ABSTRACT

The purpose of this project is to study the partially degenerate stellar models. In order to begin this study, polytropic stellar models are discussed and the Lane-Emden equation is solved by a new numerical technique. Next, the partially degenerate models are proved to reduce to polytropic models, at their limits of very high and very low degeneracy.

Following this, the partially degenerate standard model is studied. The equation of equilibrium is solved and the physical characteristics are evaluated for different values of parameters.

To complete this study, the partially degenerate standard model is discussed as a convective model, and the luminosity is evaluated.

The adiabatic exponents are estimated for a mixture of degenerate electron gas and radiation. The FORTRAN IV programes are found in the appendices of this thesis.

PARTIALLY DEGENERATE STELLAR MODELS

by

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A Thesis presented for the Degree of Master of Science in the University of St Andrews

July 1977

IT IS DEDICATED TO MY PARENTS

This is to certify that Miss Joanna Manousoyannaki was admitted as a research student under Ordinance No. 51, that she has spent seven terms full-time research in the University of St Andrews and that the present thesis embodying the result of her special research can be submitted for the degree of Master of Science.

Signed:

Dr T. R. Carson, Supervisor

University Observatory,
St Andrews.
29 June 1977.

DECLARATION

Except where reference is made to the work of others, the research described in this thesis and the composition of the thesis are my own work. No part of this work has been previously submitted for a higher degree to this or any other University. Under Ordinance No. 338 (St Andrews No. 51), I was admitted to the Faculty of Science of the University of St Andrews as a research student and I was accepted as a candidate for the degree of M.Sc.

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INTRODUCTION

The present work aims to calculate partially degenerate stellar models whose basic equilibrium equation has been firstly described in S. Chandrasekhar's book "An Introduction to the Study of Stellar Structure".

The equation of state is based upon the Fermi-Dirac statistics for an electron gas and it is also modified by the addition of the pressure due to electromagnetic radiation. The contribution from the particle pressure of the nuclei in the gas has been neglected in the present treatment.

The equilibrium equation is solved using a method of numerical integration whose accuracy is tested in the solution of the Lane-Emden equation in Chapter I. Tables with the Lane-Emden functions are produced, showing the accuracy of our method.

In Chapter II the degeneracy of the electron gas is discussed and it is shown that the partially degenerate standard model equation of equilibrium reduces to the Lane-Emden equation of index $n = 3/2$ in the case of very high degeneracy and of index $n = 3$ in the case of very low degeneracy.

A complete discussion of the numerical solution of the partially degnerate standard model equation is given in Chapter III. Our results, for various degrees of degeneracy are tabulated and also the functions $M/M(R)$., P/P_c, T/T_c, ρ/ρ_c are shown in diagrams.

In Chapter V a criterion for convection is discussed and the adiabatic exponents (gammas) are derived for the mixture of partially degenerate electron gas and radiation.

In Chapter IV the luminosity of the completely convective partially degenerate standard models is discussed and approximate values are given.

The problem of the partially degenerate standard model has been discussed by G. Wares in his paper (Ap. J. 100, 1944). He gives results for only three values of the degeneracy parameter. The Fermi-Dirac integrals have been obtained by interpolation in the tables of J. McDougall and E. C. Stoner. In the present thesis the Fermi-Dirac integrals are obtained directly as required from the very accurate rational formulae given in a paper by W. J. Cody and H. C. Thacher. The values obtained by this latter method are much more accurate than those by the former. Also, the models which we obtained by the new method are more accurate than those by G. Wares. The position of the surface for a degeneracy parameter equal to 0 is at point 9.6 according to G. Wares. The value found in this thesis is at point 9.0. This value is checked by using logarithmic variables. When using logarithmic variables the value is found to be at point 9.025.

G. Wares has shown that the partially degenerate standard models are applicable for subdwarf stars as Wolf 134 and Wolf 1037 as well as for old novae with low hydrogen content.

The equation of state for partially degenerate matter has been considered by N. D. Limber for the study of the structure of M-dwarf stars which are suggested to be completely convective insofar as their interiors are concerned.

The structure of stars of very low mass has been studied by S. S. Kumar using the equation of state of a nonrelativistic partially degenerate gas. S. Kumar proved that there is a lower limit to the mass of a main sequence star under which the star becomes completely degenerate or "black dwarf". S. Kumar also showed in a second paper that the end-product of a star of very low mass is a completely degenerate object and that the known planetary companions can be identified with the dead dwarf stars.

We can see that the partially degenerate configurations can help in the understanding of the structure of stars of very low mass and they are also important for the study of the helium-core of highly evolved red giants (P. Demarque and J. Geisler).

The scope of the present thesis is to give a detailed account of the theory of the partially degenerate standard model. Suggestions for further study can be the problem of the isothermal gas sphere and also the construction of models for highly evolved red giants.

CHAPTER I

In the first part of this chapter the general theory of the hydrostatic equilibrium of a gas sphere will be discussed and the basic formulae will be derived.

The second part is concerned with polytropic stellar models. The Lane-Emden equation is derived and its analytical properties and physical characteristics of a polytropic configuration are discussed.

In the third part, a numerical method of solution of the Lane-Emden equation is introduced. Tables are obtained giving the results of our solution for various polytropic indices.

The accuracy of the method is checked by comparing our results to the known ones from the British Association for the Advancement of Science Mathematical Tables Vol. 2, 1932.

The actual FORTRAN IV program for the numerical solution of the classical nonlinear differential equation is given in appendix I.

A. GENERAL THEORY FOR THE HYDROSTATIC EQUILIBRIUM OF A STAR

We consider the equilibrium of an isolated static mass of gas held together by its own gravitational attraction, which in the absence of rotation, or any other disturbing causes, will settle down, into a distribution of spherical symmetry.

Let r denote the radius vector, measured from the center of the configuration

$P(r)$ be the pressure at any point r,

$g(r)$ the gravitational acceleration

$\rho(r)$ the density

$M(r)$ the mass enclosed inside r.

Since we have a spherically symmetrical distribution of matter the pressure P, the density ρ and the other physical variables will be functions of r only, and

$$M(r) = \int_0^r 4\pi r^2 \rho(r)\, dr \tag{1}$$

If a volume element of gas is to be held mechanically at a certain position in the sphere, neither being expelled outward by pressure, nor falling to the center of gravitational attraction, then it will be necessary for the pressure and gravity forces to sum to zero. The gravitational force at r, is due entirely to the mass $M(r)$ interior to r, since the symmetrical shell outside r does not exert resultant attraction in its interior. Hence

$$g(r) = \frac{G\,M(r)}{r^2} \tag{2}$$

where $G = 6.67 \times 10^{-8}$ dynes $\cdot\, cm^2/gm^2$

also if Φ is the gravitational potential, we have by definition:

$$g(r) = \frac{d\Phi}{dr} = \frac{G\,M(r)}{r^2} \tag{3}$$

The radial force on a volume element due to the pressure differential dP is equal to:

$$F_p = P\,dA - (P + dP)\,dA = -dP \cdot dA \tag{4}$$

where for the volume element we have dA as the cross-sectional area, and

$$\rho \, dA \, dr = dm \tag{5}$$

the mass of the volume element.

Here, we note that since dP is negative, the pressure force is positive.

By Newton's law, the attractive force for an element of mass dm is

$$F_G = -\frac{G M(r)}{r^2} \, dm \tag{6}$$

From (4) and (5) we get

$$F_p + F_G = 0 \quad \Rightarrow \; -dP \, dA - \frac{G M(r)}{r^2} \rho(r) dA \, dr = 0$$

$$\Rightarrow \; -dP = \frac{G M(r)}{r^2} \rho(r) \, dr$$

$$\text{or} \quad \frac{dP}{dr} = -\frac{G M(r)}{r^2} \rho(r) \tag{7}$$

which is the condition for hydrostatic equilibrium. From relation (1) we get

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r) \tag{8}$$

Eliminating M(r) between (7) and (8) we get

$$\frac{d}{dr}\left(\frac{r^2}{\rho(r)} \frac{dP}{dr} \right) = -G \frac{dM(r)}{dr} = -4\pi r^2 G \rho(r)$$

$$\Rightarrow \frac{1}{r^2} \frac{d}{dr} \left(\frac{r^2}{\rho(r)} \frac{dP(r)}{dr} \right) = -4\pi G \rho(r) \tag{9}$$

From (3) $g = \frac{d\Phi}{dr}$ and (7) $\Rightarrow$

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\Phi}{dr} \right) = 4\pi G \rho(r) \tag{10}$$

which is the analogue of Poisson's equation

$$\nabla^2 \Phi = 4\pi G \rho$$

which for spherical symmetry takes the form

$$\frac{d^2 \Phi}{dr} + \frac{2}{r} \frac{d\Phi}{dr} = 4\pi G \rho \tag{11}$$

From equation (1) we can easily obtain that for $r \to o$ the mass will be proportional to r^3 i.e. $M(r) \propto r^3$ while $\left(\frac{d\Phi(r)}{dr}\right)_{r\to 0} = 0$

$\Rightarrow \Phi(r)$ finite and

its value is calculated from

$$\left(\frac{d\Phi(r)}{dr}\right)_{r=0} = \lim_{r\to 0} \frac{GM(r)}{r^2} = G\,\frac{4}{3}\,\frac{\pi r^3}{r^2}\,\rho_0(r) = \frac{4}{3}\pi G r \rho_0$$

where $\rho_0 = \rho\,(r = r_0)$

The gravitational acceleration $g(r)_{r=0} = 0$

and also $\left(\frac{dP(r)}{dr}\right)_{r=0} = 0$ when $\rho(r)$ is finite

At the boundary for $r = R$, $\Phi = \frac{GM}{R}$, $M(R) = M$.

B. POLYTROPES

The pressure in the static configuration of the gaseous sphere, is determined by the equation of state, applicable to the local conditions of the stellar interior.
The consideration of the hydrostatic equilibrium as well as that of the equation of state, do not, in themselves, determine the structure of a star. We need more conditions on the density and temperature in a stellar interior which, together with the hydrostatic equilibrium conditions will specify the stellar structure. An explicit auxiliary condition that has been found to correspond to certain idealized physical situations is of the form

$$P = K\rho^{\frac{n+1}{n}} \quad \text{or} \quad P = K\rho^{\gamma'} \tag{12}$$

where $\gamma' = \frac{n+1}{n}$

Equation (12) governs a thermodynamical polytropic change when this change is also adiabatic then $\gamma' = \gamma$. Gaseous spheres in hydrostatic equilibrium in which the pressure and density are related by (12) at each point along the radius are called polytropes.
The constants k and n (or γ) depend upon the nature of the polytrope.

An example of a stellar model which can be represented by a polytrope is the one studied by Kelvin and considered to be in a state of adiabatic-convective equilibrium. If for this model radiation pressure is of no importance to the structure of the star, then the pressure will be given by the well known relation for adiabatic changes

$$P = K\rho^{\gamma}$$

where $\gamma = 5/3$ for an ideal monatomic gas and the polytropic index $n = 3/2$

A second example of a configuration for which a polytrope can be applied is the one of Eddington's standard model. In this case, by introducing a quantity β such that

$$P_{gas} = \beta P \quad \text{and} \quad P_{radiation} = \frac{1}{3}aT^4 = (1-\beta)P$$

where P is the total pressure, we easily obtain that for a perfect gas

$$T = \left(\frac{k}{\mu H} \frac{1-\beta}{\beta}\right)^{1/3} \left(\frac{3}{a}\right)^{1/3} \rho^{1/3} \qquad \text{or}$$

$$P = \left[\left(\frac{k}{\mu H}\right)^4 \frac{3}{a} \frac{1-\beta}{\beta^4} \right] \rho^{4/3}$$

Assuming now that β is a constant throughout the star then $P = \text{const} \cdot \rho^{4/3}$ which is a polytrope of index $n = 3$

A third example of a polytropic configuration is the one of the white dwarfs where the pressure is the pressure of a completely degenerate electron gas. The pressure of such a gas is proportional to $\rho^{5/3}$ when the electron momenta are not relativistic ($p \ll mc$) and to $\rho^{4/3}$ when the electron momenta are relativistic ($p \gg mc$). Since the nuclei pressure is negligible small in comparison to the electron pressure, then the total pressure is taken to be equal to the electron pressure alone. So, a non-relativistic, completely degenerate model will be represented by a polytrope of index $n = 3/2$ $(\gamma = 5/3)$, while an extremely relativistic completely degenerate model will be represented by a polytrope of index $n = 3$ $(\gamma = 4/3)$

In the following paragraphs we shall in brief refer to the theory and the equation of equilibrium of the polytropes.

From thermodynamics we get that for a polytropic change (i.e. a quasi-statistical change for which dQ/dT = C = constant)

$$P = \rho^{\gamma'} \text{ constant},$$
$$P^{1-\gamma'} \; T^{\gamma'} = \text{constant},$$
$$T = \rho^{\gamma'-1} \text{ constant}$$

where $\gamma' = Cp - C/Cv - C$

Since the density ρ is proportional to $T^{1/\gamma'-1}$ in a polytrope of index $n = 1/\gamma'-1$, a convenient definition is $\rho = \lambda\theta^n$ (13)

where λ is a scaling parameter whose equilibrium depends upon the definition of θ .

For this representation the pressure is

$$P = K \rho^{\frac{n+1}{n}} = K \lambda^{\frac{n+1}{n}} \theta^{n+1} \qquad (14)$$

Substitute (13) and (14) in equ. (9)

$$\left[\frac{K\lambda^{\frac{1}{n}-1}(n+1)}{4\pi G}\right]\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = -\theta^n \qquad (15)$$

Introduce a unit length

$$a \equiv \left[\frac{(n+1)K}{4\pi G}\lambda^{\frac{1}{n}-1}\right]^{1/2} \qquad (16)$$

and a dimension less distance variable $\xi = r/a$ (16a) whereupon equ. (15) reduces to

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n$$

or

$$\frac{2}{\xi}\frac{d\theta}{d\xi} + \frac{d^2\theta}{d\xi^2} = -\theta^n \qquad (17)$$

This equation is called the "Lane Emden" equation for the structure of polytropes of index n. Although the problem of the gravitational equilibrium of a gas sphere was first studied by I. J. Lane and equ. (17) was first explicitly established by A. Ritter, V. R. Emden was the first to systematize the earlier work and also to include new results and extensive tables in his work Gaskugeln (1907).

Since we assume that at the surface $\rho = 0$ when $\theta = 0$ and at the center $\rho = \rho_c$ when $\theta = 1$, it is clear that we are interested in those values of the solution between 0 and 1.

The solution for θ as a function of ξ determines the structure of the polytrope except for the choice of the central density.

We can choose λ to be equal to the central density ρ_c.

It is evident that the solution of (17) must satisfy the boundary conditions

$$\theta = 1 \,, \quad \frac{d\theta}{d\xi} = 0 \quad \text{at} \quad \xi = 0 \qquad (18)$$

under these conditions in order to find $\frac{d^2\theta}{d\xi^2}$, we use de l'Hospital's rule to evaluate the term

$$\frac{2}{\xi}\frac{d\theta}{d\xi} \quad \text{at} \quad \xi \to 0$$

Indeed we have

$$\frac{2}{\xi}\frac{d\theta}{d\xi} \longrightarrow 2\,\frac{d^2\theta}{d\xi^2}$$

and (17) becomes

$$\frac{2}{\xi}\frac{d\theta}{d\xi} + \frac{d^2\theta}{d\xi^2} = 3\,\frac{d^2\theta}{d\xi^2} \quad \Rightarrow$$

$$3\,\frac{d^2\theta}{d\xi^2} = -1 \quad \Rightarrow \quad \frac{d^2\theta}{d\xi^2} = -\frac{1}{3} \quad \text{at} \quad \xi \to 0,\ \theta = 1$$

Explicity solutions of the Lane-Emden equ. for general values of n, apparently do not exist.

In order to preserve the continuity of the discussion, the explicit solutions for n = 0, 1 and 5 (Stellar Structure, S. Chandresekhar, p 91) are included where the solution is reduced to a classical function

a) $n = 0$

(17) becomes $$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -1 \quad \Rightarrow$$

$$\xi^2\frac{d\theta}{d\xi} = -\frac{1}{3}\xi^3 - C \quad \Rightarrow$$

$$\theta = D + \frac{C}{\xi} - \frac{1}{6}\xi^2$$

where -C, D are the constants for the two integrations, for $C \neq 0$ we have a singularity for $\xi \to 0$,
for $C = 0$ we get the solution

$$\theta = D - \frac{1}{6}\xi^2$$

For the boundary conditions (18) the solution reduces to

$$\theta = 1 - (\xi^2/6)$$

and for $\xi = \sqrt{6} \quad \Rightarrow \quad \theta(\xi) = 0$

A polytrope of index $n = 0$ corresponds to a constant density model.

b) $n=1$

(17) becomes (using the transformation $\theta = x/\xi$)

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi\frac{dx}{d\xi}-x\right) = -\frac{x^n}{\xi^n} \Longrightarrow$$

$$\frac{d}{d\xi}\left(\xi\frac{dx}{d\xi}-x\right) = -\frac{x^n}{\xi^n}\,\xi^2 \Longrightarrow$$

$$\xi\frac{d^2x}{d\xi^2}+\frac{dx}{d\xi}-\frac{dx}{d\xi} = -\frac{x^n}{\xi^{n-2}} \Longrightarrow$$

$$\frac{d^2x}{d\xi^2} = -\frac{x^n}{\xi^{n-1}} \quad . \text{ For } n=1 \Rightarrow \frac{d^2x}{d\xi^2} = -x$$

The general solution is $x = c\sin(\xi-\delta) \Rightarrow \theta = c\sin(\xi-\delta)/\xi$

where c and δ are the constants of integrations

For $\delta \neq 0$ we have a singularity for $\xi \to 0$

For $\delta = 0 \Rightarrow \theta = c\sin\xi/\xi$

and for the boundary conditions (18) this function has its first zero at $\xi = \pi$

c) $n=5$

Introducing $x = 1/\xi$ equ. (17) becomes

$$x^4\frac{d^2\theta}{dx^2} = -\theta^n \qquad (c1)$$

We first look for a solution of the form $\theta = a x^{\bar\omega}$ (c2)

Substituting (c2) in (c1) $\Rightarrow a\bar\omega(\bar\omega-1)\,x^{\bar\omega+2} = -a^n x^{n\bar\omega}$

valid $\forall x$, $\Rightarrow \bar\omega+2 = n\bar\omega$, $a^{n-1} = \bar\omega(1-\bar\omega)$

For $n>3$ and $\bar\omega < 1$

we have a singular solution

$$\theta_s = \left[\frac{2(n-3)}{(n-1)^2}\right]^{1/n-1} x^{2/n-1}$$

Since $\theta = a x^{\bar\omega}$ is a solution of (c1), we make the transformation

$$\Theta = A \cdot z\, x^{\bar\omega} \quad , \bar\omega = 2/n-1 \quad \Rightarrow$$

$$\frac{d\Theta}{dx} = A \frac{dz}{dx} x^{\bar\omega} + \bar\omega A z x^{\bar\omega - 1} \Rightarrow$$

$$\frac{d^2\Theta}{dx^2} = A \frac{d^2z}{dx^2} x^{\bar\omega} + A \frac{dz}{dx} \bar\omega x^{\bar\omega-1} + \bar\omega A \frac{dz}{dx} x^{\bar\omega-1} + \bar\omega(\bar\omega-1) A z x^{\bar\omega-2}$$

$$= A\left(x^{\bar\omega} \frac{d^2z}{dx^2} + 2\bar\omega \frac{dz}{dx} x^{\bar\omega-1} + \bar\omega(\bar\omega-1) z x^{\bar\omega-2} \right) \qquad (C3)$$

Using (C1), (C3) becomes $A\left(x^{\bar\omega+4} \frac{d^2z}{dx^2} + 2\omega \frac{dz}{dx} x^{\bar\omega-1+4} + \bar\omega(\bar\omega-1) z x^{\bar\omega-2+4}\right) - A^{n} z^{n} x^{n\bar\omega} = 0$

$$\text{or} \quad x^{\bar\omega+4} \frac{d^2z}{dx^2} + 2\bar\omega \frac{dz}{dx} x^{\bar\omega+3} + \bar\omega(\bar\omega-1) z x^{\bar\omega+2} - A^{n-1} z^{n} x^{\bar\omega n} = 0$$

$$\text{or} \quad x^{n\bar\omega+2} \frac{d^2z}{dx^2} + 2\bar\omega \frac{dz}{dx} x^{n\bar\omega+1} + \bar\omega(\bar\omega-1) z x^{n\bar\omega} - A^{n-1} z^{n} x^{\bar\omega n} = 0$$

$$\text{or} \quad x^2 \frac{d^2z}{dx^2} + 2\bar\omega \frac{dz}{dx} x + \bar\omega(\bar\omega-1) z - A^{n-1} z^{n} = 0 \qquad (C4)$$

We now substitute $x = 1/\zeta = e^{t}$ (C5)

$$\Rightarrow \quad t = \ln x = -\ln \zeta$$

$$\Rightarrow \quad \frac{dz}{dx} = \frac{dz}{de^{t}} = e^{-t} \frac{dz}{dt}$$

$$\Rightarrow \quad \frac{d^2z}{dx^2} = e^{-2t} \frac{d^2z}{dt^2} - e^{-2t} \frac{dz}{dt} = e^{-2t}\left(\frac{d^2z}{dt^2} - \frac{dz}{dt} \right)$$

Substituting in (C4)

$$e^{2t} e^{-2t} \frac{d^2z}{dt^2} - e^{2t} e^{-2t} \frac{dz}{dt} + 2\bar\omega \frac{dz}{dt} e^{-t} e^{t} + \bar\omega(\bar\omega-1) z - A^{n-1} z^{n} = 0$$

$$\Rightarrow \quad \frac{d^2z}{dt^2} + (2\bar\omega - 1) \frac{dz}{dt} + \bar\omega(\bar\omega-1) z - A^{n-1} z^{n} = 0 \qquad (C6)$$

For $n > 3$ we choose $A = a$, $A^{n-1} = a^{n-1} = \bar\omega(1-\bar\omega)$ and (C6)

becomes

$$\frac{d^2z}{dt^2} + (2\bar\omega - 1) \frac{dz}{dt} - \bar\omega(1-\bar\omega)(z - z^{n}) = 0$$

Since $(n-1)\bar\omega = 2 \quad \Rightarrow \quad \bar\omega = 2/n-1 \quad \Rightarrow$

$$\frac{d^2z}{dt^2} + \frac{5-n}{n-1}\,\frac{dz}{dt} - \frac{2(n-3)}{(n-1)^2}\,(z - z^n) = 0 \qquad (C7)$$

this, for $n = 5$ becomes

$$\frac{d^2z}{dt^2} - \frac{4}{4^2}\,(z - z^5) = 0$$

$$\Rightarrow \frac{d^2z}{dt^2} = \frac{1}{4}\,z(1 - z^4) \qquad (C8)$$

Multiplying both sides of (C8) by $\frac{dz}{dt}$ we get

$$\frac{dz}{dt}\cdot\frac{d^2z}{dt^2} = \frac{1}{4}\,z(1 - z^4)\,\frac{dz}{dt}$$

$$\Rightarrow \frac{1}{2}\,\frac{d}{dt}\left[\left(\frac{dz}{dt}\right)^2\right] = \frac{1}{4}\,z(1 - z^4)\,\frac{dz}{dt}$$

Integrating we get

$$\frac{1}{2}\left(\frac{dz}{dt}\right)^2 = \frac{1}{8}\,z^2 - \frac{1}{24}\,z^6 + D$$

$$\Rightarrow \left(\frac{dz}{dt}\right)^2 = \frac{1}{4}\,z^2 - \frac{1}{12}\,z^6 + 2D \qquad (C9)$$

For $z \to \pm\infty \Rightarrow \left(\frac{dz}{dt}\right)^2 \to -\infty$

which is inconsistent since $\frac{dz}{dt}$ is real

From (C9) $\Rightarrow$

$$\frac{dz}{\pm\left[2D + \frac{1}{4}z^2 - \frac{1}{12}z^6\right]^{1/2}} = dt$$

In the case of $D = 0$,

$$\frac{dz}{\pm\left[z^2\left(\frac{1}{4} - \frac{1}{12}z^4\right)\right]^{1/2}} = dt$$

$$\Rightarrow \frac{dz}{z\left(1 - \frac{1}{3}z^4\right)^{1/2}} = -\frac{1}{2}\,dt \qquad (C10)$$

We substitute $\frac{1}{3}\,z^4 = \sin^2\zeta$ (C11)

$$\Rightarrow 4\,\frac{dz}{z} = 2\,\frac{\cos\zeta}{\sin\zeta}\,d\zeta$$

(C10) becomes $\frac{1}{2}\frac{\cos\zeta}{\sin\zeta}\, d\zeta\, \frac{1}{\cos\zeta} = -\frac{1}{2}\, dt$

$$\Rightarrow \operatorname{cosec}\zeta\, d\zeta = -dt \qquad (C12)$$

Integrate (C12)

$$\Rightarrow \tan\frac{1}{2}\zeta = c e^{-t} \qquad (C13)$$

From (C13) $\Rightarrow -t = \ln\left(\frac{1}{C}\tan\frac{\zeta}{2}\right)$

$$\Rightarrow \tan\frac{\zeta}{2} = C e^{-t}$$

where C = integrating constants

From (C11) $\frac{1}{3}z^4 = \left[\frac{2\tan\zeta/2}{1+\tan^2\zeta/2}\right]^2$

$$\Rightarrow z = \pm\left[\frac{12\, c^2\, e^{-2t}}{(1+c^2 e^{-2t})^2}\right]^{1/4}$$

Recall that $\theta = \left(\frac{x}{2}\right)^{1/2} z = \left(\frac{1}{2}e^t\right)^{1/2} z$

$$\Rightarrow \theta = \pm\left[\frac{3c^2}{(1+c^2 e^{-2t})^2}\right]^{1/4}$$

The Lane-Emden function for $n=5$ is

$$\theta = \frac{1}{\left(1+\frac{1}{3}\xi^2\right)^{1/2}} \qquad (C14)$$

The solution for $n=5$ corresponds to a sphere of infinite radius.

PHYSICAL CHARACTERISTICS OF THE LANE EMDEN EQUATION

We can easily see that when the Lane Emden function $\theta(\xi)$ is known for a given polytropic index n and a fixed value for k and λ we can construct a stellar polytropic model by using the following very useful formulae.

(1) The radius R is given by (16a) (19)

$$R = a\xi_n = \left[\frac{(n+1)}{4\pi G} k \lambda^{\frac{1}{n}-1} \right]^{1/2} \xi_n$$

where ξ_n defines the zero of the Lane-Emden function θ_n .

(2) The mass is given by

$$M(\xi) = \int_0^{a\xi} 4\pi\rho r^2 dr = 4\pi a^3 \lambda \int_0^{\xi} \xi^2 \theta^n d\xi$$

$$= -4\pi a^3 \lambda \xi^2 \frac{d\theta}{d\xi} \quad (20)$$

and the total mass is

$$M = -4\pi \left[\frac{(n+1)k}{4\pi G} \right]^{3/2} \lambda^{\frac{3-n}{2n}} \left(\xi^2 \frac{d\theta}{d\xi} \right)_{\xi=\xi_n} \quad (21)$$

(3) The mean density

$$\bar{\rho}(\xi) = \frac{M(\xi)}{\frac{4}{3}\pi a^3 \xi^3} = -\frac{3}{\xi} \left(\frac{d\theta}{d\xi} \right) \lambda$$

and since λ is the central density

$$\lambda = \rho_c = -\left[\frac{\xi}{3} \frac{1}{\frac{d\theta}{d\xi}} \right]_{\xi=\xi_n} \bar{\rho} \quad (22)$$

while $\rho = \lambda \theta^n$

(4) The central pressure

$$P = k \rho^{\frac{n+1}{n}} = k \lambda^{\frac{n+1}{n}} \theta^{n+1}$$

From equ. (19) $\Rightarrow$

$$R = \left[\frac{n+1}{4\pi G} \xi_n^2 \right]^{1/2} \left(k \lambda^{\frac{1-n}{n}} \right)^{1/2} \Rightarrow$$

$$\Rightarrow K\lambda^{\frac{1-n}{n}} = \frac{4\pi R^2 G}{(n+1)\xi_n^2}$$

since $\theta=1$ at the origin

$$P_c = K\lambda^{\frac{1-n}{n}}\lambda^2 = K\lambda^{\frac{1-n}{n}}\rho_c^2 = \frac{4\pi R^2 G}{(n+1)\xi_n^2}\left[\frac{\xi_n}{3}\frac{1}{\left(\frac{d\theta}{d\xi}\right)_{\xi=\xi_n}}\right]^2\bar{\rho}^2$$

$$\Rightarrow P_c = \frac{1}{4\pi(n+1)\left(\frac{d\theta}{d\xi}\right)^2_{\xi=\xi_n}}\frac{GM^2}{R^4} \qquad (23)$$

(5) The central temperature.

This can be computed by the central pressure and central density, if we know the appropriate equation of state.

For a perfect gas, the equ. of state

$$P = \frac{1}{\beta}\frac{R}{\mu}\rho T \qquad \text{where } P = \text{total pressure}$$

and

$$P = \frac{1}{\beta}P_{gas} = \frac{1}{1-\beta}P_{radiation}$$

$$\Rightarrow T_c = \frac{\mu}{R}\frac{\beta_c P_c}{\rho_c} \qquad (24)$$

(6) The gravitational acceleration

$$g(r) = \frac{GM(r)}{r^2} = -\frac{4\pi G a^3 \lambda \xi^2 \frac{d\theta}{d\xi}}{a^2\xi^2} =$$

$$= -4\pi G a\lambda\frac{d\theta}{d\xi} = -4\pi G\left[\frac{(n+1)K}{4\pi G}\right]^{1/2}\lambda^{(1/n-1)\frac{1}{2}}\lambda\frac{d\theta}{d\xi}$$

$$= -\left[(n+1)K\right]^{1/2}(4\pi G)^{1/2}\lambda^{\frac{1}{2}(1/n+1)}\frac{d\theta}{d\xi} \qquad (25)$$

(7) The gravitational energy of a polytrope

$$\Omega = -G\int_0^R \frac{M(r)\,dM(r)}{r} = \text{(in the case of hydrostatic equilibrium)}$$

$$= \frac{1}{2}\int_0^R \Phi(r)\; dM(r)$$

where $-\dfrac{d\Phi(r)}{dr} = \dfrac{1}{\rho}\dfrac{dP}{dr}$

$$\Rightarrow \frac{1}{\rho}\frac{dP}{dr} = (n+1)\,\frac{d}{dr}\,\frac{P}{\rho}$$

$$\Rightarrow (n+1)\,\frac{P}{\rho} = \Phi(R) - \Phi(r)$$

and since $\Phi(R) = -\dfrac{GM}{R}$

$$\Rightarrow -\Phi(r) = (n+1)\,\frac{P}{\rho} + \frac{GM}{R}$$

Relation $\Omega = \dfrac{1}{2}\displaystyle\int_0^R \Phi(r)\,dM(r)$ becomes

$$-\Omega = \frac{1}{2}(n+1)\int_0^R \frac{P}{\rho}\,dM(r) + \frac{1}{2}\,\frac{GM}{R}\int_0^R dM(r)$$

But $-\Omega = 3\displaystyle\int_0^R P\,dV$, for a volume element dV

$$\Rightarrow -\Omega = -\frac{1}{2}(n+1)\,\frac{\Omega}{3} + \frac{1}{2}\,\frac{GM^2}{R}$$

$$\Rightarrow -\Omega = \frac{3}{5-n}\,\frac{GM^2}{R} \qquad (26)$$

C. A NUMERICAL SOLUTION OF THE LANE-EMDEN EQUATION

We are now proceeding to find a numerical technique for solving the Lane-Emden equation.

The purpose of this particular project is to test the accuracy of the numerical technique, by applying it to the known Lane-Emden equation, which (technique) is going to be used later for the solution of a more complicated equilibrium equation, namely the equilibrium equation of a partially degenerate stellar model.
As we shall see the accuracy of the method is up to the sixth decimal point, which is considered to be very satisfactory.

In the following section we shall give the detailed analysis of the numerical solution of the differential equation (17), for values of n between 0 and 5, and solutions over an adequate range of variables ξ will be obtained.

We first derive the Taylor series expansion of θ, $\frac{d\theta}{d\xi}$ which will be used to find the starting values of the problem for the numerical solution of the differential equation.

We first note that if $\theta(\xi)$ is a solution of the equation then $\theta(-\xi)$ is also a solution. This implies that if θ is expressed as a power series in ξ only even powers of ξ appear, that is:

$$\theta(\xi) = a_0 + a_2\xi^2 + a_4\xi^4 + \dots = \sum_{\nu=0}^{\infty} a_\nu \xi^\nu \qquad (27)$$

with for $m = 2\nu+1 \quad \forall \nu \,, \Rightarrow a_m = 0$

In order to evaluate the coefficients $a_0, a_2, a_4 \ldots$ we do the following algebraic calculations:

Let
$$\theta = a_0 + a_1\xi + a_2\xi^2 + \ldots = \sum_{\nu=0}^{\infty} a_\nu \xi^\nu \tag{28}$$

since for $\xi = 0,\ \theta = 1 \Rightarrow \theta = a_0 = 1$

$$(28) \Rightarrow \frac{d\theta}{d\xi} = a_1 + 2a_2\xi + 3a_3\xi^2 + \ldots = \sum_{\nu=1}^{\infty} \nu a_\nu \xi^{\nu-1} \tag{29}$$

from the boundary conditions we can also see that $a_1 = 0$.

The second derivative is

$$\frac{d^2\theta}{d\xi^2} = 2a_2 + 6a_3\xi + 12a_4\xi^2 + \ldots = \sum_{\nu=2}^{\infty} \nu(\nu-1)a_\nu \xi^{\nu-2} \tag{30}$$

From (29) and for $\xi \neq 0$

$$\frac{2}{\xi}\frac{d\theta}{d\xi} = 4a_2 + 6a_3\xi + 8a_4\xi^2 + 10a_5\xi^3 + 12a_6\xi^4 + $$
$$+ 16a_8 + 18a_9\xi^7 + 20a_{10}\xi^8 + 22a_{11}\xi^9 + 24a_{12}\xi^{10} + \ldots$$
$$= \sum_{\nu=2}^{\infty} 2\nu a_\nu \xi^{\nu-2} \tag{31}$$

The series (30) and (31) can be added

$$\frac{d^2\theta}{d\xi^2} + \frac{2}{\xi}\frac{d\theta}{d\xi} = 6a_2 + 12a_3\xi + 20a_4\xi^2 + 30a_5\xi^3 + 42a_6\xi^4 + $$
$$+ 72a_8\xi^6 + 90a_9\xi^7 + 110a_{10}\xi^8 + 132a_{11}\xi^9 + 156a_{12}\xi^{10} + \tag{32}$$
$$+ \ldots$$

From equ. (28) we get:

$$\theta^n = (a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3 + a_4\xi^4 + \ldots)^n \overset{a_1=0}{\underset{a_0=1}{=}}$$
$$= 1 + n(a_2\xi^2 + a_3\xi^3 + a_4\xi^4 + \ldots) + \frac{n(n-1)}{2!}(a_2\xi^2 + a_3\xi^3 + a_4\xi^4 + \ldots)^2$$
$$+ \frac{n(n-1)(n-2)}{3!}(a_2\xi^2 + a_3\xi^3 + a_4\xi^4 + \ldots)^3$$

$$+\frac{n(n-1)(n-2)(n-3)}{4!}(a_2\xi^2+a_3\xi^3+a_4\xi^4)^4+\dots$$

$$\Rightarrow \Theta^n = 1 + n a_2\xi^2 + n a_3\xi^3 + n a_4\xi^4 + n a_5\xi^5 + n a_6\xi^6 + n a_7\xi^7 + \dots$$

$$+\frac{n(n-1)}{2!}a_2^2\xi^4+\frac{n(n-1)}{2!}a_3^2\xi^6+\frac{n(n-1)}{2!}a_4^2\xi^8+\frac{n(n-1)}{2!}\xi^{10}a_5^2+\dots$$

$$+\frac{n(n-1)(n-2)}{3!}\left[a_2^3\xi^6+a_3^3\xi^9+\dots\right]$$

$$+\frac{n(n-1)(n-2)}{3!}\left[3\,a_2^2a_3\xi^7+\dots\right]$$

$$+\frac{n(n-1)(n-2)(n-3)}{4!}\left[a_2^4\xi^8+\dots\right] \qquad (33)$$

From equ. (32) and (33)

$$6a_2+1=0 \qquad \Rightarrow a_2=-\frac{1}{6}=-\frac{1}{3!}$$

$$12a_3=0 \qquad \Rightarrow a_3=0$$

$$20a_4=-na_2 \qquad \Rightarrow a_4=+\frac{n}{120}=\frac{n}{5!}$$

$$30a_5=-na_3 \qquad \Rightarrow a_5=-\frac{n}{30}\cdot 0=0$$

$$42a_6+na_4+\frac{n(n-1)}{2!}a_2^2=0 \qquad \Rightarrow a_6=\left[-\frac{n^2}{120}-\frac{n(n-1)}{2}\frac{1}{36}\right]\Big/42$$

$$\Rightarrow a_6=-\frac{n(8n-5)}{15120=3\times 7!}$$

$$56\, a_7 + n a_5 + \frac{n(n-1)}{2!}\, 2a_2 a_3 = 0 \qquad \Rightarrow \quad a_7 = 0$$

$$72\, a_8 + n a_6 + \frac{n(n-1)}{2!}\,(a_3^2 + 2a_2 a_4) + \frac{n(n-1)(n-2)}{3!}\, a_2^3$$

$$\Rightarrow a_8 = \frac{70n - 183n^2 + 122n^3}{9 \times 9! = 3265920}$$

$$90\, a_9 + n a_7 + \frac{n(n-1)}{2!}\,(2a_3 a_4 + 2a_2 a_5) + \frac{n(n-1)(n-2)}{3!}\, 2a_2^2 a_3 = 0$$

$$\Rightarrow \quad a_9 = 0$$

$$110\, a_{10} + n a_8 + \frac{n(n-1)}{2!}\left[a_4^2 + 2a_3 a_5 + 2a_2 a_6\right] + \frac{n(n-1)(n-2)}{3!}\left[3a_2^2 a_4 + 3a_3^2 a_2\right] +$$

$$+ \frac{n(n-1)(n-2)(n-3)}{4}\, a_2^4 = 0$$

$$\Rightarrow \quad a_{10} = \frac{3150n - 10805n^2 + 12642n^3 - 5032n^4}{45 \times 11! = 1796256000}$$

From the above calculations we get:

$$\theta = 1 - \frac{1}{3!}\xi^2 + \frac{n}{5!}\xi^4 - \frac{(8n^2 - 5n)}{3 \times 7!}\xi^6 + \frac{(70n - 183n^2 + 122n^3)}{9 \times 9!}\xi^8$$

$$+ \frac{n(3150n - 1080n^2 + 12642n^3 - 5032n^4)}{45 \times 11!}\xi^{10} + \ldots \tag{34}$$

$$\frac{d\theta}{d\xi} = -\frac{1}{3}\xi + \frac{n}{30}\xi^3 - \frac{(8n^2 - 5n)}{21 \times 5!}\xi^5 + \frac{(70n - 183n^2 + 122n^3)}{81 \times 7!}\xi^7$$

$$+ \frac{n(3150n - 10805n^2 + 12642n^3 - 5032n^4)}{45 \times 9! \times 11}\xi^9 + \ldots \tag{35}$$

The second derivative is calculated by (34) and (35) if we substitute them in equ. (17)

$$\frac{d^2\theta}{d\xi^2} = -\theta^n - \frac{2}{\xi}\frac{d\theta}{d\xi} \qquad (36).$$

Relations (34), (35), (36) will be used to find the starting values of the problem for specific polytropic indices $n=0(0.5)5$ by using

$$\xi_1 = 0$$

$$\xi_i = \xi_{i-1} + \Delta\xi \qquad i=2,3,\ldots$$

$\Delta\xi$ is an interval ahead, equal to all the steps of the integration. The interval $\Delta\xi$ is reduced to be as sufficient as possible for our computation with the IBM 360 computer of the University of St Andrews (256K bytes)

The following numerical method of solution of second order differential equation has been modified in an improved form from the original method which was described in "Numerical Mathematical Analysis" by J. B. Scarborough.

The principle, behind a numerical technique is that for any ordinary differential equation having numerical coefficients and initial conditions, there exists a method of solution. Starting with the initial values, the solution is tnence constructed by short steps ahead at equal intervals Δx , each step usually being checked by some method before proceeding to the next step.

The second order, nonlinear differential equation

$$\frac{d^2\theta}{d\xi^2} = -\theta^n - \frac{2}{\xi}\frac{d\theta}{d\xi} \qquad (36)$$

can be reduced to a system of first order equations by putting

$$\frac{d\theta}{d\xi} = \theta' \qquad (37)$$

The resultant equations are

$$\begin{cases} \dfrac{d\theta}{d\xi} = \theta' \\ \dfrac{d\theta'}{d\xi} = -\theta^n - \dfrac{2}{\xi}\theta' \end{cases} \qquad (38)$$

with the initial conditions $\xi = 0$, $\theta = 1$, $\theta' = 0$, $\frac{d\theta'}{d\xi} = -\frac{1}{3}$ (39)

Since the second equation involves θ directly, it is necessary to compute θ at every step throughout the computation.

We approximate θ' by a polynomial, namely the Newton's formulae for backward interpolation

$$\frac{d\theta}{d\xi} = \theta' = \theta'_n + u\Delta_1\theta'_n + \frac{u(u+1)}{2}\Delta_2\theta'_n + \frac{u(u+1)(u+2)}{3!}\Delta_3\theta'_n +$$

$$+ \frac{u(u+1)(u+2)(u+3)}{4!}\Delta_4\theta'_n + \ldots + \frac{u(u+1)(u+2)\ldots(u+n-1)}{n!}\Delta_n\theta'_n \quad (40)$$

where $u = \frac{\xi - \xi_n}{h}$ or $\xi = \xi_n + hu$, $h = \Delta\xi$ (41)

$\Delta_n\theta_n$ are the horizontal differences.

We can now integrate the polynomial over any interval.

The change of θ for any interval where $d\theta/d\xi$ is continuous is given by the formula

$$\Delta\theta = \int_{\xi_k}^{\xi_{k+1}} \left(\frac{d\theta}{d\xi}\right) d\xi = \int_{\xi_k}^{\xi_{k+1}} \theta' \, d\xi \quad (42)$$

$$\Rightarrow \Delta\theta = \int_{\xi_k}^{\xi_{k+1}} \Big[\theta'_n + \Delta_1\theta'_n u + \frac{\Delta_2\theta'_n}{2}(u^2+u) + \frac{\Delta_3\theta'_n}{6}(u^3+3u^2+2u) +$$

$$+ \frac{\Delta_4\theta'_n}{24}(u^4+6u^3+11u^2+6u) + \frac{\Delta_5\theta'_n}{120}(u^5+10u^4+35u^3+50u^2+24u) +$$

$$+ \frac{\Delta_6\theta'_n}{720}(u^6+15u^5+85u^4+225u^3+274u^2+120u)$$

$$+ \frac{\Delta_7\theta'_n}{5040}(u^7+21u^6+175u^5+735u^4+1624u^3+1764u^2+720u)\Big] d\xi \quad (43)$$

Since $\xi = \xi_m + hu \Rightarrow d\xi = h\,du \Rightarrow$

$$\Delta\theta = h\int_{u_k}^{u_{k+1}} \left[\theta'_m + \Delta_1\theta'_m + \frac{\Delta_2\theta'_m}{2}(u^2+u) + \dots\right] du \qquad (44)$$

$$\Rightarrow \Delta\theta = h\Big[\theta'_m u + \Delta_1\theta'_m \frac{u^2}{2} + \frac{\Delta_2\theta'_m}{2}\left(\frac{u^3}{3}+\frac{u^2}{2}\right) + \frac{\Delta_3\theta'_m}{6}\left(\frac{u^4}{4}+u^3+u^2\right) +$$

$$+ \frac{\Delta_4\theta'_m}{24}\left(\frac{u^5}{5}+\frac{6u^4}{4}+\frac{11u^3}{3}+\frac{6u^2}{2}\right) + \frac{\Delta_5\theta'_m}{120}\left(\frac{u^6}{6}+\frac{10u^5}{5}+\frac{35}{4}u^4+\frac{50}{3}u^3+\frac{24}{2}u^2\right)$$

$$+ \frac{\Delta_6\theta'_m}{720}\left(\frac{u^7}{7}+15\frac{u^6}{6}+17u^5+225\frac{u^4}{4}+274\frac{u^3}{3}+60u^2\right) +$$

$$+ \frac{\Delta_7\theta'_m}{5040}\left(\frac{u^8}{8}+3u^7+175\frac{u^6}{6}+147u^5+406u^4+588u^3+360u^2\right)\Big]_{u_k}^{u_{k+1}} \qquad (45)$$

We have that

$$u_{k+1} = (\xi_{n+1}-\xi_n)/h = 1$$

$$u_k = (\xi_k-\xi_k)/h = 0$$

(45) becomes:

$$\Delta\theta = I_n = h\Big[\theta'_m + \frac{1}{2}\Delta_1\theta'_m + \frac{5}{12}\Delta_2\theta'_m + \frac{3}{8}\Delta_3\theta'_m + \frac{251}{720}\Delta_4\theta'_m +$$

$$+ \frac{95}{288}\Delta_5\theta'_m + \frac{19087}{60480}\Delta_6\theta'_m\Big] \qquad (46)$$

For the interval $\xi_n - \xi_{n+1}$ the limits for u are:

$$u_{k+1} = \frac{\xi_m - \xi_n}{h} = 0$$

$$u_k = \frac{\xi_{m-1} - \xi_n}{h} = -1$$

and (45) becomes

$$\Delta\theta = I_{n-1}^{n} = h\Big[\theta'_m - \frac{1}{2}\Delta_1\theta'_m - \frac{1}{12}\Delta_2\theta'_m - \frac{1}{24}\Delta_3\theta'_m - \frac{19}{720}\Delta_4\theta'_m$$

$$- \frac{3}{160} \Delta_5 \Theta'_M - \frac{863}{60480} \Delta_6 \Theta'_M \Big] \tag{47}$$

Formulae (46) and (47) are valid if instead of Θ' we integrate Θ''. In this case we have

$$\Delta\Theta' = \int_{\xi_n}^{\xi_{n+1}} \Theta'' \, d\xi = h \Big[\Theta'' + \frac{1}{2} \Delta_1 \Theta''_M + \frac{5}{12} \Delta_2 \Theta''_M + \frac{3}{8} \Delta_3 \Theta''_M +$$

$$+ \frac{251}{720} \Delta_4 \Theta''_M + \frac{95}{288} \Delta_5 \Theta''_M + \frac{19087}{60480} \Delta_6 \Theta''_M \Big] \tag{48}$$

and for

$$\Delta\Theta' = \int_{\xi_{n-1}}^{\xi_n} \Theta'' \, d\xi = h \Big[\Theta''_M - \frac{1}{2} \Delta_1 \Theta''_M - \frac{1}{12} \Delta_2 \Theta''_M - \frac{1}{24} \Delta_3 \Theta''_M$$

$$- \frac{19}{720} \Delta_4 \Theta''_M - \frac{3}{160} \Delta_5 \Theta''_M - \frac{863}{60480} \Delta_6 \Theta''_M \Big] \tag{49}$$

Formulae (46) and (48) are used for integrating ahead. They give by extrapolation the change in Θ and Θ' respectively, for the next step ahead. This change in Θ (and Θ') added to the last already obtained, will therefore give the new Θ, Θ' at the end of the next step. The formulae are therefore used for finding the approximate change in Θ and Θ' in the next interval ahead of us, thereby enabling us to find the approximate value of Θ, Θ' at the end of that interval.

When a line in the table of corresponding values of ξ and Θ and Θ' have been finished, the first entry in the next line is computed by (46). The procedure we follow for solving equ. (36) is as follows

(1) From equations (34), (35), (36) and given the polytropic index n, we compute the starting values of Θ_j, Θ'_j, Θ''_j for $\xi_j = \xi_{j-1} + \Delta\xi$ where in our case

$j=1(1)7$ and $\Delta\xi$ varies according to the polytropic index.

(2) We form the differences for these quantities, namely

$$\Delta_1\theta'_j,\ \Delta_2\theta'_j,\ \Delta_3\theta'_j \dots \Delta_6\theta'_j$$
$$\Delta_1\theta''_j,\ \Delta_2\theta''_j,\ \Delta_3\theta''_j \dots \Delta_6\theta''_j \quad \text{and} \quad \Delta\theta_j$$

(3) Put the differences of the second derivative in formula (48) and compute $\Delta\theta'_{j+1}$ which we add in the previous value of $\theta''_{j=7}$ and get the new $\theta'_{j+1} = \theta'_j + \Delta\theta'_{j+1}$.

(4) Compute the various orders of differences for θ'_{j+1}

(5) Next we compute $\Delta\theta_{j+1}$ (for this new line) by applying (47) to the θ' quantities and get the new $\theta_{j+1} = \theta_7 + \Delta\theta_{j+1}$.

(6) We next substitute the new values of θ'_{j+1}, θ'_j to the (36) and get the new θ''_{j+1}.

(7) Then we compute the several orders of differences for this θ''_{j+1}.

(8) In order to check the new values we use (49) with the new differences of the second order derivative and get $\Delta\theta'_{j+1}$ If this $\Delta\theta'_{j+1}$ is the same as the one in step 3. It is not possible to improve the value and the result is regarded as correct.

(9) For the corrected value of $\Delta\theta'_{j+1}$ we find θ'_{j+1} and proceed as step 4 describes and onwards as steps 5, 6, 7, 8 describe until the new value of θ''_{j+2} is found.

This procedure is continued until the value of θ obtained becomes zero.

From equ. (36) we should also have in mind that

$$\frac{d\theta}{d\xi} < 0 \quad \forall \xi \text{ (i.e. } \theta(\xi) \text{ decreasing function)}$$

while $\dfrac{d^2\theta}{d\xi^2} < 0$ for $n=0 \Rightarrow \theta(\xi)$ is a concave curve and $\frac{d^2\theta}{d\xi^2} < 0$ until some value ξ after which

$\frac{d^2\theta}{d\xi^2}$ becomes positive $\Rightarrow \theta(\xi)$ is concave in the beginning and later becomes convex.

The Fortran IV program which was used for this numerical integration is to be found in Appendix I.

Tables 1 to 10 give the values of the Lane Emden function $\theta(\xi)$, its first and second derivatives $\frac{d\theta}{d\xi}$, $\frac{d^2\theta}{d\xi^2}$ the $\rho_c/\bar{\rho} = -\frac{1}{3}\xi\left(\frac{1}{\frac{d\theta}{d\xi}}\right)$ function, the $-\xi^2\frac{d\theta}{d\xi}$ mass variable function, for $n = 0, 0.5, 1, 1.0, 1.5, 2, 2.5, 3.0, 3.5, 4, 4.5$.

For $n=0$ and $n=1$ the fifth column gives the value of the exact solution

$$\theta_0 = 1 - \frac{1}{6}\xi^2$$

$$\theta_1 = \frac{\sin\xi}{\xi}$$

Figure 1 gives the graphical representation of the $\theta(\xi)$ function for four values of the polytropic index.

