$\begin{gathered}(a_1, 1_T)(a_2, 1_T) \cdots (a_m, 1_T)(1_S, b_1)(1_S, b_2) \cdots (1_S, b_n) = \\ = (a_1\, {}^{1_T}a_2 \cdots {}^{1_T}a_m\, {}^{1_T}1_S\, {}^{b_1}1_S\, {}^{b_1 b_2}1_S\, {}^{b_1 b_2 \cdots b_{m-1}}1_S, b_1 b_2 \cdots b_n) = (s,t)\end{gathered}$$

and thus C generates $S \underset{M}{\rtimes}_\phi T$. In particular, if A and B are finite, C is also finite and therefore if S and T are both finitely generated then so is $S \underset{M}{\rtimes}_\phi T$. □

The converse of Proposition 4.8 is not true even for the monoid case, as the following example shows:

Example 4.9 Let M be a non-trivial finitely generated monoid and let G be an infinite finitely presented group. By Theorem 4.3 it follows that $M\mathrm{wr}G$ is finitely generated. But $M\mathrm{wr}G$ is the semidirect product of $M^{(G)}$ by G, with respect to the wreath action (since for all $b, c \in G$ the set $\{x \in G : b = xc\}$ is finite). Now, since G is infinite, the monoid $M^{(G)}$ is not finitely generated.

But for the second factor, T, we have:

Proposition 4.10 *Let S and T be semigroups such that a semidirect product $S \rtimes_\phi T$ is defined. Then if $S \rtimes_\phi T$ is finitely generated, then T is finitely generated.*

PROOF This follows from the obvious fact that T is a homomorphic image of $S \rtimes_\phi T$. □

Whenever T is a finite semigroup we can prove the following

Proposition 4.11 [65] *Let S and T be semigroups such that a semidirect product $S \rtimes_\phi T$ is defined and suppose that T is finite. If $S \rtimes_\phi T$ is finitely generated then S finitely generated.*

PROOF Suppose that $S \rtimes_\phi T$ is generated by a (finite) set

$$\{(u_1, v_1), (u_2, v_2), \ldots, (u_k, v_k)\}$$

and let

$$U = \{u_1, u_2, \ldots, u_k\}.$$

We claim that the set

$$Z = \{\,{}^t u : u \in U, t \in T\}$$

is a generating set for S. Indeed, if $s \in S$, then for any $t \in T$, we can write

$$(s, t) = (u_{i_1}, v_{i_1})(u_{i_2}, v_{i_2}) \cdots (u_{i_h}, v_{i_h}).$$

Thus

$$s = u_{i_1}\, {}^{v_{i_1}}u_{i_2}\, {}^{v_{i_1} v_{i_2}}u_{i_3} \cdots {}^{v_{i_1} v_{i_2} \cdots v_{i_{h-1}}}u_{i_h},$$

i.e. s can be written as a product of elements of Z. Notice that Z is finite whenever both U and T are finite and hence we conclude that S is finitely generated. □

We now turn to finite presentability. From the results obtained for finite generation, and the results on the direct product from Chapter 3, we know that in general the semidirect product of two finitely presented semigroups is not necessarily finitely presented, even if it is finitely generated. However, for the monoid case we have:

Proposition 4.12 [65] *Let S and T be finitely presented monoids and suppose that a monoid semidirect product $S \rtimes^M_\phi T$ can be defined. Then $S \rtimes^M_\phi T$ is finitely presented.*

PROOF Suppose that S and T are defined by presentations $\langle A \mid R\rangle$ and $\langle B \mid Q\rangle$, respectively. Notice that the action of T on S can be lifted to an action of B^* on A^*. With this in mind, for each $a \in A$, $b \in B$, we let $\zeta(a,b) \in A^*$ be a word representing the element ${}^b a$. We will prove that then $S \bowtie^M_\phi T$ is defined by the presentation

$$\langle A, B \mid R,\ Q,\ ba = \zeta(a,b)b\ (a \in A, b \in B)\rangle. \tag{4.3}$$

We start by defining a homomorphism

$$\pi : (A \cup B)^* \to S \bowtie^M_\phi T$$

by extending the mapping

$$a \mapsto (a, 1_T)\ (a \in A);\ b \mapsto (1_S, b)\ (b \in B).$$

By Proposition 4.8, π is an epimorphism and thus we can take $A \cup B$ as a generating set for the presentation. Also a relation $u = v$ $(u, v \in (A \cup B)^*)$ will hold in $S \bowtie^M_\phi T$ if and only if $u\pi = v\pi$. Thus relations from R hold and so do the relations from Q, and if $a \in A$, $b \in B$ then

$$(ba)\pi = (1_S, b)(a, 1_T) = (1_S\zeta(a,b), b) = (\zeta(a,b), b) = (\zeta(a,b)b)\pi.$$

Next suppose that $w_1, w_2 \in (A \cup B)^*$ are such that $w_1 = w_2$ in $S \bowtie^M_\phi T$. Using relations $ba = \zeta(a,b)b$ $(a \in A,\ b \in B)$ we can write $w_1 = \alpha\beta$ where $\alpha \in A^*$ and $\beta \in B^*$, and similarly we can write $w_2 = \alpha'\beta'$ with $\alpha' \in A^*$ and $\beta' \in B^*$. Then, since $w_1 = w_2$ in $S \bowtie^M_\phi T$, we have $w_1\pi = w_2\pi$,

$$(\alpha\beta)\pi = (\alpha'\beta')\pi$$

and thus, since $\alpha, \alpha' \in A^*$, $\beta, \beta' \in B^*$

$$\alpha\pi = \alpha'\pi \text{ and } \beta\pi = \beta\pi'.$$

Now, by using relations from R and Q we can transform α in α' and β in β', respectively. Thus $w_1 = w_2$ is a consequence of the presentation (4.3). In particular, if S and T are finitely presented then so is $S \bowtie^M_\phi T$. □

For the converse of Proposition 4.12 we ask the following questions:

Question 4.13 *If the semidirect product $S \rtimes_\phi T$ is finitely presented is S necessarily finitely presented? Further, if $S \rtimes_\phi T$ is finitely presented and S is finitely generated, is S finitely presented?*

And in line with Proposition 4.10:

Question 4.14 *If the semidirect product $S \rtimes_\phi T$ is finitely presented, is T necessarily finitely presented?*

For the monoid semidirect product, Question 4.13 has a positive answer whenever the second factor, T, is finite. This is our next proposition, which is was jointly proved by Thomson, Ruškuc and the author:

Proposition 4.15 *Let S and T be monoids such that a semidirect product $S \rtimes^{M}_{\phi} T$ is defined with respect to a morphism $\phi : T \to \mathrm{End}(S)$ and suppose that T is finite. If $S \rtimes^{M}_{\phi} T$ finitely presented then S is finitely presented.*

Proof By Proposition 4.11, S is finitely generated. So let $\langle A \mid R \rangle$ be a presentation for S with respect to a finite set A and with R possibly infinite. Take $\mathcal{M}(T) = \langle T \mid Q \rangle$ to be the multiplication table presentation for T. Recall that, since T is finite, $\mathcal{M}(T)$ is a finite presentation. For every $t \in T$ and $a \in A$ choose $\zeta(t, a) \in A^*$ representing the element ${}^t a$ in $S \rtimes^{M}_{\phi} T$. Then, by the proof of Proposition 4.12, we have that $S \rtimes^{M}_{\phi} T$ is defined by the following finite presentation:

$$\langle A, T \mid R,\ Q,\ ta = \zeta(t, a)t\ (a \in A,\ t \in T) \rangle.$$

Notice that the mapping $\xi : A^* \to (A \cup T)^*$ defined by $a \mapsto a$ induces an embedding $S \hookrightarrow S \rtimes^{M}_{\phi} T$. Now, since $S \rtimes^{M}_{\phi} T$ is finitely presented, there is a finite set $R' \subseteq R$ such that $S \rtimes^{M}_{\phi} T$ is defined by

$$\langle A, T \mid R',\ Q,\ ta = \zeta(t, a)t\ (a \in A,\ t \in T) \rangle. \tag{4.4}$$

For each $t \in T$ we let $\sigma_t : A^* \to A^*$ be the homomorphism extending the mapping $a \mapsto \zeta(t, a)$, and let $\psi : T^* \to \mathrm{End}(A^*)$ be the antihomomorphism defined by $t \mapsto \sigma_t$. For $z \in A^*$ and $w \in T^*$ we write $(z)(w\psi)$ as ${}^w z$. Notice that

$$
\begin{aligned}
{}^{w_1 w_2} z &\equiv {}^{w_1}({}^{w_2} z) \text{ and} && (4.5)\\
{}^{w} z_1 z_2 &\equiv {}^{w} z_1\, {}^{w} z_2 && (4.6)
\end{aligned}
$$

$(z, z_1, z_2 \in A^*,\ w, w_1, w_2 \in T^*)$.

Now let

$$\langle A \mid {}^{t}R',\ {}^{u}a = {}^{v}a\ (a \in A,\ t \in T,\ (u = v) \in Q)\rangle, \tag{4.7}$$

where ${}^{t}R' = \{\, {}^{t}u = {}^{t}v : (u = v) \in R'\}$ and denote by S_1 the semigroup defined by this presentation. Also notice that (4.7) is a finite presentation and thus S_1 is a finitely presented semigroup.

We now prove that ψ induces a homomorphism $\phi' : T \to \mathrm{End}(S_1)$ (hence T acts on S_1). We start by proving the following

Claim *If $w \in T^*$, $t \in T$ are such that $w = t$ in T then ${}^{w}a = {}^{t}a$ in S_1 $(a \in A)$. In particular, if $w_1, w_2 \in T^*$ are such that $w_1 = w_2$ in T then ${}^{w_1}a = {}^{w_2}a$ in S_1.*

Proof We use induction on the length of w. If $|w| = 1$ then the claim is trivially true. If $|w| = 2$ then ${}^{w}a = {}^{t}a$ is a relation from (4.7) for every $a \in A$. Now suppose that $w \equiv t_1 t_2 w'$, where $t_1, t_2 \in T$ and $w' \in T^*$. Let $t_3 \in T$ be such that $t_1 t_2 = t_3$ in T (and recall that this relation is in Q) and suppose that ${}^{w'}a \equiv a_1 a_2 \cdots a_k$ for $a_1, a_2, \ldots, a_k \in A$. Then

$$\begin{array}{rcll}
{}^{w}a & \equiv & {}^{t_1 t_2 w'}a & \\
 & \equiv & {}^{t_1 t_2}(a_1 a_2 \cdots a_k) & \text{(by (4.5))} \\
 & \equiv & {}^{t_1 t_2}a_1\ {}^{t_1 t_2}a_2 \cdots {}^{t_1 t_2}a_k & \text{(by (4.6))} \\
 & = & {}^{t_3}a_1\ {}^{t_3}a_2 \cdots {}^{t_3}a_k & (({}^{t_1 t_2}a_i = {}^{t_3}a_i) \in Q, i = 1, 2, \ldots, k) \\
 & \equiv & {}^{t_3}(a_1 a_2 \cdots a_k) & \text{(by (4.6))} \\
 & \equiv & {}^{t_3}({}^{w'}a) & \\
 & \equiv & {}^{t_3 w'}a & \text{(by (4.5))} \\
 & = & {}^{t}a & \text{(induction hypothesis).}
\end{array}$$

The last statement is then an immediate consequence. □

Now we have to check that for all $t \in T$, $t\phi'$ is a homomorphism. Notice that $t\phi'$ will be the endomorphism of S_1 induced by the σ_t. Thus, we have to prove that for any relation $(u = v)$ from (4.7) and any $w \in T^*$, ${}^{w}u = {}^{w}v$ holds in S_1. So, let $w \in T^*$ and $({}^{t}u = {}^{t}v) \in {}^{t}R'$. Suppose that $wt = t_1$ for some $t_1 \in T$. Then ${}^{wt}u = {}^{t_1}u$ and ${}^{wt}v = {}^{t_1}v$ in S_1 by the Claim. Now, ${}^{t_1}u = {}^{t_1}v \in {}^{t_1}R'$ and thus ${}^{wt}u = {}^{wt}v$ holds in S_1. For ${}^{u}a = {}^{v}a$ $((u = v) \in Q)$, if $wu = wv = t$ in T we have

$${}^{wu}a = {}^{t}a = {}^{wv}a.$$

Finally, we have to check that ϕ' is well defined, in the sense that for all $z \in A^*$, $w_1, w_2 \in T^*$, $w_1 = w_2$ in T implies ${}^{w_1}z = {}^{w_2}z$ in S_1. Indeed, if $z = a_1 a_2 \cdots a_k$ $(a_1, a_2, \ldots, a_k \in A)$ we then have

$$\begin{aligned} {}^{w_1}z &\equiv {}^{w_1}a_1\ {}^{w_1}a_2 \cdots {}^{w_1}a_k && \text{(by (4.6)}\\ &= {}^{w_2}a_1\ {}^{w_2}a_2 \cdots {}^{w_2}a_k && \text{(by the Claim)}\\ &\equiv {}^{w_2}z && \text{(by (4.6)).} \end{aligned}$$

Therefore T acts on S_1 by the action determined by the homomorphism ϕ' and therefore we can consider the semidirect product $S_1\underset{M}{\rtimes}_{\phi'}T$ which is defined by the presentation

$$\langle A, T \mid {}^tR',\ {}^ua = {}^va,\ Q,\ ta = \zeta(t,a)t\ (a \in A, t \in T, (u = v) \in Q)\rangle. \tag{4.8}$$

Notice that (4.8) contains (4.4) as a subpresentation and that all relations from (4.8) hold in $S\underset{M}{\rtimes}_{\phi}T$. Thus $S\underset{M}{\rtimes}_{\phi}T \cong S_1\underset{M}{\rtimes}_{\phi'}T$. Now, the mapping $\xi : A^* \to (A \cup T)^*$ also induces an embedding from S_1 into $S\underset{M}{\rtimes}_{\phi}T \cong S_1\underset{M}{\rtimes}_{\phi'}T$. But this is exactly the same embedding that ξ induces from S into $S\underset{M}{\rtimes}_{\phi}T$. Thus $S \cong S_1$ and therefore S is finitely presented. □

We can now deduce the following corollaries:

Corollary 4.16 *Let S and T be monoids such that a semidirect product $S\underset{M}{\rtimes}_{\phi}T$ is defined and T is finite. Then $S\underset{M}{\rtimes}_{\phi}T$ is finitely presented if and only if S is finitely presented.*

PROOF One of the implications is Proposition 4.12. The other one is Proposition 4.15. □

Corollary 4.17 [65] *Let S be an infinite semigroup and let T be a finite monoid, and suppose that $\phi : T \to \mathrm{End}(S)$ is a monoid homomorphism. If the semidirect product $S \rtimes_\phi T$ is finitely presented then S is finitely presented.*

PROOF Let S^I be the monoid obtained from S by adjoining an identity I, regardless of whether S contains an identity. We may extend any given endomorphism $\alpha : S \to S$ to a monoid homomorphism $\alpha^I : S^I \to S^I$ by letting

$$s\alpha^I = s\alpha\ (s \in S) \text{ and } I\alpha^I = I.$$

Thus we can also extend the homomorphism $\phi : T \to \mathrm{End}(S)$ to a homomorphism $\phi^I : T \to \mathrm{End}(S^I)$ in the following way: if $t\phi = \alpha$ $(t \in T)$, then $t\phi^I = \alpha^I$. Clearly ϕ^I is a monoid homomorphism and $S \underset{M}{\rtimes}_{\phi^I} T$ is a monoid semidirect product. Furthermore

$$(S \underset{M}{\rtimes}_{\phi^I} T) \setminus (S \rtimes_\phi T)$$

is a finite set and thus $S \underset{M}{\rtimes}_{\phi^I} T$, by Proposition 2.7, it is finitely presented if and only if $S \rtimes_\phi T$ is finitely presented. By Proposition 4.15 it follows that S^I is finitely presented and thus S is also finitely presented. □

The following question, however, remains unanswered:

Question 4.18 *Let S be an infinite semigroup and T be a finite semigroup. If $S \rtimes_\phi T$ is finitely presented is S necessarily finitely presented?*

4.3 Action by idempotent morphism

In this section we consider the semidirect product of two semigroups S and T with respect to a homomorphism $\phi : T \to \mathrm{End}(S)$ that maps every element of T to a fixed idempotent endomorphism α of S. Notice that, in this case, multiplication in the first component does not depend on T. Indeed, the action that ϕ determines from T on S, induces a new operation on S, that we denote by $\star$, and that is given by

$$s_1 \star s_2 = s_1(s_2\alpha)$$

$(s_1, s_2 \in S)$. Clearly S is a semigroup with respect to this operation, and we denote it by $(S, \star)$. Whenever we refer to S with the usual operation we will write $(S, \cdot)$, and we write S whenever we refer to the set S itself. Also, if $X \subseteq S$ we denote by $X(S\alpha)$ the set

$$\{x(s\alpha) : x \in X, s \in S\}.$$

We start by listing some properties of $(S, \star)$ in the following

Lemma 4.19 *With the above notation we have*

(i) *if $(S, \star)$ is finitely generated then the set $S \setminus S(S\alpha)$ is finite;*

(ii) *if* $X \subseteq S$ *is a generating set for* $(S, \star)$ *then* $S(S\alpha) = X(S\alpha)$*;*

(iii) *if* $S\alpha$ *is a finite set then* $(S, \star)$ *is finitely generated if and only if* S *is finite;*

(iv) $S \rtimes_\phi T \cong (S, \star) \times T$*; and*

(v) $(S, \star)^2 = (S, \star)$ *if and only if* $S(S\alpha) = S$.

PROOF (i) If $S \backslash S(S\alpha)$ is not a finite set then $(S, \star)$ has infinitely many indecomposable elements, and since all indecomposable elements have to belong to any generating set it follows that $(S, \star)$ is not finitely generated.
(ii) Suppose that $X \subseteq S$ is a generating set for $(S, \star)$. Clearly $X(S\alpha) \subseteq S(S\alpha)$. Now, if $s_1(s_2\alpha) \in S(S\alpha)$ $(s_1, s_2 \in S)$ then, since $s_1 \in S$ and X generates $(S, \star)$, there exist $x_1, x_2, \ldots, x_n \in X$ such that

$$s_1 = x_1 \star s_2 \star \cdots \star x_n,$$

that is

$$s_1 = x_1(x_2\alpha) \cdots (x_n\alpha)$$

and therefore

$$\begin{aligned} s_1(s_2\alpha) &= x_1(x_2\alpha) \cdots (x_n\alpha)(s_2\alpha) \\ &= x_1((x_2 \cdots x_n s_2)\alpha) \in X(S\alpha) \end{aligned}$$

as required.
(iii) If $S\alpha$ is finite and $(S, \star)$ is finitely generated then, by (i) $S \backslash S(S\alpha)$ is finite and thus S has to be finite as well. The converse follows from the fact that any finite semigroup is finitely generated.
(iv) Both $S \rtimes_\phi T$ and $(S, \star) \times T$ have the same support set $S \times T$, and thus we can define a bijection $\psi : S \rtimes_\phi T \to (S, \star) \times T$ by mapping $(s, t) \in S \rtimes_\phi T$ to $(s, t) \in (S, \star) \times T$. Now, if $(s_1, t_1), (s_2, t_2) \in S \rtimes_\phi T$ then

$$\begin{aligned} [(s_1, t_1)(s_2, t_2)]\psi &= (s_1(s_2\alpha), t_1t_2)\psi \\ &= (s_1(s_2\alpha), t_1t_2) \\ &= (s_1, t_1)(s_2, t_2) \end{aligned}$$

and therefore $S \rtimes_\phi T \cong (S, \star) \times T$.

(v) Clearly $(S,\star)^2 = S(S\alpha)$, and therefore $(S,\star)^2 = (S,\star)$ if and only if $S = S(S\alpha)$. $\square$

Finite generation of $(S,\cdot)$ and $(S,\star)$ is related as follows:

Proposition 4.20 *With the above notation, $(S,\star)$ is finitely generated if and only if $(S,\cdot)$ is finitely generated and there exists a finite set $X \subseteq S$ such that $S\backslash X(S\alpha)$ is finite.*

PROOF Suppose $(S,\star)$ is generated by a finite set $X \subseteq S$. By Lemma 4.19(i) and (ii), $S\backslash X(S\alpha)$ is finite. Also, if $s \in S$ there exist $x_1, x_2, \ldots, x_n \in X$ such that

$$s = x_1 \star x_2 \star \cdots \star x_n = x_1(x_2\alpha)\cdots(x_n\alpha)$$

and thus $(S,\cdot)$ is generated by the set $X \cup (X\alpha)$, which is a finite set.

Conversely, suppose that A is a finite generating set for $(S,\cdot)$, and $X \subseteq S$ is a finite set such that $S\backslash X(S\alpha)$ is finite. We will prove that

$$Y = (S\backslash X(S\alpha)) \cup X \cup A$$

is a generating set for $(S,\star)$. Indeed, if $s \in S$ then either $s \in S\backslash X(S\alpha)$, in which case $s \in Y$, or else $s \in X(S\alpha)$. In this last case there exists $x \in X$ and $s' \in S$ such that

$$s = x(s'\alpha)$$

and if $s' = a_1a_2\cdots a_m$ for some $a_1, a_2, \ldots, a_m \in A$ then

$$\begin{aligned} s = x(s'\alpha) &= x(a_1\alpha)(a_2\alpha)\cdots(a_m\alpha) \\ &= x \star a_1 \star a_1 \star \cdots \star a_m. \end{aligned}$$

Therefore $(S,\star)$ is generated by Y, which is a finite set. $\square$

We thus have

Theorem 4.21 *Let S and T be infinite semigroups, let $\alpha : S \to S$ be an idempotent endomorphism of S, and let $\phi : T \to \mathrm{End}(S)$ be such that $t \mapsto \alpha$ for all $t \in T$. Then $S \rtimes_\phi T$ is finitely generated if and only if the following are satisfied:*

(i) *S and T are both finitely generated;*

(ii) *$T^2 = T$; and*

(iii) *there exists a finite set $X \subseteq S$ such that $X(S\alpha) = S$.*

Proof By Lemma 4.19(iv) we know that $S \rtimes_\phi T \cong (S, \star) \times T$, and thus $S \rtimes_\phi T$ is finitely generated if and only if $(S, \star) \times T$ is finitely generated. Now, if $(S, \star) \times T$ is finitely generated then, by Theorem 3.2, $(S, \star)$ and T are both finitely generated, $T^2 = T$ and $(S, \star)^2 = (S, \star)$. This implies, by Proposition 4.20, that $(S, \cdot)$ is finitely generated. Furthermore, if X is a finite generating set for $(S, \star)$ then, by Lemma 4.19(v) and (iii), $X(S\alpha) = S$.

For the converse implication notice that if (i), (ii) and (iii) hold then, by Proposition 4.20, $(S, \star)$ is finitely generated. Also $S(S\alpha) = S$ and thus $(S, \star)^2 = (S, \star)$ by Lemma 4.19(v). Using Theorem 3.2 once again we conclude that $(S, \star) \times T$ is finitely generated and thus $S \rtimes_\phi T$ is finitely generated as well. □

Theorem 4.22 *Let S be a finite semigroup and T be an infinite semigroup, let $\alpha : S \to S$ be an idempotent endomorphism of S, and let $\phi : T \to \mathrm{End}(S)$ be such that $t \mapsto \alpha$ for all $t \in T$. Then $S \rtimes_\phi T$ is finitely generated if and only if the following are satisfied:*

(i) *T is finitely generated; and*

(ii) *$S(S\alpha) = S$.*

Proof The semidirect product $S \rtimes_\phi T$ is finitely generated if and only if $(S, \star) \times T$ is finitely generated. Thus, by Theorem 3.3, $S \rtimes_\phi T$ is finitely generated if and only if T is finitely generated and $(S, \star)^2 = (S, \star)$. Using Lemma 4.19 we conclude that $S \rtimes_\phi T$ is finitely generated if and only if T is finitely generated and $S(S\alpha) = S$. □

Theorem 4.23 *Let S be an infinite semigroup and T be a finite semigroup, let $\alpha : S \to S$ be an idempotent endomorphism of S, and let $\phi : T \to \mathrm{End}(S)$ be such that $t \mapsto \alpha$ for all $t \in T$. Then $S \rtimes_\phi T$ is finitely generated if and only if the following are satisfied:*

(i) *S is finitely generated;*

(ii) $T^2 = T$;

(iii) *there exists a finite set $X \subseteq S$ such that $S \backslash X(S\alpha)$ is finite.*

PROOF The semidirect product $S \rtimes_\phi T$ is finitely generated if and only if $(S, \star) \times T$ is finitely generated. Thus, by Theorem 3.3, $S \rtimes_\phi T$ is finitely generated if and only if $T^2 = T$ and $(S, \star)$ is finitely generated. The result then follows by Proposition 4.20. $\square$

We are now interested in investigating finite presentability of $S \rtimes_\phi T$. We shall once again explore the connections between finitely presentability of $(S, \cdot)$ and $(S, \star)$.

So let $\langle X \mid R \rangle$ be a presentation for $(S, \star)$. We are going to find a presentation for $(S, \cdot)$. By the proof of Proposition 4.20 it follows that, since $(S, \star)$ is generated by the set X, $(S, \cdot)$ is generated by the set $X \cup (X\alpha)$. Thus we introduce a new alphabet $\overline{X} = \{\overline{x} : x \in X\}$ in one one correspondence with X; each $\overline{x}$ represents the element $x\alpha \in S$. We define a homomorphism $\psi : X^+ \to \overline{X}^+$ by extending the mapping

$$x\psi = \overline{x} \ (x \in X)$$

and define a mapping $\phi : X^+ \to (X \cup \overline{X})^+$ by

$$(x_1 x_2 \cdots x_n)\phi = x_1 \, \overline{x_2} \cdots \overline{x_n} = x_1(x_2 \cdots x_n)\psi$$

$(x_1, x_2, \ldots, x_n \in X)$. Notice that $w_1, w_2 \in X^+$ satisfy $w_1 = w_2$ in $(S, \star)$ if and only if $w_1\phi = w_2\phi$ in $(S, \cdot)$.

Lemma 4.24 *With the above notation we have:*

(i) *If $a, b \in X \cup \overline{X}$ then there exist $\alpha(a, b) \in X$ and $\mu(a, b) \in \overline{X}^*$ such that the relation $ab = \alpha(a, b)\mu(a, b)$ holds in $(S, \cdot)$.*

(ii) *For each $\overline{x} \in \overline{X}$, there exist $\beta(\overline{x}) \in X$ and $\nu(\overline{x}) \in \overline{X}^*$ such that the relation $\overline{x} = \beta(\overline{x})\nu(\overline{x})$ holds in $(S, \cdot)$.*

PROOF If $a, b \in X \cup \overline{X}$ then either $ab \in S \backslash X\overline{S}$ or else $ab \in X\overline{S}$. In the first case $ab \notin (S, \star)^2$ and therefore there must exist a generator $\alpha(a, b) \in X$ of $(S, \star)$ such that $ab = \alpha(a, b)$ in $(S, \cdot)$. In the second case there must exist $\alpha(a, b) \in X$ and $\mu(a, b) \in \overline{X}^+$ such that $ab = \alpha(a, b)\mu(a, b)$ in $(S, \cdot)$. Now, if $\overline{x} \in \overline{X}$, it is clear that there exist $\beta(\overline{x}) \in X$ and $\nu(\overline{x}) \in \overline{X}^*$ such that $\overline{x} = \beta(\overline{x})\nu(\overline{x})$ in $(S, \cdot)$. $\square$

For each $a, b \in X \cup \overline{X}$, $\overline{x} \in \overline{X}$ we fix $\alpha(a, b)$, $\mu(a, b)$, $\beta(\overline{x})$ and $\nu(\overline{x})$, according to Lemma 4.24.

We then have

Theorem 4.25 *With the above notation let*

$$R_1 = \{u\phi = v\phi : (u = v) \in R\},$$

$$R_2 = \{u\psi = v\psi : (u = v) \in R\},$$

$$R_3 = \{ab = \alpha(a, b)\mu(a, b) : a, b \in X \cup \overline{X}\}, \textit{and}$$

$$R_4 = \{x = \beta(x)\nu(x) : x \in \overline{X}\}.$$

Then $(S, \cdot)$ is defined by the presentation

$$\langle X, \overline{X} \mid R_1, R_2, R_3, R_4 \rangle.$$

For the proof of the theorem we will need the following

Lemma 4.26 *With the notation of Theorem 4.25, if $w \in (X \cup \overline{X})^+$ then there is a word $xw' \in X\overline{X}^*$ such that $w = xw'$ in $(S, \cdot)$ and xw' can be obtained from w by finitely many applications of relations from R_3 and R_4.*

PROOF We use induction on the length of w. If $|w| = 1$ then either $w \in X$, or else $w \in \overline{X}$. In the latter case applying a relation from R_4 yields $w = \beta(w)\nu(w) \in X\overline{X}^*$ as required. Thus the result holds for $|w| = 1$. Now, if

$$w \equiv x_1 x_2,$$

that is w is of length 2, the relation $w \equiv x_1 x_2 = \alpha(x_1, x_2)\mu(x_1, x_2)$ is in R_3 and thus the result is also true. Now suppose that the result is true for a word of length $k - 1$ and suppose that $|w| = k > 2$. Then

$$w \equiv xw'$$

where $x \in X \cup \overline{X}$ and $w' \in (X \cup \overline{X})^+$. Notice that $|w'| = k - 1$. Hence, by the induction hypothesis, there exists $x' \in X$ and $w'' \in \overline{X}^*$ such that $w' = x'w''$ in $(S, \cdot)$ and there is a finite sequence of applications of relations from R_3 that transforms w' into $x'w''$. The result follows by the length 2 case, and using the relation

$$xx' = \alpha(x, x')\mu(x, x').$$

□

PROOF[of Theorem 4.25] We suppose without loss of generality, that X does not contain redundant generators.

First we will prove that all the relations hold in $(S, \cdot)$. Let $(u = v) \in R$. This means that $u = v$ holds in $(S, \star)$, i.e. $u\phi = v\phi$ holds in $(S, \cdot)$. Now, since ψ is homomorphism and $\phi\psi = \psi$, we also have $u\psi = v\psi$ holding in $(S, \cdot)$. Therefore both sets of relations R_1 and R_2 hold in $(S, \cdot)$. The relations from R_3 and R_4 hold in $(S, \cdot)$ by Lemma 4.24.

Now suppose that $w_1 = w_2$ holds in S for some $w_1, w_2 \in (X \cup \overline{X})^+$. We have to prove that then $w_1 = w_2$ is a consequence of the relations from R_1, R_2, R_3 and R_4. By Lemma 4.26 there exists a finite sequence

$$w_1 \equiv \gamma_1, \gamma_2, \ldots, \gamma_k \equiv x_1 w_1' \in X\overline{X}^*$$

of words from $(X \cup \overline{X})^+$, in which every term is obtained from the previous one by one application of one relation from either R_3 or R_4. Similarly, there is a finite sequence

$$w_2 \equiv \gamma_1', \gamma_2', \ldots, \gamma_h' \equiv x_2 w_2' \in X\overline{X}^*.$$

Now, since $w_1 = w_2$ holds in $(S, \cdot)$, the relation

$$x_1 w_1' = x_2 w_2'$$

holds in $(S, \cdot)$. If $w_1' \equiv \overline{y_1}\,\overline{y_2} \cdots \overline{y_m}$ and $w_2' \equiv \overline{y_1'}\,\overline{y_2'} \cdots \overline{y_n'}$ then

$$x_1 w_1' \equiv x_1 \overline{y_1}\,\overline{y_2} \cdots \overline{y_m} \equiv (x_1 y_1 y_2 \cdots y_m)\phi$$

and

$$x_2 w_2' \equiv x_1 \overline{y_1'}\,\overline{y_2'} \cdots \overline{y_m'} \equiv (x_2 y_1' y_2' \cdots y_n')\phi.$$

Thus $x_1y_1y_2\cdots y_m = x_2y_1'y_2'\cdots y_n'$ holds in $(S,\star)$ and thus, by using finitely many relations from R_1 and R_2 we can transform one into the other. Therefore $w_1 = w_2$ is a consequence of the relations given. $\square$

Corollary 4.27 *With the above notation, if $(S,\star)$ if finitely presented then $(S,\cdot)$ is also finitely presented.*

If $(S,\cdot)$ is finitely presented, then the question now arises as to when is $(S,\star)$ finitely presented. From Proposition 4.20 we know that the existence of a finite set $X \subseteq S$ such that $S\backslash X(S\alpha)$ is also finite, is a necessary condition.

Question 4.28 *Is is true that $(S,\star)$ is finitely presented if and only if $(S,\cdot)$ is finitely presented and there exists a finite set $X \subseteq S$ such that $S\backslash X(S\alpha)$ is finite?*

We consider the following

Example 4.29 Let $S = C_2 \times C_2 \times C_\infty$. Then S is defined by the presentation

$$\begin{aligned}\langle a,b,c,c^{-1} \quad | \quad & a^2=1,\ b^2=1,\ cc^{-1}=c^{-1}c=1,\\ & ab=ba,\ ac=ca,\ bc=cb\rangle.\end{aligned}$$

Take $\alpha : S \to S$ to be the endomorphism of S defined by

$$a \mapsto b,\ b \mapsto b,\ c \mapsto c.$$

Then $S\alpha$ is generated by $\{b,c\}$. In fact $S\alpha = C_2 \times C_\infty$. Notice that any element of S different from the identity can be written uniquely as $a^ib^jc^k$ for $i \in \{0,1\}$, $j \in \{0,1\}$, $k \in \mathbb{Z}$, at least one of i,j,k non zero. Now consider the semigroup $(S,\star)$ as defined before. Then multiplication of two elements $a^ib^jc^k, a^mb^nc^p \in S$ in $(S,\star)$ is given by

$$(a^ib^jc^k)\star(a^mb^nc^p) = a^ib^jc^k((a^mb^nc^p)\alpha) = \begin{cases} a^ibc^{k+p} & \text{if } j+m+n = 1,3 \\ a^ic^{k+p} & \text{if } j+m+n=0,2.\end{cases}$$

We claim that then $(S,\star)$ is defined by the presentation

$$\begin{aligned}\langle a,b,c,c^{-1} \quad | \quad & a^2=ab,\ b^2=ba,\ b^2c=c,\ b^2c^{-1}=c^{-1},\\ & ab^2=a,\ b^2=cc^{-1}=c^{-1}c,\ b^3=b,\\ & ca=cb=bc,\ c^{-1}a=c^{-1}b=bc^{-1}\rangle.\end{aligned} \tag{4.9}$$

Clearly all the relations from this presentation hold in $(S,\star)$. It is thus enough to prove that given an element $a^ib^jc^k \in S$, and multiplying it by each of the generators, we can obtain an element of the same form by using the given relations in the presentation. (We choose b^2 to be the word representing the identity element of $(S,\cdot)$.) We start by noticing that if $h,k\in\mathbb{Z}$ the relations

$$c^hc^{-h}=b^2$$

and

$$c^hc^k=c^{h+k}\ (h\neq -k)$$

are consequences of the presentation (4.9). Thus, for $i\in\{0,1\}$, $j\in\{0,1\}$ and $k\in\mathbb{Z}$ (at least one of i,j,k non zero) we have

$$(a^ib^jc^k)a=a^ib^{j+1}c^k\equiv\begin{cases}a^ib^2c^k=a^ic^k & \text{if } j=1, (i\neq 0 \text{ or } k\neq 0)\\ a^ibc^k & \text{if } j=0\\ b^2 & \text{if } i=0, j=1, k=0\end{cases}$$

$$(a^ib^jc^k)b=a^ib^{j+1}c^k=\begin{cases}a^ic^k & \text{if } j=1, (i\neq 0 \text{ or } k\neq 0)\\ b^2 & \text{if } i=0, j=1, k=0\\ a^ibc^k & \text{if } j=0\end{cases}$$

$$(a^ib^jc^k)c\equiv\begin{cases}a^ib^jc^{k+1} & \text{if } k\neq -1\\ a^ib^jc^{-1}c=a^ib^jb^2=\begin{cases}a^ib^j & \text{if } (i\neq 0 \text{ or } j\neq 0), k=-1\\ b^2 & \text{if } i=0, j=0, k=-1\end{cases}\end{cases}$$

$$(a^ib^jc^k)c^{-1}\equiv\begin{cases}a^ib^jc^{k-1} & \text{if } k\neq 1\\ a^ib^jcc^{-1}=a^ib^jb^2=\begin{cases}a^ib^j & \text{if } (i\neq 0 \text{ or } j\neq 0), k=1\\ b^2 & \text{if } i=0, j=0, k=1\end{cases}\end{cases}$$

All these are consequences from presentation (4.9) and therefore it follows that (4.9) defines $(S,\star)$.

Another question arises, which in view of Theorems 3.21 and 3.22, will also be relevant for the proof of a result about finitely presentability of $S\rtimes_\phi T$.

Question 4.30 *If $(S,\cdot)$ is a stable semigroup, is $(S,\star)$ necessarily stable? Conversely, does $(S,\cdot)$ have to be stable for $(S,\star)$ to be stable?*

We now look at the special case when S contains an idempotent $e = e^2$, and an endomorphism $\alpha : S \to S$ such that $s \mapsto e$ $(s \in S\ (s \in S))$. We then have

Proposition 4.31 *With the above notation $S \rtimes_\phi T$ is finitely generated (resp. finitely presented) if and only if both S and T are finite or S is finite, $S = Se$ and T is finitely generated (resp. finitely presented).*

Proof Suppose that $S \rtimes_\phi T$ is finitely generated and let Z be a finite generating set for it. Then, for any $s \in Se$ and $t \in T$ there exist elements $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n) \in Z$ such that

$$(s,t) = (x_1, y_1)(x_2, y_2)\cdots(x_n, y_n) = (x_1 e, y_1 y_2 \cdots y_n).$$

Thus,

$$Se \subseteq \{u \in S : (\exists v \in T)(u, v) \in Z\}e \tag{4.10}$$

which implies that Se is finite, since the right hand side of (4.10) is a finite set. On the other hand, $S - Se$ must also be finite. Indeed, the elements of $S \rtimes_\phi T$ with first component in $S - Se$ are indecomposable, and therefore they must all be in Z. We conclude that S is finite. Now, by Theorem 4.22, it follows that if T is not finite then T is finitely generated and $S = S(S\alpha) = Se$.

Conversely, if both S and T are finite then $S \rtimes_\phi T$ is finite and thus is finitely generated. If S is finite, $S = Se$ and T is infinite but finitely generated then, also by Theorem 4.22, it follows that $S \rtimes_\phi T$ is finitely generated (take $X = S$). In fact, it is clear (and will be useful for what follows) that if S is finite, $S = Se$ and T is generated by a set $B \subseteq T$ then $S \rtimes_\phi T$ is generated by the set $S \times B$.