Table 1. Lane-Emden function $n=0.0$

ξ	$\theta(\xi)$	$\theta = 1-\frac{1}{6}\xi^2$	$\theta'(\xi)$	$\theta''(\xi)$	$-\xi^2\theta'(\xi)$	$\rho_c/\bar{\rho}$
0.0	0.10000000D 01	0.10000000D 01	0.0	-0.33333331D 00	0.0	0.0
0.69999985D-01	0.99918333D 00	0.99918333D 00	-0.23333328D-01	-0.33333336D 00	0.11433326D-03	0.99999998D 00
0.13999997D 00	0.99673333D 00	0.99673333D 00	-0.46666657D-01	-0.33333334D 00	0.91466609D-03	0.99999994D 00
0.20999996D 00	0.99265000D 00	0.99265000D 00	-0.69999985D-01	-0.33333333D 00	0.30869981D-02	0.99999994D 00
0.27999994D 00	0.98693334D 00	0.98693334D 00	-0.93333314D-01	-0.33333333D 00	0.73173287D-02	0.99999994D 00
0.34999993D 00	0.97958334D 00	0.97958334D 00	-0.11666664D 00	-0.33333333D 00	0.14291658D-01	0.99999994D 00
0.41999991D 00	0.97060001D 00	0.97060001D 00	-0.13999997D 00	-0.33333333D 00	0.24695985D-01	0.99999994D 00
0.48999990D 00	0.95998335D 00	0.95998335D 00	-0.16333330D 00	-0.33333333D 00	0.39216309D-01	0.99999994D 00
0.55999988D 00	0.94773336D 00	0.94773336D 00	-0.18666663D 00	-0.33333333D 00	0.58538630D-01	0.99999994D 00
0.62999987D 00	0.93385003D 00	0.93385003D 00	-0.20999996D 00	-0.33333333D 00	0.83348948D-01	0.99999994D 00
0.69999985D 00	0.91833337D 00	0.91833337D 00	-0.23333328D 00	-0.33333333D 00	0.11433326D 00	0.99999994D 00
0.76999984D 00	0.90118337D 00	0.90118337D 00	-0.25666661D 00	-0.33333333D 00	0.15217757D 00	0.99999994D 00
0.83999982D 00	0.88240005D 00	0.88240005D 00	-0.27999994D 00	-0.33333333D 00	0.19756788D 00	0.99999994D 00
0.90999981D 00	0.86198339D 00	0.86198339D 00	-0.30333327D 00	-0.33333333D 00	0.25119018D 00	0.99999994D 00
0.97999980D 00	0.83993340D 00	0.83993340D 00	-0.32666660D 00	-0.33333333D 00	0.31373047D 00	0.99999994D 00
0.10499998D 01	0.81625008D 00	0.81625008D 00	-0.34999993D 00	-0.33333333D 00	0.38587476D 00	0.99999994D 00
0.11199998D 01	0.79093342D 00	0.79093342D 00	-0.37333326D 00	-0.33333333D 00	0.46830904D 00	0.99999994D 00
0.11899998D 01	0.76398343D 00	0.76398343D 00	-0.39666658D 00	-0.33333333D 00	0.56171932D 00	0.99999994D 00
0.12599997D 01	0.73540011D 00	0.73540011D 00	-0.41999991D 00	-0.33333333D 00	0.66679158D 00	0.99999994D 00
0.13299997D 01	0.70518346D 00	0.70518346D 00	-0.44333324D 00	-0.33333333D 00	0.78421184D 00	0.99999994D 00
0.13999997D 01	0.67333347D 00	0.67333347D 00	-0.46666657D 00	-0.33333333D 00	0.91466609D 00	0.99999994D 00
0.14699997D 01	0.63985015D 00	0.63985015D 00	-0.48999990D 00	-0.33333333D 00	0.10588403D 01	0.99999994D 00
0.15399997D 01	0.60473350D 00	0.60473350D 00	-0.51333323D 00	-0.33333333D 00	0.12174206D 01	0.99999994D 00
0.16099997D 01	0.56798351D 00	0.56798351D 00	-0.53666655D 00	-0.33333333D 00	0.13910928D 01	0.99999994D 00
0.16799996D 01	0.52960020D 00	0.52960020D 00	-0.55999988D 00	-0.33333333D 00	0.15805430D 01	0.99999994D 00
0.17499996D 01	0.48958355D 00	0.48958355D 00	-0.58333321D 00	-0.33333333D 00	0.17864572D 01	0.99999994D 00
0.18199996D 01	0.44793356D 00	0.44793356D 00	-0.60666654D 00	-0.33333333D 00	0.20095214D 01	0.99999994D 00
0.18899996D 01	0.40465025D 00	0.40465025D 00	-0.62999987D 00	-0.33333333D 00	0.22504216D 01	0.99999994D 00
0.19599996D 01	0.35973360D 00	0.35973360D 00	-0.65333320D 00	-0.33333333D 00	0.25098438D 01	0.99999994D 00
0.20299996D 01	0.31318362D 00	0.31318362D 00	-0.67666653D 00	-0.33333333D 00	0.27884739D 01	0.99999994D 00
0.20999996D 01	0.26500031D 00	0.26500031D 00	-0.69999985D 00	-0.33333333D 00	0.30869981D 01	0.99999994D 00
0.21699995D 01	0.21518366D 00	0.21518366D 00	-0.72333318D 00	-0.33333333D 00	0.34061022D 01	0.99999994D 00
0.22399995D 01	0.16373368D 00	0.16373368D 00	-0.74666651D 00	-0.33333333D 00	0.37464723D 01	0.99999994D 00
0.23099995D 01	0.11065037D 00	0.11065037D 00	-0.76999984D 00	-0.33333333D 00	0.41087944D 01	0.99999994D 00
0.23799995D 01	0.55933727D-01	0.55933727D-01	-0.79333317D 00	-0.33333333D 00	0.44937545D 01	0.99999994D 00
0.24399995D 01	0.77337476D-02	0.77337473D-02	-0.81333316D 00	-0.33333333D 00	0.48422583D 01	0.99999994D 00

$\xi_0 = 2.4495082$

Table 2. Lane-Emden function $n = 0.5$

ξ	$\theta(\xi)$	$\theta'(\xi)$	$\theta''(\xi)$	$-\xi^2 \theta'(\xi)$	$\rho_c/\bar{\rho}$
0.0	0.10000000D 01	0.0	-0.33333331D 00	0.0	0.0
0.79999983D-01	0.99893350D 00	-0.26658126D-01	-0.33301331D 00	0.17061194D-03	0.10003201D 01
0.15999997D 00	0.99573607D 00	-0.53265035D-01	-0.33205268D 00	0.13635843D-02	0.10012820D 01
0.23999995D 00	0.99041383D 00	-0.79769425D-01	-0.33045003D 00	0.45947170D-02	0.10028903D 01
0.31999993D 00	0.98297707D 00	-0.10611984D 00	-0.32820285D 00	0.10866667D-01	0.10051526D 01
0.39999992D 00	0.97344015D 00	-0.13226459D 00	-0.32530762D 00	0.21162325D-01	0.10080801D 01
0.47999990D 00	0.96182161D 00	-0.15815165D 00	-0.32175971D 00	0.36438124D-01	0.10116870D 01
0.55999988D 00	0.94814416D 00	-0.18372857D 00	-0.31755333D 00	0.57617257D-01	0.10159912D 01
0.63999987D 00	0.93243472D 00	-0.20894242D 00	-0.31268139D 00	0.85582780D-01	0.10210147D 01
0.71999985D 00	0.91472449D 00	-0.23373961D 00	-0.30713546D 00	0.12111057D 00	0.10267833D 01
0.79999983D 00	0.89504895D 00	-0.25806585D 00	-0.30090552D 00	0.16516207D 00	0.10333277D 01
0.87999982D 00	0.87344799D 00	-0.28186594D 00	-0.29397986D 00	0.21827689D 00	0.10406836D 01
0.95999980D 00	0.84996594D 00	-0.30508370D 00	-0.28634480D 00	0.28116502D 00	0.10488922D 01
0.10399998D 01	0.82465167D 00	-0.32766176D 00	-0.27798444D 00	0.35439881D 00	0.10580013D 01
0.11199998D 01	0.79755869D 00	-0.34954137D 00	-0.26888027D 00	0.43846451D 00	0.10680660D 01
0.11999997D 01	0.76874529D 00	-0.37066219D 00	-0.25901077D 00	0.53375332D 00	0.10791494D 01
0.12799997D 01	0.73827465D 00	-0.39096201D 00	-0.24835084D 00	0.64055189D 00	0.10913248D 01
0.13599997D 01	0.70621499D 00	-0.41037645D 00	-0.23687108D 00	0.75903197D 00	0.11046765D 01
0.14399997D 01	0.67263982D 00	-0.42883859D 00	-0.22453694D 00	0.88923934D 00	0.11193019D 01
0.15199997D 01	0.63762809D 00	-0.44627849D 00	-0.21130752D 00	0.10310814D 01	0.11353147D 01
0.15999997D 01	0.60126449D 00	-0.46262262D 00	-0.19713407D 00	0.11843134D 01	0.11528472D 01
0.16799996D 01	0.56363977D 00	-0.47779321D 00	-0.18195791D 00	0.13485230D 01	0.11720549D 01
0.17599996D 01	0.52485110D 00	-0.49170726D 00	-0.16570771D 00	0.15231118D 01	0.11931215D 01
0.18399996D 01	0.48500251D 00	-0.50427547D 00	-0.14829559D 00	0.17072743D 01	0.12162661D 01
0.19199996D 01	0.44420551D 00	-0.51540066D 00	-0.12961165D 00	0.18999722D 01	0.12417521D 01
0.19999996D 01	0.40257975D 00	-0.52497572D 00	-0.10951590D 00	0.20999020D 01	0.12698997D 01
0.20799996D 01	0.36025393D 00	-0.53288075D 00	-0.87826135D-01	0.23054543D 01	0.13011038D 01
0.21599995D 01	0.31736701D 00	-0.53897897D 00	-0.64298655D-01	0.25146592D 01	0.13358588D 01
0.22399995D 01	0.27406973D 00	-0.54311066D 00	-0.38596360D-01	0.27251109D 01	0.13747962D 01
0.23199995D 01	0.23052685D 00	-0.54508363D 00	-0.10232338D-01	0.29338569D 01	0.14187422D 01
0.23999995D 01	0.18692028D 00	-0.54465739D 00	0.21538474D-01	0.31372253D 01	0.14688129D 01
0.24799995D 01	0.14345401D 00	-0.54151452D 00	0.57952118D-01	0.33305295D 01	0.15265822D 01
0.25599995D 01	0.10036228D 00	-0.53520167D 00	0.10132632D 00	0.35074962D 01	0.15944141D 01
0.26399994D 01	0.57925374D-01	-0.52498163D 00	0.15703653D 00	0.36589104D 01	0.16762487D 01
0.27199994D 01	0.16508970D-01	-0.50926810D 00	0.24597467D 00	0.37677675D 01	0.17803323D 01
0.27499994D 01	0.13505299D-02	-0.50090811D 00	0.32754649D 00	0.37881160D 01	0.18300091D 01

$\xi_{0.5} = 2.7526955$

Table 3. Lane-Emden function, $n = 1.0$

ξ	$\theta(\xi)$	$\theta = \sin\xi/\xi$	$\theta'(\xi)$	$\theta''(\xi)$	$-\xi^2\,\theta'(\xi)$	$\rho_c/\bar{\rho}$
0.0	0.10000000D 01	0.10000000D 01	0.0	-0.33333331D 00	0.0	0.0
0.89999981D-01	0.99865055D 00	0.99865055D 00	-0.29975700D-01	-0.33252374D 00	0.24280307D-03	0.10008104D 01
0.17999996D 00	0.99460874D 00	0.99460874D 00	-0.59805812D-01	-0.33009958D 00	0.19377075D-02	0.10032467D 01
0.26999994D 00	0.98789422D 00	0.98789421D 00	-0.89345587D-01	-0.32607491D 00	0.65132906D-02	0.10073242D 01
0.35999992D 00	0.97853955D 00	0.97853955D 00	-0.11845196D 00	-0.32047298D 00	0.15351367D-01	0.10130687D 01
0.44999991D 00	0.96659009D 00	0.96659009D 00	-0.14698436D 00	-0.31332615D 00	0.29764320D-01	0.10205165D 01
0.53999989D 00	0.95210371D 00	0.95210371D 00	-0.17480553D 00	-0.30467567D 00	0.50973272D-01	0.10297154D 01
0.62999987D 00	0.93515044D 00	0.93515044D 00	-0.20178234D 00	-0.29457143D 00	0.80087379D-01	0.10407251D 01
0.71999985D 00	0.91581208D 00	0.91581208D 00	-0.22778651D 00	-0.28307165D 00	0.11808448D 00	0.10536179D 01
0.80999983D 00	0.89418174D 00	0.89418174D 00	-0.25269534D 00	-0.27024249D 00	0.16570334D 00	0.10684800D 01
0.89999981D 00	0.87036328D 00	0.87036328D 00	-0.27639247D 00	-0.25615767D 00	0.22387781D 00	0.10854128D 01
0.98999979D 00	0.84447075D 00	0.84447075D 00	-0.29876846D 00	-0.24089797D 00	0.29282285D 00	0.11045340D 01
0.10799998D 01	0.81662767D 00	0.81662767D 00	-0.31972146D 00	-0.22455076D 00	0.37292296D 00	0.11259798D 01
0.11699998D 01	0.78696641D 00	0.78696641D 00	-0.33915776D 00	-0.20720943D 00	0.46427286D 00	0.11499070D 01
0.12599997D 01	0.75562735D 00	0.75562735D 00	-0.35699229D 00	-0.18897280D 00	0.56676072D 00	0.11764957D 01
0.13499997D 01	0.72275815D 00	0.72275815D 00	-0.37314910D 00	-0.16994454D 00	0.68006396D 00	0.12059519D 01
0.14399997D 01	0.68851286D 00	0.68851286D 00	-0.38756178D 00	-0.15023249D 00	0.80364778D 00	0.12385119D 01
0.15299997D 01	0.65305107D 00	0.65305107D 00	-0.40017377D 00	-0.12994799D 00	0.93676639D 00	0.12744460D 01
0.16199997D 01	0.61653702D 00	0.61653702D 00	-0.41093867D 00	-0.10920522D 00	0.10784670D 01	0.13140644D 01
0.17099996D 01	0.57913863D 00	0.57913863D 00	-0.41982045D 00	-0.88120463D-01	0.12275965D 01	0.13577229D 01
0.17999996D 01	0.54102662D 00	0.54102662D 00	-0.42679362D 00	-0.66811393D-01	0.13828107D 01	0.14058313D 01
0.18899996D 01	0.50237351D 00	0.50237351D 00	-0.43184332D 00	-0.45396362D-01	0.15425869D 01	0.14588621D 01
0.19799996D 01	0.46335268D 00	0.46335268D 00	-0.43496523D 00	-0.23993670D-01	0.17052374D 01	0.15173619D 01
0.20699996D 01	0.42413742D 00	0.42413742D 00	-0.43616607D 00	-0.27208337D-02	0.18689272D 01	0.15819658D 01
0.21599995D 01	0.38489995D 00	0.38489995D 00	-0.43546246D 00	0.18306119D-01	0.20316928D 01	0.16534142D 01
0.22499995D 01	0.34581051D 00	0.34581051D 00	-0.43288177D 00	0.38973359D-01	0.21914630D 01	0.17325742D 01
0.23399995D 01	0.30703645D 00	0.30703645D 00	-0.42846138D 00	0.59169939D-01	0.23460822D 01	0.18204670D 01
0.24299995D 01	0.26874126D 00	0.26874126D 00	-0.42224853D 00	0.78783459D-01	0.24933343D 01	0.19183010D 01
0.25199995D 01	0.23108381D 00	0.23108381D 00	-0.41429990D 00	0.97725707D-01	0.26309690D 01	0.20275162D 01
0.26099995D 01	0.19421741D 00	0.19421741D 00	-0.40468130D 00	0.11588327D 00	0.27567283D 01	0.21498393D 01
0.26999994D 01	0.15828907D 00	0.15828907D 00	-0.39346711D 00	0.13316811D 00	0.28683740D 01	0.22873570D 01
0.27899994D 01	0.12343874D 00	0.12343874D 00	-0.38073984D 00	0.14949310D 00	0.29637158D 01	0.24426121D 01
0.28799994D 01	0.89798606D-01	0.89798606D-01	-0.36658958D 00	0.16477754D 00	0.30406393D 01	0.26187317D 01
0.29699994D 01	0.57492416D-01	0.57492416D-01	-0.35111333D 00	0.17894759D 00	0.30971343D 01	0.28196017D 01
0.30599994D 01	0.26634904D-01	0.26634904D-01	-0.33441445D 00	0.19193668D 00	0.31313219D 01	0.30501066D 01
0.31399993D 01	0.50742303D-03	0.50742303D-03	-0.31863260D 00	0.20244332D 00	0.31415887D 01	0.32848691D 01

$\xi_1 = 3.1415918$

Table 4. Lane-Emden function, $n = 1.5$

ξ	$\theta(\xi)$	$\theta'(\xi)$	$\theta''(\xi)$	$-\xi^2\theta'(\xi)$	$\rho_c/\bar{\rho}$
0.0	0.10000000D 01	0.0	-0.33333331D 00	0.0	0.0
0.10999998D 00	0.99798516D 00	-0.36600176D-01	-0.33152139D 00	0.44286194D-03	0.10018164D 01
0.21999995D 00	0.99196254D 00	-0.72803059D-01	-0.32612194D 00	0.35236666D-02	0.10072834D 01
0.32999993D 00	0.98199736D 00	-0.10821933D 00	-0.31724310D 00	0.11785080D-01	0.10164540D 01
0.43999991D 00	0.96819686D 00	-0.14247534D 00	-0.30506130D 00	0.27583214D-01	0.10294176D 01
0.54999989D 00	0.95070822D 00	-0.17522040D 00	-0.28981588D 00	0.53006148D-01	0.10463010D 01
0.65999986D 00	0.92971561D 00	-0.20613325D 00	-0.27180183D 00	0.89791695D-01	0.10672705D 01
0.76999984D 00	0.90543664D 00	-0.23492776D 00	-0.25136101D 00	0.13928861D 00	0.10925341D 01
0.87999982D 00	0.87811827D 00	-0.26135754D 00	-0.22887224D 00	0.20239519D 00	0.11223447D 01
0.98999979D 00	0.84803221D 00	-0.28521939D 00	-0.20474063D 00	0.27954340D 00	0.11570038D 01
0.10999998D 01	0.81547002D 00	-0.30635564D 00	-0.17938661D 00	0.37069017D 00	0.11968657D 01
0.12099997D 01	0.78073805D 00	-0.32465520D 00	-0.15323505D 00	0.47532748D 00	0.12423433D 01
0.13199997D 01	0.74415232D 00	-0.34005348D 00	-0.12670487D 00	0.59250893D 00	0.12939138D 01
0.14299997D 01	0.70603344D 00	-0.35253120D 00	-0.10019948D 00	0.72089075D 00	0.13521258D 01
0.15399997D 01	0.66670174D 00	-0.36211222D 00	-0.74098299D-01	0.85878498D 00	0.14176080D 01
0.16499997D 01	0.62647268D 00	-0.36886046D 00	-0.48749617D-01	0.10042222D 01	0.14910784D 01
0.17599996D 01	0.58565268D 00	-0.37287620D 00	-0.24464943D-01	0.11550208D 01	0.15733547D 01
0.18699996D 01	0.54453531D 00	-0.37429174D 00	-0.15148581D-02	0.13088602D 01	0.16653672D 01
0.19799996D 01	0.50339803D 00	-0.37326678D 00	0.19873567D-01	0.14623545D 01	0.17681719D 01
0.20899996D 01	0.46249946D 00	-0.36998356D 00	0.39517870D-01	0.16161245D 01	0.18829661D 01
0.21999995D 01	0.42207716D 00	-0.36464192D 00	0.57279894D-01	0.17648662D 01	0.20111049D 01
0.23099995D 01	0.38234594D 00	-0.35745455D 00	0.73064241D-01	0.19074124D 01	0.21541194D 01
0.24199995D 01	0.34349675D 00	-0.34864238D 00	0.86815632D-01	0.20417884D 01	0.23137361D 01
0.25299995D 01	0.30569596D 00	-0.33843038D 00	0.98515333D-01	0.21662581D 01	0.24918954D 01
0.26399994D 01	0.26908515D 00	-0.32704378D 00	0.10817682D 00	0.22793634D 01	0.26907706D 01
0.27499994D 01	0.23378126D 00	-0.31470475D 00	0.11584081D 00	0.23799537D 01	0.29127823D 01
0.28599994D 01	0.19987709D 00	-0.30162978D 00	0.12156967D 00	0.24672099D 01	0.31606067D 01
0.29699994D 01	0.16744205D 00	-0.28802754D 00	0.12544123D 00	0.25406611D 01	0.34371704D 01
0.30799994D 01	0.13652305D 00	-0.27409766D 00	0.12754159D 00	0.26001989D 01	0.37456227D 01
0.31899993D 01	0.10714561D 00	-0.26003019D 00	0.12795628D 00	0.26460921D 01	0.40892677D 01
0.32999993D 01	0.79314811D-01	-0.24600640D 00	0.12675748D 00	0.26790086D 01	0.44714272D 01
0.34099993D 01	0.53016192D-01	-0.23220129D 00	0.12398136D 00	0.27000587D 01	0.48951768D 01
0.35199993D 01	0.28216113D-01	-0.21879007D 00	0.11957292D 00	0.27108954D 01	0.53628257D 01
0.36299992D 01	0.48608999D-02	-0.20596755D 00	0.11314186D 00	0.27140126D 01	0.58747103D 01
0.36499992D 01	0.76407929D-03	-0.20371971D 00	0.11160614D 00	0.27140547D 01	0.59722563D 01

$\xi_{1.5} = 3.6537498$

Table 5. Lane-Emden function, $n = 2$.

ξ	$\theta(\xi)$	$\theta'(\xi)$	$\theta''(\xi)$	$-\xi^2\theta'(\xi)$	$\rho_c/\bar{\rho}$
0.0	0.10000000D 01	0.	-0.33333331D 00	0.0	0.0
0.11999997D 00	0.99760345D 00	-0.39885009D-01	-0.33046237D 00	0.57434388D-03	0.10028828D 01
0.23999995D 00	0.99045502D 00	-0.79085292D-01	-0.32195691D 00	0.45553109D-02	0.10115658D 01
0.35999992D 00	0.97867681D 00	-0.11694164D 00	-0.30813240D 00	0.15155630D-01	0.10261526D 01
0.47999990D 00	0.96246728D 00	-0.15284422D 00	-0.28949223D 00	0.35215293D-01	0.10468172D 01
0.59999987D 00	0.94209406D 00	-0.18625338D 00	-0.26669649D 00	0.67051188D-01	0.10738057D 01
0.71999985D 00	0.91788447D 00	-0.21671613D 00	-0.24052251D 00	0.11234560D 00	0.11074392D 01
0.83999982D 00	0.89021439D 00	-0.24387761D 00	-0.21182055D 00	0.17207997D 00	0.11481166D 01
0.95999980D 00	0.85949607D 00	-0.26748708D 00	-0.18146864D 00	0.24651599D 00	0.11963191D 01
0.10799998D 01	0.82616555D 00	-0.28739866D 00	-0.15032966D 00	0.33522166D 00	0.12526151D 01
0.11999997D 01	0.79067025D 00	-0.30356723D 00	-0.11921396D 00	0.43713663D 00	0.13176649D 01
0.13199997D 01	0.75345737D 00	-0.31604018D 00	-0.88849138D-01	0.55066819D 00	0.13922276D 01
0.14399997D 01	0.71496339D 00	-0.32496615D 00	-0.59858471D-01	0.67380805D 00	0.14771675D 01
0.15599997D 01	0.67560520D 00	-0.33048158D 00	-0.32747958D-01	0.80425965D 00	0.15734609D 01
0.16799996D 01	0.63577271D 00	-0.33289638D 00	-0.79016383D-02	0.93956636D 00	0.16822047D 01
0.17999996D 01	0.59582339D 00	-0.33247927D 00	0.14415977D-01	0.10772324D 01	0.18046233D 01
0.19199996D 01	0.55607844D 00	-0.32954395D 00	0.34051786D-01	0.12148303D 01	0.19420773D 01
0.20399996D 01	0.51682055D 00	-0.32441639D 00	0.50951867D-01	0.13500907D 01	0.20960711D 01
0.21599995D 01	0.47829312D 00	-0.31742381D 00	0.65146690D-01	0.14809719D 01	0.22682602D 01
0.22799995D 01	0.44070066D 00	-0.30888551D 00	0.76735185D-01	0.16057098D 01	0.24604579D 01
0.23999995D 01	0.40421025D 00	-0.29910561D 00	0.85868806D-01	0.17228476D 01	0.26746398D 01
0.25199995D 01	0.36895362D 00	-0.28836767D 00	0.92736504D-01	0.18312493D 01	0.29129471D 01
0.26399994D 01	0.33502993D 00	-0.27693099D 00	0.97551194D-01	0.19300974D 01	0.31776861D 01
0.27599994D 01	0.30250879D 00	-0.26502850D 00	0.10053811D 00	0.20188803D 01	0.34713238D 01
0.28799994D 01	0.27143348D 00	-0.25286584D 00	0.10192518D 00	0.20973695D 01	0.37964786D 01
0.29999994D 01	0.24182423D 00	-0.24062152D 00	0.10193542D 00	0.21655928D 01	0.41559031D 01
0.31199993D 01	0.21368146D 00	-0.22844790D 00	0.10078122D 00	0.22238023D 01	0.45524591D 01
0.32399993D 01	0.18698878D 00	-0.21647268D 00	0.98660333D-01	0.22724426D 01	0.49890809D 01
0.33599993D 01	0.16171587D 00	-0.20480085D 00	0.95753270D-01	0.23121187D 01	0.54687259D 01
0.34799993D 01	0.13782105D 00	-0.19351684D 00	0.92221957D-01	0.23435654D 01	0.59943087D 01
0.35999992D 01	0.11525366D 00	-0.18268681D 00	0.88209289D-01	0.23676201D 01	0.65686169D 01
0.37199992D 01	0.93956044D-01	-0.17236088D 00	0.83839422D-01	0.23851978D 01	0.71942059D 01
0.38399992D 01	0.73865420D-01	-0.16257537D 00	0.79218589D-01	0.23972704D 01	0.78732692D 01
0.39599992D 01	0.54915350D-01	-0.15335486D 00	0.74436272D-01	0.24048486D 01	0.86074848D 01
0.40799991D 01	0.37037058D-01	-0.14471415D 00	0.69566578D-01	0.24089685D 01	0.93978348D 01
0.41999991D 01	0.20160493D-01	-0.13665996D 00	0.64669738D-01	0.24106806D 01	0.10244403D 02
0.43199991D 01	0.42151967D-02	-0.12919257D 00	0.59793619D-01	0.24110424D 01	0.11146149D 02
0.43499991D 01	0.36614576D-03	-0.12741694D 00	0.58582379D-01	0.24110460D 01	0.11379960D 02

$\xi_1 = 4.3528727$

Table 6. Lane-Emden function, $n=2.5$

ξ	$\theta(\xi)$	$\theta'(\xi)$	$\theta''(\xi)$	$-\xi^2\theta'(\xi)$	$\rho_c/\bar{\rho}$
0.0	0.10000000D 01	0.0	-0.33333331D 00	0.0	0.0
0.14999997D 00	0.99626052D 00	-0.49719866D-01	-0.32774582D 00	0.11186965D-02	0.10056340D 01
0.29999994D 00	0.98516637D 00	-0.97785645D-01	-0.31142449D 00	0.88007044D-02	0.10226447D 01
0.44999991D 00	0.96708419D 00	-0.14267248D 00	-0.28563071D 00	0.28891165D-01	0.10513588D 01
0.59999987D 00	0.94258894D 00	-0.18309597D 00	-0.25227283D 00	0.65914520D-01	0.10923231D 01
0.74999984D 00	0.91242742D 00	-0.21809129D 00	-0.21365872D 00	0.12267630D 00	0.11463086D 01
0.89999981D 00	0.87746337D 00	-0.24705269D 00	-0.17222277D 00	0.20011260D 00	0.12143155D 01
0.10499998D 01	0.83862623D 00	-0.26973306D 00	-0.13027488D 00	0.29738058D 00	0.12975788D 01
0.11999997D 01	0.79685459D 00	-0.28620988D 00	-0.89806734D-01	0.41214206D 00	0.13975754D 01
0.13499997D 01	0.75305657D 00	-0.29682761D 00	-0.52373011D-01	0.54096810D 00	0.15160311D 01
0.14999997D 01	0.70807240D 00	-0.30212798D 00	-0.19047690D-01	0.67978767D 00	0.16549274D 01
0.16499997D 01	0.66265078D 00	-0.30277860D 00	0.95573453D-02	0.82431440D 00	0.18165083D 01
0.17999996D 01	0.61743491D 00	-0.29950789D 00	0.33230658D-01	0.97040516D 00	0.20032856D 01
0.19499996D 01	0.57295767D 00	-0.29305114D 00	0.52076807D-01	0.11143265D 01	0.22180423D 01
0.20999996D 01	0.52964371D 00	-0.28410993D 00	0.66426161D-01	0.12529242D 01	0.24638344D 01
0.22499995D 01	0.48781678D 00	-0.27332463D 00	0.76751136D-01	0.13837054D 01	0.27439891D 01
0.23999995D 01	0.44771023D 00	-0.26125863D 00	0.83595876D-01	0.15048491D 01	0.30620990D 01
0.25499995D 01	0.40947912D 00	-0.24839176D 00	0.87522132D-01	0.16151668D 01	0.34220127D 01
0.26999994D 01	0.37321279D 00	-0.23512078D 00	0.89071074D-01	0.17140298D 01	0.38278188D 01
0.28499994D 01	0.33894705D 00	-0.22176440D 00	0.88739030D-01	0.18012806D 01	0.42838245D 01
0.29999994D 01	0.30667523D 00	-0.20857111D 00	0.86964341D-01	0.18771392D 01	0.47945267D 01
0.31499993D 01	0.27635812D 00	-0.19572835D 00	0.84122458D-01	0.19421138D 01	0.53645764D 01
0.32999993D 01	0.24793235D 00	-0.18337197D 00	0.80526689D-01	0.19969199D 01	0.59987342D 01
0.34499993D 01	0.22131750D 00	-0.17159519D 00	0.76432473D-01	0.20424109D 01	0.67018178D 01
0.35999992D 01	0.19642183D 00	-0.16045689D 00	0.72043591D-01	0.20795205D 01	0.74786422D 01
0.37499992D 01	0.17314690D 00	-0.14998883D 00	0.67519168D-01	0.21092171D 01	0.83339516D 01
0.38999992D 01	0.15139112D 00	-0.14020180D 00	0.62980711D-01	0.21324684D 01	0.92723466D 01
0.40499992D 01	0.13105255D 00	-0.13109076D 00	0.58518720D-01	0.21502152D 01	0.10298206D 02
0.41999991D 01	0.11203092D 00	-0.12263906D 00	0.54198639D-01	0.21633520D 01	0.11415610D 02
0.43499991D 01	0.94229108D-01	-0.11482177D 00	0.50066025D-01	0.21727140D 01	0.12628264D 02
0.44999991D 01	0.77554194D-01	-0.10760835D 00	0.46150951D-01	0.21790683D 01	0.13939434D 02
0.46499990D 01	0.61918118D-01	-0.10096468D 00	0.42471687D-01	0.21831079D 01	0.15351899D 02
0.47999990D 01	0.47238116D-01	-0.94854574D-01	0.39037759D-01	0.21854485D 01	0.16867922D 02
0.49499990D 01	0.33436929D-01	-0.89240916D-01	0.35852503D-01	0.21866246D 01	0.18489272D 02
0.50999989D 01	0.20442889D-01	-0.84086418D-01	0.32915320D-01	0.21870868D 01	0.20217290D 02
0.52499989D 01	0.81899042D-02	-0.79354030D-01	0.30224043D-01	0.21871945D 01	0.22053064D 02
0.53499989D 01	0.40281441D-03	-0.76415424D-01	0.28566516D-01	0.21871996D 01	0.23337342D 02

ξ_1 = 5.355[illegible]

Table 7. Lane-Emden function, $n=3$

ξ	$\theta(\xi)$	$\theta'(\xi)$	$\theta''(\xi)$	$-\xi^2\,\theta'(\xi)$	$\rho_c/\bar{\rho}$
0.0	0.10000000D 01	0.0	-0.33333331D 00	0.0	0.0
0.19999996D 00	0.99337310D 00	-0.65873834D-01	-0.32151227D 00	0.26349523D-02	0.10120354D 01
0.39999992D 00	0.97395827D 00	-0.12715769D 00	-0.28810305D 00	0.20345223D-01	0.10485665D 01
0.59999987D 00	0.94307320D 00	-0.18003960D 00	-0.23862495D 00	0.64814230D-01	0.11108664D 01
0.79999983D 00	0.90267213D 00	-0.22202762D 00	-0.18044339D 00	0.14209762D 00	0.12010514D 01
0.99999979D 00	0.85505762D 00	-0.25212925D 00	-0.12089414D 00	0.25212915D 00	0.13220729D 01
0.11999997D 01	0.80259199D 00	-0.27069007D 00	-0.65842553D-01	0.38979353D 00	0.14777044D 01
0.13999997D 01	0.74746489D 00	-0.27901916D 00	-0.19012560D-01	0.54687733D 00	0.16725251D 01
0.15999997D 01	0.69154420D 00	-0.27895496D 00	0.17974246D-01	0.71412440D 00	0.19118971D 01
0.17999996D 01	0.63630954D 00	-0.27243724D 00	0.45128393D-01	0.88285828D 00	0.22019374D 01
0.19999996D 01	0.58235062D 00	-0.26149095D 00	0.63487994D-01	0.10459634D 01	0.25494820D 01
0.21999995D 01	0.53190698D 00	-0.24757673D 00	0.74580000D-01	0.11982709D 01	0.29620438D 01
0.23999995D 01	0.48392777D 00	-0.23203457D 00	0.80033031D-01	0.13365186D 01	0.34477612D 01
0.25999995D 01	0.43913612D 00	-0.21583894D 00	0.81346741D-01	0.14590706D 01	0.40153387D 01
0.27999994D 01	0.39758880D 00	-0.19968704D 00	0.79784047D-01	0.15655458D 01	0.46739792D 01
0.29999994D 01	0.35922662D 00	-0.18404993D 00	0.76344023D-01	0.16564487D 01	0.54333069D 01
0.31999993D 01	0.32391448D 00	-0.16922393D 00	0.71779680D-01	0.17328523D 01	0.63032833D 01
0.33999993D 01	0.29147163D 00	-0.15537634D 00	0.66635685D-01	0.17961497D 01	0.72941160D 01
0.35999992D 01	0.26169355D 00	-0.14258277D 00	0.61290969D-01	0.18478719D 01	0.84161622D 01
0.37999992D 01	0.23436731D 00	-0.13085627D 00	0.55998402D-01	0.18895638D 01	0.96798288D 01
0.39999992D 01	0.20928171D 00	-0.12016910D 00	0.50918268D-01	0.19227048D 01	0.11095473D 02
0.41999991D 01	0.18623388D 00	-0.11046842D 00	0.46144861D-01	0.19486622D 01	0.12673302D 02
0.43999991D 01	0.16503303D 00	-0.10168741D 00	0.41726736D-01	0.19686674D 01	0.14423283D 02
0.45999990D 01	0.14550249D 00	-0.93752817D-01	0.37681673D-01	0.19838088D 01	0.16355060D 02
0.47999990D 01	0.12748046D 00	-0.86589993D-01	0.34007452D-01	0.19950326D 01	0.18477881D 02
0.49999990D 01	0.11081992D 00	-0.80126075D-01	0.30689451D-01	0.20031510D 01	0.20800547D 02
0.51999989D 01	0.95388067D-01	-0.74291911D-01	0.27705893D-01	0.20088524D 01	0.23331381D 02
0.53999989D 01	0.81065481D-01	-0.69023114D-01	0.25031391D-01	0.20127132D 01	0.26078214D 02
0.55999988D 01	0.67745091D-01	-0.64260531D-01	0.22639285D-01	0.20152094D 01	0.29048408D 02
0.57999988D 01	0.55331126D-01	-0.59950341D-01	0.20503137D-01	0.20167286D 01	0.32248905D 02
0.59999987D 01	0.43738054D-01	-0.56043906D-01	0.18597634D-01	0.20175798D 01	0.35686297D 02
0.61999987D 01	0.32889577D-01	-0.52497495D-01	0.16899102D-01	0.20180029D 01	0.39366947D 02
0.63999987D 01	0.22717680D-01	-0.49271925D-01	0.15385755D-01	0.20181772D 01	0.43297126D 02
0.65999986D 01	0.13161765D-01	-0.46332174D-01	0.14037776D-01	0.20182286D 01	0.47483190D 02
0.67999986D 01	0.41678513D-02	-0.43646988D-01	0.12837280D-01	0.20182359D 01	0.51931787D 02
0.68899986D 01	0.29093521D-03	-0.42514167D-01	0.12340835D-01	0.20182360D 01	0.54021194D 02

$\xi_3 = 6.8968418$

Table 8. Lane-Emden function $n = 3.5$

ξ	$\theta(\xi)$	$\theta'(\xi)$	$\theta''(\xi)$	$-\xi^2\theta'(\xi)$	$\rho_c/\bar{\rho}$
0.0	0.10000000D 01	0.0	-0.33333331D 00	0.0	0.0
0.23999998D 00	0.99049569D 00	-0.78413105D-01	-0.31368568D 00	0.45165941D-02	0.10202375D 01
0.47999996D 00	0.96308569D 00	-0.14786758D 00	-0.26053710D 00	0.34068684D-01	0.10820491D 01
0.71999994D 00	0.92076150D 00	-0.20192733D 00	-0.18814797D 00	0.10467911D 00	0.11885462D 01
0.95999992D 00	0.86761395D 00	-0.23793982D 00	-0.11262704D 00	0.21928531D 00	0.13448777D 01
0.11999999D 01	0.80793408D 00	-0.25672974D 00	-0.46157885D-01	0.36969076D 00	0.15580585D 01
0.14399999D 01	0.74552156D 00	-0.26132107D 00	0.51703545D-02	0.54187527D 00	0.18368206D 01
0.16799999D 01	0.68332913D 00	-0.25553940D 00	0.40456135D-01	0.72123429D 00	0.21914425D 01
0.19199998D 01	0.62340006D 00	-0.24301583D 00	0.61855182D-01	0.89585341D 00	0.26335729D 01
0.21599998D 01	0.56698303D 00	-0.22669615D 00	0.72659278D-01	0.10576734D 01	0.31760570D 01
0.23999998D 01	0.51471601D 00	-0.20872635D 00	0.76105301D-01	0.12022635D 01	0.38327691D 01
0.26399998D 01	0.46681024D 00	-0.19054000D 00	0.74847457D-01	0.13279874D 01	0.46184521D 01
0.28799998D 01	0.42320266D 00	-0.17301764D 00	0.70842929D-01	0.14350773D 01	0.55485663D 01
0.31199997D 01	0.38366899D 00	-0.15664663D 00	0.65432209D-01	0.15248607D 01	0.66391460D 01
0.33599997D 01	0.34790169D 00	-0.14165260D 00	0.59480105D-01	0.15992010D 01	0.79066661D 01
0.35999997D 01	0.31556043D 00	-0.12809678D 00	0.53513086D-01	0.16601340D 01	0.93679158D 01
0.38399997D 01	0.28630293D 00	-0.11594323D 00	0.47829978D-01	0.17096522D 01	0.11039884D 02
0.40799997D 01	0.25980254D 00	-0.10510327D 00	0.42582976D-01	0.17495907D 01	0.12939653D 02
0.43199996D 01	0.23575733D 00	-0.95463458D-01	0.37833543D-01	0.17815769D 01	0.15084304D 02
0.45599996D 01	0.21389377D 00	-0.86902664D-01	0.33589426D-01	0.18070189D 01	0.17490831D 02
0.47999996D 01	0.19396728D 00	-0.79301848D-01	0.29828411D-01	0.18271143D 01	0.20176072D 02
0.50399996D 01	0.17576105D 00	-0.72549354D-01	0.26513130D-01	0.18428694D 01	0.23156647D 02
0.52799996D 01	0.15908390D 00	-0.66543371D-01	0.23600023D-01	0.18551224D 01	0.26448912D 02
0.55199995D 01	0.14376783D 00	-0.61192737D-01	0.21044563D-01	0.18645669D 01	0.30068923D 02
0.57599995D 01	0.12966544D 00	-0.56416788D-01	0.18804137D-01	0.18717733D 01	0.34032418D 02
0.59999995D 01	0.11664749D 00	-0.52144701D-01	0.16839488D-01	0.18772089D 01	0.38354803D 02
0.62399995D 01	0.10460065D 00	-0.48314608D-01	0.15115309D-01	0.18812546D 01	0.43051156D 02
0.64799995D 01	0.93425450D-01	-0.44872628D-01	0.13600332D-01	0.18842193D 01	0.48136243D 02
0.67199994D 01	0.83034504D-01	-0.41771915D-01	0.12267149D-01	0.18863525D 01	0.53624539D 02
0.69599994D 01	0.73350902D-01	-0.38971770D-01	0.11091900D-01	0.18878546D 01	0.59530261D 02
0.71999994D 01	0.64306851D-01	-0.36436831D-01	0.10053905D-01	0.18888850D 01	0.65867410D 02
0.74399994D 01	0.55842478D-01	-0.34136353D-01	0.91352890D-02	0.18895697D 01	0.72649813D 02
0.76799994D 01	0.47904796D-01	-0.32043582D-01	0.83206218D-02	0.18900071D 01	0.79891181D 02
0.79199993D 01	0.40446816D-01	-0.30135208D-01	0.75965942D-02	0.18902728D 01	0.87605157D 02
0.81599993D 01	0.33426780D-01	-0.28390889D-01	0.69517232D-02	0.18904240D 01	0.95805370D 02
0.83999993D 01	0.26807495D-01	-0.26792845D-01	0.63760952D-02	0.18905028D 01	0.10450550D 03
0.86399993D 01	0.20555763D-01	-0.25325512D-01	0.58611424D-02	0.18905391D 01	0.11371930D 03
0.88799993D 01	0.14641889D-01	-0.23975239D-01	0.53994493D-02	0.18905528D 01	0.12346069D 03
0.91199992D 01	0.90392480D-02	-0.22730033D-01	0.49845865D-02	0.18905565D 01	0.13374374D 03
0.93599992D 01	0.37239167D-02	-0.21579340D-01	0.46109674D-02	0.18905570D 01	0.14458271D 03
0.95999992D 01	0.53807112D-03	-0.20903973D-01	0.43962091D-02	0.18905571D 01	0.15164579D 03

$\xi_1 = 9.5357393$

Table 9. Lane-Emden function, n = 4.

ξ	$\theta(\xi)$	$\theta'(\xi)$	$\theta''(\xi)$	$-\xi^2\theta'(\xi)$	$\rho_c/\bar{\rho}$
0.0	0.10000000D 01	0.0	-0.33333331D 00	0.0	0.0
0.44999996D 00	0.96755986D 00	-0.13859642D 00	-0.26043403D 00	0.28065770D-01	0.10822789D 01
0.89999992D 00	0.88364909D 00	-0.22318915D 00	-0.11372849D 00	0.18078318D 00	0.13441512D 01
0.13499999D 01	0.77609856D 00	-0.24611903D 00	0.18206196D-02	0.44855186D 00	0.18283833D 01
0.17999998D 01	0.66786662D 00	-0.23076968D 00	0.57453898D-01	0.74769364D 00	0.25999946D 01
0.22499998D 01	0.57058175D 00	-0.20052623D 00	0.72253935D-01	0.10151638D 01	0.37401586D 01
0.26999998D 01	0.48762063D 00	-0.16850321D 00	0.68280788D-01	0.12283882D 01	0.53411437D 01
0.31499997D 01	0.41838958D 00	-0.13995161D 00	0.58215723D-01	0.13886696D 01	0.75025923D 01
0.35999997D 01	0.36094031D 00	-0.11617894D 00	0.47571529D-01	0.15056788D 01	0.10328893D 02
0.40499997D 01	0.31314777D 00	-0.96931754D-01	0.38251474D-01	0.15899228D 01	0.13927323D 02
0.44999996D 01	0.27313061D 00	-0.81491953D-01	0.30653440D-01	0.16502118D 01	0.18406723D 02
0.49499996D 01	0.23934759D 00	-0.69105461D-01	0.24639570D-01	0.16932563D 01	0.23876546D 02
0.53999995D 01	0.21057610D 00	-0.59120709D-01	0.19930322D-01	0.17239596D 01	0.30446180D 02
0.58499995D 01	0.18585777D 00	-0.51014433D-01	0.16247607D-01	0.17458411D 01	0.38224471D 02
0.62999995D 01	0.16444292D 00	-0.44379240D-01	0.13357408D-01	0.17614117D 01	0.47319415D 02
0.67499994D 01	0.14574316D 00	-0.38901775D-01	0.11075270D-01	0.17724618D 01	0.57837970D 02
0.71999994D 01	0.12929400D 00	-0.34341655D-01	0.92598934D-02	0.17802711D 01	0.69885964D 02
0.76499994D 01	0.11472601D 00	-0.30514036D-01	0.78042867D-02	0.17857574D 01	0.83568088D 02
0.80999993D 01	0.10174308D 00	-0.27276046D-01	0.66276701D-02	0.17895811D 01	0.98987939D 02
0.85499993D 01	0.90105918D-01	-0.24516523D-01	0.56689403D-02	0.17922188D 01	0.11624812D 03
0.89999992D 01	0.79619601D-01	-0.22148336D-01	0.48816664D-02	0.17940149D 01	0.13545034D 03
0.94499992D 01	0.70124108D-01	-0.20102670D-01	0.42303530D-02	0.17952184D 01	0.15669558D 03
0.98999992D 01	0.61487090D-01	-0.18324754D-01	0.36876773D-02	0.17960088D 01	0.18008425D 03
0.10349999D 02	0.53598299D-01	-0.16770666D-01	0.32324557D-02	0.17965153D 01	0.20571631D 03
0.10799999D 02	0.46365274D-01	-0.15404925D-01	0.28481427D-02	0.17968301D 01	0.23369147D 03
0.11249999D 02	0.39709971D-01	-0.14198667D-01	0.25217211D-02	0.17970185D 01	0.26410926D 03
0.11699999D 02	0.33566119D-01	-0.13128250D-01	0.22428761D-02	0.17971258D 01	0.29706926D 03
0.12149999D 02	0.27877111D-01	-0.12174185D-01	0.20033773D-02	0.17971833D 01	0.33267110D 03
0.12599999D 02	0.22594339D-01	-0.11320307D-01	0.17966136D-02	0.17972116D 01	0.37101463D 03
0.13049999D 02	0.17675842D-01	-0.10553129D-01	0.16172403D-02	0.17972240D 01	0.41219996D 03
0.13499999D 02	0.13085224D-01	-0.98613379D-02	0.14609098D-02	0.17972285D 01	0.45632747D 03
0.13949999D 02	0.87907691D-02	-0.92353907D-02	0.13240645D-02	0.17972298D 01	0.50349785D 03
0.14399999D 02	0.47647169D-02	-0.86671988D-02	0.12037772D-02	0.17972300D 01	0.55381207D 03
0.14849999D 02	0.98266588D-03	-0.81498729D-02	0.10976261D-02	0.17972301D 01	0.60737135D 03
0.14969999D 02	0.12520772D-04	-0.80197373D-02	0.10714413D-02	0.17972301D 01	0.62221480D 03

$\xi_4 = 14.971551$

Table 10. Lane - Emden function, $n=4.5$

ξ	$\theta(\xi)$	$\theta'(\xi)$	$\theta''(\xi)$	$-\xi^2\theta'(\xi)$	$\rho_c/\bar{\rho}$
0.0	0.10000000D 01	0.0	-0.33333331D 00	0.0	0.0
0.89999995D 00	0.88553313D 00	-0.21619395D 00	-0.98226593D-01	0.17511708D 00	0.13876427D 01
0.17999999D 01	0.68124795D 00	-0.21430747D 00	0.60343293D-01	0.69435611D 00	0.27997155D 01
0.26999998D 01	0.51548199D 00	-0.15394257D 00	0.63337053D-01	0.11222412D 01	0.58463353D 01
0.35999998D 01	0.39981117D 00	-0.10622296D 00	0.42856262D-01	0.13766494D 01	0.11296992D 02
0.44999997D 01	0.31926296D 00	-0.75074503D-01	0.27496038D-01	0.15202585D 01	0.19980150D 02
0.53999997D 01	0.26138415D 00	-0.54967898D-01	0.17972008D-01	0.16028637D 01	0.32746382D 02
0.62999996D 01	0.21830990D 00	-0.41621914D-01	0.12152026D-01	0.16519736D 01	0.50454185D 02
0.71999996D 01	0.18522340D 00	-0.32448267D-01	0.85068475D-02	0.16821180D 01	0.73963880D 02
0.80999995D 01	0.15911290D 00	-0.25927929D-01	0.61462913D-02	0.17011312D 01	0.10413480D 03
0.89999995D 01	0.13803286D 00	-0.21152974D-01	0.45657394D-02	0.17133907D 01	0.14182401D 03
0.98999994D 01	0.12068376D 00	-0.17563859D-01	0.34745627D-02	0.17214336D 01	0.18788579D 03
0.10799999D 02	0.10617022D 00	-0.14804361D-01	0.27001474D-02	0.17267805D 01	0.24317155D 03
0.11699999D 02	0.93857932D-01	-0.12640587D-01	0.21370092D-02	0.17303697D 01	0.30852995D 03
0.12599999D 02	0.83286313D-01	-0.10914558D-01	0.17185834D-02	0.17327950D 01	0.38480712D 03
0.13499999D 02	0.74113554D-01	-0.95168186D-02	0.14016854D-02	0.17344400D 01	0.47284703D 03
0.14399999D 02	0.66081041D-01	-0.83697792D-02	0.11575677D-02	0.17355572D 01	0.57349177D 03
0.15299999D 02	0.58989730D-01	-0.74172979D-02	0.96664049D-03	0.17363151D 01	0.68758185D 03
0.16199999D 02	0.52684053D-01	-0.66179986D-02	0.81526861D-03	0.17368273D 01	0.81595656D 03
0.17099999D 02	0.47040753D-01	-0.59408767D-02	0.69377740D-03	0.17371715D 01	0.95945424D 03
0.17999999D 02	0.41960951D-01	-0.53623490D-02	0.59518155D-03	0.17374009D 01	0.11189125D 04
0.18899999D 02	0.37364441D-01	-0.48642315D-02	0.51435677D-03	0.17375519D 01	0.12951685D 04
0.19799999D 02	0.33185500D-01	-0.44323290D-02	0.44748909D-03	0.17376500D 01	0.14890590D 04
0.20699999D 02	0.29369770D-01	-0.40554338D-02	0.39170186D-03	0.17377126D 01	0.17014207D 04
0.21599999D 02	0.25871917D-01	-0.37246054D-02	0.34479882D-03	0.17377517D 01	0.19330905D 04
0.22499999D 02	0.22653834D-01	-0.34326431D-02	0.30508421D-03	0.17377754D 01	0.21849050D 04
0.23399999D 02	0.19683265D-01	-0.31736969D-02	0.27123511D-03	0.17377893D 01	0.24577013D 04
0.24299999D 02	0.16932721D-01	-0.29429750D-02	0.24220948D-03	0.17377971D 01	0.27523166D 04
0.25199998D 02	0.14378636D-01	-0.27365229D-02	0.21717925D-03	0.17378013D 01	0.30695884D 04
0.26099998D 02	0.12000691D-01	-0.25510541D-02	0.19548082D-03	0.17378034D 01	0.34103545D 04
0.26999998D 02	0.97812729D-02	-0.23838197D-02	0.17657834D-03	0.17378043D 01	0.37754529D 04
0.27899998D 02	0.77050428D-02	-0.22325059D-02	0.16003597D-03	0.17378047D 01	0.41657220D 04
0.28799998D 02	0.57585767D-02	-0.20951546D-02	0.14549677D-03	0.17378048D 01	0.45820002D 04
0.29699998D 02	0.39300782D-02	-0.19700995D-02	0.13266663D-03	0.17378049D 01	0.50251263D 04
0.30599998D 02	0.22091385D-02	-0.18559155D-02	0.12130168D-03	0.17378049D 01	0.54959391D 04
0.31499998D 02	0.58653810D-03	-0.17513783D-02	0.11119863D-03	0.17378049D 01	0.59952775D 04
0.31799998D 02	0.66081387D-04	-0.17184893D-02	0.10808109D-03	0.17378049D 01	0.61689077D 04

$\xi_{4.5}$ = 31.83895149

ξ	$\theta(\xi)$	$\theta = 1/(1+\frac{1}{3}\xi^2)^{1/2}$	$\theta'(\xi)$	$\theta''(\xi)$	$-\xi^2\,\theta'(\xi)$	$\rho_c/\bar{\rho}$
0.0	0.10000000D 01	0.10000000D 01	0.0	-0.33333331D 00	0.0	0.0
0.29999994D 00	0.98532928D 00	0.98532928D 00	-0.95663019D-01	-0.29101378D 00	0.86096681D-02	0.10453357D 01
0.59999987D 00	0.94491120D 00	0.94491120D 00	-0.16873412D 00	-0.19083032D 00	0.60744256D-01	0.11852964D 01
0.89999981D 00	0.88735655D 00	0.88735655D 00	-0.20961176D 00	-0.84358254D-01	0.16978545D 00	0.14312170D 01
0.11999997D 01	0.82199499D 00	0.82199499D 00	-0.22216079D 00	-0.50036736D-02	0.31991141D 00	0.18004972D 01
0.14999997D 01	0.75592901D 00	0.75592901D 00	-0.21597971D 00	0.41138957D-01	0.48595415D 00	0.23150316D 01
0.17999996D 01	0.69337532D 00	0.69337532D 00	-0.20001211D 00	0.61969543D-01	0.64803898D 00	0.29998175D 01
0.20999996D 01	0.63628484D 00	0.63628484D 00	-0.18032365D 00	0.67443187D-01	0.79522695D 00	0.38819080D 01
0.23999995D 01	0.58520582D 00	0.58520582D 00	-0.16033037D 00	0.64974060D-01	0.92350255D 00	0.49896958D 01
0.26999994D 01	0.53994933D 00	0.53994933D 00	-0.14167768D 00	0.59051487D-01	0.10328298D 01	0.63524457D 01
0.29999994D 01	0.50000008D 00	0.50000008D 00	-0.12500003D 00	0.52083348D-01	0.11249998D 01	0.79999958D 01
0.32999993D 01	0.46473949D 00	0.46473949D 00	-0.11041328D 00	0.45237724D-01	0.12024001D 01	0.99625668D 01
0.35999992D 01	0.43355506D 00	0.43355506D 00	-0.97794387D-01	0.39011548D-01	0.12674147D 01	0.12270640D 02
0.38999992D 01	0.40588755D 00	0.40588755D 00	-0.86928153D-01	0.33562429D-01	0.13221767D 01	0.14954875D 02
0.41999991D 01	0.38124649D 00	0.38124649D 00	-0.77579240D-01	0.28888175D-01	0.13684972D 01	0.18046060D 02
0.44999991D 01	0.35921067D 00	0.35921067D 00	-0.69524656D-01	0.24919235D-01	0.14078737D 01	0.21575074D 02
0.47999990D 01	0.33942218D 00	0.33942218D 00	-0.62566310D-01	0.21564236D-01	0.14415272D 01	0.25572861D 02
0.50999989D 01	0.32157838D 00	0.32157838D 00	-0.56533954D-01	0.18731167D-01	0.14704475D 01	0.30070416D 02
0.53999989D 01	0.30542367D 00	0.30542367D 00	-0.51283834D-01	0.16336271D-01	0.14954360D 01	0.35098771D 02
0.56999988D 01	0.29074196D 00	0.29074196D 00	-0.46695673D-01	0.14306962D-01	0.15171418D 01	0.40688984D 02
0.59999987D 01	0.27735015D 00	0.27735015D 00	-0.42669262D-01	0.12581964D-01	0.15360928D 01	0.46872137D 02
0.62999987D 01	0.26509262D 00	0.26509262D 00	-0.39121195D-01	0.11110283D-01	0.15527196D 01	0.53679328D 02
0.65999986D 01	0.25383659D 00	0.25383659D 00	-0.35981998D-01	0.98498065D-02	0.15673752D 01	0.61141670D 02
0.68999986D 01	0.24346837D 00	0.24346837D 00	-0.33193678D-01	0.87658723D-02	0.15803504D 01	0.69290284D 02

Additional table of the first points of the Lane-Emden function for $n=5$.