Now, suppose that $S \rtimes_\phi T$ is finitely presented, and T is infinite. From the above we know that S is finite, $S = Se$ and T is finitely generated. Let B be a finite generating set for T and let $\langle S \times B \mid R\rangle$ be a finite presentation for $S \rtimes_\phi T$. Also let

$$\pi : (S \times B)^+ \to B^+$$

be the homomorphism extending the mapping

$$(s,t) \mapsto t.$$

We will prove that T is then defined by the presentation

$$\langle B \mid u\pi = v\pi\ ((u = v) \in R)\rangle. \tag{4.11}$$

If $(u, v) \in R$ and

$$u \equiv (s_1, b_1)(s_2, b_2) \cdots (s_m, b_m) \in (S \times B)^+,$$

$$v \equiv (s'_1, b'_1)(s'_2, b'_2) \cdots (s'_n, b'_n) \in (S \times B)^+$$

then

$$u\pi \equiv b_1 b_2 \cdots b_m \text{ and } v\pi \equiv b'_1 b'_2 \cdots b'_n.$$

Now, since $u = v$ in $S \rtimes_\phi T$, it follows that $u\pi = v\pi$ in T, as required. If $w_1, w_2 \in B^+$ are such that $w_1 = w_2$ holds in T, $w_1 \equiv b_1 b_2 \cdots b_m \in B^+$ and $w_2 \equiv b'_1 b'_2 \cdots b'_n \in B^+$, then the relation

$$(e, b_1)(e, b_2) \cdots (e, b_m) = (e, b'_1)(e, b'_2) \cdots (e, b'_n)$$

holds in $S \rtimes_\phi T$. Hence this relation must be a consequence of the relations from R, that is, there is an elementary sequence

$$(e, b_1)(e, b_2) \cdots (e, b_m) \equiv \gamma_1, \gamma_2, \cdots, \gamma_k \equiv (e, b'_1)(e, b'_2) \cdots (e, b'_n)$$

($\gamma_i \in (S \times B)^+$, $1 \le i \le k$) of applications of relations from R. But if γ_i ($1 < i \le k$) is obtained from γ_{i-1} by application of a relation $(u = v) \in R$ then, since π is a homomorphism, $\gamma_i\pi$ is obtained from $\gamma_{i-1}\pi$ by application of the relation $u\pi = v\pi$. It therefore follows that $w_1 = w_2$ is a consequence of (4.11). This concludes the proof that the presentation (4.11) defines T.

Finally suppose that S is finite, $S = Se$ and T is finitely presented. Consider the semigroup T^I, obtained from T by adjoining an identity I (regardless of whether T has an identity). Define a homomorphism $\overline{\phi}$ by extending ϕ as follows:

$$\begin{aligned} \overline{\phi} \ : \ & T^I \to \mathrm{End}(S) \\ & t \mapsto t\phi \ (t \in T) \\ & I \mapsto \overline{1}_S \end{aligned}$$

(where $\overline{1}_S$ denotes the identity endomorphism of S). Now,

$$S \rtimes_{\overline{\phi}} T^I = (S \rtimes_\phi T) \,\dot\cup\, \{(s, I) : s \in S\}$$

and since S is a finite set, by Proposition 2.7, $S \rtimes_\phi T$ is finitely presented if and only if $S \rtimes_{\overline{\phi}} T^I$ is finitely presented. Also, if $B \subseteq T$ is a generating set for T,

$B \cup \{I\}$ is a generating set for T^I, and it therefore follows that $S \times (B \cup \{I\})$ is a generating set for $S \rtimes_{\overline{\phi}} T^I$. However, for any $(s, b) \in S \times (B \cup \{I\})$ we have

$$(s, b) = (s, I)(e, b)$$

and thus $\{(s, I), (e, b) : s \in S, b \in B\}$ is a generating set for $S \rtimes_{\overline{\phi}} T^I$. So let $\langle S \mid R \rangle$ be a presentation for S, and let $\langle B \mid Q \rangle$ be a presentation for T. We will prove that

$$\langle S, B \mid R, Q, bs = b \ (s \in S, b \in B) \rangle \tag{4.12}$$

defines $S \rtimes_{\overline{\phi}} T^I$. Clearly all relations R, Q hold in $S \rtimes_{\overline{\phi}} T^I$. Relations $bs = b$ $(s \in S, b \in B)$ also hold. Now, given two words $w_1, w_2 \in (S \cup B)^+$, such that $w_1 = w_2$ holds in $S \rtimes_{\overline{\phi}} T^I$, we can use relations $bs = b$ $(s \in S, b \in B)$ to write w_1 and w_2 as $\alpha_1\beta_1$ and $\alpha_2\beta_2$ respectively $(\alpha_1, \alpha_2 \in S^+, \beta_1, \beta_2 \in B^+)$. It is then clear that $\alpha_1\beta_1 = \alpha_2\beta_2$ implies that $\alpha_1 = \alpha_2$ in S and $\beta_1 = \beta_2$ in T. Hence these relations are consequences of R and Q, respectively. It follows that $w_1 = w_2$ is a consequence of the presentation (4.12). Thus (4.12) defines $S \rtimes_{\overline{\phi}} T^I$, which is therefore finitely presented. □

Example 4.32 If $e = 0$ is a right zero of the semigroup S then $S \neq Se$ and therefore $S \rtimes_\phi T$ is finitely generated (resp. finitely presented) if and only if both S and T are finite.

Remark 4.33 Notice that $S = Se$ if and only if e is a right identity of the semigroup S. Thus that is the only situation when $S \rtimes_\phi T$ can be finitely generated or finitely presented with T infinite.

Chapter 5

Bruck–Reilly extensions of monoids

In this chapter we are concerned with finite generation and finite presentability of Bruck–Reilly extensions of monoids.

So, let M be a monoid with identity 1_M and let $\theta : M \to M$ be an endomorphism of M. On the set $\mathbb{N}_0 \times M \times \mathbb{N}_0$ (where $\mathbb{N}_0$ denotes the set of non-negative integers) we define a binary operation by

$$(m, a, n)(p, b, q) = (m - n + t, (a\theta^{t-n})(b\theta^{t-p}), q - p + t) \tag{5.1}$$

where $t = \max(n, p)$, and θ^0 denotes the identity map on M. The set $\mathbb{N}_0 \times M \times \mathbb{N}_0$ together with this operation is a monoid with identity $(0, 1_M, 0)$, which we denote by $\mathrm{BR}(M, \theta)$ and call the *Bruck–Reilly extension of M determined by θ.*

This construction is a generalization of constructions by Bruck [12], Reilly [44], and Munn [38]. Bruck's construction considers the special case when θ maps all elements to the identity of M, in which case the obtained extension is a simple monoid, and is used to prove that every semigroup embeds in a simple monoid ([12, Theorem 8.3]). On the other hand Reilly's construction considers the case when the monoid M is a group; the monoid obtained is a bisimple inverse ω-semigroup and, conversely, every bisimple inverse ω-semigroup is a Bruck–Reilly extension of its group of units ([44, Theorems 2.2 and 3.5])). Finally, Munn considers Bruck–Reilly extensions with respect to endomorphisms that map the monoid into its group of units and hence gives a structure theorem for a special class of simple inverse semigroups ([38, Theorem 3.3]).

We begin our considerations in Section 5.1 by exhibiting a presentation for the Bruck–Reilly extension of a monoid. Then, in Section 5.2 we study Bruck–Reilly extensions of groups, characterizing finitely generated and finitely presented ones. In Section 5.3 we consider Bruck–Reilly extensions of arbitrary monoids and proof that the result for finite generation given in Section 5.2 generalizes for this case (with a different proof, however). Finally, in Section 5.4, we compare monoid presentations and inverse monoid presentations for a Bruck–Reilly extension of a group.

The results presented in Sections 5.2 and 5.4 appear in [6]. The results from Section 5.3 appear here for the first time.

5.1 A known presentation and other opening remarks

Consider the Bruck–Reilly extension $\mathrm{BR}(M,\theta)$ of a monoid M with respect to a homomorphism θ of M. We are interested in finding a presentation for $\mathrm{BR}(M,\theta)$. The first step is to find a generating set for the extension given a generating set for M.

Lemma 5.1 [29, Lemma 4.1] *With the above notation, if M is generated by a set A then $\mathrm{BR}(M,\theta)$ is generated by the set*

$$X = \{(0,a,0) \mid a \in A\} \cup \{(0,1_M,1),(1,1_M,0)\}.$$

PROOF Let $(m,x,n) \in \mathrm{BR}(M,\theta)$ be arbitrary. Notice that we can write (m,x,n) as

$$(m,x,n) = (m,1_M,0)(0,x,0)(0,1_M,n).$$

Also

$$(m,1_M,0) = (m-1,1_M,0)(1,1_M,0) = (1,1_M,0)^m,$$

$$(0,1_M,n) = (0,1_M,n-1)(0,1_M,1) = (0,1_M,1)^n.$$

Now, since x can be written as a product $a_1a_2\cdots a_k$ of elements of A and

$$(0,x,0) = (0,a_1a_2\cdots a_k,0) = (0,a_1,0)(0,a_2,0)\cdots(0,a_k,0)$$

we finally have

$$(m,x,n) = (1,1_M,0)^m(0,a_1,0)(0,a_2,0)\cdots(0,a_k,0)(0,1_M,1)^n.$$

Therefore X generates $\mathrm{BR}(M,\theta)$. $\square$

Remark 5.2 Notice that M embeds into $\mathrm{BR}(M,\theta)$ by the mapping $x \mapsto (0,x,0)$ $(x \in M)$.

Now, if $M = A^*/\rho$ is the monoid defined by the presentation $\langle A \mid R\rangle$ every endomorphism $\theta : M \to M$ can be 'lifted' to an endomorphism $\overline{\theta} : A^* \to A^*$, in the sense that for every word $w \in A^*$ we have

$$(w/\rho)\theta = (w\overline{\theta})/\rho.$$

To see this it is enough to choose arbitrary words $a\overline{\theta}$, representing $(a/\rho)\theta$ $(a \in A)$, and then extend the mapping $a \mapsto a\overline{\theta}$ to an endomorphism $A^* \to A^*$. In line with the convention of identifying w and w/ρ, we shall also identify θ and $\overline{\theta}$. With this in mind, we have the following

Proposition 5.3 [29, Theorem 4.2] *Let M be a monoid defined by a presentation $\langle A \mid R\rangle$ and let θ be an endomorphism of M. Then the Bruck–Reilly extension $\mathrm{BR}(M,\theta)$ is defined by the presentation*

$$\langle A, b, c \mid R,\ bc = 1,\ ba = (a\theta)b,\ ac = c(a\theta)\ (a \in A)\rangle. \tag{5.2}$$

Proof We follow the proof from [29] (an alternative proof can be found in [33]). So, let

$$\phi : (A \cup \{x,y\})^* \to \mathrm{BR}(M,\theta)$$

be the monoid homomorphism extending the mapping

$$a\phi = (0,a,0)\ (a \in A),\ b\phi = (0,1_M,1),\ c\phi = (1,1_M,0).$$

By Lemma 5.1, ϕ is an epimorphism. Further, a relation $u = v$ holds in $\mathrm{BR}(M,\theta)$ if and only if $u\phi = v\phi$. Now, since all the relations from R hold in M they also hold in $\mathrm{BR}(M,\theta)$ since for any $x \in M$

$$x\phi = (0,x,0).$$

The remaining relations also hold in $\mathrm{BR}(M,\theta)$ since

$$\begin{aligned}(0,1_M,1)(1,1_M,0) &= (0,1_M,0),\\ (0,1_M,1)(0,a,0) &= (0,a\theta,1)=(0,a\theta,0)(0,1_M,1), \text{ and}\\ (0,a,0)(1,1_M,0) &= (1,a\theta,0)=(1,1_M,0)(0,a\theta,0).\end{aligned}$$

Thus, all the relations from the presentation (5.2) hold in $\mathrm{BR}(M,\theta)$.

Next we will prove that given a non-empty word $w \in (A \cup \{x,y\})^*$ there is a word $c^i\alpha b^j$ ($\alpha \in A^*$, $i,j \geq 0$) such that $w = c^i\alpha b^j$ is a consequence of (5.2). Indeed, if $|w| = 1$ then either $w \equiv a$ for some $a \in A$, $w \equiv b$ or $w \equiv c$. In any of these cases w is already a word of the desired form. Now suppose that the claim is true for any word of length less than k ($k > 1$) and let $w \in (A \cup \{b,c\})^*$ be a word of length k. Then $w \equiv w'x$ where $w' \in (A \cup \{b,c\})^*$ is a word of length $k-1$ and $x \in A \cup \{b,c\}$. Thus, by the induction hypothesis, there is $\alpha \in A^*$, $i,j \geq 0$ such that $w' = c^i\alpha b^j$ is a consequence of (5.2). Thus the relation

$$w = c^i\alpha b^j x$$

is also a consequence of (5.2). Now, there are three possibilities: either $x \in A$, $x \equiv b$ or $x \equiv c$. If $x \in A$ then j applications of the relation $bx = (x\theta)b$ yield $w = c^i\alpha(x\theta^j)b^j$. If $x \equiv b$ then $w \equiv c^i\alpha b^{j+1}$. Finally, if $x \equiv c$, then either $j = 0$, and by applying relations $a_ic = c(a_i\theta)$ ($1 \leq i \leq k$ with $\alpha \equiv a_1a_2\cdots a_k$) we obtain $w = c^{i+1}(\alpha\theta)$, or else $w \equiv c^i\alpha b^{j-1}$. This concludes the proof of the claim.

Now let $w_1, w_2 \in (A \cup \{b,c\})^*$ be two words such that $w_1 = w_2$ holds in S. We have to prove that this relation is a consequence of the presentation (5.2). From above we know that there are $\alpha, \beta \in A^*$, $i,j,k,l \geq 0$ such that

$$w_1 = c^i\alpha b^j \text{ and } w_2 = c^k\beta b^l$$

are consequences of (5.2). Since $w_1 = w_2$ we also have

$$c^i\alpha b^j = c^k\beta b^l$$

in $\mathrm{BR}(M,\theta)$, which means that

$$(c^i\alpha b^j)\phi = (c^k\beta b^l)\phi,$$

i.e. $(i,\alpha,j) = (k,\beta,l)$ and thus $i = k$, $j = l$ and $\alpha = \beta$ in M. But then this implies that $\alpha = \beta$ is a consequence of R and therefore

$$c^i\alpha b^j = c^k\beta b^l$$

is a consequence of (5.2). $\square$

For the purposes of our study we will consider the presentation (5.2) as the definition of the Bruck–Reilly extension $\mathrm{BR}(M,\theta)$.

It follows from (5.2) that if a monoid is finitely generated (resp. finitely presented) then so is every Bruck–Reilly extension of it. The converses are not true though, as the following two examples show.

Example 5.4 The free group F on an infinite set $\{x_i : i \geq 1\}$ of generators has finitely generated Bruck–Reilly extensions. Indeed, the mapping $x_i \mapsto x_{i+1}$ $(i \geq 1)$ extends to a homomorphism of groups $\theta : F \to F$. Since $x_{i+1} \equiv x_1\theta^i = b^i x_1 c^i$, the Bruck–Reilly extension $\mathrm{BR}(F,\theta)$ is generated by $\{x_1, x_1^{-1}, b, c\}$, and so it is finitely generated.

Example 5.5 [57] Consider the group G defined by the (group) presentation

$$\langle a_1, a_2, a_3, a_4 \mid a_1^{2^i} a_2^{2^i} = a_3^{2^i} a_4^{2^i} \ (i \geq 0)\rangle.$$

This is the free product with amalgamation of free groups $\langle a_1, a_2 \mid\rangle$ and $\langle a_3, a_4 \mid\rangle$ along the subgroup generated by $\{a_1^{2^i} a_2^{2^i} : i \geq 0\}$ (isomorphic to the subgroup generated by $\{a_3^{2^i} a_4^{2^i} : i \geq 0\}$). This subgroup has infinite rank since the given set of generators is Nielson reduced. Therefore G is not finitely presented (see, for example, [10]).

Now, let $\theta : G \to G$ be an endomorphism extending the mapping $a_j \mapsto a_j^2$ $(j = 1,2,3,4)$. Then, by (5.2), the Bruck–Reilly extension of G with respect to θ is defined by the following (monoid) presentation

$$\begin{aligned}\langle a_1, a_2, a_3, a_4, a_1', a_2', a_3', a_4', b, c \ \mid \ & a_j a_j' = a_j' a_j = 1,\ a_1^{2^i} a_2^{2^i} = a_3^{2^i} a_4^{2^i},\ bc = 1,\\ & ba_j = a_j^2 b,\ a_j c = c a_j^2 \ (j = 1,2,3,4)\rangle.\end{aligned}$$

In fact, this presentation is equivalent to

$$\begin{aligned}\langle a_1, a_2, a_3, a_4, a_1', a_2', a_3', a_4', b, c \ \mid \ & a_j a_j' = a_j' a_j = 1,\ a_1 a_2 = a_3 a_4,\ bc = 1,\\ & ba_j = a_j^2 b,\ a_j c = c a_j^2 \ (j = 1,2,3,4)\rangle.\end{aligned} \tag{5.3}$$

To prove this we have to show that the relations $a_1^{2^i}a_2^{2^i} = a_3^{2^i}a_4^{2^i}$ $(i \geq 0)$ are a consequence of (5.3). We will use induction on i. If $i = 1$ then

$$a_1a_2 = a_3a_4$$

is one of the relations in (5.3). Now, suppose $a_1^{2^{i-1}}a_2^{2^{i-1}} = a_3^{2^{i-1}}a_4^{2^{i-1}}$ $(i \geq 1)$ is a consequence of (5.3). Then

$$\begin{array}{rcll} a_1^{2^i}a_2^{2^i} & = & a_1^{2^i}bc\,a_2^{2^i} & (bc = 1) \\ & = & b\,a_1^{2^{i-1}}a_2^{2^{i-1}}c & (a_1^2b = ba_1 \text{ and } ca_2^2 = a_2c) \\ & = & b\,a_3^{2^{i-1}}a_4^{2^{i-1}}c & (\text{induction hypothesis}) \\ & = & a_3^{2^i}a_4^{2^i} & (a_1^2b = ba_1,\ ca_2^2 = a_2c \text{ and } bc = 1). \end{array}$$

Thus, $\mathrm{BR}(G, \theta)$ is a finitely presented Bruck–Reilly extension of a non-finitely presented group.

The following questions thus arise:

Question 5.6 *Under which conditions is a Bruck–Reilly extension* $\mathrm{BR}(M, \theta)$ *of a monoid M, determined by an endomorphism θ of M, finitely generated?*

Question 5.7 *Under which conditions is a Bruck–Reilly extension* $\mathrm{BR}(M, \theta)$ *of a monoid M, determined by an endomorphism θ of M, finitely presented?*

Example 5.8 Let M be a monoid and let $\overline{1}_M$ be the identity endomorphism of M. Then, it is an easy consequence (5.1) that the Bruck–Reilly extension $\mathrm{BR}(M, 1_M)$ of M is isomorphic to the direct product of M with the bicyclic monoid. Therefore, by the results in Chapter 3, $\mathrm{BR}(M, \overline{1}_M)$ is finitely generated (resp. finitely presented) if and only if M is finitely generated (resp. finitely presented).

Remark 5.9 It is worth remarking the similarity between the presentations for a Bruck–Reilly extension and an HNN extension of a group. Indeed, for a group G (defined by a monoid presentation $\langle A \mid R \rangle$), subgroups $H = \langle X \rangle$ and K and an isomorphism $\theta : H \to K$, the HNN extension of G relative to H, K and θ is given by

$$\langle A, b, c \mid R,\ bc = 1,\ cb = 1,\ bx = (x\theta)b\ (x \in X) \rangle,$$

which should be compared to (5.2). Also, it is known that, provided G is finitely presented, the above HNN extension is finitely presented if and only if H is finitely generated. This is a relatively easy consequence of the normal form theorem for HNN extension (see [34] and [10, Exercise VI.4(i)]) and should be compared to Theorem 5.15.

Before we end this section we will list some basic properties of Bruck–Reilly extensions, which will be used throughout the text.

Lemma 5.10 *Let $S = \mathrm{BR}(M, \theta)$ be a Bruck–Reilly extension of a monoid M defined by a presentation $\langle A \mid R\rangle$.*

(i) Every word $w \in (A \cup \{b, c\})^*$ is equal in S to a word of the form $c^i \alpha b^j$, where $\alpha \in A^*$, $i, j \geq 0$.

(ii) If $w \equiv c^i \alpha b^j$ and $w' \equiv c^k \beta b^l$ are words in $(A \cup \{b, c\})^*$ ($\alpha, \beta \in A^*$, $i, j \geq 0$) then the word ww' is equal in S to

$$c^{i-j+t}(a\theta^{t-j})(b\theta^{t-k})b^{l-k+t}$$

where $t = \max\{j, k\}$.

(iii) For all $i, j, k, l \geq 0$ and all $\alpha, \beta \in A^*$ we have

$$c^i \alpha b^j = c^k \beta b^l \Leftrightarrow i = k,\ j = l,\ \text{and } \alpha = \beta \text{ in } M.$$

(iv) For all $i, j, k, l \geq 0$ and all $\alpha, \beta \in A^*$ we have

$$\begin{aligned} c^i \alpha b^j \, \mathcal{R}^S c^k \beta b^l &\Leftrightarrow i = k \text{ and } \alpha \, \mathcal{R}^M \beta; \\ c^i \alpha b^j \, \mathcal{L}^S c^k \beta b^l &\Leftrightarrow j = l \text{ and } \alpha \, \mathcal{L}^M \beta; \\ c^i \alpha b^j \, \mathcal{H}^S c^k \beta b^l &\Leftrightarrow i = k,\ j = l \text{ and } \alpha \, \mathcal{H}^M \beta. \end{aligned}$$

(v) There is a (unique) epimorphism π from S onto the bicyclic monoid B (with presentation $\langle b, c \mid bc = 1\rangle$), such that $b\pi = b$, $c\pi = c$ and $a\pi = 1_B$ ($a \in A$).

(vi) The monoid M embeds in S and a word $w \in (A \cup \{b, c\})^*$ represents an element of M if and only if it is equal to some word from A^*, i.e. if and only if $w\pi = 1_B$.

PROOF Observations (i) and (iii) follow from the proof of Proposition 5.3, and (ii) is a consequence of (5.1).

For (iv) we follow the proof given in [28]. So, if $\alpha, \beta \in A^*$ and $i, j, k, l \geq 0$ are such that $c^i\alpha b^j \, \mathcal{R}^S \, c^k\beta b^l$ then there exists $\gamma, \gamma' \in A^*$, $r, s, t, u \geq 0$ such that

$$(c^i\alpha b^j)(c^r\gamma b^s) = c^k\beta b^l \text{ and } (c^k\beta b^l)(c^t\gamma' b^u) = c^i\alpha b^j$$

in S and thus, by (ii),

$$k = i - j + \max\{j, r\} \text{ and } i = k - l\max\{l, t\}.$$

In particular $k \geq i$, $i \geq k$ and therefore $i = k$. It then follows that $j \geq r$, $l \geq t$. Again, by (ii), $\alpha(\gamma\theta^{j-r}) = \beta$ and $\beta(\gamma'\theta^{l-t}) = \alpha$ in M, that is

$$\alpha\mathcal{R}^M\beta.$$

Conversely, suppose that $\alpha\mathcal{R}^M\beta$. Then there are $\gamma, \gamma' \in A^*$ such that

$$\beta = \alpha\gamma \text{ and } \alpha = \beta\gamma'$$

in M. Thus, in S

$$\begin{aligned}(c^i\alpha b^j)(c^j\gamma b^l) &= c^i\beta b^l\\(c^i\beta b^l)(c^l\gamma' b^j) &= c^i\alpha b^j\end{aligned}$$

from what we conclude that $c^i\alpha b^j \, \mathcal{R}^S \, c^i\beta b^l$.

For the $\mathcal{L}$–relation we proceed analogously. Thus, if $\alpha, \beta \in A^*$ and $i, j, k, l \geq 0$ are such that $c^i\alpha b^j \, \mathcal{L}^S \, c^k\beta b^l$ it follows that there exists $\gamma, \gamma' \in A^*$, $r, s, t, u \geq 0$ such that

$$(c^r\gamma b^s)(c^i\alpha b^j) = c^k\beta b^l \text{ and } (c^t\gamma' b^u)(c^k\beta b^l) = c^i\alpha b^j$$

and thus

$$l = j - i + \max\{s, i\} \text{ and } j = l - k + \max\{u, k\}.$$

In particular $l \geq j$, $j \geq l$ and thus $j = l$ which also implies that $i \geq s$ and $k \geq u$. Hence, by (ii) we have $(\gamma\theta^{i-s})\alpha = \beta$ and $(\gamma'\theta^{k-u})\beta = \alpha$ in M which means that $\alpha\mathcal{L}^M\beta$. Now, if $\alpha\mathcal{L}^M\beta$ then there is $\gamma, \gamma \in A^*$ such that

$$\beta = \gamma\alpha \text{ and } \alpha = \gamma'\beta$$

in M. So in S we have

$$\begin{aligned}(c^k\gamma b^i)(c^i\alpha b^l) &= c^k\beta b^l, \text{ and}\\ (c^i\gamma' b^k)(c^k\beta b^l) &= c^i\alpha b^l\end{aligned}$$

which means that $c^i\alpha b^l \mathcal{L}^s c^k\beta b^l$.

The result for the $\mathcal{H}$–relation follows as an immediate consequence of the above.

Now, if B denotes the bicyclic monoid, we define a mapping

$$\pi : \mathrm{BR}(M,\theta) \to B$$

inductively by

$$b\pi = b,\ c\pi = c,\ a\pi = 1_B\ (a \in A)$$

and for $w \equiv w'x$ with $w' \in (A \cup \{b,c\})^*$, $x \in A \cup \{b,c\}$

$$w\pi = (w'x)\pi = (w'\pi)(x\pi).$$

Since B is defined by the presentation $\langle b, c \mid bc = 1\rangle$ it follows that π is a well-defined onto mapping. It is also clear from the definition of π that it is a homomorphism.

For (vi) recall that, by Remark 5.2, M embeds in S. Thus, if $w \in A^*$ then w represents an element from M, and by (v), $w\pi = 1_B$. Conversely, if $w \in (A \cup \{x,y\})^*$ represents an element from M, then it has to be equal to a word in A^*, and therefore it also follows that $w\pi = 1_B$. □

Throughout the rest of this chapter we fix π to be the epimorphism described in Lemma 5.10(v).

5.2 Bruck–Reilly extensions of groups

In this section we answer Questions 5.6 and 5.7 for Bruck–Reilly extensions of groups. We start by proving the following

Proposition 5.11 *Let G be a group defined by a presentation $\langle A \mid R\rangle$, let θ be an endomorphism of G and let $S = \mathrm{BR}(G,\theta)$ be the Bruck–Reilly extension of G with respect to θ. Then G is isomorphic to the group of units of S via the mapping $a \mapsto a$ $(a \in A)$.*

PROOF By Remark 5.2 we know that G embeds in S, via the mapping

$$\psi : G \hookrightarrow S$$

$$a \mapsto a.$$

Clearly $G\psi$ is a group which is contained in the group of units of S. Now, if b was an invertible element of $\mathrm{BR}(G,\theta)$ then there would exist $x \in \mathrm{BR}(G,\theta)$ such that $bx = xb = 1$. By multiplying both sides of the equality $xb = 1$ by c we would obtain $xbc = c$ and hence $x = c$. This would imply that $cb = 1$ which is not true. Therefore b is not invertible in $\mathrm{BR}(G,\theta)$. A similar argument proves that c is not invertible either. Thus it follows that $G\psi$ is the group of units of S. The last part is then immediate. □

We will call the words in $(A \cup \{b,c\})^*$ which represent elements of G the *group words*. Examples of groups words are all the words from A^*. Notice though that there are group words which are not in A^*: bc and bac $(a \in A)$ are such examples.

Proposition 5.11 exhibits the connection between the problem here considered and the wider problem of finding presentations, and finite generation and presentability conditions, for a subgroup of a given semigroup or monoid. We have described that problem and a general solution in Section 2.2 and we will use those results to answer Questions 5.6 and 5.7.

Thus, we have the following characterization of finitely generated Bruck–Reilly extensions of groups:

Theorem 5.12 *Let G be a group and let $\theta : G \to G$ be an endomorphism. The Bruck–Reilly extension $S = \mathrm{BR}(G,\theta)$ is finitely generated if and only if there exists a finite subset $A_0 \subseteq G$ such that G is generated (as a monoid) by the set $\bigcup_{i=0}^{\infty} A_0\theta^i$.*

PROOF First suppose that there exists a finite set $A_0 \subseteq G$ such that G is generated by the set $A = \bigcup_{i=0}^{\infty} A_0\theta^i$. We are going to prove that S is generated by the set $Y = A_0 \cup \{b,c\}$. It is enough to show that any $a\theta^i$ $(a \in A_0,\ i \geq 1)$ can be written as a product of elements of Y. Indeed, using relations from the presentation (5.2), we have that $a\theta^i = b^i a c^i$. Therefore S is finitely generated.

Conversely suppose that S is finitely generated. If G is generated by a (possibly infinite) set A, then S is generated by $A \cup \{b, c\}$. Since S is finitely generated, there exists a finite subset $A_0 \subseteq A$ such that S is generated by $A_0 \cup \{b, c\}$.

We can now use Theorem 2.10, which gives a (Schreier type) generating set for any subgroup of a monoid, to obtain generators for G. For this we need $\mathcal{H}$–classes in the $\mathcal{R}$–class of G. They are $H_i = Gb^{i-1}$ $(i \geq 1)$ (by Lemma 5.10 (iv)). We also need elements r_i, r_i' such that $Gr_i = H_i$ and $H_i r_i' = G$. One possible choice is $r_i = b^{i-1}$ and $r_i' = c^{i-1}$ $(i \geq 1)$. Notice that this choice of r_i, r_i' satisfies the condition stated in the second part of Lemma 2.9. Using this, an immediate application of Theorem 2.10 yields the following generating set

$$\begin{aligned} \{b^i a c^i, b^i b c^{i+1}, b^{i+1} c c^i \ : \ a \in A_0, \ i \geq 0\} \backslash \{1_G\} &= \{b^i a c^i : a \in A_0, i \geq 0\} \\ &= \{a\theta^i \ : \ a \in A_0, \ i \geq 0\} \\ &= \textstyle\bigcup_{i=0}^{\infty} A_0 \theta^i \end{aligned}$$

for G, completing the proof. □

Example 5.13 Let F be the free group on an infinite set $\{x_i : i \geq 1\}$ of generators. Example 5.4 exhibits a finitely generated Bruck–Reilly extension of F. We are now going to build a non finitely generated one.

So, let $\theta : F \to F$ be the endomorphism of F extending the mapping

$$\begin{aligned} x_1 &\mapsto 1, \\ x_i &\mapsto x_{i-1} \ (i \geq 2). \end{aligned}$$

Now, suppose that there exists a finite set $A_0 \subseteq F$ such that F is generated by the set $A = \bigcup_{k \geq 0} A_0 \theta^k$. Notice that θ does not increase the length of a word. Also, if $x_j \in F$ $(j \geq 1)$ is such that for any word $x_i \in A_0$, $i < j$, then certainly $x_j \notin A$. Moreover, since x_j is a word of length 1 it cannot be written as a product of other generators from F. This contradicts the fact that A is a generating set for F. Therefore, by Theorem 5.12, it follows that $BR(F, \theta)$ is not finitely generated.

For finite presentability we have the following:

Theorem 5.14 *Let G be a group and $\theta : G \to G$ an endomorphism. Then the Bruck–Reilly extension $S = \mathrm{BR}(G, \theta)$ is finitely presented if and only if G can be defined by a presentation $\langle A \mid R\rangle$, where A is finite and*

$$R = \bigcup_{k=0}^{\infty} \overline{R}\theta^k = \{u\theta^k = v\theta^k \ : \ k \geq 0,\ (u = v) \in \overline{R}\},$$

for some finite set of relations $\overline{R} \subseteq A^* \times A^*$.

Notice that an immediate consequence of the above theorem is that if S is finitely presented then G is finitely generated. In fact, proving this is the main ingredient of the proof of the Theorem 5.14:

Theorem 5.15 *Let $S = \mathrm{BR}(G, \theta)$ be a Bruck–Reilly extension of a group G. If S is finitely presented then G is finitely generated.*

Proof Since S is finitely presented it is, in particular, finitely generated, and so, by Theorem 5.12, there exists a finite subset $A_0 = \{a_1, a_2, \ldots, a_t\} \subseteq G$ such that $\{a_i\theta^j \ : \ 1 \leq i \leq t,\ j \geq 0\}$ generates G.

Let $A = \{a_{i,j} \ : \ 1 \leq i \leq t,\ j \geq 0\}$ be a new alphabet, where $a_{i,j}$ represents $a_i\theta^j$. In particular $a_{i,0}$ represents a_i and from here on we shall identify these two symbols.

Let $\langle A \mid R\rangle$ be a presentation for G in terms of the generating set A. The presentation (5.2) for S can now be written as

$$\langle A, b, c \mid R,\ bc = 1,\ ba_{i,j} = a_{i,j+1}b,\ a_{i,j}c = ca_{i,j+1}\ (1 \leq i \leq t,\ j \geq 0)\rangle. \qquad (5.4)$$

We can remove the redundant generators $a_{i,j}$ $(1 \leq i \leq t,\ j \geq 1)$ from the presentation, by using $a_{i,j} = b^j a_{i,0} c^j$. Thus, keeping in mind that $a_{i,0} = a_i$, we obtain an equivalent presentation

$$\begin{aligned} \langle a_1, a_2, \ldots, a_t, b, c \ \mid \ & R',\ bc = 1,\ b^{j+1}a_i c^j = b^{j+1}a_i c^{j+1} b, \\ & b^j a_i c^{j+1} = cb^{j+1}a_i c^{j+1}\ (1 \leq i \leq t,\ j \geq 0)\rangle, \end{aligned} \qquad (5.5)$$

where R' is obtained from R by replacing every occurrence of $a_{i,j}$ by $b^j a_i c^j$ $(1 \leq i \leq t,\ j \geq 1)$.

Presentation (5.5) is an infinite presentation on a finite number of generators. Since S is finitely presented, there is a finite subset of the set of relations of (5.5) that defines S, i.e. there is a finite subset $\overline{R'} \subseteq R'$ and an integer $s \geq 1$ such that

$$\begin{aligned}\langle a_1, a_2, \ldots, a_t, b, c \mid\ & \overline{R'},\ bc = 1,\ b^{j+1}a_ic^j = b^{j+1}a_ic^{j+1}b,\\ & b^ja_ic^{j+1} = cb^{j+1}a_ic^{j+1}\ (1 \leq i \leq t,\ 0 \leq j \leq s-1)\rangle\end{aligned}$$

defines S.

We can now add the redundant relations $R'\backslash\overline{R'}$, and we can also add the redundant generators $a_{i,k} = b^ka_ic^k$ $(1 \leq i \leq t,\ k \geq 1)$ to obtain

$$\begin{aligned}\langle A, b, c \mid\ & R',\ bc = 1,\ b^{j+1}a_ic^j = b^{j+1}a_ic^{j+1}b,\ b^ja_ic^{j+1} = cb^{j+1}a_ic^{j+1},\\ & a_{i,k} = b^ka_ic^k\ (1 \leq i \leq t,\ 0 \leq j \leq s-1,\ k \geq 1)\rangle.\end{aligned}$$

Now, replacing each occurrence of $b^ka_ic^k$ by $a_{i,k}$ $(1 \leq i \leq t,\ k \geq 1)$, and noting that the relations $a_{i,k} = b^ka_ic^k$ for $k < s$ are consequences of $ba_{i,j} = a_{i,j+1}b$ $(j < s)$ and $bc = 1$, we obtain

$$\begin{aligned}\langle A, b, c \mid\ & R,\ bc = 1,\ ba_{i,j} = a_{i,j+1}b,\ a_{i,j}c = ca_{i,j+1},\\ & a_{i,s+n} = b^{s+n}a_ic^{s+n}\ (1 \leq i \leq t,\ 0 \leq j \leq s-1,\ n \geq 0)\rangle,\end{aligned} \tag{5.6}$$

which still must define S.

Now, let H be the subgroup of G generated by $\{a_{i,j} : 1 \leq i \leq t,\ 0 \leq j \leq s-1\}$. We are going to prove that $H = G$. Suppose to the contrary that H is a proper subgroup of G and let $a_{i,s+n}$ be a generator of G which is not in H.

From (5.4) the relation $ba_{i,s+n} = a_{i,s+n+1}b$ holds in S. Hence it has to be a consequence of any presentation for S. However we will prove that it is not a consequence of (5.6), and thus obtain a contradiction.

Claim *Any word obtained from $ba_{i,s+n}$ by applying relations from (5.6) is of the form ubv, where both u and v are group words and v does not represent an element of H.*

Proof Note that $ba_{i,s+n}$ is of the desired form. We now prove that an application of a relation from (5.6) to a word $w_1 \equiv ubv$ of the desired form yields another word $w_2 \equiv u'bv'$ of the same form. There are several situations to be considered.

If we apply a relation to the subword u of w_1, then u' is obtained from u by an application of one relation from (5.6), so that $u = u'$ in G, and $v \equiv v'$. The case where the relation is applied to v is analogue. Notice that the applied relation cannot affect u, b and v simultaneously: indeed, the only possible such relations are the ones of the form $a_{i,s+n} = b^{s+n}a_ic^{s+n}$, in which case u would have to end with a b, contradicting the fact that it is a group word (by Lemma 5.10(vi)). Now, the only relation that we can apply to the subword ub (involving b) is $a_{i,j+1}b = ba_{i,j}$, provided $u \equiv u_0a_{i,j+1}$ for some group word u_0. In this case $u' \equiv u_0$ and $v' \equiv a_{i,j}v$ which are clearly group words. Moreover, since $a_{i,j}$ represents an element of H and v does not, we have that $a_{i,j}v$ does not represent an element from H either. Therefore, in this case, w_2 is of the desired form. Finally consider the case when w_2 is obtained from w_1 by applying a relation to bv (again involving b). This relation cannot be $bc = 1$ because v, being a group word, does not start with a c. Also it cannot be a relation of the type $a_{i,s+n} = b^{s+n}a_ic^{s+n}$ because in such a case v would have the word $b^{s+n-1}a_ic^{s+n}$ as a prefix, and hence it would not be a group word. So the applied relation is $ba_{i,j} = a_{i,j+1}b$, where $v \equiv a_{i,j}v_0$. Notice that then v_0 is a group word which does not represent an element from H. Hence $w_2 \equiv ua_{i,j+1}bv_0$, and both $ua_{i,j+1}$ and v_0 are group words and v_0 does not represent an element of H, i.e. w_2 has the desired form. □

Note that the word $a_{i,s+n+1}b$, which is equal to $ba_{i,s+n}$, does not have the above form, because the empty word represents an element of H. This contradiction is caused by our assumption that $G \neq H$. Therefore $G = H$, and, in particular, G is finitely generated. This completes the proof of the theorem. □

Proof[of Theorem 5.14] Suppose that G is defined by the presentation $\langle A \mid R\rangle$ with respect to a finite generating set A and with $R = \bigcup_{k=0}^{\infty} \overline{R}\theta^k$ for a finite set $\overline{R}$. We will prove that then S is defined by the presentation

$$\langle A, b, c \mid \overline{R},\ bc = 1,\ ba = (a\theta)b,\ ac = c(a\theta)\ (a \in A)\rangle. \qquad (5.7)$$

Thus we have to prove that the relations $u\theta^k = v\theta^k$ $(k \geq 1,\ (u = v) \in \overline{R})$ are consequences of (5.7). Clearly, for any word w, we have $w\theta = bwc$ as a consequence of $ba = (a\theta)b$ and $bc = 1$. Using this fact, we see that for every $(u = v) \in R$ we have

$$u\theta^k = b^kuc^k = b^kvc^k = v\theta^k$$

and hence $u\theta^k = v\theta^k$. Therefore S is finitely presented.