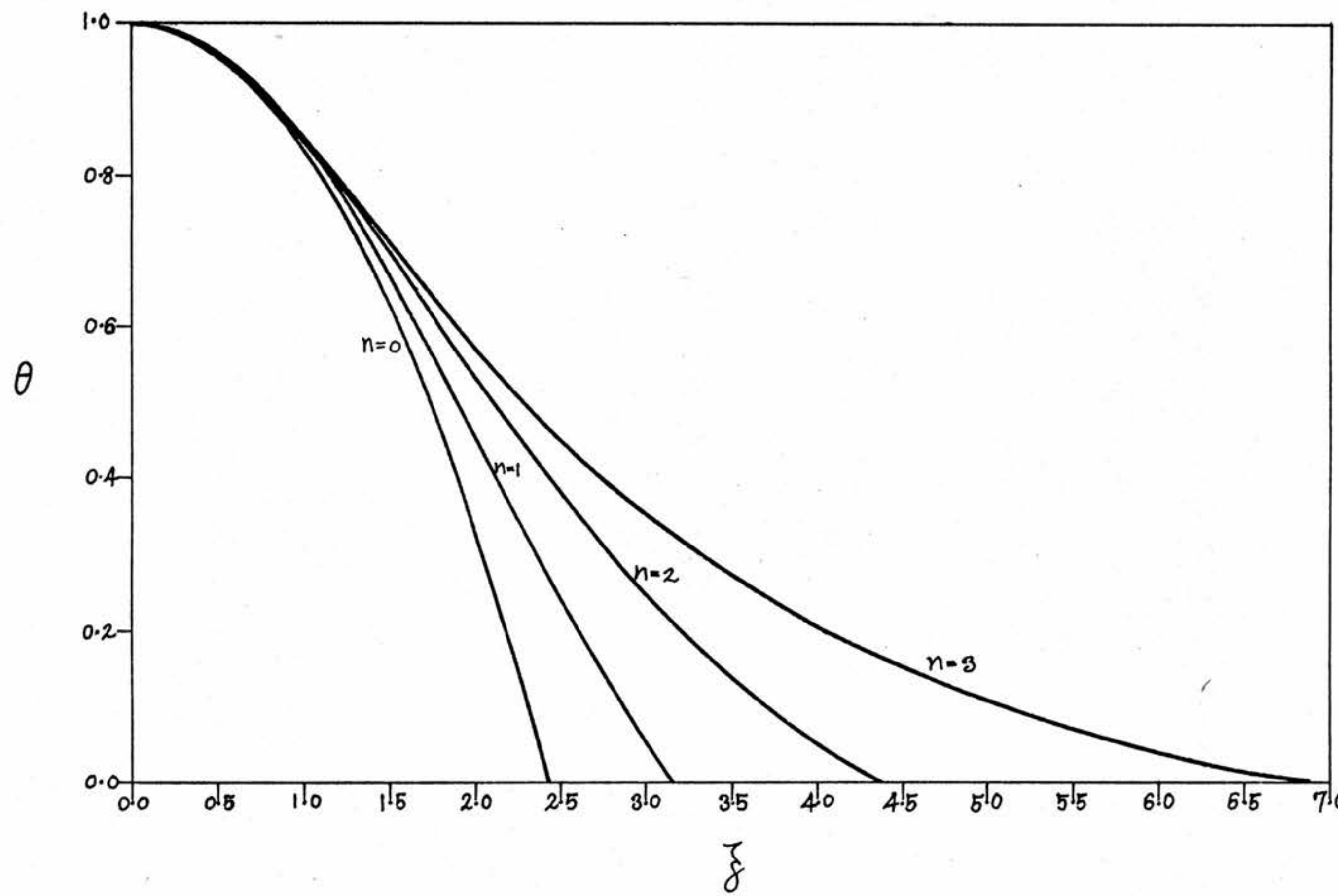


Fig. 1. The Lane-Emden functions for four corresponding values of n.

CHAPTER II

In the first section of this chapter we shall briefly refer to the electron degeneracy. Secondly, we shall discuss the equation of equilibrium of a completely degenerate electron gas and we shall see how the completely degenerate models reduce to polytropic models of polytropic index $n = 3/2$ in the case of non relativistic momenta and of index $n=3$ in the case of extreme relativistic momenta. The discussion is based on S. Chandrasekhar's book "Stellar Structure" Chapter 11.

The third part will be concerned with the partially degenerate electron gas and we shall restrict ourselves in the non relativistic equation of state. Assuming a standard model, we shall prove that the partially degenerate standard models also reduce to polytropic models of polytropic index $n=3/2$ in the case of very high degeneracy and of index $n=3$ in the case of very low degeneracy.

(A) GENERAL DISCUSSION OF ELECTRON DEGENERACY

The completely degenerate models of stars and the partially degenerate stellar models are based on the Fermi-Dirac equation of state of an electron gas and are used to study stars that are at high densities and high temperatures.

We can generally describe the situation by saying that the matter is fully ionised and is also sufficiently dense that the free electrons may be partially or fully degenerate but not dense enough for the heavy particles to be degenerate. We, thus, assume that the stellar material is a two-components neutral plasma made up of nuclei and free electrons. We assume that the two components do not interact with each other.

In deriving the necessary equation of state we only consider the electron-component of the plasma.

The meaning of electron degeneracy is, as follows:

Electrons can only be described by antisymmetrical wave functions and only then do they obey the Pauli's exclusion principle which states that no two electrons can be described by the same set of quantum numbers. As a result, this principle limits the number of free electrons which can have energies in some range about some energy in a gas of free electrons. According to the Pauli principle, not more than one electron can occupy a unit cell h^3 in phase space. Because of the spin of the electron, two electrons can occupy each such cell provided that their spins are in opposite direction.

The maximum possible number of electrons in the momentum range which can be in the box of V is then

$$V.\frac{8\pi p^2}{h^3}dp$$

The term "degeneracy" is used to describe the extent to which the available unit cells in phase space are actually occupied by electrons.

Electrons, therefore, as antisymmetric particles or fermions (particles with integral spin) must obey the Fermi-Dirac distribution function, which is derived under the assumptions of thermodynamic equilibrium but which may hold under more general conditions, of weakly interacting constituent elements, and of systems of particles which cannot be permanently distinguished one from another.

For the above assumptions the statistical mechanics gives the formula for the Fermi-Dirac systems

$$n_e(p)dp = \frac{8\pi p^2}{h^3} dp \cdot \frac{1}{e^{\alpha+\theta E}+1} \qquad (1)$$

as the number of electrons per unit volume having momenta between p and (p + dp).

Where $-\alpha$ is the degeneracy parameter and is equal to

$-\alpha = \frac{\mu_c}{kT}$ where μ_c = the chemical potential or

$-\alpha = \frac{G_t}{NkT}$ where G_t = thermodynamical potential

(B) DISCUSSION OF COMPLETELY DEGENERATE STELLAR MODELS AS POLYTROPIC STARS

The probability factor $\frac{1}{e^{\alpha+\beta E}+1}$ is equal to 1 in the case of complete degeneracy. In this case the total number of electrons per unit volume is

$$n_e = \int_0^{p_F} n_e(p)\,dp = \frac{8\pi}{3h^3}\, p_F^3$$

where $p_F = \left(\frac{3h^3 n_e}{8\pi}\right)^{1/3}$ is the highest momentum occupied by the electrons (it is often called the Fermi threshhold).

To calculate the pressure in a degenerate electron gas, we recall that, by definition, the pressure is the rate of transfer of momentum across an ideal surface of unit area in the gas and is given by the formula

$$P = \frac{1}{3}\int_0^{\infty} n_e(p)\, p\, v_p\, dp$$

where $n_e(p)$ depends upon the type of particles and the Quantum Statistics, while the relation of v_p to p depends upon relativistic considerations. If we take non relativistic mechanics we have

$$E = \frac{1}{2}mv^2 = \frac{p^2}{2m} \Rightarrow P = \frac{8\pi}{3h^3 m}\int_0^{p_F} p^4\,dp \Rightarrow$$

$$P = \frac{8\pi}{15mh^3}\left(\frac{3h^3 n_e}{8\pi}\right)^{5/3} = \frac{h^2}{20m}\left(\frac{3}{\pi}\right)^{2/3} N_0^{5/3}\left(\frac{\rho}{\mu_e}\right)^{5/3} \Rightarrow$$

$$P = 1.0036\times 10^{13}\left(\frac{\rho}{\mu_e}\right)^{5/3} \text{ dynes/cm}^2$$

where N_0 is the Avogadro's number and

μ_e is the mean molecular weight per free electrons and is defined by the relation

$$\rho = n_e \mu_e H$$

where H is the hydrogen atom mass $\simeq 1/$Avogadro's number

For a completely ionized gas we know that

$$n_e = \frac{\rho}{H} \sum_i \frac{x_i Z_i}{A_i}$$

where x_i is the relative mass abundance of the element of atomic number Z_i and atomic weight A_i

$$\Rightarrow \sum_i \frac{x_i Z_i}{A_i} = \frac{1}{\mu_e}$$

$\Rightarrow \frac{1}{\mu_e}$ is the average number of free ionization electrons per unit atomic weight, or

μ_e is the average atomic weight per free ionisation electron.

If X, Y are the hydrogen and helium abundances, assuming that $\frac{Z_i}{A_i} \simeq \frac{1}{2}$

$$\Rightarrow \frac{1}{\mu_e} = X + \frac{1}{2} Y + \frac{1}{2}(1-X-Y) = \frac{1}{2}(1+X)$$

Since $0 \le X \le 1$, it follows that for a completely ionized matter μ_e always lies between $1 \le \mu_e \le 2$

If μ_e is a constant then the equation of state for a non relativistic degenerate electron gas is a polytropic relation of index $n = 3/2$

For relativistic degeneracy we first take the variation of mass with velocity:

$$v_p = \frac{p}{m_o}\left(1+\frac{p^2}{m_o^2c^2}\right)^{-1/2} \quad , \quad m = \frac{m_o}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} .$$

The pressure integral becomes

$$P = \frac{1}{3}\int n_e(p)\, v_p\, p\, dp = \frac{8\pi}{3m_o h^3}\int_0^{p_F} \frac{p^4\,dp}{\left(1+\frac{p^2}{m_o^2c^2}\right)^{1/2}}$$

The above integral can be solved (Chandrasekhar Chap. 10) by introducing the substitution

$$\sinh\theta = p/mc \quad , \quad \sinh\theta_F = p_F/mc$$

by which we obtain

$$P_e = \frac{8\pi m^4c^5}{3h^3}\int_0^{\theta_F} \sinh^4\theta\, d\theta =$$

$$= \frac{8\pi m^4c^5}{3h^3}\left[\frac{3}{8}\theta_F - \frac{3}{16}\sinh 2\theta_F + \frac{1}{4}\sinh^3\theta_F\cosh\theta_F\right]$$

Letting $x = \sinh\theta_F = p_F/mc$ and defining the function

$$f(x) = x(x^2+1)^{1/2}(2x^2-3) + 3\ln(x+\sqrt{1+x^2})$$

we may write for the electron pressure

$$P_e = \frac{\pi m^4c^5}{3h^3} f(x) = A f(x) = 6.002\times10^{22} f(x) \text{ dynes/cm}^2 \quad \text{(a)}$$

From the relations $\rho = n_e \mu_e \frac{1}{N_o}$ and $n_e = \frac{8\pi}{3h^3} p_F^3$

$$\Rightarrow \quad \rho = \left[\frac{8\pi m^3c^3\mu_e}{3h^3N_o}\right]x^3 = Bx^3 \quad \text{(b)}$$

where $B = 9.736 \times 10^5 \cdot \mu_e$ (c.g.s)

The function $f(x)$ has the following behaviour for $x \to 0$ and for $x \to \infty$

For $x \to 0$, $f(x) \to \frac{8}{5} x^5$

for $x \to \infty$, $f(x) \to 2x^4$

The equations (a) and (b) represent parametrically the equation of state of a highly degenerate electron gas. From the asymptotic forms of $f(x)$ it follows that the exact variation of the pressure with density is

$P = k\rho^{5/3}$ at low densities (non relativistic)

$P = k\rho^{4/3}$ at high densities (extremely relativistic)

The equation of equilibrium of this completely degenerate matter in equilibrium under its own gravitation is

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho \qquad \text{(c)}$$

By substituting P_e and ρ in the basic differential equation (c) we get

$$\frac{A}{B}\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{x^3}\frac{df(x)}{dr}\right) = -4\pi G B x^3$$

From the definition of $f(x)$ $\Rightarrow$ $\frac{df(x)}{dr} = \frac{8x^4}{(x^2+1)^{1/2}}\frac{dx}{dr}$

$$\Rightarrow \frac{1}{x^3}\frac{df(x)}{dr} = \frac{8x}{(x^2+1)^{1/2}}\frac{dx}{dr} = 8\frac{d}{dr}(x^2+1)^{1/2}$$

$$\Rightarrow \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}(x^2+1)^{1/2}\right) = -\frac{\pi G B^2}{2A}x^3$$

We now define the dimensionless variable $y^2 = x^2+1$

Then $\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dy}{dr}\right) = -\frac{\pi G B^2}{2A}(y^2-1)^{3/2}$

If x_0 is the value of x at the center then y_0 is the corresponding value of y at the center. We now introduce the new variables

$$r = a\eta$$
$$y = y_0 \phi$$

where a is a scale length $\quad a = \left(\frac{2A}{\pi G}\right)^{1/2} \frac{1}{B y_0}$

The equation of equilibrium becomes

$$\frac{1}{\eta^2} \frac{d}{d\eta} \left(\eta^2 \frac{d\phi}{d\eta} \right) = - \left(\phi^2 - \frac{1}{y_0^2} \right)^{3/2}$$

The boundary conditions at the center are

$$\phi = 1 \; , \qquad \frac{d\phi}{d\eta} = 0$$

The outer boundary is defined at the point where the density becomes zero.

At the center $\quad \rho_0 = B x_0^3 = B \left(y_0^2 - 1 \right)^{3/2}$

At the boundary $\quad \rho = \rho_0 \frac{y_0^3}{(y_0^2 - 1)^{3/2}} \left(\phi^2 - \frac{1}{y_0^2} \right)^{3/2}$

At the non-relativistic limit $\quad x \to 0 \quad$ or $\quad y \to 1$

and $\quad \phi = \frac{y}{y_0} = \frac{(1+x^2)^{1/2}}{(1+x_0^2)^{1/2}} \simeq 1 + \frac{1}{2} x^2 - \frac{1}{2} x_0^2 + \dots$

and $\quad \frac{d\phi}{d\eta} = \frac{1}{2} \frac{d}{d\eta} x^2$

Put $\quad \phi^2 - \frac{1}{y_0^2} = \theta \qquad$ then $\qquad \phi = 1 - \frac{1}{2} (x_0^2 - \theta)$

At the origin $\quad \theta(0) = x_0^2$

Introduce $\quad \xi = 2^{1/2} \eta$

The equation of equilibrium reduces to

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = - \theta^{3/2}$$

which is the differential equation of a polytropic star of index $n = 3/2$. Hence a polytropic degenerate white dwarf star is a polytropic star of index $n = 3/2$

In the extreme relativistic limit $x \to \infty$, $y \to \infty$ and the equation of equilibrium reduces to

$$\frac{1}{\eta^2} \frac{d}{d\eta} \left(\eta^2 \frac{d\phi}{d\eta} \right) = -\phi^3$$

which is the differential equation of a polytropic star of index 3. Hence, an extremely relativistic degenerate white dwarf is a polytropic star of index 3.

(C) PARTIALLY DEGENERATE STELLAR MODELS

We first recall the general formulae for an electron gas.

The total number of particles is

$$N = \frac{8\pi V}{h^3} \int_0^\infty \frac{p^2}{e^{\alpha+\theta E}+1}\, dp \tag{2}$$

(or else $N/V = n_e$ = number density of free ionization electrons). The total energy corresponding to the distribution is

$$U = \frac{8\pi V}{h^3} \int_0^\infty \frac{E p^2}{e^{\alpha+\theta E}+1}\, dp \tag{3}$$

and

$$PV = \frac{8\pi V}{3h^3} \int_0^\infty \frac{p^3}{e^{\alpha+\theta E}+1} \frac{\partial E}{\partial p}\, dp \tag{4}$$

However, for the astronomical applications we are going to consider here, it is permissible to neglect the relativistic effects. Therefore, we can write

$$E = p^2/2m \quad \Rightarrow \quad dp = \frac{m}{p}\, dE$$

$$(2) \Rightarrow \quad N = \frac{4\pi V}{h^3} (2m)^{3/2} \int_0^\infty \frac{E^{1/2}}{e^{\alpha+\theta E}+1}\, dE \tag{2a}$$

$$(3) \Rightarrow \quad U = \frac{4\pi V}{h^3} (2m)^{3/2} \int_0^\infty \frac{E^{3/2}}{e^{\alpha+\theta E}+1}\, dE \tag{3a}$$

$$(4) \Rightarrow \quad PV = \frac{2}{3} \frac{4\pi V}{h^3} (2m)^{3/2} \int_0^\infty \frac{E^{3/2}}{e^{\alpha+\theta E}+1}\, dE \tag{4a}$$

Put $\theta E = u$ and $\alpha = -\ln A$ and define the integral U_ν by

$$U_\nu = \frac{1}{\Gamma(\nu+1)} \int_0^\infty \frac{u^\nu\, du}{\frac{1}{A} e^u + 1} \tag{5}$$

Equations 2a, 3a, 4a can be written

$$N = \frac{2V}{h^3} (2\pi m kT)^{3/2} \, U_{1/2} \tag{6}$$

$$PV = \frac{2}{3} U = \frac{2V}{h^3} (2\pi m kT)^{3/2} \, kT \, U_{3/2} \tag{7}$$

The above treatment leads to an equation of state applicable for stellar models which are too degenerate at the center for the perfect gas law to apply but not massive enough for the central density to be high enough for the white dwarf models to be valid. For these stars of such a small mass the relativistic effects can be neglected. Under these circumstances the equ. (7) provides the equation of state. Considering for the time being contributions only from the electron gas:

$$P_{gas} = \frac{2}{h^3} (2\pi m)^{3/2} (kT)^{5/2} \, U_{3/2} \tag{8}$$

The density is given by equ. (6) as $\rho = n_e \mu_e H$ (9)

$$\rho = \frac{2}{h^3} (2\pi m)^{3/2} (kT)^{3/2} \, U_{1/2} \, \mu_e H \tag{10}$$

where μ_e = mean molecular weight per free electron.

In the present treatment we consider μ_e as a constant throughout the model.

Assuming that $U_{1/2}$, $U_{3/2}$ are known functions of Λ, we get the equation of state in terms of Λ. We shall consider next the standard model equilibrium configuration as it is built on the equation of state (8).

We know that the ratio of the gas pressure to the total pressure β, depends upon the distance from the center of the star. However, for this specific configuration we assume that β is constant i.e. that the gas pressure is a constant fraction of the total pressure throughout the star.

$$P = \frac{1}{\beta} P_{gas} = \frac{1}{1-\beta} P_{rad} = \frac{1}{1-\beta} \frac{a}{3} T^4 \quad (13)$$

where $P = P_{gas} + P_{radiation}$ is the total pressure. For $\beta = 0$ we have radiation pressure only

For $\beta \to 1$ radiation pressure is negligible .

We adopt here the notation of S. Chandrasekhar "An Introduction to the Study of Stellar Structure" Chapter X1

$$Q_1 = \frac{2}{h^3} (2\pi m)^{3/2} , \qquad Q_2 = k^4 \frac{3}{a} \frac{1-\beta}{\beta} \quad (14)$$

equ. (8) is written $\quad P_{gas} = Q_1 (kT)^{5/2} U_{3/2} \quad$ (15)

from equas. (13) and (14) $(kT)^4 = Q_2 P_{gas}$ and substituting equ. (15)

$$(kT)^4 = Q_2 Q_1 (kT)^{5/2} U_{3/2} \Rightarrow (kT)^{3/2} = Q_2 Q_1 U_{3/2} \quad (16)$$

$$\Rightarrow T = Q_2^{2/3} . Q_1^{2/3} . k^{-1} . U_{3/2}^{2/3} \quad (17)$$

From equ. (15) and (16) $\Rightarrow P_{gas} = Q_1^{8/3} Q_2^{5/3} U_{3/2}^{8/3}$ (18)

$\Rightarrow$ the total pressure is $P = \frac{P_{gas}}{\beta} = Q_1^{8/3} Q_2^{5/3} U_{3/2}^{8/3} \beta^{-1}$ (19)

equation (10) is written

$$\rho = Q_1 (kT)^{3/2} \mu_e H U_{3/2} \quad (20)$$

from equas. (15) and (20) $\Rightarrow \rho = Q_1^2 Q_2 \mu_e H U_{1/2} U_{3/2}$ (21)

Equations 17, 18, 21 give respectively the temperature, pressure and density of the partially degenerate partial model as functions of the exponential of the degeneracy parameter, the relative radiation pressure $\frac{1-\beta}{\beta}$ and the mean molecular weight per free electron μ_e , the pressure and temperature being independent of μ_e.

(D) THE EQUATION OF EQUILIBRIUM FOR THE PARTIALLY DEGENERATE STANDARD MODEL

By putting equ. 19, 21 in the equation (9) Chapter I of hydrostatic equilibrium

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = -4\pi G\rho$$

we get

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{U_{1/2}U_{3/2}}\frac{d}{dr}U_{3/2}^{2/3}\right) = -4\pi G\beta(\mu_e H)^2 Q_1^{4/3} Q_2^{1/3} U_{1/2}U_{3/2} \quad (22)$$

For the Fermi-Dirac integral we know that

$$\frac{d}{d\Lambda}U_\nu = \frac{d}{d\Lambda}\frac{1}{\Gamma(\nu+1)}\int_0^\infty \frac{u^\nu du}{\frac{1}{\Lambda}e^u+1} = \frac{1}{\Gamma(\nu+1)}\frac{1}{\Lambda^2}\int_0^\infty \frac{e^u u^\nu}{(\frac{1}{\Lambda}e^u+1)^2}du$$

$$= \frac{1}{\nu\Gamma(\nu)}\frac{1}{\Lambda}\int_0^\infty \frac{\frac{1}{\Lambda}e^u u^\nu du}{(\frac{1}{\Lambda}e^u+1)^2} = \frac{1}{\Gamma(\nu)}\frac{1}{\Lambda}\int_0^\infty \frac{u^{\nu-1}}{(\frac{1}{\Lambda}e^u+1)}du = \frac{1}{\Lambda}U_{\nu-1} \quad (23)$$

Equ. (22) is simplified by (23) to the form:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\,\frac{8}{3}\,\frac{U_{3/2}^{7/3}U_{1/2}}{U_{1/2}U_{3/2}}\,\frac{1}{\Lambda}\frac{d\Lambda}{dr}\right) = -4\pi G\beta(\mu_e H)^2 Q_1^{4/3}Q_2^{1/3}U_{1/2}U_{3/2} \quad (24)$$

or

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 U_{3/2}^{2/3}\frac{d\log\Lambda}{dr}\right) = -\frac{3\pi}{2}G\beta Q_1^{4/3}Q_2^{1/3}(\mu_e H)^2 U_{3/2}U_{1/2} \quad (25)$$

Let

$$r = a\zeta = \left(\frac{2}{3\pi G\beta Q_1^{4/3}Q_2^{1/3}(\mu_e H)^2}\right)^{1/2}\zeta \quad (26)$$

Equ. (25) reduces to

$$\frac{1}{\zeta^2}\frac{d}{d\zeta}\left(\zeta^2 U_{3/2}^{2/3}\frac{d\log\Lambda}{d\zeta}\right) = -U_{3/2}U_{1/2} \quad (27)$$

which is the equation of equilibrium for the standard model of a partially degenerate configuration .

or in an equivalent form:

$$\frac{1}{\Lambda}\frac{d^2\Lambda}{d\zeta^2} + \frac{1}{\Lambda^2}\left(\frac{d\Lambda}{d\zeta}\right)^2\left\{\frac{2}{3}\frac{U_{1/2}}{U_{3/2}} - 1\right\} + \frac{2}{\zeta}\frac{1}{\Lambda}\frac{d\Lambda}{d\zeta} = -U_{3/2}^{1/3}U_{1/2} \quad (28)$$

If we put $\Lambda = \Lambda_0\lambda$ where Λ_0 is a constant and λ normalized variable $0 \leqq \lambda \leqq 1$ we get

$$\frac{d}{d\zeta}U_\nu(\Lambda_0\lambda) = \frac{1}{\Lambda_0\lambda}U_{\nu-1}(\Lambda_0\lambda)\Lambda_0\frac{d\lambda}{d\zeta} = \frac{1}{\lambda}U_{\nu-1}(\Lambda_0\lambda)\frac{d\lambda}{d\zeta} \quad (29)$$

and from equ. (28) we have:

$$\frac{1}{\vartheta}\frac{d^2\vartheta}{d\eta^2}+\left(\frac{d\vartheta}{d\eta}\right)^2\frac{1}{\vartheta^2}\left\{\frac{2}{3}\frac{U_{1/2}}{U_{3/2}}-1\right\}+\frac{2}{\eta\vartheta}\frac{d\vartheta}{d\eta}=-U_{3/2}^{1/3}U_{1/2} \tag{30}$$

or

$$\vartheta\frac{d^2\vartheta}{d\eta^2}+\left(\frac{d\vartheta}{d\eta}\right)^2\left\{\frac{2}{3}\frac{U_{1/2}}{U_{3/2}}-1\right\}+\frac{2}{\eta}\vartheta\frac{d\vartheta}{d\eta}=-\vartheta^2U_{3/2}^{1/3}U_{1/2} \tag{31}$$

for $\vartheta\neq 0$

The required solution for equation 27 (or 30) is the function $\Lambda(\eta)$ or for a chosen value of Λ_0 the function $\vartheta(\eta)$ where the boundary conditions are:

$$\text{for the}\begin{cases}\text{center} & \eta=0,\ \Lambda(0)=\Lambda_0,\ \vartheta=1,\ \dfrac{d\vartheta}{d\eta}=0\\ \text{boundary} & \eta=1,\ \Lambda(1)\to 0,\ \vartheta\to 0\end{cases} \tag{32}$$

In order to derive the second derivative at the center, we need to evaluate the term

$$\frac{2}{\eta}\vartheta\frac{d\vartheta}{d\eta}\qquad\text{for}\qquad \vartheta\to 1,\ \eta\to 0,\ \frac{d\vartheta}{d\eta}\to 0$$

Using de l'Hospital's rule $\Rightarrow$

$$\frac{2}{\eta}\frac{d\vartheta}{d\eta}\to 2\frac{d^2\vartheta}{d\eta^2} \tag{33}$$

from equs. (31) and (33), and for the initial values (32)

we get
$$\frac{d^2\vartheta}{d\eta^2}+2\frac{d^2\vartheta}{d\eta^2}=-U_{3/2}^{1/3}U_{1/2}\qquad\text{or}\qquad\frac{d^2\vartheta}{d\eta^2}=-\frac{1}{3}U_{3/2}^{1/3}U_{1/2} \tag{34}$$

By continuous differentiations of the above relation (31) we can also find the higher order derivatives. By using de l'Hospital's rule we are able to evaluate the derivatives at the origin initial values (32).

Indeed the third derivative is going to be

$$\vartheta\frac{d^3\vartheta}{d\eta^3}-\frac{2}{3\vartheta}\left(\frac{d\vartheta}{d\eta}\right)^3\left\{U_{3/2}^{-2}\,U_{1/2}^{2}-U_{3/2}^{-1}\,U_{-1/2}\right\}+\frac{4}{3}\frac{d\vartheta}{d\eta}\frac{d^2\vartheta}{d\eta^2}\left\{-\frac{3}{4}+U_{3/2}^{-1}U_{1/2}\right\}+$$

$$+\frac{2}{\eta}\left[\left(\frac{d\vartheta}{d\eta}\right)^2-\frac{\vartheta}{\eta}\frac{d\vartheta}{d\eta}+\vartheta\frac{d^2\vartheta}{d\eta^2}\right]=-2\vartheta\frac{d\vartheta}{d\eta}U_{3/2}^{1/3}U_{1/2}\left[1+\frac{1}{6}U_{3/2}^{-1}U_{1/2}+\frac{1}{2}U_{-1/2}U_{1/2}^{-1}\right] \tag{35}$$

For $\eta=0$, $\frac{d\lambda}{d\eta}=0$, $\lambda \to 1$ equ. (35) becomes:

$$\lambda \frac{d^3\lambda}{d\eta^3} = -\frac{2}{\eta}\left[\left(\frac{d\lambda}{d\eta}\right)^2 - \frac{\lambda}{\eta}\frac{d\lambda}{d\eta} + \lambda\frac{d^2\lambda}{d\eta^2} \right] \tag{36}$$

the terms in the r.h.s. are of an indeterminate form, so by using de l'Hospital's rule we have

$$\begin{aligned} \lambda \frac{d^3\lambda}{d\eta^3} &\to -2\left[3\frac{d\lambda}{d\eta}\frac{d^2\lambda}{d\eta^2} - \frac{1}{\eta}\left(\frac{d\lambda}{d\eta}\right)^2 + \frac{\lambda}{\eta^2}\frac{d\lambda}{d\eta} - \frac{\lambda}{\eta}\frac{d^2\lambda}{d\eta^2} + \lambda\frac{d^3\lambda}{d\eta^3} \right] = \\ &= -2\left[3\frac{d\lambda}{d\eta}\frac{d^2\lambda}{d\eta^2} - \frac{1}{\eta}\left\{ \left(\frac{d\lambda}{d\eta}\right)^2 - \frac{\lambda}{\eta}\frac{d\lambda}{d\eta} + \lambda\frac{d^2\lambda}{d\eta^2} \right\}\right] - 2\lambda\frac{d^3\lambda}{d\eta^3} = \\ &= -\frac{2}{\eta}\left\{ \left(\frac{d\lambda}{d\eta}\right)^2 - \frac{\lambda}{\eta}\frac{d\lambda}{d\eta} + \lambda\frac{d^2\lambda}{d\eta^2} \right\} - 2\lambda\frac{d^3\lambda}{d\eta^3} \end{aligned} \tag{37}$$

We note here that the terms in the brackets are equal to $\lambda \frac{d^3\lambda}{d\eta^3}$, from equ. (36).

$$(37) \Rightarrow \lambda \frac{d^3\lambda}{d\eta^3} = -\lambda \frac{d^3\lambda}{d\eta^3} \Rightarrow \frac{d^3\lambda}{d\eta^3} = 0 \text{ at } \frac{d\lambda}{d\eta}=0,\ \eta=0,\ \lambda\to 1 \tag{38}$$

We notice here that we expect all the odd order derivatives to be zero at the origin.

Because, since equ. 27 receives solution of the form $\Lambda(\eta)$ and $\Lambda(-\eta)$ then for a Taylor's expansion about the origin $\eta=0$ we expect to get only the even powers of η which means that the derivatives of odd orders must be zero at the origin.

By differentiation of equation 35 we find the fourth derivative:

$$\begin{aligned} &\lambda\frac{d^4\lambda}{d\eta^4} + \frac{d\lambda}{d\eta}\frac{d^3\lambda}{d\eta^3}\left\{\frac{4}{3}U_{3/2}^{-1}U_{1/2}\right\} + \frac{2}{3\lambda^2}\left(\frac{d\lambda}{d\eta}\right)^4\left\{U_{3/2}^{-2}U_{1/2}^{2} + 2U_{3/2}^{-3}U_{1/2}^{3} - 3U_{3/2}^{-2}U_{-1/2}U_{1/2} - \right. \\ &\left. - U_{3/2}^{-1}U_{-1/2} + U_{3/2}^{-1}U_{-3/2}\right\} + \frac{10}{3\lambda}\left(\frac{d\lambda}{d\eta}\right)^2\frac{d^2\lambda}{d\eta^2}\left\{-U_{3/2}^{-2}U_{1/2}^{2} + U_{3/2}^{-1}U_{-1/2}\right\} + \left(\frac{d^2\lambda}{d\eta^2}\right)^2\left\{\frac{4}{3}\cdot\right. \\ &\left.\cdot U_{3/2}^{-1}U_{1/2} - 1\right\} - \frac{4}{\eta^2}\left\{\left(\frac{d\lambda}{d\eta}\right)^2 + \lambda\frac{d^2\lambda}{d\eta^2} - \frac{\lambda}{\eta}\frac{d\lambda}{d\eta}\right\} + \frac{2}{\eta}\left\{3\frac{d\lambda}{d\eta}\frac{d^2\lambda}{d\eta^2} + \lambda\frac{d^3\lambda}{d\eta^3}\right\} = \\ &= -\left\{\lambda\frac{d^2\lambda}{d\eta^2} + \left(\frac{d\lambda}{d\eta}\right)^2\right\}\left\{\frac{1}{3}U_{1/2}^{2}U_{3/2}^{-2/3} + U_{-1/2}U_{3/2}^{1/3} + 2U_{1/2}U_{3/2}^{1/3}\right\} - \left(\frac{d\lambda}{d\eta}\right)^2\left\{U_{-1/2}U_{1/2}U_{3/2}^{-2/3} - \right. \\ &\left. \frac{2}{9}U_{1/2}^{3}U_{3/2}^{-5/3} + U_{-3/2}U_{3/2}^{1/3} + 2U_{-1/2}U_{3/2}^{1/3} + \frac{2}{3}U_{1/2}^{2}U_{3/2}^{-2/3}\right\}. \end{aligned} \tag{39}$$

Relation (39) is the general form of the fourth derivative. In order to evaluate $\frac{d^4\lambda}{d\eta^4}$ at the origin

we first consider the terms whose limits at the origin are of an indeterminate form $(0/0$ or $0\cdot\infty)$

We have the following:

$$X_4 = \frac{2}{\eta}\left[3\frac{d\lambda}{d\eta}\frac{d^2\lambda}{d\eta^2} + \lambda\frac{d^3\lambda}{d\eta^3} - \frac{2}{\eta}\left\{\lambda\frac{d^2\lambda}{d\eta^2} + \left(\frac{d\lambda}{d\eta}\right)^2\right\} + \frac{2}{\eta^2}\lambda\frac{d\lambda}{d\eta}\right] \Rightarrow$$

(l'Hospital's rule) $$2\left[4\frac{d\lambda}{d\eta}\frac{d^3\lambda}{d\eta^3} + \lambda\frac{d^4\lambda}{d\eta^4} + 3\left(\frac{d^2\lambda}{d\eta^2}\right)^2\right] - \frac{4}{\eta}\left[3\frac{d\lambda}{d\eta}\frac{d^2\lambda}{d\eta^2} + \lambda\frac{d^3\lambda}{d\eta^3} - \frac{2}{\eta}\left\{\lambda\frac{d^2\lambda}{d\eta^2} + \left(\frac{d\lambda}{d\eta}\right)^2\right\} + \frac{2}{\eta^2}\lambda\frac{d\lambda}{d\eta}\right] \Rightarrow$$

$$X_4 = 2\left[4\frac{d\lambda}{d\eta}\frac{d^3\lambda}{d\eta^3} + \lambda\frac{d^4\lambda}{d\eta^2} + 3\left(\frac{d^2\lambda}{d\eta^2}\right)^2\right] - 2X_4 \Rightarrow$$

$$X_4 = \frac{2}{3}\left[4\frac{d\lambda}{d\eta}\frac{d^3\lambda}{d\eta^3} + \lambda\frac{d^4\lambda}{d\eta^4} + 3\left(\frac{d^2\lambda}{d\eta^2}\right)^2\right] \qquad (40)$$

For $\eta=0$, $\frac{d\lambda}{d\eta}=0$, $\lambda=1$ equ. (39) becomes

$$\frac{d^4\lambda}{d\eta^4} = -\frac{3}{5}\left[\left\{1+\frac{4}{3}u_{3/2}^{-1}u_{1/2}\right\}\left(\frac{d^2\lambda}{d\eta^2}\right)^2 + \left(\frac{d^2\lambda}{d\eta^2}\right)\left\{\frac{1}{3}u_{3/2}^{-2/3}u_{1/2}^{2} + 2u_{3/2}^{1/3}u_{1/2} + u_{3/2}^{1/3}u_{-1/2}\right\}\right] \qquad (41)$$

By differentiation of equ. 39 we find the fifth derivative as

$$\lambda \frac{d^5\lambda}{d\eta^5} + \frac{d\lambda}{d\eta}\frac{d^4\lambda}{d\eta^4}\left\{\frac{4}{3}U_{1/2}U^{-1}_{3/2} + 1\right\} + \frac{d^2\lambda}{d\eta^2}\frac{d^3\lambda}{d\eta^3}\left\{4U_{1/2}U^{-1}_{3/2} - 2\right\} + \left\{\frac{14}{3}\lambda\left(\frac{d\lambda}{d\eta}\right)^2\frac{d^3\lambda}{d\eta^3} + \frac{24}{3\lambda}\right.$$

$$\left.\frac{d\lambda}{d\eta}\left(\frac{d^2\lambda}{d\eta^2}\right)^2 - \frac{10}{3\lambda^2}\left(\frac{d\lambda}{d\eta}\right)^3\frac{d^2\lambda}{d\eta^2}\right\}\left\{U_{-1/2}U^{-1}_{3/2} - U^{-2}_{3/2}U^2_{1/2}\right\} + \left\{\frac{8}{3\lambda^2}\left(\frac{d\lambda}{d\eta}\right)^3\frac{d^2\lambda}{d\eta^2} - \frac{4}{3\lambda^2}\left(\frac{d\lambda}{d\eta}\right)^5\right\}\left\{\left(\frac{U_{1/2}}{U_{3/2}}\right)^2 - \left(\frac{U_{-1/2}}{U_{3/2}}\right)\right.$$

$$\left. + \frac{U_{-3/2}}{U_{3/2}} + 2\left(\frac{U_{1/2}}{U_{3/2}}\right)^3 - 3U^{-2}_{3/2}U_{-1/2}U_{1/2}\right\} + \frac{2}{3\lambda^3}\left(\frac{d\lambda}{d\eta}\right)^5\left[\frac{U_{1/2}}{U_{3/2}}\left\{-2\left(\frac{U_{1/2}}{U_{3/2}}\right)^2 + 3\frac{U_{-1/2}}{U_{3/2}} - \frac{U_{-3/2}}{U_{1/2}} - \frac{U_{-3/2}}{U_{3/2}} + \right.\right.$$

$$\left.\left.\frac{U_{-5/2}}{U_{1/2}} - 6\left(\frac{U_{1/2}}{U_{3/2}}\right)^3\right\}\right] + \frac{10}{3\lambda^2}\left(\frac{d\lambda}{d\eta}\right)^3\frac{d^2\lambda}{d\eta^2}\left\{\left(\frac{U_{1/2}}{U_{3/2}}\right)^3 - 3U_{1/2}U_{-1/2}U^{-2}_{3/2} + \frac{U_{-3/2}}{U_{3/2}}\right\} - \frac{2}{\eta}\left\{\frac{9}{\eta}\frac{d\lambda}{d\eta}\frac{d^2\lambda}{d\eta^2} - 3\right.$$

$$\left(\frac{d^2\lambda}{d\eta^2}\right)^2 - 4\frac{d\lambda}{d\eta}\frac{d^3\lambda}{d\eta^3} + \frac{3\lambda}{\eta}\frac{d^3\lambda}{d\eta^3} - \lambda\frac{d^4\lambda}{d\eta^4} - \frac{6}{\eta^2}\left(\frac{d\lambda}{d\eta}\right)^2 - \frac{6\lambda}{\eta^2}\frac{d^2\lambda}{d\eta^2} + \left.\frac{6\lambda}{\eta^3}\frac{d\lambda}{d\eta}\right\} = -\left\{3\frac{d\lambda}{d\eta}\frac{d^2\lambda}{d\eta^2} + \lambda\right.$$

$$\left.\frac{d^3\lambda}{d\eta^3}\right\}\left\{\frac{1}{3}U^2_{1/2}U^{-2/3}_{3/2} + U_{-1/2}U^{1/3}_{3/2} + 2U_{1/2}U^{1/3}_{3/2}\right\} - \left\{+3\frac{d\lambda}{d\eta}\frac{d^2\lambda}{d\eta^2} + \frac{1}{\lambda}\left(\frac{d\lambda}{d\eta}\right)^3\right\}\left\{U_{1/2}U_{-1/2}U^{-2/3}_{3/2} - \right.$$

$$\left.\frac{2}{9}U^{-5/3}_{3/2}U^3_{1/2} + U_{-3/2}U^{1/3}_{3/2} + 2U_{-1/2}U^{1/3}_{3/2} + \frac{2}{3}U^2_{1/2}U^{-2/3}_{3/2}\right\} - \frac{2}{\lambda}\left(\frac{d\lambda}{d\eta}\right)^3\left\{\frac{4}{3}U_{1/2}U_{-3/2}U^{-2/3}_{3/2} + \right.$$

$$U^2_{-1/2}U^{-2/3}_{3/2} - \frac{4}{3}U_{-1/2}U^{2/3}_{1/2}U^{-5/3}_{3/2} + \frac{10}{27}U^4_{1/2}U^{-8/3}_{3/2} + U_{-5/2}U^{1/3}_{3/2} + 2U_{-3/2}U^{1/3}_{3/2} + 2$$

$$\left.U_{1/2}U_{-1/2}U^{-2/3}_{3/2} - \frac{4}{9}U^3_{1/2}U^{-5/3}_{3/2}\right\} \qquad (42)$$

Using the same steps as we did for the third derivative we get

$$\frac{d^5\lambda}{d\eta^5} = 0 \quad \text{at the origin.} \qquad (43)$$

We differentiate expression 42, to find the sixth derivative. At the origin this reduces to:

$$\lambda\frac{d^6\lambda}{d\eta^6} + \frac{d^2\lambda}{d\eta^2}\frac{d^4\lambda}{d\eta^4}\left(\frac{16}{3}\frac{U_{1/2}}{U_{3/2}} - 1\right) + \frac{24}{3\lambda}\left(\frac{d^2\lambda}{d\eta^2}\right)^3\left\{\frac{U_{-1/2}}{U_{3/2}} - \left(\frac{U_{1/2}}{U_{3/2}}\right)^2\right\} + \frac{72}{\eta^3}\frac{d\lambda}{d\eta}\frac{d^2\lambda}{d\eta^2} - \frac{24}{\eta^2}\left\{\left(\frac{d^2\lambda}{d\eta^2}\right) + \right.$$

$$\left.\frac{2\lambda}{\eta^2}\right\}\frac{d^2\lambda}{d\eta^2} - \frac{32}{\eta^2}\frac{d\lambda}{d\eta}\frac{d^3\lambda}{d\eta^3} + \frac{20}{\eta}\frac{d^2\lambda}{d\eta^2}\frac{d^3\lambda}{d\eta^3} + \frac{10}{\eta}\frac{d\lambda}{d\eta}\frac{d^4\lambda}{d\eta^4} + \frac{2\lambda}{\eta}\left\{\frac{12}{\eta^2}\frac{d^3\lambda}{d\eta^3} - \frac{4}{\eta}\frac{d^4\lambda}{d\eta^4} + \frac{d^5\lambda}{d\eta^5} - \right.$$

$$\left.\frac{24}{\eta^3}\left(\frac{d\lambda}{d\eta}\right)^2 + \frac{24}{\eta^4}\frac{d\lambda}{d\eta}\right\} = -\lambda\frac{d^4\lambda}{d\eta^4}\left[\frac{1}{3}U^2_{1/2}U^{-2/3}_{3/2} + U_{-1/2}U^{1/3}_{3/2} + 2U_{1/2}U^{1/3}_{3/2}\right] -$$

$$3\left(\frac{d^2\lambda}{d\eta^2}\right)^2\left\{U_{1/2}^2U_{3/2}^{-2/3}+3U_{-1/2}U_{3/2}^{1/3}+2U_{1/2}U_{3/2}^{1/3}+U_{-1/2}U_{1/2}U_{3/2}^{-2/3}-\frac{2}{9}\right.$$
$$\left.U_{1/2}^3U_{3/2}^{-5/3}+U_{-3/2}U_{3/2}^{1/3}\right\} \qquad (44)$$

We consider the terms whose limits, at the origin, are of an indeterminate form.

$$X_6=\frac{2}{\eta}\left[10\frac{d^2\lambda}{d\eta^2}\frac{d^3\lambda}{d\eta^3}+5\frac{d\lambda}{d\eta}\frac{d^4\lambda}{d\eta^4}+\lambda\frac{d^5\lambda}{d\eta^5}-\frac{4}{\eta}\left\{3\left(\frac{d^2\lambda}{d\eta^2}\right)^2+4\frac{d\lambda}{d\eta}\frac{d^3\lambda}{d\eta^3}+\lambda\frac{d^4\lambda}{d\eta^4}\right\}+\frac{12}{\eta^2}\left\{\right.\right.$$
$$\left.\left.3\frac{d^2\lambda}{d\eta^2}\frac{d\lambda}{d\eta}+\lambda\frac{d^3\lambda}{d\eta^3}\right\}-\frac{24}{\eta^3}\left\{\lambda\frac{d^2\lambda}{d\eta^2}+\lambda\frac{d\lambda}{d\eta}\right\}+\frac{24}{\eta^4}\lambda\frac{d\lambda}{d\eta}\right]\text{(l'Hospital's rule)}\longrightarrow$$

$$30\frac{d^2\lambda}{d\eta^2}\frac{d^4\lambda}{d\eta^4}+20\left(\frac{d^3\lambda}{d\eta^3}\right)^2+12\frac{d\lambda}{d\eta}\frac{d^5\lambda}{d\eta^5}+2\lambda\frac{d^6\lambda}{d\eta^6}-4X_6$$

Putting $\quad \frac{d\lambda}{d\eta}=\frac{d^3\lambda}{d\eta^3}=\frac{d^5\lambda}{d\eta^5}=0 \Rightarrow X_6=6\frac{d^2\lambda}{d\eta^2}\frac{d^4\lambda}{d\eta^4}+\frac{2}{5}\lambda\frac{d^6\lambda}{d\eta^6}$ $\qquad$ (45)

Relation (44) becomes now:

$$\frac{d^6\lambda}{d\eta^6}=-\frac{5}{7}\left[\frac{d^2\lambda}{d\eta^2}\frac{d^4\lambda}{d\eta^4}\left\{\frac{16}{3}U_{1/2}U_{3/2}^{-1}+5\right\}+8\left(\frac{d^2\lambda}{d\eta^2}\right)^3\left\{\frac{U_{-1/2}}{U_{3/2}}-\left(\frac{U_{1/2}}{U_{3/2}}\right)^2\right\}+\right.$$
$$\frac{d^4\lambda}{d\eta^4}\left\{\frac{1}{3}U_{1/2}^2U_{3/2}^{-2/3}+U_{-1/2}U_{3/2}^{1/3}+2U_{1/2}U_{3/2}^{1/3}\right\}+3\left(\frac{d^2\lambda}{d\eta^2}\right)^2\left\{U_{1/2}^2U_{3/2}^{-2/3}+3U_{-1/2}U_{3/2}^{1/3}+\right.$$
$$\left.\left.+2U_{1/2}U_{3/2}^{1/3}+U_{-1/2}U_{1/2}U_{3/2}^{-2/3}-\frac{2}{9}U_{3/2}^{-5/3}U_{1/2}^3+U_{-3/2}U_{3/2}^{1/3}\right\}\right] \qquad (46)$$

at the origin.

Using the relations 34, 38, 41, 43, 46 and the initial values 32 we can derive the solution $\lambda(\eta)$ of our fundamental differential equation as a Taylor's power series with center $\eta=0$

$$\lambda(\eta)=\lambda(\eta_0)+\frac{\eta^2}{2!}\left(\frac{d^2\lambda}{d\eta^2}\right)_0+\frac{\eta^4}{4!}\left(\frac{d^4\lambda}{d\eta^4}\right)_0+\frac{\eta^6}{6!}\left(\frac{d^6\lambda}{d\eta^6}\right)+\ldots \qquad (47)$$

where $\lambda(\eta_0)=1$

$$\frac{d\lambda}{d\eta}=\eta\left(\frac{d^2\lambda}{d\eta^2}\right)_0+\frac{\eta^3}{6}\left(\frac{d^4\lambda}{d\eta^4}\right)_0+\frac{\eta^5}{120}\left(\frac{d^6\lambda}{d\eta^6}\right)_0+\ldots \qquad (48)$$

the second derivate is given by the equation (31).

$$\frac{d^2\lambda}{d\eta^2} = \lambda u_{3/2}^{1/3} u_{1/2} - \left(\frac{d\lambda}{d\eta}\right)^2 \frac{1}{\lambda} \left\{ \frac{2}{3} \frac{u_{1/2}}{u_{3/2}} - 1 \right\} - \frac{2}{\eta}\left(\frac{d\lambda}{d\eta}\right) \qquad (49)$$

Using the relations 47, 48, 49 as starting series for a numerical integration, discussed in the next chapter, for solving the differential equation (27).

(E) THE LIMITING CASES OF VERY LOW AND VERY HIGH CENTRAL DEGENERACY

In this section we shall prove that the partially degenerate standard model reduces to the classical standard model or Lane Emden polytrope of $n=3$ as $\Lambda \ll 1$ or $\alpha \gg 0$ and, in the opposite limiting case, as $\Lambda \gg 1$ to a Lane Emden polytrope of index $n=3/2$ which is the limiting case of small central density for completely degenerate configurations (white dwarf configurations) for $1-b \sim 10^{-4}$ or less.
Of course, the solution in the white dwarf case is not the Lane-Emden function $\Theta_{3/2}$

Case (i) $\Lambda \ll 1$

From equ. (5) we get
$$U_\nu(\Lambda) = \frac{1}{\Gamma(\nu+1)} \int_0^\infty \frac{u^\nu du}{\frac{1}{\Lambda} e^u + 1} \tag{E1}$$

or
$$I_\nu(\alpha) = \int_0^\infty \frac{u^\nu du}{e^{u+\alpha} + 1} \qquad \text{where} \quad \alpha = -\ln \Lambda \tag{E2}$$

From statistical mechanics and for $\Lambda \ll 1 \Rightarrow \alpha \gg 0$, we know that we can expand $I_\nu(\alpha)$ as a series of the form:

$$I_\nu(\alpha) = \Gamma(\nu+1)\, e^{-\alpha} \left[1 - \frac{e^{-\alpha}}{2^{\nu+1}} + \frac{e^{-2\alpha}}{3^{\nu+1}} - \frac{e^{-3\alpha}}{4^{\nu+1}} + \cdots \right] \tag{E3}$$

or

$$I_{1/2}(\alpha) = \frac{\sqrt{\pi}}{2} e^{-\alpha} \left[1 - \frac{e^{-\alpha}}{2\sqrt{2}} + \frac{e^{-2\alpha}}{3\sqrt{3}} - \frac{e^{-3\alpha}}{4\sqrt{4}} + \cdots \right] \tag{E4}$$

$$I_{3/2}(\alpha) = \frac{3\sqrt{\pi}}{4} e^{-\alpha} \left[1 - \frac{e^{-\alpha}}{2^2\sqrt{2}} + \frac{e^{-2\alpha}}{3^2\sqrt{3}} - \frac{e^{-3\alpha}}{4^2\sqrt{4}} + \cdots \right] \tag{E5}$$

from (1) and (2) $\Rightarrow U_\nu(\Lambda) = \dfrac{1}{\Gamma(\nu+1)} I_\nu(-\ln \Lambda)$

$$\Rightarrow U_{1/2}(\Lambda) = \frac{1}{\Gamma(3/2)} I_{1/2}(\alpha) \tag{E6}$$

$$U_{3/2}(\Lambda) = \frac{1}{\Gamma(5/2)} I_{3/2}(\alpha) \tag{E7}$$

From (E5) and (E3) $\Rightarrow U_{1/2}(\Lambda) = e^{-\alpha}\left[1 - \dfrac{e^{-\alpha}}{2\sqrt{2}} + \cdots \right]$ (E8)

from (E6) and (E4) $\Rightarrow U_{3/2}(\Lambda) = e^{-\alpha}\left[1 - \dfrac{e^{-\alpha}}{2^2\sqrt{2}} + \cdots \right]$ (E9)

or
$$U_\nu = \Lambda - \frac{\Lambda^2}{2^{\nu+1}} + \frac{\Lambda^3}{3^{\nu+1}} + \cdots \tag{E10}$$

equations (E8), (E9), (E10) for $\Lambda \ll 1$ will reduce to

$$U_0 = \Lambda \qquad \text{(E11)}$$

The equation of equilibrium

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\, U_{3/2}^{2/3}\,\frac{1}{\Lambda}\,\frac{d\Lambda}{d\xi}\right) = -\,U_{3/2}(\Lambda)\,U_{1/2}(\Lambda)$$

can be written, by (E11), as

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\,\Lambda^{-1/3}\,\frac{d\Lambda}{d\xi}\right) = -\Lambda^2 \qquad \text{(E12)}$$

Let $\Theta = \Lambda^{2/3}$ and $\xi = \sqrt{3/2}\,\xi$ (E13)

equ. (E12) $\Longrightarrow \dfrac{1}{\xi^2}\dfrac{d}{d\xi}\left(\xi^2\,\dfrac{d\Theta}{d\xi}\right) = -\Theta^3$ (E14)

which is the Lane-Emden equation of index $n=3$.