Conversely, suppose that S is finitely presented. We are going to obtain a presentation for G from a finite presentation for S. First let $\langle A \mid R\rangle$ be any presentation for G with A finite (such a presentation exists since, by the previous theorem, G is finitely generated). Since S is finitely presented there exists a finite subset $\overline{R} \subseteq R$ such that

$$\langle A, b, c \mid \overline{R},\ bc = 1,\ ba = (a\theta)b,\ ac = c(a\theta)\ (a \in A)\rangle$$

defines S. Once again we are going to use a method described in Section 2.2, this time for finding a presentation for a subgroup of a monoid. As in the proof of Theorem 5.12 we have $H_i = Gb^{i-1}$, $r_i \equiv b^{i-1}$ and $r_i' \equiv c^{i-1}$ $(i \geq 1)$. We introduce a new alphabet

$$B = \{[k, a], [k, b], [k + 1, c] : k \geq 1, a \in A\}$$

representing the generators $b^k ac^k$, $b^k bc^{k+1} (= 1)$ and $b^{k+1} cc^k (= 1)$ of G. Now let $\overline{R}_k$ be a copy of $\overline{R}$, obtained by replacing each occurrence of a by $[k, a]$ $(a \in A)$. For brevity we write $[k, a_1][k, a_2]...[k, a_n]$ as $[k, a_1a_2 \cdots a_n]$ for $a_1, a_2, \ldots, a_n \in A$. Thus

$$\overline{R}_k = \{[k, u] = [k, v]\ :\ (u = v) \in \overline{R}\}.$$

Applying Theorem 2.11 we obtain the following presentation for G:

$$\begin{aligned}\langle B \ \mid\ & \overline{R}_k,\ [k, b][k + 1, c] = 1, [k, b][k + 1, x] = [k, x\theta][k, b],\\ & [k + 1, x][k + 1, c] = [k + 1, c][k, x\theta],\ [k, b] = 1\ (k \geq 1,\ a \in A)\rangle.\end{aligned}$$

Clearly from $[k, b] = 1$ we obtain $[k + 1, c] = 1$ and hence we can eliminate these generators obtaining relations

$$[k + 1, a] = [k, a\theta] = \cdots = [1, a\theta^k],$$

and also

$$\overline{R}_k = \{[1, u\theta^{k-1}] = [1, v\theta^{k-1}]\ :\ (u = v) \in \overline{R}\}.$$

Renaming each generator $[1, a]$ to a we obtain the following presentation for G:

$$\langle A \mid u\theta^k = v\theta^k\ (k \geq 0,\ (u = v) \in \overline{R})\rangle.$$

This ends the proof of the theorem. □

The following example exhibits a non finitely presented Bruck–Reilly extension of a finitely generated group:

Example 5.16 Let G be the group considered in Example 5.5. Recall that G is a non finitely presented group. In Example 5.5 we have exhibited a finitely presented Bruck–Reilly extension of G. We will now build a non finitely presented one.

We can extend the mapping

$$a_1 \mapsto a_3, \ a_2 \mapsto a_4, \ a_3 \mapsto a_1 \text{ and } a_4 \mapsto a_2$$

to an endomorphism θ of G. The Bruck–Reilly extension $\mathrm{BR}(G, \theta)$ is then generated by the set

$$A = \{a_1, a_2, a_3, a_4, a_1', a_2', a_3', a_4', b, c\}.$$

Suppose that G is defined by a presentation $\langle A \mid R \rangle$, where $R = \bigcup_{k \geq 0} \overline{R}\theta^k$ for a finite set $\overline{R} \subseteq A^* \times A^*$. Notice that θ preserves length of words from A^*. Thus, the length of the words in R is bounded. Therefore R has to be a finite set. This contradicts the fact that G is not finitely presented.

Remark 5.17 We note that the presentation given in Theorem 5.14 for G, although infinite in general, is recursive (that is there is an algorithm that decides whether a relation is in the presentation).

Remark 5.18 Related to the questions considered in this section, is the more general question of whether there exists a finitely presented monoid with a non finitely generated group of units. By Proposition 5.11, the Bruck–Reilly extension of a given group has that group as its group of units. Thus Theorem 5.15 gives a positive answer to the referred question in this particular case.

5.3 Bruck–Reilly extensions of monoids

We now consider Bruck–Reilly extensions of arbitrary monoids. The results from Section 5.2 do not immediately generalize to this case, since we have used the theory presented in Section 2.2 which is only valid for the group case. Nevertheless most results are still true: the remaining open question is the analogous of Theorem 5.15.

We start by proving the following

Lemma 5.19 *A word $w \in (A \cup \{b, c\})^*$ represents an element from M if and only if*

$$w \equiv \alpha_1(b\beta_1c)\alpha_2 \cdots \alpha_n(b\beta_nc)\alpha_{n+1} \tag{5.8}$$

for unique $\alpha_1, \alpha_2, \ldots, \alpha_{n+1} \in A^*$ and $\beta_1, \beta_2, \ldots, \beta_n \in (A \cup \{b, c\})^*$ representing elements from M. In particular, if $w \in (A \cup \{b, c\})^*$ represents an element from M and w' is a subword of w also representing an element from M then one of the two following situations has to hold

(i) there exist $\alpha, \delta \in (A \cup \{b, c\})^*$ such that $\alpha\pi = 1_B$, $\delta\pi = 1_B$ and $w \equiv \alpha w'\delta$; or

(ii) there exist $\alpha, \beta, \gamma, \delta \in (A \cup \{b, c\})^*$ such that $\alpha\pi = 1_B$, $\delta\pi = 1_B$, $(\beta w'\gamma)\pi = 1_B$ and $w \equiv \alpha\, b\beta w'\gamma c\, \delta$.

Proof If $w \in (A \cup \{b, c\})^*$ is of the form (5.8) then clearly $w\pi = 1_B$ and hence w represents an element from M.

Conversely suppose that $w \in (A \cup \{b, c\})^*$ represents an element from M. We will use induction on the number of occurrences of b in w. If $n(w, b) = 0$ then $w \in A^*$ for otherwise, if c occurs in w, then $w\pi \neq 1_B$. Now, suppose that the result is true for any word containing fewer than $n(w, b)$ occurrences of b. Write $w \equiv \alpha b\beta$ where $\alpha \in A^*$. Let $\beta' c$ be the smallest prefix of β such that $(b\beta' c)\pi = 1_B$ (such β' exists necessarily since $(b\beta)\pi = 1_B$) and let $\beta \equiv \beta' c\gamma$. Then $w \equiv \alpha b\beta' c\gamma$ and $\gamma\pi = 1_B$. Also $n(\gamma, b) < n(w, b)$ and hence the result follows by induction.

Now let $w \in (A \cup \{b, c\})^*$ represent an element from M and let w' be a subword of w also representing an element from M. Let $w \equiv \alpha w'\delta$. If $\alpha\pi = 1_B$ and $\delta\pi = 1_B$ then we are in case (i). Otherwise let $\alpha \equiv \alpha' b\beta$, where $\alpha', \beta \in (A \cup \{b, c\})^*$ (possibly empty) and α' is the longest prefix of α satisfying $\alpha'\pi = 1_B$. Let $\delta \equiv \gamma c\delta'$, where $\gamma, \delta' \in (A \cup \{b, c\})^*$ and δ' is the longest suffix of δ satisfying $\delta'\pi = 1_B$. Notice that such words exist, $(b\beta w'\gamma c)\pi = 1_B$ and that $w \equiv \alpha'\, b\beta w'\gamma c\, \delta'$. Thus we are in case (ii). □

The following generalizes Theorem 5.12 for the case of monoids:

Theorem 5.20 *Let M be a monoid and $\theta : M \to M$ an endomorphism. The Bruck–Reilly extension $S = \mathrm{BR}(M,\theta)$ is finitely generated if and only if there exists a finite subset $A_0 \subseteq M$ such that M is generated by the set $A = \bigcup_{k\geq 0} A_0\theta^k$.*

PROOF Suppose that there exists a finite set $A_0 \subseteq M$ such that $A = \bigcup_{k\geq 0} A_0\theta^k$ generates M. Let $A_0 = \{a_1, \ldots, a_t\}$ and denote $a_i\theta^j \in A$ by $a_{i,j}$ (notice that all elements from M can be written in this way). We will show that any $a_{i,j} \in (A_0 \cup \{b,c\})^*$ $(1 \leq i \leq t,\ j \geq 1)$. Let i,j be arbitrary. Then $a_{i,j} = b^j a_i c^j$. So $a_{i,j} \in (A_0 \cup \{b,c\})^*$ and therefore S is generated by $A_0 \cup \{b,c\}$. In particular, S is finitely generated.

Conversely suppose that S is finitely generated and let Y be a generating set for M. Let $A_0 \cup \{b,c\}$ be a finite generating set for S with $A_0 \subseteq Y$. We claim that $A = \bigcup_{k\geq 0} A_0\theta^k \subseteq Y^*$ is a generating set for M. Let $w \in (A_0 \cup \{b,c\})^*$ be any word representing an element of M. By induction on the number $n(w,b)$ of occurrences of the letter b in w we prove that w can be written as a product of elements of A. If $n(w,b) = 0$ then $w \in A_0^* \subseteq A^*$. Otherwise, using Lemma 5.19, we can write $w \equiv \alpha b\beta c\gamma$, with $\alpha \in A_0^*$ and β, γ words representing elements of M. By induction β equals a word $\beta_1 \in A^*$, and γ equals a word $\gamma_1 \in A^*$. Also $b\beta c = b\beta_1 c = \beta_1\theta \in A^*$ and hence $w = \alpha(\beta_1\theta)\gamma \in A^*$. Therefore A generates M.
$\square$

Notice that the homomorphism $\theta : A^* \to A^*$ can be extended to $(A \cup \{b,c\})^*$ by defining

$$b\theta = b \text{ and } c\theta = c.$$

Using this fact we prove that the following necessary condition for a Bruck–Reilly of extension to be finitely presented still holds for monoids:

Theorem 5.21 *Let M be a monoid and $\theta : M \to M$ an endomorphism. If the Bruck–Reilly extension $S = \mathrm{BR}(M,\theta)$ is finitely presented and M is generated by a set A, then M is defined by a presentation $\langle A \mid R\rangle$ where $R = \bigcup_{k\geq 0} \overline{R}\theta^k$, for some finite set of relations $\overline{R}$.*

PROOF From Theorem 5.20 it follows that there is a finite subset $A_0 \subseteq M$ such that M is generated by the set $A = \bigcup_{k\geq 0} A_0\theta^k$. Clearly from (5.2), $a\theta^j$ $(a \in A,\, j \geq 1)$ are redundant generators for $\mathrm{BR}(M,\theta)$, since $a\theta^j = b^j a c^j$. Hence, $\mathrm{BR}(M,\theta)$ is defined by the presentation

$$\begin{aligned}\langle A_0, b, c \ \mid\ & R',\, bc = 1,\, b^{j+1}ac^j = b^{j+1}ac^{j+1}b,\\ & b^j ac^{j+1} = cb^{j+1}ac^{j+1}\ (a \in A_0,\, j \geq 0)\rangle\end{aligned}$$

where R' is obtained from R by replacing occurrences of $a\theta^j$ $(a \in A,\, j \geq 1)$ by $b^j ac^j$. Now, $A_0 \cup \{b, c\}$ is a finite generating set and since $\mathrm{BR}(M,\theta)$ is finitely presented it follows that there exists a finite set $\overline{R'} \subseteq R'$ and an integer $s \geq 0$ such that the $\mathrm{BR}(M,\theta)$ is defined by

$$\begin{aligned}\langle A_0, b, c \ \mid\ & \overline{R'},\, bc = 1,\, b^{j+1}ac^j = b^{j+1}ac^{j+1}b,\\ & b^j ac^{j+1} = cb^{j+1}ac^{j+1}\ (a \in A_0,\, 0 \leq j \leq s)\rangle.\end{aligned}$$

We can now restore the generating set $A \cup \{b, c\}$ reintroducing generators $a' \in A\backslash A_0$. We denote by a_j the generator representing $b^j ac^j$ $(a \in A_0,\, j \geq 1)$ (and denote a by a_0). We thus obtain

$$\begin{aligned}\langle A, b, c \ \mid\ & \overline{R},\, bc = 1,\, ba_j = (a_j\theta)b,\, a_jc = c(a_j\theta),\\ & a_{s+n} = b^{s+n}ac^{s+n}\ (a \in A_0,\, 0 \leq j \leq s,\, n \geq 0)\rangle\end{aligned} \tag{5.9}$$

where $\overline{R}$ is obtained from $\overline{R'}$ by substituting the missing generators. Notice that $\overline{R} \subseteq R$ and that it is a finite set. We are going to prove that $\langle A \mid \bigcup_{k\geq 0} \overline{R}\theta^k\rangle$ is a presentation for M.

We define a mapping

$$\phi : \{w \in (A \cup \{b, c\})^* \,:\, w\pi = 1_B\} \rightarrow A^*$$

inductively by

$$\begin{aligned}\alpha &\mapsto \alpha \text{ if } \alpha \in A^*\\ w \equiv \alpha b\beta c\gamma &\mapsto \alpha\,((\beta\theta)\phi)\,(\gamma\phi) \text{ where } \alpha \in A^*,\, \beta\pi = \gamma\pi = 1_B.\end{aligned}$$

This is a well defined mapping (notice that in particular, after a finite number of steps it reduces to the first case since there are only finitely many occurrences of b in w). Also, if $w \equiv \alpha\beta$ and $\alpha\pi = 1_B$, $\beta\pi = 1_B$ then

$$w\phi \equiv (\alpha\phi)(\beta\phi). \tag{5.10}$$

Moreover, we have:

Claim 1 *The following hold:*

$$\theta\phi = \phi\theta \text{ and } \phi^2 = \phi.$$

PROOF To see this we have to prove that for any $w \in (A \cup \{b, c\})^*$ with $w\pi = 1_B$, both $w\theta\phi \equiv w\phi\theta$ and $w\phi^2 \equiv w\phi$. We will use induction on $n(w, b)$, the number of occurrences of b in w. If $n(w, b) = 0$ then $w \in A^*$ and also $w\theta \in A^*$. Thus we have

$$w\phi\theta \equiv w\theta \equiv w\theta\phi$$

and

$$w\phi^2 \equiv w\phi.$$

Now, let $n(w, b) = k$ and suppose that the result is valid for any word with less than k occurrences of b. By Lemma 5.19

$$w \equiv \alpha(b\beta c)\gamma$$

for $\alpha \in A^*$, $\beta, \gamma \in (A \cup \{b, c\})^*$, $\beta\pi, \gamma\pi = 1_B$. We therefore have

$$\begin{aligned}
w\phi\theta &\equiv (\alpha(\beta\theta\phi)(\gamma\phi))\theta && \text{(definition of } \phi) \\
&\equiv (\alpha\theta)(\beta\theta\phi\theta)(\gamma\phi\theta) && (\theta \text{ is a homomorphism}) \\
&\equiv (\alpha\theta)(\beta\theta\theta\phi)(\gamma\theta\phi) && \text{(induction hypothesis)} \\
&\equiv ((\alpha\theta)b(\beta\theta)c(\gamma\theta))\phi && \text{(definition of } \phi) \\
&\equiv (\alpha b\beta c\gamma)\theta\phi && (\theta \text{ is a homomorphism}) \\
&\equiv w\theta\phi
\end{aligned}$$

and

$$\begin{aligned}
w\phi^2 &\equiv (\alpha(\beta\theta\phi)(\gamma\phi))\phi && \text{(definition of } \phi) \\
&\equiv (\alpha\phi)(\beta\theta\phi^2)(\gamma\phi^2) && \text{(by (5.10))} \\
&\equiv \alpha(\beta\theta\phi)(\gamma\phi) && \text{(induction hypothesis)} \\
&\equiv (\alpha b\beta c\gamma)\phi && \text{(definition of } \phi) \\
&\equiv w\phi.
\end{aligned}$$

□

Now let $(u = v) \in R\backslash\overline{R}$. In particular $u = v$ holds in $\mathrm{BR}(M, \theta)$ and hence there is an elementary sequence

$$u \equiv \xi_1, \xi_2, \ldots, \xi_k \equiv v$$

from u to v with respect to (5.9). We will prove that

$$u \equiv \xi_1\phi, \xi_2\phi, \ldots, \xi_k\phi \equiv v$$

is an elementary sequence from u to v with respect to $\langle A \mid \bigcup_{k\geq 0} \overline{R}\theta^k\rangle$. Notice that $u \equiv u\phi$ and $v \equiv v\phi$.

We start by proving the following

Claim 2 *Let $w \in (A \cup \{b, c\})^*$ with $w\pi = 1_B$, u a subword of w with $u\pi = 1_B$, $u = v$ a relation that holds in $\mathrm{BR}(M, \theta)$ and w' a word obtained from w by replacing u by v. Then there exists $m \geq 0$ such that $w'\phi$ is obtained from $w\phi$ by applying the relation $u\theta^m\phi = v\theta^m\phi$.*

Proof If w' is obtained from w by applying the relation $u = v$ then u is a subword of w. By Lemma 5.19 there are two cases to be considered. If there exist $\alpha, \delta \in (A \cup \{b, c\})^*$ such that $w \equiv \alpha u \delta$ and $\alpha\pi = 1_B$, $\delta\pi = 1_B$ then

$$w\phi \equiv \alpha\phi\, u\phi\, \delta\phi$$

and

$$w'\phi \equiv \alpha\phi\, v\phi\, \delta\phi.$$

Thus $w'\phi$ can be obtained from $w\phi$ by applying the relation $u\phi = v\phi$. Otherwise, there exist $\alpha, \beta, \gamma, \delta \in (A \cup \{b, c\})^*$ such that $w \equiv \alpha\, b\beta u\gamma c\, \delta$ and $\alpha\pi = 1_B$, $\delta\pi = 1_B$ and $(\beta u\gamma)\pi = 1_B$. In this case

$$w\phi \equiv (\alpha\phi)(b\beta u\gamma c)\phi(\delta\phi).$$

Let $n(\beta, b)$ be the number of occurrences of b in β, $n(\beta, c)$ the number of occurrences of c in β and $m(\beta) = 1 + n(\beta, b) - n(\beta, c)$ (notice that $m(\beta) \geq 1$ since β is the prefix of a word representing an element from M). We will use induction on $m(\beta)$ to prove that $(b\beta v\gamma c)\phi$ can be obtained from $(b\beta u\gamma c)\phi$ by using the relation $u\theta^{m(\beta)}\phi = v\theta^{m(\beta)}\phi$. If $m(\beta) = 1$ then $\beta\pi = 1_B$ and hence also $\gamma\pi = 1_B$. Therefore,

$$(b\beta u\gamma c)\phi \equiv (\beta u\gamma)\theta\phi \equiv (\beta\theta\phi)(u\theta\phi)(\gamma\theta\phi)$$

and the result follows. Now suppose that the result is valid for $m = k-1$. Since $(\beta u\gamma)\pi = 1_B$ and $u\pi = 1_B$ we can use the last part of Lemma 5.19 again. The fact that $m > 1$ implies that $\beta\pi \neq 1_B$ and hence there exists $\beta', \beta'', \gamma', \gamma''$ such that $\beta \equiv \beta' b\beta''$, $\gamma \equiv \gamma'' c\gamma'$, $\beta'\pi = 1_B$, $\gamma'\pi = 1_B$, $(\beta'' u\gamma'')\pi = 1_B$ and

$$\beta u\gamma \equiv \beta'\, b\beta'' u\gamma'' c\,\gamma'.$$

Also

$$\begin{array}{rcll}
(b\beta u\gamma c)\phi & \equiv & (\beta u\gamma)\theta\phi & (\text{definition of } \phi)\\
& \equiv & (\beta'\, b\beta'' u\gamma'' c\,\gamma')\theta\phi & (\beta \equiv \beta' b\beta'',\ \gamma \equiv \gamma'' c\gamma')\\
& \equiv & (\,(\beta'\theta)\,(b\beta'' u\gamma'' c)\theta\,(\gamma'\theta)\,)\phi & (\theta \text{ is a homomorphism})\\
& \equiv & (\beta'\theta\phi)\,(b\,\beta'' u\gamma''\, c)\theta\phi\,(\gamma'\theta\phi) & (\text{by } (5.10))\\
& \equiv & (\beta'\theta\phi)\,(b\,\beta'' u\gamma''\, c)\phi\theta\,(\gamma'\theta\phi) & (\theta\phi = \phi\theta).
\end{array}$$

The fact that $m(\beta'') = k-1$ allows us to apply the induction hypothesis and thus to conclude that $(b\,\beta'' v\gamma''\, c)\phi$ can be obtained from $(b\,\beta'' u\gamma''\, c)\phi$ by applying the relation $u\theta^{k-1}\phi = v\theta^{k-1}\phi$. So $(b\,\beta'' v\gamma''\, c)\phi\theta$ can be obtained from $(b\,\beta'' u\gamma''\, c)\phi\theta$ by applying $u\theta^k\phi = v\theta^k\phi$. Thus $w'\phi$ can also be obtained from $w\phi$ by applying the relation $u\theta^k\phi = v\theta^k\phi$. This ends the proof of the claim. □

So, if ξ_i is obtained from ξ_{i-1} by applying a relation $(u = v) \in \overline{R}$ then it follows from Claim 2 that $\xi_i\phi$ is obtained from $\xi_{i-1}\phi$ by applying the relation $u\theta^m\phi = v\theta^m\phi$ for some $m > 1$. But since $u, v \in A^*$, also $u\theta^m, v\theta^m \in A^*$ and hence $u\theta^m\phi \equiv u\theta^m$ and $v\theta^m\phi \equiv v\theta^m$. Therefore, $\xi_i\phi$ is obtained from $\xi_{i-1}\phi$ by applying a relation from $\bigcup_{k\geq 0} \overline{R}\theta^k$.

Now suppose that ξ_i is obtained from ξ_{i-1} by applying one of the relations $bc = 1$ or $a_{s+n} = b^{s+n}ac^{s+n}$ $(a \in A_0,\ n \geq 0)$. Notice that $(bc)\phi \equiv 1\phi$ and $a_{s+n}\phi \equiv (b^{s+n}ac^{s+n})\phi$ and hence, this together with Claim 2 implies that in this case $\xi_{i-1}\phi \equiv \xi_i\phi$.

Now, if ξ_i is obtained from ξ_{i-1} by applying a relation $ba = (a\theta)b$ $(a \in A_0)$ (without loss of generality we assume that the left hand side is replaced by the right hand side) then ba is a subword of ξ_{i-1}. Thus, there exist $\alpha, \beta, \gamma \in (A \cup \{b, c\})^*$ with $(ba\beta c)\pi = 1_B$ and

$$\xi_{i-1} \equiv \alpha ba\beta c\delta.$$

Hence ξ_i is obtained from ξ_{i-1} by replacing $ba\beta c$ by $(a\theta)b\beta c$. But then, by Claim 2, there exists $m \geq 0$ such that $\xi_i\phi$ is obtained from $\xi_{i-1}\phi$ by applying the relation $(ba\beta c)\theta^m\phi = ((a\theta)b\beta c)\theta^m\phi$. However,

$$\begin{array}{rcll}
(ba\beta c)\theta^m\phi & \equiv & (ba\beta c)\phi\theta^m & (\theta\phi = \phi\theta) \\
& \equiv & ((a\beta)\theta)\phi\theta^m & (\text{definition of } \phi) \\
& \equiv & ((a\theta)\,(\beta\theta))\phi\theta^m & (\theta \text{ is a homomorphism}) \\
& \equiv & ((a\theta\phi)\,(b\beta c)\phi)\phi\theta^m & (\text{definition of } \phi) \\
& \equiv & ((a\theta)b\beta c)\phi\theta^m & (\phi^2 = \phi) \\
& \equiv & ((a\theta)b\beta c)\theta^m\phi & (\theta\phi = \phi\theta)
\end{array}$$

Thus $\xi_{i-1}\phi \equiv \xi_i\phi$.

Similarly if ξ_i is obtained from ξ_{i-1} by applying a relation $ac = c(a\theta)$ $(a \in A_0)$, $\xi_{i-1}\phi \equiv \xi_i\phi$. Indeed, assuming without loss of generality that the left hand side of the relation is replaced by its right hand side, ac is a subword of ξ_{i-1}. Hence there exist $\alpha, \beta, \delta \in (A \cup \{b, c\})^*$ such that $(b\beta ac)\pi = 1_B$ and

$$\xi_{i-1} \equiv \alpha b\beta ac\delta.$$

Thus, ξ_i is obtained from ξ_{i-1} by replacing $b\beta ac$ by $b\beta c(a\theta)$. By Claim 2, there exists $m > 0$ such that $\xi_i\phi$ is obtained from $\xi_{i-1}\phi$ by applying the relation $(b\beta ac)\theta^m\phi = (b\beta c(a\theta))\theta^m\phi$. But then

$$\begin{array}{rcll}
(b\beta ac)\theta^m\phi & \equiv & (b\beta ac)\phi\theta^m & (\theta\phi = \phi\theta) \\
& \equiv & ((\beta a)\theta)\phi\theta^m & (\text{definition of } \phi) \\
& \equiv & ((\beta\theta)\,(a\theta))\phi\theta^m & (\theta \text{ is a homomorphism}) \\
& \equiv & ((b\beta c)\phi\,(a\theta\phi))\phi\theta^m & (\text{definition of } \phi) \\
& \equiv & (b\beta c(a\theta))\phi\theta^m & (\phi^2 = \phi) \\
& \equiv & (b\beta c(a\theta))\theta^m\phi & (\theta\phi = \phi\theta)
\end{array}$$

Thus $\xi_{i-1}\phi \equiv \xi_i\phi$.

Therefore there exists an elementary sequence form $u\phi$ to $v\phi$ with respect to $\langle A \mid \bigcup_{k\geq 0} \overline{R}\theta^k\rangle$. This completes the proof. $\square$

Also, the converse part of Theorem 5.14 holds in the case of monoids (the proof of the result for the group case, in Section 5.2, is valid for this case).

Theorem 5.22 *Let M be a finitely generated monoid defined by a presentation $\langle A \mid R\rangle$ where A is finite and $R = \bigcup_{k\geq 0} R_0\theta^k$, for some finite set of relations R_0. Let $\theta : M \to M$ be an endomorphism. Then the Bruck–Reilly extension $S = \mathrm{BR}(M,\theta)$ is finitely presented.*

The key question which we have not yet answered in this general case is:

Question 5.23 *If $\mathrm{BR}(M,\theta)$ is a finitely presented Bruck–Reilly extension of a monoid M, with respect to an endomorphism θ of M, is M necessarily finitely generated?*

We can now prove the following

Corollary 5.24 *Let M be a monoid and θ an endomorphism of M. Suppose that θ satisfies one of the following conditions:*

(i) its image is finite.

(ii) it preserves length.

(iii) it is an endomorphism of finite order.

Then Bruck–Reilly extension $\mathrm{BR}(M,\theta)$ if finitely generated (resp. finitely presented) if and only if M is finitely generated (resp. finitely presented).

PROOF From Proposition 5.3 it follows that if M is finitely generated (resp. finitely presented) then so is any Bruck–Reilly extension of it.

For the converse start by noticing that if the image of θ is finite, then given a finite set $X_0 \subseteq M$, the set $X = \bigcup_{k\geq 0} X_0\theta^k$ is contained in the set $X_0 \cup (M\theta)$, which is a finite set, and hence X is a also a finite set. If θ preserves length the length of all elements in M is bounded by the maximum length of words in X_0, and therefore X can only contain a finite number of elements. Finally, if θ is of finite order h, then

$$X = \bigcup_{k\geq 0} X_0\theta^k = \bigcup_{0\leq k\leq h} X_0\theta^k$$

and hence X is also finite.

So, suppose that $\mathrm{BR}(M, \theta)$ is finitely generated. Then, by Theorem 5.20, it follows that there exists a finite set $A_0 \subseteq M$ such that M is generated by $A = \bigcup_{k\geq 0} A_0\theta^k$. From the above considerations A is finite in any of the cases (i), (ii) or (iii) and hence M is finitely generated.

On the other hand, if $\mathrm{BR}(M, \theta)$ is finitely presented then, in particular, it is finitely generated. Thus, M is generated by a finite set A. Furthermore, by Theorem 5.21, it follows that M is defined by a presentation $\langle A \mid R \rangle$, where $R = \bigcup_{k\geq 0} \overline{R}\theta^k$ for a finite set $\overline{R} \subseteq A^* \times A^*$. But then, whenever θ satisfies (i), (ii) or (iii), it follows that R is a finite set, which proves that M is finitely presented. □

We then have the following immediate corollary for Bruck extensions:

Corollary 5.25 *A Bruck–Reilly extension* $\mathrm{BR}(M, \theta)$ *of a monoid* M *with respect to the endomorphism that maps all elements to identity of* M *is finitely generated (resp. finitely presented) if and only if* M *is finitely generated (resp. finitely presented).*

5.4 Inverse monoid presentations

We start by proving the following known fact:

Proposition 5.26 *The Bruck–Reilly extension* $S = BR(M, \theta)$ *of a monoid* M *is an inverse monoid if and only if* M *is an inverse monoid.*

Proof We follow the proof given in [28]. So start by supposing that M is an inverse monoid. Let A be a (monoid) generating set for M and let $\alpha \in A^*$, $i, j \geq 0$. Since

$$(c^i\alpha b^j)(c^j\alpha^{-1}b^i)(c^i\alpha b^j) = c^i\alpha b^j \text{ and}$$

(where α^{-1} denotes the inverse of α in M) it follows that every element of S has an inverse. Thus S is regular.

Now, for any idempotent $e \in M$ and $i \geq 0$, $c^i e b^i$ is an idempotent of S. In fact these are the only idempotents of S. Indeed, if $\alpha \in A^*$, $i, j \geq 0$ is such that $c^i \alpha b^j$ is idempotent of S this means that

$$(c^i \alpha b^j)^2 = c^{i-j+t}(\alpha\theta^{t-j})(\alpha\theta^{t-i})b^{j-i+t} = c^i \alpha b^j,$$

where $t = \max\{i, j\}$. Thus, by Lemma 5.10(iii), $i = i - j + t$ and $j = j - i + t$. Thus $i = j$ and $\alpha^2 = \alpha$ in M.

So, consider two idempotents $c^i e b^i, c^j f b^j$ of S ($e, f \in A^*$, $e^2 = e$. $f^2 = f$ in M and $i, j \geq 0$). Without loss of generality we assume that $i \geq j$. Then

$$(c^i e b^i)(c^j f b^j) = c^i e(f\theta^{i-j})b^j \text{ and}$$

$$(c^j f b^j)(c^i e b^i) = c^i (f\theta^{i-j}) e b^j.$$

Since f is an idempotent, so is $f\theta^{i-j}$ and therefore

$$e(f\theta^{i-j}) = (f\theta^{i-j})e$$

in M, since idempotents in M commute. We thus conclude that idempotents also commute in S. Therefore S is an inverse monoid.

Conversely suppose that S is an inverse monoid. Let $\alpha \in A^*$, $i, j \geq 0$ and suppose that the inverse of $c^i \alpha b^j$ in S is $c^k \beta b^l$ for some $\beta \in A^*$, $k, l \geq 0$. This means that

$$(c^i \alpha b^j)(c^k \beta b^l) = c^{i-j+t}(\alpha\theta^{t-j})(\beta\theta^{t-k})b^{l-k+t},$$

($t = \max\{j, k\}$) is an idempotent of S which is $\mathcal{R}^S$-related to $c^i \alpha b^j$ and $\mathcal{L}^S$-related to $c^k \beta b^l$. Thus, by Lemma 5.10(iv)

$$i = i - j + t = l - k + t = l$$

and so $i = l$ and $j = k (= t)$. The fact that $c^i \alpha b^j$ and $c^j \beta b^i$ are inverse from each other now yields

$$c^i \alpha b^j = (c^i \alpha b^j)(c^j \beta b^i)(c^i \alpha b^j) = c^i \alpha\beta\alpha b^j \text{ and}$$

$$c^j \beta b^i = (c^j \beta b^i)(c^i \alpha b^j)(c^j \beta c^i) = c^i \beta\alpha\beta b^i$$

and so $\alpha = \alpha\beta\alpha$ (and $\beta = \beta\alpha\beta$ in M). Therefore M is regular. Finally, given two idempotents e, f of M, since $c^0 e b^0$ and $c^0 f b^0$ commute in S, then so do e and f in M. Therefore M is an inverse monoid. $\square$

An immediate consequence of Proposition 5.26 is that it makes sense to consider inverse monoid presentations for S and ask when S is finitely presented as an inverse monoid. Recall that there are inverse monoids which are finitely presented as inverse monoids but are not finitely presented as (ordinary) monoids (see Example 2.4). By way of contrast, an inverse monoid with finitely many $\mathcal{R}$– and $\mathcal{L}$–classes is finitely presented as an inverse monoid if and only if it is finitely presented as a monoid (see Proposition 2.5). Here we show that the analogous result holds for Bruck–Reilly extensions of groups although they have infinitely many $\mathcal{R}$– and $\mathcal{L}$–classes.

Theorem 5.27 *Let $S = \mathrm{BR}(M, \theta)$ be a Bruck–Reilly extension of an inverse monoid M. Then S is finitely presented as an inverse monoid if and only if S is finitely presented as a monoid.*

PROOF Since S is inverse, a monoid presentation for S also defines S when considered as an inverse monoid presentation. Hence if S is finitely presented as a monoid it is also finitely presented as an inverse monoid.

So suppose that S is finitely presented as an inverse monoid. Thus S is defined by an inverse monoid presentation $\langle A, b \mid R\rangle$, where $A \subseteq M$ and R are finite sets. We can add redundant relations to the above presentation to obtain

$$\langle A, b \mid R,\ aa^{-1} = a^{-1}a = 1,\ bb^{-1} = 1,\ ba = (a\theta)b,\ ab^{-1} = b^{-1}(a\theta)\ (a \in A)\rangle$$

which still defines S as an inverse monoid. To obtain a monoid presentation for S we have to adjoin generators $A^{-1} = \{a^{-1}\ :\ a \in A\}$ (a set in one-one correspondence with A, disjoint from it) and b^{-1}; we also have to add relations $ww^{-1}w = w$ and $ww^{-1}zz^{-1} = zz^{-1}ww^{-1}$ $(w, z \in (A \cup A^{-1} \cup \{b, b^{-1}\})^*)$. Thus S is defined by the (infinite) monoid presentation

$$\begin{aligned}\langle A, A^{-1}, b, b^{-1} \ \mid\ & R,\ aa^{-1} = a^{-1}a = 1,\ bb^{-1} = 1,\ ba = (a\theta)b,\ ab^{-1} = b^{-1}(a\theta),\\ & ww^{-1}w = w,\ ww^{-1}zz^{-1} = zz^{-1}ww^{-1}\\ & (a \in A,\ w, z \in (A \cup A^{-1} \cup \{b, b^{-1}\})^*)\rangle.\end{aligned}$$

Now, notice that for any $w \in (A \cup A^{-1} \cup \{b, b^{-1}\})^*$ we have $w = b^{-i}a_1a_2\cdots a_tb^j$ for some $i, j \geq 0$ and some $a_1, \ldots, a_t \in A \cup A^{-1}$ as a consequence of the relations

$ba = (a\theta)b$, $ab^{-1} = b^{-1}(a\theta)$ and $bb^{-1} = 1$. Therefore

$$ww^{-1} = b^{-i}a_1 \cdots a_t b^j b^{-j} a_t^{-1} \cdots a_1^{-1} b^i = b^i b^{-i}$$

as a consequence of $aa^{-1} = a^{-1}a = 1$ and $bb^{-1} = 1$. It now follows immediately that

$$ww^{-1}w = b^{-i}b^i b^{-i} a_1 \cdots a_t b^j = b^{-i}a_1 \cdots a_t b^j = w.$$

Also, for $z = b^{-k}a_1' \cdots a_s' b^l$, we have $zz^{-1} = b^{-k}b^k$ and hence

$$ww^{-1}zz^{-1} = b^{-i}b^i b^{-k}b^k = b^{-m}b^m = b^{-k}b^k b^{-i}b^i = zz^{-1}ww^{-1},$$

where $m = \max(i, k)$. Therefore, these relations are redundant and hence S is defined by the following finite monoid presentation

$$\begin{aligned}\langle A, A^{-1}, b, b^{-1} \ \mid \ & R,\ aa^{-1} = a^{-1}a = 1,\ bb^{-1} = 1, \\ & ba = (a\theta)b,\ ab^{-1} = b^{-1}(a\theta)\ (a \in A)\rangle.\end{aligned}$$

Thus S is finitely presented as a monoid. □

Chapter 6

Unions of semigroups

In this chapter we consider semigroup decompositions, in particular we consider semigroups that are disjoint unions of some of its subsemigroups. In general, such a decomposition does not correspond to a semigroup construction, in the sense that given a family of semigroups there is no 'recipe' to build a semigroup which is the union of those semigroups.

Well known examples of semigroups that can be decomposed as unions of semigroups are completely regular semigroups and Clifford semigroups. Indeed, completely regular semigroups are precisely unions of groups as well as semilattices of completely simple semigroups; these semigroups have been widely considered (see, for example [40]). On the other hand, Clifford semigroups are precisely semilattices of groups as well as strong semilattices of groups.

Thus, given a semigroup S which is a disjoint union of a finite family $(S_i)_{i\in I}$ of its subsemigroups, we consider the following questions:

(Q1) If all S_i $(i \in I)$ are finitely generated, is S finitely generated?

(Q2) If S is finitely generated, are all S_i $(i \in I)$ finitely generated?

(Q3) If all S_i $(i \in I)$ are finitely presented, is S finitely presented?

(Q4) If S is finitely presented, are all S_i $(i \in I)$ finitely presented?

We will also consider infinite unions.

In Section 6.1 we introduce some definitions and some results that we will be using. In Section 6.2 we answer the above questions for the general case of unions of semigroups and in Sections 6.3, 6.4 and 6.5 we consider bands of semigroups, bands of monoids and semilattices and strong semilattices of semigroups, respectively. Finally, in Section 6.6, we summarize the results obtained in a table.

The results from this chapter will appear in [3].

6.1 Preliminaries

The ordering relation $\leq_{\mathcal{R}}$, in the case of bands, is given by

$$\alpha \leq_{\mathcal{R}} \beta \Longleftrightarrow \alpha\beta = \alpha$$

(and we denote it by $\leq$). The following fact is well known (see, for example, [28, Theorem 4.5.3]).

Lemma 6.1 *Let Y be a finitely generated band. Then Y is finite.*

Now, let S be a semigroup which is a disjoint union of some of its subsemigroups. Then S is called a *union of semigroups*. Suppose further that there exists a band Y and a family $(S_\alpha)_{\alpha\in Y}$ of subsemigroups of S, indexed by Y, such that $S = \dot{\bigcup}_{\alpha\in Y} S_\alpha$ and $S_\alpha S_\beta \subseteq S_{\alpha\beta}$, for all $\alpha, \beta \in Y$. Then S is called the *band of semigroups* $(S_\alpha)_{\alpha\in Y}$ and is denoted by $\mathcal{B}[Y, S_\alpha]$. This means that the equivalence relation ρ on S with equivalence classes S_α $(\alpha \in Y)$ is a congruence, and that S/ρ is a band isomorphic to Y. If Y is a semilattice then S is called a *semilattice of semigroups* and is denoted by $S = \mathcal{S}[Y, S_\alpha]$. If $S = \dot{\bigcup}_{\alpha\in Y} S_\alpha$, with Y a semilattice, is such that for any $\alpha, \beta \in Y$ with $\alpha\beta = \beta$ there exists a homomorphism $\lambda_{\alpha,\beta} : S_\alpha \to S_\beta$ satisfying

$$x\lambda_{\alpha,\alpha} = x \ (\alpha \in Y,\ x \in S_\alpha), \tag{6.1}$$

$$\lambda_{\alpha,\beta}\,\lambda_{\beta,\gamma} = \lambda_{\alpha,\gamma} \ (\alpha, \beta, \gamma \in Y,\ \alpha\beta = \beta,\ \text{and}\ \beta\gamma = \gamma), \tag{6.2}$$

$$xy = (x\lambda_{\alpha,\alpha\beta})(y\lambda_{\beta,\alpha\beta}) \ (x \in S_\alpha,\ y \in S_\beta,\ \text{and}\ \alpha, \beta \in Y), \tag{6.3}$$

then S is necessarily a semilattice of the semigroups S_α. In this case S is called a *strong semilattice of semigroups*, and is denoted by $S = \mathcal{S}[Y, S_\alpha, \lambda_{\alpha,\beta}]$. We say that S is a *union of monoids* (resp. *band of monoids*, *semilattice of monoids*, *strong semilattice of monoids*), if $S = \dot{\bigcup}_{\alpha\in Y} S_\alpha$ is a union of semigroups (resp.

band of semigroups, semilattice of semigroups, strong semilattice of semigroups) and all S_α ($\alpha \in Y$) are monoids.

Remark 6.2 A strong semilattice of semigroups (or monoids) is a semigroup construction. Indeed given a family S_α of semigroups indexed in a semilattice Y, and given morphisms $\lambda_{\alpha,\beta} : S_\alpha \to S_\beta$ (for $\alpha, \beta \in Y$, $\alpha\beta = \beta$) satisfying (6.1), (6.2) and (6.3), $S = \dot{\bigcup}_{\alpha\in Y} S_\alpha$ is a semigroup, with multiplication given by

$$st = (s\lambda_{\alpha,\alpha\beta})(t\lambda_{\beta,\alpha\beta})$$

for $s \in S_\alpha$, $t \in S_\beta$, $\alpha, \beta \in Y$.

6.2 General unions of semigroups, monoids and groups

If all the subsemigroups S_i ($i \in I$) are groups we have a positive answer to all the questions (Q1)–(Q4). This is a straightforward consequence of the following

Proposition 6.3 [57, Theorem 4.1] *Let S be a regular semigroup with finitely many $\mathcal{L}$- and $\mathcal{R}$-classes. Then S is finitely presented if and only if all maximal subgroups of S are finitely presented.*

In the general case, when the semigroup is a finite disjoint union of a family of its subsemigroups, question (Q1) still has an affirmative answer. This is an immediate corollary of the following obvious fact.

Proposition 6.4 *Let S be a semigroup which is a disjoint union of a family $(S_i)_{i\in I}$ of its subsemigroups. If each S_i is generated by a set A_i ($i \in I$), with $A_i \cap A_j = \emptyset$ ($i, j \in I, i \neq j$), then S is generated by $\dot{\bigcup}_{i\in I} A_i$.*

However, all the other questions have negative answers. The following example deals with (Q2) and (Q4).

Example 6.5 Let S be the monoid defined by the (monoid) presentation

$$\langle a, b, c \mid ab = ba = 1,\ ac = c,\ c^2 = c\rangle.$$

The submonoid S_1 of S generated by $\{a, b\}$ is defined by the presentation $\langle a, b \mid ab = ba = 1\rangle$ (i.e. S_1 is the infinite cyclic group). Let $S_2 = S \backslash S_1$. We claim that

$$S_2 = \{ca^i, cb^i : i \geq 0\}.$$

Clearly the intersection of $\{ca^i, cb^i : i \geq 0\}$ with S_1 is empty. On the other hand, if $w \in \{a, b, c\}^*$ is a word representing an element of S_2, then it contains at least one occurrence of c, and hence there exist words $\alpha \in \{a, b, c\}^*$, $\beta \in \{a, b\}^*$ such that $w \equiv \alpha c \beta$. Now, from the presentation for S, c is a right zero, and hence $w \equiv \alpha c \beta = c\beta$. Also, from the relations $ab = ba = 1$ it follows that there is $i \geq 0$ such that β represents the same element of S as a^i or b^i. Hence $w = ca^i$ or $w = cb^i$, for some $i \geq 0$. This concludes the proof of the claim. Thus $S = S_1 \dot{\cup} S_2$. Moreover, S_2 is an ideal of S, and hence, in particular, it is a semigroup. However, S_2 is not finitely generated (nor finitely presented) since it is an infinite right zero semigroup.

Remark 6.6 The semigroup S from Example 6.5 can also be viewed as a disjoint union of the infinite cyclic group and infinitely many copies of the trivial semigroup. Thus, the finiteness of I is not necessary for a union of semigroups $S = \dot{\bigcup}_{i \in I} S_i$ to be finitely generated or finitely presented.

The negative answer to the question (Q3) is perhaps somewhat more surprising.

Example 6.7 Let T be the semigroup defined by the following multiplication table

	a	b	c	0
a	0	0	0	0
b	a	b	b	0
c	a	c	c	0
0	0	0	0	0

and let U be any finitely presented infinite semigroup. The semigroup T comes from Example 3.29, and it thus follows that the semigroup $S = T \times U$ is not finitely presented.

Clearly $S_1 = \{c\} \times U$ and $S_2 = \{a, b, 0\} \times U$ are subsemigroups of S and $S = S_1 \dot{\cup} S_2$. The semigroup S_1 is finitely presented since it is isomorphic to U, and from Example 3.29 it also follows that S_2 is finitely presented.

Remark 6.8 The questions (Q2), (Q3) and (Q4) have negative answers even if all S_i $(i \in I)$ are monoids. This can be seen by considering the semigroups S from Examples 6.5 and 6.7, and then successively adjoining two new identity elements to S. This yields the monoid $(S^I)^I$ which is a disjoint union of two semigroups isomorphic to S_1^I and S_2^I respectively. The assertions now follow from Proposition 2.7.

6.3 Bands of semigroups

In this section we consider questions (Q1)–(Q4) for bands of semigroups. Question (Q1) is answered affirmatively by Proposition 6.4.

In the Example 6.5, the semigroup S is a semilattice (and hence a band as well) of the semigroups S_1 and S_2. Thus both questions (Q2) and (Q4) have negative answers. However we have the following

Proposition 6.9 *If $S = \mathcal{B}[Y, S_\alpha]$ is a finitely generated band of semigroups, then Y must be finite.*

Proof Indeed, Y, being a homomorphic image of S, is finitely generated, and hence it is finite by Lemma 6.1. $\square$

We now turn to question (Q3).

Theorem 6.10 *Let $S = \mathcal{B}[Y, S_\alpha]$ be a band of semigroups. Suppose that for each $\alpha \in Y$ the semigroup S_α is defined by the presentation $\langle A_\alpha \mid R_\alpha \rangle$, with $A_\alpha \cap A_\beta = \emptyset$ for $\alpha \neq \beta$. For each $a, b \in \bigcup_{\alpha \in Y} A_\alpha$, with $ab \in S_\gamma$, fix a word $\zeta(a, b)$ on the alphabet A_γ representing the element ab. Then, the semigroup S is defined by the presentation*

$$\langle \bigcup_{\alpha \in Y} A_\alpha \mid \bigcup_{\alpha \in Y} R_\alpha,\ ab = \zeta(a, b)\ (a, b \in \bigcup_{\alpha \in Y} A_\alpha) \rangle. \tag{6.4}$$

In order to prove this theorem we first need some technical lemmas. The notation will be the same as in the theorem. Also we write $A = \bigcup_{\alpha \in Y} A_\alpha$. Recall that, in a band, $\alpha \geq \beta$ means $\alpha\beta = \beta$.

Lemma 6.11 *For each word $u \equiv xu'$, with $x \in A_\alpha$, $u' \in A^*$, there exist $\alpha_1, \alpha_2, \ldots, \alpha_k \in Y$ and words $u_1 \in A^+_{\alpha_1}, \ldots, u_k \in A^+_{\alpha_k}$, with $\alpha \geq \alpha_1 \geq \cdots \geq \alpha_k$, such that the relation $u = u_1 \ldots u_k$ is a consequence of the relations $ab = \zeta(a,b)$ $(a, b \in A)$.*

Proof We use induction on the length of u. If $|u| = 1$ then take $k = 1$ and $u_1 \equiv u$. Next assume that the result is true for any word of length n, and let u be a word of length $n+1$. Put $u \equiv a_1 a_2 u'$, with $a_1 \in A_{\alpha_1}$, $a_2 \in A_{\alpha_2}$ and $u' \in A^*$. The relation

$$u = \zeta(a_1, a_2)u'$$

is a consequence of the relations $ab = \zeta(a,b)$ $(a, b \in A)$. Now put $\zeta(a_1, a_2) \equiv u_1 x$, with $u_1 \in A^*_{\alpha_1\alpha_2}$ and $x \in A_{\alpha_1\alpha_2}$. Then $|xu'| = n$ and the result follows from the induction hypothesis and the fact that $\alpha_1 \geq \alpha_1\alpha_2$. □

Lemma 6.12 *If $u \in A^+_\alpha$ and $v \in A^+_\beta$, with $\alpha \geq \beta$ $(\alpha, \beta \in Y)$, then there exists $w \in A^+_\beta$ such that the relation $uv = w$ is a consequence of the relations $ab = \zeta(a,b)$ $(a, b \in A)$.*

Proof We use induction on the length of u. Put $v \equiv yv'$, with $y \in A_\beta$ and $v' \in A^*_\beta$. If $|u| = 1$ it is sufficient to let $w \equiv \zeta(u, y)v'$. Now suppose that the result is true for every word of length $n \geq 1$, and let $u \in A^+_\alpha$ be such that $|u| = n+1$. Put $u \equiv u'x$, with $u' \in A^+_\alpha$ and $x \in A_\alpha$. Then, using the relation $xy = \zeta(x,y)$, we have

$$uv \equiv u'xyv' = u'\zeta(x,y)v'.$$

Since $|u'| = n$ and $\zeta(x,y) \in A^+_{\alpha\beta} = A^+_\beta$, we can apply the induction hypothesis to $u'(\zeta(x,y)v')$, and obtain $u'\zeta(x,y)v' = w_1$ with $w_1 \in A^+_\beta$ as a consequence of relations $ab = \zeta(a,b)$ $(a, b \in A)$. □

Lemma 6.13 *If $u_1 \in A^+_{\alpha_1}, \ldots, u_k \in A^+_{\alpha_k}$, $(\alpha_1, \alpha_2, \ldots, \alpha_k \in Y)$ with $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_k$, then there exists $w \in A^+_{\alpha_k}$ such that the relation $u_1 \cdots u_k = w$ is a consequence of the relations $ab = \zeta(a,b)$ $(a, b \in A)$.*

PROOF We use induction on k. If $k = 1$ there is nothing to prove. Suppose now that the result is true for some k and let $u_1 \in A^+_{\alpha_1}, \ldots, u_{k+1} \in A^+_{\alpha_{k+1}}$ with $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_k \geq \alpha_{k+1}$. Then there is a word $w_1 \in A^+_{\alpha_k}$ such that the relation $u_1 \cdots u_k = w_1$ is a consequence of the relations $ab = \zeta(a, b)$ $(a, b \in A)$. Also by Lemma 6.12, there is a word $w \in A^+_{\alpha_{k+1}}$ such that $w_1 u_{k+1} = w$, and the result follows. □

PROOF [of Theorem 6.10] By Proposition 6.4, S is generated by the set $A = \bigcup_{\alpha \in Y} A_\alpha$. Clearly, all relations from presentation (6.4) hold in S.

Now, let u and v be two words on the alphabet A representing the same element of S. By Lemmas 6.11 and 6.13 there exist $\beta, \gamma \in Y$, $\overline{u} \in A^+_\beta$ and $\overline{v} \in A^+_\gamma$ such that the relations $u = \overline{u}$ and $v = \overline{v}$ are consequences of the relations $ab = \zeta(a, b)$ $(a, b \in A)$. Since S satisfies the relations $ab = \zeta(a, b)$ $(a, b \in A)$, it also satisfies the relation $\overline{u} = \overline{v}$, so that $\beta = \gamma$. Thus, the relation $\overline{u} = \overline{v}$ holds in S_β, and therefore must be a consequence of R_β. We conclude that $u = v$ is a consequence of the relations of (6.4), which completes the proof. □

Remark 6.14 Notice that the presentation given in Theorem 6.10 is still valid if the band Y is infinite. In that case, however, the obtained presentation is infinite as well.

As an immediate consequence of Theorem 6.10 we have:

Corollary 6.15 *Let $S = \mathcal{B}[Y, S_\alpha]$ be a band of semigroups. If Y is finite and each S_α $(\alpha \in Y)$ is finitely presented, then S is also finitely presented.*

Remark 6.16 For the particular case of a strong semilattice of monoids an analogous result to Theorem 6.10 is presented in [29]. Thus, let S be a strong semilattice of monoids S_α $(\alpha \in Y)$, with morphisms $\lambda_{\alpha,\beta}$ $(\alpha \geq \beta,\ \alpha, \beta \in Y)$. Suppose that each S_α $(\alpha \in Y)$ is defined by a semigroup presentation $\langle A_\alpha \mid R_\alpha \rangle$, with $A_\alpha \cap A_\beta = \emptyset$ for $\alpha \neq \beta$ and let $e_\alpha \in A^+_\alpha$ be a word representing the identity of S_α. Then [29, Theorem 5.1] states that S is defined by the presentation

$$\begin{aligned}\langle \bigcup_{\alpha \in Y} A_\alpha \ \mid \ \bigcup_{\alpha \in Y} R_\alpha,\ & e_\alpha e_\beta = e_{\alpha\beta}\ (\alpha, \beta \in Y, \alpha \neq \beta), \\ & e_\beta a = a e_\beta = a\lambda_{\alpha,\beta}\ (a \in A_\alpha, \alpha \geq \beta) \rangle\end{aligned}$$

which is different from the one given above.

6.4 Bands of monoids

We now turn our attention to the case when $S = \mathcal{B}[Y, S_\alpha]$ and all S_α ($\alpha \in Y$) are monoids. The identity of the monoid S_α ($\alpha \in Y$) will be denoted by 1_α.

Affirmative answers to questions (Q1) and (Q3) follow from Sections 6.2 and 6.3, respectively. The affirmative answer to question (Q2) follows from the next two results.

Theorem 6.17 *Let $S = \mathcal{B}[Y, S_\alpha]$ be a band of monoids generated by a set A. Then, for each $\xi \in Y$, the monoid S_ξ is generated by the set*

$$X = \{1_\xi a 1_\beta \,:\, a \in S_\alpha \cap A,\ \alpha, \beta \in Y,\ \xi\alpha\beta = \xi\}.$$

PROOF From the definition of a band of monoids we have that X is contained in S_ξ, since $\xi\alpha\beta = \xi$. Also,

$$S_\xi = \{s 1_\beta \,:\, s \in S_\alpha,\ \alpha, \beta \in Y,\ \alpha\beta = \xi\},$$

since on one hand $S_\alpha S_\beta \subseteq S_\xi$ whenever $\alpha\beta = \xi$ and on the other hand for any $s \in S_\xi$ we have $s = s1_\xi$ and $\xi\xi = \xi$. So, let $s1_\beta \in S_\xi$ with $s \in S_\alpha$, $\alpha, \beta \in Y$, such that $\alpha\beta = \xi$. If $s \in A$ then $s1_\beta = 1_\xi s 1_\beta \in X$. Now suppose that $s = s_1 a$ with $s_1 \in S_{\alpha_1}$, $a \in S_{\alpha_2} \cap A$. Thus $\alpha_1\alpha_2 = \alpha$. Note that $a1_\beta \in S_{\alpha_2\beta}$ and $s_1 1_{\alpha_2\beta} \in S_\xi$, so that

$$s1_\beta = s_1 a 1_\beta = s_1 1_{\alpha_2\beta} a 1_\beta = (s_1 1_{\alpha_2\beta})(1_\xi a 1_\beta).$$

Hence, using induction on the length of s as a product of elements of A, we conclude that $s1_\beta$ is a product of elements of X. □

Corollary 6.18 *A band of monoids $S = \mathcal{B}[Y, S_\alpha]$ is finitely generated if and only if Y is finite and every S_α ($\alpha \in Y$) is finitely generated.*

PROOF Assume that S is finitely generated and let A be a finite generating set for it. By Proposition 6.9, Y is finite. With both A and Y finite, the generating set X for S_ξ given in Theorem 6.17 is also finite. The converse is Proposition 6.4. □

In order to answer question (Q4) we will use the results introduced in Section 2.2 on the general theory of presentations for subsemigroups. Thus, we are going to find a presentation for an arbitrary component of a band of monoids.

Let $S = \mathcal{B}[Y, S_\alpha]$ be a band of monoids and let $\langle A \mid R \rangle$ be a presentation for S. Let us fix $\xi \in Y$. We aim to find a presentation for the monoid S_ξ. For each $\alpha \in Y$, fix a word $e_\alpha \in A^+$ representing the identity 1_α of S_α. Then, recalling Theorem 6.17, the set

$$X = \{e_\xi a e_\beta \,:\, a \in S_\alpha \cap A,\, \alpha, \beta \in Y,\, \xi\alpha\beta = \xi\} \subseteq A^+$$

is a generating set for S_ξ.

The definition of a band of monoids implies that there is a left action and a right action of S on Y given as follows: for each $s \in S$ and $\alpha \in Y$ let $s\alpha = \beta\alpha$ and $\alpha s = \alpha\beta$, where $\beta \in Y$ is such that $s \in S_\beta$. We have $(\alpha s)\gamma = \alpha(s\gamma)$, for every $\alpha, \gamma \in Y$ and $s \in S$.

Consider the alphabet

$$B = \{[a, \beta] \,:\, a \in S_\alpha \cap A,\, \alpha, \beta \in Y,\, \xi\alpha\beta = \xi\}$$

representing the generating set X of S_ξ. Let $\pi : B^+ \to A^+$ be the unique homomorphism extending the mapping

$$\pi \,:\, [a, \beta] \mapsto e_\xi a e_\beta \tag{6.5}$$

for all $[a, \beta] \in B$.

We now look for a rewriting mapping. First let

$$\psi : \{(w, \beta) \in A^+ \times Y : \xi w\beta = \xi\} \to B^+$$

be defined inductively by

$$\begin{aligned} &(a, \beta)\psi = [a, \beta], && \text{if } a \in A,\ \beta \in Y,\ \xi a\beta = \xi \quad \text{and} \\ &(wa, \beta)\psi = (w, a\beta)\psi\, [a, \beta], && \text{if } w \in A^+,\ a \in A,\ \beta \in Y,\ \xi wa\beta = \xi. \end{aligned} \tag{6.6}$$

This definition extends to

$$(w_1 w_2, \beta)\psi \equiv (w_1, w_2\beta)\psi\, (w_2, \beta)\psi \tag{6.7}$$

for all $w_1, w_2 \in A^+$, $\beta \in Y$ such that $\xi w_1 w_2 \beta = \xi$. To see this, first notice that $\xi w_1 w_2 \beta = \xi$ and $(w_2\beta)(w_2\beta) = w_2\beta$ together imply $\xi w_2 \beta = \xi$. Then (6.7) can be proved by induction on the length of w_2: the case of length one is the above definition of ψ and if the result is true for words of length $n \geq 1$, and $w_2 \equiv w_2' x$ ($w_2' \in A^*$, $a \in A$) has length $n+1$ then

$$\begin{aligned}
(w_1 w_2, \beta)\phi &\equiv (w_1 w_2' a, \beta)\psi \\
&\equiv (w_1 w_2', a\beta)\psi\,[a, \beta] && \text{(by (6.6))} \\
&\equiv (w_1, w_2' a\beta)\psi\,(w_2', a\beta)\psi\,[a, \beta] && \text{(induction)} \\
&\equiv (w_1, w_2' a\beta)\psi\,(w_2' a, \beta)\psi && \text{(by (6.6))}.
\end{aligned}$$

Now let

$$\phi : \mathfrak{L}(A, S_\xi) \to B^+$$

be given by

$$w\phi = (w, \xi)\psi$$

for every $w \in \mathfrak{L}(A, S_\xi)$.

Lemma 6.19 *With the above notation, the relations*

$$(w, \beta)\psi\pi = e_\xi w e_\beta$$

hold in S for all $w \in A^+$, $\beta \in Y$ such that $\xi w \beta = \xi$. In particular, ϕ is a rewriting mapping.

Proof We prove the first statement by induction on the length of w. For the case of length one we have

$$(a, \beta)\psi\pi \equiv [a, \beta]\pi \equiv e_\xi a e_\beta,$$

immediately from the definitions of ψ and π. The inductive step is:

$$\begin{aligned}
(wa, \beta)\psi\pi &\equiv ((w, a\beta)\psi\,[a, \beta])\pi && \text{(by (6.6))} \\
&\equiv ((w, a\beta)\psi\pi)\,([a, \beta]\pi) && (\pi \text{ is a homomorphism}) \\
&= e_\xi w e_{a\beta} e_\xi a e_\beta && \text{(induction, (6.5))} \\
&= e_\xi w e_{a\beta} a e_\beta && (\text{since } e_\xi w e_{a\beta} \in S_\xi) \\
&= e_\xi w a e_\beta, && (\text{since } a e_\beta \in S_{a\beta})
\end{aligned}$$

completing induction. Using the first statement, for $w \in \mathcal{L}(A, S_\xi)$ we have

$$w\phi\pi \equiv (w, \xi)\psi\pi = e_\xi w e_\xi = w,$$

proving that ϕ is indeed a rewriting mapping. $\square$

By the previous lemma and Theorem 2.8, we know that S_ξ is defined by the infinite presentation with generators B and relations

$$[a, \beta] = (e_\xi a e_\beta)\phi, \tag{6.8}$$

$$(w_1 w_2)\phi = w_1\phi\, w_2\phi, \tag{6.9}$$

$$(w_3 u w_4)\phi = (w_3 v w_4)\phi, \tag{6.10}$$

where $[a, \beta] \in B$, $w_1, w_2 \in \mathcal{L}(A, S_\xi)$, $(u = v) \in R$, $w_3, w_4 \in A^*$ and $w_3 u w_4 \in \mathcal{L}(A, S_\xi)$. We denote this presentation by $\langle B \mid (6.8),\ (6.9),\ (6.10)\rangle$. Notice now that from (6.7) we always have

$$(w_1 w_2)\phi \equiv (w_1 w_2, \xi)\psi \equiv (w_1, w_2\xi)\psi\, (w_2, \xi)\psi \equiv (w_1, \xi)\psi\, (w_2, \xi)\psi \equiv w_1\phi\, w_2\phi,$$

for all $w_1, w_2 \in \mathcal{L}(A, S_\xi)$. In other words, relations (6.9) are trivially satisfied, and hence $\langle B \mid (6.8),\ (6.10)\rangle$ is also a presentation for S_ξ.

We now use this infinite presentation to find a new one, which is finite when we start with a finite presentation $\langle A \mid R\rangle$ for S.

Theorem 6.20 *Let $S = \mathcal{B}[Y, S_\alpha]$ be a band of monoids, let $\langle A \mid R\rangle$ be a presentation for S, and let $\xi \in Y$ be arbitrary. As above, let $e_\alpha \in A^+$ $(\alpha \in Y)$ be a word representing the identity of S_α, let*

$$B = \{[a, \beta] \,:\, a \in S_\alpha \cap A,\ \alpha, \beta \in Y,\ \xi\alpha\beta = \xi\}$$

be a new alphabet, and let ψ be defined by (6.6). Then the monoid S_ξ is defined by the (semigroup) presentation with set of generators B and relations

$$(e_\xi a e_\beta, \xi)\psi = [a, \beta] \quad (a \in A,\ \beta \in Y,\ \xi a\beta = \xi), \tag{6.11}$$

$$(u, \gamma\xi)\psi = (v, \gamma\xi)\psi \quad ((u = v) \in R,\ \gamma \in Y,\ \xi u\gamma\xi = \xi). \tag{6.12}$$

PROOF First notice that relations (6.11) are precisely relations (6.8). Next we prove that all relations (6.12) hold in S_ξ.

So, let $(u = v) \in R$ and $\gamma \in Y$ be such that $\xi u\gamma\xi = \xi$. The words $(u, \gamma\xi)\psi, (v, \gamma\xi)\psi \in B^+$ represent the elements $(u, \gamma\xi)\psi\pi$ and $(v, \gamma\xi)\psi\pi$, respectively. By Lemma 6.19, $(u, \gamma\xi)\psi\pi = e_\xi u e_{\gamma\xi}$ and $(v, \gamma\xi)\psi\pi = e_\xi v e_{\gamma\xi}$. Since the relation $u = v$ holds in S, it follows that $(u, \gamma\xi)\psi = (v, \gamma\xi)\psi$ holds in S_ξ. Hence $\langle B \mid (6.11),\ (6.10),\ (6.12)\rangle$ is a presentation for S_ξ.

Finally, we prove that relations (6.10) are consequences of (6.12), and so $\langle B \mid (6.11),\ (6.12)\rangle$ is a presentation for S_ξ. Let $w_3, w_4 \in A^*$ and $(u = v) \in R$ be such that $w_3uw_4 \in \mathcal{L}(A, S_\xi)$. Suppose that $w_3, w_4 \in A^+$. Then

$$(w_3uw_4)\phi \equiv (w_3uw_4, \xi)\psi \equiv (w_3, uw_4\xi)\psi\,(u, w_4\xi)\psi\,(w_4, \xi)\psi.$$

Since $u = v$ is satisfied in S, we also have $w_3vw_4 \in \mathcal{L}(A, S_\xi)$ and

$$(w_3vw_4)\phi \equiv (w_3, vw_4\xi)\psi\,(v, w_4\xi)\psi, (w_4, \xi)\psi.$$

The elements $uw_4\xi, vw_4\xi \in Y$ are equal, and hence $(w_3, uw_4\xi)\psi \equiv (w_3, vw_4\xi)\psi$. Also, $(u, w_4\xi)\psi = (v, w_4\xi)\psi$ is one of the relations (6.12), proving that the relation $(w_3uw_4)\phi = (w_3vw_4)\phi$ is a consequence of (6.12). The cases when one or both of w_3, w_4 are empty are dealt with analogously. □

From Corollary 6.15 and Theorem 6.20 we obtain:

Corollary 6.21 *A band of monoids $S = \mathcal{B}[Y, S_\alpha]$ is finitely presented if and only if Y is finite and every monoid S_α ($\alpha \in Y$) is finitely presented.*

In the case when S is a semilattice of monoids we are able to deduce a simpler presentation for S_ξ, not involving the rewriting mapping:

Corollary 6.22 *Let $S = \mathcal{B}[Y, S_\alpha]$ be a semilattice of monoids, let $\langle A \mid R\rangle$ be a presentation for S, let $\xi \in Y$ be arbitrary, and let $e_\xi \in A^+$ be a word representing the identity of S_ξ. The monoid S_ξ is defined by the (monoid) presentation*

$$\langle A_\xi \mid e_\xi = 1,\ u = v\,((u = v) \in R,\ u \in A_\xi^+)\rangle, \tag{6.13}$$

where $A_\xi = \{a \in A : a\xi = \xi\}$.

PROOF By Theorem 6.20, we have that $\langle B \mid (6.11),\ (6.12)\rangle$ is a presentation for S_ξ. Notice that, since Y is a semilattice, if a word $w \in A^+$ satisfies $w\xi = \xi$ then $w \in A_\xi^+$; in particular $e_\xi \in A_\xi^+$. Also, if $w \equiv a_1 \cdots a_n \in A_\xi^+$, then

$$(w, \xi)\psi = [a_1, \xi] \cdots [a_n, \xi]. \tag{6.14}$$

Consider relations (6.11). Note that if $a \in A$, $\beta \in Y$ and $\xi a \beta = \xi$ then $e_\xi a e_\beta \xi = \xi$, so that $(e_\xi a e_\beta, \xi)\psi$ has the form (6.14). Therefore, each relation (6.11) has the form $[a, \beta] = [a_1, \xi] \ldots [a_n, \xi]$, for some $a_1, \ldots, a_n \in A_\xi^+$. So, for $\beta \neq \xi$, we can eliminate generator $[a, \beta]$ as a product of generators from $\{[a, \xi] : a \in A_\xi\}$.

Next observe that, by Lemma 6.19, in S we have

$$(e_\xi, \xi)\psi\pi = e_\xi e_\xi e_\xi = e_\xi.$$

Therefore the (monoid) relation $(e_\xi, \xi)\psi = 1$ holds in S_ξ, and by adding this relation to our presentation we obtain a monoid presentation for S_ξ.

Now we replace $[a, \xi]$ by a; with this in mind (6.14) becomes $(w, \xi)\psi \equiv w$. Thus the remaining relations (6.11) become $e_\xi a e_\xi = a$, and so can be eliminated, since they are consequences of $e_\xi = 1$. Finally, we look at relations (6.12) and note that we have $\xi u \gamma \xi = \xi$ if and only if $\gamma\xi = \xi$ and $u\xi = \xi$, i.e. if and only if $\gamma\xi = \xi$ and $u \in A_\xi^+$. Hence these relations can be written more simply as $u = v$, for $(u = v) \in R$, $u \in A_\xi^+$. Therefore (6.13) is in fact a presentation for S_ξ, as required. □

6.5 Semilattices and strong semilattices of semigroups

In this section we consider semilattices and strong semilattices of semigroups.

If $S = \mathcal{S}[Y, S_\alpha]$ is a semilattice of semigroups the answers to questions (Q1)–(Q4) follow from Section 6.3 and Example 6.5.

To deal with the case of strong semilattices we use the results about bands of monoids from the previous section, and the device of adjoining identities.

Theorem 6.23 *A strong semilattice of semigroups $S = \mathcal{S}[Y, S_\alpha, \lambda_{\alpha,\beta}]$ is finitely generated (resp. presented) if and only if Y is finite and every semigroup S_α $(\alpha \in Y)$ is finitely generated (resp. presented).*

PROOF Let $S = \mathcal{S}[Y, S_\alpha, \lambda_{\alpha,\beta}]$ be a strong semilattice of semigroups. By Proposition 6.4 and Corollary 6.15, if Y is finite and every semigroup S_α $(\alpha \in Y)$ is finitely generated (resp. presented) then S is finitely generated (resp. presented).

Let us consider the monoids S_α^I $(\alpha \in Y)$, obtained from S_α by adjoining an identity 1_α. For each $\alpha, \beta \in Y$, with $\alpha\beta = \beta$, define $\lambda_{\alpha,\beta}^I : S_\alpha^I \to S_\beta^I$ by $s\lambda_{\alpha,\beta}^I = s\lambda_{\alpha,\beta}$ for all $s \in S_\alpha$, and $1_\alpha\lambda_{\alpha,\beta}^I = 1_\beta$. In this way we obtain the strong semilattice of monoids $T = \mathcal{S}[Y, S_\alpha^I, \lambda_{\alpha,\beta}^I]$.

If S is finitely generated (respectively, finitely presented) then, by Proposition 6.9, it follows that Y is finite. This, in turn, implies that $T\backslash S$ is finite, and hence T is finitely generated (respectively, finitely presented) by Proposition 2.7. Since T is a semilattice (and hence a band as well) of monoids, we have, by Corollaries 6.18 and 6.21, that all S_α^I $(\alpha \in Y)$ are finitely generated (respectively, finitely presented). Finally, using once more Proposition 2.7, we conclude that all S_α $(\alpha \in Y)$ are finitely generated (respectively, finitely presented). $\square$

6.6 Summary

In the following table we summarize the answers to questions (Q1)–(Q4) obtained throughout this chapter. It can be seen that we have answered all the considered questions for finite unions.

	(Q1) all S_i f.g. $\Rightarrow S$ f.g.	**(Q2)** S f.g. $\Rightarrow$ all S_i f.g.	**(Q3)** all S_i f.p. $\Rightarrow S$ f.p.	**(Q4)** S f.p. $\Rightarrow$ all S_i f.p.
unions of semigroups	YES (Prop. 6.4)	NO (Example 6.5)	NO (Example 6.7)	NO (Example 6.5)
unions of monoids	YES (Prop. 6.4)	NO (Remark 6.8)	NO (Remark 6.8)	NO (Remark 6.8)
unions of groups	YES (Prop. 6.3)	YES (Prop. 6.3)	YES (Prop. 6.3)	YES (Prop. 6.3)
bands of semigroups	YES (Prop. 6.4)	NO (Example 6.5)	YES (Cor. 6.15)	NO (Example 6.5)
bands of monoids	YES (Prop. 6.4)	YES (Cor. 6.18)	YES (Cor. 6.15)	YES (Cor. 6.21)
semilattices of semigroups	YES (Prop. 6.4)	NO (Example 6.5)	YES (Cor. 6.15)	NO (Example 6.5)
strong semilattices of semigroups	YES (Prop. 6.4)	YES (Thm. 6.23)	YES (Cor. 6.15)	YES (Thm. 6.23)

Chapter 7

Finitely generated commutative semigroups

In this chapter we study decidability and algorithmic questions concerning finitely generated commutative semigroups.

Commutative semigroups have been widely studied. The first cornerstone in the study of finitely generated commutative semigroups was Rédei's result [42] stating that a finitely generated commutative semigroup is finitely presented. Another result of major importance is the fact that every finitely generated commutative semigroup has a solvable word problem. A proof of this result is given by Gilman [21] who attributes it to Biryukov [11]. Grillet has also studied this class of semigroups (see for example [23, 24, 26]). More recently, work by Rosales, García Sanchez, García García and Urbano Blanco explores many decidability questions for commutative semigroups (see, for example, [48, 50, 51, 52, 53]).

The aim of this chapter is to explore the computability of some structure aspects of finitely generated commutative semigroups. Section 7.1 is of an introductory character. Section 7.2 describes a known decision procedure to decide finiteness of a commutative semigroup and presents a combinatorial method to determine its size. Section 7.3 introduces some further important notation and known results for the study of finitely generated commutative semigroups. In Section 7.4 we examine the computability of Grillet's result that every archimedean component has a cancellative power. Section 7.5 and 7.6 deal with the study of the poset of $\mathcal{H}$–classes. In Section 7.7 we study the Schützenberger groups of a finitely generated commutative semigroup. Finally, Section 7.8 is concerned with the computability of the Grillet representation. In Appendix B we exhibit the code implementing some of the developed algorithms in GAP.

Some of the results in Section 7.5 appear in [4]. All the other results in the chapter are joint work with Robertson, Ruškuc and Solomon.

7.1 Presentations and the word problem: known results

Let FC_A (or simply FC) be the free commutative semigroup on a finite set $A = \{a_1, a_2, \ldots, a_n\}$. This is the semigroup defined by the presentation

$$\langle A \mid a_i a_j = a_j a_i \ (a_i, a_j \in A)\rangle.$$

The free commutative monoid on A is FC_A^1, i.e. FC_A with an identity adjoined (we denote the identity of FC_A^1 by ε). We will refer to the elements of FC_A as *commutative words*, or simply *words*. For $w \in FC_A$ and a generator $a \in A$ we denote by $n(w, a)$ the number of occurrences of the generator a in the underlying word of w in A^+. Notice that two words $w, w' \in A^+$ are equal in FC_A if and only if $n(w, a) = n(w', a)$ for all $a \in A$. Hence every commutative word in FC_A can be written uniquely as a product of the form

$$\prod_{a \in A} a^{k_a} \in A^+$$

($k_a \geq 0$). Therefore we shall represent $w \in FC_A$ by such a product, and whenever we refer to a commutative word we will assume it is given in such a way. Moreover, we write $w \equiv w'$ whenever $w, w' \in FC_A$ are equal. Since in this chapter we deal with commutative semigroups, this should not lead to confusion.

Next we define a commutative presentation to be a pair $\langle A \mid R\rangle_{\mathfrak{C}}$, where $R \subseteq FC_A \times FC_A$. The (commutative) semigroup defined by this presentation is (isomorphic to) FC_A/ρ, where ρ is the minimum congruence on $FC_A \times FC_A$ containing R. It is well known that any commutative semigroup is a homomorphic image of a free commutative semigroup and admits a commutative presentation (see, for example, [27]). Also, it is easy to see that if a semigroup is defined by a commutative presentation $\langle A \mid R \rangle_{\mathfrak{C}}$, then it is defined as a semigroup by the presentation $\langle A \mid R, ab = ba \ (a, b \in A) \rangle$ (where the commutative words in R are now considered in A^+). We say that a (commutative) semigroup is *finitely presented as a commutative semigroup* if it can be defined by a finite commutative semigroup presentation. Notice that a semigroup is finitely presented as a commutative semigroup if and only if it is finitely presented as a semigroup. Also, given a semigroup presentation defining a commutative semigroup, we can obtain a commutative presentation by simply considering the words as words in the free commutative semigroup. *Commutative monoid presentations* and the notion of *finitely presented as a commutative monoid* are defined similarly. There are another two common descriptions of the free commutative semigroup that we shall introduce now.

For a (finite) set X the free commutative semigroup FC_X can be regarded as the multiplicative subsemigroup of the polynomial subring of $R[X]$ (where R is a ring with identity) consisting of all the monomials

$$X^a = \prod_{x \in X} x^{a_x}$$

($a = (a_x)_{x \in X}$, $a_x \in \mathbb{N}_0$, with at least one of a_x non zero ($x \in X$)). Multiplication in FC_X is then given by

$$X^a X^b = X^{a+b}$$

where $(a+b)_x = a_x + b_x$ ($x \in X$). The set X is a generating set for FC_X. In this case the study of presentations is related to the study of ideals in $R[X]$. Such an approach is taken in [27] and emphasizes the connections between commutative semigroups and algebraic geometry (see also [35]).

On the other hand, it is easy to see that the free commutative monoid on an alphabet of size n is (isomorphic to) $\mathbb{N}^n$ (with addition), while the free commutative semigroup on such an alphabet is isomorphic to $\mathbb{N}^n \backslash \{(0, 0, \ldots, 0)\}$ (where $(0, 0, \ldots, 0)$ is the n-tuple with all entries equal to zero). Thus, the elements of the free commutative monoid (or semigroup) can be seen as vectors of length n, over the integers, with addition. Presentations is in this case are related to the study of congruences in $\mathbb{N}^n$. This is the approach taken in, for example, [49] and explores the connections with linear algebra.

Although we will use mainly our first description of the free commutative semigroup, and consequently of commutative presentations, it would be not possible, nor desirable, to forget about the other two. All the three approaches are descriptions of the same object and each of them is connected with a different area of mathematics. The study of commutative semigroups will benefit from all these different views.

We now introduce some further notation. So let S be a commutative semigroup defined by a presentation $\langle A \mid R\rangle_{\mathcal{C}}$. In FC_A we define a partial order $\leq$ by

$$w_1 \leq w_2 \iff (\forall a \in A)(n(w_1, a) \leq n(w_2, a))$$

$(w_1, w_2 \in FC_A)$. A well known result about this ordering is Dickson's Lemma. A proof can be found, for example, in [49].

Theorem 7.1 [Dickson's Lemma] *Let FC_A be the free commutative semigroup over a finite alphabet A. Then any subset of FC_A of incomparable elements, under $\leq$, is finite.*

For a pair of words $w_1, w_2 \in FC_A$ we denote by $w_1 \vee w_2$ their least common multiple, i.e.

$$n(w_1 \vee w_2, a) = \max\{n(w_1, a), n(w_2, a)\} \ (\forall\, a \in A)$$

and by $l(w_1, w_2)$ the word in FC_A such that

$$w_1 l(w_1, w_2) \equiv w_1 \vee w_2,$$

i.e.

$$n(l(w_1, w_2), a) = \max\{n(w_2, a) - n(w_1, a), 0\}.$$

Also, if $w \in FC_A$, $(u = v) \in R$ and $w \geq u$ we denote by $wu^{-1}v$ the word obtained from w by application of the relation $(u = v)$ (from left to right). Recall that $w = wu^{-1}v$, that is, w and $wu^{-1}v$ represent the same element of S.

The following is known as Rédei's Theorem and plays a major role in what will follow. It was first proved by Rédei [42]. A proof can also be found in [27].

Proposition 7.2 *Congruences on a finitely generated free commutative semigroup are finitely generated, and therefore every finitely generated commutative semigroup is finitely presented.*

We have mentioned earlier that finitely generated commutative semigroups have a solvable word problem. We will describe such a solution in general. We start by fixing a reduction ordering $\preccurlyeq$ on FC_A (i.e. $\preccurlyeq$ is linear order which admits no infinite descending chains and whenever $u \preccurlyeq v$, $aub \preccurlyeq avb$ for any $a, b \in FC_A$ ($u, v \in FC_A$)). Now, for each element $s \in S$ there is a minimum word $w_s \in FC_A$ (with respect to the $\preccurlyeq$) that represents s. We say that w_s is the *canonical form of* s, and for any word $z \in FC_A$ representing s we also say that w_s is the *canonical form* of z. We denote by W_S the set consisting of all the canonical forms and define a mapping

$$\begin{array}{rcccl} \mu & : & FC_A & \to & W_S \\ & & z & \mapsto & z\mu \end{array}$$

where $z\mu$ is the canonical form of $z \in FC_A$.