In this point of our investigation, it is also worth stating that the limiting cases provide a valuable check of the mathematical analysis in the derivation of the second, fourth and sixth derivatives of $\lambda(\xi)$ and in the series expansion.

Indeed, from equation (34), Chapter I we have that for a polytropic index $n=3$ the function $\Theta_3(\xi)$ is:

$$\Theta = \Theta_3 = 1 - \frac{1}{3!}\xi^2 + \frac{3}{5!}\xi^4 - \frac{(15+72)}{3\times 7!}\xi^6 + \ldots \qquad \text{(E15)}$$

If we consider the Taylor's expansion of Θ i.e.

$$\Theta = 1 + \frac{d^2\Theta}{d\xi^2}\frac{\xi^2}{2!} + \frac{d^4\Theta}{d\xi^4}\frac{\xi^4}{4!} + \ldots \qquad \text{(E16)}$$

from (E15) and (E16) we get that

$$\frac{d^2\Theta}{d\xi^2} = -\frac{1}{3} \quad , \quad \frac{d^4\Theta}{d\xi^4} = \frac{3}{5} \quad , \quad \frac{d^6\Theta}{d\xi^6} = -\frac{19}{7} \qquad \text{(E17)}$$

Since $\Lambda = \Theta^{3/2}$

$$\frac{d\Lambda}{d\xi} = \frac{3}{2}\,\Theta^{1/2}\,\frac{d\Theta}{d\xi}$$

$$\frac{d^2\Lambda}{d\xi^2} = \frac{3}{2}\,\Theta^{1/2}\,\frac{d^2\Theta}{d\xi^2} + \frac{3}{4}\,\Theta^{-1/2}\left(\frac{d\Theta}{d\xi}\right)^2$$

$$\frac{d^3\Lambda}{d\xi^3} = \frac{3}{2}\Theta^{1/2}\frac{d^3\Theta}{d\xi^3} + \frac{9}{4}\Theta^{-1/2}\frac{d\Theta}{d\xi}\frac{d^2\Theta}{d\xi^2} - \frac{3}{8}\Theta^{-3/2}\left(\frac{d\Theta}{d\xi}\right)^3$$

$$\frac{d^4\Lambda}{d\xi^4} = \frac{3}{2}\Theta^{1/2}\frac{d^4\Theta}{d\xi^4} + 3\Theta^{-1/2}\frac{d\Theta}{d\xi}\frac{d^3\Theta}{d\xi^3} - \frac{9}{4}\Theta^{-3/2}\left(\frac{d\Theta}{d\xi}\right)^2\frac{d^2\Theta}{d\xi^2} + \frac{9}{4}\Theta^{-1/2}\left(\frac{d^2\Theta}{d\xi^2}\right)^2 + \frac{9}{16}\Theta^{-5/2}\left(\frac{d\Theta}{d\xi}\right)^4$$

$$\frac{d^5\Lambda}{d\xi^5} = \frac{3}{2}\Theta^{1/2}\frac{d^5\Theta}{d\xi^5} + \frac{15}{4}\Theta^{-1/2}\frac{d\Theta}{d\xi}\left[\frac{d^4\Theta}{d\xi^4} - \Theta^3\frac{d\Theta}{d\xi}\frac{d^3\Theta}{d\xi^3} + \frac{3}{2}\Theta^5\left(\frac{d\Theta}{d\xi}\right)^2\frac{d^2\Theta}{d\xi^2}\right] + \frac{15}{2}\Theta^{-1/2}\frac{d^2\Theta}{d\xi^2}\left[\frac{d^3\Theta}{d\xi^3} - \frac{3}{4}\Theta^3\left(\frac{d^2\Theta}{d\xi^2}\right)^2\right] - \frac{45}{8}\Theta^{-3/2}\frac{d\Theta}{d\xi}\left(\frac{d^2\Theta}{d\xi^2}\right)^2 + \frac{36}{16}\Theta^{-5/2}\left(\frac{d\Theta}{d\xi}\right)^3\frac{d^2\Theta}{d\xi^2}$$

$$\frac{d^6\Lambda}{d\xi^6} = \frac{3}{2}\Theta^{1/2}\frac{d^6\Theta}{d\xi^6} + \frac{18}{4}\Theta^{-1/2}\frac{d\Theta}{d\xi}\frac{d^5\Theta}{d\xi^5} - \frac{45}{8}\Theta^{-3/2}\left(\frac{d\Theta}{d\xi}\right)^2\frac{d^4\Theta}{d\xi^4} - \frac{210}{4}\Theta^{-3/2}\frac{d\Theta}{d\xi}\frac{d^2\Theta}{d\xi^2}\frac{d^3\Theta}{d\xi^3} + \frac{15}{2}\Theta^{-1/2}\left(\frac{d^3\Theta}{d\xi^3}\right)^2 - \frac{675}{32}\Theta^{-7/2}\left(\frac{d\Theta}{d\xi}\right)^4\frac{d^2\Theta}{d\xi^2} + \frac{513}{16}\Theta^{-5/2}\left(\frac{d\Theta}{d\xi}\right)^2\left(\frac{d^2\Theta}{d\xi^2}\right)^2 - \frac{45}{8}\Theta^{-3/2}\left(\frac{d^2\Theta}{d\xi^2}\right)^3 + \frac{315}{64}\Theta^{-9/2}\left(\frac{d\Theta}{d\xi}\right)^6 + \frac{36}{16}\left[-\frac{5}{2}\Theta^{-7/2}\frac{d\Theta}{d\xi}\frac{d^2\Theta}{d\xi^2} + \Theta^{5/2}\frac{d^3\Theta}{d\xi^3}\right]\left(\frac{d\Theta}{d\xi}\right)^3 + \frac{45}{4}\Theta^{-1/2}\frac{d^2\Theta}{d\xi^2}\frac{d^4\Theta}{d\xi^4}$$

$$\frac{d^7\Lambda}{d\xi^7} = \cdots$$

Substituting $\frac{d\Theta}{d\xi}, \frac{d^3\Theta}{d\xi^3}, \frac{d^5\Theta}{d\xi^5}, \frac{d^7\Theta}{d\xi^7}$ with zero in the above derivatives we get:

$$\frac{d\Lambda}{d\xi} = 0$$

$$\frac{d^2\Lambda}{d\xi^2} = \frac{3}{2}\Theta^{1/2}\frac{d^2\Theta}{d\xi^2} \qquad \text{(E18)}$$

$$\frac{d^3\Lambda}{d\xi^3} = 0$$

$$\frac{d^4\Lambda}{d\xi^4} = \frac{3}{2}\Theta^{1/2}\frac{d^4\Theta}{d\xi^4} + \frac{9}{4}\Theta^{-1/2}\left(\frac{d^2\Theta}{d\xi^2}\right)^2 \qquad \text{(E19)}$$

$$\frac{d^5\Lambda}{d\xi^5} = 0$$

$$\frac{d^6\Lambda}{d\xi^6} = \frac{3}{2}\Theta^{1/2}\frac{d^6\Theta}{d\xi^6} + \frac{45}{4}\Theta^{-1/2}\frac{d^2\Theta}{d\xi^2}\frac{d^4\Theta}{d\xi^4} - \frac{45}{8}\Theta^{-3/2}\left(\frac{d^2\Theta}{d\xi^2}\right)^3 \qquad \text{(E20)}$$

Substituting $\frac{d^2\Theta}{d\xi^2}, \frac{d^4\Theta}{d\xi^4}, \frac{d^6\Theta}{d\xi^6}$, with their values from (E17)

we get that; as $\xi \to 0$ and $\Theta \to 1$

$$\frac{d^2\Lambda}{d\xi^2} = \frac{3}{2}\left(-\frac{1}{3}\right) = -\frac{1}{2} \qquad \text{(E21)}$$

$$\frac{d^4\Lambda}{d\bar{\eta}^4} = \frac{3}{2}\cdot\frac{3}{5} + \frac{9}{4}\left(\frac{-1}{3}\right)^2 = \frac{23}{20} \qquad \text{(E22)}$$

$$\frac{d^6\Lambda}{d\bar{\eta}^6} = \frac{3}{2}\left(-\frac{19}{7}\right) + \frac{45}{4}\left(-\frac{1}{3}\right)\left(\frac{3}{5}\right) - \frac{45}{8}\left(-\frac{1}{27}\right) = -\frac{1027}{168} \qquad \text{(E23)}$$

Recalling that $\bar{\eta} = \sqrt{\frac{2}{3}}\,\eta$ (E13)

from (E13) and (E18) $\Rightarrow$ $$\frac{d^2\Lambda}{d\eta^2} = \frac{d^2\Lambda}{d\bar{\eta}^2}\left(\sqrt{\frac{2}{3}}\right)^2 = -\frac{1}{3} \qquad \text{(E24)}$$

(E13) and (E19) $\Rightarrow$ $$\frac{d^4\Lambda}{d\eta^4} = \frac{d^4\Lambda}{d\bar{\eta}^4}\left(\sqrt{\frac{2}{3}}\right)^4 = \frac{23}{45} \qquad \text{(E25)}$$

(E13) (E20) $\Rightarrow$ $$\frac{d^6\Lambda}{d\eta^6} = \frac{d^6\Lambda}{d\bar{\eta}^6}\left(\sqrt{\frac{2}{3}}\right)^6 = -\frac{1027}{567} \qquad \text{(E26)}$$

We can now verify our results of equations (39), (41) and (46) checking whether the derivatives reduce to the values from (E24), (E25), (E26) for $\Lambda \ll 1$

From equ. (34) we have

$$\frac{d^2\Lambda}{d\eta^2} = -\frac{1}{3}U_{3/2}^{1/3}U_{1/2} = -\frac{1}{3}\Lambda^{4/3} \to -\frac{1}{3} \quad \text{as } \Lambda \to 1$$

from equ. (41)

$$\frac{d^4\Lambda}{d\eta^4} = -\frac{3}{5}\left[\left\{\frac{4}{3}U_{1/2}U_{3/2}^{-1} + 1\right\}\left(\frac{d^2\Lambda}{d\eta^2}\right) + \left\{\frac{1}{3}U_{3/2}^{-2/3}U_{1/2}^{2} + 2U_{3/2}^{1/3}U_{1/2} + U_{3/2}^{1/3}U_{-1/2}\right\}\frac{d^2\Lambda}{d\eta^2}\right]$$

$$= -\frac{3}{5}\left[\frac{7}{27} - \frac{1}{3}\left\{\frac{1}{3} + 3\right\}\right]\Lambda^{8/3} = -\frac{3}{5}\left(-\frac{23}{27}\right)\Lambda^{8/3} =$$

$$= \frac{23}{45}\Lambda^{8/3} \longrightarrow \frac{23}{45} \qquad \text{as } \Lambda \to 1$$

from equ. (46)

$$\frac{d^6\Lambda}{d\eta^6} = -\frac{5}{7}\left[\frac{d^2\Lambda}{d\eta^2}\frac{d^4\Lambda}{d\eta^4}\left\{\frac{16}{3}U_{1/2}U_{3/2}^{-1} + 5\right\} + 8\left(\frac{d^2\Lambda}{d\eta^2}\right)^3\left\{U_{-1/2}U_{3/2}^{-1} - U_{3/2}^{-2}U_{1/2}^{2}\right\} + \frac{d^4\Lambda}{d\eta^4}\left\{\frac{1}{3}U_{1/2}^{2}U_{3/2}^{-2/3} + \right.\right.$$

$$\left. + U_{-1/2}U_{3/2}^{1/3} + 2U_{1/2}U_{3/2}^{1/3}\right\} + 3\left(\frac{d^2\Lambda}{d\eta^2}\right)^2\left\{U_{1/2}^{2}U_{3/2}^{-2/3} + 3U_{-1/2}U_{3/2}^{1/3} + 2U_{1/2}U_{3/2}^{1/3} + U_{-1/2}U_{1/2}U_{3/2}^{-2/3} - \frac{2}{9}\right.$$

$$\left.\left. U_{3/2}^{-5/3}U_{1/2}^{3} + U_{-3/2}U_{3/2}^{1/3}\right\}\right] = -\frac{5}{7}\left(-\frac{1}{3}\,\frac{23}{45}\,\frac{31}{3} + \frac{23}{45}\,\frac{10}{3} + \frac{70}{27}\right)\Lambda^4 \to -\frac{1027}{567} \quad \text{as } \Lambda \to 1$$

Case (ii) $\Lambda \gg 1$

In the case of very large Λ we can obtain an asymptotic expansion of the integral U_ν by applying Sommerfeld's lemma

Sommerfeld's lemma states

If $\phi(u)$ is a sufficiently regular function which vanishes for u = o then we have the asymptotic formula

$$\int_0^\infty \frac{du}{\frac{1}{\Lambda}e^{u}+1}\,\frac{d\phi(u)}{du} = \phi(u_0) + 2\left[C_2\,\phi''(u_0) + C_4\,\phi''''(u_0) + \dots\right]$$

where $u_0 = \log\Lambda$ and $C_2, C_4 ..$ are numerical coefficients defined by $C_\nu = 1 - \frac{1}{2^\nu} + \frac{1}{3^\nu} - \frac{1}{4^\nu} + \dots = \sum_{k=1}^{\infty}(-1)^{k+1}k^{-\nu}$

In our case the function $\phi(u)$ will be $\phi(u) = u^{\nu+1}$

$$U_\nu = \frac{1}{\Gamma(\nu+1)}\int_0^\infty \frac{u^\nu}{\frac{1}{\Lambda}e^{u}+1}\,du = \frac{1}{\Gamma(\nu+2)}\int_0^\infty \frac{du}{\frac{1}{\Lambda}e^{u}+1}\,\frac{d}{du}\left(u^{\nu+1}\right) \tag{E27}$$

Then by the lemma we find that:

$$U_\nu = \frac{(\log\Lambda)^{\nu+1}}{\Gamma(\nu+2)}\left[1 + 2\left\{C_2\,\frac{(\nu+1)\nu}{(\log\Lambda)^2} + C_4\,\frac{\nu(\nu-1)(\nu-2)(\nu+1)}{(\log\Lambda)^4} + \dots\right\}\right] \tag{E28}$$

The coefficients $C_\nu = 1 - \frac{1}{2^\nu} + \frac{1}{3^\nu} - \frac{1}{4^\nu} + \dots = \left[1 - 2^{-(\nu-1)}\right]\zeta(\nu)$

where $\zeta(\nu) \equiv \sum_{\nu=1}^{\infty}\frac{1}{r^\nu}$ is the Riemann zeta function

From tables we get $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(4) = \frac{\pi^4}{90}$ ..

Finally we get

$$U_{1/2} = \frac{4}{3\sqrt{\pi}}(\log\Lambda)^{3/2}\left[1 + \frac{\pi^2}{8(\log\Lambda)^2} + \dots\right] \tag{E29}$$

$$U_{3/2} = \frac{8}{15\sqrt{\pi}}(\log\Lambda)^{5/2}\left[1 + \frac{5\pi^2}{8(\log\Lambda)^2} + \dots\right] \tag{E30}$$

From equations (E29), (E30) and for $\Lambda \gg 1$

$$U_{1/2} = \frac{4}{3\sqrt{\pi}}(\log\Lambda)^{3/2} \tag{E31}$$

$$U_{3/2} = \frac{8}{15\sqrt{\pi}}(\log\Lambda)^{5/2} \tag{E32}$$

Under these conditions the equation of equilibrium becomes

$$\frac{1}{\eta^2}\frac{d}{d\eta}\left[\eta^2 (\log\Lambda)^{5/3}\frac{d\log\Lambda}{d\eta}\right] = -\Gamma^{-1/3}(7/2)\,\Gamma^{-1}(5/2)\,(\log\Lambda)^4 \quad \text{or}$$

$$\frac{1}{\eta^2}\frac{d}{d\eta}\left[\eta^2\frac{d}{d\eta}(\log\Lambda)^{8/3}\right] = -\frac{64}{9}\left(\frac{1}{15\pi^2}\right)^{1/3}(\log\Lambda)^4 \qquad \text{(E33)}$$

Let $(\log\Lambda)^{8/3} = \Theta$ and $\eta = a\xi$ where $a = \sqrt{\frac{9}{64}(15\pi^2)^{1/3}}$ (E34)

the equation (E33) becomes

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left[\xi^2\frac{d\Theta}{d\xi}\right] = -\Theta^{3/2} \qquad \text{(E35)}$$

which is the Lane Emden polytrope of index $n = 3/2$

Following the same steps as in case (i) we can also verify the formulae (34), (41), (46) of the second, fourth and sixth derivatives by checking their validity in this particular case.

Indeed, for $n = 3/2$ from equation (34) Chapter I we obtain:

$$\Theta = \Theta_{3/2} = 1 - \frac{1}{6}\xi^2 + \frac{1}{80}\xi^4 - \frac{1}{1440}\xi^6 + \ldots \Rightarrow \qquad \text{(E36)}$$

$$\frac{d^2\Theta}{d\xi^2} = -\frac{1}{3}\ , \quad \frac{d^4\Theta}{d\xi^4} = \frac{3}{10}\ , \quad \frac{d^6\Theta}{d\xi^6} = -\frac{1}{2} \qquad \text{(E37)}$$

and since

$$\log\Lambda = \Theta^{3/8} \Rightarrow \frac{1}{\Lambda}\frac{d\Lambda}{d\xi} = \frac{3}{8}\Theta^{-5/8}\frac{d\Theta}{d\xi} \Rightarrow$$

$$\frac{1}{\Lambda}\frac{d^2\Lambda}{d\xi^2} - \frac{1}{\Lambda^2}\left(\frac{d\Lambda}{d\xi}\right)^2 = \frac{3}{8}\Theta^{-5/8}\frac{d^2\Theta}{d\xi^2} - \frac{15}{64}\Theta^{-13/8}\left(\frac{d\Theta}{d\xi}\right)^2$$

$$\frac{1}{\Lambda}\left[\frac{d^3\Lambda}{d\xi^3} - \frac{3}{\Lambda}\frac{d\Lambda}{d\xi}\frac{d^2\Lambda}{d\xi^2} + \frac{2}{\Lambda^2}\left(\frac{d\Lambda}{d\xi}\right)^3\right] = \frac{3}{8}\Theta^{-5/8}\frac{d^3\Theta}{d\xi^3} - \frac{45}{64}\Theta^{-13/8}\frac{d\Theta}{d\xi}\frac{d^2\Theta}{d\xi^2} + \frac{195}{512}\Theta^{-21/8}\left(\frac{d\Theta}{d\xi}\right)^3$$

$$\frac{1}{\Lambda}\left[\frac{d^4\Lambda}{d\xi^4} - \frac{3}{\Lambda}\left(\frac{d^2\Lambda}{d\xi^2}\right)^2\right] - \frac{2}{\Lambda^2}\frac{d\Lambda}{d\xi}\left[2\frac{d^3\Lambda}{d\xi^3} - \frac{6}{\Lambda}\frac{d\Lambda}{d\xi}\frac{d^2\Lambda}{d\xi^2} + \frac{3}{\Lambda^2}\left(\frac{d\Lambda}{d\xi}\right)^3\right] = \frac{3}{8}\Theta^{-5/8}\frac{d^4\Theta}{d\xi^4} - \frac{45}{64}\Theta^{-13/8}$$

$$\left(\frac{d^2\Theta}{d\xi^2}\right)^2 - \frac{5}{8}\Theta^{-13/8}\frac{d\Theta}{d\xi}\left[\frac{3}{8}\frac{d^3\Theta}{d\xi^3} - \frac{117}{32}\Theta^{-21/13}\frac{d\Theta}{d\xi}\frac{d^2\Theta}{d\xi^2} + \frac{819}{512}\Theta^{-29/13}\left(\frac{d\Theta}{d\xi}\right)^3\right]$$

the fifth derivatives will be given

$$\frac{1}{\Lambda}\frac{d^5\Lambda}{d\xi^5}-\frac{5}{\Lambda^2}\frac{d\Lambda}{d\xi}\frac{d^4\Lambda}{d\xi^4}-\frac{10}{\Lambda^2}\frac{d^2\Lambda}{d\xi^2}\frac{d^3\Lambda}{d\xi^3}+\frac{20}{\Lambda^3}\left(\frac{d\Lambda}{d\xi}\right)^2\frac{d^3\Lambda}{d\xi^3}+\frac{30}{\Lambda^3}\frac{d\Lambda}{d\xi}\left(\frac{d^2\Lambda}{d\xi^2}\right)^2-$$

$$\frac{60}{\Lambda^4}\left(\frac{d\Lambda}{d\xi}\right)^3\frac{d^2\Lambda}{d\xi^2}+\frac{24}{\Lambda^5}\left(\frac{d\Lambda}{d\xi}\right)^5=\frac{3}{8}\Theta^{-5/8}\frac{d^5\Theta}{d\xi^5}-\frac{75}{64}\Theta^{-13/8}\frac{d\Theta}{d\xi}\frac{d^4\Theta}{d\xi^4}+$$

$$\frac{1950}{512}\Theta^{-21/8}\left(\frac{d\Theta}{d\xi}\right)^2\frac{d^3\Theta}{d\xi^3}-\frac{150}{64}\Theta^{-13/8}\frac{d^2\Theta}{d\xi^2}\frac{d^3\Theta}{d\xi^3}-\frac{40950}{4096}\Theta^{-29/8}\left(\frac{d\Theta}{d\xi}\right)^3\frac{d^2\Theta}{d\xi^2}+$$

$$\frac{2925}{512}\Theta^{-21/8}\frac{d\Theta}{d\xi}\left(\frac{d^2\Theta}{d\xi^2}\right)^2+\frac{59585}{98304}\Theta^{-37/8}\left(\frac{d\Theta}{d\xi}\right)^5$$

Putting $\frac{d\Theta}{d\xi}, \frac{d^3\Theta}{d\xi^3}, \frac{d^5\Theta}{d\xi^5}$ equal to zero, the above derivatives become:

$$\frac{d\Lambda}{d\xi}=0$$

$$\frac{1}{\Lambda}\frac{d^2\Lambda}{d\xi^2}=\frac{3}{8}\Theta^{-5/8}\frac{d^2\Theta}{d\xi^2} \qquad \text{(E38)}$$

$$\frac{d^3\Lambda}{d\xi^3}=0$$

$$\frac{1}{\Lambda}\frac{d^4\Lambda}{d\xi^4}=\frac{3}{8}\Theta^{-5/8}\frac{d^4\Theta}{d\xi^4}-\frac{45}{64}\Theta^{-13/8}\left(\frac{d^2\Theta}{d\xi^2}\right)^2+\frac{3}{\Lambda^2}\left(\frac{d^2\Lambda}{d\xi^2}\right)^2 \qquad \text{(E39)}$$

$$\frac{d^5\Lambda}{d\xi^5}=0$$

$$\frac{1}{\Lambda}\frac{d^6\Lambda}{d\xi^6}=\frac{3}{8}\Theta^{-5/8}\frac{d^6\Theta}{d\xi^6}-\frac{225}{64}\Theta^{-13/8}\frac{d^4\Theta}{d\xi^4}\frac{d^2\Theta}{d\xi^2}+\frac{2925}{512}\Theta^{-21/8}\left(\frac{d^2\Theta}{d\xi^2}\right)^3+$$

$$+\frac{15}{\Lambda^2}\frac{d^2\Lambda}{d\xi^2}\frac{d^4\Lambda}{d\xi^4}-\frac{30}{\Lambda^3}\left(\frac{d^2\Lambda}{d\xi^2}\right)^3 \qquad \text{(E40)}$$

We also know that for $\Theta\to 1, \xi\to 0$ then $\lambda\to 1, \eta\to 0$ and the derivatives

$$\frac{d^2\Theta}{d\eta^2}=-\frac{1}{3} \qquad \frac{d^4\Theta}{d\eta^4}=\frac{3}{10} \qquad \frac{d^6\Theta}{d\eta^6}=-\frac{1}{2}$$

and recalling that $\eta=a\xi$ where $a=\sqrt{\frac{9}{64}(15\pi^2)^{1/3}}$

the derivatives (E38), (E39), (E40) become

$$\frac{d^2\lambda}{d\eta^2}=-\frac{1}{8}A^2 \qquad \text{where} \quad A=\frac{8}{3}\left(\frac{1}{\pi\sqrt{15}}\right)^{1/3}=\frac{1}{a}$$

$$\frac{d^4\lambda}{d\eta^4} = \frac{13}{160} A^4 \tag{E39}$$

$$\frac{d^6\lambda}{d\eta^6} = -\frac{217}{512\cdot 3} A^6 \tag{E40}$$

We can check now our results from Part (D), namely the second, fourth and sixth derivatives. From the relations (34), (41), (46) we get

$$\frac{d^2\lambda}{d\eta^2} = -\frac{1}{3} U_{3/2}^{1/3} U_{1/2}$$

$$\frac{d^4\lambda}{d\eta^4} = -\frac{3}{5}\left[\left\{1+\frac{4}{3}\frac{U_{1/2}}{U_{3/2}}\right\}\left(\frac{d^2\lambda}{d\eta^2}\right)^2 + \left\{\frac{1}{3}U_{1/2}^2 U_{3/2}^{-2/3} + 2U_{3/2}^{1/3}U_{1/2} + U_{3/2}^{1/3}U_{-1/2}\right\}\frac{d^2\lambda}{d\eta^2}\right]$$

$$\frac{d^6\lambda}{d\eta^6} = -\frac{5}{7}\left[\frac{d^2\lambda}{d\eta^2}\frac{d^4\lambda}{d\eta^4}\left\{\frac{16}{3}U_{1/2}U_{3/2}^{-1}+5\right\} + 8\left(\frac{d^2\lambda}{d\eta^2}\right)^3\left\{U_{-1/2}U_{3/2}^{-1} - U_{3/2}^{-2}U_{1/2}^2\right\} + \right.$$

$$\frac{d^4\lambda}{d\eta^4}\left\{\frac{1}{3}U_{1/2}^2 U_{3/2}^{-2/3} + U_{-1/2}U_{3/2}^{1/3} + 2U_{1/2}U_{3/2}^{1/3}\right\} + 3\left(\frac{d^2\lambda}{d\eta^2}\right)^2\left\{U_{1/2}^2 U_{3/2}^{-2/3} + \right.$$

$$3U_{-1/2}U_{3/2}^{1/3} + 2U_{1/2}U_{3/2}^{1/3} + U_{-1/2}U_{1/2}U_{3/2}^{-2/3} - \frac{2}{9}U_{3/2}^{-5/3}U_{1/2}^3 +$$

$$\left.\left. U_{-3/2}U_{3/2}^{1/3}\right\}\right]$$

If we now substitute $U_\nu = \dfrac{(\log \Lambda)^{\nu+1}}{\Gamma(\nu+2)}$ in the above relations, we can easily verify that:

$$\frac{d^2\lambda}{d\eta^2} = -\frac{1}{3}\Gamma^{-1}\left(\frac{5}{2}\right)\Gamma^{-1/3}\left(\frac{7}{2}\right) = -\frac{1}{8}A^2$$

$$\frac{d^4\lambda}{d\eta} = \frac{13}{160} A^4$$

$$\frac{d^6\lambda}{d\eta^6} = \frac{217}{512\cdot 3} A^6$$

We thus proved that the general equation of a partially degenerate standard model can be considered as a polytropic model in the cases of very low and very high degeneracy.

We also derived the necessary formulae for the series expansion of the variables of our basic equation which (formulae) will be used for the numerical integration through the same analysis as the Lane Emden equation as described in Chapter I.

CHAPTER III

In this chapter we shall discuss the numerical solution of the fundamental equation of equilibrium of the partially degenerate standard models.

The numerical integration applied is the same as the one applied for the solution of the Lane-Emden equation in Chapter I.

The range of the values for the exponential Λ, of the degeneracy parameter α, has been chosen from 0.005 to 100.0.
For the initial and any other values of the degneracy parameter the values of the Fermi-Dirac integrals were obtained from the tables of W. J. Cody and H. C. Thacher.

We next derive the relations for the mass, radius, pressure, density and temperature for the standard partially degenerate model.

Our results are tabulated in the tables (11) to (21). The tabulated quantities are: $\xi, \theta(\xi)$, the ratio of the temperature to the central temperature, the ratio of the density to the central density, the ratio of the pressure to the central pressure and the ratio of the mass to the total mass.

Diagrams are obtained for the above functions and for $\Lambda_0 = 0.1, 1$ and 2 using the facilities of the G.I.L. 6011 plotter of St Andrews.

(A) NUMERICAL SOLUTION OF THE EQUILIBRIUM EQUATION OF THE PARTIALLY DEGENERATE STANDARD MODEL

From the discussion in Chapter II we can see that the fundamental differential equation for the hydrostatic equilibrium of a partially degenerate standard model reduces to

$$\frac{1}{\eta^2}\frac{d}{d\eta}\left(\eta^2 \, U_{3/2}^{2/3} \, \frac{d\log\Lambda}{d\eta}\right) = -U_{3/2}(\Lambda)\,U_{1/2}(\Lambda) \tag{1}$$

and (1) reduces to:

$$\lambda\frac{d^2\lambda}{d\eta^2} + \left(\frac{d\lambda}{d\eta}\right)^2\left\{\frac{2}{3}\frac{U_{1/2}}{U_{3/2}} - 1\right\} + \frac{2}{\eta}\lambda\frac{d\lambda}{d\eta} = -\lambda^2 U_{3/2}^{1/3} U_{1/2} \tag{2}$$

Equation (2) requires as solution a function $\lambda(\eta)$ for a chosen value of Λ_0 with the initial conditions:

$$\begin{cases} \eta = 0 \;, \; \lambda = 1, \; \dfrac{d\lambda}{d\eta} = 0 \;, \; \dfrac{d^2\lambda}{d\eta^2} = -\dfrac{1}{3} U_{3/2}^{1/3}(\Lambda_0)\,U_{1/2}(\Lambda_0) \\ \eta = 1, \; \lambda \to 0 \end{cases} \tag{3}$$

The starting values of $\lambda(\eta)$ and $\left(\frac{d\lambda}{d\eta}\right)$ for the numerical integration are found using the Taylor's power series with center $\eta_0 = 0$ for $\lambda(\eta)$ and $\left(\frac{d\lambda}{d\eta}\right)$.

$$\lambda(\eta) = \lambda(\eta_0) + \frac{\eta^2}{2!}\left(\frac{d^2\lambda}{d\eta^2}\right)_0 + \frac{\eta^4}{4!}\left(\frac{d^4\lambda}{d\eta^4}\right)_0 + \frac{\eta^6}{6!}\left(\frac{d^6\lambda}{d\eta^6}\right) + \dots \tag{4}$$

with

$$\lambda(\eta_0) = 1$$

$$\left(\frac{d^2\lambda}{d\eta^2}\right)_0 = -\frac{1}{3} U_{3/2}^{1/3}(\Lambda_0)\,U_{1/2}(\Lambda_0)$$

$$\left(\frac{d^4\lambda}{d\eta^4}\right)_0 = -\frac{3}{5}\left[\left\{1 + \frac{4}{3}\frac{U_{1/2}}{U_{3/2}}\right\}\left(\frac{d^2\lambda}{d\eta^2}\right)^2 + \left(\frac{d^2\lambda}{d\eta^2}\right)\left\{\frac{1}{3}U_{3/2}^{-2/3}U_{1/2}^2 + 2U_{3/2}^{1/3}U_{1/2} + U_{3/2}^{1/3}U_{-1/2}\right\}\right]$$

$$\left(\frac{d^6\lambda}{d\eta^6}\right)_0 = -\frac{5}{7}\left[\frac{d^2\lambda}{d\eta^2}\frac{d^4\lambda}{d\eta^4}\left\{\frac{16}{3}\frac{U_{1/2}}{U_{3/2}}+5\right\}+8\left(\frac{d^2\lambda}{d\eta^2}\right)^3\left\{\frac{U_{-1/2}}{U_{3/2}}-\left(\frac{U_{1/2}}{U_{3/2}}\right)^2\right\}\right.$$

$$+\left(\frac{d^4\lambda}{d\eta^4}\right)\left\{\frac{1}{3}U_{1/2}^2U_{3/2}^{-2/3}+U_{-1/2}U_{3/2}^{1/3}+2U_{1/2}U_{3/2}^{1/3}\right\}+3\left(\frac{d^2\lambda}{d\eta^2}\right)^2$$

$$\left\{U_{1/2}^2U_{3/2}^{-2/3}+3U_{-1/2}U_{3/2}^{1/3}+2U_{1/2}U_{3/2}^{1/3}+U_{-1/2}U_{1/2}U_{3/2}^{-1/3}-\frac{2}{9}\right.$$

$$\left.\left.U_{3/2}^{-5/3}U_{1/2}^3+U_{-3/2}U_{3/2}^{1/3}\right\}\right].$$

$$\frac{d\lambda}{d\eta} = \eta\left(\frac{d^2\lambda}{d\eta^2}\right)_0+\frac{\eta^3}{6}\left(\frac{d^4\lambda}{d\eta^4}\right)_0+\frac{\eta^5}{120}\left(\frac{d^6\lambda}{d\eta^6}\right)_0+\dots \tag{5}$$

and the second derivative is given by

$$\frac{d^2\lambda}{d\eta^2} = -\lambda\,U_{3/2}^{1/3}U_{1/2}-\left(\frac{d\lambda}{d\eta}\right)^2\frac{1}{\lambda}\left\{\frac{2}{3}\frac{U_{1/2}}{U_{3/2}}-1\right\}-\frac{2}{\eta}\left(\frac{d\lambda}{d\eta}\right) \tag{6}$$

Using the relations (4), (5) and (6) we find the seven starting values for our numerical integration. The method is the same as in Chapter I and the Fortran IV program is described in Appendix II.

For any Λ_0 the required values for the Fermi-integrals $U_{3/2}(\Lambda_0)$, $U_{1/2}(\Lambda_0)$ $U_{-1/2}(\Lambda_0)$ for the determination of $\left(\frac{d^2\lambda}{d\eta^2}\right)_0$, $\left(\frac{d^4\lambda}{d\eta^4}\right)_0$ and $\left(\frac{d^6\lambda}{d\eta^6}\right)_0$ are obtained from the tables of W. J. Cody and H. C. Thacher

the same for the $U_{3/2}(\Lambda)$, $U_{1/2}(\Lambda)$, $U_{-1/2}(\Lambda)$ for any value of Λ throughout the integration using a subroutine described in appendix III.

For $U_{-3/2}(\Lambda_0)$ we use the property of the Fermi-Dirac integrals, namely

$$U_{-3/2}(\Lambda) = \Lambda.\frac{d}{d\Lambda}U_{-1/2} = \lambda\frac{dU_{-1/2}}{d\lambda}.$$

The chosen interval for each integration is as small as possible for best accuracy. In the tables we only give a number of values suitable to the space of this presentation.

Tables (11) to (21) give the partially degenerate standard model function for various values of the exponential of the degeneracy parameter Λ_0.

Figure 2 gives the partially degenerate standard model function $\chi(\zeta)$ for four values of Λ_0.

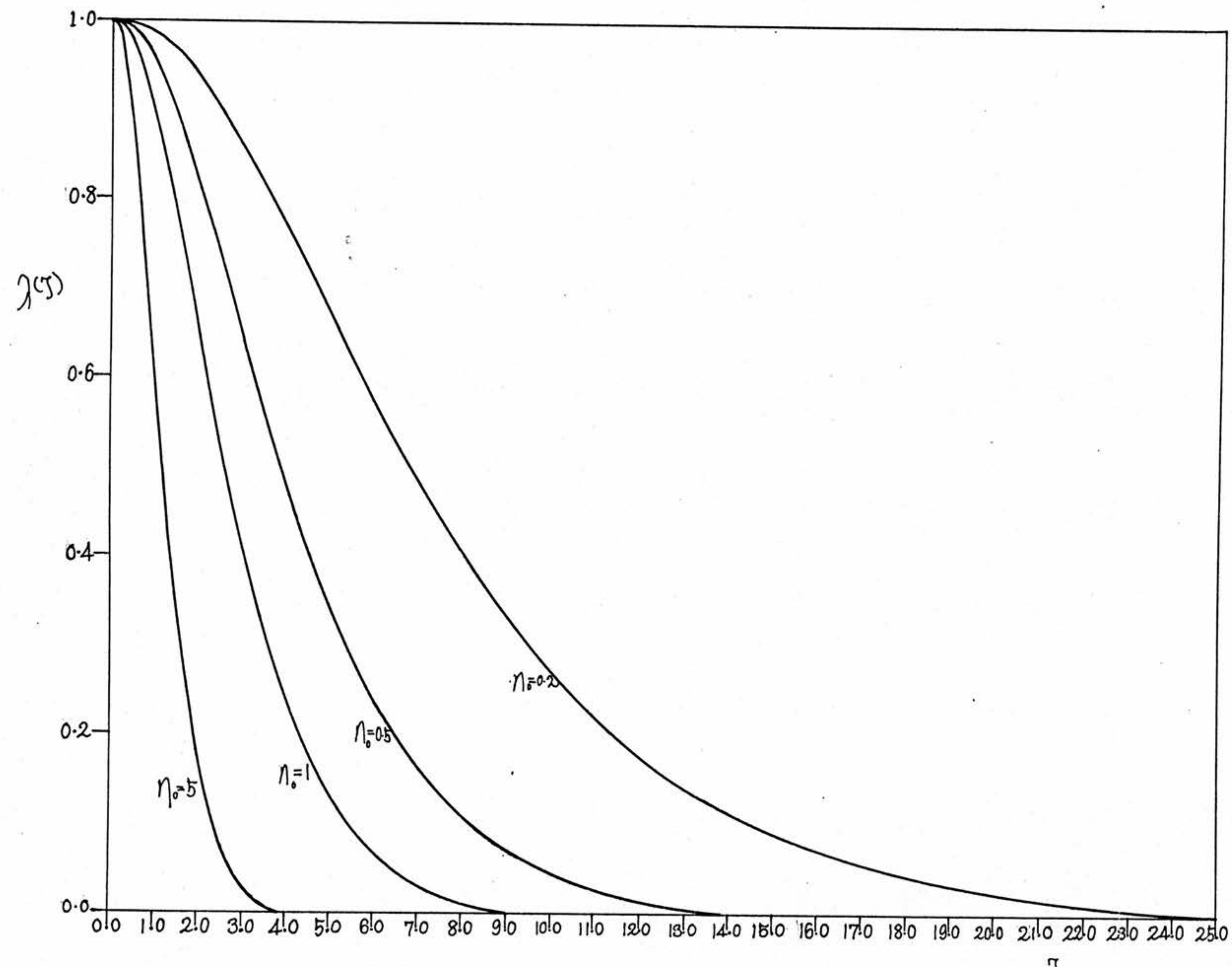


Fig. 2. The partially Degenerate Standard Model function for various η_0.

(B) DERIVATION OF THE RELATIONS FOR THE PHYSICAL CHARACTERISTICS OF THE PARTIALLY DEGENERATE STANDARD MODEL

(a) Mass:

The mass enclosed in a sphere of radius $r = a\eta$ is given by:

$$M(r) = \int_0^r 4\pi r^2 \rho \, dr = 4\pi a^3 \int_0^\eta \rho \eta^2 \, d\eta \tag{1}$$

where

$$a = \left[\frac{2}{3\pi G \beta Q_1^{4/3} Q_2^{1/3} (\mu_e H)^2} \right]^{1/2} \tag{2}$$

$$\rho = Q_1^2 Q_2 \mu_e H U_{1/2} U_{3/2} \tag{3}$$

Hence the relation (1) becomes:

$$M(r) = 4\pi a^3 Q_1^2 Q_2 \mu_e H \int_0^\eta - \left[\frac{d}{d\eta} \eta^2 U_{3/2}^{2/3} \frac{1}{\lambda} \frac{d\lambda}{d\eta} \right] d\eta \Rightarrow$$

$$M(\eta) = 4\pi a^3 Q_1^2 Q_2 \mu_e H \left\{ -\eta^2 \frac{1}{\lambda} \frac{d\lambda}{d\eta} U_{3/2}^{2/3} \right\} \tag{4}$$

or

$$M(\eta) = C_M \left(\frac{1-\beta}{\beta^4} \right)^{1/2} \mu_e^{-2} \left\{ -\eta^2 \frac{1}{\lambda} \frac{d\lambda}{d\eta} U_{3/2}^{2/3} \right\}$$

where

$$C_M = \left(\frac{2}{3\pi G} \right)^{3/2} \frac{4\pi}{H^2} k^2 \left(\frac{3}{a} \right)^{1/2}$$

$$C_M = 4.87127 \odot \tag{5}$$

b) Radius:

The radius at each point is given by:

$$R = a\eta = \left[\frac{2}{3\pi G \beta Q_1^{4/3} Q_2^{1/3} (\mu_e H)^2} \right]^{1/2} \eta \tag{6}$$

or

$$R = C_R \left(\frac{1}{1-\beta} \right)^{1/6} \frac{1}{\beta^{1/3}} \frac{1}{\mu_e} \eta$$

where

$$C_R = \left(\frac{2}{3\pi G} \right)^{1/2} \frac{1}{Q_1^{2/3}} \left(\frac{a}{3k^4} \right)^{1/6} \frac{1}{H}$$

$$C_R = 0.0102336 \odot \tag{7}$$

c) Density:

The density is given as:

$$\rho = Q_1^2 \, Q_2 \, \mu_e H \, U_{1/2} \, U_{3/2}$$

where

$$Q_1 = \frac{2}{h^3} (2\pi m)^{3/2}$$

$$Q_2 = K^4 \, \frac{3}{a} \, \frac{1-\beta}{\beta}$$

$$\Rightarrow \qquad \rho = C_\rho \left(\frac{1-\beta}{\beta}\right) \mu_e U_{1/2} U_{3/2} \tag{8}$$

with

$$C_\rho = Q_1^2 \, K^4 \, \frac{3}{a} \, H$$

$$C_\rho = 2.1377 \times 10^6 \quad gm/cm^3 \tag{9}$$

d) Pressure:

The pressure is given by:

$$P = \frac{1}{\beta} P_{gas} = \frac{1}{\beta} Q_1^{8/3} \, Q_2^{5/3} \, U_{3/2}^{8/3} \tag{10}$$

or

$$P = C_P \, \frac{(1-\beta)^{5/3}}{\beta^{8/3}} \, U_{3/2}^{8/3}$$

where

$$C_P = Q_1^{8/3} \left(\frac{K^4}{a} 3\right)^{5/3}$$

$$C_P = 7.2684 \times 10^{23} \quad dynes/cm^2 \tag{11}$$

e) Temperature:

The temperature is given by :

$$T = Q_2^{2/3} \, Q_1^{2/3} \, U_{3/2}^{2/3} \, K^{-1} \tag{12}$$

or

$$T = C_T \left(\frac{1-\beta}{\beta}\right)^{2/3} U_{3/2}^{2/3}$$

with $$C_T = Q_1^{2/3} \; K^{5/3} \left(\frac{3}{a}\right)^{2/3}$$

$$C_T = 4.1224 \times 10^{9} \; {}^{\circ}K \qquad (13)$$

From the above relations we realize that the values of β and μ_e together have a strong effect on the values of the mass, radius and density, while the temperature and pressure depend only upon β.
We expect that the radiation pressure is negligible so $1-\beta$ must be very small, and that the configurations do not contain considerable amounts of hydrogen so μ_e will be about 2.
Typical values for the degeneracy parameter $\alpha = -\log\Lambda$ will be around $\alpha = 0$ for the partially degenerate models will lie between their degeneracy-non degeneracy values in the ($\log\rho$, $\log T$) plane, as it is shown in appendix IV.
Tables (22) to (27) give the values of the Mass, the Radius and the central values of the pressure temperature and density of models with $\Lambda_0 = 0.1, 0.2, 0.5, 1, 2, 5$.
and for trial values of $\mu_e = 1, 1.5$ and 2, and $\log(1-\beta)$ in the range of $(-1, -6)$

The variation of the mass and the radius for the above values of μ_e and β is shown in the diagrams (7) and (6), for $\Lambda_0 = 1$.

Figures 3, 4 and 5 give the characteristic functions of a partially degenerate model for Λ_0 = 0.1, 1, 2 correspondingly.