The set W_S is in one-one onto correspondence with S, that is, it contains one and only one representative for each element of S. Thus, there is an injective mapping from S to W_S; we also denote this mapping by μ and it will be clear from the context to which one we will be referring. Notice that given two words $w, w' \in FC_A$ they represent the same element of S if and only if they have the same canonical form, that is, if and only if $\mu(w) = \mu(w')$. So, if we can compute μ, then we have an effective algorithm to solve the word problem.

A presentation $\langle A \mid R \rangle_{\mathcal{C}}$ is a *(commutative) rewriting system*, with respect to the reduction ordering $\preccurlyeq$ on FC_A, if for each pair $(u = v) \in R$,

$$u \succ v.$$

Each $(u = v) \in R$ is called a *rewriting rule* and is written as $u \to v$ (and may only be applied from left to right). If by applying the rule $u \to v$ to a word $w \in FC_A$ a word $w' \in FC_A$ is obtained (i.e. $w' \equiv wu^{-1}v$) then we write $w \to_R w'$. If $w' \in FC_A$ is obtained from w by applying several (or no) rules from R then we write $w \xrightarrow{*}_R w'$, or simply $w \xrightarrow{*} w'$. This process is called *rewriting.* If $u \in FC_A$ is a word to which no rule can be applied then u is said to be *reduced.* A rewriting system is said to be *confluent* if for any words $u, v, z \in FC_A$ satisfying $z \xrightarrow{*} u$ and $z \xrightarrow{*} v$, there exists $w \in FC_A$ such that $u \xrightarrow{*} w$ and $v \xrightarrow{*} w$. Confluent rewriting systems are also called *confluent presentations.* Clearly, if $\langle A \mid R \rangle$ is a confluent rewriting system then a word w is reduced if and only if $\mu(w) \equiv w$. Thus, to obtain the canonical form of an arbitrary word $w \in FC_A$, given a confluent rewriting system, it is enough to keep applying rules while possible. Whenever we obtain a reduced word, that word is the canonical form.

Theorem 7.3 [21] *Every finitely generated commutative semigroup has a finite commutative confluent rewriting system. Given a finitely generated commutative semigroup defined by a finite (commutative) presentation such a confluent rewriting system can be effectively computed. In particular every finitely generated commutative semigroup has a solvable word problem.*

The computation of a commutative confluent rewriting system is based on the Knuth–Bendix procedure. (A description of this procedure can be found in [63].) Commutative rewriting systems have also been referred to as *vector replacement systems* (see [17]).

All that has been said for finitely generated commutative semigroups also follows for finitely generated commutative monoids.

Remark 7.4 Rewriting systems can actually be introduced in the more general context of non commutative semigroups. It is clear that if a semigroup has a confluent rewriting system then it has a solvable word problem. Thus, since there are finitely presented semigroups that do not have a solvable word problem, it follows that not all semigroups have finite confluent rewriting systems. A result by Diekert [17] however states that every finitely generated commutative semigroup has a finite (non commutative) confluent rewriting system. To obtain a (non commutative) confluent rewriting system from a commutative one it is not enough to adjoin the rules $ba \to ab$ for each pair of generators $a, b \in A$ with $ba \succ ab$. In general, a larger generating set is needed.

From now on all the semigroups considered are commutative, unless otherwise stated. Also we will be considering commutative presentations and commutative rewriting systems. Thus, we shall drop the adjective commutative whenever no confusion may arise and write $\langle A \mid R \rangle$ instead of $\langle A \mid R \rangle_{\mathcal{C}}$.

7.2 Finiteness and size

Given a finitely generated commutative semigroup S, one of the first questions that one may want to ask about it is whether it is finite. In general, for an arbitrary semigroup given by a presentation, finiteness is not decidable. However, it is well known that the same question has a positive answer for commutative semigroups. We present here a proof for that fact.

Theorem 7.5 *Let S be a finitely generated semigroup defined by a confluent presentation $\langle A \mid R\rangle$. Then S is finite if and only if for all $a \in A$ there is $k > 0$ such that $(a^k \to v) \in R$ for some $v \in FC_A$.*

Proof First suppose that S is finite. Then, for each generator $a \in A$ there exist $k > 0$ and a word $v \in FC_A$ such that $(a^k \to v) \in R$ for otherwise all the powers of a would be distinct, which contradicts the finiteness of S.

Conversely suppose that there exists a rule $(a^{k_a} \to v)$ for each generator $a \in A$ ($k_a > 0$, $v \in FC_A$). Hence, for each $w \in FC_A$, there is a word $w' \in FC_A$, such that $w = w'$ and $n(w', a) < k_a$ for each $a \in A$. That is, if $\alpha \in FC_A$ is the word satisfying $n(\alpha, a) = k_a$ for each $a \in A$ then the set

$$T = \{w \in FC_A \ : \ w < \alpha\}$$

contains at least one word representing each element of S. Since this set is finite it follows that the semigroup S is finite as well. □

Corollary 7.6 *Finiteness of finitely generated commutative semigroups is decidable.*

Notice that the decidability procedure described in Theorem 7.5 does not depend of some enumeration of the semigroup, but only in examining its presentation. Further, the set T in the proof gives a bound for the size of a finite commutative semigroup. Indeed it is clear that

$$\mid T \mid = (\prod_{a \in A} n(\alpha, a)) - 1 > \mid S \mid .$$

In fact, if we know that the semigroup is finite we can enumerate its elements to find its size. This might not be very easy though if the semigroup is very large.

We shall see that a combinatorial argument will be enough. We start by proving the following

Lemma 7.7 *Let S be a finite commutative semigroup defined by a confluent presentation $\langle A \mid R\rangle$ with $R = \{u_i \to v_i : 1 \le i \le r\}$ and let T be defined as above. For each $(u_i \to v_i) \in R$ ($1 \le i \le r$) let T_i be the intersection of the ideal of FC_A generated by u_i with T, that is*

$$T_i = \{w \in T : w \ge u_i\}.$$

Then a word $w \in FC_A$ is reduced if and only if $w \in T \backslash (T_1 \cup T_2 \cup \cdots \cup T_r)$.

Proof Suppose that $w \in FC_A$ is reduced. Then $w \in T$. Suppose further that $w \in T_i$ for some i ($1 \le i \le r$). Then

$$w \equiv u_i y$$

for some $y \in FC_A$. But then we can reduce w by applying the rule $u_i \to v_i$ and this contradicts the fact that w is reduced.

Conversely suppose that $w \in T$ is not reduced. Then there exists $u_i \to v_i \in R$ ($1 \le i \le r$) such that $w \ge u_i$, that is, there is $y \in FC_A$ such that

$$w \equiv u_i y.$$

In particular $w \in T_i$. □

Lemma 7.8 *With the above notation we have*

$$\mid T_1 \cup \cdots \cup T_k \mid = \sum_{j=1}^{k} \mid T_j \mid - \sum_{\substack{1 \le j,l \le k \\ j \ne l}} \mid T_j \cap T_l \mid + \cdots + (-1)^{k+1} \mid T_1 \cap \cdots \cap T_k \mid$$

with

$$\mid T_i \mid = \prod_{a \in A} (n(\alpha, a) - n(u_i, a))$$

and for $k > 1$

$$\mid T_{j_1} \cap T_{j_2} \cap \cdots \cap T_{j_k} \mid = \prod_{a \in A} (n(\alpha, a) - \max\{n(u_l, a) : l = j_1, j_2, \ldots, j_k\}).$$

(The element α is as in the proof of Theorem 7.5.)

PROOF The first equality is an application of the inclusion exclusion principle and other equalities follow from

$$\begin{aligned} T_i &= \{w \in T : w \geq u_i\} \\ &= \{w \in FC_A : \alpha > w \geq u_i\} \\ &= \{w \in FC_A : n(\alpha, a) > n(w, a) \geq n(u_i, a)\ (\forall\, a \in A)\} \end{aligned}$$

and

$$\begin{aligned} T_{j_1} \cap T_{j_2} \cap \cdots \cap T_{j_k} &= \{w \in T\ :\ w \geq u_{j_l},\ (\forall\, 1 \leq l \leq k)\} \\ &= \{w \in FC_A\ :\ n(\alpha, a) > n(w, a) \geq \\ &\quad \geq \max\{n(u_{j_l}, a) : 1 \leq l \leq k\}\ (\forall\, a \in A)\} \end{aligned}$$

□

Proposition 7.9 *Let S be a finite commutative semigroup defined by a confluent presentation $\langle A \mid R\rangle$. Let T and T_u be defined as above. Then*

$$\mid S \mid = \mid T \mid - \mid \bigcup_{(u \to v) \in R} T_u \mid . \tag{7.1}$$

Furthermore, the size of S can be computed by a combinatorial argument.

PROOF The formula (7.1) for the size of S follows from Lemma 7.7. The fact that those sizes are effectively computable follows from Lemma 7.8. Therefore we can compute the size of S by a combinatorial argument, that is, we do not need to enumerate S. □

Example 7.10 Let S be the finitely generated commutative semigroup defined by the presentation

$$\langle a, b \mid a^4 = a,\ b^3 = b,\ a^2 b = ab\rangle.$$

It is easy to see that this is a confluent presentation. Then, by Theorem 7.5, S is a finite semigroup. In this case we have

$$T = \{a^i b^j\ :\ 0 \leq i < 4,\ 0 \leq j < 3,\ i \text{ and } j \text{ not both } 0\}.$$

Thus $| T | = 11$. Now,

$$T_{a^4} = \emptyset$$
$$T_{b^3} = \emptyset$$
$$T_{a^2b} = \{a^i b^j \; : \; 2 \le i < 4, \; 1 \le j \le 3\}$$

and thus

$$| S | = | T | - | T_{a^2b} | = 11 - 4 = 7.$$

We thus conclude that S has seven elements. Indeed $S = \{a, a^2, a^3, b, b^2, ab, ab^2\}$.

In the next example we illustrate the main idea behind the combinatorial argument presented above.

Example 7.11 Let S be the finitely generated commutative monoid defined by the presentation

$$S = \langle x, y, z \mid x^2y = xy, \; xy^3 = xy, \; xyz^4 = xy, \; x^5 = x, \; y^5 = y, \; z^5 = z \rangle.$$

This is a confluent presentation for S. From Theorem 7.5 we know that S is a finite monoid. For counting purposes, we see from the presentation that we can draw S in the three dimensional space, inside a $5 \times 5 \times 5$ cube. This cube is represented in Figure 7.1. The elements of the semigroup are unitary cubes and we have to remove the cubes corresponding to the ideals arising from the relations. By counting we conclude that the monoid has size 53.

7.3 The $\mathcal{H}$–relation and the archimedean relation: definitions and known results

The five Green's relations that can be defined on a semigroup S, in terms of its ideal structure, coincide in a commutative semigroup. We choose to denote the unique one by $\mathcal{H}$. So, for a commutative semigroup S, $s\mathcal{H}t$ if s and t generate the same principal ideal of S. And we have the usual preorder on S: $s \le_{\mathcal{H}} t$ if s belongs to the principal ideal of S generated by t. We call this ordering the *divisibility ordering* on S and, when no confusion arises we denote it by $\le$.

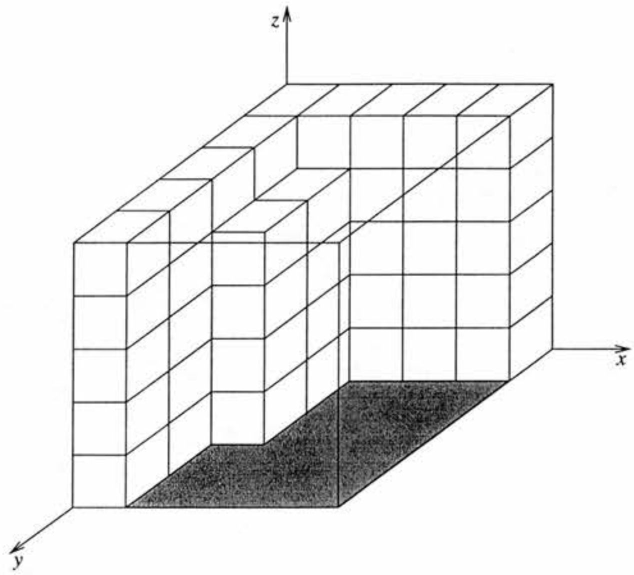

Figure 7.1: The cube from Example 7.11

(Notice that this ordering, when defined on a free commutative semigroup, is exactly the opposite of the one defined in Section 7.1.)

Now, if S is a commutative semigroup, then $\mathcal{H}$ is a congruence, and therefore $S/\mathcal{H}$ is a commutative semigroup. Moreover, the ordering relation $\leq_{\mathcal{H}}$ induces a partial ordering on $S/\mathcal{H}$, which will also be denote by $\leq$. Thus we will refer to $S/\mathcal{H}$ as the poset of $\mathcal{H}$-classes of S.

There is another congruence that will play a major role in the theory of commutative semigroups: the smallest semilattice congruence on S. This will be denoted by $\mathcal{N}$ and is defined as follows:

$$a\mathcal{N}b \iff (\exists\, m, n > 0)\ a^m \leq b \text{ and } b^n \leq a$$

for $a, b \in S$. The congruence $\mathcal{N}$ also induces a preorder relation $\leq_{\mathcal{N}}$ on S, namely

$$a \leq_{\mathcal{N}} b \iff (\exists\, m > 0)\ a^m \leq b.$$

The $\mathcal{N}$-classes of a commutative semigroup are known as its *archimedean components* and, as expected, $S/\mathcal{N}$ is a semilattice (with respect to the ordering $\leq_{\mathcal{N}}$) whose elements are the archimedean components of S. Furthermore, $S/\mathcal{N}$ is the largest semilattice homomorphic image of S.

Remark 7.12 If S is a finitely generated commutative semigroup given by a presentation $\langle A \mid R \rangle$ then $S/\mathcal{N}$ is defined by the presentation $\langle A \mid R,\ a^2 = a\ (a \in A)\rangle$, since $S/\mathcal{N}$ is the largest semilattice homomorphic image of S. In particular it is decidable whether two elements $s, t \in S$ are $\mathcal{N}$-related.

Proposition 7.13 [27, Proposition IV.9.5] *A finitely generated commutative semigroup S has finitely many archimedean components.*

Proof This is an immediate consequence from the fact that $S/\mathcal{N}$ is a finitely generated semilattice, and thus it is finite. □

A finitely generated commutative semigroup S is *group complete* (or simply *complete*) if every archimedean component has an idempotent, or equivalently, every element of S contains a power in a subgroup of S. Finite commutative semigroups are examples of complete semigroups.

It is easy to see that each archimedean component contains at most one idempotent. Indeed, if e, f are two idempotents of the same archimedean component then $e\mathcal{N}f$ and hence $e \leq f \leq e$ so $e = f$ (since $e\mathcal{H}f$ implies $e = f$). It thus follows from Proposition 7.13 that a finitely generated commutative semigroup contains only finitely many idempotents. In [52] it was proved that it is in fact possible to find the set of idempotents of a finitely generated commutative semigroup. That is,

Proposition 7.14 *There is an algorithm that given a finitely generated commutative semigroup (defined by a presentation) returns its set of idempotents.*

We will denote the algorithm that calculates the idempotent set of a semigroup by `Idempotents`.

The following establishes a connection between the existence of idempotents in archimedean components and the $\mathcal{H}$-relation.

Proposition 7.15 *Let S be a finitely generated commutative group-complete semigroup. Then $S/\mathcal{H}$ is finite. Moreover $\mathcal{H}$ is the equality on S if and only if all the subgroups of S are trivial.*

A proof for the first statement of the above proposition may be found in [26]. The second statement is from [27].

We close this section with the following

Proposition 7.16 [26, Proposition 2.4] *Every finitely generated commutative semigroup S can be embedded into a complete finitely generated commutative semigroup T.*

7.4 The cancellative power of an archimedean component

Archimedean components of finitely generated commutative semigroups have the following property:

Proposition 7.17 [26, Theorem 2.5] *Let S be a finitely generated commutative semigroup and let A be an archimedean component of S. Then there exists $m > 0$ such that A^m is a cancellative semigroup.*

In this section we will prove that it is possible to compute the integer m from Proposition 7.17. We will follow the proof of Proposition 7.17 given in [27], proving that each step is computable. This result will be used in in Section 7.7.

The first step is to decompose the semigroup S. Such a decomposition will rely on the connection between congruences on the free commutative semigroup and ideals of $\mathbb{Z}[X]$. Appendix C contains the basic definitions and results that we will be using.

So let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set and let FC_X be the free commutative semigroup on X. In this section we will identify FC_X with $\mathbb{N}^n \backslash \{(0, 0, \ldots, 0\}$ as described in Section 7.1. The set $\{b_i \in \mathbb{N}^n : 1 \leq i \leq n\}$ (where each b_i has all entries equal to zero, except the ith entry which is one) is then a generating set for $FC_X = \mathbb{N}^n \backslash \{(0, 0, \ldots, 0\}$. Now, to each element $a = (a_1, a_2, \ldots, a_n) \in FC_X$, corresponds a monomial $X^a = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ in the polynomial ring $\mathbb{Z}[X]$. Each ideal I of $\mathbb{Z}[X]$ induces a congruence $\mathcal{C}_I$ on FC_X, namely

$$a \mathcal{C}_I b \iff X^a - X^b \in I$$

$(a, b \in FC_X)$; we refer to $\mathcal{C}_I$ as the *congruence associated to* I. Conversely, we have

Lemma 7.18 [Preston] *Let X be a finite set and FC_X be the free commutative semigroup on X. Let $\mathcal{C}$ be a congruence on FC_X and let $I(\mathcal{C})$ be the ideal of $\mathbb{Z}[X]$ generated by $\{X^a - X^b : a\mathcal{C}b\}$. Then $a\mathcal{C}b$ if and only if $X^a - X^b \in I(\mathcal{C})$.*

Proof We follow the proof in [27]. From the definition we have that if $a\mathcal{C}b$ then $X^a - X^b \in I(\mathcal{C})$. Conversely, since $\mathcal{C}$ is a congruence

$$I = \{ \sum_{1 \le j \le r} m_j(X^{a_j} - X^{b_j}) \, : \, r > 0, m_j > 0, a_j \ne b_j, a_j\mathcal{C}b_j \}.$$

So assume that $a \ne b$ and $X^a - X^b \in I(\mathcal{C})$. From what we have said above,

$$X^a - X^b = \sum_{1 \le j \le r} m_j(X^{a_j} - X^{b_j}) \tag{7.2}$$

with $m_j > 0$, $a_j \ne b_j$ and $a_j\mathcal{C}b_j$ $(1 \le j \le r)$. Since $X^a - X^b \ne 0$ we certainly have $r > 0$. We show by induction on $m = \sum_{1 \le j \le r} m_j > 0$ that $a\mathcal{C}b$.

If $m = 1$, then $r = 1$, $m_1 = 1$, and $a = a_1\mathcal{C}b_1 = b$. Now let $m > 1$. The coefficient of X^a in the right hand side of (7.2) is $\sum_{a_j = a} m_j - \sum_{b_j = a} m_j = 1$. Therefore $a = a_k$ for some k. If $b = b_k$, then $a\mathcal{C}b$. Otherwise the coefficient of X_{b_k} in the right hand side of (7.2) is $\sum_{a_j = b_k} m_j - \sum_{b_j = b_k} m_j = 0$, since $b_k \ne a, b$. Therefore there exists l such that $b_k = a_l$. In the right hand side of (7.2) we can then replace $(X^{a_k} - X^{b_k}) + (X^{a_l} - X^{b_l})$ by $(X^{a_k} - X^{b_l})$ (or by 0 if $a_k = b_l$). This decreases m by 1 (or by 2). The result follows by induction. □

In the next lemma we show how we can effectively obtain a basis for the ideal given a generating set for the congruence.

Lemma 7.19 *With the notation as above, if ρ is a generating set for $\mathcal{C}$, then $I(\mathcal{C})$ is generated by the set $\{X^a - X^b : (a, b) \in \rho\}$. In particular $I(\mathcal{C})$ membership is decidable.*

Proof To prove the first part of the lemma it is enough to show that any polynomial of the form $X^a - X^b \in I(\mathcal{C})$ with $a\mathcal{C}b$ is in the ideal generated by the set $\{X^a - X^b : (a,b) \in \rho\}$. So let $X^a - X^b \in I(\mathcal{C})$ with $a\mathcal{C}b$. Since $a\mathcal{C}b$, there exist a_i, u_i, v_i $(1 \leq i \leq k)$ such that

$$a \equiv a_1u_1\mathcal{C}a_1v_1 \equiv a_2v_2\mathcal{C}\cdots\mathcal{C}a_{k-1}v_{k-1} \equiv a_ku_k \equiv b$$

with $(u_i, v_i) \in \rho$. We will prove by induction on k, that

$$X^a - X^b = \sum_{i=1}^{k-1} X^{a_i}(X^{u_i} - X^{v_i})$$

and hence it follows that the set $\{X^a - X^b : (a,b) \in \rho\}$ generates the ideal $I(\mathcal{C})$. Notice that if $a \neq b$ then $k \geq 2$. So, if $k = 2$ then

$$X^a - X^b = X^{a_1}(X^{u_1} - X^{v_1})$$

as claimed. Now suppose that the result is true for $k = h - 1$. Hence

$$\begin{aligned} \sum_{i=1}^{h} X^{a_i}(X^{u_i} - X^{v_i}) &= \sum_{i=1}^{h-1} X^{a_i}(X^{u_i} - X^{v_i}) + X^{a_h}(X^{u_h} - X^{v_h}) \\ &= X^{a_1}X^{u_1} - X^{a_{h-1}}X^{v_{h-1}} + X^{a_h}X^{u_h} - X^{a_h}X^{v_h} \\ &= X^{a_1}X_{u_1} - X^{a_h}X^{v_h} = X^a - X^b \end{aligned}$$

and thus the result is also true for $k = h$. The fact that we can decide membership in the ideal $I(\mathcal{C})$ follows from Theorem C.2. □

Remark 7.20 Conversely, given an ideal $I \subseteq \mathbb{Z}[X]$ one may ask whether we can compute a (finite) generating set for its associated congruence $\mathcal{C}_I$. In general we do not know how to find such a generating set from a basis for an arbitrary ideal. It is not difficult to see, however, that if an ideal $I \subseteq \mathbb{Z}[X]$ is given by a basis of the form $G = \{X^{a_j} - X^{b_j} : j \in J\}$, then the congruence $\mathcal{C}_I$ is generated by the set $\{(a_j, b_j) : j \in J\}$. Indeed, if $X^a - X^b \in G$ then $a\mathcal{C}_Ib$ and it is enough to prove that $\mathcal{C}_I$ is generated by the set $\{(a,b) : a = \alpha+\gamma, b = \beta+\gamma, X^\alpha - X^\beta \in G, \gamma \geq 0\}$. So suppose that $a\mathcal{C}_Ib$. Then we know that $X^a - X^b \in I$. Hence

$$X^a - X^b = \sum_{1 \leq j \leq r} m_j(X^{a_j} - X^{b_j})$$

where $a_j = \alpha_j + \gamma_j$, $b_j = \beta_j + \gamma_j$, $m_j > 0$, $X^{\alpha_j} - X^{\beta_j} \in G$ or $X^{\beta_j} - X^{\alpha_j} \in G$. In particular notice that $a_j\mathcal{C}_Ib_j$. The proof of Lemma 7.18 shows that $a\mathcal{C}_Ib$.

We will now prove the decomposition result for finitely generated commutative semigroups. First we recall that a semigroup S is said to be *nilsemigroup* if it has a zero and every element has a power equal to zero. If furthermore there is an integer $m > 0$ such that $S^m = \{s_1 s_2 \cdots s_m : s_k \in S, 1 \leq i \leq k\}$ equals $\{0\}$ then the semigroup is *nilpotent* (for a finitely generated commutative semigroup this is equivalent to the existence of an integer m' such that for every $s \in S$, $s^{m'} = 0$). A semigroup S is a *cancellative semigroup* if for every $c, x, y \in S$, $cx = cy$ implies $x = y$. Finally, a semigroup S is a *subelementary semigroup* if $S = C \cup N$ is a disjoint union of a cancellative semigroup C and a nilsemigroup (or trivial semigroup) N, such that N is an ideal of S, the zero element of N is the zero element of S, and every element of C is cancellative in S (i.e. $cx = cy$ implies $x = y$ for every $x, y \in S$ and $c \in C$). A *subdirect product* of a family $(S_j)_{j \in J}$ of semigroups is (isomorphic to) a semigroup P, which is a subsemigroup of the direct product $\prod_{j \in J} S_j$, such that for each $j \in I$ the projection $\pi_j : P \to S_j$, $(x_i)_{i \in I} \mapsto x_j$ is an epimorphism.

Proposition 7.21 [27, Proposition IV.9.4] *Every finitely generated commutative semigroup S is a subdirect product of finitely many cancellative semigroups (which may be missing), finitely many nilsemigroups (which may also be missing), and finitely many subelementary semigroups.*

Proof We follow the proof from [27, Proposition IV.9.4]. So, let $\mathcal{C}$ be a congruence in FC_X such that $S = FC_X/\mathcal{C}$. Let $I(\mathcal{C})$ the ideal of $\mathbb{Z}[X]$ associated to $\mathcal{C}$, as described above. Since $\mathbb{Z}[X]$ is noetherian, I is the intersection $I = Q_1 \cap Q_2 \cap \cdots \cap Q_r$ of finitely many primary ideals, by Theorem C.3.

We will show that given a primary ideal Q, if its associated congruence is $\mathcal{Q}$ then the semigroup $T = FC_X/\mathcal{Q}$ is either a nilsemigroup, a cancellative semigroup, or a subelementary semigroup. It will therefore follow that S is a subdirect product of these types of semigroups.

So let Q be a primary ideal, $\mathcal{Q}$ its associated congruence and P its radical. Let $\pi : FC_X \to T$ be the homomorphism from the free commutative semigroup FC_X onto T and let $a, b, c \in FC_X$. If $X^a \in Q$ then $X^a X^b - X^a \in Q$ and so $(a\pi)(b\pi) = a\pi$ for all $b \in FC_X$ and thus $a\pi$ is a zero of T. If $X^a \in P$ then X^a has a power in Q and hence $a\pi$ is nilpotent in T; these elements form an ideal of T since P is an ideal of $\mathbb{Z}[X]$. Finally, if $X^a \notin P$, then $(a\pi)(b\pi) = (a\pi)(c\pi)$ implies $X^a(X^b - X^c) \in Q$ and hence since Q is primary and no power of X^a is in Q, $X^a - X^b \in Q$. Hence $b\pi = c\pi$ and $a\pi$ is cancellative in T. Thus either T is nil or trivial (if $X^a \in P$ for all $a \in FC_X$), or T is cancellative (if $X^a \notin P$ for all $a \in FC_X$) or subelementary (otherwise). $\square$

Lemma 7.22 *Let FC_X be the free commutative semigroup on a finite set X, let $Q \in \mathbb{Z}[X]$ be a primary ideal, let $P \in \mathbb{Z}[X]$ be its associated prime ideal and $\mathcal{Q}$ its associated congruence, and let $T = FC_X/\mathcal{Q}$. Then either T is cancellative or we can calculate $m > 0$ such that T^m is zero or is a cancellative semigroup with a zero adjoined.*

PROOF The semigroup T is cancellative if and only if $X^{b_i} \notin P$ for every generator $b_i \in FC_X$ $(1 \leq i \leq n)$. Indeed, from the proof of Proposition 7.21, it follows that if $a \in FC_X$ is such that $X^a \notin P$ then $a\pi$ is cancellative in T. So, if $a = \sum_{i=1}^n l_i b_i$ and $c, d \in FC_X$ are such that

$$(a\pi)\,(c\pi) = (a\pi)\,(d\pi)$$

then

$$(\prod_{i=1}^n (b_i\pi)^{l_i})(c\pi) = (\prod_{i=1}^n (b_i\pi)^{l_i})(d\pi)$$

and hence

$$c\pi = d\pi$$

since each $b_i\pi$ $(1 \leq i \leq n)$ is cancellative in T. Therefore $a\pi$ is cancellative in T. On the other hand if there exists j $(1 \leq j \leq n)$ such that $X^{b_j} \in P$ then T contains a zero element and therefore it cannot be a cancellative semigroup. Furthermore, it is decidable whether T is a cancellative semigroup.

We will now prove that T is a nilsemigroup if and only if for all generators b_i of FC_X $(1 \leq i \leq n)$, $X^{b_i} \in P$. Indeed, if one of the generators e does not satisfy this condition then, by the proof of Proposition 7.21 $e\pi$ is not nilpotent in T and thus T is not a nilsemigroup. Conversely, if all generators satisfy this condition and $a \in FC_X$ is such that

$$a = \sum_{i=1}^n l_i b_i$$

then

$$X^a = \prod_{i=1}^{n} X^{l_i b_i}$$

and since each $X^{b_i} \in P$ so does X^a. Therefore, we can decide whether T is a nilsemigroup. Now, there exists $m_i > 0$ such that $(b_i\pi)^{m_i}$ is zero in T $(1 \leq i \leq n)$. This means that $X^{m_i b_i} \in Q$. We can find those integers $m_1, m_2, \ldots, m_n > 0$ by inspection. Further, if

$$m > \sum_{i=1}^{n} m_i$$

then for any

$$a = \sum_{i=1}^{n} l_i b_i \in FC_X$$

with $\sum_{i=1}^{n} l_i > m$ (i.e. a is a sum of more then m generators) there exists j $(1 \leq j \leq n)$ such that $l_j > m_j$ and hence $a\pi$ is zero. We have proved that $T^m = \{0\}$.

Finally, if T is not nil nor cancellative then it is subelementary by the proof of Proposition 7.21. Therefore $T = C \cup N$, where C is a cancellative semigroup and N is a nilsemigroup. Without loss of generality let $\{b_1, b_2, \ldots, b_k\}$ $(0 < k < n)$ be the set $\{b_i : 1 \leq i \leq n, X^{b_i} \in P\}$. For each b_j $(1 \leq j \leq k)$ determine m_j such that $X^{m_j b_j} \in Q$ (from what we have said above we can calculate such integers m_j) and let $m > m_1 + m_2 + \cdots + m_k$. Now let $a \in FC_X$ be such that $a\pi \in N$,

$$a = (\sum_{j=1}^{k} l_j b_j) + c,$$

where $c\pi \in C$, $l_j \in \mathbb{N}$ (not all zero) and $\sum_{j=1}^{k} l_j > m$. Then by the same argument as above there is at least one l_j $(1 \leq j \leq k)$ satisfying $l_j > m_j$ and thus $a\pi = 0$.
□

This proposition gives rise to the following

Algorithm 7.23 `CancellativeOrNilPowerOfSemigroup`
INPUT: X a finite non-empty set, $Q \subseteq \mathbb{Z}[X]$ a primary ideal and $P \subseteq \mathbb{Z}[X]$ its associated prime ideal
OUTPUT: $m > 0$ such that $(FC_X/\mathcal{Q})^m$ is cancellative, zero or cancellative with a zero adjoined

```
if (∀i ∈ {1, 2, ..., n}) X^{b_i} ∉ P then
    m ← 1
else
    for all i ∈ {1, 2, ..., n} do
        if X^{b_i} ∈ P then
            m_i ← 1
            while X^{m_i b_i} ∉ Q do
                m_i ← m_i + 1
            end while
        end if
    end for
    m ← Σ_{i=1}^{n} m_i
end if
return m
```

Finally we have

Proposition 7.24 [27, Proposition IV.9.6] *Let S be a finitely generated commutative semigroup and A be an archimedean component of S. Then A^m is cancellative for some $m > 0$. Furthermore, there is an algorithm that given S, defined by a finite presentation, returns $m > 0$ such that for any archimedean component A of S, A^m is cancellative.*

PROOF Let S be a finitely generated commutative semigroup defined by a presentation $\langle X \mid R \rangle$ and let $I \in \mathbb{Z}[X]$ be the ideal associated to this presentation. By Theorem C.4 we can find a primary decomposition $Q_1 \cap Q_2 \cap \cdots \cap Q_r$ of I and for each Q_j $(1 \leq j \leq r)$ we can also obtain its associated prime ideal P_j. Each of these ideals Q_j $(1 \leq j \leq r)$ gives rise to a semigroup T_j which is, by Proposition 7.21, either nil, cancellative or subelementary. By Lemma 7.22 we can find integers $m_j > 0$ $(1 \leq j \leq r)$ such that $T_j^{m_j}$ is either cancellative or is zero or is cancellative with a zero adjoined (if T_j itself is cancellative take $m_j = 1$).

Now, let A be an archimedean component of S. Then the projection of A in T_j is either contained in C_j or N_j (where $T_j = C_j \cup N_j$, C_j is either empty or is a cancellative semigroup and N_j is either empty or is a nilsemigroup $(1 \leq j \leq r)$).

Hence A is, up to isomorphism, contained in the direct product of finitely many cancellative semigroups and finitely many nilpotent semigroups. Now, if we take $m = \max\{m_i : 1 \leq i \leq r\}$ we certainly have that A^m is contained in the direct product of some cancellative semigroups and some trivial semigroups. Therefore A^m is cancellative. Clearly, from all that was said, there is a constructive way of determining m. □

Thus we are ready to present the main algorithm. We assume there is an algorithm `PrimaryDecompWithPrimes` available, that given an ideal $I \subseteq \mathbb{Z}[X]$ returns a set of pairs (Q, P) where Q are the components of the primary decomposition of I and P is the prime ideal associated to Q (the existence of such algorithm follows from Theorem C.4).

Algorithm 7.25 `CancellativePowerOfArchimedeanComponent`

INPUT: $S = \langle X \mid R \rangle$

OUTPUT: $m > 0$ such that for any archimedean component A of S, A^m is a cancellative semigroup

1: $I \leftarrow$ ideal of $\mathbb{Z}[X]$ generated by $\{X^a - X^b : (a, b) \in R\}$
2: $\{(Q_j, P_j) : 1 \leq j \leq r\} \leftarrow$ `PrimaryDecompWithPrimes`(I)
3: **for all** $j \in \{1, 2, \ldots, r\}$ **do**
4: $\quad m_j \leftarrow$`CancellativeOrNilPowerOfSemigroup`(X, Q_j, P_j)
5: **end for**
6: $m \leftarrow \max\{m_j : 1 \leq j \leq r\}$
7: **return** m

Example 7.26 Let S be the commutative semigroup defined by the presentation

$$\langle a, b \mid a^3b^3 = a^2b^2 \rangle.$$

This semigroup has three archimedean components, each of them containing infinitely many $\mathcal{H}$–classes. The poset of $\mathcal{H}$–classes is represented in Figure 7.2. Now, the ideal $I \subseteq \mathbb{Z}[x, y]$ associated to the congruence generated by $a^2b^2 = a^3b^3$ is the principal ideal of $\mathbb{Z}[x, y]$ generated by $x^3y^3 - x^2y^2$. By using Singular [22], and computing a primary decomposition of I considered as an ideal in $\mathbb{Q}[x, y]$, we obtain the following ideals of $\mathbb{Q}[x, y]$:

$$Q_1 = P_1 = \langle xy - 1 \rangle,$$

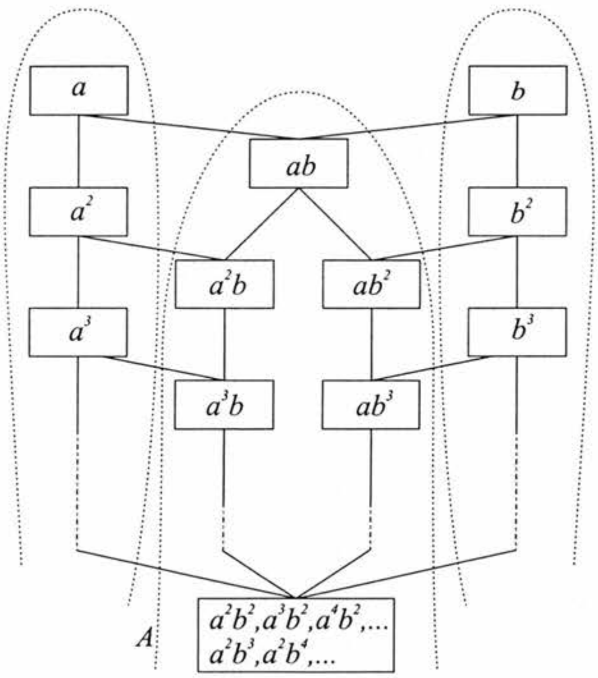

Figure 7.2: The poset of $\mathcal{H}$–classes of the semigroup S from Example 7.26

$$Q_2 = \langle x^2 \rangle, P_2 = \langle x \rangle, \text{ and}$$
$$Q_3 = \langle y^2 \rangle, P_3 = \langle y \rangle.$$

(where Q_1, Q_2, Q_3 are the primary ideals and P_1, P_2, P_3 are their associated prime ideals, respectively). Since the basis of $Q_1, Q_2, Q_3, P_1, P_2, P_3$ are in $\mathbb{Z}[x, y]$ it follows that we can consider them as ideals in $\mathbb{Z}[x, y]$ and, by Section C.4, this is a primary decomposition for I in $\mathbb{Z}[x, y]$. Now, if T_1, T_2, T_3 are the semigroups obtained by taking the quotient of $FC_{\{x,y\}}$ by the associated congruence of Q_1, Q_2, Q_3, respectively, then from the proof of Lemma 7.22 it follows that T_1 is a cancellative semigroup, and T_2^2, T_3^2 are cancellative semigroups with zero adjoined. Therefore, by applying Algorithm 7.25 we conclude that the square power of each of the archimedean components is cancellative. Notice that the bottom archimedean component A is not cancellative, and hence A^2 is a proper cancellative subsemigroup of A.

7.5 An algorithm to decide $\mathcal{H}$–equality

In this section we will prove that it is possible to decide whether two elements of a finitely generated commutative semigroup are $\mathcal{H}$–related. In order to do that we will solve the ideal membership problem for finitely generated commutative semigroups, that is, given an ideal and an element of the semigroup we will show how to decide whether the element belongs to the ideal. Clearly this will give a solution to the former problem.

So, let S be a finitely generated commutative semigroup defined by a presentation $\langle A \mid R \rangle$. Let FC_A be the free commutative semigroup on A. For a subset $B \subseteq S$ we denote by I_B the ideal of S generated by B, and by J_B the subset of FC_A consisting of all words representing elements of I_B.

Lemma 7.27 *With the above notation we have:*

(i) J_B is an ideal of FC_A; and

(ii) J_B is generated by its minimal elements, with respect to the ordering $\leq$ defined on FC_A. In particular, J_B is a finitely generated ideal.

Proof Let $w \in FC_A$ and let $x \in J_B$. Then x represents an element of S that belongs to I_B and thus xw also represents an element of I_B. Therefore $xw \in J_B$ and J_B is an ideal of FC_A. Now, it is clear that the minimal elements (with respect to $\leq$) generate J_B. It then follows by Dickson's Lemma (Theorem 7.1) that such a set of minimal elements of J_B has to be finite and thus J_B is a finitely generated ideal of FC_A. □

We refer to the set of minimal elements of J_B, as the *basis of the ideal* I_B and denote that set by Z_B. Notice that once we have such a basis Z_B, the membership problem for I_B is solved: given an element $s \in S$, with canonical form $s\mu \in FC_A$, $s \in I_B$ if and only if $s\mu \in J_B$, that is, if and only if there exists $b \in Z_B$ such that $b \leq s\mu$. This last condition can be effectively checked since Z_B is a finite set. The following algorithm calculates Z_B:

Algorithm 7.28 [4, Algorithm 1] `IdealBasis`
Input: $S = \langle A \mid R \rangle$; $B \subseteq S$.
Output: $Z_B \subseteq FC_A$, a basis for I_B.