TABLE II a. PARTIALLY DEGENERATE STANDARD MODEL FUNCTIONS

$\eta_0 = 0.005$

ξ	$\theta(\xi)$	$\frac{\rho}{\rho_c} = \frac{[U_{1/2}(\eta)U_{3/2}(\eta)]}{[U_{1/2}(\eta_0)U_{3/2}(\eta_0)]}$	$\frac{T}{T_c} = \left[\frac{U_{3/2}(\eta)}{U_{3/2}(\eta_0)}\right]^{2/3}$	$\frac{P_r}{P_{r_c}} = \left[\frac{U_{3/2}(\eta)}{U_{3/2}(\eta_0)}\right]^{8/3}$	$\frac{M}{M(R)} = \frac{\left[\xi^2 \frac{1}{d(\xi)} U_{3/2}^{2/3} \theta'(\xi)\right]}{\left[\xi^2 \frac{1}{d(\xi)} U_{3/2}^{2/3} \theta'(\xi)\right]_{d \to 0}}$
0.0	0.10000000D 01	0.10000000D 01	0.10000000D 01	0.10000000D 01	0.0
0.50000000D 00	0.99996445D 00	0.99997632D 00	0.99992899D 00	0.99990529D 00	0.27837422D-06
0.10000000D 02	0.98593290D 00	0.99060800D 00	0.97209982D 00	0.96295794D 00	0.21896165D-02
0.19500000D 02	0.94808408D 00	0.96511223D 00	0.89898677D 00	0.86758352D 00	0.15493538D-01
0.29000000D 02	0.89050610D 00	0.92566224D 00	0.79323066D 00	0.73419297D 00	0.47298341D-01
0.38500000D 02	0.81881636D 00	0.87532755D 00	0.67078144D 00	0.58705992D 00	0.10020733D 00
0.48000000D 02	0.73901964D 00	0.81753048D 00	0.54652701D 00	0.44669992D 00	0.17217641D 00
0.57500000D 02	0.65655455D 00	0.75555876D 00	0.43145554D 00	0.32589152D 00	0.25796840D 00
0.67000000D 02	0.57572586D 00	0.69223280D 00	0.33183239D 00	0.22961953D 00	0.35104292D 00
0.76500000D 02	0.49952764D 00	0.62974903D 00	0.24985839D 00	0.15727879D 00	0.44514319D 00
0.86000000D 02	0.42974559D 00	0.56966554D 00	0.18496007D 00	0.10531251D 00	0.53524852D 00
0.95500000D 02	0.36720058D 00	0.51297621D 00	0.13506219D 00	0.69245112D-01	0.61791446D 00
0.10500000D 03	0.31202533D 00	0.46022372D 00	0.97537185D-01	0.44861751D-01	0.69119296D 00
0.11450000D 03	0.26391128D 00	0.41161707D 00	0.69784953D-01	0.28706072D-01	0.75434274D 00
0.12400000D 03	0.22230094D 00	0.36713539D 00	0.49519513D-01	0.18167924D-01	0.80748434D 00
0.13350000D 03	0.18652356D 00	0.32661064D 00	0.34866016D-01	0.11379461D-01	0.85128670D 00
0.14300000D 03	0.15588332D 00	0.28978916D 00	0.24353955D-01	0.70522703D-02	0.88671996D 00
0.15250000D 03	0.12971188D 00	0.25637495D 00	0.16863971D-01	0.43201891D-02	0.91487904D 00
0.16200000D 03	0.10739630D 00	0.22605859D 00	0.11561247D-01	0.26114667D-02	0.93686762D 00
0.17150000D 03	0.88391017D-01	0.19853561D 00	0.78318462D-02	0.15536533D-02	0.95372839D 00
0.18100000D 03	0.72220187D-01	0.17351733D 00	0.52285796D-02	0.90650876D-03	0.96640580D 00
0.19050000D 03	0.58474635D-01	0.15073670D 00	0.34278149D-02	0.51626960D-03	0.97573036D 00

TABLE 11 Q. PARTIALLY DEGENERATE STANDARD MODEL FUNCTIONS

$\eta_0 = 0.005$

ξ	$\lambda(\xi)$	$\frac{\rho}{\rho_c} = \frac{[U_{1/2}(\eta)\, U_{3/2}(\eta)]}{[U_{1/2}(\eta_0)\, U_{3/2}(\eta_0)]}$	$\frac{T}{T_c} = \left[\frac{U_{3/2}(\eta)}{U_{3/2}(\eta_0)}\right]^{2/3}$	$\frac{P_r}{P_{rc}} = \left[\frac{U_{3/2}(\eta)}{U_{3/2}(\eta_0)}\right]^{8/3}$	$\frac{M}{M(R)} = \frac{\left[\xi^2 \frac{1}{\lambda(\xi)} U_{3/2}^{2/3} \lambda'(\xi)\right]}{\left[\xi^2 \frac{1}{\lambda(\xi)} U_{3/2}^{2/3} \lambda'(\xi)\right]_{\lambda \to 0}}$
0.20000000D 03	0.46806094D-01	0.12995068D 00	0.21963451D-02	0.28517728D-03	0.98241616D 00
0.20950000D 03	0.36920485D-01	0.11094075D 00	0.13666016D-02	0.15148341D-03	0.98706618D 00
0.21900000D 03	0.28571175D-01	0.93511935D-01	0.81841384D-03	0.76466095D-04	0.99018137D 00
0.22850000D 03	0.21552840D-01	0.77491329D-01	0.46572961D-03	0.36058964D-04	0.99217143D 00
0.23800000D 03	0.15696252D-01	0.62726191D-01	0.24701510D-03	0.15480911D-04	0.99336568D 00
0.24750000D 03	0.10864348D-01	0.49081968D-01	0.11834352D-03	0.58034842D-05	0.99402332D 00
0.25700000D 03	0.69501320D-02	0.36440352D-01	0.48431483D-04	0.17633207D-05	0.99434281D 00
0.26650000D 03	0.38778955D-02	0.24697447D-01	0.15077780D-04	0.37205694D-06	0.99447000D 00
0.27600000D 03	0.16130326D-02	0.13762078D-01	0.26087597D-05	0.35870497D-07	0.99450525D 00
0.28550000D 03	0.21171062D-03	0.35542727D-02	0.44940169D-07	0.15958965D-09	0.99451087D 00
0.28850000D 03	0.10240963D-04	0.47186341D-03	0.10515533D-09	0.49575554D-13	0.10000000D 01

TABLE 12 a. PARTIALLY DEGENERATE STANDARD MODEL FUNCTIONS

$\Lambda_0 = 0.01$

η	$\theta(\eta)$	$\frac{\rho}{\rho_c} = \frac{[U_{1/2}(\eta)U_{3/2}(\eta)]}{[U_{1/2}(\eta_0)U_{3/2}(\eta_0)]}$	$\frac{T}{T_c} = \left[\frac{U_{3/2}(\eta)}{U_{3/2}(\eta_0)}\right]^{2/3}$	$\frac{P_r}{P_{r_c}} = \left[\frac{U_{3/2}(\eta)}{U_{3/2}(\eta_0)}\right]^{8/3}$	$\frac{M}{M(R)} = \frac{\left[\eta^2 \frac{1}{\theta(\eta)} U_{3/2}^{2/3} \theta'(\eta)\right]}{\left[\eta^2 \frac{1}{\theta(\eta)} U_{3/2}^{2/3} \theta'(\eta)\right]_{\theta' \to 0}}$
0.0	0.10000000D 01	0.10000000D 01	0.10000000D 01	0.10000000D 01	0.0
0.30000000D 00	0.99996782D 00	0.99997858D 00	0.99993580D 00	0.99991471D 00	0.23975701D-06
0.48000000D 01	0.99181258D 00	0.99454379D 00	0.98373462D 00	0.97835349D 00	0.97246984D-03
0.93000000D 01	0.96978795D 00	0.97979053D 00	0.94063838D 00	0.92158015D 00	0.68856620D-02
0.13800000D 02	0.93528630D 00	0.95645087D 00	0.87505880D 00	0.83685617D 00	0.21546397D-01
0.18300000D 02	0.89036458D 00	0.92562358D 00	0.79320723D 00	0.73407054D 00	0.47389909D-01
0.22800000D 02	0.83747729D 00	0.88865147D 00	0.70196928D 00	0.62362858D 00	0.85245763D-01
0.27300000D 02	0.77919704D 00	0.84698814D 00	0.60785517D 00	0.51464705D 00	0.13443269D 00
0.31800000D 02	0.71797255D 00	0.80207609D 00	0.51625173D 00	0.41386857D 00	0.19309841D 00
0.36300000D 02	0.65595494D 00	0.75525099D 00	0.43105831D 00	0.32536091D 00	0.25867670D 00
0.40800000D 02	0.59490058D 00	0.70767856D 00	0.35466377D 00	0.25080969D 00	0.32834098D 00
0.45300000D 02	0.53614215D 00	0.66032295D 00	0.28815275D 00	0.19011909D 00	0.39937218D 00
0.49800000D 02	0.48061002D 00	0.61394102D 00	0.23161996D 00	0.14207144D 00	0.46940570D 00
0.54300000D 02	0.42888431D 00	0.56909481D 00	0.18449705D 00	0.10489110D 00	0.53655896D 00
0.58800000D 02	0.38126057D 00	0.52617487D 00	0.14583517D 00	0.76651483D-01	0.59946332D 00
0.63300000D 02	0.33781680D 00	0.48542842D 00	0.11451986D 00	0.55526584D-01	0.65723109D 00
0.67800000D 02	0.29847440D 00	0.44698800D 00	0.89417578D-01	0.39919363D-01	0.70938627D 00
0.72300000D 02	0.26304927D 00	0.41089795D 00	0.69464731D-01	0.28505985D-01	0.75578127D 00
0.76800000D 02	0.23129235D 00	0.37713744D 00	0.53713780D-01	0.20230135D-01	0.79651427D 00
0.81300000D 02	0.20292013D 00	0.34563953D 00	0.41350305D-01	0.14272294D-01	0.83185541D 00
0.85800000D 02	0.17763660D 00	0.31630633D 00	0.31692135D-01	0.10009945D-01	0.86218566D 00
0.90300000D 02	0.15514827D 00	0.28902075D 00	0.24178658D-01	0.69777641D-02	0.88794874D 00
0.94800000D 02	0.13517394D 00	0.26365537D 00	0.18355671D-01	0.48322193D-02	0.90961522D 00
0.99300000D 02	0.11745058D 00	0.24007879D 00	0.13859111D-01	0.33221203D-02	0.92765680D 00

TABLE 12b. PARTIALLY DEGENERATE STANDARD MODEL FUNCTIONS

$\eta_0 = 0.01$

η	$\lambda(\eta)$	$\frac{\rho}{\rho_c} = \frac{[u_{1/2}(\eta)\, u_{3/2}(\eta)]}{[u_{1/2}(\eta_0)\, u_{3/2}(\eta_0)]}$	$\frac{T}{T_c} = \left[\frac{u_{3/2}(\eta)}{u_{3/2}(\eta_0)}\right]^{2/3}$	$\frac{P_r}{P_{r_c}} = \left[\frac{u_{3/2}(\eta)}{u_{3/2}(\eta_0)}\right]^{8/3}$	$\frac{M}{M(R)} = \frac{\left[\eta^2 \frac{1}{\lambda(\eta)} u_{3/2}^{2/3}(\eta)\lambda'(\eta)\right]}{\left[\eta^2 \frac{1}{\lambda(\eta)} u_{3/2}^{2/3}(\eta)\lambda'(\eta)\right]_{\lambda\to 0}}$
0.10380000D 03	0.10173629D 00	0.21816027D 00	0.10399514D-01	0.22651806D-02	0.94252893D 00
0.10830000D 03	0.87811525D-01	0.19777279D 00	0.77481200D-02	0.15299116D-02	0.95465974D 00
0.11280000D 03	0.75478909D-01	0.17879509D 00	0.57249661D-02	0.10219332D-02	0.96444359D 00
0.11730000D 03	0.64562332D-01	0.16111287D 00	0.41889506D-02	0.67378465D-03	0.97223812D 00
0.12180000D 03	0.54905581D-01	0.14461945D 00	0.30297168D-02	0.43742840D-03	0.97836352D 00
0.12630000D 03	0.46370750D-01	0.12921596D 00	0.21611103D-02	0.27878204D-03	0.98310337D 00
0.13080000D 03	0.38836589D-01	0.11481131D 00	0.15159616D-02	0.17375559D-03	0.98670641D 00
0.13530000D 03	0.32196898D-01	0.10132191D 00	0.10419560D-02	0.10539344D-03	0.98938888D 00
0.13980000D 03	0.26359028D-01	0.88671338D-01	0.69838287D-03	0.61820599D-04	0.99133714D 00
0.14430000D 03	0.21242548D-01	0.76789887D-01	0.45358616D-03	0.34770927D-04	0.99271025D 00
0.14880000D 03	0.16778103D-01	0.65614094D-01	0.28297139D-03	0.18534834D-04	0.99364262D 00
0.15330000D 03	0.12906522D-01	0.55086272D-01	0.16744965D-03	0.92081779D-05	0.99424639D 00
0.15780000D 03	0.95782579D-02	0.45154041D-01	0.92224536D-04	0.41570633D-05	0.99461376D 00
0.16230000D 03	0.67532916D-02	0.35769872D-01	0.45847001D-04	0.16370793D-05	0.99481897D 00
0.16680000D 03	0.44018983D-02	0.26890669D-01	0.19478942D-04	0.52288546D-06	0.99492020D 00
0.17130000D 03	0.25072414D-02	0.18477364D-01	0.63194857D-05	0.11656286D-06	0.99496125D 00
0.17580000D 03	0.10732009D-02	0.10494555D-01	0.11578575D-05	0.12129873D-07	0.99497292D 00
0.18030000D 03	0.15671507D-03	0.29101653D-02	0.24689746D-07	0.71725041D-10	0.99497763D 00
0.18180000D 03	0.10064306D-04	0.46669503D-03	0.10182704D-09	0.47438709D-13	0.10000000D 01

TABLE 12 γ. PARTIALLY DEGENERATE STANDARD MODEL FUNCTIONS

$\Lambda_0 = 0.1$

η	$\lambda(\eta)$	$\frac{\rho}{\rho_c} = \frac{[u_{1/2}(\eta)\, u_{3/2}(\eta)]}{[u_{1/2}(\eta_0)\, u_{3/2}(\eta_0)]}$	$\frac{T}{T_c} = \left[\frac{u_{3/2}(\eta)}{u_{3/2}(\eta_0)}\right]^{2/3}$	$\frac{P_r}{P_{r_c}} = \left[\frac{u_{3/2}(\eta)}{u_{3/2}(\eta_0)}\right]^{8/3}$	$\frac{M}{M(R)} = \frac{\left[\eta^2 \frac{1}{\lambda(\eta)} u_{3/2}^{2/3} \lambda'(\eta)\right]}{\left[\eta^2 \frac{1}{\lambda(\eta)} u_{3/2}^{2/3} \lambda'(\eta)\right]_{\lambda \to 0}}$
0.0	0.10000000D 01	0.10000000D 01	0.10000000D 01	0.10000000D 01	0.0
0.80000000D 00	0.99525953D 00	0.99639002D 00	0.99077507D 00	0.98761798D 00	0.42230884D-03
0.16000000D 01	0.98124046D 00	0.98766141D 00	0.96373188D 00	0.95155159D 00	0.33228899D-02
0.24000000D 01	0.95853036D 00	0.97261013D 00	0.92067932D 00	0.89486014D 00	0.10912178D-01
0.32000000D 01	0.92804674D 00	0.95220370D 00	0.86436451D 00	0.82209019D 00	0.24909304D-01
0.40000000D 01	0.89095520D 00	0.92704635D 00	0.79813145D 00	0.73859406D 00	0.46396538D-01
0.48000000D 01	0.84857323D 00	0.89783711D 00	0.72554438D 00	0.64981577D 00	0.75767181D-01
0.56000000D 01	0.80227312D 00	0.86532575D 00	0.65004069D 00	0.56068451D 00	0.11276295D 00
0.64000000D 01	0.75339499D 00	0.83027119D 00	0.57466098D 00	0.47520382D 00	0.15658127D 00
0.72000000D 01	0.70317769D 00	0.79340591D 00	0.50187942D 00	0.39626142D 00	0.20602459D 00
0.80000000D 01	0.65271114D 00	0.75540843D 00	0.43353408D 00	0.32563223D 00	0.25966273D 00
0.88000000D 01	0.60290976D 00	0.71688443D 00	0.37084006D 00	0.26411721D 00	0.31598462D 00
0.96000000D 01	0.55450447D 00	0.67835623D 00	0.31446004D 00	0.21175389D 00	0.37352357D 00
0.10400000D 02	0.50804891D 00	0.64025926D 00	0.26460596D 00	0.16804424D 00	0.43094860D 00
0.11200000D 02	0.46393530D 00	0.60294415D 00	0.22115025D 00	0.13216258D 00	0.48712141D 00
0.12000000D 02	0.42241599D 00	0.56668257D 00	0.18373081D 00	0.10312396D 00	0.54112316D 00
0.12800000D 02	0.38362708D 00	0.53167558D 00	0.15184122D 00	0.79907410D-01	0.59225744D 00
0.13600000D 02	0.34761187D 00	0.49806306D 00	0.12490218D 00	0.61537174D-01	0.64003670D 00
0.14400000D 02	0.31434243D 00	0.46593342D 00	0.10231433D 00	0.47129808D-01	0.68415836D 00
0.15200000D 02	0.28373847D 00	0.43533287D 00	0.83494509D-01	0.35915845D-01	0.72447606D 00
0.16000000D 02	0.25568323D 00	0.40627395D 00	0.67898573D-01	0.27244334D-01	0.76096977D 00
0.16800000D 02	0.23003631D 00	0.37874301D 00	0.55033884D-01	0.20576846D-01	0.79371732D 00
0.17600000D 02	0.20664378D 00	0.35270659D 00	0.44464462D-01	0.15475853D-01	0.82286891D 00
0.18400000D 02	0.18534587D 00	0.32811680D 00	0.35811157D-01	0.11590822D-01	0.84862529D 00
0.19200000D 02	0.16598266D 00	0.30491574D 00	0.28748735D-01	0.86440989D-02	0.87121972D 00
0.20000000D 02	0.14839812D 00	0.28303894D 00	0.23001220D-01	0.64177850D-02	0.89090358D 00

TABLE 12 δ. PARTIALLY DEGENERATE STANDARD MODEL FUNCTIONS

$\Lambda_0 = 0.1$

ξ	$\lambda(\xi)$	$\frac{\rho}{\rho_c} = \frac{[u_{1/2}(\eta)\, u_{3/2}(\eta)]}{[u_{1/2}(\eta_0)\, u_{3/2}(\eta_0)]}$	$\frac{T}{T_c} = \left[\frac{u_{3/2}(\eta)}{u_{3/2}(\eta_0)}\right]^{2/3}$	$\frac{P_r}{P_{r_c}} = \left[\frac{u_{3/2}(\eta)}{u_{3/2}(\eta_0)}\right]^{8/3}$	$\frac{M}{M(R)} = \frac{\left[\xi^2 \frac{1}{\lambda(\xi)} u_{3/2}^{2/3} \lambda'(\xi)\right]}{\left[\xi^2 \frac{1}{\lambda(\xi)} u_{3/2}^{2/3} \lambda'(\xi)\right]_{\lambda \to 0}}$
0.20800000D 02	0.13244285D 00	0.26241821D 00	0.18336445D-01	0.47421612D-02	0.90793523D 00
0.21600000D 02	0.11797583D 00	0.24298373D 00	0.14560443D-01	0.34358542D-02	0.92257165D 00
0.22400000D 02	0.10486542D 00	0.22466568D 00	0.11512054D-01	0.25476948D-02	0.93506239D 00
0.23200000D 02	0.92989807D-01	0.20739545D 00	0.90579653D-02	0.18501093D-02	0.94564533D 00
0.24000000D 02	0.82237049D-01	0.19110649D 00	0.70882854D-02	0.13338356D-02	0.95454422D 00
0.24800000D 02	0.72504845D-01	0.17573492D 00	0.55126855D-02	0.95374608D-03	0.96196658D 00
0.25600000D 02	0.63700141D-01	0.16121986D 00	0.42570800D-02	0.67557675D-03	0.96810349D 00
0.26400000D 02	0.55738613D-01	0.14750373D 00	0.32608135D-02	0.47338289D-03	0.97312916D 00
0.27200000D 02	0.48544094D-01	0.13453229D 00	0.24742956D-02	0.32757201D-03	0.97720131D 00
0.28000000D 02	0.42047989D-01	0.12225471D 00	0.18570278D-02	0.22338965D-03	0.98046182D 00
0.28800000D 02	0.36188684D-01	0.11062346D 00	0.13759673D-02	0.14975785D-03	0.98303744D 00
0.29600000D 02	0.30919988D-01	0.99594249D-01	0.10041754D-02	0.98387008D-04	0.98504078D 00
0.30400000D 02	0.26165617D-01	0.89125897D-01	0.71970560D-03	0.63098118D-04	0.98657121D 00
0.31200000D 02	0.21908726D-01	0.79180169D-01	0.50469038D-03	0.39306692D-04	0.98771588D 00
0.32000000D 02	0.18101493D-01	0.69721625D-01	0.34459377D-03	0.23630387D-04	0.98855062D 00
0.32800000D 02	0.14709826D-01	0.60717451D-01	0.22759959D-03	0.13591112D-04	0.98914088D 00
0.33600000D 02	0.11704069D-01	0.52137291D-01	0.14411172D-03	0.73891545D-05	0.98954260D 00
0.34400000D 02	0.90589519D-02	0.43953087D-01	0.86346025D-04	0.37321448D-05	0.98980297D 00
0.35200000D 02	0.67536556D-02	0.36138918D-01	0.47997358D-04	0.17056960D-05	0.98996120D 00
0.36000000D 02	0.47722352D-02	0.28670851D-01	0.23967835D-04	0.67571478D-06	0.99004922D 00
0.36800000D 02	0.31046817D-02	0.21526791D-01	0.10145132D-04	0.21474266D-06	0.99009222D 00
0.37600000D 02	0.17494735D-02	0.14686351D-01	0.32216024D-05	0.46521847D-07	0.99010931D 00
0.38400000D 02	0.72064895D-03	0.81307254D-02	0.54667088D-06	0.43703746D-08	0.99011407D 00
0.39200000D 02	0.77747470D-04	0.18426420D-02	0.63630562D-08	0.11528314D-10	0.99036261D 00
0.39400000D 02	0.59852540D-05	0.33345629D-03	0.37710306D-10	0.12363976D-13	0.10000000D 01

TABLE 13a. PARTIALLY DEGENERATE STANDARD MODEL FUNCTIONS

$\Lambda_0 = 0.2$

η	$\vartheta(\eta)$	$\frac{\rho}{\rho_c} = \frac{[U_{1/2}(\eta) U_{3/2}(\eta)]}{[U_{1/2}(\eta_0) U_{3/2}(\eta_0)]}$	$\frac{T}{T_c} = \left[\frac{U_{3/2}(\eta)}{U_{3/2}(\eta_0)}\right]^{2/3}$	$\frac{P_r}{P_{r_c}} = \left[\frac{U_{3/2}(\eta)}{U_{3/2}(\eta_0)}\right]^{8/3}$	$\frac{M}{M(R)} = \frac{\left[\xi^2 \frac{1}{\alpha(\xi)} U_{3/2}^{2/3} \vartheta'(\xi)\right]}{\left[\xi^2 \frac{1}{\alpha(\xi)} U_{3/2}^{1/3} \vartheta'(\xi)\right]_{\alpha \to 0}}$
0.0	0.10000000D 01	0.10000000D 01	0.10000000D 01	0.10000000D 01	0.0
0.10000000D 00	0.99981957D 00	0.99988355D 00	0.99965608D 00	0.99953608D 00	0.31205824D-05
0.80000000D 00	0.98855035D 00	0.99259427D 00	0.97828156D 00	0.97070627D 00	0.15771633D-02
0.15000000D 01	0.96060052D 00	0.97437645D 00	0.92617427D 00	0.90137995D 00	0.10061211D-01
0.22000000D 01	0.91805526D 00	0.94625103D 00	0.84935319D 00	0.80172651D 00	0.30144519D-01
0.29000000D 01	0.86387223D 00	0.90970299D 00	0.75593193D 00	0.68485600D 00	0.64430073D-01
0.36000000D 01	0.80144510D 00	0.86650068D 00	0.65451793D 00	0.56373682D 00	0.11321347D 00
0.43000000D 01	0.73417608D 00	0.81850714D 00	0.55282426D 00	0.44883908D 00	0.17477751D 00
0.50000000D 01	0.66513815D 00	0.76751570D 00	0.45679407D 00	0.34701628D 00	0.24602132D 00
0.57000000D 01	0.59686596D 00	0.71512965D 00	0.37029862D 00	0.26154108D 00	0.32319330D 00
0.64000000D 01	0.53127327D 00	0.66269188D 00	0.29528746D 00	0.19286232D 00	0.40253819D 00
0.71000000D 01	0.46966900D 00	0.61125966D 00	0.23219524D 00	0.13960588D 00	0.48075468D 00
0.78000000D 01	0.41283510D 00	0.56161329D 00	0.18042440D 00	0.99483323D-01	0.55524209D 00
0.85000000D 01	0.36113218D 00	0.51428631D 00	0.13878390D 00	0.69955303D-01	0.62417029D 00
0.92000000D 01	0.31460849D 00	0.46960618D 00	0.10582739D 00	0.48633553D-01	0.68642865D 00
0.99000000D 01	0.27309752D 00	0.42773726D 00	0.80081563D-01	0.33474115D-01	0.74150940D 00
0.10600000D 02	0.23629762D 00	0.38872099D 00	0.60180391D-01	0.22832463D-01	0.78936812D 00
0.11300000D 02	0.20383272D 00	0.35251064D 00	0.44930034D-01	0.15441507D-01	0.83028877D 00
0.12000000D 02	0.17529573D 00	0.31899953D 00	0.33328331D-01	0.10355258D-01	0.86476764D 00
0.12700000D 02	0.15027794D 00	0.28804317D 00	0.24557782D-01	0.68838452D-02	0.89342125D 00

TABLE 13 Q. PARTIALLY DEGENERATE STANDARD MODEL FUNCTIONS

$\eta_0 = 0.2$

y	$\vartheta(y)$	$\frac{\rho}{\rho_c} = \frac{[U_{1/2}(\eta)U_{3/2}(\eta)]}{[U_{1/2}(\eta_0)U_{3/2}(\eta_0)]}$	$\frac{T}{T_c} = \left[\frac{U_{3/2}(\eta)}{U_{3/2}(\eta_0)}\right]^{2/3}$	$\frac{P_r}{P_{r_c}} = \left[\frac{U_{3/2}(\eta)}{U_{3/2}(\eta_0)}\right]^{5/3}$	$\frac{M}{M(R)} = \frac{\left[y^2 \frac{1}{\vartheta(y)} U_{3/2}^{2/3} \vartheta'(y)\right]}{\left[y^2 \frac{1}{\vartheta(y)} U_{3/2}^{2/3} \vartheta'(y)\right]_{\vartheta \to 0}}$
0.13400000D 02	0.12838759D 00	0.25947580D 00	0.17965307D-01	0.45330258D-02	0.91691809D 00
0.14100000D 02	0.10926065D 00	0.23312236D 00	0.13037169D-01	0.29534970D-02	0.93593067D 00
0.14800000D 02	0.92566042D-01	0.20880693D 00	0.93738474D-02	0.19009927D-02	0.95110424D 00
0.15500000D 02	0.78007408D-01	0.18635827D 00	0.66672880D-02	0.12061337D-02	0.96303769D 00
0.16200000D 02	0.65322498D-01	0.16561340D 00	0.46814655D-02	0.75228541D-03	0.97227360D 00
0.16900000D 02	0.54281303D-01	0.14641966D 00	0.32363936D-02	0.45961946D-03	0.97929428D 00
0.17600000D 02	0.44683494D-01	0.12863576D 00	0.21953019D-02	0.27380893D-03	0.98452190D 00
0.18300000D 02	0.36355620D-01	0.11213199D 00	0.14545366D-02	0.15809528D-03	0.98832103D 00
0.19000000D 02	0.29148329D-01	0.96790140D-01	0.93570596D-03	0.87765787D-04	0.99100267D 00
0.19700000D 02	0.22933802D-01	0.82502952D-01	0.57962814D-03	0.46331753D-04	0.99282882D 00
0.20400000D 02	0.17603529D-01	0.69173496D-01	0.34169742D-03	0.22896004D-04	0.99401725D 00
0.21100000D 02	0.13066546D-01	0.56714416D-01	0.18835292D-03	0.10346051D-04	0.99474622D 00
0.21800000D 02	0.92483241D-02	0.45047152D-01	0.94395756D-04	0.41178462D-05	0.99515877D 00
0.22500000D 02	0.60907036D-02	0.34101168D-01	0.40954971D-04	0.13523148D-05	0.99536675D 00
0.23200000D 02	0.35538702D-02	0.23813210D-01	0.13947347D-04	0.32156782D-06	0.99545436D 00
0.23900000D 02	0.16236853D-02	0.14126608D-01	0.29119307D-05	0.39824689D-07	0.99548134D 00
0.24600000D 02	0.34092613D-03	0.49906451D-02	0.12839763D-06	0.62033679D-09	0.99551055D 00
0.24900000D 02	0.41973312D-04	0.12350989D-02	0.19462412D-08	0.23270610D-11	0.99999999D 00

TABLE 14a. PARTIALLY DEGENERATE STANDARD MODEL FUNCTIONS

$\eta_0 = 0.5$

ξ	$\theta(\xi)$	$\frac{\rho}{\rho_c} = \left[\frac{U_{1/2}(\eta)\,U_{3/2}(\eta)}{U_{1/2}(\eta_0)\,U_{3/2}(\eta_0)}\right]$	$\frac{T}{T_c} = \left[\frac{U_{3/2}(\eta)}{U_{3/2}(\eta_0)}\right]^{2/3}$	$\frac{P_r}{P_{rc}} = \left[\frac{U_{3/2}(\eta)}{U_{3/2}(\eta_0)}\right]^{8/3}$	$\frac{M}{M(R)} = \frac{\left[\xi^2 \frac{1}{\theta(\xi)} U_{3/2}^{2/3}\,\theta'(\xi)\right]}{\left[\xi^2 \frac{1}{\theta(\xi)} U_{3/2}^{2/3}\,\theta'(\xi)\right]_{\theta\to 0}}$
0.0	0.10000000D 01	0.10000000D 01	0.10000000D 01	0.10000000D 01	0.0
0.10000000D 00	0.99944622D 00	0.99965666D 00	0.99900378D 00	0.99862733D 00	0.16343472D-04
0.60000000D 00	0.98034249D 00	0.98776054D 00	0.96487912D 00	0.95193369D 00	0.34574410D-02
0.11000000D 01	0.93610721D 00	0.95981805D 00	0.88769136D 00	0.84870287D 00	0.20269949D-01
0.16000000D 01	0.87160655D 00	0.91802308D 00	0.77983607D 00	0.71025521D 00	0.57758342D-01
0.21000000D 01	0.79321664D 00	0.86539419D 00	0.65653626D 00	0.56086192D 00	0.11800745D 00
0.26000000D 01	0.70755345D 00	0.80529638D 00	0.53201884D 00	0.42055525D 00	0.19816115D 00
0.31000000D 01	0.62045083D 00	0.74099790D 00	0.41693721D 00	0.30148659D 00	0.29204744D 00
0.36000000D 01	0.53638501D 00	0.67534828D 00	0.31750525D 00	0.20802298D 00	0.39224513D 00
0.41000000D 01	0.45833427D 00	0.61060674D 00	0.23598793D 00	0.13901015D 00	0.49182132D 00
0.46000000D 01	0.38793925D 00	0.54840395D 00	0.17185735D 00	0.90448734D-01	0.58537008D 00
0.51000000D 01	0.32580622D 00	0.48979595D 00	0.12301562D 00	0.57552073D-01	0.66936064D 00
0.56000000D 01	0.27183233D 00	0.43536606D 00	0.86756018D-01	0.35926801D-01	0.74200410D 00
0.61000000D 01	0.22548561D 00	0.38534025D 00	0.60376132D-01	0.22048437D-01	0.80288070D 00
0.66000000D 01	0.18601550D 00	0.33969407D 00	0.41493493D-01	0.13315338D-01	0.85251096D 00
0.71000000D 01	0.15259557D 00	0.29824109D 00	0.28158440D-01	0.79117082D-02	0.89197501D 00
0.76000000D 01	0.12441150D 00	0.26070034D 00	0.18851590D-01	0.46192006D-02	0.92262067D 00
0.81000000D 01	0.10070954D 00	0.22674453D 00	0.12427925D-01	0.26433079D-02	0.94586272D 00
0.86000000D 01	0.80818820D-01	0.19603253D 00	0.80446187D-02	0.14767707D-02	0.96305812D 00
0.91000000D 01	0.64157648D-01	0.16823003D 00	0.50915558D-02	0.80096703D-03	0.97543732D 00
0.96000000D 01	0.50230776D-01	0.14302175D 00	0.31323266D-02	0.41841660D-03	0.98407352D 00

TABLE 14b. PARTIALLY DEGENERATE STANDARD MODEL FUNCTIONS

$\Lambda_0 = 0.5$

η	$\lambda(\eta)$	$\frac{\rho}{\rho_c} = \left[\frac{U_{1/2}(\eta)\,U_{3/2}(\eta)}{U_{1/2}(\eta_0)\,U_{3/2}(\eta_0)}\right]$	$\frac{T}{T_c} = \left[\frac{U_{3/2}(\eta)}{U_{3/2}(\eta_0)}\right]^{2/3}$	$\frac{P_r}{P_{r_c}} = \left[\frac{U_{3/2}(\eta)}{U_{3/2}(\eta_0)}\right]^{8/3}$	$\frac{M}{M(R)} = \frac{\left[\eta^2 \frac{1}{\lambda(\eta)} U_{3/2}^{2/3}\, \lambda'(\eta)\right]}{\left[\eta^2 \frac{1}{\lambda(\eta)} U_{3/2}^{2/3}\, \lambda'(\eta)\right]_{\lambda \to 0}}$
0.10100000D 02	0.38622223D-01	0.12011775D 00	0.18574579D-02	0.20817537D-03	0.98987558D 00
0.1C6CCCC0D 02	0.28986472D-01	0.99256029D-01	0.10488958D-02	0.97057321D-04	C.99359459D 00
0.11100000D 02	0.21039757D-01	0.8C202725D-01	0.55377151D-03	0.41376837D-04	0.99583718D 00
0.11600000D 02	0.14552548D-01	C.6275C890D-01	0.26538104D-03	0.155C5309D-04	0.99708161D 00
0.12100000D 02	0.93442424D-02	0.46718560D-01	0.10956639D-03	0.47638574D-05	C.99769402D 00
0.12600000D 02	0.52818818D-02	0.31946474D-01	0.35045576D-04	0.10415804D-05	C.99794365D 00
0.13100000D 02	0.22885690D-02	0.18295728D-01	0.65845810D-05	0.11204694D-06	C.99801712D 00
0.13600000D 02	0.39223887D-03	0.56457631D-02	0.19351724D-06	0.10159965D-08	0.99828784D 00
0.138CCCCCD 02	0.26760413D-04	0.94267104D-03	0.90083679D-09	0.78966490D-12	C.1CCC0000D 01

TABLE 15 a. PARTIALLY DEGENERATE STANDARD MODEL FUNCTIONS

$\Lambda_0 = 1$

ξ	$\theta(\xi)$	$\frac{\rho}{\rho_c} = \frac{[U_{1/2}(\eta)\,U_{3/2}(\eta)]}{[U_{1/2}(\eta_0)\,U_{3/2}(\eta_0)]}$	$\frac{T}{T_c} = \left[\frac{U_{3/2}(\eta)}{U_{3/2}(\eta_0)}\right]^{2/3}$	$\frac{P_r}{P_{r_c}} = \left[\frac{U_{3/2}(\eta)}{U_{3/2}(\eta_0)}\right]^{8/3}$	$\frac{M}{M(R)} = \frac{\left[\xi^2 \frac{1}{\theta(\xi)} U_{3/2}^{2/3}\, \theta'(\xi)\right]}{\left[\xi^2 \frac{1}{\theta(\xi)} U_{3/2}^{2/3}\, \theta'(\xi)\right]_{\xi \to 0}}$
0.0	0.10000000D 01	0.10000000D 01	0.10000000D 01	0.10000000D 01	0.0
0.20000000D-01	0.99995136D 00	0.99997139D 00	0.99991863D 00	0.99988806D 00	0.40754127D-06
0.24000000D 00	0.99302988D 00	0.99589287D 00	0.98836211D 00	0.98367488D 00	0.69933838D-03
0.46000000D 00	0.97472877D 00	0.98503951D 00	0.95801914D 00	0.94148996D 00	0.48329952D-02
0.68000000D 00	0.94593036D 00	0.96775211D 00	0.91091048D 00	0.87711714D 00	0.15149162D-01
0.90000000D 00	0.90796740D 00	0.94455804D 00	0.85003651D 00	0.79600480D 00	0.33706736D-01
0.11200000D 01	0.86249291D 00	0.91613513D 00	0.77902084D 00	0.70443227D 00	0.61690283D-01
0.13400000D 01	0.81133190D 00	0.88326791D 00	0.70171604D 00	0.60865459D 00	0.99354080D-01
0.15600000D 01	0.75633643D 00	0.84680082D 00	0.62183553D 00	0.51419300D 00	0.14608945D 00
0.17800000D 01	0.69926123D 00	0.80759296D 00	0.54266113D 00	0.42537424D 00	0.20058781D 00
0.20000000D 01	0.64167095D 00	0.76647814D 00	0.46685333D 00	0.34514388D 00	0.26105988D 00
0.22200000D 01	0.58488267D 00	0.72423254D 00	0.39636789D 00	0.27511434D 00	0.32546978D 00
0.24400000D 01	0.52994160D 00	0.68155153D 00	0.33246446D 00	0.21577239D 00	0.39174961D 00
0.26600000D 01	0.47762397D 00	0.63903540D 00	0.27578182D 00	0.16676341D 00	0.45797145D 00
0.28800000D 01	0.42845922D 00	0.59718344D 00	0.22645192D 00	0.12718389D 00	0.52246650D 00
0.31000000D 01	0.38276378D 00	0.55639470D 00	0.18422760D 00	0.95836961D-01	0.58389124D 00
0.33200000D 01	0.34067982D 00	0.51697388D 00	0.14860529D 00	0.71429149D-01	0.64124826D 00
0.35400000D 01	0.30221405D 00	0.47914056D 00	0.11893106D 00	0.52705125D-01	0.69387263D 00
0.37600000D 01	0.26727332D 00	0.44304042D 00	0.94484663D-01	0.38527825D-01	0.74139575D 00
0.39800000D 01	0.23569526D 00	0.40875696D 00	0.74540942D-01	0.27916548D-01	0.78369696D 00
0.42000000D 01	0.20727345D 00	0.37632310D 00	0.58410935D-01	0.20056012D-01	0.82085144D 00
0.44200000D 01	0.18177709D 00	0.34573187D 00	0.45466307D-01	0.14287583D-01	0.85307993D 00
0.46400000D 01	0.15896577D 00	0.31694594D 00	0.35151241D-01	0.10091178D-01	0.88070390D 00
0.48600000D 01	0.13860013D 00	0.28990571D 00	0.26985497D-01	0.70636335D-02	0.90410774D 00

TABLE 15l. PARTIALLY DEGENERATE STANDARD MODEL FUNCTIONS

$\Lambda_0 = 1$

η	$\lambda(\eta)$	$\frac{\rho}{\rho_c} = \frac{[u_{1/2}(\eta)\, u_{3/2}(\eta)]}{[u_{1/2}(\eta_0)\, u_{3/2}(\eta_0)]}$	$\frac{T}{T_c} = \left[\frac{u_{3/2}(\eta)}{u_{3/2}(\eta_0)}\right]^{2/3}$	$\frac{P_r}{P_{r_c}} = \left[\frac{u_{3/2}(\eta)}{u_{3/2}(\eta_0)}\right]^{8/3}$	$\frac{M}{M(R)} = \frac{\left[\eta^2 \frac{1}{\lambda(\eta)} u_{3/2}^{2/3} \lambda'(\eta)\right]}{\left[\eta^2 \frac{1}{\lambda(\eta)} u_{3/2}^{2/3} \lambda'(\eta)\right]_{\lambda \to 0}}$
0.50800000D 01	0.12044913D 00	0.26453613D 00	0.20561749D-01	0.48971239D-02	0.92370855D 00
0.53000000D 01	0.10429476D 00	0.24075220D 00	0.15539595D-01	0.33595578D-02	0.93993286D 00
0.55200000D 01	0.89934731D-01	0.21846326D 00	0.11637939D-01	0.22777955D-02	0.95319971D 00
0.57400000D 01	0.77183779D-01	0.19757641D 00	0.86269054D-02	0.15238472D-02	0.96390886D 00
0.59600000D 01	0.65873969D-01	0.17799895D 00	0.63199950D-02	0.10038547D-02	0.97243315D 00
0.61800000D 01	0.55854327D-01	0.15964028D 00	0.45668864D-02	0.64948778D-03	0.97911402D 00
0.64000000D 01	0.46990068D-01	0.14241310D 00	0.32470755D-02	0.41133996D-03	0.98425951D 00
0.66200000D 01	0.39161585D-01	0.12623435D 00	0.22644006D-02	0.25392841D-03	0.98814376D 00
0.68400000D 01	0.32263352D-01	0.11102566D 00	0.15424225D-02	0.15194782D-03	0.99100787D 00
0.70600000D 01	0.26202824D-01	0.96713641D-01	0.10205860D-02	0.87488711D-04	0.99306132D 00
0.72800000D 01	0.20899429D-01	0.83230009D-01	0.65106555D-03	0.47986695D-04	0.99448395D 00
0.75000000D 01	0.16283716D-01	0.70511487D-01	0.39619856D-03	0.24719551D-04	0.99542816D 00
0.77200000D 01	0.12296751D-01	0.58499693D-01	0.22641028D-03	0.11711578D-04	0.99602103D 00
0.79400000D 01	0.88899064D-02	0.47140932D-01	0.11854654D-03	0.49384849D-05	0.99636662D 00
0.81600000D 01	0.60253407D-02	0.36385957D-01	0.54539881D-04	0.17528158D-05	0.99654800D 00
0.83800000D 01	0.36778769D-02	0.26189710D-01	0.20346244D-04	0.47046123D-06	0.99662919D 00
0.86000000D 01	0.18404445D-02	0.16511060D-01	0.50998538D-05	0.74319180D-07	0.99665697D 00
0.88200000D 01	0.54233066D-03	0.73125525D-02	0.44313835D-06	0.28594142D-08	0.99666274D 00
0.90000000D 01	0.65428507D-05	0.38467893D-03	0.64516148D-10	0.21897507D-13	0.99999999D 00

TABLE 16a. PARTIALLY DEGENERATE STANDARD MODEL FUNCTIONS

$\eta_0 = 2.$

y	$\theta(y)$	$\frac{\rho}{\rho_c} = \frac{[U_{1/2}(\eta)U_{3/2}(\eta)]}{[U_{1/2}(\eta_0)U_{3/2}(\eta_0)]}$	$\frac{T}{T_c} = \left[\frac{U_{3/2}(\eta)}{U_{3/2}(\eta_0)}\right]^{2/3}$	$\frac{P_r}{P_{r_c}} = \left[\frac{U_{3/2}(\eta)}{U_{3/2}(\eta_0)}\right]^{2/3}$	$\frac{M}{M(R)} = \frac{\left[y^2 \frac{1}{\theta(y)} U_{3/2}^{2/3} \theta'(y)\right]}{\left[y^2 \frac{1}{\theta(y)} U_{3/2}^{2/3} \theta'(y)\right]_{y\to 0}}$
0.0	0.10000000D 01	0.10000000D 01	0.10000000D 01	0.10000000D 01	0.0
0.10000000D-01	0.99997521D 00	0.99998647D 00	0.99996245D 00	0.99994865D 00	0.13831169D-06
0.16000000D 00	0.99367985D 00	0.99654380D 00	0.99043979D 00	0.98624944D 00	0.56328269D-03
0.31000000D 00	0.97655363D 00	0.98711409D 00	0.96464286D 00	0.94944677D 00	0.40326134D-02
0.46000000D 00	0.94935384D 00	0.97194056D 00	0.92400811D 00	0.89240098D 00	0.12841325D-01
0.61000000D 00	0.91324136D 00	0.95140609D 00	0.87072111D 00	0.81934143D 00	0.28905254D-01
0.76000000D 00	0.86967801D 00	0.92601217D 00	0.80750108D 00	0.73530581D 00	0.53462911D-01
0.91000000D 00	0.82030621D 00	0.89635254D 00	0.73734978D 00	0.64553028D 00	0.87001915D-01
0.10600000D 01	0.76682684D 00	0.86308358D 00	0.66329948D 00	0.55489733D 00	0.12927380D 00
0.12100000D 01	0.71088942D 00	0.82689417D 00	0.58819055D 00	0.46752078D 00	0.17938540D 00
0.13600000D 01	0.65400412D 00	0.78847680D 00	0.51450049D 00	0.38650658D 00	0.23594531D 00
0.15100000D 01	0.59748110D 00	0.74850215D 00	0.44423529D 00	0.31388707D 00	0.29723911D 00
0.16600000D 01	0.54239707D 00	0.70759799D 00	0.37888319D 00	0.25069610D 00	0.36140781D 00
0.18100000D 01	0.48958646D 00	0.66633337D 00	0.31942260D 00	0.19713670D 00	0.42660850D 00
0.19600000D 01	0.43965157D 00	0.62520817D 00	0.26637083D 00	0.15279171D 00	0.49114319D 00
0.21100000D 01	0.39298599D 00	0.58464761D 00	0.21985841D 00	0.11683634D 00	0.55354904D 00
0.22600000D 01	0.34980546D 00	0.54500122D 00	0.17971489D 00	0.88224886D-01	0.61264987D 00
0.24100000D 01	0.31018157D 00	0.50654532D 00	0.14555470D 00	0.65837667D-01	0.66757342D 00
0.25600000D 01	0.27407466D 00	0.46948820D 00	0.11685483D 00	0.48584744D-01	0.71774153D 00
0.27100000D 01	0.24136375D 00	0.43397702D 00	0.93020020D-01	0.35470569D-01	0.76284150D 00
0.28600000D 01	0.21187231D 00	0.40010575D 00	0.73433415D-01	0.25627154D-01	0.80278633D 00
0.30100000D 01	0.18538929D 00	0.36792331D 00	0.57493307D-01	0.18324429D-01	0.83767031D 00

TABLE 16 b. PARTIALLY DEGENERATE STANDARD MODEL FUNCTIONS

$\eta_0 = 2$

ξ	$\lambda(\xi)$	$\frac{\rho}{\rho_c} = \left[\frac{U_{1/2}(\eta)U_{3/2}(\eta)}{U_{1/2}(\eta_0)U_{3/2}(\eta_0)}\right]$	$\frac{T}{T_c} = \left[\frac{U_{3/2}(\eta)}{U_{3/2}(\eta_0)}\right]^{2/3}$	$\frac{P_r}{P_{rc}} = \left[\frac{U_{3/2}(\eta)}{U_{3/2}(\eta_0)}\right]^{8/3}$	$\frac{M}{M_R} = \frac{\left[\xi^2 \frac{1}{\lambda(\xi)} U_{3/2}^{2/3} \lambda'(\xi)\right]}{\left[\xi^2 \frac{1}{\lambda(\xi)} U_{3/2}^{2/3} \lambda'(\xi)\right]_{\lambda \to 0}}$
0.31600000D 01	0.16168565D 00	0.33744170D 00	0.44637398D-01	0.12965707D-01	0.86772489D 00
0.33100000D 01	0.14052684D 00	0.30864344D 00	0.34356959D-01	0.90746397D-02	0.89327816D 00
0.34600000D 01	0.12168179D 00	0.28148834D 00	0.26203254D-01	0.62783117D-02	0.91471970D 00
0.36100000D 01	0.10492904D 00	0.25591944D 00	0.19788447D-01	0.42895752D-02	0.93247161D 00
0.37600000D 01	0.90060651D-01	0.23186790D 00	0.14782881D-01	0.28904386D-02	0.94696580D 00
0.39100000D 01	0.76884465D-01	0.20925725D 00	0.10910186D-01	0.19174465D-02	0.95862689D 00
0.40600000D 01	0.65225114D-01	0.18800662D 00	0.79413257D-02	0.12493777D-02	0.96785999D 00
0.42100000D 01	0.54924178D-01	0.16803343D 00	0.56883289D-02	0.79723058D-03	0.97504261D 00
0.43600000D 01	0.45839771D-01	0.14925538D 00	0.39981955D-02	0.49627364D-03	0.98051971D 00
0.45100000D 01	0.37845783D-01	0.13159198D 00	0.27472731D-02	0.29986021D-03	0.98460120D 00
0.46600000D 01	0.30830946D-01	0.11496558D 00	0.18362428D-02	0.17469180D-03	0.98756127D 00
0.48100000D 01	0.24697854D-01	0.99302179D-01	0.11857656D-02	0.97238070D-04	0.98963891D 00
0.49600000D 01	0.19362065D-01	0.84531862D-01	0.73277584D-03	0.51060253D-04	0.99103929D 00
0.51100000D 01	0.14751367D-01	0.70589077D-01	0.42737461D-03	0.24828542D-04	0.99193567D 00
0.52600000D 01	0.10805404D-01	0.57412751D-01	0.23025720D-03	0.10865123D-04	0.99247158D 00
0.54100000D 01	0.74758924D-02	0.44946289D-01	0.11060440D-03	0.40810936D-05	0.99276304D 00
0.55600000D 01	0.47280555D-02	0.33137489D-01	0.44367699D-04	0.12058120D-05	0.99290084D 00
0.57100000D 01	0.25449381D-02	0.21938418D-01	0.12884185D-04	0.23164472D-06	0.99295272D 00
0.58600000D 01	0.94090158D-03	0.11305259D-01	0.17641197D-05	0.16335156D-07	0.99296540D 00
0.60100000D 01	0.32464372D-04	0.11984375D-02	0.21021895D-08	0.20628269D-11	0.99368220D 00
0.60200000D 01	0.99530400D-05	0.54490009D-03	0.19759671D-09	0.88159420D-13	0.10000000D 01

TABLE 17 a. PARTIALLY DEGENERATE STANDARD MODEL FUNCTIONS

$\eta_0 = 5.$

J.	$\lambda(J)$	$\frac{\rho}{\rho_c} = \left[\frac{U_{1/2}(\eta)U_{3/2}(\eta)}{U_{1/2}(\eta_0)U_{3/2}(\eta_0)}\right]$	$\frac{T}{T_c} = \left[\frac{U_{3/2}(\eta)}{U_{3/2}(\eta_0)}\right]^{2/3}$	$\frac{P_r}{P_{rc}} = \left[\frac{U_{3/2}(\eta)}{U_{3/2}(\eta_0)}\right]^{8/3}$	$\frac{M}{M(R)} = \frac{\left[J^2 \frac{1}{\lambda(J)} U_{3/2}^{2/3} \lambda'(J)\right]}{\left[J^2 \frac{1}{\lambda(J)} U_{3/2}^{2/3} \lambda'(J)\right]}$
0.0	0.10000000D 01	0.10000000D 01	0.10000000D 01	0.10000000D 01	0.0
0.10000000D-01	0.99994408D 00	0.99997314D 00	0.99992794D 00	0.99989567D 00	0.41205299D-06
0.11000000D 00	0.99326255D 00	0.99675616D 00	0.99132186D 00	0.98709071D 00	0.54560662D-03
0.21000000D 00	0.97572505D 00	0.98824263D 00	0.96876149D 00	0.95379645D 00	0.37443103D-02
0.31000000D 00	0.94807069D 00	0.97460725D 00	0.93327609D 00	0.90223548D 00	0.11779307D-01
0.41000000D 00	0.91142806D 00	0.95612523D 00	0.88644179D 00	0.83571936D 00	0.26429369D-01
0.51000000D 00	0.86722249D 00	0.93316068D 00	0.83025655D 00	0.75827560D 00	0.48935455D-01
0.61000000D 00	0.81706618D 00	0.90615163D 00	0.76698913D 00	0.67422501D 00	0.79924479D-01
0.71000000D 00	0.76264453D 00	0.87559270D 00	0.69901904D 00	0.58777334D 00	0.11939514D 00
0.81000000D 00	0.70561096D 00	0.84201663D 00	0.62868389D 00	0.50267124D 00	0.16676213D 00
0.91000000D 00	0.64749962D 00	0.80597564D 00	0.55814770D 00	0.42197723D 00	0.22094828D 00
0.10100000D 01	0.58966140D 00	0.76802363D 00	0.48929998D 00	0.34793625D 00	0.28050950D 00
0.11100000D 01	0.53322498D 00	0.72870031D 00	0.42368991D 00	0.28196628D 00	0.34377569D 00
0.12100000D 01	0.47908117D 00	0.68851666D 00	0.36249565D 00	0.22472902D 00	0.40899106D 00
0.13100000D 01	0.42788676D 00	0.64794509D 00	0.30652574D 00	0.17626016D 00	0.47444138D 00
0.14100000D 01	0.38008282D 00	0.60741050D 00	0.25624510D 00	0.13612270D 00	0.53855724D 00
0.15100000D 01	0.33592223D 00	0.56728527D 00	0.21181917D 00	0.10356366D 00	0.59998917D 00
0.16100000D 01	0.29550178D 00	0.52788664D 00	0.17316797D 00	0.77654035D-01	0.65765326D 00
0.17100000D 01	0.25879513D 00	0.48947656D 00	0.14002377D 00	0.57402256D-01	0.71074932D 00
0.18100000D 01	0.22568388D 00	0.45226349D 00	0.11198671D 00	0.41837667D-01	0.75875608D 00
0.19100000D 01	0.19598520D 00	0.41640565D 00	0.88574711D-01	0.30065458D-01	0.80140909D 00
0.20100000D 01	0.16947505D 00	0.38201557D 00	0.69265608D-01	0.21297353D-01	0.83866685D 00

TABLE 17 a. PARTIALLY DEGENERATE STANDARD MODEL FUNCTIONS

$\Lambda_0 = 5$

ξ	$\theta(\xi)$	P/P_c	T/T_c	P_r/P_{rc}	$M/M(R)$
0.21100000D 01	0.14590690D 00	0.349165150D 00	0.53530361D-01	0.14863631D-01	0.87067090D 00
0.22100000D 01	0.12502615D 00	0.317891210D 00	0.40857686D-01	0.10212108D-01	0.89770397D 00
0.23100000D 01	0.10658078D 00	0.288201000D 00	0.30770823D-01	0.68989550D-02	0.92014989D 00
0.24100000D 01	0.90328793D-01	0.260077640D 00	0.22837677D-01	0.45752349D-02	0.93845732D 00
0.25100000D 01	0.76043045D-01	0.233485020D 00	0.16675681D-01	0.29719222D-02	0.95310874D 00
0.26100000D 01	0.63514168D-01	0.208372420D 00	0.11952708D-01	0.18852212D-02	0.96459517D 00
0.27100000D 01	0.52551955D-01	0.184678470D 00	0.83852229D-02	0.11632323D-02	0.97339647D 00
0.28100000D 01	0.42985744D-01	0.162334580D 00	0.57346743D-02	0.69445730D-03	0.97996684D 00
0.29100000D 01	0.34664097D-01	0.141267850D 00	0.38029044D-02	0.39826732D-03	0.98472482D 00
0.30100000D 01	0.27454104D-01	0.121403400D 00	0.24271650D-02	0.21723248D-03	0.98804697D 00
0.31100000D 01	0.21240521D-01	0.102666320D 00	0.14751473D-02	0.11109980D-03	0.99026440D 00
0.32100000D 01	0.15925002D-01	0.849830910D-01	0.84028892D-03	0.52159264D-04	0.99166164D 00
0.33100000D 01	0.11425685D-01	0.682827710D-01	0.43751131D-03	0.21739318D-04	0.99247699D 00
0.34100000D 01	0.76775873D-02	0.524977850D-01	0.19946052D-03	0.75956559D-05	0.99290406D 00
0.35100000D 01	0.46348093D-02	0.375645490D-01	0.73266318D-04	0.19911964D-05	0.99309397D 00
0.36100000D 01	0.22774956D-02	0.234239210D-01	0.17800790D-04	0.30105076D-06	0.99315795D 00
0.37100000D 01	0.63642622D-03	0.100216270D-01	0.13960499D-05	0.10086822D-07	0.99317081D 00
0.37800000D 01	0.21817236D-04	0.105794530D-02	0.16432789D-08	0.12527208D-11	0.10000001D 01

TABLE 18a. PARTIALLY DEGENERATE STELLAR MODEL FUNCTIONS

$\Lambda_0 = 10$

ξ	$\lambda(\xi)$	$\frac{\rho}{\rho_c} = \left[\frac{U_{1/2}(\Lambda)\,U_{3/2}(\Lambda)}{U_{1/2}(\Lambda_0)\,U_{3/2}(\Lambda_0)}\right]$	$\frac{T}{T_c} = \left[\frac{U_{3/2}(\Lambda)}{U_{3/2}(\Lambda_0)}\right]^{2/3}$	$\frac{P_r}{P_{rc}} = \left[\frac{U_{3/2}(\Lambda)}{U_{3/2}(\Lambda_0)}\right]^{8/3}$	$\frac{M}{M(R)} = \frac{\left[\xi^2 \frac{1}{\lambda(\xi)} U_{3/2}^{2/3}\,\lambda'(\xi)\right]}{\left[\xi^2 \frac{1}{\lambda(\xi)} U_{3/2}^{2/3}\,\lambda'(\xi)\right]_{\lambda\to 0}}$
0.0	0.10000000D 01	0.10000000D 01	0.10000000D 01	0.10000000D 01	0.0
0.50000000D-02	0.99997645D 00	0.99998986D 00	0.99997341D 00	0.99995946D 00	0.10004944D-06
0.70000000D-01	0.99539786D 00	0.99801567D 00	0.99480312D 00	0.99208628D 00	0.27368334D-03
0.13500000D 00	0.98301574D 00	0.99264329D 00	0.98081355D 00	0.97089631D 00	0.19465685D-02
0.20000000D 00	0.96319194D 00	0.98393826D 00	0.95839476D 00	0.93728443D 00	0.62424045D-02
0.26500000D 00	0.93649380D 00	0.97200581D 00	0.92816232D 00	0.89263817D 00	0.14246118D-01
0.33000000D 00	0.90366196D 00	0.95698819D 00	0.89092751D 00	0.83873803D 00	0.26849386D-01
0.39500000D 00	0.86556989D 00	0.93906120D 00	0.84765901D 00	0.77763465D 00	0.44709120D-01
0.46000000D 00	0.82317878D 00	0.91842990D 00	0.79943894D 00	0.71151504D 00	0.68221288D-01
0.52500000D 00	0.77749146D 00	0.89532387D 00	0.74741600D 00	0.64257036D 00	0.97511099D-01
0.59000000D 00	0.72950846D 00	0.86599198D 00	0.69275878D 00	0.57287654D 00	0.13243925D 00
0.65500000D 00	0.68018935D 00	0.84269701D 00	0.63661197D 00	0.50429641D 00	0.17262276D 00
0.72000000D 00	0.63042103D 00	0.81371029D 00	0.58005753D 00	0.43840883D 00	0.21746786D 00
0.78500000D 00	0.58099419D 00	0.78330636D 00	0.52408278D 00	0.37646681D 00	0.26621186D 00
0.85000000D 00	0.53258817D 00	0.75175813D 00	0.46955620D 00	0.31938360D 00	0.31797052D 00
0.91500000D 00	0.48576372D 00	0.71933223D 00	0.41721145D 00	0.26774303D 00	0.37178726D 00
0.98000000D 00	0.44096263D 00	0.68628510D 00	0.36763927D 00	0.22182904D 00	0.42668132D 00
0.10450000D 01	0.39851305D 00	0.65285951D 00	0.32128679D 00	0.18166827D 00	0.48169172D 00
0.11100000D 01	0.35863879D 00	0.61928181D 00	0.27846288D 00	0.14707994D 00	0.53591533D 00
0.11750000D 01	0.32147149D 00	0.58575973D 00	0.23934862D 00	0.11772758D 00	0.58853742D 00
0.12400000D 01	0.28706413D 00	0.55248094D 00	0.20401120D 00	0.93168558D-01	0.63885414D 00
0.13050000D 01	0.25540492D 00	0.51961219D 00	0.17242026D 00	0.72898319D-01	0.68628697D 00