$X \leftarrow$ minimal elements of $\{z\mu : z \in B\}$ with respect to $\leq$
$V_0 \leftarrow \emptyset$
$V_1 \leftarrow X$
$i \leftarrow 1$
while $V_i \neq V_{i-1}$ **do**
    $V_{i+1} \leftarrow V_i$
    for all $x \in V_i$ **do**
        for all $(a, b) \in R$ **do**
            $V_{i+1} \leftarrow$ minimal elements of $V_{i+1} \cup \{(x \vee a)a^{-1}b\}$ with respect to $\leq$
            $V_{i+1} \leftarrow$ minimal elements of $V_{i+1} \cup \{(x \vee b)b^{-1}a\}$ with respect to $\leq$
        end for
    end for
    $i \leftarrow i + 1$
end while
return V_i

Proposition 7.29 [4, Lemma 1] *Let S be a finitely generated commutative semigroup defined by a presentation $\langle A \mid R\rangle$ and let B be a finite subset of S. Then Algorithm 7.28 calculates a basis for I_B*

Proof Let $X = \{z\mu : z \in B\}$ (i.e. X is the set of minimum representatives of the elements of B with respect to the total ordering $\prec$ defined in FC_A) and assume, without loss of generality, that all the elements of X are incomparable with respect to $\leq$.

Let $u \in FC_A$ represent an element of I_B. We have to show that there is some $k > 0$ such that there exists $y \in V_k$ with $y \leq u$. Now, since u represents an element of I_B, there is $x \in X$, $v \in FC_A^1$ and $k > 0$ such that

$$xv \equiv u_1, u_2, \ldots, u_k \equiv u,$$

where u_{i+1} is obtained from u_i $(1 \leq i \leq k-1)$ by application of one relation from R. We will show by induction on k that $y \leq u$ for some $y \in V_k$.

If $k = 1$ then since $xv \equiv u$, we have $x \leq u$ and we can choose $y \equiv x \in V_1$. Now assume that the hypothesis holds for $k-1$. The words u, u_{k-1} and u_k all represent the same element of S, and such an element is in the ideal of S generated by B.

Therefore, by the induction hypothesis, there exists $y \in V_{k-1}$ with $y \leq u_{k-1}$. If u_{k-1} is transformed into u_k by application of a relation $(a = b) \in R$ (without loss of generality we assume that the relation is applied from left to right) this means that $u_k \equiv u_{k-1}a^{-1}b$. Step 9 ensures that there exists $y' \in V_k$ such that $y' \leq (y \vee a)a^{-1}b$. But since $a, y \leq u_{k-1}$, we certainly have $y \vee a \leq u_{k-1}$ whence

$$y' \leq (y \vee a)a^{-1}b \leq u_{k-1}a^{-1}b = u_k$$

as required. □

Algorithm 7.30 `IdealMembership`

INPUT: $S = \langle A \mid R \rangle$; $B \subseteq S$, $s \in S$;

OUTPUT: true if $s \in S$ is in the ideal of S generated by B; `false` otherwise.

1: $Z_B \leftarrow$ `IdealBasis`(S, B)
2: **if** $(\exists z \in Z_B)(z \leq s\mu)$ **then**
3:     **return** `true`
4: **else**
5:     **return** `false`
6: **end if**

Corollary 7.31 *Let S be a finitely generated commutative semigroup defined by a presentation $\langle A \mid R \rangle$ and let B be a finite subset of S. Then it is decidable whether $s \in S$ belongs to I_B, and Algorithm 7.30 is such a decision procedure.*

So far we can decide whether an element s of S belongs to an ideal I_B of S generated by a set $B \subseteq S$. If this is the case then it means that there are elements $z \in B$ and $t \in S^1$ such that $s = tz$. We are now interested in finding those elements t and z. Clearly finding z is part of the algorithm for I_B membership testing. So, in fact, the question is whether we can find the element that multiplied by z gives s, i.e. we are looking for a division algorithm in S. It will suffice to modify Algorithm 7.28, keeping track of how the basis elements of I_B are obtained.

Algorithm 7.32 `IdealBasisWithDivision`

INPUT: $S = \langle A \mid R \rangle$; $B \subseteq S$.

OUTPUT: $W \subseteq FC_A \times X \times FC_A^1$ such that $\{v \in FC_A : \exists (v, x, w) \in W\} = Z_B$, and for each $(v, x, w) \in W$, $v = xw$ in S.

$X \leftarrow$ minimal elements of $\{z\mu : z \in B\}$ with respect to $\leq$
$V_0 \leftarrow \emptyset$
$V_1 \leftarrow \{(x, x, \varepsilon) : x \in X\}$
$i \leftarrow 1$
while $V_i \neq V_{i-1}$ **do**
    $V_{i+1} \leftarrow V_i$
    for all $(v, x, w) \in V_i$ **do**
        for all $(a, b) \in R$ **do**
            $V_{i+1} \leftarrow V_{i+1} \cup \{((v \vee a)a^{-1}b, x, w\, l(v, a))\}$
            $V_{i+1} \leftarrow V_{i+1} \cup \{((v \vee b)b^{-1}a, x, w\, l(v, b))\}$
        end for
    end for
    $V_{i+1} \leftarrow$ minimal elements of V_{i+1} with respect to the first component and $\leq$
    $i \leftarrow i + 1$
end while
return V_i

Proposition 7.33 *Let S be a finitely generated commutative semigroup defined by a presentation $\langle A \mid R \rangle$ and let B be a finite subset of S. Then Algorithm 7.32 calculates a basis Z_B for I_B and for each element $z \in Z_B$ it calculates elements $x \in X$ and $w \in FC_A^1$ such that z and xw represent the same element of S.*

PROOF The fact that Algorithm 7.32 calculates a basis for I_B follows from Proposition 7.29. It remains to prove that if $(v, x, w) \in V_k$ $(k \geq 0)$ then $v = xw$. We will prove the result by induction on k.

The cases $k = 0$ and $k = 1$ hold trivially. Now let $k > 1$ and let $(v, x, w) \in V_k$. Then there exists $(v', x', w') \in V_{k-1}$ and $(a, b) \in R$ (or $(b, a) \in R$) such that $v \equiv (v' \vee a)a^{-1}b$, $x \equiv x'$ and $w \equiv w'\, l(v, a)$. By the induction hypothesis we know that $v' = xw'$, and thus we have

$$v \equiv (v' \vee a)a^{-1}b = v' \vee a \equiv v'\, l(v', a) = xw'\, l(v', a) \equiv xw$$

as required. $\square$

We are now ready to build a division algorithm:

Algorithm 7.34 `Division`

INPUT: $S = \langle A \mid R\rangle$; $B \subseteq S$, $s \in S$;

OUTPUT: $(x, w) \in FC_A \times FC_A^1$ such that $x \in B\mu$ and $s\mu = xw$; `fail` if such element does not exits.

1: $W \leftarrow$`IdealBasisWithDivision`(S, B)
2: **if** $(\exists\ (z, x, w) \in W)(s\mu \geq z)$ **then**
3: $\quad u \leftarrow$ element of FC_A^1 such that $s\mu \equiv zu$
4: $\quad$ **return** (x, wu)
5: **else**
6: $\quad$ **return** `fail`
7: **end if**

Proposition 7.35 *Let S be a finitely generated commutative semigroup defined by a presentation $\langle A \mid R\rangle$, let B be a finite subset of S, and let $s \in S$. Then, if $s \in I_B$, Algorithm 7.34 returns $x \in B\mu$ and $v \in FC_A^1$ such that $s\mu = xv$. If $s \notin I_B$ then it fails.*

PROOF Since $s \in I_B$ if and only if $s\mu \in J_B$, if this is the case then in Z_B we can find an element $z \in FC_A$ such that $s\mu \geq z$ and determine $u \in FC_A^1$ such that $s\mu = zu$. By Proposition 7.33 we know that then there is $(z, x, w) \in W$ such that $z = xw$ in S. But then we have that $s\mu = xwu$ as well. $\square$

Corollary 7.36 *Let S be a finitely generated commutative semigroup defined by a presentation $\langle A \mid R\rangle$ and let B be a finite subset of S. Then if $s \in S$ belongs to I_B then there is an algorithm that returns $b \in B$ and $t \in S^1$ such that $s = bt$.*

By applying our algorithms to principal ideals of S we obtain the aimed result for the $\mathcal{H}$–relation:

Corollary 7.37 *Let S be a finitely generated commutative semigroup. Then $\mathcal{H}$–equality on S is decidable. Further, if $s \leq_{\mathcal{H}} t$ $(s, t \in S)$ than there is an algorithm that returns $v \in S^1$ such that $s = tv$.*

7.6 Looking for a presentation for $S/\mathcal{H}$

We are now interested in finding a presentation for $S/\mathcal{H}$ since we know it is a finitely presented semigroup. If we have such a presentation then testing $\mathcal{H}$–equality is reduced to the word problem in $S/\mathcal{H}$. Now, there is a straightforward naive procedure to get such a presentation starting with a presentation for S. So, for each pair of elements in S check whether they are $\mathcal{H}$–related. If they are $\mathcal{H}$–related add the pair to the set of relations. Eventually we will get $S/\mathcal{H}$. Moreover, since $S/\mathcal{H}$ is finitely presented, a presentation for $S/\mathcal{H}$ will be obtained after a finite number of steps and every step is computable. The problem though is that we do not have a stopping condition for this procedure, that is, we do not know how to decide whether we already have a presentation for $S/\mathcal{H}$. So we have the following

Question 7.38 *Let S be a finitely generated commutative semigroup. Is is decidable whether S is $\mathcal{H}$–trivial? How can we decide it?*

Although we do not have an answer for this question, we will be able to prove some partial results which will be of interest and that will bring some light into this problem. We start by proving that it is actually decidable whether a semigroup has a finite poset of $\mathcal{H}$–classes.

Proposition 7.39 *Let S be a finitely generated commutative semigroup. Then the poset of $\mathcal{H}$–classes $S/\mathcal{H}$ is finite if and only if $|E(S)| = |S/\mathcal{N}|$, that is, if and only if S is a group complete semigroup.*

PROOF For the direct implication suppose that A is an archimedean component of S which contains only finitely many $\mathcal{H}$–classes. We will prove that in this case A contains an idempotent. So let H be a minimal $\mathcal{H}$–class of A. Then $H^2 = H$ and hence H is a group and therefore it contains an idempotent. Therefore A contains an idempotent. The converse implication follows from Proposition 7.15. □

The proof of Proposition 7.39 implies that if an archimedean component has finitely many $\mathcal{H}$–classes then it contains an idempotent. The converse is not true. Indeed we have the following

Example 7.40 Let S be the semigroup from Example 7.26. Then S is defined by the presentation

$$\langle a, b \mid a^3b^3 = a^2b^2 \rangle$$

and it has three archimedean components. Although the bottom archimedean component A contains the element a^2b^2 which is an idempotent, it contains infinitely many $\mathcal{H}$–classes. Indeed, the $\mathcal{H}$–classes of the elements of the form a^ib and ab^i ($i \geq 1$) are all distinct and are all in A. The poset of $\mathcal{H}$–classes of S is represented in Figure 7.2.

From Proposition 7.14 we know that we can effectively calculate the set of idempotents of S. Also, we know that $S/\mathcal{N}$ is finite and by Remark 7.12 and Proposition 7.9 we know how to calculate its size. We thus have:

Corollary 7.41 *It is decidable whether a finitely generated commutative semigroup has a finite poset of $\mathcal{H}$–classes.*

Algorithm 7.42 `HasFinitePosetOfHClasses`
INPUT: $S = \langle A \mid R \rangle$
OUTPUT: `true` if $S/\mathcal{H}$ is finite; `false` otherwise.

$E(S) \leftarrow$ `Idempotents`(S)
if $\mid E(S) \mid = |S/\mathcal{N} \mid$ **then**
  return `true`
else
  return `false`
end if

Now, for a semigroup with a finite poset of $\mathcal{H}$–classes we have:

Proposition 7.43 *Let S be a finitely generated commutative semigroup which has finitely many $\mathcal{H}$–classes. Given a (finite) presentation for S, there is a constructive way of computing a presentation for $S/\mathcal{H}$. In particular it is possible to enumerate the $\mathcal{H}$–classes of S.*

In order to prove this proposition we need the following

Lemma 7.44 *Let S be a finitely generated commutative semigroup. An $\mathcal{H}$-class H of S is trivial if and only if for all $a \in A \cup \mathrm{Stab}(H)$ and for an element $h \in H$ we have $h = ha$. In particular, if H is an $\mathcal{H}$-class containing an idempotent e then H is trivial if and only if, for all generators a of S ($a \in A$), $ea = e$ or ea and e are not $\mathcal{H}$-related.*

PROOF The first statement is obvious and has the second statement as an immediate consequence. □

A semigroup is said to be *group free* if all $\mathcal{H}$-classes which are groups are trivial.

Lemma 7.45 *Let S be a finitely generated commutative semigroup. Then there is an algorithm that given a presentation for S returns a presentation for the group free homomorphic image of S, with the same poset of $\mathcal{H}$-classes as S.*

PROOF By Lemma 7.44 we only have to ensure that for each generator a ($a \in A$) and idempotent e of S, if $ea\mathcal{H}e$ then $ea = e$. Now, by Proposition 7.14 we can enumerate all the idempotents of S, and by Corollary 7.37 we can decide whether ea and e are $\mathcal{H}$-related ($e \in E(S)$, $a \in A$). It is also clear that by adding relations $ea = a$ whenever $ea\mathcal{H}e$ ($e \in E(S)$, $a \in A$) the semigroup obtained has the same poset of $\mathcal{H}$-classes as S. □

PROOF[of Proposition 7.43] From Proposition 7.15 it follows that a group free complete semigroup is $\mathcal{H}$-trivial. Now, by Proposition 7.39 the semigroup $S/\mathcal{H}$ is finite if and only if it is group complete. Hence, the presentation for a group free homomorphic image of S given in Lemma 7.45 is in fact a presentation for $S/\mathcal{H}$. □

Algorithm 7.46 `PresentationForGroupFreeHomormorphicImage`
INPUT: $S = \langle A \mid R \rangle$
OUTPUT: $\langle A \mid R' \rangle$ a presentation for a group free homomorphic image of S with the same poset of $\mathcal{H}$-classes.

1: $E(S) \leftarrow$ `Idempotents`(S)
2: $R' \leftarrow R$

for all $e \in E(S)$ **do**
    for all $a \in A$ **do**
        if $ea\mathcal{H}e$ **then**
            $R' \leftarrow R' \cup \{ea = e\}$
        end if
    end for
end for
return $\langle A \mid R' \rangle$

Example 7.47 Let S be the monoid defined by the presentation

$$\langle x, y, a \mid xy = 1,\ x^9a^2 = a^2,\ x^3a^3 = a^3,\ a^4 = a^3 \rangle. \tag{7.3}$$

This is an infinite semigroup, with two archimedean components, A_1 and A_2. The empty word and the word a^3 both represent idempotents of S. Thus, since the size of the set of idempotents of S is bounded by the size of its archimedean semilattice we conclude that these are the only two idempotents of S. At this point we conclude that $S/\mathcal{H}$ is finite.

Given (7.3), the output of `PresentationForGroupFreeHomormorphicImage` is the presentation

$$\begin{aligned} \langle x, y, a \quad \mid \quad & xy = 1,\ x^9a^2 = a^2,\ x^3a^3 = a^3,\ a^4 = a^3 \\ & x = 1,\ y = 1,\ a^4 = a^3,\ xa^3 = a^3,\ ya^3 = a^3 \rangle \end{aligned}$$

and thus we know that this is a presentation for $S/\mathcal{H}$. By simplifying the above presentation we conclude that $S/\mathcal{H}$ is defined by the (monoid) presentation

$$\langle a \mid a^4 = a^3 \rangle$$

that is, $S/\mathcal{H}$ is the cyclic monoid of order 4. The poset of $\mathcal{H}$–classes of S is represented in Figure 7.3.

The question remains as how to find a presentation for $S/\mathcal{H}$ in the case it is an infinite semigroup.

Example 7.48 Let S be the semigroup defined by the presentation

$$\langle x, a \mid x^4a = a,\ a^4 = a^3 \rangle.$$

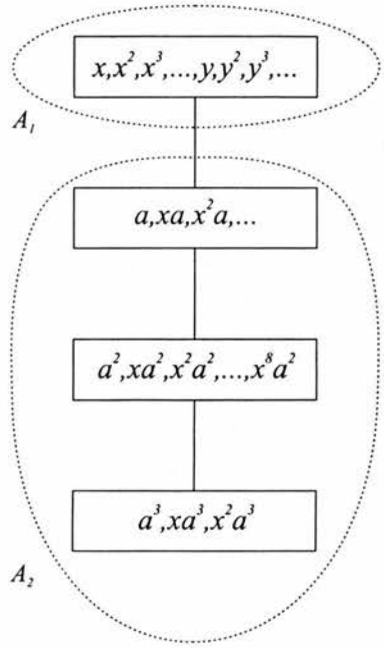

Figure 7.3: The poset of $\mathcal{H}$–classes of the semigroup S from Example 7.47

Clearly S is an infinite semigroup and it contains a unique idempotent: a^3. Therefore S has an infinite poset of $\mathcal{H}$–classes. Now, from Algorithm 7.46 we obtain the following presentation a group free homomorphic image of S:

$$\langle x, a, b \mid x^4 a = a,\ a^4 = a^3,\ xa^3 = a^3 \rangle \tag{7.4}$$

which does not define an $\mathcal{H}$–trivial semigroup since xa and a are $\mathcal{H}$–related but distinct in the semigroup defined by the presentation (7.4). In this case the poset of $\mathcal{H}$–classes is in Figure 7.4.

Example 7.49 Let S be the semigroup from Example 7.26. From Algorithm 7.46 we obtain the following presentation for a group free homomorphic image of S:

$$\langle a, b \mid a^3b^3 = a^2b^2,\ a^3b^2 = a^2b^2,\ a^2b^3 = a^2b^2 \rangle$$

Although S has an infinite poset of $\mathcal{H}$–classes this is indeed a presentation for $S/\mathcal{H}$.

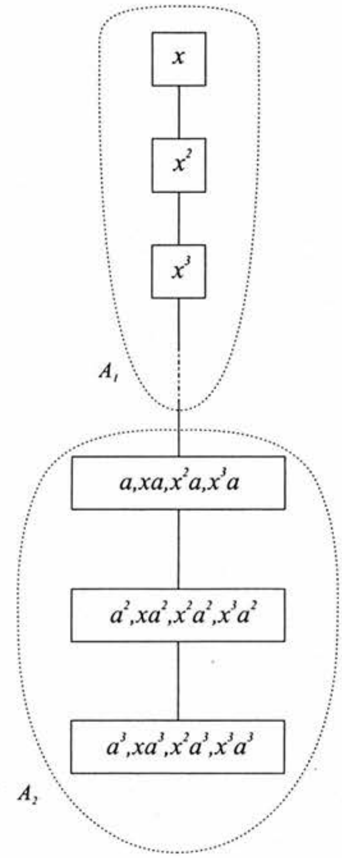

Figure 7.4: The poset of $\mathcal{H}$–classes of the semigroup S from example 7.48

7.7 Schützenberger groups

Schützenberger groups were introduced in [60, 61] and reflect the group–like structure of an $\mathcal{H}$–class of a semigroup. Thus, to each $\mathcal{H}$–class H of a (not necessarily commutative) semigroup S a group $\Gamma(H)$ is associated.

Let S be a monoid (not necessarily commutative) and let H be an $\mathcal{H}$–class of S. We define $\mathrm{Stab}(H)$ to be the set of all elements of S that stabilize H on the right, that is

$$\mathrm{Stab}(H) = \{s \in S : Hs \subseteq H\}.$$

This set is called the *(right) stabilizer* of H in S and it is a submonoid of S. Next we define a relation $\sigma(H)$ on $\mathrm{Stab}(H)$ by

$$(s,t) \in \sigma(H) \iff (\forall\ h \in H)\ hs = ht$$

(i.e. s and t act in the same way on h). This relation is a congruence and it is called the *Schützenberger congruence* of H. The quotient $\Gamma(H) = \mathrm{Stab}(H)/\sigma(H)$ is a group called the *(right) Schützenberger group* of H. Often we identify an element $s \in \mathrm{Stab}(H) \subseteq S$ with the element it represents in $\Gamma(H)$.

We prove some elementary results about Schützenberger groups.

Lemma 7.50 *Let $\Gamma(H)$ be a Schützenberger group as defined above, and let $h_0 \in H$ be an arbitrary element of H. Then*

(i) *$\Gamma(H)$ acts regularly on H; in particular $|\Gamma(H)| = |H|$;*

(ii) *if H is a maximal subgroup of S then $\Gamma(H) \cong H$;*

(iii) $\mathrm{Stab}(H) = \{s \in S : h_0 s \mathcal{H} h_0\}$*; and*

(iv) $\sigma(H) = \{(s,t) \in \mathrm{Stab}(H) \times \mathrm{Stab}(H) : h_0 s = h_0 t\}$.

PROOF To prove (i) notice that there is an action of $\Gamma(H)$ on H given by

$$(x, h) \mapsto x \cdot h = xh \tag{7.5}$$

($x \in \Gamma(H)$, $h \in H$ and xh denotes multiplication in S). If $x, y \in \Gamma(H)$ are such that there exists $h \in H$ such that $xh = yh$ then by the definition of $\sigma(H)$ it follows that x and y are equal in $\Gamma(H)$. Thus the action (7.5) is a regular action.

Now suppose that H is a maximal subgroup of S. Then $H \subseteq \mathrm{Stab}(H)$ and if $h_1, h_2, h \in H$ are such that $h_1 h = h_2 h$ then $h_1 = h_2$ since H is a group. It therefore follows that H and $\Gamma(H)$ are isomorphic and this proves (ii).

For (iii), if $s \in \mathrm{Stab}(H)$ then $sh_0 \in H$ and hence $sh_0 \mathcal{H} h_0$; conversely notice that if $s \in S$ is such that $h_0 s \mathcal{H} h_0$ then $h_0 s \in \mathrm{Stab}(H)$.

As for (iv) let $h \in H$ and let $u \in S$ be such that $h = uh_0$. Then if $s, t \in \mathrm{Stab}(H)$ are such that $h_0 s = h_0 t$ then

$$hs = uh_0 s = uh_0 t = ht$$

and the result follows. $\square$

Similarly we can define the *left Schützenberger group* of H, which is denoted by $\Gamma_l(H)$. In fact $\Gamma(H) \cong \Gamma_l(H)$. A detailed introduction to Schützenberger groups can be found in [32].

Remark 7.51 Clearly, if S is a commutative semigroup, then the left and the right Schützenberger groups of an $\mathcal{H}$–class coincide.

For an arbitrary semigroup S we define the Schützenberger group of an $\mathcal{H}$–class H of S to be the Schützenberger group of H considered as an $\mathcal{H}$–class of S^1 (where S^1 denotes S if S contains an identity, or S with an identity adjoined otherwise). We will therefore, without loss of generality, study Schützenberger groups of monoids, and the results will follow for semigroups.

7.7.1 A presentation

Let S be a finitely generated commutative monoid defined by a presentation $\langle A \mid R\rangle_{\mathcal{C}}$. Let H be an $\mathcal{H}$–class H of S and fix a word $h \in A^*$ representing an element of H. Denote by A_ω the set $A \cap \mathrm{Stab}(H)$ and notice that A_ω is a generating set for $\Gamma(H)$. We are now going to exhibit a presentation for $\Gamma(H)$. In order to do this we will work in A^*, rather then FC_A^1, since we will apply a result from [58] which obtains a (non commutative) monoid presentation for $\Gamma(H)$. (Therefore, in this section, we distinguish between commutative semigroup presentations and (non commutative) semigroup presentations.)

Since $\langle A \mid R\rangle_{\mathcal{C}}$ is a commutative monoid presentation for S, S is defined by the semigroup presentation

$$\langle A \mid R,\ ab = ba\ (a, b \in A)\rangle;$$

we denote by $\overline{R}$ the set $R \cup \{ab = ba : a, b \in A\}$ so that S is defined by the semigroup presentation $\langle A \mid \overline{R}\rangle$. Let $\{H_i : i \in I\}$ be the set $\{H' \in S/\mathcal{H} : (\exists \in S)(sH' \subseteq H)\}$. There is an action $(s, i) \mapsto s * i$ of S on the set $I \cup \{0\}$ (where $0 \notin I$) given by

$$s * i = \begin{cases} j & \text{if } i, j \in I \text{ and } sH_i \subseteq H_j\text{; and} \\ 0 & \text{otherwise.} \end{cases} \tag{7.6}$$

(Clearly, this action defines an action from A^* on $I \cup \{0\}$ which we shall denote in the same way.) For each $i \in I$ we choose a word $h_i \in A^*$ representing an element of H_i and without loss of generality we assume that I has two distinguished elements, 1 and ω, such that

$$1_S \in H_1,\ h_1 \equiv \varepsilon,\ H = H_\omega,\ h_\omega \equiv h.$$

For each $a \in A$ and $i \in I$ such that $a * i \neq 0$, we have that $ah_i \in aH_i \subseteq H_{a*i}$. Therefore we can choose words $(a,i)\tau \in A^*$ such that the relations

$$ah_i = h_{a*i}(a,i)\tau \ (a \in A,\ i \in I)$$

hold in S. We fix these words as follows: we let $(a,\omega)\tau \equiv a$ $(a \in A_\omega)$ and for $(a,i)\tau$ we take a word in A^* representing the commutative word given as output of Algorithm 7.34 with input $(S, \{h_i\}, a)$ $(a \in A,\ i \in I,\ i \neq \omega,\ a * i \neq 0)$. We then extend the mapping $(a,i) \mapsto (a,i)\tau$ to a mapping

$$\tau : \{(w,i) \in A^* \times I : w * i \neq 0\} \to A^*$$

$$(\varepsilon, i)\tau = \varepsilon,\ (wa,i)\tau = (w, a*i)\tau\,(a,i)\tau.$$

With the above notation [58, Theorem 4.2] reads:

Theorem 7.52 [58, Theorem 4.2] *With the above notation, the group $\Gamma(H)$ is defined by the (non commutative) presentation*

$$\begin{aligned}\langle A_\omega \ \mid\ & u = v\ ((u=v) \in \overline{R},\ u,v \in \mathrm{Stab}(H)), & (7.7)\\ & (u,i)\tau = (v,i)\tau\ ((u=v) \in \overline{R},\ i \in I,\ u*i \in I), & (7.8)\\ & (a,\omega)\tau = a\ (a \in \mathrm{Stab}(H)), & (7.9)\\ & (h,1)\tau = 1\rangle. & (7.10)\end{aligned}$$

In our case however we are able to obtain a simpler presentation for $\Gamma(H)$.

Proposition 7.53 *With the above notation, the group $\Gamma(H)$ is defined by the (non commutative) presentation*

$$\begin{aligned}\langle A_\omega \ \mid\ & u = v\ ((u=v) \in \overline{R},\ u,v \in A_\omega^*), & (7.11)\\ & (u,1)\tau = (v,1)\tau\ ((u=v) \in R,\ u*1 \in I), & (7.12)\\ & (ab,i)\tau = (ba,i)\tau\ (a,b \in A_\omega) & (7.13)\\ & (h,1)\tau = 1\rangle. & (7.14)\end{aligned}$$

Proof We will prove that this presentation and the one given in Theorem 7.52 are equivalent. (We treat them as group presentations.) Start by noticing that by the way we chose $(a,\omega)\tau$ it follows that relations (7.9) are trivial. Also notice that relations $(w_1w_2, 1)\tau = (w_2w_1, 1)\tau$ $(w_1, w_2 \in A^*,\ w_1w_2 * 1 \in I)$ are a consequence of the set of relations (7.13). Next we will prove that the set of relations (7.8) is a consequence of the set of relations (7.12) and (7.13). Indeed, for any $i \in I$, we have $i = h_i * 1$ since $h_iH_1 \subseteq H_i$. Now let $(u = v) \in R$, $i \in I$ and $u * i \in I$. Then

$$\begin{array}{rcll}
(u,i)\tau & \equiv & (u, h_i * 1)\tau & (i = h_i * 1) \\
& = & (u, h_i * 1)\tau\, (h_i, 1)\tau\, ((h_i, 1)\tau)^{-1} & \\
& \equiv & (uh_i, 1)\tau\, ((h_i, 1)\tau)^{-1} & (\text{definition of } \tau) \\
& = & (h_iu, 1)\tau\, ((h_i, 1)\tau)^{-1} & ((ab, i)\tau = (ba, i)\tau\, (a, b \in A_\omega)) \\
& \equiv & (h_i, u * i)\tau\, (u, 1)\tau\, ((h_i, 1)\tau)^{-1} & (\text{definition of } \tau) \\
& = & (h_i, v * i)\tau\, (v, 1)\tau\, ((h_i, 1)\tau)^{-1} & (u * 1 = v * 1,\ (u, 1)\tau = (v, 1)\tau) \\
& \equiv & (h_iv, 1)\tau\, ((h_i, 1)\tau)^{-1} = (v, i)\tau &
\end{array}$$

which completes the proof. □

Notice that the presentations from both Theorem 7.52 and Proposition 7.53 are not finite in general. However $\Gamma(H)$ is a finitely generated group (A_ω is a finite generating set for it) and since $\Gamma(H)$ is also commutative it follows that $\Gamma(H)$ is a finitely presented. Therefore, by Proposition 2.2, it follows that it is presented by a presentation with generators A_ω and relations (7.11), (7.12), (7.14) and a finite subset of the relations (7.13).

Question 7.54 *With the above notation, is the group $\Gamma(H)$ defined by the finite (non commutative) presentation*

$$\langle A_\omega \mid (7.11), (7.12), (7.14), \tag{7.15}$$

$$(ab, 1)\tau = (ba, 1)\tau\ (a, b \in A_\omega)\rangle? \tag{7.16}$$

In all the examples studied in this chapter, the set of relations (7.13) is indeed a consequence of the set of relations (7.16).

Remark 7.55 If Question 7.54 has a positive answer not only we have a finite presentation for the Schützenberger group of an $\mathcal{H}$–class of a commutative semigroup, as well as this presentation is effectively computable. To see this let S be a finitely generated commutative semigroup given by a presentation, let s be an element of S, let H_s be the $\mathcal{H}$-class of s and consider the Schützenberger group

$\Gamma(H_s)$. Let $\langle A \mid R\rangle_{\mathcal{C}}$ be a finite presentation for S. Then the presentation given in Question 7.54 is clearly a finite presentation. Also, it follows from the fact that we can check membership to $\mathcal{H}$–classes, that we can find the set A_ω.

We now just have to prove that we can determine the set of relations (7.12), (7.14), and (7.16). Notice that if $w \in A_\omega^*$ is such that $w \equiv a_1a_2\cdots a_k$, where $a_j * 1 \in I$ $(1 \leq j \leq k)$, then

$$\begin{aligned}(w,1)\tau &\equiv (a_1a_2a_3\cdots a_k,1)\tau\\ &\equiv (a_1,a_2a_3\cdots a_k*1)\tau\,(a_2,a_3\cdots a_k*1)\tau\cdots(a_{k-1},a_k*1)\tau\,(a_k,1)\tau.\end{aligned}$$

So, to write down any of the relations (7.12), (7.14) or (7.16) we only have to calculate finitely many $(a,i)\tau$ for some $a \in A$, $i \in I$ and $a * i \neq 0$. This can be done as described in the definition of τ. This completes the proof that the presentation from Question 7.54 can be effectively determined.

Finally, by considering the words in the presentation obtained as commutative words, we obtain a commutative presentation for $\Gamma(H)$.

Example 7.56 Let S be the semigroup from Example 7.48. Notice that for all $i \geq 1$, $\mathrm{Stab}(H_{x^i}) = \emptyset$ and thus the $\mathcal{H}$–class of x^i $(i \geq 1)$ has trivial Schützenberger group. As for the $\mathcal{H}$–classes of the powers of a, $\mathrm{Stab}(H_{a^i}) = \{x^j : j \geq 1\}$, $(i \geq 1)$ that is, the Schützenberger group of H_{a^i} $(i \geq 1)$ is generated by x. It is then easy to see that it is defined by the presentation

$$\langle x \mid x^4 = 1\rangle,$$

i.e C_4, the cyclic group of order 4.

7.7.2 Finitely many Schützenberger groups

Although in general a finitely generated commutative semigroup may have infinitely many $\mathcal{H}$–classes, we will see in this section that they will give rise to only finitely many non-isomorphic Schützenberger groups. That is, a finitely generated commutative semigroup has only finitely many 'essentially distinct' $\mathcal{H}$–classes.

We will start by proving the following

Lemma 7.57 *Let S and T be finitely generated commutative semigroups and let $\phi : S \hookrightarrow T$ be an embedding. Let $a \in S$, let H_a be the $\mathcal{H}$-class of a in S and let $\Gamma(H_a)$ be the Schützenberger group of H_a. Then there exists an embedding $\psi : \Gamma(H_a) \hookrightarrow \Gamma(H_{a\phi})$ given by $x \mapsto x\phi$ ($x \in \Gamma(H_a)$), where $H_{a\phi}$ and $\Gamma(H_{a\phi})$ denote the $\mathcal{H}$-class of $a\phi$ in T and its Schützenberger group, respectively.*

PROOF We have to show that ψ is well defined, it is a homomorphism and that it is one-one.

To see that is well defined we first prove that that $(\mathrm{Stab}(H_a))\phi \subseteq \mathrm{Stab}(H_{a\phi})$. So, let $y \in (\mathrm{Stab}(H_a))\phi$. Then there exists $x \in \mathrm{Stab}(H_a)$ such that $y = x\phi$. In particular $xH_a \subseteq H_a$. Thus

$$y\,(H_a\phi) = (x\phi)(H_a\phi) = (xH_a)\phi \subseteq H_a\phi \subseteq H_{a\phi}$$

and also

$$y\,(H_a\phi) \subseteq yH_{a\phi}.$$

Now, $yH_{a\phi}$ is contained in some $\mathcal{H}$-class of T, and thus, since $y\,(H_a\phi) \subseteq H_{a\phi}$, we conclude that

$$yH_{a\phi} \subseteq H_{a\phi}.$$

Next we prove that if $x_1, x_2 \in \mathrm{Stab}(H_a)$ are such that $x_1 = x_2$ in $\Gamma(H_a)$ then $x_1\psi = x_2\psi$ in $\Gamma(H_{a\phi})$. Indeed, if $x_1 = x_2$ in $\Gamma(H_a)$ then $ax_1 = ax_2$. Thus

$$(a\phi)(x_1\psi) = (a\phi)(x_1\phi) = (ax_1)\phi = (ax_2)\phi = (a\phi)(x_2\psi)$$

which, by Lemma 7.50, implies that $x_1\psi = x_2\psi$ in $\Gamma(H_{a\phi})$. Therefore ψ is well defined.

The fact that ψ is a homomorphism is an immediate consequence from ϕ being a homomorphism.

Now, to see that ψ is one-one let $x_1, x_2 \in \mathrm{Stab}(H_a)$ be such that $x_1\psi = x_2\psi$ in $\Gamma(H_{a\phi})$. We have to show that then $x_1 = x_2$ in $\Gamma(H_a)$. The fact that $x_1\psi = x_2\psi$ implies that for any $h \in H_{a\phi}$ we have

$$h(x_1\psi) = h(x_2\psi).$$

In particular, if $h \in (H_a)\phi$ and $h' \in H_a$ is such that $h = h'\phi$, then

$$(h'\phi)(x_1\phi) = (h'\phi)(x_2\phi)$$

and thus

$$(h'x_1)\phi = (h'x_2)\phi$$

and it follows that $h'x_1 = h'x_2$ since ϕ is an embedding. Therefore $x_1 = x_2$ in $\Gamma(H_a)$ and the result is proved. □

Theorem 7.58 *Let S be a finitely generated commutative semigroup. Then S has only finitely many non-isomorphic Schützenberger groups.*

PROOF Let S be a finitely generated commutative semigroup. From Proposition 7.16 it follows that there exists a group complete semigroup T and an embedding $\phi : S \hookrightarrow T$. Now, from Proposition 7.39 we know that T has only finitely many $\mathcal{H}$–classes, and therefore it has only finitely many Schützenberger groups.

By Lemma 7.57, any Schützenberger group of S can be embedded into a Schützenberger group of T, that is the Schützenberger groups of S are (isomorphic to) subgroups of the Schützenberger groups of T. Now, since the Schützenberger groups of T are finitely generated abelian groups, they only have finitely many non-isomorphic subgroups, and hence the result follows. □

Remark 7.59 Notice that, as an easy consequence of Dickson's Lemma, it follows that even if a finitely generated commutative semigroup S has infinitely many $\mathcal{H}$–classes, there can be no infinite sequence of pairwise unrelated ones (with respect to the ordering defined on $S/\mathcal{H}$). This fact can be used to provide an alternative proof for Theorem 7.58. In fact, given an infinite sequence of $\mathcal{H}$–classes of S, with non isomorphic Schützenberger groups, an infinite subsequence of it must form a descending chain. Moreover, there will be a descending chain of $\mathcal{H}$–classes

$$H_1 > H_2 > \cdots$$

all of which have the same stabilizer, since there are only finitely many distinct stabilizers. But then there is a sequence of onto homomorphisms on the respective Schützenberger groups

$$\Gamma(H_1) \to \Gamma(H_2) \to \cdots$$

and if FC is the free commutative group on the same number of generators as $\Gamma(H_1)$ we have

$$FC \twoheadrightarrow \Gamma(H_1) \twoheadrightarrow \Gamma(H_2) \twoheadrightarrow \cdots,$$

an infinite sequence of quotients of the free commutative semigroup. Hence there exist normal subgroups of FC, $N_1, N_2, \ldots$, such that $FC/N_i \cong \Gamma(H_i)$ $(i \geq 1)$. This defines an infinite sequence $N_1 \lhd N_2 \lhd \cdots$ of normal subgroups of FC, which must terminate. Hence, only finitely many of the $\Gamma(H_i)$ $(i \geq 1)$ are distinct.

7.7.3 Some remarks on finding Schützenberger groups

If the poset of $\mathcal{H}$–classes of a finitely generated commutative semigroups is finite we have seen in Section 7.6 that we can enumerate the $\mathcal{H}$–classes and hence we can enumerate the Schützenberger groups of that semigroup. Now, we have proved in Section 7.7.2 that even if the poset of $\mathcal{H}$–classes is infinite, there are only finitely many distinct such groups. Hence it makes sense to attempt to enumerate all the finitely many Schützenberger groups in general. Notice that this question might be related to the one of finding a presentation for $S/\mathcal{H}$. Indeed, if we find (finitely many) $\mathcal{H}$–classes that give rise to all the Schützenberger groups, we might be able to localize a finite family of $\mathcal{H}$–classes that would suffice to 'kill' to obtain $S/\mathcal{H}$. However, it is not clear that this is the case, nor have we succeeded in the task of finding such classes. Nevertheless we have

Lemma 7.60 *Let S be a finitely generated commutative monoid, let A be an archimedean component and let H be an $\mathcal{H}$-class of S, contained in A. Then if $m > 0$ and $A^m \cap H \neq \emptyset$ then $H \subseteq A^m$.*

PROOF Suppose that $A^m \cap H \neq \emptyset$ and let $a \in A^m \cap H$. Let $h \in H$. Then $h = ax$ for some $x \in S$. Recall that A is a subsemigroup of S and A^m is an ideal of A. Also $x \in B$ for some archimedean component B above A. Hence, if $a \in A^m$, that is $a = a_1 a_2 \ldots a_m$ $(a_i \in A)$, then $xa_1 \in A$ and therefore $h = xa = xa_1 a_2 \ldots a_m \in A^m$. □

Theorem 7.61 *Let S be a finitely generated commutative monoid and let A be an archimedean component of S which does not contain an idempotent. Let $m > 0$ be such that $C = A^m$ is cancellative. Then if H_1, H_2 are $\mathcal{H}$-classes of S contained in C then $\Gamma(H_1) \cong \Gamma(H_2)$.*

PROOF Let H_1 and H_2 be two $\mathcal{H}$–classes contained in C and suppose that $\Gamma(H_1) \not\cong \Gamma(H_2)$. Let $h_1 \in H_1$ and $h_2 \in H_2$. Then there exists $w \in S$ such that $h_1w = h_1$, $h_2w \neq h_2$. On the other hand, since both $h_1, h_2 \in A$, there exist $m, n > 0$ such that $h_1^m = h_2q$, $h_2^n = h_1r$ for some $q, r \in S$. Then $h_2qw = h_1^mw = h_1^m = h_2q$ and hence $h_2wq = h_2q$ and also $(h_2w)(qh_2) = h_2(qh_2)$. Notice that $h_2w, qh_2, h_2 \in C$ and hence by cancellativity it follows that $h_2w = h_2$ which is a contradiction. $\square$

From Section 7.4 we know how to calculate an integer such that for any archimedean component A of S, A^m is cancellative. Once we have such a power we can determine the unique Schützenberger group of all $\mathcal{H}$–classes inside A^m. Notice however that outside the cancellative power infinitely many $\mathcal{H}$–classes may be found. Indeed, observing Example 7.26 once again we can see that if A is the bottom archimedean component, then A^2 is cancellative. But $A - A^2$, i.e. the complement of A^2 in A, contains infinitely many $\mathcal{H}$–classes. In that example, though, all such $\mathcal{H}$–classes have trivial Schützenberger groups. We shall see in the next example that this is not the case in general.