TABLE 18 ℓ. PARTIALLY DEGENERATE STANDARD MODEL FUNCTIONS

$\Lambda_0 = 10.$

J	$\lambda(J)$	$\frac{\rho}{\rho_c} = \left[\frac{U_{1/2}(\eta)U_{3/2}(\eta)}{U_{1/2}(\eta_0)U_{3/2}(\eta_0)}\right]$	$\frac{T}{T_c} = \left[\frac{U_{3/2}(\eta)}{U_{3/2}(\eta_0)}\right]^{2/3}$	$\frac{Pr}{Pr_c} = \left[\frac{U_{3/2}(\eta)}{U_{3/2}(\eta_0)}\right]^{8/3}$	$\frac{M}{M(R)} = \frac{\left[J^2 \frac{1}{\lambda(J)} U_{3/2}^{2/3} \lambda'(J)\right]}{\left[J^2 \frac{1}{\lambda(J)} U_{3/2}^{2/3} \lambda'(J)\right]_{\lambda \to 0}}$
0.13700000D 01	0.22643080D 00	0.48729844D 00	0.14446503D 00	0.56387171D-01	0.73038870D 00
0.14350000D 01	0.20003986D 00	0.45566384D 00	0.11997179D 00	0.43110047D-01	0.77084425D 00
0.15000000D 01	0.17610235D 00	0.42481168D 00	0.98720313D-01	0.32567622D-01	0.80746419D 00
0.15650000D 01	0.15447026D 00	0.39482547D 00	0.80458886D-01	0.24300820D-01	0.84017463D 00
0.16300000D 01	0.13498510D 00	0.36577015D 00	0.64917603D-01	0.17899188D-01	0.86900384D 00
0.16950000D 01	0.11748434D 00	0.33769348D 00	0.51819555D-01	0.13004422D-01	0.89406696D 00
0.17600000D 01	0.10180630D 00	0.31062763D 00	0.40889944D-01	0.93102356D-02	0.91554985D 00
0.18250000D 01	0.87793924D-01	0.28459079D 00	0.31863181D-01	0.65596970D-02	0.93369292D 00
0.18900000D 01	0.75297409D-01	0.25958893D 00	0.24488106D-01	0.45409332D-02	0.94877579D 00
0.19550000D 01	0.64176041D-01	0.23561742D 00	0.18531557D-01	0.30819813D-02	0.96110313D 00
0.20200000D 01	0.54299311D-01	0.21266267D 00	0.13780526D-01	0.20453401D-02	0.97099208D 00
0.20850000D 01	0.45547513D-01	0.19070375D 00	0.10043154D-01	0.13226272D-02	0.97876145D 00
0.21500000D 01	0.37811944D-01	0.16971379D 00	0.71488139D-02	0.82960063D-03	0.98472273D 00
0.22150000D 01	0.30994845D-01	0.14966134D 00	0.49474937D-02	0.50169429D-03	0.98917279D 00
0.22800000D 01	0.25009165D-01	0.13051163D 00	0.33086963D-02	0.29013323D-03	0.99238832D 00
0.23450000D 01	0.19778285D-01	0.11222760D 00	0.21200079D-02	0.15863513D-03	0.99462162D 00
0.24100000D 01	0.15235771D-01	0.94770927D-01	0.12854852D-02	0.80667993D-04	0.99609773D 00
0.24750000D 01	0.11325328D-01	0.78102838D-01	0.72396434D-03	0.37210719D-04	0.99701264D 00
0.25400000D 01	0.80011517D-02	0.62184854D-01	0.36737884D-03	0.14953379D-04	0.99753238D 00
0.26050000D 01	0.52291735D-02	0.46979433D-01	0.15914373D-03	0.48711559D-05	0.99779279D 00
0.26700000D 01	0.29903611D-02	0.32450562D-01	0.52648869D-04	0.11088939D-05	0.99789995D 00
0.27350000D 01	0.12900915D-02	0.18564374D-01	0.98865192D-05	0.11877431D-06	0.99793092D 00
0.28000000D 01	0.19586497D-03	0.52900809D-02	0.22920478D-06	0.78316068D-09	0.99794078D 00
0.28250000D 01	0.38683470D-05	0.38660995D-03	0.89495844D-10	0.22340585D-13	0.10000000D 01

TABLE 19a. PARTIALLY DEGENERATE STANDARD MODEL FUNCTIONS

$\Lambda_0 = 15$

η	$\vartheta(\eta)$	$\frac{\rho}{\rho_c} = \frac{[U_{1/2}(\eta)\, U_{3/2}(\eta)]}{[U_{1/2}(\eta_0)\, U_{3/2}(\eta_0)]}$	$\frac{T}{T_c} = \left[\frac{U_{3/2}(\eta)}{U_{3/2}(\eta_0)}\right]^{2/3}$	$\frac{P_r}{P_{r_c}} = \left[\frac{U_{3/2}(\eta)}{U_{3/2}(\eta_0)}\right]^{8/3}$	$\frac{M}{M(R)} = \frac{\left[\eta^2 \frac{1}{\vartheta(\eta)} U_{3/2}^{2/3}\, \vartheta'(\eta)\right]}{\left[\eta^2 \frac{1}{\vartheta(\eta)} U_{3/2}^{2/3}\, \vartheta'(\eta)\right]_{\vartheta \to 0}}$
0.0	0.10000000D 01	0.10000000D 01	0.10000000D 01	0.10000000D 01	0.0
0.50000000D-02	0.99996910D 00	0.99998755D 00	0.99996771D 00	0.99995021D 00	0.13754973D-06
0.45000000D-01	0.99750119D 00	0.99899230D 00	0.99738820D 00	0.99597527D 00	0.10011850D-03
0.85000000D-01	0.99111941D 00	0.99640995D 00	0.99071360D 00	0.98571697D 00	0.67202437D-03
0.12500000D 00	0.98091559D 00	0.99225467D 00	0.98002885D 00	0.96937678D 00	0.21234258D-02
0.16500000D 00	0.96703507D 00	0.98654911D 00	0.96546896D 00	0.94727230D 00	0.48402735D-02
0.20500000D 00	0.94967261D 00	0.97932419D 00	0.94721607D 00	0.91982653D 00	0.91776080D-02
0.24500000D 00	0.92906699D 00	0.97061875D 00	0.92549539D 00	0.88755386D 00	0.15451238D-01
0.28500000D 00	0.90549464D 00	0.96047912D 00	0.90057036D 00	0.85104342D 00	0.23930633D-01
0.32500000D 00	0.87926236D 00	0.94895858D 00	0.87273707D 00	0.81094061D 00	0.34833225D-01
0.36500000D 00	0.85069971D 00	0.93611684D 00	0.84231816D 00	0.76792755D 00	0.48320260D-01
0.40500000D 00	0.82015107D 00	0.92201935D 00	0.80965636D 00	0.72270351D 00	0.64494282D-01
0.44500000D 00	0.78796797D 00	0.90673665D 00	0.77510783D 00	0.67596577D 00	0.83398268D-01
0.48500000D 00	0.75450159D 00	0.89034365D 00	0.73903563D 00	0.62839207D 00	0.10501637D 00
0.52500000D 00	0.72009607D 00	0.87291889D 00	0.70180324D 00	0.58062484D 00	0.12927619D 00
0.56500000D 00	0.68508240D 00	0.85454378D 00	0.66376857D 00	0.53325794D 00	0.15605241D 00
0.60500000D 00	0.64977329D 00	0.83530186D 00	0.62527841D 00	0.48682610D 00	0.18517162D 00
0.64500000D 00	0.61445906D 00	0.81527804D 00	0.58666345D 00	0.44179728D 00	0.21641818D 00
0.68500000D 00	0.57940438D 00	0.79455787D 00	0.54823402D 00	0.39856781D 00	0.24954075D 00
0.72500000D 00	0.54484616D 00	0.77322688D 00	0.51027661D 00	0.35746034D 00	0.28425944D 00
0.76500000D 00	0.51099226D 00	0.75136987D 00	0.47305106D 00	0.31872432D 00	0.32027312D 00
0.80500000D 00	0.47802110D 00	0.72907032D 00	0.43678865D 00	0.28253860D 00	0.35726687D 00
0.84500000D 00	0.44608200D 00	0.70640982D 00	0.40169081D 00	0.24901587D 00	0.39491926D 00

TABLE 19B. PARTIALLY DEGENERATE STANDARD MODEL FUNCTIONS

$\eta_0 = 15.$

ξ	$\lambda(\xi)$	$\frac{\rho}{\rho_c} = \left[\frac{U_{1/2}(\eta)U_{3/2}(\eta)}{U_{1/2}(\eta_0)U_{3/2}(\eta_0)}\right]$	$\frac{T}{T_c} = \left[\frac{U_{3/2}(\eta)}{U_{3/2}(\eta_0)}\right]^{2/3}$	$\frac{P_r}{P_{r_c}} = \left[\frac{U_{3/2}(\eta)}{U_{3/2}(\eta_0)}\right]^{8/3}$	$\frac{M}{M(R)} = \frac{\left[\xi^2 \frac{1}{\lambda(\xi)} U_{3/2}^{2/3} \lambda'(\xi)\right]}{\left[\xi^2 \frac{1}{\lambda(\xi)} U_{3/2}^{2/3} \lambda'(\xi)\right]_{\lambda \to 0}}$
0.88500000D 00	0.41529613D 00	0.68346753D 00	0.36792865D 00	0.21820851D 00	0.43290919D 00
0.92500000D 00	0.38575804D 00	0.66031976D 00	0.33564306D 00	0.19011540D 00	0.47092233D 00
0.96500000D 00	0.35753751D 00	0.63703951D 00	0.30494543D 00	0.16468938D 00	0.50865682D 00
0.10050000D 01	0.33068169D 00	0.61369618D 00	0.27591890D 00	0.14184494D 00	0.54582831D 00
0.10450000D 01	0.30521746D 00	0.59035528D 00	0.24861992D 00	0.12146579D 00	0.58217413D 00
0.10850000D 01	0.28115381D 00	0.56707817D 00	0.22308026D 00	0.10341223D 00	0.61745667D 00
0.11250000D 01	0.25848425D 00	0.54392193D 00	0.19930920D 00	0.87527891D-01	0.65146588D 00
0.11650000D 01	0.23718914D 00	0.52093921D 00	0.17729587D 00	0.73645869D-01	0.68402099D 00
0.12050000D 01	0.21723793D 00	0.49817821D 00	0.15701173D 00	0.61594102D-01	0.71497141D 00
0.12450000D 01	0.19859127D 00	0.47568264D 00	0.13841301D 00	0.51199937D-01	0.74419695D 00
0.12850000D 01	0.18120285D 00	0.45349186D 00	0.12144320D 00	0.42293950D-01	0.77160756D 00
0.13250000D 01	0.16502116D 00	0.43164051D 00	0.10603531D 00	0.34712753D-01	0.79714151D 00
0.13650000D 01	0.14999100D 00	0.41015929D 00	0.92114189D-01	0.28301567D-01	0.82076515D 00
0.14050000D 01	0.13605480D 00	0.38907464D 00	0.79598559D-01	0.22915641D-01	0.84247023D 00
0.14450000D 01	0.12315378D 00	0.36840903D 00	0.68402888D-01	0.18421346D-01	0.86227185D 00
0.14850000D 01	0.11122887D 00	0.34818111D 00	0.58439113D-01	0.14696745D-01	0.88020619D 00
0.15250000D 01	0.10022156D 00	0.32840596D 00	0.49618131D-01	0.11631735D-01	0.89632789D 00
0.15650000D 01	0.90074486D-01	0.30909530D 00	0.41851109D-01	0.91278804D-02	0.91070749D 00
0.16050000D 01	0.80732017D-01	0.29025766D 00	0.35050586D-01	0.70979863D-02	0.92342883D 00
0.16450000D 01	0.72140603D-01	0.27189872D 00	0.29131391D-01	0.54654898D-02	0.93458650D 00
0.16850000D 01	0.64249092D-01	0.25402148D 00	0.24011392D-01	0.41637264D-02	0.94428337D 00
0.17250000D 01	0.57008947D-01	0.23662650D 00	0.19612072D-01	0.31351187D-02	0.95262826D 00
0.17650000D 01	0.50374383D-01	0.21971220D 00	0.15858968D-01	0.23303285D-02	0.95973374D 00
0.18050000D 01	0.44302449D-01	0.20327503D 00	0.12681968D-01	0.17074054D-02	0.96571416D 00

TABLE 19γ. PARTIALLY DEGENERATE STANDARD MODEL FUNCTIONS

$\Lambda_0 = 15$

$\mathcal{J}$	$\lambda(\mathcal{J})$	$\frac{\rho}{\rho_c} = \left[\frac{u_{1/2}(\Lambda)\,u_{3/2}(\Lambda)}{u_{1/2}(\Lambda_0)\,u_{3/2}(\Lambda_0)}\right]$	$\frac{T}{T_c} = \left[\frac{u_{3/2}(\Lambda)}{u_{3/2}(\Lambda_0)}\right]^{2/3}$	$\frac{P_r}{P_{r_c}} = \left[\frac{u_{3/2}(\Lambda)}{u_{3/2}(\Lambda_0)}\right]^{8/3}$	$\frac{M}{M(R)} = \frac{\left[\mathcal{J}^2 \frac{1}{\lambda(\mathcal{J})} u_{3/2}^{2/3} \lambda'(\mathcal{J})\right]}{\left[\mathcal{J}^2 \frac{1}{\lambda(\mathcal{J})} u_{3/2}^{2/3} \lambda'(\mathcal{J})\right]_{\lambda \to 0}}$
0.18450000D 01	0.38753071D-01	0.18730977D 00	0.10015502D-01	0.12309551D-02	0.97068380D 00
0.18850000D 01	0.33689048D-01	0.17180969D 00	0.77986322D-02	0.87134706D-03	0.97475524D 00
0.19250000D 01	0.29076030D-01	0.15676683D 00	0.59750544D-02	0.60397268D-03	0.97803793D 00
0.19650000D 01	0.24882481D-01	0.14217217D 00	0.44930363D-02	0.40856295D-03	0.98063695D 00
0.20050000D 01	0.21079636D-01	0.12801584D 00	0.33052975D-02	0.26856875D-03	0.98265192D 00
0.20450000D 01	0.17641463D-01	0.11428731D 00	0.23688463D-02	0.17060538D-03	0.98417611D 00
0.20850000D 01	0.14544642D-01	0.10097557D 00	0.16447852D-02	0.10395992D-03	0.98529574D 00
0.21250000D 01	0.11768592D-01	0.88069287D-01	0.10980922D-02	0.60158734D-04	0.98608936D 00
0.21650000D 01	0.92955513D-02	0.75556982D-01	0.69738940D-03	0.32591114D-04	0.98662740D 00
0.22050000D 01	0.71107785D-02	0.63427158D-01	0.41470300D-03	0.16184596D-04	0.98697190D 00
0.22450000D 01	0.52029594D-02	0.51668465D-01	0.22522310D-03	0.71269409D-05	0.98717625D 00
0.22850000D 01	0.35650109D-02	0.40269839D-01	0.10706671D-03	0.26297869D-05	0.98728512D 00
0.23250000D 01	0.21957649D-02	0.29220672D-01	0.41048935D-04	0.72905773D-06	0.98733441D 00
0.23650000D 01	0.11039669D-02	0.18511035D-01	0.10465324D-04	0.11741513D-06	0.98735138D 00
0.24050000D 01	0.32078481D-03	0.81320555D-02	0.88910818D-06	0.43732351D-08	0.98735674D 00
0.24350000D 01	0.62570531D-05	0.58958385D-03	0.33911854D-09	0.12083274D-12	0.10000000D 01

TABLE 20α. PARTIALLY DEGENERATE STANDARD MODEL FUNCTIONS

$\Lambda_0 = 25$

η	$\vartheta(\eta)$	$\frac{\rho}{\rho_c} = \left[\frac{U_{1/2}(\eta)\,U_{3/2}(\eta)}{U_{1/2}(\eta_0)\,U_{3/2}(\eta_0)}\right]$	$\frac{T}{T_c} = \left[\frac{U_{3/2}(\eta)}{U_{3/2}(\eta_0)}\right]^{2/3}$	$\frac{P_r}{P_{r_c}} = \left[\frac{U_{3/2}(\eta)}{U_{3/2}(\eta_0)}\right]^{8/3}$	$\frac{M}{M(R)} = \frac{\left[\eta^2 \frac{1}{\vartheta(\eta)} U_{3/2}^{2/3}\,\vartheta'(\eta)\right]}{\left[\eta^2 \frac{1}{\vartheta(\eta)} U_{3/2}^{2/3}\,\vartheta'(\eta)\right]_{\eta\to 0}}$
0.0	0.10000000D 01	0.10000000D 01	0.10000000D 01	0.10000000D 01	0.0
0.50000000D-02	0.99995781D 00	0.99998436D 00	0.99995992D 00	0.99993744D 00	0.20010976D-06
0.55000000D-01	0.99491039D 00	0.99810959D 00	0.99516243D 00	0.99245978D 00	0.26557885D-03
0.10500000D 00	0.98159925D 00	0.99312908D 00	0.98248312D 00	0.97279827D 00	0.18337332D-02
0.15500000D 00	0.96042107D 00	0.98509387D 00	0.96222818D 00	0.94169546D 00	0.58257506D-02
0.20500000D 00	0.93199425D 00	0.97408572D 00	0.93488043D 00	0.90030304D 00	0.13247960D-01
0.25500000D 00	0.89712395D 00	0.96021530D 00	0.90107940D 00	0.85010879D 00	0.24946134D-01
0.30500000D 00	0.85675835D 00	0.94361995D 00	0.86159538D 00	0.79284534D 00	0.41569110D-01
0.35500000D 00	0.81193977D 00	0.92446086D 00	0.81729923D 00	0.73038884D 00	0.63543703D-01
0.40500000D 00	0.76375457D 00	0.90291983D 00	0.76912949D 00	0.66465581D 00	0.91061936D-01
0.45500000D 00	0.71328539D 00	0.87919584D 00	0.71805877D 00	0.59750637D 00	0.12408074D 00
0.50500000D 00	0.66156889D 00	0.85350124D 00	0.66506102D 00	0.53066040D 00	0.16233350D 00
0.55500000D 00	0.60956099D 00	0.82605802D 00	0.61108126D 00	0.46563141D 00	0.20535195D 00
0.60500000D 00	0.55811093D 00	0.79709399D 00	0.55700908D 00	0.40368091D 00	0.25249667D 00
0.65500000D 00	0.50794459D 00	0.76683912D 00	0.50365672D 00	0.34579373D 00	0.30299354D 00
0.70500000D 00	0.45965639D 00	0.73552204D 00	0.45174239D 00	0.29267314D 00	0.35597401D 00
0.75500000D 00	0.41370895D 00	0.70336688D 00	0.40187896D 00	0.24475286D 00	0.41051639D 00
0.80500000D 00	0.37043899D 00	0.67059033D 00	0.35456792D 00	0.20222240D 00	0.46568608D 00
0.85500000D 00	0.33006791D 00	0.63739915D 00	0.31019820D 00	0.16506161D 00	0.52057263D 00
0.90500000D 00	0.29271543D 00	0.60398810D 00	0.26904924D 00	0.13308027D 00	0.57432223D 00
0.95500000D 00	0.25841492D 00	0.57053816D 00	0.23129758D 00	0.10595927D 00	0.62616444D 00

TABLE 20 b. PARTIALLY DEGENERATE STANDARD MODEL FUNCTIONS

$\Lambda_0 = 25.$

ξ	$\lambda(\xi)$	$\frac{\rho}{\rho_c} = \left[\frac{U_{1/2}(\eta)\,U_{3/2}(\eta)}{U_{1/2}(\eta_0)\,U_{3/2}(\eta_0)}\right]$	$\frac{T}{T_c} = \left[\frac{U_{3/2}(\eta)}{U_{3/2}(\eta_0)}\right]^{2/3}$	$\frac{P_r}{P_{rc}} = \left[\frac{U_{3/2}(\eta)}{U_{3/2}(\eta_0)}\right]^{8/3}$	$\frac{M}{M(R)} = \frac{\left[\xi^2 \frac{1}{\lambda(\xi)} U_{3/2}^{2/3} \lambda'(\xi)\right]}{\left[\xi^2 \frac{1}{\lambda(\xi)} U_{3/2}^{2/3} \lambda'(\xi)\right]_{\lambda\to 0}}$
0.10050000D 01	0.22712921D 00	0.53721530D 00	0.19702609D 00	0.83290155D-01	0.67543251D 00
0.10550000D 01	0.19876583D 00	0.50416954D 00	0.16623516D 00	0.64611023D-01	0.72157729D 00
0.11050000D 01	0.17319119D 00	0.47153446D 00	0.13885484D 00	0.49437213D-01	0.76417472D 00
0.11550000D 01	0.15024304D 00	0.43942702D 00	0.11475754D 00	0.37286127D-01	0.80292750D 00
0.12050000D 01	0.12974122D 00	0.40794768D 00	0.93770387D-01	0.27696070D-01	0.83766172D 00
0.12550000D 01	0.11149655D 00	0.37718086D 00	0.75686995D-01	0.20239459D-01	0.86831920D 00
0.13050000D 01	0.95317846D-01	0.34719567D 00	0.60278224D-01	0.14531068D-01	0.89494675D 00
0.13550000D 01	0.81017445D-01	0.31804640D 00	0.47301613D-01	0.10232042D-01	0.91768200D 00
0.14050000D 01	0.68415187D-01	0.28977412D 00	0.36509585D-01	0.70508060D-02	0.93673992D 00
0.14550000D 01	0.57341263D-01	0.26240743D 00	0.27656118D-01	0.47413820D-02	0.95239633D 00
0.15050000D 01	0.47638053D-01	0.23596384D 00	0.20502110D-01	0.31001468D-02	0.96497249D 00
0.15550000D 01	0.39161196D-01	0.21045103D 00	0.14819448D-01	0.19615739D-02	0.97481999D 00
0.16050000D 01	0.31780089D-01	0.18586823D 00	0.10393933D-01	0.11934966D-02	0.98230638D 00
0.16550000D 01	0.25377976D-01	0.16220760D 00	0.70272032D-02	0.69228569D-03	0.98780206D 00
0.17050000D 01	0.19851801D-01	0.13945554D 00	0.45378303D-02	0.37821927D-03	0.99166842D 00
0.17550000D 01	0.15111986D-01	0.11759405D 00	0.27617407D-02	0.19122384D-03	0.99424745D 00
0.18050000D 01	0.11082324D-01	0.96602053D-01	0.15521326D-02	0.87085559D-04	0.99585276D 00
0.18550000D 01	0.77002728D-02	0.76456656D-01	0.77902922D-03	0.34171340D-04	0.99676210D 00
0.19050000D 01	0.49182238D-02	0.57134438D-01	0.32859799D-03	0.10655963D-04	0.99721134D 00
0.19550000D 01	0.27072178D-02	0.38612807D-01	0.10234506D-03	0.22229337D-05	0.99738983D 00
0.20050000D 01	0.10683084D-02	0.20871718D-01	0.16276541D-04	0.18977280D-06	0.99743732D 00
0.20550000D 01	0.85823063D-04	0.38971200D-02	0.10640936D-06	0.23066242D-09	0.99771264D 00
0.20650000D 01	0.60049959D-05	0.66188879D-03	0.52150258D-09	0.19192979D-12	0.10000000D 01

TABLE 21 α . PARTIALLY DEGENERATE STANDARD MODEL FUNCTIONS

$\Lambda_0 = 50$

y	$\lambda(y)$	$\frac{\rho}{\rho_c}$ $\left[\frac{U_{1/2}(\Lambda)\,U_{3/2}(\Lambda)}{U_{1/2}(\Lambda_0)\,U_{3/2}(\Lambda_0)}\right]$	$\frac{T}{T_c}$ $\left[\frac{U_{3/2}(\Lambda)}{U_{3/2}(\Lambda_0)}\right]^{2/3}$	$\frac{P}{P_c} = \left[\frac{U_{3/2}(\Lambda)}{U_{3/2}(\Lambda_0)}\right]^{8/3}$	$\frac{M}{M(R)} = \frac{\left[y^2 \frac{1}{\lambda(y)} U_{3/2}^{2/3} \lambda'(y)\right]}{\left[y^2 \frac{1}{\lambda(y)} U_{3/2}^{2/3} \lambda'(y)\right]}$
0.0	0.10000000D 01	0.10000000D 01	0.10000000D 01	0.10000000D 01	0.0
0.40000000D-02	0.99996068D 00	0.99998695D 00	0.99996700D 00	0.99994789D 00	0.15632257D-06
0.36000000D-01	0.99682106D 00	0.99894357D 00	0.99733031D 00	0.99578098D 00	0.11377880D-03
0.68000000D-01	0.98871196D 00	0.99623611D 00	0.99050735D 00	0.98502923D 00	0.76364771D-03
0.10000000D 00	0.97577531D 00	0.99187872D 00	0.97958393D 00	0.96790846D 00	0.24125798D-02
0.13200000D 00	0.95823491D 00	0.98589406D 00	0.96469650D 00	0.94475892D 00	0.54982681D-02
0.16400000D 00	0.93638906D 00	0.97831309D 00	0.94602924D 00	0.91603373D 00	0.10422474D-01
0.19600000D 00	0.91160085D 00	0.96917471D 00	0.92381011D 00	0.88228381D 00	0.17541336D-01
0.22800000D 00	0.88128675D 00	0.95852535D 00	0.89830604D 00	0.84413991D 00	0.27157070D-01
0.26000000D 00	0.84890395D 00	0.94641947D 00	0.86981747D 00	0.80229272D 00	0.39511305D-01
0.29200000D 00	0.81393715D 00	0.93291401D 00	0.83867226D 00	0.75747182D 00	0.54780209D-01
0.32400000D 00	0.77688519D 00	0.91807777D 00	0.80521929D 00	0.71042447D 00	0.73071503D-01
0.35600000D 00	0.73824834D 00	0.90198076D 00	0.76982188D 00	0.66189504D 00	0.94423404D-01
0.38800000D 00	0.69851644D 00	0.88459848D 00	0.73285115D 00	0.61260589D 00	0.11880545D 00
0.42000000D 00	0.65815846D 00	0.86631123D 00	0.69467964D 00	0.56324043D 00	0.14612111D 00
0.45200000D 00	0.61761375D 00	0.84689833D 00	0.65567516D 00	0.51442865D 00	0.17621203D 00

TABLE 21 b. PARTIALLY DEGENERATE STANDARD MODEL FUNCTIONS

$\Lambda_0 = 50$

ξ	$\theta(\xi)$	$\frac{\rho}{\rho_c} = \left[\frac{U_{3/2}(\eta)\,U_{1/2}(\eta)}{U_{3/2}(\eta_0)\,U_{1/2}(\eta_0)}\right]$	$\frac{T}{T_c} = \left[\frac{U_{3/2}(\eta)}{U_{3/2}(\eta_0)}\right]^{2/3}$	$\frac{P}{P_c} = \left[\frac{U_{3/2}(\eta)}{U_{3/2}(\eta_0)}\right]^{8/3}$	$\frac{M}{M(R)} = \frac{\left[\xi^2 \frac{1}{\theta(\xi)} U_{3/2}^{2/3}\, \theta'(\xi)\right]}{\left[\xi^2 \frac{1}{\theta(\xi)} U_{3/2}^{2/3}\, \theta'(\xi)\right]_{\theta\to 0}}$
0.484000000D 00	0.577284930D 00	0.826547390D 00	0.616195140D 00	0.466735790D 00	0.208863680D 00
0.516000000D 00	0.537532810D 00	0.805343620D 00	0.576581590D 00	0.420653940D 00	0.243812240D 00
0.548000000D 00	0.498673080D 00	0.783374020D 00	0.537156640D 00	0.376596900D 00	0.280752370D 00
0.580000000D 00	0.460974620D 00	0.760725770D 00	0.498218840D 00	0.334898040D 00	0.319345610D 00
0.612000000D 00	0.424659520D 00	0.737485550D 00	0.460040260D 00	0.295810810D 00	0.359229240D 00
0.644000000D 00	0.389904290D 00	0.713738700D 00	0.422864220D 00	0.259511790D 00	0.400025130D 00
0.676000000D 00	0.356842220D 00	0.689569110D 00	0.386903940D 00	0.226105600D 00	0.441348530D 00
0.708000000D 00	0.325566520D 00	0.665058230D 00	0.352341760D 00	0.195631510D 00	0.482816430D 00
0.740000000D 00	0.295134120D 00	0.640284760D 00	0.319329080D 00	0.168071010D 00	0.524055330D 00
0.772000000D 00	0.268569770D 00	0.615324270D 00	0.287986920D 00	0.143356150D 00	0.564708380D 00
0.804000000D 00	0.242870350D 00	0.590248760D 00	0.258406920D 00	0.121378150D 00	0.604441500D 00
0.836000000D 00	0.219009160D 00	0.565126370D 00	0.230652920D 00	0.101995840D 00	0.642948690D 00
0.868000000D 00	0.196940080D 00	0.540021120D 00	0.204762710D 00	0.850439000D-01	0.679956260D 00
0.900000000D 00	0.176550150D 00	0.514992720D 00	0.180750250D 00	0.703403550D-01	0.715226070D 00
0.932000000D 00	0.157919910D 00	0.490096410D 00	0.158607890D 00	0.576934230D-01	0.748557800D 00
0.964000000D 00	0.140813160D 00	0.465382890D 00	0.138308890D 00	0.469074560D-01	0.779790170D 00
0.996000000D 00	0.125193270D 00	0.440898240D 00	0.119809860D 00	0.377879840D-01	0.808801330D 00
0.102800000D 01	0.110968940D 00	0.416683970D 00	0.103053210D 00	0.301458440D-01	0.835508430D 00
0.106000000D 01	0.980476370D-01	0.392777020D 00	0.879696020D-01	0.238004180D-01	0.859866400D 00
0.109200000D 01	0.863372530D-01	0.369209990D 00	0.744801780D-01	0.185820510D-01	0.881866120D 00
0.112400000D 01	0.757475520D-01	0.346010760D 00	0.624987540D-01	0.143337140D-01	0.901532090D 00

TABLE 21 γ. PARTIALLY DEGENERATE STANDARD MODEL FUNCTIONS

$\Lambda_0 = 50$

ξ	$\theta(\xi)$	$\frac{\rho}{\rho_c} = \left[\frac{U_{1/2}(\eta)\,U_{3/2}(\eta)}{U_{1/2}(\eta_0)\,U_{3/2}(\eta_0)}\right]$	$\frac{T}{T_c} = \left[\frac{U_{3/2}(\eta)}{U_{3/2}(\eta_0)}\right]^{2/3}$	$\frac{P}{P_c} = \left[\frac{U_{3/2}(\eta)}{U_{3/2}(\eta_0)}\right]^{8/3}$	$\frac{M}{M(R)} = \frac{\left[\xi^2 \frac{1}{\theta(\xi)} U_{3/2}^{2/3} \theta'(\xi)\right]}{\left[\xi^2 \frac{1}{\theta(\xi)} U_{3/2}^{2/3} \theta'(\xi)\right]_{\theta \to 0}}$
0.115600000D 01	0.661912380D-01	0.323203261D 00	0.519337660D-01	0.109120200D-01	0.918919560D 00
0.118800000D 01	0.575847810D-01	0.300808490D 00	0.426900610D-01	0.818767740D-02	0.934111490D 00
0.122000000D 01	0.498490210D-01	0.278841770D 00	0.346704760D-01	0.604549340D-02	0.947215300D 00
0.125200000D 01	0.429095820D-01	0.257316140D 00	0.277771880D-01	0.438398000D-02	0.958359280D 00
0.128400000D 01	0.366971230D-01	0.236241320D 00	0.219129110D-01	0.311475510D-02	0.967685630D 00
0.131600000D 01	0.311474740D-01	0.215624090D 00	0.169818410D-01	0.216167110D-02	0.975355440D 00
0.134800000D 01	0.262016710D-01	0.195468720D 00	0.128904320D-01	0.145985460D-02	0.981535350D 00
0.138000000D 01	0.218059320D-01	0.175777340D 00	0.954799970D-02	0.954667310D-03	0.986398790D 00
0.141200000D 01	0.179115890D-01	0.156550340D 00	0.686716200D-02	0.600643240D-03	0.990121360D 00
0.144400000D 01	0.144750140D-01	0.137786830D 00	0.476415090D-02	0.360438750D-03	0.992877150D 00
0.147600000D 01	0.114575700D-01	0.119485120D 00	0.315901450D-02	0.203824260D-03	0.994835310D 00
0.150800000D 01	0.882563200D-02	0.101643170D 00	0.197574180D-02	0.106736640D-03	0.996156530D 00
0.154000000D 01	0.655075660D-02	0.842592240D-01	0.114234550D-02	0.504046480D-04	0.996989750D 00
0.157200000D 01	0.461013500D-02	0.673323490D-01	0.590940720D-03	0.205539790D-04	0.997468920D 00
0.160400000D 01	0.298765570D-02	0.508631510D-01	0.257860240D-03	0.669289420D-05	0.997709970D 00
0.163600000D 01	0.167641420D-02	0.348546290D-01	0.838498180D-04	0.147585210D-05	0.997808050D 00
0.166800000D 01	0.685640740D-03	0.193134820D-01	0.143850050D-04	0.139137280D-06	0.997835250D 00
0.170000000D 01	0.704789240D-04	0.425331180D-02	0.154465310D-06	0.327273300D-09	0.998134540D 00
0.170400000D 01	0.299986240D-04	0.240727330D-02	0.280143460D-07	0.335817610D-10	0.999999960D 00

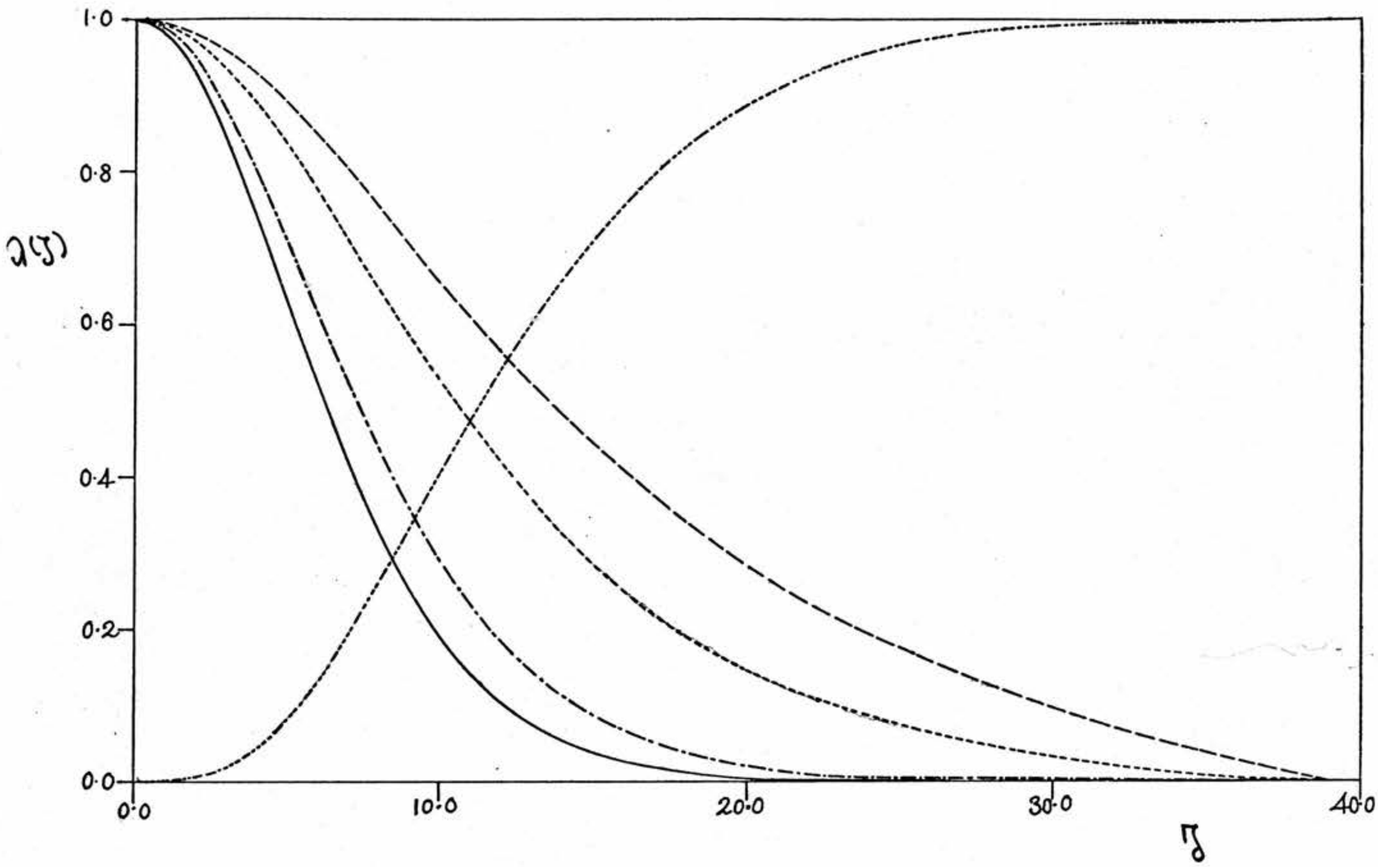


Figure 3

The Partially Degenerate Standard Model functions for $n_0 = 0.1$

- ·········· = $M/M(R)$
- ———— = $P_r/P_{r.c.}$
- —·—·— = ρ/ρ_c
- - - - - - = T/T_c
- — — — = $\theta(\zeta)$

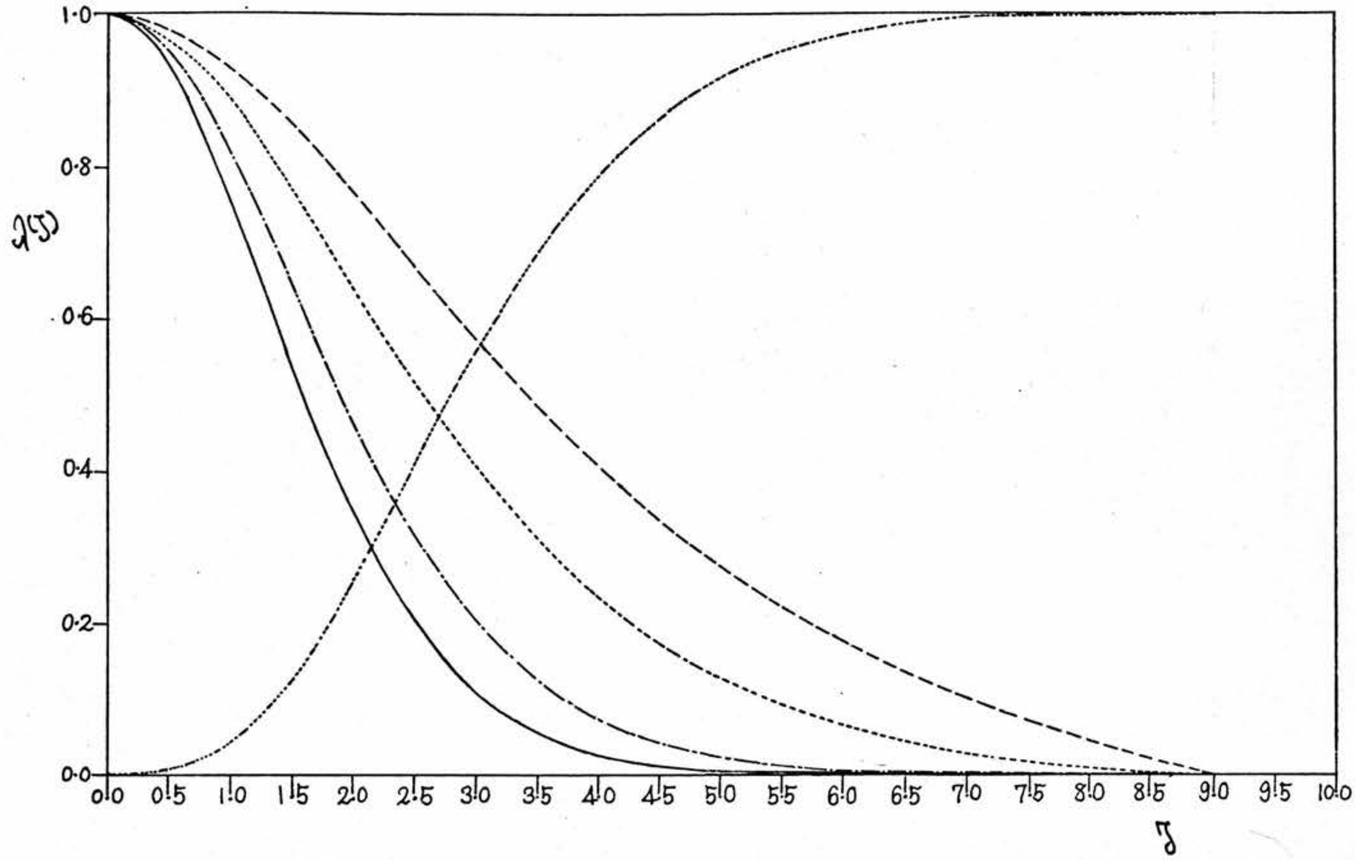


Figure 4.

The partially degenerate standard model characteristic functions for $\Lambda_0 = 1$.

- $M/M(R)$
- $\rho_r/\rho_{r.c.}$
- ρ/ρ_c
- T/T_c
- $\vartheta(\zeta)$

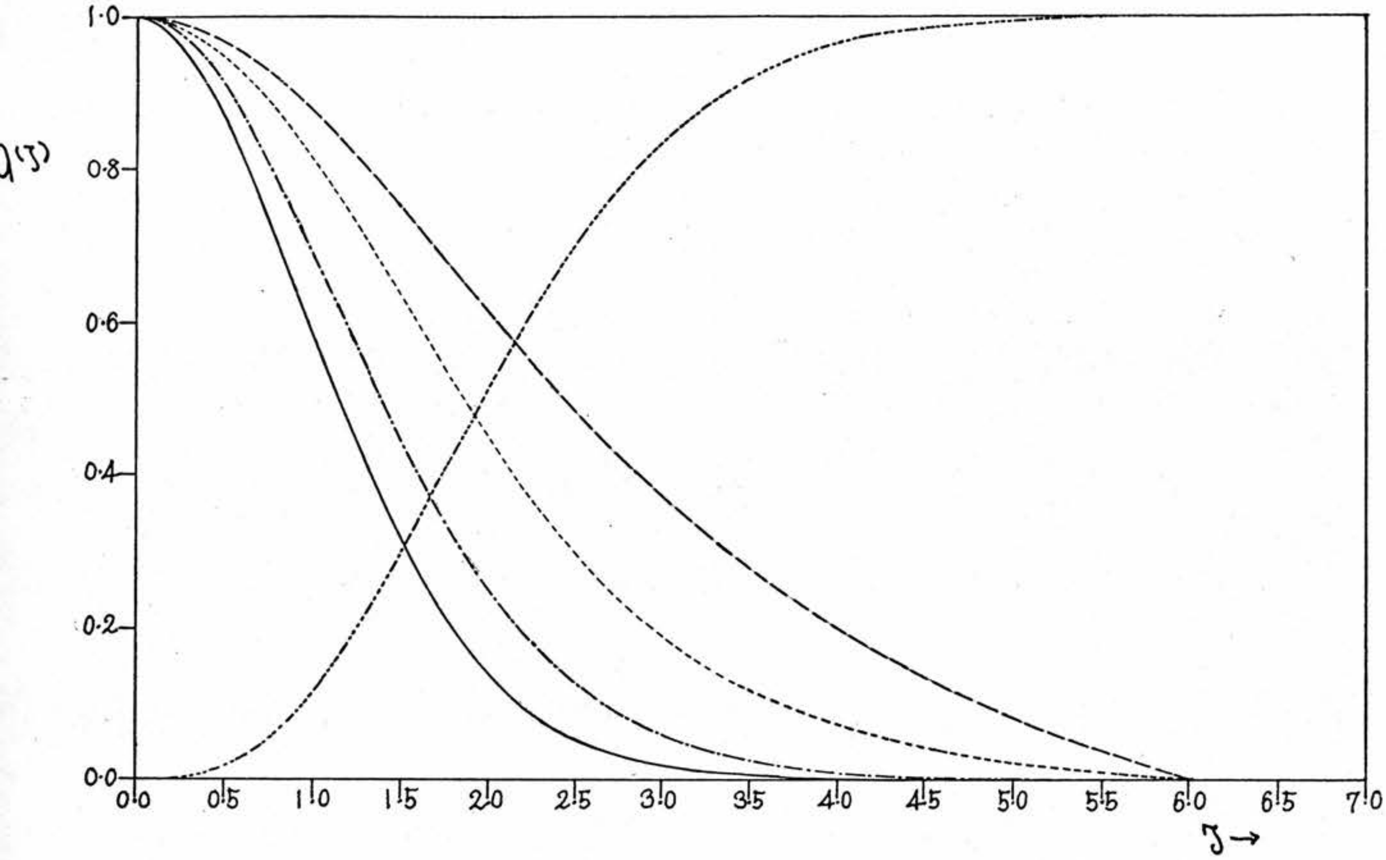
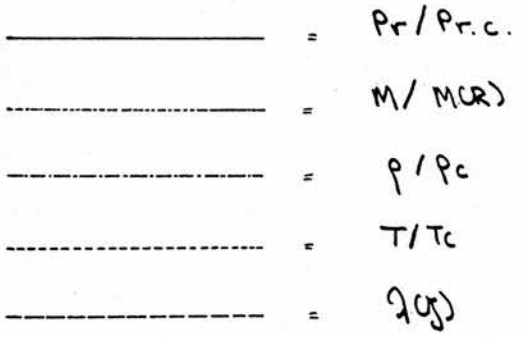


Figure 5.

The partially degenerate standard model characteristic functions for $\Lambda_0 = 2$.



The boundary-values found by G. Wares, for the three numerical integrations he considered, are as follows:

$\psi_0 (= -\alpha = \ln \Lambda_0)$	$\xi (= \zeta)$	Mass variable
0	9.75789	4.3271
2	3.45971	6.4143
5	1.4617	14.742

The boundary values found from our integration for the above three cases are:-

Λ_0	ζ	Mass variable
1	9.0096405	4.33
7.389	3.1932509	6.42
148.41	1.3481148	14.76

As we can see the difference in the mass variable is small since the mass at the surface fringe is very small, but the difference in ζ (or ξ) is quite substantial.

We can also see that our solutions give values of much closer to zero than Wares's integration

ψ_0	$\psi (= \ln \Lambda)$	
0	-7.86857	
2	-5.62393	(G.Wares's)
5	-4.2456	

while our solution gives

Λ_0	$\Lambda(\zeta)$		$\ln(\Lambda(\zeta))$
1	0.65428507	D-5	-11.94
7.39	0.90865127	D-5	-11.61
148.41	0.59232673	D-5	-12.04

DEGENERACY PARAMETER=0.1

	BETA	CENT.TEMPER.	CENT.DENS.	CENT.PRESSURE	MASS/M*	RADIUS/R*	GRAV.ACC.
μ_e=1	0.900000	0.20292813D 09	0.22564287D 04	0.42677059D 20	0.71886937D 01	0.58808634D 00	0.56958633D 06
	0.950000	0.12331092D 09	0.10688347D 04	0.11637556D 20	0.45621839D 01	0.64831447D 00	0.29743572D 06
	0.990000	0.41028014D 08	0.20512988D 03	0.71309598D 18	0.18787310D 01	0.83620000D 00	0.73626822D 05
	0.995000	0.25759371D 08	0.10204954D 03	0.22161398D 18	0.13151456D 01	0.93702789D 00	0.41045029D 05
	0.999500	0.55330188D 07	0.10159009D 02	0.47174273D 16	0.41214914D 00	0.13733015D 01	0.59884490D 04
	0.999900	0.18917612D 07	0.20309890D 01	0.32232372D 15	0.18417126D 00	0.17955725D 01	0.15653365D 04
	0.999950	0.11916952D 07	0.10154437D 01	0.10151212D 15	0.13021572D 00	0.20154284D 01	0.87845689D 03
	0.999990	0.40754319D 06	0.20308062D 00	0.69425977D 13	0.58229583D-01	0.26354608D 01	0.22973245D 03
	0.999995	0.25673527D 06	0.10153980D 00	0.21867530D 13	0.41174121D-01	0.29581997D 01	0.12893212D 03
μ_e=1.5	0.900000	0.20292813D 09	0.33846431D 04	0.42677059D 20	0.31949750D 01	0.39205756D 00	0.56958633D 06
	0.950000	0.12331092D 09	0.16032520D 04	0.11637556D 20	0.20276373D 01	0.43220965D 00	0.29743572D 06
	0.990000	0.41028014D 08	0.30769483D 03	0.71309598D 18	0.83499157D 00	0.55746666D 00	0.73626822D 05
	0.995000	0.25759371D 08	0.15307431D 03	0.22161398D 18	0.58450916D 00	0.62468526D 00	0.41045029D 05
	0.999000	0.87860500D 07	0.30492280D 02	0.14996889D 17	0.25931134D 00	0.81578447D 00	0.10677324D 05
	0.999500	0.55330188D 07	0.15238513D 02	0.47174273D 16	0.18317740D 00	0.91553436D 00	0.59884490D 04
	0.999900	0.18917612D 07	0.30464834D 01	0.32232372D 15	0.81853893D-01	0.11970484D 01	0.15653365D 04
	0.999950	0.11916952D 07	0.15231655D 01	0.10151212D 15	0.57873655D-01	0.13436185D 01	0.87845689D 03
	0.999990	0.40754319D 06	0.30462092D 00	0.69425977D 13	0.25879815D-01	0.17569738D 01	0.22973245D 03
	0.999995	0.25673527D 06	0.15230970D 00	0.21867530D 13	0.18299610D-01	0.19721331D 01	0.12893212D 03
μ_e=2	0.900000	0.20292813D 09	0.45128574D 04	0.42677059D 20	0.17971734D 01	0.29404317D 00	0.56958633D 06
	0.950000	0.12331092D 09	0.21376693D 04	0.11637556D 20	0.11405460D 01	0.32415724D 00	0.29743572D 06
	0.990000	0.41028014D 08	0.41025977D 03	0.71309598D 18	0.46968276D 00	0.41810000D 00	0.73626822D 05
	0.995000	0.25759371D 08	0.20409908D 03	0.22161398D 18	0.32878640D 00	0.46851395D 00	0.41045029D 05
	0.999000	0.87860500D 07	0.40656373D 02	0.14996889D 17	0.14586263D 00	0.61183835D 00	0.10677324D 05
	0.999500	0.55330188D 07	0.20318018D 02	0.47174273D 16	0.10303729D 00	0.68665077D 00	0.59884490D 04
	0.999900	0.18917612D 07	0.40619779D 01	0.32232372D 15	0.46042815D-01	0.89778627D 00	0.15653365D 04
	0.999950	0.11916952D 07	0.20308874D 01	0.10151212D 15	0.32553931D-01	0.10077142D 01	0.87845689D 03
	0.999990	0.40754319D 06	0.40616123D 00	0.69425977D 13	0.14557396D-01	0.13177304D 01	0.22973245D 03
	0.999995	0.25673527D 06	0.20307960D 00	0.21867530D 13	0.10293530D-01	0.14790998D 01	0.12893212D 03

Table 22. Partially Degenerate Standard Model for Λ_0 = 0.1

DEGENERACY PARAMETER=0.2

	BETA	CENT.TEMPER.	CENT.DENS.	CENT.PRESSURE	MASS/M*	RADIUS/R*	GRAV.ACC.
$\mu_e=1$	0.900000	0.31863445D 09	0.17201032D 05	0.25941572D 21	0.18392141D 01	0.19369511D 00	0.13433425D 07
	0.950000	0.19362080D 09	0.81478571D 04	0.70739760D 20	0.11672264D 01	0.21353215D 00	0.70148812D 06
	0.990000	0.64421522D 08	0.15637302D 04	0.43346077D 19	0.48066988D 00	0.27541508D 00	0.17364539D 06
	0.995000	0.40446946D 08	0.77793611D 03	0.13470973D 19	0.33647759D 00	0.30862427D 00	0.96802766D 05
	0.999000	0.13795713D 08	0.15496425D 03	0.91159720D 17	0.14927474D 00	0.40303637D 00	0.25181965D 05
	0.999500	0.86878564D 07	0.77443364D 02	0.28675237D 17	0.10544760D 00	0.45231757D 00	0.14123475D 05
	0.999900	0.29704128D 07	0.15482477D 02	0.19592690D 16	0.47119879D-01	0.59139889D 00	0.36917724D 04
	0.999950	0.18711805D 07	0.77408513D 01	0.61704906D 15	0.33315454D-01	0.66381173D 00	0.20717992D 04
	0.999990	0.63991770D 06	0.15481083D 01	0.42201103D 14	0.14897932D-01	0.86802874D 00	0.54181315D 03
	0.999995	0.40312156D 06	0.77405030D 00	0.13292343D 14	0.10534323D-01	0.97432767D 00	0.30408034D 03
$\mu_e=1.5$	0.900000	0.31863445D 09	0.12900774D 05	0.25941572D 21	0.32697139D 01	0.25826014D 00	0.13433425D 07
	0.950000	0.19362080D 09	0.61108928D 04	0.70739760D 20	0.20750691D 01	0.28470953D 00	0.70148812D 06
	0.990000	0.64421522D 08	0.11727976D 04	0.43346077D 19	0.85452424D 00	0.36722010D 00	0.17364539D 06
	0.995000	0.40446946D 08	0.58345208D 03	0.13470973D 19	0.59818238D 00	0.41149902D 00	0.96802766D 05
	0.999000	0.13795713D 08	0.11622319D 03	0.91159720D 17	0.26537732D 00	0.53738183D 00	0.25181965D 05
	0.999500	0.86878564D 07	0.58082523D 02	0.28675237D 17	0.18746240D 00	0.60309009D 00	0.14123475D 05
	0.999900	0.29704128D 07	0.11611858D 02	0.19592690D 16	0.83768674D-01	0.78853186D 00	0.36917724D 04
	0.999950	0.18711805D 07	0.58056385D 01	0.61704906D 15	0.59227474D-01	0.88508231D 00	0.20717992D 04
	0.999990	0.63991770D 06	0.11610813D 01	0.42201103D 14	0.26485213D-01	0.11573717D 01	0.54181315D 03
	0.999995	0.40312156D 06	0.58053772D 00	0.13292343D 14	0.18727686D-01	0.12991036D 01	0.30408034D 03
$\mu_e=2$	0.900000	0.31863445D 09	0.86005159D 04	0.25941572D 21	0.73568563D 01	0.38739021D 00	0.13433425D 07
	0.950000	0.19362080D 09	0.40739286D 04	0.70739760D 20	0.46689055D 01	0.42706430D 00	0.70148813D 06
	0.990000	0.64421522D 08	0.78186508D 03	0.43346077D 19	0.19226795D 01	0.55083016D 00	0.17364539D 06
	0.995000	0.40446946D 08	0.38896805D 03	0.13470973D 19	0.13459104D 01	0.61724853D 00	0.96802767D 05
	0.999000	0.13795713D 08	0.77482125D 02	0.91159720D 17	0.59709897D 00	0.80607275D 00	0.25181965D 05
	0.999500	0.86878564D 07	0.38721682D 02	0.28675237D 17	0.42179041D 00	0.90463514D 00	0.14123475D 05
	0.999900	0.29704128D 07	0.77412384D 01	0.19592690D 16	0.18847952D 00	0.11827978D 01	0.36917724D 04
	0.999950	0.18711805D 07	0.38704257D 01	0.61704906D 15	0.13326182D 00	0.13276235D 01	0.20717992D 04
	0.999950	0.18711805D 07	0.38704257D 01	0.61704906D 15	0.13326182D 00	0.13276235D 01	0.20717992D 04
	0.999990	0.63991770D 06	0.77405417D 00	0.42201103D 14	0.59591728D-01	0.17360575D 01	0.54181316D 03

Table 23 . Partially Degenerate standard Model for $\Lambda_0 = 0.2$

DEGENERACY PARAMETER=0.5

	BETA	CENT.TEMPER.	CENT.DENS.	CENT.PRESSURE	MASS/M*	RADIUS/R*	GRAV.ACC.
$\mu_e=1$	0.900000	0.56964870D 09	0.47205552D 05	0.26500473D 22	0.77149820D 01	0.21469819D 00	0.45863788D 07
	0.950000	0.34615164D 09	0.22360525D 05	0.72263820D 21	0.48961835D 01	0.23668624D 00	0.23949886D 07
	0.990000	0.11517159D 09	0.42914138D 04	0.44279951D 20	0.20162736D 01	0.30527936D 00	0.59285207D 06
	0.995000	0.72310292D 08	0.21349245D 04	0.13761200D 20	0.14114278D 01	0.34208955D 00	0.33049950D 06
	0.999000	0.24663717D 08	0.42527525D 03	0.93123720D 18	0.62616501D 00	0.44673911D 00	0.85975085D 05
	0.999500	0.15531987D 08	0.21253125D 03	0.29293034D 18	0.44232261D 00	0.50136405D 00	0.48219702D 05
	0.999900	0.53104484D 07	0.42489246D 02	0.20014807D 17	0.19765443D 00	0.65552649D 00	0.12604274D 05
	0.999950	0.33452614D 07	0.21243561D 02	0.63034313D 16	0.13974880D 00	0.73579132D 00	0.70734378D 04
	0.999990	0.11440329D 07	0.42485422D 01	0.43110309D 15	0.62492558D-01	0.96215234D 00	0.18498324D 04
	0.999995	0.72069317D 06	0.21242605D 01	0.13578721D 15	0.44188465D-01	0.10799777D 01	0.10381764D 04
$\mu_e=1.5$	0.900000	0.56964870D 09	0.70808328D 05	0.26500473D 22	0.34288777D 01	0.14313213D 00	0.45863745D 07
	0.950000	0.34615164D 09	0.33540787D 05	0.72263820D 21	0.21760795D 01	0.15779082D 00	0.23949864D 07
	0.990000	0.11517159D 09	0.64371208D 04	0.44279951D 20	0.89612077D 00	0.20351958D 00	0.59285152D 06
	0.995000	0.72310292D 08	0.32023867D 04	0.13761200D 20	0.62730064D 00	0.22805970D 00	0.33049920D 06
	0.999000	0.24663717D 08	0.63791287D 03	0.93123720D 18	0.27829530D 00	0.29782607D 00	0.85975005D 05
	0.999500	0.15531987D 08	0.31879688D 03	0.29293034D 18	0.19658764D 00	0.33424270D 00	0.48219657D 05
	0.999900	0.53104484D 07	0.63733869D 02	0.20014807D 17	0.87846333D-01	0.43701766D 00	0.12604262D 05
	0.999950	0.33452614D 07	0.31865341D 02	0.63034313D 16	0.62110520D-01	0.49052755D 00	0.70734312D 04
	0.999990	0.11440329D 07	0.63728133D 01	0.43110309D 15	0.27774444D-01	0.64143489D 00	0.18498307D 04
	0.999995	0.72069317D 06	0.31863907D 01	0.13578721D 15	0.19639300D-01	0.71998511D 00	0.10381754D 04
$\mu_e=2$	0.900000	0.56964870D 09	0.94411104D 05	0.26500473D 22	0.19287419D 01	0.10734909D 00	0.45863703D 07
	0.950000	0.34615164D 09	0.44721049D 05	0.72263820D 21	0.12240436D 01	0.11834312D 00	0.23949841D 07
	0.990000	0.11517159D 09	0.85828277D 04	0.44279951D 20	0.50406747D 00	0.15263968D 00	0.59285097D 06
	0.995000	0.72310292D 08	0.42698489D 04	0.13761200D 20	0.35285628D 00	0.17104477D 00	0.33049889D 06
	0.999000	0.24663717D 08	0.85055049D 03	0.93123720D 18	0.15654096D 00	0.22336956D 00	0.85974925D 05
	0.999500	0.15531987D 08	0.42506250D 03	0.29293034D 18	0.11058045D 00	0.25068203D 00	0.48219612D 05
	0.999900	0.53104484D 07	0.84978492D 02	0.20014807D 17	0.49413516D-01	0.32776324D 00	0.12604251D 05
	0.999950	0.33452614D 07	0.42487121D 02	0.63034313D 16	0.34937135D-01	0.36789566D 00	0.70734246D 04
	0.999990	0.11440329D 07	0.84970844D 01	0.43110309D 15	0.15623110D-01	0.48107617D 00	0.18498290D 04
	0.999995	0.72069317D 06	0.42485209D 01	0.13578721D 15	0.11047096D-01	0.53998883D 00	0.10381745D 04

Table 24. Partially Degenerate Standard Model for $\Lambda_0 = 0.5$

DEGENERACY PARAMETER=1.0

	BETA	CENT.TEMPER.	CENT.DENS.	CENT.PRESSURE	MASS/M*	RADIUS/R*	GRAV.ACC.
$\psi_e=1$	0.900000	0.86644041D 09	0.15760890D 06	0.14183384D 23	0.53988178D 01	0.14002056D 00	0.75458268D 07
	0.950000	0.52649953D 09	0.74656845D 05	0.38676500D 22	0.34262692D 01	0.15436059D 00	0.39404008D 07
	0.990000	0.17517695D 09	0.14328081D 05	0.23699183D 21	0.14109554D 01	0.19909524D 00	0.97540129D 06
	0.995000	0.10998456D 09	0.71280405D 04	0.73651660D 20	0.98769425D 00	0.22310188D 00	0.54376072D 06
	0.999000	0.37513720D 08	0.14199000D 04	0.49840979D 19	0.43818016D 00	0.29135159D 00	0.14145219D 06
	0.999500	0.23624281D 08	0.70559483D 03	0.15677998D 19	0.30953024D 00	0.32697656D 00	0.79334414D 05
	0.999900	0.80772362D 07	0.14186219D 03	0.10712175D 18	0.13831540D 00	0.42751727D 00	0.20737432D 05
	0.999950	0.50881705D 07	0.70927549D 02	0.33736752D 17	0.97793978D-01	0.47986390D 00	0.11637715D 05
	0.999990	0.17400835D 07	0.14184942D 02	0.23073176D 16	0.43731297D-01	0.62749066D 00	0.30434740D 04
	0.999995	0.10961803D 07	0.70924357D 01	0.72675014D 15	0.30922388D-01	0.70433326D 00	0.17080808D 04
$\psi_e=1.5$	0.900000	0.86644041D 09	0.23641334D 06	0.14183384D 23	0.23994746D 01	0.93347039D-01	0.75458268D 07
	0.950000	0.52649953D 09	0.11198527D 06	0.38676500D 22	0.15227863D 01	0.10290706D 00	0.39404008D 07
	0.990000	0.17517695D 09	0.21492122D 05	0.23699183D 21	0.62709131D 00	0.13273016D 00	0.97540129D 06
	0.995000	0.10998456D 09	0.10692061D 05	0.73651660D 20	0.43897522D 00	0.14873459D 00	0.54376072D 06
	0.999000	0.37513720D 08	0.21298499D 04	0.49840979D 19	0.19474674D 00	0.19423440D 00	0.14145219D 06
	0.999500	0.23624281D 08	0.10643922D 04	0.15677998D 19	0.13756900D 00	0.21798437D 00	0.79334414D 05
	0.999900	0.80772362D 07	0.21279329D 03	0.10712175D 18	0.61473512D-01	0.28501152D 00	0.20737432D 05
	0.999950	0.50881705D 07	0.10639132D 03	0.33736752D 17	0.43463990D-01	0.31990927D 00	0.11637715D 05
	0.999990	0.17400835D 07	0.21277414D 02	0.23073176D 16	0.19436132D-01	0.41832710D 00	0.30434740D 04
	0.999995	0.10961803D 07	0.10638654D 02	0.72675014D 15	0.13743283D-01	0.46955550D 00	0.17080808D 04
$\psi_e=2$	0.900000	0.86644041D 09	0.31521779D 06	0.14183384D 23	0.13497045D 01	0.70010279D-01	0.75458268D 07
	0.950000	0.52649953D 09	0.14931369D 06	0.38676500D 22	0.85656730D 00	0.77180295D-01	0.39404008D 07
	0.990000	0.17517695D 09	0.28656163D 05	0.23699183D 21	0.35273886D 00	0.99547619D-01	0.97540129D 06
	0.995000	0.10998456D 09	0.14256081D 05	0.73651660D 20	0.24692356D 00	0.11155094D 00	0.54376072D 06
	0.999000	0.37513720D 08	0.28397999D 04	0.49840979D 19	0.10954504D 00	0.14567580D 00	0.14145219D 06
	0.999500	0.23624281D 08	0.14191897D 04	0.15677998D 19	0.77382560D-01	0.16348828D 00	0.79334414D 05
	0.999900	0.80772362D 07	0.28372438D 03	0.10712175D 18	0.34578850D-01	0.21375864D 00	0.20737432D 05
	0.999950	0.50881705D 07	0.14185510D 03	0.33736752D 17	0.24448494D-01	0.23993195D 00	0.11637715D 05
	0.999990	0.17400835D 07	0.28369885D 02	0.23073176D 15	0.10932824D-01	0.31374533D 00	0.30434740D 04
	0.999995	0.10961803D 07	0.14184871D 02	0.72675014D 15	0.77305969D-02	0.35216663D 00	0.17080808D 04

Table 25. Partially Degenerate Standard Model for $\Lambda_0=1.0$

DEGENERACY PARAMETER=2.0

	BETA	CENT.TEMPER.	CENT.DENS.	CENT.PRESSURE	MASS/M*	RADIUS/R*	GRAV.ACC.
$\mu_e=1$	0.900000	0.12842966D 10	0.47632550D 06	0.68467788D 23	0.91973342D 01	0.93658196D-01	0.28731755D 08
	0.950000	0.78041323D 09	0.22562787D 06	0.18670398D 23	0.58369339D 01	0.10325008D 00	0.15003609D 08
	0.990000	0.25965913D 09	0.43302318D 05	0.11440363D 22	0.24036797D 01	0.13317259D 00	0.37139723D 07
	0.995000	0.16302655D 09	0.21542359D 05	0.35554041D 21	0.16826191D 01	0.14923037D 00	0.20704425D 07
	0.999000	0.55605374D 08	0.42912207D 04	0.24059854D 20	0.74647627D 00	0.19488184D 00	0.53859834D 06
	0.999500	0.35017508D 08	0.21445370D 04	0.75682771D 19	0.52731047D 00	0.21871099D 00	0.30207651D 06
	0.999900	0.11972626D 08	0.42873582D 03	0.51711136D 18	0.23563177D 00	0.28596155D 00	0.78960578D 05
	0.999950	0.75420306D 07	0.21435719D 03	0.16285822D 18	0.16660016D 00	0.32097563D 00	0.44312174D 05
	0.999990	0.25792696D 07	0.42869723D 02	0.11138169D 17	0.74499897D-01	0.41972153D 00	0.11588439D 05
	0.999995	0.16248326D 07	0.21434754D 02	0.35082582D 16	0.52678855D-01	0.47112069D 00	0.65037486D 04
$\mu_e=1.5$	0.900000	0.12842966D 10	0.71448824D 06	0.68467788D 23	0.40877041D 01	0.62438797D-01	0.28731755D 08
	0.950000	0.78041323D 09	0.33844180D 06	0.18670398D 23	0.25941929D 01	0.68833389D-01	0.15003609D 08
	0.990000	0.25965913D 09	0.64953477D 05	0.11440363D 22	0.10683021D 01	0.88781728D-01	0.37139723D 07
	0.995000	0.16302655D 09	0.32313539D 05	0.35554041D 21	0.74783073D 00	0.99486912D-01	0.20704425D 07
	0.999000	0.55605374D 08	0.64368310D 04	0.24059854D 20	0.33176723D 00	0.12992123D 00	0.53859834D 06
	0.999500	0.35017508D 08	0.32168055D 04	0.75682771D 19	0.23436021D 00	0.14580732D 00	0.30207651D 06
	0.999900	0.11972626D 08	0.64310373D 03	0.51711136D 18	0.10472523D 00	0.19064104D 00	0.78960578D 05
	0.999950	0.75420306D 07	0.42871438D 03	0.16285822D 18	0.41650040D-01	0.16048782D 00	0.44312174D 05
	0.999990	0.25792696D 07	0.64304585D 02	0.11138169D 17	0.33111065D-01	0.27981435D 00	0.11588439D 05
	0.999995	0.16248326D 07	0.32152132D 02	0.35082582D 16	0.23412825D-01	0.31408046D 00	0.65037486D 04
$\mu_e=2$	0.900000	0.12842966D 10	0.95265099D 06	0.68467788D 23	0.22993335D 01	0.46829098D-01	0.28731755D 08
	0.950000	0.78041323D 09	0.45125573D 06	0.18670398D 23	0.14592335D 01	0.51625041D-01	0.15003609D 08
	0.990000	0.25965913D 09	0.86604636D 05	0.11440363D 22	0.60091993D 00	0.66586296D-01	0.37139723D 07
	0.995000	0.16302655D 09	0.43084718D 05	0.35554041D 21	0.42065479D 00	0.74615184D-01	0.20704425D 07
	0.999000	0.55605374D 08	0.85824414D 04	0.24059854D 20	0.18661907D 00	0.97440922D-01	0.53859834D 06
	0.999500	0.35017508D 08	0.42890740D 04	0.75682771D 19	0.13182762D 00	0.10935549D 00	0.30207651D 06
	0.999900	0.11972626D 08	0.85747164D 03	0.51711136D 18	0.58907943D-01	0.14298078D 00	0.78960578D 05
	0.999950	0.75420306D 07	0.42871438D 03	0.16285822D 18	0.41650040D-01	0.16048782D 00	0.44312174D 05
	0.999990	0.25792696D 07	0.85739447D 02	0.11138169D 17	0.18624974D-01	0.20986076D 00	0.11588439D 05
	0.999995	0.16248326D 07	0.42869509D 02	0.35082582D 16	0.13169714D-01	0.23556034D 00	0.65037486D 04

Table 26. Partially Degenerate Standard Model for $\Lambda_0 = 2.0$

DEGENERACY PARAMETER=5.0

	BETA	CENT.TEMPER.	CENT.DENS.	CENT.PRESSURE	MASS/M*	RADIUS/R*	GRAV.ACC.
$\mu_e=1$	0.900000	0.20560660D 10	0.17199654D 07	0.44975191D 24	0.11147160D 02	0.58808634D-01	0.88322997D 08
	0.950000	0.12493851D 10	0.81472047D 06	0.12264230D 24	0.70743580D 01	0.64831447D-01	0.46121919D 08
	0.990000	0.41569548D 09	0.15636049D 06	0.75149572D 22	0.29132574D 01	0.83620000D-01	0.11416955D 08
	0.995000	0.26099372D 09	0.77787382D 05	0.23354775D 22	0.20393327D 01	0.93702789D-01	0.63646542D 07
	0.999000	0.89020183D 08	0.15495184D 05	0.15804461D 21	0.90472848D 00	0.12236767D 00	0.16556809D 07
	0.999500	0.56060499D 08	0.77437163D 04	0.49714576D 20	0.63909975D 00	0.13733015D 00	0.92859983D 06
	0.999900	0.19167309D 08	0.15481237D 04	0.33968064D 19	0.28558547D 00	0.17955725D 00	0.24272916D 06
	0.999950	0.12074246D 08	0.77402315D 03	0.10697848D 19	0.20191923D 00	0.20154284D 00	0.13621806D 06
	0.999990	0.41292241D 07	0.15479844D 03	0.73164519D 17	0.90293799D-01	0.26354608D 00	0.35623500D 05
	0.999995	0.26012396D 07	0.77398832D 02	0.23045082D 17	0.63846719D-01	0.29581997D 00	0.19992881D 05
$\mu_e=1.5$	0.900000	0.20560660D 10	0.25799482D 07	0.44975191D 24	0.49542932D 01	0.39205756D-01	0.88322997D 08
	0.950000	0.12493851D 10	0.12220807D 07	0.12264230D 24	0.31441591D 01	0.43220965D-01	0.46121919D 08
	0.990000	0.41569548D 09	0.23454074D 06	0.75149572D 22	0.12947810D 01	0.55746666D-01	0.11416955D 08
	0.995000	0.26099372D 09	0.11668107D 06	0.23354775D 22	0.90637008D 00	0.62468526D-01	0.63646542D 07
	0.999000	0.89020183D 08	0.23242776D 05	0.15804461D 21	0.40210155D 00	0.81578447D-01	0.16556809D 07
	0.999500	0.56060499D 08	0.11615575D 05	0.49714576D 20	0.28404433D 00	0.91553436D-01	0.92859983D 06
	0.999900	0.19167309D 08	0.23221856D 04	0.33968064D 19	0.12692687D 00	0.11970484D 00	0.24272916D 06
	0.999950	0.12074246D 08	0.11610347D 04	0.10697848D 19	0.89741878D-01	0.13436189D 00	0.13621806D 06
	0.999990	0.41292241D 07	0.23219766D 03	0.73164519D 17	0.40130577D-01	0.17569738D 00	0.35623500D 05
	0.999995	0.26012396D 07	0.11609825D 03	0.23045082D 17	0.28376320D-01	0.19721331D 00	0.19992881D 05
$\mu_e=2.0$	0.900000	0.20560660D 10	0.34399309D 07	0.44975191D 24	0.27867899D 01	0.29404317D-01	0.88322997D 08
	0.950000	0.12493851D 10	0.16294409D 07	0.12264230D 24	0.17685895D 01	0.32415724D-01	0.46121919D 08
	0.990000	0.41569548D 09	0.31272099D 06	0.75149572D 22	0.72831434D 00	0.41810000D-01	0.11416955D 08
	0.995000	0.26099372D 09	0.15557476D 06	0.23354775D 22	0.50983317D 00	0.46851395D-01	0.63646542D 07
	0.999000	0.89020183D 08	0.30990368D 05	0.15804461D 21	0.22618212D 00	0.61183835D-01	0.16556809D 07
	0.999500	0.56060499D 08	0.15487433D 05	0.49714576D 20	0.15977494D 00	0.68665077D-01	0.92859983D 06
	0.999900	0.19167309D 08	0.30962474D 04	0.33968064D 19	0.71396367D-01	0.89778627D-01	0.24272916D 06
	0.999950	0.12074246D 08	0.15480463D 04	0.10697848D 19	0.50479807D-01	0.10077142D 00	0.13621806D 06
	0.999990	0.41292241D 07	0.30959687D 03	0.73164519D 17	0.22573450D-01	0.13177304D 00	0.35623500D 05
	0.999995	0.26012396D 07	0.15479766D 03	0.23045082D 17	0.15961680D-01	0.14790598D 00	0.19992881D 05

Table 27. Partially Degenerate Standard Model for $\alpha_0 = 5.0$

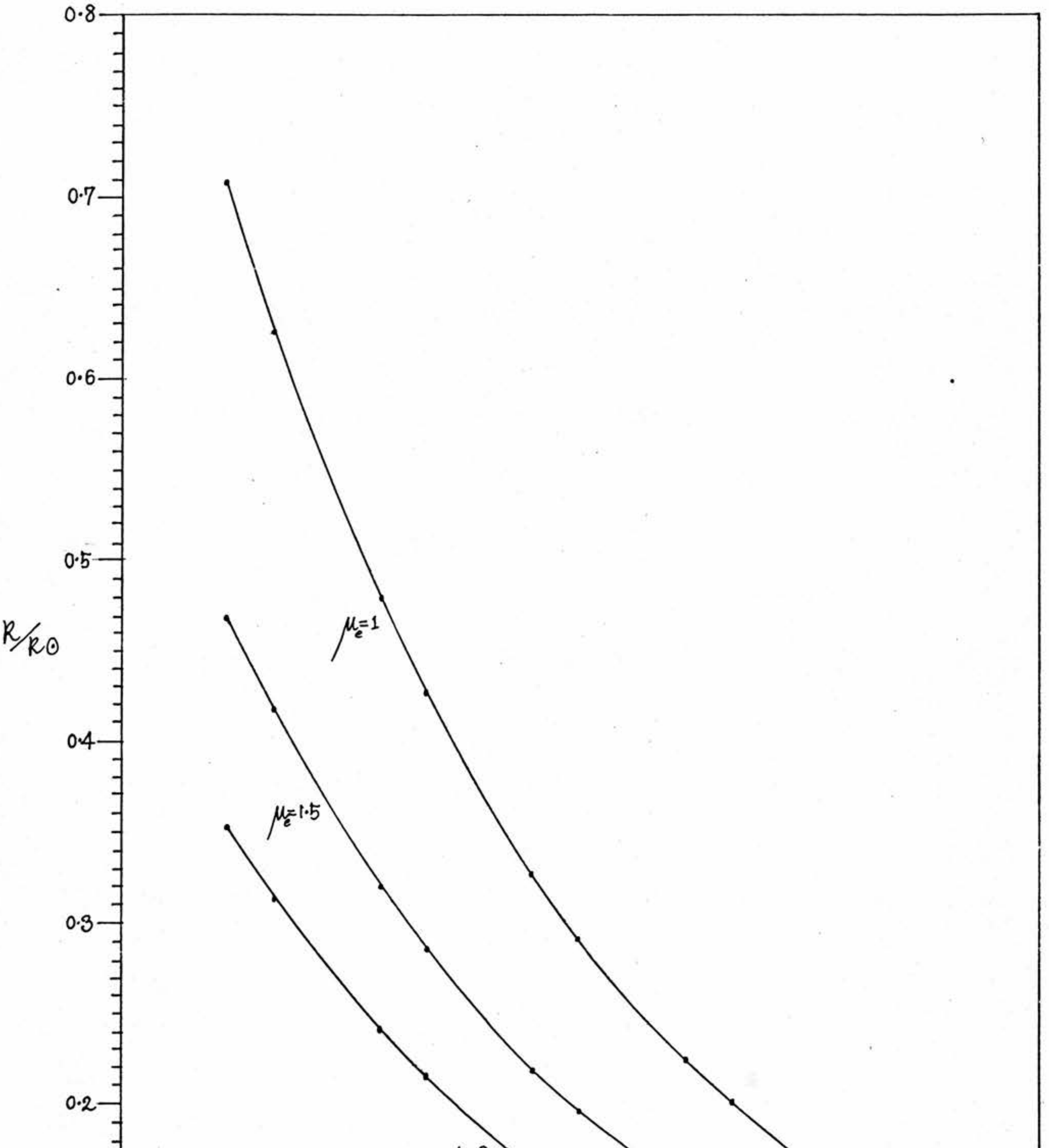


FIGURE 6.

$\Lambda_0 = 1$

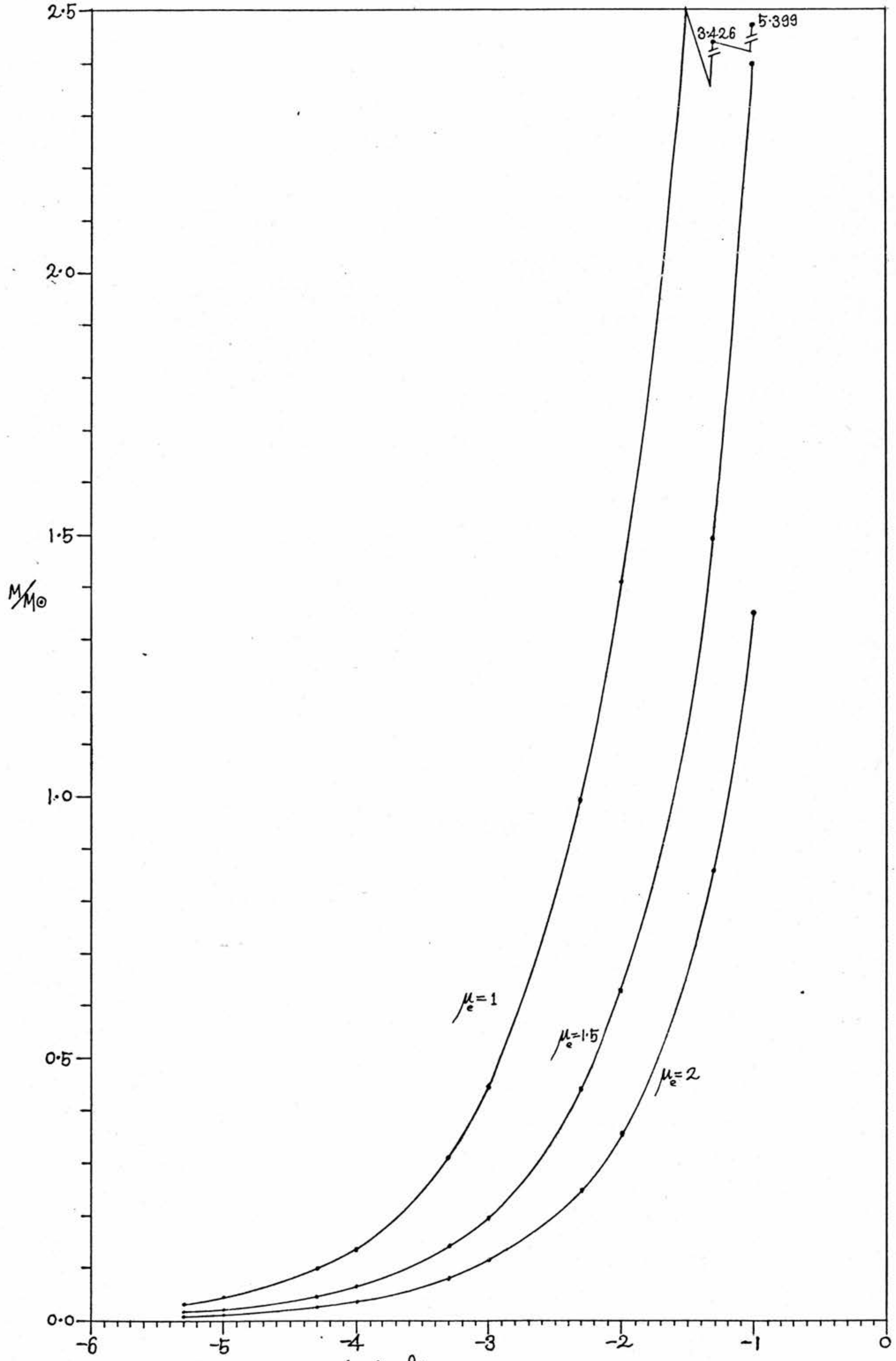


FIGURE 7. $\Lambda_0 = 1$

CHAPTER IV

The object of the first part of this chapter is to investigate a criterion for convection in the case of the partially degenerate stellar models.

In the second part we compute formulae for the adiabatic exponents (gammas) $\Gamma_1, \Gamma_2, \Gamma_3$ in the case of a mixture of black body radiation and a partially degenerate perfect gas. We will see that the formulae which give $\Gamma_1, \Gamma_2, \Gamma_3$ depend upon β (the ratio of the gas pressure to the total pressure) and also upon the degeneracy parameter. In this analysis we treat the partially degenerate gas, as a monatomic gas with $\gamma = 5/3$

In the third part of this chapter, tables of Γ_1, Γ_2 and Γ_3 will be obtained for different values of the degeneracy parameter. The recorded values have been computed at the center and the surface of each partially degenerate configuration.

(A) CRITERION FOR CONVECTION

We expect models of such small masses as in the case of partially degenerate configurations poor in hydrogen to be completely convective.

We shall investigate this problem by establishing a criterion for convection instability for this particular case. The stability condition can easily be expressed in terms of the temperature gradient (under the assumptions of constant chemical composition and no existence of energy sources).

Because the temperature decreases radially, it is also clear that the stability condition demands that the temperature decrement for a radial adiabatic displacement be greater than the temperature decrement of the environment. Thus a layer is stable if

$$\left| \left(\frac{dT}{dr}\right)_{star} \right| < \left| \left(\frac{dT}{dr}\right)_{adiab.} \right| \tag{1}$$

$\Rightarrow$ If the temperature changes too rapidly with distance, instability towards convection exists.

The adiabatic gradient $\left(\frac{dT}{dr}\right)_{adiab.}$ is defined by the second adiabatic exponent

(definition: $\frac{\Gamma_2 - 1}{\Gamma_2} := \left(\frac{d\ln T}{d\ln P}\right)_{adiab.}$

The adiabatic relation between P and T is written in the form

$$\left(\frac{dT}{dr}\right)_{ad} = \frac{\Gamma_2 - 1}{\Gamma_2} \frac{T}{P} \left(\frac{dP}{dr}\right)_{star} \tag{2}$$

where the pressure gradient $\frac{dP}{dr}$ is obtained from

the hydrostatic equilibrium equation $-\frac{1}{\rho}\frac{dP}{dr} = G\frac{M(r)}{r^2}\frac{\rho}{\rho}$

From (1) and (2) and as long as both gradients are negative, the algebraic condition for stability is

$$\left(\frac{dT}{dr}\right)_{star} > \left(1 - \frac{1}{\Gamma_2}\right) \frac{T}{P} \left(\frac{dP}{dr}\right)_{star} \tag{3}$$

This condition has to be checked at each point of the model.

The adiabatic exponent Γ_2 for a mixture of partially degenerate gas and radiation is derived in part B.

For the case of standard model we also recall that $P = \frac{aT^4}{3(1-\beta)}$ and β is a constant.

$$\Rightarrow \frac{d\ln P}{dr} = 4\frac{d\ln T}{dr} \qquad (4)$$

From the formulae for the pressure and density of the configuration we can also derive the exponent $\frac{d\ln P}{d\ln T}$ explicitly

The pressure is given by

$$P = \frac{2}{h^3}(2\pi m)^{3/2}(kT)^{5/2}\, U_{3/2}(\eta)\,\frac{1}{\beta}$$

Since β is a constant throughout the model we have

$$P = \text{const} \, . \, T^{5/2}\, U_{3/2}(\eta)$$

and by logarithimic differentiation

$$\frac{d\ln P}{d\ln T} = \frac{5}{2} + \frac{d\ln U_{3/2}(\eta)}{d\ln T} = \frac{5}{2} + \frac{U_{1/2}}{U_{3/2}}\,\frac{T}{\eta}\,\frac{d\eta}{dT} \qquad (5)$$

The temperature is given by

$$(kT)^{3/2} = \frac{2}{h^3}(2\pi m)^{3/2}\left(k^4\,\frac{3}{a}\,\frac{1-\beta}{\beta}\right) U_{3/2}(\eta)$$

by differentiation we get: $$\frac{\eta}{T}\frac{dT}{d\eta} = \text{const} \, . \, \frac{2}{3}\, U_{3/2}^{-1/3}\, U_{1/2}\,\frac{1}{T} \qquad (6)$$

from (5) and (6) we have

$$\frac{d\ln P}{d\ln T} = \frac{5}{2} + \frac{3}{2}\,\frac{T}{U_{3/2}^{2/3}\cdot\text{const}} = \frac{5}{2} + \frac{3}{2} = 4 \qquad (7)$$

From the above expression we get the temperature gradient as

$$\frac{1}{T}\frac{dT}{dr} = -\frac{1}{4}\frac{1}{P}\frac{GM(r)}{r^2}\rho(r) \Rightarrow \left(\frac{dT}{dr}\right)_{star} = -\frac{1}{4}\frac{T}{P}g(r)\rho(r) \qquad (8)$$

From (3) and (8) we can see that the instability condition takes the form

$$-\frac{1}{4}\frac{T}{P}g(r)\rho(r) < \frac{\Gamma_2 - 1}{\Gamma_2}\,\frac{T}{P}\left(\frac{dP}{dr}\right)_{star}$$

or

$$\frac{\Gamma_2}{\Gamma_2 - 1} < 4 \qquad (9)$$

We shall see in the following paragraphs that inequality (9) is valid throughout the model.

(B) DERIVATION OF THE ADIABATIC EXPONENT Γ_2

The adiabatic exponents $\Gamma_1, \Gamma_2, \Gamma_3$ are defined by the relations

$$\Gamma_1 := -\left(\frac{d\ln P}{d\ln V}\right)_{ad} = \left(\frac{d\ln P}{d\ln \rho}\right)_{ad} \tag{1}$$

$$\frac{\Gamma_2}{\Gamma_2 - 1} := \left(\frac{d\ln P}{d\ln T}\right)_{ad} \tag{2}$$

$$\Gamma_3 - 1 := -\left(\frac{d\ln T}{d\ln V}\right)_{ad} = \left(\frac{d\ln T}{d\ln \rho}\right)_{ad} \tag{3}$$

We consider systems which are in thermodynamic equilibrium (systems which are in chemical and thermal equilibrium, definitions by Cox).
This assumption means that the second law of thermodynamics is valid, hence we have $dS = dQ/T = 0$ if $dQ = 0$. Therefore, an adiabatic change i.e. a change for which $dQ=0$ is an isentropic change as well i.e. $dS=0$
We consider now an adiabatic, quasi-statistical change in an enclosure containing radiation and matter in the form of a degenerate electron gas.
The internal energy of such a system is $U = U_{rad} + U_{gas}$ (4)

In general when two or more systems are brought into contact the energy is not additive.

However, if two or more systems are isolated from each other adiabatically, then by definition the energy of the system is equal to the sum of energies.

According to the electromagnetic theory

$$E_{rad} = 3\,P_{rad} = aT^4 \tag{5}$$

according to Quantum statistic, for a non relativistic partially degenerate electron gas $E_{gas} = \frac{3}{2} PV = \frac{3}{2} V \frac{2}{h^3} (2\pi m)^{3/2} (kT)^{5/2} U_{3/2}(\eta)$ (6)

Similarly, for the pressure

$$P = P_{rad} + P_g = \frac{1}{3} aT^4 + \frac{2}{h^3} (2\pi m)^{3/2} (kT)^{5/2} U_{3/2}(\eta) \tag{7}$$

In order to derive the relations for the three gammas we regard the internal energy of the system as function of any two of the set P,V,T. Generally, the thermodynamic functions can be written as functions of any two of the three variables, P, V, T. We should recall here that the volume $V = \mu/\rho$ (8)

where μ = mean molecular weight
and ρ is considered as a function of T and Λ in the following analysis.

We shall avoid, in the following, the relations under constant pressure, since pressure is an additive quantity.
From the definitions we can easily see that

$\frac{\Gamma_2}{\Gamma_2 - 1} = \frac{\Gamma_1}{\Gamma_3 - 1}$ (9) which means that only two of the three gammas are independent.

From (1) $\Rightarrow \Gamma_1 = -\frac{V}{P}\left(\frac{dP}{dV}\right)_{ad}$ (10)

from (3) $\Rightarrow \Gamma_3 - 1 = -\frac{V}{T}\left(\frac{dT}{dV}\right)_{ad}$ (11)

from (10) and (11) $\Rightarrow \frac{\Gamma_2}{\Gamma_2 - 1} = \frac{T}{P}\frac{\left(\frac{dP}{dV}\right)_{ad}}{\left(\frac{dT}{dV}\right)_{ad}}$ (12)

for a quasi-statical change we have

$$dQ = dE + PdV = \left(\frac{\partial E}{\partial P}\right)_V dP + \left(\frac{\partial E}{\partial V}\right)_P dV + PdV \qquad (13)$$

for an adiabatic change $dQ = 0$ and

from (13) $\Rightarrow$

$$\left(\frac{dP}{dV}\right)_S = -\frac{P + \left(\frac{\partial E}{\partial V}\right)_P}{\left(\frac{\partial E}{\partial P}\right)_V} \qquad (14)$$

similarly we can see that when E = E(T, V) a quasi-statical adiabatic change will lead to

$$\left(\frac{dT}{dV}\right)_S = -\frac{P + \left(\frac{\partial E}{\partial V}\right)_T}{\left(\frac{\partial E}{\partial T}\right)_V} \qquad (15)$$

Substituting (14) and (15) in (12) we get:

$$\frac{\Gamma_2}{\Gamma_2 - 1} = \frac{T}{P}\frac{\left[P + \left(\frac{\partial E}{\partial V}\right)_P\right]\left(\frac{\partial E}{\partial T}\right)_V}{\left[P + \left(\frac{\partial E}{\partial V}\right)_T\right]\left(\frac{\partial E}{\partial P}\right)_V} = \frac{T}{P}\left(\frac{\partial P}{\partial T}\right)_V \frac{P + \left(\frac{\partial E}{\partial V}\right)_P}{P + \left(\frac{\partial E}{\partial V}\right)_T} \qquad (16)$$

But, for $E = E(T,V)$ and $T(V, P)$ we have

$$dE = \left(\frac{\partial E}{\partial T}\right)_V dT + \left(\frac{\partial E}{\partial V}\right)_T dV = \left(\frac{\partial E}{\partial T}\right)_V \left[\left(\frac{\partial T}{\partial P}\right)_V dP + \left(\frac{\partial T}{\partial V}\right)_P dV\right] + \left(\frac{\partial E}{\partial V}\right)_T dV$$

$$\Rightarrow \left(\frac{\partial E}{\partial V}\right)_P = \left(\frac{\partial E}{\partial T}\right)_V \left(\frac{\partial T}{\partial V}\right)_P + \left(\frac{\partial E}{\partial V}\right)_T \qquad (17)$$

For $V = V(T, P)$ and $P = P(T, V)$ $\Longrightarrow$

$$dV = \left(\frac{\partial V}{\partial T}\right)_P dT + \left(\frac{\partial V}{\partial P}\right)_T dP = \left(\frac{\partial V}{\partial T}\right)_P dT + \left(\frac{\partial V}{\partial P}\right)_T \left[\left(\frac{\partial P}{\partial T}\right)_V dT + \left(\frac{\partial P}{\partial V}\right)_T dV\right] \Longrightarrow$$

$$\left(\frac{\partial V}{\partial T}\right)_V = \left(\frac{\partial V}{\partial T}\right)_P + \left(\frac{\partial V}{\partial P}\right)_T \left(\frac{\partial P}{\partial T}\right)_V = 0 \quad \Rightarrow$$

$$\left(\frac{\partial T}{\partial V}\right)_P = - \frac{1}{\left(\frac{\partial V}{\partial P}\right)_T \left(\frac{\partial P}{\partial T}\right)_V} = - \frac{\left(\frac{\partial P}{\partial V}\right)_T}{\left(\frac{\partial P}{\partial T}\right)_V} \qquad (18)$$

because $\left(\frac{\partial V}{\partial P}\right)_T \left(\frac{\partial P}{\partial V}\right)_T = 1$

By inserting (18) in (17) and (17) in (16) we get:

$$\frac{\Gamma_2}{\Gamma_2 - 1} = \frac{T}{P}\left(\frac{\partial P}{\partial T}\right)_V \frac{P + \left[\left(\frac{\partial E}{\partial V}\right)_T - \left(\frac{\partial E}{\partial T}\right)_V \left(\frac{\partial P}{\partial V}\right)_T \Big/ \left(\frac{\partial P}{\partial T}\right)_V\right]}{P + \left(\frac{\partial E}{\partial V}\right)_T} \quad \text{or} \qquad (19)$$

$$\frac{\Gamma_2}{\Gamma_2 - 1} = \frac{T}{P}\left(\frac{\partial P}{\partial T}\right)_V - \frac{T}{P} \frac{\left(\frac{\partial E}{\partial T}\right)_V \left(\frac{\partial P}{\partial V}\right)_T}{P + \left(\frac{\partial E}{\partial V}\right)_T} \qquad (20)$$

To find : $\left(\frac{\partial E}{\partial V}\right)_T$, $\left(\frac{\partial E}{\partial T}\right)_V$:

$$dE = d\left(\frac{3}{2}P_g V + aT^4 V\right) = \left(\frac{3}{2}P_g + 3P_r\right) dV + \left(\frac{3}{2}V + 3V\right) d(P_g + P_r) =$$

$$= \left(3P_r + \frac{3}{2}P_g\right) dV + V\left[3\frac{\partial P_r}{\partial T} + \frac{3}{2}\frac{\partial P_g}{\partial T}\right] dT + \frac{3}{2}V\left(\frac{\partial P_g}{\partial \rho}\right) d\rho \qquad (21)$$

to find $d\rho$:

$$dP = \left(\frac{\partial P}{\partial T}\right)_\rho dT + \left(\frac{\partial P}{\partial \rho}\right)_T d\rho \quad \Rightarrow$$

$$d\rho = \left\{ dP - \left(\frac{\partial P}{\partial T}\right)_\rho dT \right\} \Big/ \left(\frac{\partial P}{\partial \rho}\right)_T \qquad (22)$$

Inserting (22) in (21) and taking the partial derivatives we have

$$\left(\frac{\partial E}{\partial V}\right)_T = 3p_r + \frac{3}{2}p_g - \frac{3}{2}p\,\frac{\left(\frac{\partial p_g}{\partial \Lambda}\right)_T}{\left(\frac{\partial p}{\partial \Lambda}\right)_T} \tag{23}$$

$$\left(\frac{\partial E}{\partial T}\right)_V = V\left[3\frac{\partial p_r}{\partial T} + \frac{3}{2}\frac{\partial p_g}{\partial T}\right] + \frac{3}{2}V\left(\frac{\partial p_g}{\partial \Lambda}\right)\frac{-\left(\frac{\partial p}{\partial T}\right)_\Lambda}{\left(\frac{\partial p}{\partial \Lambda}\right)_T} \tag{24}$$

To find $\left(\frac{\partial p_r}{\partial T}\right), \left(\frac{\partial p_g}{\partial T}\right)_\Lambda, \left(\frac{\partial p_g}{\partial \Lambda}\right)_T$:

$$dP = dp_r + dp_g = \left(\frac{\partial p_r}{\partial T}\right)dT + \left(\frac{\partial p_g}{\partial T}\right)_\Lambda dT + \left(\frac{\partial p_g}{\partial \Lambda}\right)_T d\Lambda =$$

$$\left[\left(\frac{\partial p_r}{\partial T}\right) + \left(\frac{\partial p_g}{\partial T}\right)_\Lambda\right]dT + \left(\frac{\partial p_g}{\partial \Lambda}\right)_T \frac{\left(d\rho - \left(\frac{\partial \rho}{\partial T}\right)_\Lambda dT\right)}{\left(\frac{\partial \rho}{\partial \Lambda}\right)_T} \tag{25}$$

From (25) $\Longrightarrow$

$$\left(\frac{\partial P}{\partial T}\right)_V = \left(\frac{\partial p_r}{\partial T}\right) + \left(\frac{\partial p_g}{\partial T}\right)_\Lambda - \left(\frac{\partial p_g}{\partial \Lambda}\right)_T \frac{\left(\frac{\partial \rho}{\partial T}\right)_\Lambda}{\left(\frac{\partial \rho}{\partial \Lambda}\right)_T} \tag{26}$$

and

$$\left(\frac{\partial P}{\partial V}\right)_T = \left(\frac{\partial p_g}{\partial \Lambda}\right)_T \frac{(-\mu/V^2)}{(\partial \rho/\partial \Lambda)_T} \tag{27}$$

By inserting relations (23), (24), (26), (27) into (20) we get:

$$\frac{\Gamma_2}{\Gamma_2 - 1} = \frac{T}{P}\left[\left(\frac{\partial p_r}{\partial T}\right) + \left(\frac{\partial p_g}{\partial T}\right)_\Lambda - \left(\frac{\partial p_g}{\partial \Lambda}\right)_T \frac{\left(\frac{\partial \rho}{\partial T}\right)_\Lambda}{\left(\frac{\partial \rho}{\partial \Lambda}\right)_T}\right] - \frac{T}{P}\left[\left[\left\{V\left(3\frac{\partial p_r}{\partial T} + \frac{3}{2}\left(\frac{\partial p_g}{\partial T}\right)_\Lambda\right) + \frac{3}{2}V\left(\frac{\partial p_g}{\partial \Lambda}\right)_T \cdot \frac{\left(-\frac{\partial \rho}{\partial T}\right)_\Lambda}{\left(\frac{\partial \rho}{\partial \Lambda}\right)_T}\right\}\left\{\left(\frac{\partial p_g}{\partial \Lambda}\right)_T \frac{\left(-\frac{\mu}{V^2}\right)}{\left(\frac{\partial \rho}{\partial \Lambda}\right)_T}\right\}\right] \Big/ \left[P + 3p_r + \frac{3}{2}p_g - \frac{3}{2}\rho\left(\frac{\partial p_g}{\partial \Lambda}\right)_T \frac{1}{\left(\frac{\partial \rho}{\partial \Lambda}\right)_T}\right]\right] \tag{28}$$

We can easily calculate now the above partial derivatives from the formulae for the radiation pressure, the partially degenerate electron gas pressure and density

$$\left(\frac{\partial p_r}{\partial T}\right) = \frac{\partial}{\partial T}\left(\frac{1}{3}aT^4\right) = 4\frac{p_r}{T} \tag{29}$$

$$\left(\frac{\partial p_g}{\partial T}\right)_\Lambda = \frac{5}{2}\,\frac{2}{h^3}(2\pi m)^{3/2}\cdot K^{5/2}\cdot T^{3/2}\,U_{3/2} = \frac{5}{2}\,\frac{p_g}{T} \tag{30}$$

$$\left(\frac{\partial p_g}{\partial \Lambda}\right)_T \cdot \frac{(\partial \rho/\partial T)_\Lambda}{(\partial \rho/\partial \Lambda)_T} = \frac{2}{h^3}(2\pi m)^{3/2}(kT)^{5/2}\frac{1}{\Lambda}U_{1/2}\cdot\frac{\frac{3}{2}\cdot\frac{2}{h^3}(2\pi m)^{3/2}K^{3/2}T^{1/2}U_{1/2}\mu_e H}{\frac{2}{h^3}(2\pi m)^{3/2}K^{3/2}T^{3/2}\frac{1}{\Lambda}U_{-1/2}\mu H}$$

$$= \frac{3}{2}\,\frac{p_g}{T}\,\frac{U_{1/2}^2}{U_{3/2}U_{-1/2}} \tag{31}$$

$$\left(\frac{\partial p_g}{\partial \Lambda}\right)_T \frac{-\frac{\mu}{v^2}}{\left(\frac{\partial p}{\partial \Lambda}\right)_T} = -\frac{1}{V} \rho \frac{KT}{\mu_{eff}} \frac{U_{1/2}}{U_{-1/2}} = -\frac{1}{V} P_g \frac{U_{1/2}^2}{U_{-1/2} U_{3/2}} \tag{32}$$

We put the relations 29–32 in 28 and get:

$$\frac{\Gamma_2}{\Gamma_2 - 1} = +\frac{T}{P}\left[\frac{1}{T}\left\{4P_r + \frac{5}{2}P_g - \frac{3}{2}P_g \frac{U_{1/2}^2}{U_{3/2}U_{-1/2}}\right\}\right] - \frac{T}{P}\left[V\left[\frac{1}{T}\left\{12P_r + \frac{15}{4}P_g - \frac{9}{4}P_g\frac{U_{1/2}^2}{U_{3/2}U_{-1/2}}\right\}\right.\right.$$

$$\left.\left.\left\{-\frac{1}{V}\left(P_g\frac{U_{1/2}^2}{U_{3/2}U_{-1/2}}\right)\right\}\right] \Big/ 4P_r + \frac{5}{2}P_g - \frac{3}{2}P_g\frac{U_{1/2}^2}{U_{3/2}U_{1/2}}\right] \tag{33}$$

If we now substitute $P_r = (1-\beta)P$ and $P_g = \beta P$, $\frac{U_{1/2}^2}{U_{3/2}U_{-1/2}} \equiv A \Rightarrow$

$$\frac{\Gamma_2}{\Gamma_2 - 1} = 4(1-\beta) + \frac{5}{2}\beta - \frac{3}{2}\beta A + \left[\left[12(1-\beta) + \frac{15}{4}\beta - \frac{9}{4}\beta A\right]\beta A \Big/ 4(1-\beta) + \frac{5}{2}\beta - \frac{3}{2}\beta A\right] \tag{34}$$

The above relation gives the adiabatic exponent Γ_2 as a function of β and Λ .

We would expect the above formula for $\frac{\Gamma_2}{\Gamma_2 - 1}$ to reduce in its classical values for a monatomic gas of $\gamma = 5/3$ in the two cases for slight and high degeneracy. For a monatomic gas we know that

$$\frac{\Gamma_2}{\Gamma_2 - 1} = \frac{-3\beta^2 - 24\beta + 32}{2(4-3\beta)} = n+1 \begin{cases} = 4 & \text{when } n=3 \text{ (non-degenerate case)} \\ = 5/2 & \text{when } n=3/2 \text{ (extr. degeneracy)} \end{cases}$$

$$\Rightarrow \frac{4}{3} < \Gamma_2 < \frac{5}{2}$$

Indeed, equ. (34) for $\Lambda \ll 1 \Rightarrow U_\nu = \Lambda$ becomes:

$$\frac{\Gamma_2}{\Gamma_2 - 1} = 4(1-\beta) + \frac{5}{2}\beta - \frac{3}{2}\beta + \left[12(1-\beta) + \frac{15}{4}\beta - \frac{9}{4}\beta\right]\beta \Big/ 4(1-\beta) + \frac{5}{2}\beta - \frac{3}{2}\beta \Rightarrow$$

$$= \frac{\left[4(1-\beta) + \beta^2\right]^2 + 12\beta(1-\beta) + \frac{3}{2}\beta^2}{4(1-\beta) + \beta} = \frac{16 - 32\beta + 16\beta^2 + \beta^2 + 8\beta - 8\beta^2 + 12\beta - 12\beta^2 + 3/2\beta^2}{4(1-\beta) + \beta}$$

$$= \frac{-6\beta^2 + 3\beta^2 - 24\beta + 32}{2\left[4(1-\beta) + \beta\right]} = \frac{-3\beta^2 - 24\beta + 32}{2\left[4(1-\beta) + \beta\right]}$$

Where n is defined by the relation

$$\frac{dT}{dr} = \frac{1}{(n+1)_{ad}} \frac{T}{P} \frac{dP}{dr}$$

$$(n+1)_{ad} = \frac{32 - 24\beta - 3\beta^2}{8 - 6\beta} = \frac{\Gamma_2}{\Gamma_2 - 1}$$

which has the physical meaning that the temperature gradient corresponds to adiabatic change of matter and radiation $(dS = 0)$ as above . If the absolute value of radiation temperature gradient

$$\frac{dT}{dr} = -\frac{1}{\lambda} \frac{L(r)}{4\pi r^2} \quad \text{where} \quad \lambda = \frac{4acT^3}{3\rho\kappa}$$

is greater than $\frac{dT}{dr} = \frac{1}{(n+1)_{ad}} \frac{T}{P} \frac{dP}{dr}$ matter is unstable for convection.