Example 7.62 Let S be the monoid defined by the presentation

$$\begin{aligned}\langle a, b, x, y \mid\ & xy = 1,\ a^2bx^8 = a^2b,\ a^3bx^4 = a^3b,\ a^4bx^2 = a^4b, \\ & ab^2x^3 = ab^2,\ a^3b^2 = a^2b^2,\ a^2b^3 = a^2b^2\rangle.\end{aligned}$$

Then S is a finitely generated commutative monoid which has four archimedean components. It is represented in Figure 7.5.

The top archimedean component A_1, which contains x and y, consists only of one $\mathcal{H}$–class which is in fact a group isomorphic to $\mathbb{Z}$. The archimedean components A_2 and A_3 of a and b, respectively, have infinitely many $\mathcal{H}$–classes, all of which have Schützenberger group isomorphic to $\mathbb{Z}$. Finally, the bottom archimedean component A_4 contains an idempotent, namely a^2b^2 which is a zero of the semigroup. The $\mathcal{H}$–class of the zero is A_4^2, the cancellative power of A_4. Nevertheless, A_4 contains infinitely many $\mathcal{H}$–classes, which in fact can be found outside its cancellative power. The top such $\mathcal{H}$–class is the class of ab which has Schützenberger group $\mathbb{Z}$ as well. The $\mathcal{H}$–classes of a^2b, a^3b, a^ib ($i \geq 4$) have Schützenberger groups C_8 (i.e. the cyclic group of order 8), C_4 and C_2, respectively. The $\mathcal{H}$–classes of the elements ab^i ($i \geq 2$) all have C_3 as their Schützenberger group.

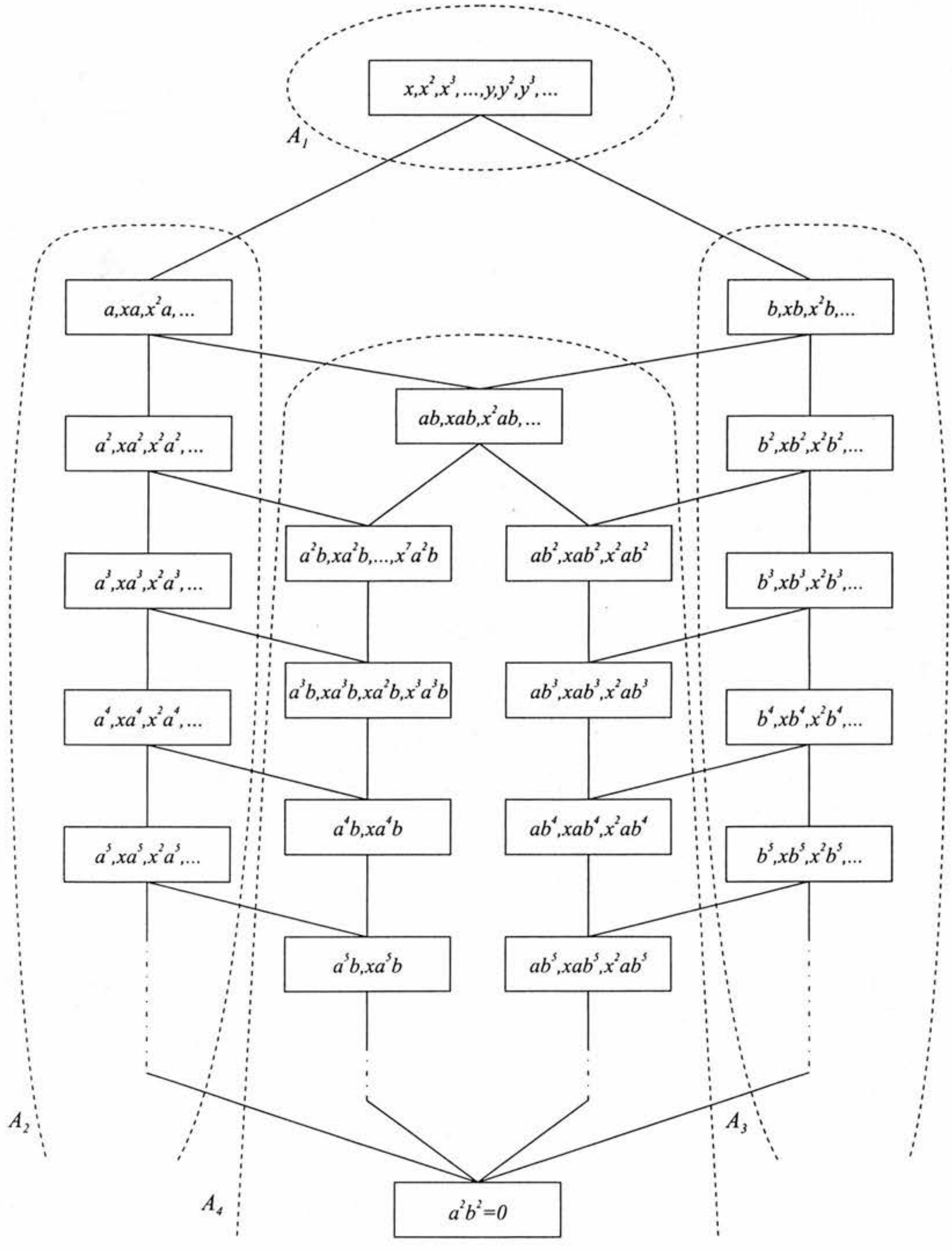

Figure 7.5: The poset of $\mathcal{H}$–classes of the monoid S from Example 7.62

It is now not difficult to construct another example where the bottom archimedean component does not have an idempotent.

Example 7.63 Let S be the monoid defined by the presentation

$$\begin{aligned}\langle a, b, x, y \mid\ & xy = 1,\\ & x^8a^2b = a^2b,\ x^4a^3b = a^3b,\ x^2a^4b = a^4b,\\ & x^3ab^2 = ab^2,\ a^3b^3 = a^2b^2\rangle.\end{aligned}$$

The structure of S is similar to the one of the monoid in Example 7.62, although this time there are infinitely many $\mathcal{H}$–classes inside the cancellative power of the bottom archimedean component. All these classes have trivial Schützenberger group.

7.8 The Grillet representation

Grillet [25] showed how to decompose a commutative semigroup in terms of the Schützenberger groups of its $\mathcal{H}$–classes. Computationally this is an important result, since, provided we know how to compute such decomposition, we will be able to perform our computations regarding S in its Schützenberger groups, which being abelian groups are 'easy' to compute in. In this section we study the computability of such a decomposition. In fact, provided we can compute the finitely many Schützenberger groups of a semigroup, we can compute this decomposition.

So, let S be a finitely generated commutative semigroup and let $\{H_i : i \in I\}$ be the set of $\mathcal{H}$–classes of S. We denote by Γ_i the Schützenberger group of H_i $(i \in I)$. On the set I we define a (commutative) multiplication by

$$ij = k \text{ if } H_iH_j \subseteq H_k$$

and a partial order $\succcurlyeq$ by

$$i \succcurlyeq j \text{ if and only if } H_i \geq H_j$$

$(i, j, k \in I)$. There is an action $(s, i) \mapsto s * i$ of S on I by

$$s * i = j \text{ if } i, j \in I \text{ and } sH_i \subseteq H_j.$$

Lemma 7.64 *With the above notation let H_i, H_j $(i, j \in I)$ be $\mathcal{H}$–classes of S such that $i \succcurlyeq j$. Then* $\mathrm{Stab}(H_i) \subseteq \mathrm{Stab}(H_j)$ *and this inclusion induces a homomorphism $\gamma_j^i : \Gamma_i \to \Gamma_j$. Furthermore, $\gamma_j^i \gamma_k^j = \gamma_k^i$ for any $i, j, k \in I$ such that $i \succcurlyeq j \succcurlyeq k$.*

Proof Notice that $s \in \mathrm{Stab}(H_i)$ $(i \in I)$ if and only if $s * i = i$. Now, if $i, j \in I$ are such that $i \succcurlyeq j$ then this means that there is $k \in I$ such that $j = ik$. So let $s \in \mathrm{Stab}(H_i)$. Then

$$s * j = s * (ik) = (s * i)k = ik = j.$$

Thus $\mathrm{Stab}(H_i) \subseteq \mathrm{Stab}(H_j)$.

Now consider the mapping $\gamma_j^i : \Gamma_i \to \Gamma_j$ defined by $s \mapsto s$ $(s \in \mathrm{Stab}(H_i))$. We have to prove that if $s, t \in \mathrm{Stab}(H_i)$ and $s = t$ in Γ_i then $s = t$ in Γ_j. Now, $s = t$ in Γ_i means that for any $h \in H_i$

$$hs = ht \tag{7.17}$$

in S. Once again, since $i \succcurlyeq j$, there exists $k \in I$ such that $j = ik$ and hence if $h \in H_i$ and $h' \in H_k$ then $hh' \in H_j$. Thus

$$(hh')s = h'(hs) \overset{\text{by (7.17)}}{=} h'(ht) = (hh')t$$

which means that $s = t$ in Γ_j, as required. Therefore γ_j^i is well defined and it is obviously a homomorphism.

As for the last claim, notice that clearly $\gamma_j^i \gamma_k^j : \Gamma_i \to \Gamma_k$ and if $s \in \mathrm{Stab}(H_i)$ then

$$s\gamma_j^i \gamma_k^j = s\gamma_k^i = s$$

since $\mathrm{Stab}(H_i) \subseteq \mathrm{Stab}(H_j) \subseteq \mathrm{Stab}(H_k)$. □

It follows from Lemma 7.64 that if $s \in \Gamma_i$ and $i \succcurlyeq j$ then s can be considered as an element of Γ_j as well.

Now, in each $\mathcal{H}$–class H_i we fix an arbitrary element $h_i \in H_i$ $(i \in I)$. These elements have the following property:

Lemma 7.65 *With the above notation $h_i h_j \mathcal{H} h_{ij}$ for any $i, j \in I$.*

PROOF From the definition of multiplication in I, $H_iH_j \subseteq H_{ij}$ and thus $h_ih_j \in H_{ij}$ and the result follows. $\square$

Since the action of Γ_i on H_i ($i \in I$) is regular for each $i, j \in I$ there exists a unique $\alpha_{i,j} \in \Gamma_{ij}$ such that

$$h_ih_j = \alpha_{i,j}h_{ij}.$$

(Recall that we denote the action of Γ_i on H_i ($i \in I$) by multiplication.) Moreover, for given $i, j \in I$ we can calculate $\alpha_{i,j}$ by the procedure described in Corollary 7.36. The next lemma will be used later.

Lemma 7.66 *With the above notation, for any $i, j, k \in I$ we have*

(i) $\alpha_{j,i} = \alpha_{i,j}$; *and*

(ii) $\alpha_{i,j}\alpha_{ij,k} = \alpha_{i,jk}\alpha_{j,k}$.

PROOF By definition $\alpha_{i,j} \in \Gamma_{ij} = \Gamma_{ji}$. Also $h_ih_j = h_jh_i$ and $h_{ij} = h_{ji}$. Therefore $\alpha_{j,i} = \alpha_{i,j}$.

For the second part notice that both left and right hand side are elements of Γ_{ijk} and thus it is enough to show they both act in the same way on h_{ijk}. So, for the left hand side we have

$$\begin{aligned}
(\alpha_{i,j}\alpha_{ij,k})\, h_{ijk} &= \alpha_{i,j}\,(\alpha_{ij,k}h_{ijk}) \\
&= \alpha_{i,j}\,(h_{ij}h_k) \\
&= (\alpha_{i,j}h_{ij})\, h_k \\
&= h_ih_jh_k
\end{aligned}$$

and for the right hand side

$$\begin{aligned}
(\alpha_{i,jk}\alpha_{j,k})\, h_{ijk} &= \alpha_{j,k}\,(\alpha_{i,jk}h_{ijk}) \\
&= \alpha_{j,k}\,(h_ih_{jk}) \\
&= h_i\,(\alpha_{j,k}h_jk) \\
&= h_ih_jh_k
\end{aligned}$$

as required. $\square$

Proposition 7.67 *With the above notation, let $E[S]$ be the set*

$$\{(x,i) : i \in I, x \in \Gamma_i\}$$

and define a multiplication on it by

$$(x,i)(y,j) = (x\alpha_{i,j}y, ij).$$

Then $E[S]$ is a commutative semigroup isomorphic to S. Moreover, any commutative semigroup S can be represented in this way and such a representation can be effectively computed whenever all the Schützenberger groups of S are known.

PROOF We will start by proving that $E[S]$ is a semigroup. So let $(x,i),(y,j),(z,k) \in E[S]$. Then $x\alpha_{i,j}y \in \Gamma_{ij}$ and thus $(x,i)(y,j) \in E[S]$. Also

$$\begin{aligned}
[(x,i)(y,j)](z,k) &= (x\sigma_{i,j}y, ij)(z,k) \\
&= (x\sigma_{i,j}y\sigma_{ij,k}z, ijk) \\
&= (xyz\sigma_{i,j}\sigma_{ij,k}, ijk) && \text{(by commutativity)} \\
&= (xyz\sigma_{i,jk}\sigma_{j,k}, ijk) && \text{(by Lemma 7.66)} \\
&= (x\sigma_{i,jk}y\sigma_{j,k}z, ijk) && \text{(by commutativity)} \\
&= (x,i)(y\sigma_{j,k}z, jk) \\
&= (x,i)[(y,j)(z,k)].
\end{aligned}$$

and

$$\begin{aligned}
(x,i)(y,j) &= (x\sigma_{i,j}y, ij) \\
&= (x\sigma_{j,i}y, ij) && \text{(by Lemma 7.66)} \\
&= (y\sigma_{j,i}x, ji) && \text{(by commutativity)} \\
&= (y,j)(x,i).
\end{aligned}$$

Therefore $E[S]$ is a commutative semigroup.

Now, define a mapping $\psi : E[S] \to S$ by

$$(x,i) \mapsto xh_i.$$

This is a well-defined mapping. We will show that ψ is an isomorphism. So let

$(x,i),(y,j) \in E[S]$. Then

$$\begin{aligned}
[(x,i)(y,j)]\psi &= (x\alpha_{ij}y, ij)\psi \\
&= (x\alpha_{ij}y)h_{ij} \\
&= xy(\alpha_{i,j}h_{ij}) && \text{(by commutativity)} \\
&= xyh_ih_j && \text{(by the definition of } \alpha_{i,j}\text{)} \\
&= (xh_i)\,(yh_j) && \text{(by commutativity)} \\
&= (x,i)\psi\,(y,j)\psi
\end{aligned}$$

and therefore ψ is a homomorphism. Now, suppose that $(x,i)\psi = (y,j)\psi$. Then this means that $i = j$ and $xh_i = yh_i$. But then $x, y \in \mathrm{Stab}(H_i)$ and, since they act in the same way on i, $x = y$ in Γ_i and thus $(x,i) = (y,i)$. Now, if $s \in S$ let H_i be its $\mathcal{H}$–class. Then, since Γ_i acts regularly on H_i, there exists $x \in \Gamma_i$ such that $xh_i = y$. Thus $(x,i)\psi = s$. Therefore ψ is an isomorphism.

Clearly, the isomorphism ψ can be effectively computed and so can $E(S)$.

The fact that any commutative semigroup can be represented in this way follows from the way we have constructed $E(S)$. □

We call the semigroup $E[S]$, defined as in Proposition 7.67, the *Grillet representation* of S.

Remark 7.68 In the language of categories, the semigroup $E[S]$ is an abelian group coextension of the semigroup $S/\mathcal{H}$ by the Schützenberger functor of S. For details see [27].

Appendix A

Implementation of algorithms for direct products in GAP

Section 3.5 describes algorithms that, given a finite semigroup S, decide whether it preserves finite generation and finite presentability in direct products of semigroups. These algorithms have been implemented in GAP and used to compute some of the examples that appear in Section 3.4. In here we reproduce the code implemented in GAP (v.4.2).

A.1 Some auxiliary functions

In GAP there is available functionality to work with Greens Relations. A function `Elements` (or `AsSortedList`), that given a finite semigroup S returns a list of its elements, is also available. We also need `GreensRMaximalElementsOfSemigroup` and `GreensLMaximalElementsOfSemigroup`. We implement them as attributes of a semigroup. In this way they are only computed once and then stored with the semigroup. Hence, if we need them more then once we do not have to repeat the computation.

```
DeclareAttribute("GreensRMaximalElementsOfSemigroup",IsSemigroup);
DeclareAttribute("GreensLMaximalElementsOfSemigroup",IsSemigroup);

#########################################################################
##
#M  GreensRMaximalElementsOfSemigroup(<S>)
```

```
##
##  for a finite semigroup <S>;
##  returns the set of all R-maximal elements of <S>.
##
InstallMethod(GreensRMaximalElementsOfSemigroup,
"for a finite semigroup", true,
[IsFinite and IsSemigroup], 0,
function(S)
local rclasses,                 # the set of rclasses of S
      rmaximal,                 # the set of maximal elements of S
      i,j;                      # loop variables

  # calculate the Greens' R-classes of the semigroup
  rclasses := GreensRClasses(S);

  # start with an empty list of R-maximal elements
  rmaximal :=[];

  # look for the maximal R-classes
  # whenever one is found add its elements to <rmaximal>

  # so we loop through the R-classes of the semigroup and
  # check if each of them is maximal
  for i in [1..Length(rclasses)] do
    j :=1;
    while j in [1..Length(rclasses)] do

      if j<>i then

        if IsGreensLessThanOrEqual(rclasses[i],rclasses[j]) then
          # this means that the i-th jclass is not maximal
          # so we make j big enough to take us out of the loop
          j := Length(rclasses)+1;
        fi;

      fi;
      j:=j+1;
    od;

    if j=Length(rclasses)+1 then
      # i-th R class is maximal (otherwise j=Length(rclasses)+2)
      # hence all elements of the i-th class are R-maximal
      Append(rmaximal,Elements(rclasses[i]));
```

```
    fi;

  od;

  return rmaximal;

end);

#####################################################################
##
#F  GreensLMaximalElementsOfSemigroup(<S>)
##
##  for a finite semigroup <S>;
##  returns the set of all L-maximal elements of <S>.
##
InstallMethod(GreensLMaximalElementsOfSemigroup,
"for a finite semigroup", true,
[IsFinite and IsSemigroup],0,
function(S)
local lclasses,                   # the set of lclasses of S
      lmaximal,                   # the set of maximal elements of S
      i,j;                        # loop variables

  # start by calculating the Greens' L-classes of the semigroup
  lclasses := GreensLClasses(S);

  # start with an empty list of L-maximal elements
  lmaximal :=[];

  # look for the maximal L-classes
  # whenever one is found add its elements to <lmaximal>

  # so we loop through the L-classes of the semigroup and
  # check if each of them is maximal
  for i in [1..Length(lclasses)] do
    j :=1;
    while j in [1..Length(lclasses)] do

      if j<>i then

        if IsGreensLessThanOrEqual(lclasses[i],lclasses[j]) then
          # this means that the i-th jclass is not maximal
          # so we make j big enough to take us out of the loop
```

```
        j := Length(lclasses)+1;
      fi;

    fi;
    j:=j+1;
  od;

  if j=Length(lclasses)+1 then
    # i-th L class is maximal (otherwise j=Length(lclasses)+2)
    # hence all elements of the i-th class are L-maximal
    Append(lmaximal,Elements(lclasses[i]));
  fi;

od;

return lmaximal;

end);
```

A.2 The main code

We now present the implementations of the actual code respecting stability of finite semigroups. In here we use the package GRAPE [64], which is a system to compute with graphs within GAP.

```
DeclareProperty("IsSemigroupWithoutIndecomposableElements",
 IsSemigroup);
DeclareGlobalFunction("StabilityGraphOfElement");
DeclareProperty("IsStableSemigroup",IsSemigroup);

#####################################################################
##
#P  IsSemigroupWithoutIndecomposableElements(s)
##
##  for a finite semigroup <S>;
##  retuns true if and only if the semigroup does not have
##  indecomposable elements, ie, iff <S^2=S>.
##
InstallMethod(IsSemigroupWithoutIndecomposableElements,
"for a finite semigroup", true,
[IsFinite and IsSemigroup], 0,
```

```
function(s )
local gens,            # generators of the semigroup
      els,             # list of elements of the semigroup
      table;           # multiplication table of the semigroup

  # it is enough to check that each generator is decomposable
  gens := GeneratorsOfSemigroup(s);
  els := AsSortedList(s);
  table := MultiplicationTable(els);

  # <S> has no indecomposable elements iff each generator is the
  # product of any two elements of <S>
  return ForAll(gens, a->
    ForAny([1..Length(els)], i->
    ForAny([1..Length(els)],j -> a=els[table[i][j]]) ) );

end);


##########################################################################
##
#F  StabilityGraphOfElement(S,x)
##
##  for a finite semigroup <S> and an element <x> of the semigroup
##  returns the stability graph of <x>.
##
InstallGlobalFunction(StabilityGraphOfElement,
function(S,x)
local rmaximal,    # sets of L-maximal elements of <S>
      lmaximal,    # sets of R-maximal elements of <S>
      els,         # list of elements of the semigroup
      a,b,c,       # elements of the semigroup
      vertices,    # set of vertices of the graph
      triv,        # the trivial group
      rel,         # relation between vertices connected by an edge
      Gamma;       # the stability graph of <x>

  els := AsSortedList(S);
  rmaximal := GreensRMaximalElementsOfSemigroup(S);
  lmaximal := GreensLMaximalElementsOfSemigroup(S);

  # start by computing the set of vertices of the graph
  vertices := [];
```

```
  # a vertice is a triple (a,b,c) where <a> in <rmaximal>,
  # <c> in <lmaximal> and <abc=x>
  for a in rmaximal do
    for c in lmaximal do
      for b in els do
        if a*b*c=x then
          Add(vertices, [a,b,c]);
        fi;
      od;
    od;
  od;

  # now we are going to use the package Grape to create the graph

  # we use the trivial group acting on the vertices (OnPoints) and
  # rel gives the condition for an edge to exist between vertices:
  # there is an edge between vertices <[a1,b1,c1]>,<[a2,b2,c2]> if
  # they are distinct and
  # (a1=a2 and b1*c1=b2*c2) or (a1*b1=a2*b2 and c1=c2)
  triv := TrivialGroup();

  rel :=  function(x,y) return x<>y and
        ((vertices[x][1]=vertices[y][1] and
          vertices[x][2]*vertices[x][3]=
                                    vertices[y][2]*vertices[y][3])
        or
         (vertices[x][1]*vertices[x][2]=
                                    vertices[y][1]*vertices[y][2]
          and vertices[x][3]=vertices[y][3])); end;

  Gamma := Graph(triv,[1..Length(vertices)],OnPoints,rel,true);
  AssignVertexNames(Gamma,vertices);

  return Gamma;

end);

#####################################################################
##
#F  IsStableSemigroup(<S>)
##
##  for a finite semigroup <S>;
##  returns true if the semigroup is stable and false otherwise.
```

```
##
InstallMethod(IsStableSemigroup,
"for a finite semigroup", true,
[IsFinite and IsSemigroup], 0,
function(S)

  local els,                  # list of elements of the semigroup
        x;                    # loop variable

  # if the semigroup has indecomposable elements then
  # it is certainly not stable
  if not IsSemigroupWithoutIndecomposableElements(S) then
    return false;
  fi;

  els := AsSortedList(S);

  # check if each of the stability graphs of the semigroup is
  # connected
  for x in els do
    if not IsConnectedGraph(StabilityGraphOfElement(S,x)) then
      return false;
    fi;
  od;

  return true;

end);

DeclareSynonym("PreservesFiniteGenerationInDirectProducts",
               IsSemigroupWithoutIndecomposableElements);
DeclareSynonym("PreservesFinitePresentabilityInDirectProducts",
               IsStableSemigroup);
```

A.3 Examples

We now present some GAP sessions where the implementations of Section A.2 were used to find whether certain semigroups preserved, or not finite generation and finite presentability in direct products. Our semigroups are normaly given by multiplication tables. A multiplication table, for GAP is a $n \times n$ matrix, whose entries are in the set $\{1, \ldots, n\}$. Most of our examples of Section 3.4 have entries in $\{0, .., n-1\}$. We have to move the zero to the first position and add one to all entries of the multiplication table.

Example A.1 This is S_2 from Example 3.33.

```
gap> m := [ [ 1, 1, 1, 1, 1 ], [ 1, 1, 1, 1, 1 ],
> [ 1, 1, 1, 1, 1 ], [ 1, 2, 3, 4, 4 ], [ 1, 2, 3, 5, 5 ] ];;
gap> S := SemigroupByMultiplicationTable(m);
<semigroup with 5 generators>
gap> IsStableSemigroup(S);
false
gap> els := Elements(S);
[ m1, m2, m3, m4, m5 ]
gap> Gamma1 := StabilityGraphOfElement(S,els[1]);;
gap> IsConnectedGraph(Gamma1);
true
gap> Gamma2 := StabilityGraphOfElement(S,els[2]);;
gap> IsConnectedGraph(Gamma);
false
```

Example A.2 This is S from Example 3.28.

```
gap> m:= [ [ 1, 1, 1, 1, 1, 1 ], [ 1, 2, 2, 2, 5, 1 ],
> [ 1, 3, 3, 3, 5, 1 ], [ 1, 2, 3, 4, 5, 1 ],
> [ 1, 1, 1, 1, 1, 1 ], [ 1, 1, 1, 1, 1, 6 ] ];;
gap> s := SemigroupByMultiplicationTable(m);
<semigroup with 6 generators>
gap> IsStableSemigroup(s);
true
gap>
```

Browsing through all semigroups of order 4 we found that Example 3.29 and Example 3.30 were the only non stable ones which had no indecomposable elements.

Appendix B

Implementation of algorithms for commutative semigroups in GAP

In Chapter 7 we develop some algorithms to compute with commutative semigroups. Some of these algorithms have been implemented in GAP. In fact, some of these implementations have been made available at the GAP web page, in the form of deposited code. That package is called commsemi, it is an α-version, and it has been developed jointly by Andrew Solomon and the author. We have made use of these implementations to produce the examples that appear in Chapter 7.

In here we reproduce the code implemented in GAP (v.4.2) (this is based in commsemi, but it includes some more functionality). In Section B.5 we have an example of a session of GAP using our code.

This does not pretend to be a final version of the implementation of our algorithms for commutative semigroups. Indeed, not only there are important pieces of functionality missing, as some issues about how to program the whole code efficiently should be addressed in the future. To name an example, a commutative version of the Knuth–Bendix procedure would be most desirable, as it would certainly perform better in our cases than the Knuth–Bendix now available in GAP for a general finitely presented semigroup. The algorithm to calculate the set of idempotents of a finitely generated commutative semigroup from Rosales, García Sánchez and García García [52] is also an indispensable tool. Finally, as mentioned in Chapter 7 we have used Singular [22] to perform the calculations dealing with ideals. This would, of course, make the implementation in GAP of our algorithms described in Section 7.4 not possible. Fortunately, a package linking Singular with GAP is being developed by Willem De Graff, and hence we hope that in some future we will be able to tie all this code nicely together.

B.1 Auxiliary functions

We start by presenting some auxiliary functions. They will deal with the task of transforming words and elements to vectors and vice versa, as well as add commutating relations to a given finitely presented semigroup.

```
DeclareGlobalFunction("AssocWordToVector");
DeclareGlobalFunction("ElementOfFpSemigroupAsVector");
DeclareGlobalFunction("VectorToElementOfCommutativeFpSemigroup");
DeclareGlobalFunction("VectorToAssocWord");
DeclareAttribute("EpimorphismAbelianization", IsSemigroup);
DeclareAttribute("Abelianization", IsSemigroup);

#####################################################################
##
##  IsVectorDivLessThanOrEqual(<u>,<v>)
##
##  for two vector <u>, <v>;
##  returns true if each entry of u is less then or equal to the
##  correspondent entry of v; false if each entry of u is greater
##  then the correspondent entry of v; fail otherwise (i.e. if
##  the words are not comparable)
##
BindGlobal("IsVectorDivLessThanOrEqual",
function(u,v)
  return ForAll([1..Length(v)], i-> u[i]<=v[i]);
end);

#####################################################################
##
#M  EpimorphismAbelianization(<S>)
##
##  for an fp semigroup <S>;
##  returns an epimorphism from <S> to abelian homomorphic image.
##
InstallMethod(EpimorphismAbelianization,
```

```
"for an fp semigroup", true, [IsFpSemigroup],0,
function(s)
  local f,      # free semigroup
        i,j,    # loop variables
        r,      # relations
        x,      # generators of semigroup
        map,    # the function which computes the epimorphism
        epi,    # the epimorphism
        cs;     # the abelian homomorphic image

  # get the free semigroup underlying s and the relations of s
  f:=FreeSemigroupOfFpSemigroup(s);
  x:=GeneratorsOfSemigroup(f);
  r:=ShallowCopy(RelationsOfFpSemigroup(s));

  # add relations to make it a semilattice
  # relations added are: xy=yx for x,y in f
  for i in [1..Length(x)] do
    for j in [1..i-1] do
      Add(r,[x[j]*x[i],x[i]*x[j]]);
    od;
  od;

  # create the semigroup
  cs:=f/r;
  # which is certainly commutative
  SetIsCommutative(cs,true);

  # define a mapping
  map:=  g -> ElementOfFpSemigroup( cs , UnderlyingElement(g) );
  # and create the epimorphism
  epi := MagmaHomomorphismByFunctionNC(s, cs, map);

  return epi;
end);

#############################################################################
##
#M  Abelianization(<S>)
##
##  for a semigroup <S>;
##  returns its largest commutative homomorphic image.
##
```

```
InstallMethod(Abelianization,
"<S>", true, [IsSemigroup],0,
function(s)

  #if gap already knows that s is a commutative semigroup,
  #it returns s itself
  if HasIsCommutative(s) and IsCommutative(s) then
    return s;
  fi;

  return Range(EpimorphismAbelianization(s));
end);

#################################################################
##
#F  AssocWordToVector (<f>,<w>)
##
##  for a free semigroup <f> and an associative word <w> in <f>;
##  returns a vector where entry i corresponds to the number of
##  times that generator i appears in the word
##
InstallGlobalFunction(AssocWordToVector,
function(f,w)
  local i,      # loop variable
        k,      # generator of syllable
        m,      # number of generators
        n,      # number of syllables
        vector; # resulting vector

  # make sure w is from f
  if not (w in f) then
    Error("Assoc word not in semigroup");
  fi;

  m := Length(GeneratorsOfSemigroup(f));
  vector:=[];

  #we start by having zeros in all entries
  for i in [1..m] do
    Add(vector,0);
  od;

  #run through the syllables of the word
```

```
  #and add the exponent to the entry of the vector
  #correponding to the syllable generator
  n := NumberSyllables(w);
  for i in [1..n] do
    k:=GeneratorSyllable(w,i);
    vector[k]:=vector[k]+ExponentSyllable(w,i);
  od;

  return vector;
end);

#############################################################################
##
#B  VectorRulesOfCommutativeSemigroupRws(<rws>)
##
##  for a rewriting system <rws> of a commutative semigroup;
##  returns the rules of <rws> in the form of vectors, ignoring
##  the trivial rules arising from the commutative relations.
##
BindGlobal("VectorRulesOfCommutativeSemigroupRws",
function(rws)
  local f,        # the free semigroup of the rws
        rules,    # rules of rws
        u,        # a rule
        vrules,   # the rules as vectors
        v;        # a vector rule;

  f := FreeSemigroupOfRewritingSystem(rws);
  rules := Rules(rws);
  vrules := [];
  for u in rules do
    v := [AssocWordToVector(f,u[1]),AssocWordToVector(f,u[2])];
    if not (v[1]=v[2]) then
      Add(vrules,v);
    fi;
  od;

  return Set(vrules);
end);

#############################################################################
##
#F  ElementOfFpSemigroupAsVector (<e>)
```

```
##
##  for an element <e> of an fp semigroup;
##  returns a vector representing it.
##
InstallGlobalFunction(ElementOfFpSemigroupAsVector,
function(e)
  local s,    # the semigroup to which s belongs
        f,    # the free semigroup underlying s
        v;    # the vector

  s := CollectionsFamily(FamilyObj(e))!.wholeSemigroup;
  f := FreeSemigroupOfFpSemigroup(s);
  v := AssocWordToVector(f,UnderlyingElement(e));

  return v;
end);

#################################################################
##
#F  VectorToAssocWord(<f>,<v>)
##
##  for a free semigroup <f> and a vector <v>;
##  returns a word in <f> that <v> represents.
##
InstallGlobalFunction(VectorToAssocWord,
function(f,v)
  local t,              # type of generators of semigroup
        x,              # generators of semigroup
        e;              # the element that v represents

  x:=GeneratorsOfSemigroup(f);

  # make sure that v represents an element of s
  if (Length(x)<>Length(v) or
      v=ListWithIdenticalEntries(Length(x),0)) then
    Error("Wrong vector length or null vector");
  fi;

  # create the element
  t:=TypeObj(x[1]);
  e := ObjByVector(t,v);

  return e;
```

```
end);

#####################################################################
##
#F  VectorToElementOfCommutativeFpSemigroup(<S>,<v>)
##
##  for a commutative semigroup <S> and a vector <v>;
##  returns the element of <S> that <v> represents.
##
InstallGlobalFunction(VectorToElementOfCommutativeFpSemigroup,
function(s,v)
  local f,          # the free semigroup underlying s
        fam,        # the family to which the new element belongs
        e;          # the element of s that e represents

  f := FreeSemigroupOfFpSemigroup(s);
  fam := FamilyObj( GeneratorsOfSemigroup(s)[1] );
  e := ElementOfFpSemigroup(fam,VectorToAssocWord(f,v));

  return e;
end);
```

B.2 Finiteness and Size

We now present the code to check whether a commutative semigroup is finite and to calculate its size. This follows from Section 7.2.

```
DeclareGlobalFunction(
    "VectorOfSupOfEntriesOfElementsOfCommutativeFpSemigroup");

#####################################################################
##
#F  VectorOfSupOfEntriesOfElementsOfCommutativeFpSemigroup(<S>)
##
##  for a commutative fp semigroup <S>;
##  returns a vector which entries greater than any entry of a
##  ie, sups of entries of elements of the semigroup.
##
InstallGlobalFunction(
VectorOfSupOfEntriesOfElementsOfCommutativeFpSemigroup,
function(s)
```

```
  local rws,          # reduced confluent rws for s
        r,            # set of vector rules of crws
        vector,       # the vector
        n,            # number of generators of the semigroup
        list;         # subslist of the list of rules

  # obtain the rules of a confluent rws for s as vectors
  # (ommiting the trivial rules arising from commutativity)
  rws := ReducedConfluentRewritingSystem(s);
  r := VectorRulesOfCommutativeSemigroupRws(rws);

  vector := [];
  n := Length(r[1][1]);

  # now we choose the rules whose lhs is a power of a generator
  list := Filtered([1..Length(r)],
      k -> ForAny([1..n],i->
                  ForAll([1..n],j->i=j or r[k][1][j]=0)));
  # the vector we are looking for is the sum of the corresponding
  # vectors of lhs of the rules to which list refers
  if IsEmpty(list) then
    return fail;
  else
    vector := Sum(List([1..Length(list)], i-> r[list[i]][1]));
  fi;

  # if the vector has any non-zero entries then  it
  # means that all powers of a certain generator are different
  # and the semigroup is infinite
  # Hence we return fail
  if ForAny([1..n],i->vector[i]=0) then
    return fail;
  fi;


  return vector;
end);

####################################################################
##
#M  IsFinite( <S> )
##
##  for a commutative fp semigroup <S>;
```

```
## returns true if <S> is finite; returns false otherwise.
##
InstallMethod( IsFinite,"a commutative fp commutative semigroup",
true, [ IsFpSemigroup and IsCommutative ], 0,
function(S)
  return
  VectorOfSupOfEntriesOfElementsOfCommutativeFpSemigroup(S)<>fail;
end);

#####################################################################
##
#B  SizeOfFpCommutativeSemigroup(<S>)
##
## for an fp commutative semigroup <S>;
## returns the size of <S>.
##
BindGlobal("SizeOfFpCommutativeSemigroup",
function(s)
  local auxsize,              # size of union of ideals
        auxsum,               # sum of all auxsize
        gens_of_ideals,       # auxiliary function
        i,j,U,                # loop variables
        idealgenerators,      # set of ideal generators
        idgens,               # subset of ideal generators
        i_subsets_of_n,       # subsets of {1..n} with i elements
        rws,                  # rewriting system
        r,                    # set of vector rules
        rhovector,            # vector of sup of entries
        size_of_intersection_of_ideals,
        t;                    # upper bound for size of semigroup

  ###################################################################
  # given a set of vector rules this function returns
  # the vectors which are generators of ideals of
  # non-reduced words (with more than one non-zero compoenent)
  gens_of_ideals:=function(r)
    local listgen,  # list of gens of ideals of non reduced words
          u,        # a vector relation
          n;        # number of generators of the semigroup

    n := Length(r[1][1]);
    listgen:=[];
    for u in r do
```

```
    if u[1]<>u[2] then
      if not Number([1..n],i->u[1][i]<>0) in [0..1] then
        Add(listgen,u[1]);
      fi;
    fi;
  od;

  return Set(listgen);
end;

###################################################################
# given the vector of sup of entries and a set of generators of
# ideals it calculates the size of the intersection of those
# ideals
size_of_intersection_of_ideals:=function(rhovector,idealgens)
  local auxcount,             # auxiliary variable
        auxmax,               # auxiliary variable
        i,k,                  # loop variables
        vectorofmax;

  # compute the vector which entries are the maximum of
  # correponding entries of all gens of the considered ideals
  vectorofmax := List([1..Length(rhovector)], i->
     Maximum(List([1..Length(idealgens)],k->idealgens[k][i])));

  # then count how many elements are in the intersection
  auxcount:=1;
  for i in [1..Length(rhovector)] do
    auxcount:=auxcount*(rhovector[i]-vectorofmax[i]);
  od;

  return auxcount;
end;

###################################################################
# actual function

# check if the semigroup is finite or infinite
if not(IsFinite(s)) then
  return infinity;
fi;

# obtain the rules of a confluent rws for s as vectors
```

```
# (ommiting the trivial rules arising from commutativity)
rws := ReducedConfluentRewritingSystem(s);
r := VectorRulesOfCommutativeSemigroupRws(rws);

# we know that the semigroup is finite
# we compute the vector of sup of entries, call it 'rhovector'
rhovector:=
    VectorOfSupOfEntriesOfElementsOfCommutativeFpSemigroup(s);

# from the rhovector we consider the set T
# The size of T is a upper bound to the size of S
# We compute the size, t, of T:
t:=1;
for i in rhovector do
  t:=t*i;
od;
# now we have all elements, and also the empty word
t:=t-1;

# We know calculate the size of the union of all ideals of
# non-reduced words (these are principal ideals, generated by
# the vectors corresponding to the lhs of relations of the
# rewriting system for s.
# We do this by inclusion/exclusion principle
auxsize:=0;
idealgenerators:=gens_of_ideals(r);

# the following cycle is the inclusion/exclusion principle
# (we have as many ideals as ideal generators)
for i in [1..Length(idealgenerators)] do
  #add sum of sizes of all intersections of i ideals
  auxsum:=0;
  i_subsets_of_n:=
    Combinations(Set([1..Length(idealgenerators)]),i);
  for U in i_subsets_of_n do
    idgens:=[];
    for j in U do
      Add(idgens,idealgenerators[j]);
    od;
    auxsum:=auxsum+
             size_of_intersection_of_ideals(rhovector,idgens);
  od;
  auxsize:=auxsize+(-1)^(i+1)*auxsum;
```

```
  od;

  # the size of S is just t-auxsize
  return t-auxsize;
end);

#############################################################################
##
#M  Size( <S> )
##
##  for a commutative fp semigroup <S>;
##  returns the size of <S>.
##
InstallMethod( Size, "for an fp commutative semigroup", true,
[ IsFpSemigroup and IsCommutative ], 0,
function(s)
   return SizeOfFpCommutativeSemigroup(s);
end);
```

B.3 Ideal membership

This section deals with ideal membership. This follows from Section 7.5.

```
DeclareAttribute("BasisOfSemigroupIdeal",IsSemigroupIdeal);
DeclareAttribute("VectorBasisOfSemigroupIdealWithFactors",
                  IsSemigroupIdeal);
DeclareOperation("VectorBasisOfSemigroupIdeal",
                  [IsSemigroupIdeal]);
DeclareOperation("Division",[IsSemigroupIdeal,
                  IsElementOfFpSemigroup]);

#############################################################################
##
#M  BasisOfSemigroupIdeal( <I> )
##
##  for an ideal <I> of a commutative fp semigroup;
##  returns a basis for <I>.
##
InstallMethod(BasisOfSemigroupIdeal,
"for a commutative fp semigroup and a list of elements", true,
[IsSemigroupIdeal],0,
function(id)
```

```
  local s,          # the semigroup
        vbasis,     # a basis of id as vectors
        basis;      # the basis

  # get the parent semigroup
  s := Parent(id);

  # this only works for commutative fp semigroups
  if not (IsCommutative(s) and IsFpSemigroup(s)) then
    TryNextMethod();
  fi;

  # obtain a vector basis for id and transform it into a
  # set of words
  vbasis := VectorBasisOfSemigroupIdealWithFactors(id);
  basis := List([1..Length(vbasis)],i-> VectorToAssocWord(
              FreeSemigroupOfFpSemigroup(s),vbasis[i][1]));

  return basis;
end);

#####################################################################
##
#M  VectorBasisOfSemigroupIdealWithFactors( <I> )
##
##  for an ideal <I> of a commutative fp semigroup;
##  it returns a vector basis for <I> together with factors.
##
InstallMethod(VectorBasisOfSemigroupIdealWithFactors,
"for a semigroup ideal", true,
[IsSemigroupIdeal and IsCommutative],0,
function(id)
  local s,            # the semigroup
        rws,          # rewriting system
        r,            # set of rules
        u,            # a rule
        V,oldV,       # set containing the basis (set of triples)
        v,            # a triple
        gamma,delta,  # vectors
        y,            # a vector
        Y,            # generators of id as vectors
        minimize;     # auxiliary function
```

```
##################################################################
# auxiliary function to minimize the obtained set
minimize:=function(x)

  local xx ,              # subset of x
        j,                # loop variable
        nonredundant;     # the result

  # we create the subset of x containg no two elements with
  # the same first entry
  xx := ShallowCopy(x);
  for j in [1..Length(xx)] do
    if ForAny([1..j-1],i->IsBound(xx[i]) and
                          xx[i][1]=xx[j][1]) then
      Unbind(x[j]);
    fi;
  od;
  xx := Set(x);

  # we start with an empty set and keep adding the gens that
  # are not divisible by anything else
  nonredundant:=[];

  # the following tests if the generator is divisible by
  # anything else
  for j in [1..Length(xx)] do

    if not( ForAny([1..Length(xx)],i->i<>j
        and IsVectorDivLessThanOrEqual(xx[i][1],xx[j][1]))) then
      # and add it to the set
      Add(nonredundant,xx[j]);
    fi;

  od;
  return nonredundant;
end;

##################################################################
# actual function

# get the parent semigroup and transform the generators of the
# ideal into vectors
s := Parent(id);
```

```
  Y := List([1..Length(GeneratorsOfMagmaIdeal(id))],i->
                        ElementOfFpSemigroupAsVector(
                        GeneratorsOfMagmaIdeal(id)[i]));

  # obtain the rules of a confluent rws for s as vectors
  # (ommiting the trivial rules arising from commutativity)
  rws := ReducedConfluentRewritingSystem(s);
  r := VectorRulesOfCommutativeSemigroupRws(rws);

  # start with an empty set
  oldV := [];
  V := [];
  # add the vector representative of the generators of the ideal
  for y in Y do
    Add(V,[ShallowCopy(y),y,List([1..Length(y)],i->0)]);
  od;

  # we obtain new generators for the ideal
  while oldV <> V do
    oldV := ShallowCopy(V);
    for v in V do
      for u in r do
        delta := List([1..Length(v[1])],i->
                           Maximum(v[1][i],u[1][i]))-v[1];
        Add(V, [v[1]+delta-u[1]+u[2], v[2],v[3]+delta]);
        gamma := List([1..Length(v[1])],i->
                           Maximum(v[1][i],u[2][i]))-v[1];
        Add(V, [v[1]+gamma-u[2]+u[1], v[2],v[3]+gamma]);
        V := minimize(V);
      od;
    od;
  od;

  return V;
end);

#############################################################################
##
#M  \in( <e>, <id> )
##
##  for an ideal <I> of a commutative fp semigroup and an element
##  <e> of the semigroup;
##  returns true if <e> is in <I>; returns false otherwise.
```

```
##
InstallMethod(\in,
"for an element and an ideal of a commutative fp semigroup",true,
[IsElementOfFpSemigroup, IsSemigroupIdeal and IsCommutative],0,
function(x,id)
  return Division(id,x)<>fail;
end);

#####################################################################
##
#M  Division( <I>, <e> )
##
##  for an ideal <I> of a commutative fp semigroup and an element
##  <e> of the semigroup;
##  returns [d] if <e> is equal (in the semigroup) to an element
##  <d> in the basis of <I>;
##  returns [x,w] such that e=xw if <e> is in <id> but not in
##  its basis ;
##  returns fail if <e> is not in <I>.
##
InstallMethod(Division,
"for an ideal and an element of a commutative fp semigroup",true,
[IsSemigroupIdeal and IsCommutative, IsElementOfFpSemigroup],0,
function(id,e)
  local s,        # the semigroup
        basis,    # a vector basis for id
        x,        # an element from basis
        v,        # a vector representing e
        d;        # the answer

  # get the parent smg, a basis for id, and a vector repres. e
  s := Parent(id);
  basis := VectorBasisOfSemigroupIdealWithFactors(id);
  v := ElementOfFpSemigroupAsVector(e);

  # look for an element in the basis whose first entry divides v
  for x in basis do
    if IsVectorDivLessThanOrEqual(x[1],v) then
      d := [VectorToElementOfCommutativeFpSemigroup(s,x[2]),];
      if v<>x[1] then
        d[2]:=VectorToElementOfCommutativeFpSemigroup(s,
                                                  (v-x[1])+x[3]);
      fi;
```

```
      return d;
    fi;
  od;

  return fail;
end);
```

B.4 $\mathcal{H}$ and $\mathcal{N}$ relations

In this section we present code associated with the $\mathcal{H}$ ans $\mathcal{N}$ relation in a finitely generated commutative semigroup. This follows from Sections 7.3 and 7.7.

```
DeclareAttribute(
  "EpimorphismToLargestSemilatticeHomomorphicImage",IsSemigroup);
DeclareOperation("LargestSemilatticeHomomorphicImage",
  [IsSemigroup and IsCommutative]);
DeclareOperation("ArchimedeanRelation",
  [IsSemigroup and IsCommutative]);
DeclareAttribute("StabilizerOfGreensClass",IsGreensClass);

#############################################################################
##
#M  IsGreensLessThanOrEqual( <C1>, <C2> )
##
##  for any Green's classes of a commutative fp semigroup;
##  returns true if the class <H1> is less than or equal to the
##  class <H2> with respect to the order defined on Green's
##  classes.
##
InstallMethod( IsGreensLessThanOrEqual,
"for greens classes of an fp commutative semigroup",
IsIdenticalObj, [IsGreensClass, IsGreensClass], 5,
function( h1, h2 )
  local s,       # the semigroup of which h1 and h2 are classes
        f,       # the free semigroup underlying s
        idbasis, # basis of the ideal of s generated a rep of h2
        v1,      # vector representing a representative of h1
        z;       # an element of idbasis

  # get the parent semigroup
  s := Parent( h1 );
```

```
  f := FreeSemigroupOfFpSemigroup(s);
  if not (IsFpSemigroup(s))  then
    TryNextMethod();
  fi;

  # this method only works for commutative semigroups
  if not (IsCommutative(s)) then
    TryNextMethod();
  fi;

  # get the ideal basis of the ideal generated by a rep of h2
  idbasis := BasisOfSemigroupIdeal( MagmaIdealByGenerators(
                                 s, [Representative(h2)]));
  # and get a vector representing a representative of h1
  v1 := ElementOfFpSemigroupAsVector(Representative(h1));

  # h1 is less then or equal to h2 iff
  # there is an element in idbasis which divides h1
  return ForAny(idbasis, z-> IsVectorDivLessThanOrEqual(
                             AssocWordToVector(f, z ),v1));

end);

#####################################################################
##
#M  EpimorphismToLargestSemilatticeHomomorphicImage(s)
##
##  for a commutative fp semigroup <S>;
##  returns an epimorphism to the largest semilattice homomorphic
##  image of <S> (i.e. the quotient of <S> by the archimedean
##  relation).
##
InstallMethod(EpimorphismToLargestSemilatticeHomomorphicImage,
"for an fp commutative semigroup", true,
[IsFpSemigroup and IsCommutative],
function(s)
  local f,    # free semigroup
        r,    # relations
        x,    # generators of semigroup
        j,    # loop variable
        sl,   # the semilattice homomorphic image
        map,  # the function which computes the epimorphism
        epi;  # the epimorphism
```

```
  # get the free semigroup underlying s and the relations of s
  f:=FreeSemigroupOfFpSemigroup(s);
  x:=GeneratorsOfSemigroup(f);
  r:=ShallowCopy(RelationsOfFpSemigroup(s));

  # add relations to make it a semilattice
  # relations added are: x^2=x for x in f
  for j in [1..Length(x)] do
    Add(r,[x[j]^2,x[j]]);
  od;

  # create the semigroup
  sl := f/r;
  # which is certainly commutative
  SetIsCommutative(sl,true);

  # define a mapping
  map:=  g -> ElementOfFpSemigroup( sl, UnderlyingElement(g) );
  # and create the epimorphism
  epi := MagmaHomomorphismByFunctionNC(s, sl, map);

  return epi;

end);

###################################################################
##
#M  LargestSemilatticeHomomorphicImage(<S>)
##
##  for a commutative semigroup or monoid <S>;
##  returns an epimorphism to the largest semilattice homomorphic
##  image of <S> (i.e. the quotient of <S> by the archimedean
##  relation).
##
InstallMethod(LargestSemilatticeHomomorphicImage,
"for a commutative semigroup", true,
[IsSemigroup and IsCommutative],0,
function(s)
  return
    Range(EpimorphismToLargestSemilatticeHomomorphicImage(s));
end);
```

```
#############################################################################
##
#M  ArchimedeanRelation(<S>)
##
##  for a finite commutative fp semigroup <S>;
##  returns the the archimedean relation on <S>.
##
InstallMethod(ArchimedeanRelation,
"for a finite commutative fp semigroup", true,
[IsSemigroup and IsCommutative and IsFinite],0,
function(s)
  local epi,        # epimorphism to largest homom. image of s
        sl,         # largest semilattice homomorphic image of s
        x,          # an archimedean component
        partition;  # the archimedean components

  # this only works for fp semigroups
  if not IsFpSemigroup(s) then
    TryNextMethod();
  fi;

  partition := [];

  # get the epimorphism to largest semilattice homomorphic image
  epi := EpimorphismToLargestSemilatticeHomomorphicImage(s);
  sl := Range(epi);
  # the archimedean compoments are the preimages of the
  # elements of the range of epi
  for x in Elements(sl) do
    Append(partition,[PreImagesElm(epi,x)]);
  od;

  return EquivalenceRelationByPartitionNC(s,partition);

end);

#############################################################################
##
#M  StabilizerOfGreensClass( <H> )
##
##  for a greens class <H> of a commutative fp semigroup;
##  returns the stabilizer of <H>;
```

```
## returns fail if no element stabilises <H>.
##
InstallMethod(StabilizerOfGreensClass,
"for a greens class of a commutative semigroup", true,
[IsGreensClass], 0,
function( h )
local s,        # the parent semigroup
      gens,     # generators of s
      a,        # a generator of the semigroup
      x,        # an element of h
      stab;     # the stabilizer of h

  # get the parent semigroup
  s := Parent(h);
  # only works for commutative fp semigroups
  if not
    (IsCommutative(s) and IsFpSemigroup(s)) then
    TryNextMethod();
  fi;

  gens:= GeneratorsOfSemigroup(s);
  # get a representative of h
  x:= Representative(h);

  stab:=[];
  for a in gens do
    if (a*x) in h then
      Add(stab, a);
    fi;
  od;

  if stab=[] then
    return fail;
  else
    return Subsemigroup(s,stab);
  fi;
end);
```

B.5 An example session

We now present some GAP sessions where the functionality implemented in the previous sections was used to investigate some commutative semigroups.

Example B.1 This is the monoid from Example 7.11.

```
gap> f := FreeMonoid("x","y","z");;
gap> x := f.1;; y := f.2;; z := f.3;; e := One(f);;
gap> r := [[x^2*y,x*y],[x*y^3,x*y],[x*y*z^4,x*y],[x^5,x],
> [y^5,y],[z^5,z]];;
gap> s := Abelianization(f/r);;
gap> Size(s);
53
```

Example B.2 This example deals with the semigroup S of Example 7.26.

```
gap> f := FreeSemigroup("a","b");;
gap> a := f.1;; b:=f.2;;
gap> r := [[a^3*b^3,a^2*b^2]];
[ [ a^3*b^3, a^2*b^2 ] ]
gap> s := Abelianization( f/r );;
gap> Size(s);
infinity
gap> sl := LargestSemilatticeHomomorphicImage( s );
<finitely presented semigroup on the generators [ a, b ]>
gap> Size(sl);
3
```

Example B.3 This is from Example 7.47.

```
gap> f := FreeMonoid("x","y","a");;
gap> e := One(f);; x := f.1;; y:=f.2;; a := f.3;;
gap> r := [[x*y,e],[x^9*a^2,a^2],[x^3*a^3,a^3],[a^4,a^3]];
[ [ x*y, <identity ...> ], [ x^9*a^2, a^2 ], [ x^3*a^3, a^3 ],
> [ a^4, a^3 ] ]
gap> s := Abelianization(f/r);;
gap> x := GeneratorsOfSemigroup(s);;
gap> SetOne(s,x[1]);
gap> e := GeneratorsOfSemigroup(s)[1];;
gap> x := GeneratorsOfSemigroup(s)[2];;
gap> y := GeneratorsOfSemigroup(s)[3];;
gap> a := GeneratorsOfSemigroup(s)[4];;
```

```
gap> # check whether a^2 is in the ideal generated by x
gap> id := SemigroupIdealByGenerators(s,[x]);
<SemigroupIdeal with 1 generators>
gap> a^2 in id;
true
gap> # check whether a and ax are h-related
gap> hax := GreensHClassOfElement(s,a*x);
{a*x}
gap> ha := GreensHClassOfElement(s,a);
{a}
gap> IsGreensLessThanOrEqual(hax,ha);
true
gap> IsGreensLessThanOrEqual(ha,hax);
true
```

Appendix C

Some results on polynomial rings

In this appendix we introduce some results on polynomial rings that have been used in Chapter 7. All the definitions can be found in a basic text book as [30]. The results on Gröbner basis can be found in [1].

C.1 Basic definitions

A *ring* is a nonempty set R, together with two binary operations (denoted as addition and multiplication), which is an abelian group with respect to the first operation, a monoid with respect to the second operation and for any $a, b, c \in R$

$$a(b + c) = ab + ac \text{ and } (a + b)c = ac + bc.$$

The additive identity element of a ring is called *zero*, and denoted 0_R and the multiplicative identity is the *identity* of the ring, and is denoted by 1_R. When it is clear to which ring we are referring to we denote 0_R and 1_R by 0 and 1, respectively. If R is a commutative semigroup with respect to the second operation then R is a *commutative ring*. If $R \backslash \{0\}$ is a group, with respect to multiplication, then R is a *field*.

All the rings considered are commutative.

A subset $I \subseteq R$ is an *ideal* of R if I is a ring (i.e. a subring of R) and for every $r \in R$ and $i \in I$, $ri \in I$. Given $J \subseteq R$, J *generates* the ideal I if I is the intersection of all the ideals of R that contain J; in this case we write $I = \langle J \rangle$. The set J is also called a *basis* for I. If I is generated by a simple element $a \in R$

then I is called a *principal ideal.* A ring in which all the ideals are principal is called a *principal ideal domain.* The ring $\mathbb{Z}$ of integers is an example of a principal ideal domain.

The ring R is said to *satisfy the ascending chain condition on ideals* if for every chain $I_1 \subseteq I_2 \subseteq \cdots$ of ideals of R, there is an integer m such that $I_j = I_m$ for all $j \geq m$. If R satisfies the ascending chain condition on ideals, then R is said to be *noetherian.* Examples of noetherian rings are the ring of integers $\mathbb{Z}$, and any ring of polynomials over a noetherian ring.

Now consider the set $\{x_1, x_2, \ldots, x_n\}$ of indeterminates and let $R[X]$ denote the commutative ring of polynomials in $x_1, x_2, \ldots, x_n$ with coefficients in R. A term in $x_1, x_2, \ldots, x_n$ is a product of the form

$$\tau = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$ the *degree vector of* τ and $\deg(\tau) = \sum_{j=1}^{n} \alpha_j$ the total degree of τ. We write $\tau = X^\alpha$. Each polynomial $f(x_1, x_2, \ldots, x_n) = f(X) \in R[X]$ is a finite sum

$$\sum_{i=1}^{r} c_i X^{\alpha_i}$$

where $c_i \in R \backslash \{0\}$ is the *coefficient* and $\alpha_i \in \mathbb{N}^n \backslash \{(0, 0, \ldots, 0)\}$ is the *degree vector* of the ith term of f. The product $c_i X^{\alpha_i}$ is called the ith *monomial* of the polynomial f.

Let $\preccurlyeq$ be a total ordering on $\mathbb{N}^n$ satisfying

(i) $\alpha \succ (0, 0, \ldots, 0)$ for all $\alpha \in \mathbb{N}^n \backslash \{(0, 0, \ldots, 0)\}$; and

(ii) for all $\alpha, \beta, \gamma \in \mathbb{N}^n$ if $\alpha \prec \beta$ then $\alpha + \gamma \prec \beta + \gamma$.

Such an ordering on $\mathbb{N}^n$ induces an ordering on the terms in $R[X]$: $x^\alpha \prec x^\beta$ if and only if $\alpha \prec \beta$; the induced ordering is called an *admissible term ordering* on $R[X]$ and it is also denote by $\preccurlyeq$. For a polynomial $f(X) = \sum_{i=1}^{r} c_i X^{\alpha_i}$ we always assume that $\alpha_1 \succ \alpha_2 \succ \cdots \succ \alpha_n$ and call $\operatorname{lt}(f) = X^{\alpha_1}$ the *leading term*, $\operatorname{lc}(f) = c_1$ the *leading coefficient* and $\operatorname{lm}(f) = c_1 X^{\alpha_1}$ the *leading monomial* of f.

C.2 Membership problem and Gröbner basis

Given a polynomial $f \in R[X]$ we might want to know whether $f \in I$; this is called the *membership problem for I*. In order to solve this problem we need some more definitions.

Let I be an ideal in $R[X]$ and let $\preccurlyeq$ be an admissible term ordering on $R[X]$. A finite set $\{f_1, f_2, \ldots, f_r\} \subseteq R[X]$ of polynomials is called a *Gröbner basis* of I (with respect to $\preccurlyeq$) if

(i) $\{f_1, f_2, \ldots, f_r\}$ is a basis for I; and

(ii) $\{\mathrm{lt}(f_1), \mathrm{lt}(f_2), \ldots, \mathrm{lt}(f_r)\}$ is a basis for the *leading term coefficient ideal* containing the leading terms of all $f \in I$.

A Gröbner basis is *reduced* if none of its polynomials has a term divisible by the leading term of some other polynomial in the basis. The membership problem for I is easily solved given a Gröbner basis for it. Now we have

Proposition C.1 [1, Corollary 4.1.17] *With the above notation let $I \subseteq R[X]$ be a non zero ideal. Then I has a finite Gröbner basis.*

And [1, Theorem 4.2.8] and [1, Algorithm 4.1.8] imply the following

Theorem C.2 *If I is an ideal of the polynomial ring $\mathbb{Z}[X]$ then there is an algorithm that computes a Gröbner basis for I. In particular, the membership problem for I is solvable.*

C.3 Primary decomposition

Let R be a commutative ring. An ideal P of R is said to be a *prime ideal* if

(i) $R \neq P$; and

(ii) for any ideals $I, J \in R$, $IJ \subseteq P$ implies $A \subseteq P$ or $J \subseteq P$.

Also, P is a *primary ideal* if for any $a, b \in R$, $ab \in P$ and $a \notin P$ imply that there is $n > 0$ such that $b^n \in P$. Clearly every prime ideal is primary.

For an arbitrary ideal I of R we define the *radical* of I to be the ideal

$$\mathrm{Rad}(I) = \bigcup_{P \in \mathcal{P}} P,$$

where $\mathcal{P}$ is the set of all prime ideals containing I. If the set of such prime ideals is empty then $\mathrm{Rad}(I) = R$. By [30, Theorem VIII.2.6]

$$\mathrm{Rad}(I) = \{r \in R : (\exists n > 0)\, r^n \in I\}.$$

and [30, Theorem VIII.2.9] states that $\mathrm{Rad}(I)$ is a prime ideal. Thus if Q is a primary ideal in a commutative ring R, then the radical P of Q is called *associated prime ideal* of Q. The associated prime ideal of a primary ideal is unique.

We say that an ideal I in R has a *primary decomposition* if

$$I = Q_1 \cap Q_2 \cap \cdots \cap Q_r$$

where all Q_i $(1 \leq i \leq r)$ are primary ideals. It is now natural to ask which ideals have a primary decomposition.

Theorem C.3 [1, Theorem 4.6.9] *Every ideal in a noetherian ring has a primary decomposition.*

Theorem C.4 [20, Proposition 8.5] *Let R be a principal ideal domain and let I be an ideal of the polynomial ring $R[X]$. Then it is possible to compute a primary decomposition for I.*

An algorithm to compute a primary decomposition of a principal ideal domain is also given in [20]. Seidenberg [62] also gives an algorithm to compute a primary decomposition. In fact, its computations also return the associated primes.

C.4 Computations

Our interest in the theory presented in this appendix is computational, in the sense that we will need effective ways of deciding ideal membership and calculating primary decompositions. Since we are using these results as a tool for developing our own theory, it will be good if an implementation could be found.

The ideals that we will be considering are in the polynomial ring $\mathbb{Z}[X]$ and indeed, the papers to which we have referred throughout the text describe algorithms for these computations. However, we do not know about an available implementation of them. Commutative algebra packages widely available only work for polynomials over fields. However, due to the specificity of the ideals we consider we will be able to use such packages.

So let I be an ideal of $\mathbb{Z}[X]$. Then we can consider I as a subset of $\mathbb{Q}[X]$ and in this case it generates an ideal $J_I \in \mathbb{Q}[X]$. By computing in J_I we will obtain answers for I. As mentioned before, computing in $\mathbb{Q}[X]$ is in practice easier since there are more readily available systems that do so.

So let $I \subseteq \mathbb{Q}[X]$ be an ideal and suppose that a basis B for I is given such that, for any $f \in B$

$$f = X^\alpha - X^\beta$$

with $\alpha, \beta \in \mathbb{N}^n \backslash \{(0, 0, \ldots, 0\}$. Notice that $B \subseteq \mathbb{Z}[X]$. Each $f \in B$ is called a *pure difference binomial* and I is called a *pure difference ideal* (this is a special case of binomial ideals). It turns out that a Gröbner basis for a pure difference ideal consists only of pure difference binomials (see [31] and [16]). Furthermore, a Gröbner basis for a pure difference ideal over $\mathbb{Q}[X]$ is also a Gröbner basis for the correspondent ideal over $\mathbb{Z}[X]$. Thus we can use any program that calculates Gröbner basis of ideals in $\mathbb{Q}[X]$ to calculate Gröbner basis for our ideals.

As for primary ideals we obviously have the following

Lemma C.5 *Let P be an ideal over the polynomial ring $\mathbb{Z}[X]$ and suppose that the ideal of $\mathbb{Q}[X]$ generated by P is primary (resp. prime). Then P is primary (resp. prime).*

Thus if given an ideal $I \subseteq \mathbb{Z}[X]$ we consider the ideal in $\mathbb{Q}[X]$ generated by I and calculate a primary decomposition for it, then if all the ideals are given by generators in $\mathbb{Z}[X]$ then we also have a primary decomposition of I in $\mathbb{Z}[X]$.

We have used the Computational Algebra Package Singular [22] to compute Gröbner basis and primary decompositions.

Table of notation

$\leq_{\mathcal{H}}$	the preorder induced by $\mathcal{H}$ for a commutative semigroup	139
I_X	the free inverse semigroup on X	11
$\mathcal{J}$	Green's relation	5
$\mathcal{J}^S$	Green's relation on S	5
J_a	the $\mathcal{J}$–class of a	5
$\leq_{\mathcal{J}}$	the preorder induced by $\mathcal{J}$	5
$l(w_1, w_2)$	the quotient of $w_1 \vee w_2$ by w_1	133
$\mathcal{L}$	Green's relation	5
$\mathcal{L}^S$	Green's relation on S	5
L_a	the $\mathcal{L}$–class of a	5
$\leq_{\mathcal{L}}$	the preorder induced by $\mathcal{L}$	5
$\mathfrak{L}(S)$	the set of $\mathcal{L}$–maximal elements of S	37
$\mathcal{L}[A, T]$	the set of all words from A^+ representing elements of T	16
$\mathcal{M}(S)$	the multiplication table presentation of S	10
$\mathcal{M}(G; I, \Lambda; P)$	Rees matrix semigroup	28
$n(w, a)$	number of occurrences of a in the word w	8
$\mathbb{N}$	the set natural numbers (positive integers)	21
$\mathbb{N}_0$	the set of non negative integers	4
$\mathcal{N}$	the archimedean relation	140
$\leq_{\mathcal{N}}$	the archimedean preorder	140
$p_{\lambda i}$	an entry on the sandwich matrix	28
$\mathbb{Q}$	the set of rational numbers	
$\mathbb{Q}[X]$	the ring of polynomials with coefficients in $\mathbb{Q}$	
$\mathcal{R}$	Green's relation	5
$\mathcal{R}^S$	Green's relation on S	5
R_a	the $\mathcal{R}$–class of a	5
$\leq_{\mathcal{R}}$	the preorder induced by $\mathcal{R}$	5
$\mathfrak{R}(S)$	the set of $\mathcal{R}$–maximal elements of S	37
$\operatorname{rank}(S)$	the rank of S	23
$\operatorname{rank}(S : T)$	the rank of S modulo T	28
${}^s x$	the result of s acting on x	4
$\prod_{i=1}^n S_i$	the direct product of the family $(S_i)_{i \in I}$	54
S^1	semigroup with an identity adjoined	2

S^I	semigroup with an identity adjoined regardless of whether it contains an identity	2
S^0	semigroup with a zero adjoined	2
$S\phi$	image of S under ϕ	3
SL_A	the free semilattice on A	26
S^T	cartesian product of $\lvert T\rvert$ copies of S	66
$S^{(T)}$	direct product of $\lvert T\rvert$ copies of S	66
$S \hookrightarrow P$	S embeds in T	4
$S \cong T$	semigroups S and T are isomorphic	4
$S \times T$	the direct product of S and T	21
$S \rtimes_\phi T$	semidirect product of S and T with respect to ϕ	65
$S \underset{M}{\rtimes}_\phi T$	monoid semidirect product of S and T with respect to ϕ	65
$S\mathrm{Wr}T$	the unrestricted wreath product of S by T	66
$S\mathrm{wr}T$	the (restricted) wreath product of S by T	67
$\mathcal{S}[Y, S_\alpha]$	semilattice of semigroups	116
$\mathcal{S}[Y, S_\alpha, \lambda_{\alpha,\beta}]$	strong semilattice of semigroups	116
$\mathrm{Stab}(H)$	the stabilizer of H	161
T_X	the full transformation semigroup on X	5
T_n	the full transformation semigroup on $\{1, 2, \ldots, n\}$	5
$\lvert X\rvert$	the number of elements in the set X	66
$u \to v$	rule of a rewriting system	134
$\lvert w\rvert$	the length of a word	8
$wu^{-1}v$	the word obtained by applying the rule $u \to v$ to w	133
$w_1 \equiv w_2$	the words w_1 and w_2 are identical	9
	the words w_1 and w_2 are equal as commutative words	131
$w_1 = w_2$	the words w_1 and w_2 represent the same element	9
$w_1 \vee w_2$	the least common multiple of the words w_1 and w_2	133
$\mathbb{Z}$	the set of all integers	
$\mathbb{Z}[X]$	the ring of polynomials with coefficients in $\mathbb{Z}$	

Bibliography

[1] W.W. Adams and P. Loustaunau. *An Introduction to Gröbner Bases*. Graduate Studies in Mathematics. American Mathematical Society, USA, 1994.

[2] I.M. Araújo. Finite presentability of semigroup constructions. *Internat. J. Algebra Comput.*, to appear.

[3] I.M. Araújo, M.J.J. Branco, V.H. Fernandes, G.M.S. Gomes, and N. Ruškuc. On generators and relations for unions of semigroups. *Semigroup Forum*, to appear.

[4] I.M. Araújo, A.V. Kelarev, and A. Solomon. Effective identification of commutative algebras which are principal ideal rings with identity. Submitted.

[5] I.M. Araújo and N. Ruškuc. On finite presentability of direct products of semigroups. *Algebra Colloquium*, 7:83–91, 2000.

[6] I.M. Araújo and N. Ruškuc. Finite presentability of Bruck–Reilly extensions of groups. *J. Algebra*, to appear.

[7] I.M. Araújo and A. Solomon. Computing with semigroups in GAP - a tutorial. In *Proceedings of the international conference on semigroups, Braga (Portugal)*, to appear.

[8] H. Ayik and N. Ruškuc. Generators and relations of Rees matrix semigroups. *Proc. Edinb. Math. Soc.*, 42:481–495, 1999.

[9] G. Baumslag. Wreath products and finitely presented groups. *Math. Z.*, 75:22–28, 1961.

[10] G. Baumslag. *Topics in Combinatorial Group Theory.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1993.

[11] A.P. Biryukov. Some algorithmic problems for finitely defined commutative semigroups. *Siberian Math. J.*, 8:384–391, 1967.

[12] R.H. Bruck. *A Survey of Binary Systems*, volume 20 of *Ergebnisse der Math., Neue Folge.* Springer, Berlin, 1958.

[13] C.M. Campbell, E.F. Robertson, N. Ruškuc, and R.M. Thomas. Reidemeister-Schreier type rewriting for semigroups. *Semigroup Forum*, 51:47–62, 1995.

[14] C.M. Campbell, E.F. Robertson, N. Ruškuc, and R.M. Thomas. On subsemigroups of finitely presented semigroups. *J. Algebra*, 180:1–21, 1996.

[15] C.M. Campbell, E.F. Robertson, N. Ruškuc, and R.M. Thomas. Presentations for subsemigroups - applications to ideals of semigroups. *J. Pure Appl. Algebra*, 124:47–64, 1998.

[16] D. Cox, J. Little, and D. O'Shea. *Ideals, Varieties and Algorithms: an introduction to computational algebraic geometry and commutative algebra.* Springer, Berlin, 1992.

[17] V. Diekert. Commutative monoids have complete presentations by free (noncommutative) monoids. *Theoret. Comput. Sci.*, 46:319–327, 1986.

[18] G.E. Forsythe. SWAC computes 126 distinct semigroups of order 4. *Proc. Amer. Math. Soc.*, 6:443–447, 1955.

[19] The GAP Group, Aachen, St Andrews. *GAP – Groups, Algorithms, and Programming, Version 4.2*, 2000. (`http://www-gap.dcs.st-and.ac.uk/~gap`).

[20] P. Gianni, B. Trager, and G. Zacharias. Gröbner bases and primary decomposition of polynomial ideals. *J. Symbolic Comput.*, 6:149–167, 1988.

[21] R.H. Gilman. Presentations of groups and monoids. *J. Algebra*, 57:544–554, 1979.

[22] G.-M. Greuel, G. Pfister, and H. Schönemann. Singular version 1.2 user manual. In *Reports On Computer Algebra*, number 21. Centre for Computer Algebra, University of Kaiserslautern, June 1998. `http://www.mathematik.uni-kl.de/~zca/Singular`

[23] P.A. Grillet. The free envelope of a finitely generated commutative semigroup. *Trans. Amer. Math. Soc.*, 149:665–682, 1970.

[24] P.A. Grillet. Subdirect decompositions of finitely generated commutative semigroups. *Semigroup Forum*, 4:242–247, 1972.

[25] P.A. Grillet. Left coset extensions. *Semigroup Forum*, 7:200–263, 1974.

[26] P.A. Grillet. A completion theorem for finitely generated commutative semigroups. *J. Algebra*, 34:25–53, 1975.

[27] P.A. Grillet. *Semigroups: an Introduction to the Structure Theory*, volume 193 of *Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 1995.

[28] J.M. Howie. *Fundamentals of Semigroup Theory*. Clarendon Press, Oxford, 1995.

[29] J.M. Howie and N. Ruškuc. Constructions and presentations for monoids. *Comm. Algebra*, 22(15):6209–6224, 1994.

[30] T.W. Hungerford. *Algebra*, volume 73 of *Graduate Texts in Mathematics*. Springer-Verlag, New York–Heidelberg–Berlin, 1980.

[31] U. Koppenhagen and E.W. Mayr. An optimal algorithm for constructing the reduced Gröbner basis of binomial ideals. *J. Simbolic Comput.*, 28:317–338, 1999.

[32] G. Lallement. *Semigroups and Combinatorial Applications*. Pure & Applied Mathematics. A Wiley-Interscience Publication. John Wiley & Sons, New York-Chichester-Brisbane, 1979.

[33] T.G. Lavers. Presentations of general products of monoids. *J. Algebra*, 204:733–741, 1998.

[34] R.C. Lyndon and P.E. Schupp. *Combinatorial Group Theory*. Springer, New York, 1976.

[35] E.W. Mayr. Some complexity results for polynomial ideals. *J. Complexity*, 13:303–325, 1997.

[36] J.C. Meakin. An invitation to inverse semigroup theory. In K.P. Shum and P.C. Yuen (eds.), editors, *Proccedings of the SEAMS conference on ordered structures and algebra of computer languages*, pages 91–115, Singapore, 1993. World Scientific.

[37] J.D.P. Meldrum. *Wreath Products of Groups and Semigroups*, volume 74 of *Pitman Monographs and Surveys in Pure and Applied Mathematics.* Longman, England, 1995.

[38] W.D. Munn. On simple inverse semigroups. *Semigroup Forum*, 1:63–74, 1970.

[39] M. Petrich. *Inverse Semigroups.* Pure and Applied Mathematics. John Wiley & Sons, New York, 1984.

[40] M. Petrich and N.R. Reilly. *Completely Regular Semigroups*, volume 23 of *Canadian Mathematical Society Series of Monographs and Advanced Texts.* John Wiley & Sons, New York, 1999.

[41] R.J. Plemmons. There are 15973 semigroups of order 6. *Math. Algorithms*, 2:2–17, 1967.

[42] L. Rédei. Die verallgemeinerung der schreierschen erweiterungstheorie. *Acta Sci. Math. (Szeged)*, 14:252–273, 1952.

[43] D. Rees. On semi-groups. *Proc. Cambridge Philos. Soc.*, 36:387–400, 1940.

[44] N.R. Reilly. Bisimple ω-semigrops. *Proc. Glasgow Math. Assoc.*, 7:160–167, 1966.

[45] E.F. Robertson, N. Ruškuc, and M.R. Thomson. Finite generation and other finiteness conditions for wreath products of semigroups. In preparation.

[46] E.F. Robertson, N. Ruškuc, and M.R. Thomson. Finite generation and presentability of wreath products of monoids. Submitted.

[47] E.F. Robertson, N. Ruškuc, and J. Wiegold. Generators and relations of direct produts of semigroups. *Trans. Amer. Math. Soc.*, 350(7):2665–2685, 1998.

[48] J.C. Rosales. Function minimum associated to a congruence on integral n-tuple space. *Semigroup Forum*, 51:87–95, 1995.

[49] J.C. Rosales and P.A. García Sánchez. *Finitely Generated Commutative Monoids.* Nova Science Publishers, Inc., Commack, NY, 1999.

[50] J.C. Rosales and P.A. García Sanchez. On normal affine semigroups. *Linear Algebra and its Applications*, 286:175–186, 1999.

[51] J.C. Rosales and P.A. García Sanchez. Presentations for subsemiroups of finitely generated commutative semigroups. *Israel J. Math.*, 113:269–283, 1999.

[52] J.C. Rosales, P.A. García Sánchez, and J.I. García García. Commutative ideal extensions of abelian groups. *Semigroup Forum*, to appear.

[53] J.C. Rosales, P.A. García Sánchez, and J.M. Urbano Blanco. On presentations of commutative monoids. *Internat. J. Algebra Comput*, 9:539–553, 1999.

[54] N. Ruškuc. On the rank of completely 0-simple semigroups. *Math. Proc. Camb. Phil. Soc.*, 116:325–338, 1994.

[55] N. Ruškuc. *Semigroup Presentations.* PhD thesis, University of St Andrews, 1995.

[56] N. Ruškuc. On large subsemigroups and finiteness conditions of semigroups. *Proc. London Math. Soc.*, 76(3):383–405, 1998.

[57] N. Ruškuc. Presentations for subgroups of monoids. *J. Algebra*, 220(1):365–380, 1999.

[58] N. Ruškuc. On finite presentability of monoids and their Schützenberger groups. *Pacific J. Math*, 195(2):487–509, 2000.

[59] B.M. Schein. Free inverse semigroups are not finitely presentable. *Acta Math. Acad. Sci. Hung.*, 26:41–52, 1975.

[60] M.P. Schützenberger. D-représentation des demi-groupes. *C. R. Acad. Sci. Paris*, 244:1994–1996, 1957.

[61] M.P. Schützenberger. Sur la représentation monomiale des demi-groupes. *C. R. Acad. Sci. Paris*, 246:865–867, 1958.

[62] A. Seidenberg. Constructions in a polynomial ring over the ring of integers. *Amer. J. Math.*, 100(4):685–703, 1978.

[63] C.C. Sims. *Computation with Finitely Presented Groups.* Encyclopedia of mathematics. University Press, Cambridge, 1994.

[64] Leonard H. Soicher. GRAPE: a system for computing with graphs and groups. In *Groups and computation (New Brunswick, NJ, 1991)*, pages 287–291. Amer. Math. Soc., Providence, RI, 1993.

[65] R.M. Thomson. *Finiteness conditions of wreath products of semigroups and related properties of diagonal acts.* PhD thesis, University of St Andrews, in preparation.

Index