A temperature gradient which is very slightly larger than the adiabatic one is sufficient for the convection to transport the energy flux $L(r)$ in the stellar interior . $(n+1)_{ad} = 2.5$ in case of negligible radiation pressure.

(C) DERIVATION OF THE ADIABATIC EXPONENT Γ_1 :

Following the same analysis we can find the other two adiabatic exponents:

$$\Gamma_1 := -\left(\frac{d\ln P}{d\ln V}\right)_{ad} = -\frac{V}{P}\left(\frac{dP}{dV}\right)_{ad} \tag{1}$$

an adiabatic quasi-statistical change is of the form:

$$dQ = dE + PdV = \left(\frac{\partial E}{\partial P}\right)_V dP + \left(\frac{\partial E}{\partial V}\right)_P dV + P\,dV = 0$$

$$\Rightarrow \left(\frac{dP}{dV}\right)_S = -\frac{P + \left(\frac{\partial E}{\partial V}\right)_P}{\left(\frac{dE}{dP}\right)_V}$$

(1) becomes

$$\Gamma_1 = \frac{V}{P}\left[\frac{P + \left(\frac{\partial E}{\partial V}\right)_P}{\left(\frac{\partial E}{\partial P}\right)_V}\right] \tag{2}$$

$$dE = \left(\frac{\partial E}{\partial T}\right)_V dT + \left(\frac{\partial E}{\partial V}\right)_T dV = \left(\frac{\partial E}{\partial T}\right)_V \left[\left(\frac{\partial T}{\partial P}\right)_V dP + \left(\frac{\partial T}{\partial V}\right)_P dV\right] + \left(\frac{\partial E}{\partial V}\right)_T dV \Rightarrow$$

$$\left(\frac{\partial E}{\partial V}\right)_P = \left(\frac{\partial E}{\partial T}\right)_V \left(\frac{\partial T}{\partial V}\right)_P + \left(\frac{\partial E}{\partial V}\right)_T \tag{3}$$

$$\left(\frac{\partial E}{\partial P}\right)_V = \left(\frac{\partial E}{\partial T}\right)_V \left(\frac{\partial T}{\partial P}\right)_V \tag{4}$$

$$dV = \left(\frac{\partial V}{\partial T}\right)_P dT + \left(\frac{\partial V}{\partial P}\right)_T dP = \left(\frac{\partial V}{\partial T}\right)_P dT + \left(\frac{\partial V}{\partial P}\right)_T \left[\left(\frac{\partial P}{\partial T}\right)_V dT + \left(\frac{\partial P}{\partial V}\right)_T dV\right] \Rightarrow$$

$$\left(\frac{\partial V}{\partial T}\right)_V = \left(\frac{\partial V}{\partial T}\right)_P + \left(\frac{\partial V}{\partial P}\right)_T \left(\frac{\partial P}{\partial T}\right)_V = 0 \Rightarrow \left(\frac{\partial V}{\partial T}\right)_P = -\left(\frac{\partial V}{\partial P}\right)_T \left(\frac{\partial P}{\partial T}\right)_V \Rightarrow$$

$$\left(\frac{\partial T}{\partial V}\right)_P = -\frac{1}{\left(\frac{\partial V}{\partial P}\right)_T \left(\frac{\partial P}{\partial T}\right)_V} = -\frac{\left(\partial P/\partial V\right)_T}{\left(\partial P/\partial T\right)_V} \tag{5}$$

Insert (3), (4), (5) in (2) $\Rightarrow$

$$\Gamma_1 = \frac{V}{P}\,\frac{P + \left(\frac{\partial E}{\partial T}\right)_V \left\{-\frac{(\partial P/\partial V)_T}{(\partial P/\partial T)_V}\right\} + \left(\frac{\partial E}{\partial V}\right)_T}{\left(\frac{\partial E}{\partial T}\right)_V \left(\frac{\partial T}{\partial P}\right)_V} \tag{6}$$

$$\Gamma_1 = \frac{V}{P}\,\frac{P + \left(\frac{\partial E}{\partial V}\right)_T}{\left(\frac{\partial E}{\partial T}\right)_V \left(\frac{\partial T}{\partial P}\right)_V} - \frac{V}{P}\left(\frac{\partial P}{\partial V}\right)_T \tag{7}$$

We shall now calculate the partial derivatives involved in equ. (7):

$$dE = (3p_r + \frac{3}{2}p_g)\,dV + V\left[3\frac{\partial p_r}{\partial T} + \frac{3}{2}\frac{\partial p_g}{\partial T}\right]dT + \frac{3}{2}V\left(\frac{\partial p_g}{\partial \Lambda}\right)\frac{d\rho - (\partial\rho/\partial T)_\Lambda\,dT}{(\partial\rho/\partial\Lambda)_T} \Rightarrow$$

$$\left(\frac{\partial E}{\partial V}\right)_T = 3p_r + \frac{3}{2}p_g - \frac{3}{2}\rho\frac{(\partial p_g/\partial\Lambda)_T}{(\partial\rho/\partial\Lambda)_T} \quad (8)$$

and

$$\left(\frac{\partial E}{\partial T}\right)_V = V\left[3\left(\frac{\partial p_r}{\partial T}\right) + \frac{3}{2}\left(\frac{\partial p_g}{\partial T}\right)\right] - \frac{3}{2}V\left(\frac{\partial p_g}{\partial\Lambda}\right)\frac{(\partial\rho/\partial T)_\Lambda}{(\partial\rho/\partial\Lambda)_T} \quad (9)$$

$$dP = \left(\frac{\partial P}{\partial T}\right)_\Lambda dT + \left(\frac{\partial P}{\partial\Lambda}\right)_T d\Lambda \quad (10)$$

$$d\rho = \left(\frac{\partial\rho}{\partial T}\right)_\Lambda dT + \left(\frac{\partial\rho}{\partial\Lambda}\right)_T d\Lambda \quad (11)$$

from (11) $\Rightarrow d\Lambda = \left\{ d\rho - \left(\frac{\partial\rho}{\partial T}\right)_\Lambda dT \right\} \Big/ \left(\frac{\partial\rho}{\partial\Lambda}\right)_T$ (12)

we substitute (12) in relation (10) and get

$$dP = \left(\frac{\partial P}{\partial T}\right)_\Lambda dT + \left(\frac{\partial P}{\partial\Lambda}\right)_T \left\{ d\rho - \left(\frac{\partial\rho}{\partial T}\right)_\Lambda dT \right\} \Big/ \left(\frac{\partial\rho}{\partial\Lambda}\right)_T \Rightarrow$$

$$\left(\frac{\partial P}{\partial V}\right)_T = \left(\frac{\partial P}{\partial\Lambda}\right)_T (-\rho^2) \Big/ \left(\frac{\partial\rho}{\partial\Lambda}\right)_T \quad (13)$$

$$\left(\frac{\partial P}{\partial T}\right)_V = \left(\frac{\partial p_g}{\partial T}\right)_\Lambda + \left(\frac{\partial p_r}{\partial T}\right) - \left(\frac{\partial p_g}{\partial\Lambda}\right)_T \left(\frac{\partial\rho}{\partial T}\right)_\Lambda \Big/ \left(\frac{\partial\rho}{\partial\Lambda}\right)_T \quad (14)$$

By substituting relations (8), (9), (13), (14) into relation (7) we finally get

$$\Gamma_1 = \frac{1}{P}\,\frac{(4p_r + \frac{5}{2}p_g - \frac{3}{2}p_g A)^2}{12p_r + \frac{15}{4}p_g - \frac{9}{4}p_g A} + \frac{1}{P}p_g A$$

for $p_r = (1-\beta)P$ and $p_g = \beta P$, Γ_1 becomes:

$$\Gamma_1 = \frac{\left[4(1-\beta) + \frac{5}{2}\beta - \frac{3}{2}\beta A\right]^2}{12(1-\beta) + \frac{15}{4}\beta - \frac{9}{4}\beta A} + \beta A \quad (15)$$

for $\Lambda \ll 1$ (slight degeneracy) Γ_1 becomes

$$\Gamma_1 = \frac{\left[4(1-\beta) + \beta\right]^2}{12(1-\beta) + \frac{3}{2}\beta} + \beta = \frac{2(4-3\beta)^2 + 24\beta(1-\beta) + 3\beta^2}{24(1-\beta) + 3\beta} \quad (16)$$

for $\Lambda \gg 1 \Rightarrow$

$$\frac{U_{1/2}^2}{U_{-1/2}U_{3/2}} = \frac{\left[(\log\Lambda)^{1/2+1}\frac{1}{\Gamma(\frac{1}{2}+2)}\right]^2}{(\log\Lambda)^{-1/2}(\log\Lambda)^{3/2+1}\frac{1}{\Gamma(3/2)}\cdot\frac{1}{\Gamma(7/2)}} = \frac{(\log\Lambda)^3}{(\log\Lambda)^3}\frac{\frac{15\pi}{16}}{\frac{9\pi}{16}} = \frac{5}{3}$$

From (15) $\Rightarrow$

$$\Gamma_1 = \frac{\left[4(1-\beta)+\frac{5}{2}\beta-\frac{3}{2}\beta\cdot\frac{5}{3}\right]^2}{12(1-\beta)+\frac{15}{4}\beta-\frac{9}{4}\beta\frac{5}{3}} + \frac{5}{3}\beta = \frac{[4(1-\beta)]^2}{12(1-\beta)}+\frac{5}{3}\beta \Rightarrow$$

$$\Gamma_1 = \frac{4}{3}\frac{(1-\beta)^2}{(1-\beta)}+\frac{5}{3}\beta \qquad (17)$$

For $\beta=0 \Rightarrow \Gamma_1 = 4/3$

For $\beta=1 \Rightarrow \Gamma_1 = 5/3$

(D) DERIVATION OF THE ADIABATIC EXPONENT Γ_3.

For the third adiabatic exponent we get:

$$\Gamma_3 - 1 := -\left(\frac{d\ln T}{d\ln V}\right)_{ad} = -\frac{V}{T}\left(\frac{dT}{dV}\right)_{ad} \tag{1}$$

$$dQ = \left(\frac{\partial E}{\partial T}\right)_V dT + \left(\frac{\partial E}{\partial V}\right)_T dV + P\,dV = 0 \Rightarrow$$

$$\left(\frac{dT}{dV}\right)_{ad} = -\frac{P + (\partial E/\partial V)_T}{(\partial E/\partial T)_V} \tag{2}$$

$$(1) \Rightarrow \qquad \Gamma_3 - 1 = \frac{V}{T}\,\frac{P + (\partial E/\partial V)_T}{(\partial E/\partial T)_V} \tag{3}$$

from the previous calculations we take:

$$\left(\frac{\partial E}{\partial V}\right)_T = 3P_r + \frac{3}{2}P_g - \frac{3}{2}P\,(\partial P_g/\partial n)_T \,/\, (\partial P/\partial n)_T \tag{4}$$

$$\left(\frac{\partial E}{\partial T}\right)_V = V\left[3\,\frac{\partial P_r}{\partial T} + \frac{3}{2}\left(\frac{\partial P_g}{\partial T}\right)_n\right] - \frac{3}{2}V\left(\frac{\partial P_g}{\partial n}\right)_T \frac{(\partial P/\partial T)_n}{(\partial P/\partial n)_T} \tag{5}$$

(3) becomes :

$$\Gamma_3 - 1 = \frac{1}{T}\,\frac{4P_r + \frac{5}{2}P_g - \frac{3}{2}P_g\frac{U_{1/2}^2}{U_{3/2}U_{-1/2}}}{\frac{1}{T}\left[12P_r + \frac{15}{4}P_g - \frac{9}{4}P_g\frac{U_{1/2}^2}{U_{-1/2}U_{3/2}}\right]} = \frac{4(1-\beta) + \frac{5}{2}\beta - \frac{3}{2}\beta A}{12(1-\beta) + \frac{15}{4}\beta - \frac{9}{4}\beta A} \tag{6}$$

$$\Gamma_3 - 1 = \frac{4-3\beta}{12(1-\beta) + 3/2\,\beta} = \frac{2(4-3\beta)}{24-21\beta} \qquad (\text{for } \lambda \ll 1)$$

relation (6) is exactly the classical relation for $\Gamma_3 - 1$ for a monatomic gas of $\gamma = 5/3$

$$\Gamma_3 - 1 = \frac{(4-3\beta)(\gamma-1)}{\beta + 12(\gamma-1)(1-\beta)} .$$

We can also check our relations for Γ_1, $\frac{\Gamma_2}{\Gamma_2 - 1}$, $\Gamma_3 - 1$

by the formula $\frac{\Gamma_2 - 1}{\Gamma_2}\cdot\Gamma_1 = \Gamma_3 - 1$

Indeed, we get:

$$\frac{\Gamma_2 - 1}{\Gamma_2}\cdot\Gamma_1 = \frac{4(1-\beta) + 5/2\,\beta - 3/2\,\beta A}{\left[4(1-\beta) + \frac{5}{2}\beta A - \frac{3}{2}\beta A\right]^2 + 12A(1-\beta) + \frac{15}{4}\beta^2 A - \frac{9}{4}\beta^2 A^2} .$$

$$\cdot \left[\frac{\left[4(1-\beta) + \frac{5}{2}\beta - \frac{3}{2}\beta A \right]^2}{12(1-\beta) + \frac{15}{4}\beta - \frac{9}{4}\beta A} + \beta A \right] =$$

$$\frac{4(1-\beta) + \frac{5}{2}\beta - \frac{3}{2}\beta A}{\left[4(1-\beta) + \frac{5}{2}\beta - \frac{3}{2}\beta A \right]^2 + 12A\beta(1-\beta) + \frac{15}{4}\beta^2 A - \frac{9}{4}\beta^2 A^2} \cdot$$

$$\cdot \frac{\left[4(1-\beta) + \frac{5}{2}\beta - \frac{3}{2}\beta A \right]^2 + 12A\beta(1-\beta) + \frac{15}{4}A\beta^2 - \frac{9}{4}\beta^2 A^2}{12(1-\beta) + \frac{15}{4}\beta - \frac{9}{4}\beta A} =$$

$$= \frac{4(1-\beta) + \frac{5}{2}\beta - \frac{3}{2}\beta A}{12(1-\beta) + \frac{15}{4}\beta - \frac{9}{4}\beta A} = \Gamma_3 - 1 \qquad (6)$$

From the above discussion, it is obvious that in the case of the partially degenerate stellar configurations the adiabatic exponents depend upon the degree of degeneracy Λ and also upon β, the ratio of the gas pressure to the total pressure.

We compute $\Gamma_1, \Gamma_2, \Gamma_3$ at each step of our numerical integration of the basic differential equation (28) Chapter II. We can see that the values of the adiabatic exponents vary along the configuration.

In the tables below, we record the surface and the boundary values of $\Gamma_1, \Gamma_2, \Gamma_3$ for various degrees of degeneracy and β's.

$1-\beta$	$\Gamma_1(\Lambda,\beta)$	$\frac{\Gamma_2}{\Gamma_2-1}(\Lambda,\beta)$	$\Gamma_3(\Lambda,\beta)$	
10^{-1}	1.56664135 1.7047630	3.1288664 2.7538469	1.5006329 1.6190479	center surface
10^{-2}	1.6511034 1.9457678	2.5835562 2.0968913	1.6390817 1.9279297	
10^{-3}	1.6650118 1.9940656	2.5086330 2.0099658	1.6637128 1.9920894	
10^{-4}	1.6665001 1.9994030	2.5008662 2.0009973	1.6663692 1.9992033	
10^{-5}	1.6666500 1.9999424	2.5000866 2.0000976	1.6666369 1.9999224	

TABLE 28. ADIABATIC EXPONENTS FOR A PARTIALLY DEGENERATE ELECTRON GAS OF $\Lambda_0 = 0.5$

$1-\beta$	$\Gamma_1(\eta,\beta)$	$\frac{\Gamma_2}{\Gamma_2-1}(\eta,\beta)$	$\Gamma_3(\eta,\beta)$	
10^{-1}	1.5691151 1.7047810	3.1771259 2.7538588	1.4938788 1.6190517	center surface
10^{-2}	1.6511883 1.9458002	2.5922973 2.0968611	1.6369595 1.9279586	
10^{-3}	1.6650128 1.9941028	2.5095688 2.0099288	1.6634657 1.9921261	
10^{-4}	1.6665001 1.9994407	2.5009604 2.0009596	1.6663441 1.9992409	
10^{-5}	1.6666500 1.9999801	2.5000961 2.0000599	1.6666344 1.9999601	

TABLE 29. ADIABATIC EXPONENTS FOR A PARTIALLY DEGENERATE ELECTRON GAS OF $\Lambda_0 = 1$

$1-\beta$	$\Gamma_1(\Lambda,\beta)$	$\frac{\Gamma_2}{\Gamma_2-1}(\Lambda,\beta)$	$\Gamma_3(\Lambda,\beta)$	
10^{-1}	1.5730163 1.7047674	3.2491880 2.7538498	1.4841260 1.6190488	center surface
10^{-2}	1.6513219 1.9457756	2.6061693 2.096884	1.6336203 1.9279367	
10^{-3}	1.6650144 1.9940746	2.5110664 2.0099569	1.6630706 1.9920982	
10^{-4}	1.6665001 1.9994121	2.5011113 2.0009882	1.6663039 1.9992123	
10^{-5}	1.6666500 1.9999515	2.5001112 2.0000885	1.6666304 1.9999315	

TABLE 30. ADIABASTIC EXPONENTS FOR A PARTIALLY DEGENERATE ELECTRON GAS WITH $\Lambda_0 = 2.0$

CHAPTER V

LUMINOSITY OF A COMPLETELY CONVECTIVE STELLAR MODEL

In a completely convective star, in which most of the flux is carried by convection, the luminosity, or the rate of radiation of energy, which is carried outward through a sphere of radius r is given by the mixing length theory as:

$$L = (\pi/\sqrt{2})\, r^2 Q\, g^{1/2} (\rho/T^{1/2})\, c_p\, l^2 (\Delta\nabla T)^{3/2} \qquad (1)$$

where g = local gravitational acceleration of the star

$Q = 1$ if μ constant

$\Delta\nabla T = (-dT/dr) - (-dT/dr)_{adiab}$

c_p = specific heat per unit mass at constant pressure

l = mixing length, i.e. the characteristic distance at which the moving elements dissolve and merge smoothly into the surroundings, giving any excess energy they possess or absorbing any defect

PROOF OF THE FORMULA FOR CONVECTIVE LUMINOSITY

If δT is the temperature difference between the element and its surroundings then the excess energy per unit volume is $\rho c_p \delta T$

$$\delta T = \left[\left(-\frac{dT}{dr}\right)_{\text{mean surrounding}} - \left(-\frac{dT}{dr}\right)_{\text{individual convective elements}} \right] \Delta r =$$

$$T\left[-\left(\frac{d\ln T}{dr}\right)_{s.} - \left(-\frac{d\ln T}{dr}\right)_{el} \right] = T\frac{d\ln P}{dr}\left[\left(-\frac{d\ln T}{d\ln P}\right)_{ms} - \left(-\frac{d\ln T}{d\ln P}\right)_{el} \right] =$$

$$-T\,\frac{g\rho}{P}\left[\nabla_{M.S.} - \nabla_E \right] \qquad (2)$$

The energy flux transported by elements moving with velocity $\bar{v}$ is

$$\rho c_p \delta T \bar{v} = \pi F_{conv} \quad (3)$$

A simplification is made hre by averaging, over all elements, the paths of travel

We set $\Delta r = \frac{\ell}{2}$

(3) becomes

$$\pi F_{conv} = \frac{1}{2} \frac{g \rho^2 c_p T \bar{v}}{P} \ell (\nabla_{m.r} - \nabla_{el}) \quad (4)$$

We introduce the pressure scale height λ_p :

$$\frac{1}{\lambda_p} = - \frac{d \ln P}{dr} = \frac{g \rho}{P}$$

(4) becomes

$$\pi F_{conv} = \frac{1}{2} \rho c_p \bar{v} T \left(\frac{\ell}{\lambda_p} \right) (\nabla - \nabla_E) \quad (5)$$

We need an expression for the velocity $\bar{v}$.

If $\delta\rho$ is the density difference between the element and its surroundings, the buoyant force is

$$f = - g \delta \rho \quad (6)$$

For perfect gas :

$$PV = T \Rightarrow P \frac{\mu}{\rho} = T \Rightarrow$$

$$\log P + \log \mu - \log T = \log \rho \Rightarrow \frac{d\rho}{\rho} = \frac{dP}{P} - \frac{dT}{T} + \left(\frac{\partial \log \mu}{\partial \log T} \right)_P \frac{dT}{T} \Rightarrow$$

$$\frac{d\rho}{\rho} = \frac{dP}{P} - Q \frac{dT}{T} \quad (7)$$

where $Q = 1 - \left(\frac{\partial \log \mu}{\partial \log T} \right)_P = 1$ when μ constant

Inasmuch pressure equilibrium exists,

(7) becomes: $\frac{dp}{p} = -Q \frac{dT}{T}$

or $$\frac{\delta p}{p} = -Q \frac{\delta T}{T} \qquad (8)$$

(6) becomes then,

$$f = g\rho\, Q \frac{\delta T}{T} = g \frac{Q\rho}{T}\left[\left(-\frac{dT}{dr}\right)_{ms} - \left(-\frac{dT}{dr}\right)_{el}\right]\Delta r \qquad (9)$$

Integrating (9) over some displacement $\frac{\ell}{2}$, the work done by f is

$$W = \frac{1}{2}\frac{gQ\rho}{T}\left[\left(-\frac{dT}{dr}\right)_{m.s.} - \left(-\frac{dT}{dr}\right)_{el}\right]\frac{\ell^2}{4}$$

from (2)

$$= \frac{1}{8}(g\rho Q \lambda_p)(\nabla - \nabla_E)\frac{\ell^2}{\lambda_p}$$

We estimate that half of this work will end up as the Kinetic energy of the element and the other half will be lost to friction with the other neighbouring elements.

Therefore:

$$\frac{1}{2}\rho\bar{v}^2 = \frac{1}{16}(\rho g Q \lambda_p)(\nabla - \nabla_{el})\left(\frac{\ell}{\lambda_p}\right)^2 \qquad (10)$$

$$\Rightarrow \quad \bar{v} = \frac{1}{2\sqrt{2}}(gQ\lambda_p)^{1/2}(\nabla - \nabla_E)^{1/2}\frac{\ell}{\lambda_p} \qquad (11)$$

This analysis is not physically valid if $\bar{v} > v_{sound}$

If this happens the assumption of pressure equilibrium between the convective element and its surroundings would not be a realistic condition.

Substituting this result in (5)

$$\pi F_{conv} = \frac{1}{4\sqrt{2}}(gQ\lambda_p)^{1/2}(\rho c_p T)\left(\frac{\ell}{\lambda_p}\right)^2(\nabla - \nabla_E)^{3/2} \Rightarrow$$

$$\pi F_{conv} = \frac{1}{4\sqrt{2}} g^{1/2} Q^{1/2}(\lambda_p)^{1/2}(\rho c_p T)\left(\frac{\ell}{\lambda_p}\right)^2\left(\frac{\lambda_p}{T}\right)^{3/2}(\Delta\nabla T)^{3/2}$$

$$= \frac{1}{4\sqrt{2}} Q^{1/2} g^{1/2} c_p (\rho/T^{1/2})\,\ell^2(\Delta\nabla T)^{3/2} \Rightarrow \qquad (12)$$

$$L_{conv} = \pi \cdot F \cdot 4\pi r^2 = \frac{\pi}{\sqrt{2}} r^2 Q^{1/2} g^{1/2}\left(\frac{\rho}{T^{1/2}}\right) c_p \ell^2 (\Delta\nabla T)^{3/2} \qquad (13)$$

Equation (13) gives us an approximate value for the luminosity of completely convective models.

Unlike the luminosity of a radiative model which depends upon the temperature gradient through the relation

$$L_{rad} \simeq -4\pi r^2 \frac{4ac}{3} \frac{T^3}{\kappa\rho} \frac{dT}{dr}$$

the $L_{conv} = (\Delta\nabla T)^{3/2}$ depends upon the excess of the temperature gradient over the adiabatic gradient.

Relation (13) is also based on the assumption that the convective elements move adiabatically.

One of the fundamental uncertainties in the theory of mixing length is the question of how to choose an appropriate value of ℓ.

The usual prescription is to use the local pressure scale height, or else the density pressure scale height but the procedure of choosing the mixing length is rather an arbitrary one.

In the following tables we calculate the luminosity using (13) and the pressure scale height

$$\frac{1}{\lambda_p} := \frac{g\rho}{P} = -\frac{d\ln P}{dr}$$ at each point

of the model.

Characteristic values of the convective luminosity are obtained for a model with $\Lambda_0 = 1$,

values of $1-b$ in the range of 10^{-1} to 10^{-4}

and values of μ_e (mean molecular wt. per free electron) 1, 1.5 and 2.0.

	$1-\beta$	$L_{conv}/L_\odot$		$L^*_{conv}/L_\odot$		$\lambda_P = -1/\frac{d\ln P}{dr}$	
$\psi_c=1$	10^{-1}	0.199	(10^{-1})	0.414	(10^{-1})	0.276	(10^{8})
	10^{-2}	0.617	(10^{-3})	0.354	(10^{-3})	0.392	(10^{8})
	10^{-3}	0.148	(10^{-4})	0.640	(10^{-5})	0.572	(10^{8})
$\psi_c=1.5$	10^{-1}	0.700	(10^{-2})	0.225	(10^{-1})	0.183	(10^{8})
	10^{-2}	0.336	(10^{-3})	0.193	(10^{-3})	0.260	(10^{8})
	10^{-3}	0.803	(10^{-5})	0.349	(10^{-5})	0.382	(10^{8})
$\psi_c=2$	10^{-1}	0.455	(10^{-2})	0.146	(10^{-1})	0.137	(10^{8})
	10^{-2}	0.218	(10^{-3})	0.125	(10^{-3})	0.195	(10^{8})
	10^{-3}	0.521	(10^{-5})	0.226	(10^{-5})	0.282	(10^{8})

TABLE 31. LUMINOSITY OF COMPLETELY CONVECTIVE PARTIALLY DEGENERATE STELLAR MODEL OF $\Lambda_0=1$

The last column shows the pressure scale height at the surface $L^*_{conv}/L_\odot$ is the mixing length luminosity (as $L_{conv}/L_\odot$ is) with $\left|\frac{dT}{dr}\right| = \frac{T_c}{R}$.

RADIATIVE METHOD OF EVALUATION OF THE LUMINOSITY BY THE USE OF OPACITY TABLES

Another method for an approximate evaluation of the luminosity of a completely convective stellar model is based upon an opacity law as

$$K = Ko\ \rho^{m}\ T^{n}$$

Given the opacity tables we try to fit the above relations in the tables. To do that we need to solve the system of the equations.

$$\begin{aligned} K_1 &= Ko\ \rho_1^{m}\ T_1^{n} \\ K_2 &= Ko\ \rho_2^{m}\ T_2^{n} \\ K_3 &= Ko\ \rho_3^{m}\ T_3^{n} \end{aligned} \qquad (1)$$

and to find the Ko, m and n for the given triplet of values ρ, T and K.

We next consider a relation for the density and the temperature of our models of the form

$$\rho = \rho a\ (R-r)$$

$$T = Ta\ (R-r)$$

where $\rho_a = \left|\frac{D\rho}{DR}\right|$ and $T_a = \left|\frac{DT}{DR}\right|$

It is easy now to find the optical depth from the relation

$$\tau = \int K \rho\, dr$$

$$= \int K_o \left[\rho_a (R-r)\right]^{m+1} \left[T_a (R-r)\right]^{n} dr$$

$$= K_o \rho_a^{m+1} T_a^{n} \int_r^R (R-r)^{m+n+1} dr \qquad \Rightarrow$$

$$\tau(r) = K_o \rho_a^{m+1} T_a^{n} \frac{(R-r)^{m+n+2}}{m+n+2}$$

For $\tau(r) = 2/3 \Rightarrow$

$$(R-r) = \left[\frac{2}{3} \frac{m+n+2}{K_0 \rho_a^{m+1} T_a^n} \right]^{1/m+n+2}$$

$$T(\tau = 2/3) = T_a \left[\frac{2}{3} \frac{m+n+2}{K_0 \rho_a^{m+1} T_a^n} \right]^{1/m+n+2} \quad (2)$$

We thus, get the actual temperature at the photosphere which is the effective temperature T_{eff}.

We can now find a value for the luminosity from the classical formula

$$L = 4\pi\sigma R^2 T_{eff}^4 \quad (3)$$

Relation (3) for the luminosity is justified according to the theory that the radiation from an actual stellar atmosphere has approximately the same character as the radiation which would be emitted by a black body surface whose temperature is T_{eff}. Since, by definition, the photosphere is the layer from which the energy being transferred up from the interior is radiated into space, the material above the photosphere must be predominantly in radiative equilibrium.

In equation (3), R is the radius of the surface (or photosphere) of the star and σ is the Stefan-Beltzmann constant.

APPLICATION:

We use Cox's tables for a Limber I mixture of X = 1.0, Y = 0.0, Z = 0.0 for a partially degenerate standard model of $\Lambda_0 = 1$, $\mu_e = 2$, $1-\beta = 10^{-2}$

The last two points of our integration give the surface values for the radius, temperature and density as:

$$\begin{matrix} 0.6765787060 & 10^{10} \\ 0.6842638081 & 10^{10} \end{matrix} \Rightarrow \quad -\Delta R = 0.7688396835 \times 10^{8}$$

$$\begin{matrix} 0.14251739 & 10^{7} \\ 0.71696870 & 10^{6} \end{matrix} \Rightarrow \quad -\frac{\Delta T}{\Delta R} = 0.01 \quad (=T_a)$$

$$\begin{matrix} 0.17487025 & 10^{-1} \\ 0.22266105 & 10^{-2} \end{matrix} \Rightarrow \quad -\frac{\Delta \rho}{\Delta R} = 1.985 \times 10^{-10} \quad (= \rho_a)$$

From the opacity tables a system of 3 equations as the system (1), corresponding to the above values will be one with

$$
\begin{array}{lll}
T_1 = 5 \times 10^5 & T_2 = 2 \times 10^5 & T_3 = 1 \times 10^5 \\
\rho_1 = 1 \times 10^{-3} & \rho_2 = 1 \times 10^{-4} & \rho_3 = 1 \times 10^{-5} \\
K_1 = 2.14 & K_2 = 9.46 & K_3 = 36.8
\end{array}
\qquad (4)
$$

The system (1) is equivalent to

$$
\begin{aligned}
\ln\left(\frac{K_1}{K_2}\right) &= m \ln\left(\frac{\rho_1}{\rho_2}\right) + n \ln\left(\frac{T_1}{T_2}\right) \\
\ln\left(\frac{K_2}{K_3}\right) &= m \ln\left(\frac{\rho_2}{\rho_3}\right) + n \ln\left(\frac{T_2}{T_3}\right) \\
K_1 &= K_0 \, \rho_1^{m} \, T_1^{n}
\end{aligned}
\qquad (5)
$$

Substituting the set of values (4) in system (5) we get

$m = -0.422$ $\qquad n = -0.560$ $\qquad K_0 = 207.73$

From the relation (2) and by substituting the known quantities we get a very small value for the effective temperature inconsistent with the physical situation in our models.

From the opacity tables we should expect values of and as $m = 1$ and $n = -3.5$.

APPENDIX I

PROGRAM FOR THE NUMERICAL SOLUTION OF THE LANE-EMDEN EQUATION FOR $n=0.0(0.5)4.5$

The Fortran IV program for the solution of the Lane-Emden equation as described in Chapter I is given, as well as the subroutine for the computation of the exact values of the solution for $n=0.0, 1.5, 5.0$

```
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION V(700)
      DIMENSION DY(705),D1YP(705),D2YP(705),D3YP(705),D4YP(705)
      DIMENSION D1YPP(705),D2YPP(705),D4YPP(705),D3YPP(705)
      DIMENSION CY(705),CYP(705),CYPP(705)
      DIMENSION CDY(705)
      DIMENSION CD1YP(705),CD2YP(705),CD3YP(705),CD4YP(705)
      DIMENSION D5YP(705),CD5YP(705),D5YPP(705)
      DIMENSION X(705),Y(705),YP(705),YPP(705)
      DIMENSION D6YP(705),CD6YP(705),D6YPP(705)
       DIMENSION VARM(700),RORA(700)
      CALL CLEUND
      DOUBLE PRECISION N
       DO 77 IN=1,11
      N=FLOAT(IN-1)/2.0
      WRITE(6,50) N
 50   FORMAT(1H1,18H POLYTROPIC INDEX=,F5.2)
      I=1
      DX=0.03
      Y(I)=1.0
      X(I)=0.0
      YP(I)=0.0
      YPP(I)=-1.0/3.0
      DO 10 I=1,7
      II=I-1
      XS=X(I)*X(I)
      V(I)=1.0-XS*(1.0/6.0-XS*(N/120.0-XS*((8.0*N-5.0)*N/15120.0-XS
     2*((70.0*N-183.0*N**2+122.0*N**3)/3265920.0+XS*((2800*N-8865*N**2+
     39929*N**3-3909*N**4)/1796256000.0)))))
      Y(I)=1.0-XS*(1.0/6.0-XS*(N/120.0-XS*((8.0*N-5.0)*N/15120.0-XS
     2*((70.0*N-183.0*N**2+122.0*N**3)/3265920.0+XS*((3150.0*N-10805.0
     3*N**2+12642.0*N**3-5032.0*N**4)/1796256000.0)))))
      CY(I)=Y(I)
      YP(I)=-X(I)*(1.0/3.0-XS*(N/30.0-XS*(8.0*N**2-5.0*N)/2520.0-
     2XS*((70.0*N-183.0*N**2+122.0*N**3)/408240.0+XS*((3150.0*N-10805.0
     3*N**2+12642.0*N**3-5032.0*N**4)/179625600.0))))
      CYP(I)=YP(I)
      IF (I-1)25,25,20
   20 YPP(I)=-Y(I)**N-2.0/X(I)*YP(I)
      CYPP(I)=YPP(I)
```

```
      DY(I)=Y(I)-Y(II)
      D1YP(I)=YP(I)-YP(II)
      CD1YP(I)=D1YP(I)
      D1YPP(I)=YPP(I)-YPP(II)
      IF(I-2)21,21,22
   22 D2YP(I)=D1YP(I)-D1YP(II)
      CD2YP(I)=D2YP(I)
      D2YPP(I)=D1YPP(I)-D1YPP(II)
      IF(I-3)21,21,23
   23 D3YP(I)=D2YP(I)-D2YP(II)
      CD3YP(I)=D3YP(I)
      D3YPP(I)=D2YPP(I)-D2YPP(II)
      IF(I-4)21,21,24
   24 D4YP(I)=D3YP(I)-D3YP(II)
      CD4YP(I)=D4YP(I)
      D4YPP(I)=D3YPP(I)-D3YPP(II)
      IF(I-5)21,21,26
   26 D5YP(I)=D4YP(I)-D4YP(II)
      CD5YP(I)=D5YP(I)
      D5YPP(I)=D4YPP(I)-D4YPP(II)
      IF(I-6)21,21,27
   27 D6YP(I)=D5YP(I)-D5YP(II)
      CD6YP(I)=D6YP(I)
      D6YPP(I)=D5YPP(I)-D5YPP(II)
   25 CONTINUE
   21 CONTINUE
      CALL SUB(N,X(I),I,EY)
      VARM(I)=X(I)**2*YP(I)
      RCRA(I)=1.0/3.0*X(I)*(1.0/YP(I))
      X(I+1)=X(I)+DX
      WRITE(6,3)I,X(I),Y(I),YP(I),YPP(I),VARM(I),RCRA(I)
 3    FORMAT(I4,6D20.8)
  10  CONTINUE
 13   DO 88 I=8,700
      II=I-1
      X(I)=X(II)+DX
      D1YP(I)=DX*(YPP(II)+1.0/2.0*D1YPP(II)+5.0/12.0*D2YPP(II)+
     23.0/8.0*D3YPP(II)+251.0/720.0*D4YPP(II)+95.0/288.0*D5YPP(II)
     3+19087.0/60480.0*D6YPP(II))
```

```
      YP(I)=YP(II)+D1YP(I)
      D2YP(I)=D1YP(I)-D1YP(II)
      D3YP(I)=D2YP(I)-D2YP(II)
      D4YP(I)=D3YP(I)-D3YP(II)
      D5YP(I)=D4YP(I)-D4YP(II)
      D6YP(I)=D5YP(I)-D5YP(II)
      DY(I)=DX*(YP(I)-1.0/2.0*D1YP(I)-1.0/12.0*D2YP(I)-
     21.0/24.0*D3YP(I)-19.0/720.0*D4YP(I)-3.0/160.0*D5YP(I)
     3-863.0/60480.0*D6YP(I))
      Y(I)=Y(II)+DY(I)
      IF(Y(I))99,99,98
   98 YPP(I)=-Y(I)**N-2.0/X(I)*YP(I)
      D1YPP(I)=YPP(I)-YPP(II)
      D2YPP(I)=D1YPP(I)-D1YPP(II)
      D3YPP(I)=D2YPP(I)-D2YPP(II)
      D4YPP(I)=D3YPP(I)-D3YPP(II)
      D5YPP(I)=D4YPP(I)-D4YPP(II)
      D6YPP(I)=D5YPP(I)-D5YPP(II)
C     CHECKING   FORMULA
      CD1YP(I)=DX*(YPP(I)-1.0/2.0*D1YPP(I)-1.0/12.0
     2*D2YPP(I)-1.0/24.0*D3YPP(I)-19.0/720.0*D4YPP(I)
     3-3.0/160.0*D5YPP(I)-863.0/60480.0*D6YPP(I))
      CYP(I)=CYP(II)+CD1YP(I)
C     CORRECTED VALUE FOR THETA PRIME. DIFFERENCES
      YP(I)=CYP(I)
      VARM(I)=X(I)**2*YP(I)*(-1.0)
      RCRA(I)=1.0/3.0*X(I)*(1.0/YP(I))*(-1.0)
      CD2YP(I)=CD1YP(I)-CD1YP(II)
      CD3YP(I)=CD2YP(I)-CD2YP(II)
      CD4YP(I)=CD3YP(I)-CD3YP(II)
      CD5YP(I)=CD4YP(I)-CD4YP(II)
      CD6YP(I)=CD5YP(I)-CD5YP(II)
      CDY(I)=DX*(CYP(I)-1.0/2.0*CD1YP(I)-1.0/12.0*CD2YP(I)-1.0/24.0
     2*CD3YP(I)-19.0/720.0*CD4YP(I)-3.0/160.0*CD5YP(I)
     3-863.0/60480.0*CD6YP(I))
C     CORRECTED VALUE FOR THETA. DIFFERENCES
      CY(I)=CY(II)+CDY(I)
      IF(CY(I))99,99,97
C     CORRECTED VALUE FOR THETA DOUBLE PRIME.
```

```
   97 CYPP(I)=-CY(I)**N-2.C/X(I)*CYP(I)
      YPP(I)=CYPP(I)
      C1YPP(I)=CYPP(I)-CYPP(II)
      C2YPP(I)=C1YPP(I)-C1YPP(II)
      D3YPP(I)=D2YPP(I)-C2YPP(II)
      C4YPP(I)=D3YPP(I)-C3YPP(II)
      C5YPP(I)=C4YPP(I)-C4YPP(II)
      C6YPP(I)=D5YPP(I)-C5YPP(II)
C     SUBROLTINE TC CCMPLTE THE EXACT VALLES CF THE LANE-EMDEN EQU.FCR N=C:
      CALL SUB(N,X(I),I,EY)
      CYPP(I)=YPP(I)
      WRITE(6,3)I,X(I),Y(I),YP(I),YPP(I),VARM(I),RCRA(I)
   88 CCNTINUE
   99 CCNTINUE
      CONTINUE
 77   CCNTINUE
      STCP
      END
```

```
      SUBRCLTINE SLB(N,X,I,EY)
      DCLBLE PRECISICN N,X,EY
      EY=0.C
      IF(N)80,80,81
   80 EY=1.C-X**2/6.C
      GO TO 84
   81 IF(N-1.C)84,83,82
   83 IF(I-1)86,86,87
   86 EY=1.C
      GC TC 84
   87 EY=DSIN(X)/X
      GC TC 84
   82 IF(N-5.C)84,85,84
   85 EY=1.0/DSQRT(1.C+X**2/3.C)
   84 CCNTINLE
      RETURN
      END
```

APPENDIX II

PROGRAM FOR THE NUMERICAL SOLUTION OF THE PARTIALLY DEGENERATE STANDARD MODEL FUNCTION

Fortran IV program for the numerical integration of the partially degenerate standard model function $\lambda(\eta)\,(=Y(I))$ as described in Chapter II.

```
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION AMASS(700),BMASS(700),ARAD(700),RAD(700)
      DIMENSION D1YPP(705),D2YPP(705),D3YPP(705),D4YPP(705)
      DIMENSION D5YP(705),D6YP(705),D5YPP(705),D6YPP(705)
      DIMENSION CDY(705),CD1YP(705),CD2YP(705),CD3YP(705)
      DIMENSION CD4YP(705),CD5YP(705),CD6YP(705)
      DIMENSION DY(705),D1YP(705),D2YP(705),D3YP(705),D4YP(705)
      DIMENSION Y(705),YP(705),YPP(705)
      DIMENSION CY(705),CYP(705),CYPP(705)
      DIMENSION Z(707)
      CALL CLEUND
      DO  99 J=1,36
      READ(5,100)AN,DZ
      WRITE(6,100)AN,DZ
  100 FORMAT(2D10.3)
      IF (AN)99,99,94
   94 CONTINUE
      PI=3.141592653589793
      GRAV=6.67D-8
      BOLTZ=1.379D-16
      PROTON=1.672D-24
      ALFA=7.55D-15
      PLANK=6.62D-27
      ELMASS=9.105D-28
      CONST1=2.0/PLANK*(2.0*PI*ELMASS)**(3.0/2.0)/PLANK**2
      CONST2=(BOLTZ**2)*3.0/ALFA*(BOLTZ**2)
      CONST3=DSQRT(2.0/(3.0*PI*GRAV))
      CONST4=1.0/(CONST1**(2.0/3.0)*CONST2**(1.0/6.0)*PROTON)
      I=1
      Z(I)=0.0
      Y(I)=1.0
      YP(I)=0.0
      X=DLOG(AN)
      C=X/100.0
      IF(AN.EQ.1.0)GO TO 11
   11 C=1.0/100.0
      CONTINUE
      CALL FDID(-1,X+C/2,UXP)
      CALL FDID(-1,X-C/2,UXM)
```

```
      UMT=(UXP-UXM)/C
      CALL FDID(+1,X,UPH)
      CALL FDID (+3,X,UTH)
      CALL FDID (-1,X,UMH)
      YPP(I)=-1.0/3.0*UPH*UTH**(1.0/3.0)
      A2=-1.0/3.0*UPH*UTH**(1.0/3.0)
      A41=(1.0+4.0/3.0*UPH*(1.0/UTH))*A2**2
      A42=(1.0/3.0*(1.0/UTH)**(2.0/3.0)*UPH**2+UPH*UTH**(1.0/3.0)*2
     2+UMH*UTH**(1.0/3.0))
      A4=-3.0/5.0*(A41+A2*A42)
      SS=A2*A4*(16.0/3.0*UPH*(1.0/UTH)+5.0)
      ST=8.0*A2**3*(UMH*(1.0/UTH)-(UPH/UTH)**2)
      SU=A4*(1.0/3.0*UPH**2*(1.0/UTH)**(2.0/3.0)
     2+UMH*UTH**(1.0/3.0)+2.0*UPH*UTH**(1.0/3.0))
      SV=3.0*A2**2*(UPH**2*(1.0/UTH)**(2.0/3.0)+3.0*UMH
     2*UTH**(1.0/3.0)
     3+2*UPH*UTH**(1.0/3.0)+UMH*(1.0/UTH)**(2.0/3.0)*UPH
     4-2.0/9.0*(1.0/UTH)**(5.0/3.0)*UPH**3+UTH**(1.0/3.0)*UMT)
      A6=-5.0/7.0*(SS+ST+SU+SV)
      PGDG=UTH*UPH
      AMVAR=Z(I)**2*(1.0/Y(I))*UTH**(2.0/3.0)*YP(I)
      WRITE(6,3)Z(I),Y(I),YP(I),YPP(I),PCDG,AMVAR
3     FORMAT(6D16.8)
 4    FORMAT(2D20.8)
      DO 10 I=2,8
      II=I-1
      Z(I)=Z(II)+DZ
      ZS=Z(I)*Z(I)
      Y(I)=1.0+ZS*(A2/2.0+ZS*(A4/24.0+ZS*A6/720.0))
      CY(I)=Y(I)
      YP(I)=Z(I)*(A2+ZS*(A4/6.0+ZS*A6/120.0))
      CYP(I)=YP(I)
      X=DLOG(AN*Y(I))
      CALL FDID (+3,X,UTH)
      CALL FDID(+1,X,UPH)
   20 YPP(I)=-YP(I)**2*(-1.0/Y(I)+2.0/3.0/Y(I)*(1.0/UTH)*UPH)
     2-YP(I)*2.0/Z(I)-Y(I)*UPH*UTH**(1.0/3.0)
      DY(I)=Y(I)-Y(II)
      D1YP(I)=YP(I)-YP(II)
```

```
      CD1YP(I)=D1YP(I)
      D1YPP(I)=YPP(I)-YPP(II)
      IF(I-2)21,21,22
   22 D2YP(I)=D1YP(I)-D1YP(II)
      CD2YP(I)=D2YP(I)
      D2YPP(I)=D1YPP(I)-D1YPP(II)
      IF(I-3)21,21,23
   23 D3YP(I)=D2YP(I)-D2YP(II)
      CD3YP(I)=D3YP(I)
      D3YPP(I)=D2YPP(I)-D2YPP(II)
      IF(I-4)21,21,24
   24 D4YP(I)=D3YP(I)-D3YP(II)
      CD4YP(I)=D4YP(I)
      D4YPP(I)=D3YPP(I)-D3YPP(II)
      IF(I-5)21,21,26
   26 D5YP(I)=D4YP(I)-D4YP(II)
      CD5YP(I)=D5YP(I)
      D5YPP(I)=D4YPP(I)-D4YPP(II)
      IF(I-6)21,21,27
   27 D6YP(I)=D5YP(I)-D5YP(II)
      CD6YP(I)=D6YP(I)
      D6YPP(I)=D5YPP(I)-D5YPP(II)
   25 CONTINUE
   21 CONTINUE
      T=CONST1**(1.0/3.0)*UTH**(2.0/3.0)*CONST1**(1.0/3.0)/BCLTZ
     2*CONST2**(1.0/3.0)*CONST2**(1.0/3.0)
      P=ALFA/3.0*T**4
      D=CONST1*UPH*UTH*CONST2*CONST1*PROTON
      CALL FDID (+3,X,UTH)
 1004 AMASS(I)=-(2.0/3.0)**(3.0/2.0)*4.0/DSQRT(PI)*DSQRT(CONST2)
     2*1.0/PROTON**2*1.0/GRAV**(3.0/2.0)*Z(I)**2*1.0/ Y(I)*YP(I)*UTH
     3**(2.0/3.0)
      BMASS(I)=AMASS(I)/1.985D33
      ARAD(I)=CONST3*CONST4*Z(I)
      RAD(I)=ARAD(I)/6.951D10
      AMVAR=Z(I)**2*(1.0/Y(I))*UTH**(2.0/3.0)*YP(I)
      PGDG=UTH*UPH
      WRITE(6,3)Z(I),Y(I),YP(I),YPP(I),PGDG,AMVAR
      CYPP(I)=YPP(I)
```

```
10 CONTINUE
   DO 88 I=9,700
   II=I-1
   Z(I)=Z(II)+DZ
   D1YP(I)=DZ*(YPP(II)+1.0/2.0*D1YPP(II)+5.0/12.0*D2YPP(II)+
  23.0/8.0*D3YPP(II)+251.0/720.0*D4YPP(II)+95.0/288.0*D5YPP(II)
  3+19087.0/60480.0*D6YPP(II))
   YP(I)=YP(II)+D1YP(I)
   D2YP(I)=D1YP(I)-D1YP(II)
   D3YP(I)=D2YP(I)-D2YP(II)
   D4YP(I)=D3YP(I)-D3YP(II)
   D5YP(I)=D4YP(I)-D4YP(II)
   D6YP(I)=D5YP(I)-D5YP(II)
   DY(I)=DZ*(YP(I)-1.0/2.0*D1YP(I)-1.0/12.0*D2YP(I)-
  21.0/24.0*D3YP(I)-19.0/720.0*D4YP(I)-3.0/160.0*D5YP(I)
  3-863.0/60480.0*D6YP(I))
   Y(I)=Y(II)+DY(I)
   IF(Y(I))99,99,96
96 CONTINUE
   X=DLOG(AN*Y(I))
   CALL FDID (+3,X,UTH)
   CALL FDID(+1,X,UPH)
   IF(Y(I))99,99,98
98 YPP(I)=-YP(I)**2*(-1.0/Y(I)+2.0/3.0/Y(I)*(1.0/UTH)*UPH)
  2-YP(I)*2.0/Z(I)-Y(I)*UPH*UTH**(1.0/3.0)
   D1YPP(I)=YPP(I)-YPP(II)
   D2YPP(I)=D1YPP(I)-D1YPP(II)
   D3YPP(I)=D2YPP(I)-D2YPP(II)
   D4YPP(I)=D3YPP(I)-D3YPP(II)
   D5YPP(I)=D4YPP(I)-D4YPP(II)
   D6YPP(I)=D5YPP(I)-D5YPP(II)
   CD1YP(I)=DZ*(YPP(I)-1.0/2.0*D1YPP(I)-1.0/12.0
  2*D2YPP(I)-1.0/24.0*D3YPP(I)-19.0/720.0*D4YPP(I)
  3-3.0/160.0*D5YPP(I)-863.0/60480.0*D6YPP(I))
   CYP(I)=CYP(II)+CD1YP(I)
   CD2YP(I)=CD1YP(I)-CD1YP(II)
   CD3YP(I)=CD2YP(I)-CD2YP(II)
   CD4YP(I)=CD3YP(I)-CD3YP(II)
   CD5YP(I)=CD4YP(I)-CD4YP(II)
```

```
      CD6YP(I)=CD5YP(I)-CD5YP(II)
      CDY(I)=DZ*(CYP(I)-1.0/2.0*CD1YP(I)-1.0/12.0*CD2YP(I)-1.0/24.0
     2*CD3YP(I)-19.0/720.0*CD4YP(I)-3.0/160.0*CD5YP(I)
     3-863.0/60480.0*CD6YP(I))
      CY(I)=CY(II)+CDY(I)
      Y(I)=CY(I)
      IF(CY(I))99,99,97
   97 IF(YP(I))95,99,99
   95 X=DLOG(AN*Y(I))
      CALL FDID (+3,X,UTH)
      CALL FDID (+1,X,UMH)
      CYPP(I)=-CYP(I)**2*(-1.0/CY(I)+2.0/3.0/CY(I)* (1.0/UTH)*UPH)
     2-CYP(I)*(2.0/Z(I))-CY(I)*UPH*UTH**(1.0/3.0)
      YPP(I)=CYPP(I)
      D1YPP(I)=CYPP(I)-CYPP(II)
      D2YPP(I)=D1YPP(I)-D1YPP(II)
      D3YPP(I)=D2YPP(I)-D2YPP(II)
      D4YPP(I)=D3YPP(I)-D3YPP(II)
      D5YPP(I)=D4YPP(I)-D4YPP(II)
      D6YPP(I)=D5YPP(I)-D5YPP(II)
      CD1YP(I)=DZ*(CYPP(I)-1.0/2.0*D1YPP(I)-1.0/12.0
     2*D2YPP(I)-1.0/24.0*D3YPP(I)-19.0/720.0*D4YPP(I)
     3-3.0/160.0*D5YPP(I)-863.0/60480.0*D6YPP(I))
      D1YP(I)=CD1YP(I)
      CALL FDID(+1,X,UPH)
      CALL FDID (+3,X,UTH)
      T=CONST1**(1.0/3.0)*UTH**(2.0/3.0)*CONST1**(1.0/3.0)/BOLTZ
     2*CONST2**(1.0/3.0)*CONST2**(1.0/3.0)
      P=ALFA/3.0*T**4
      D=CONST1*UPH*UTH*CONST2*CONST1*PROTON
 1005 AMASS(I)=-(2.0/3.0)**(3.0/2.0)*4.0/DSQRT(PI)*DSQRT(CONST2)
     2*1.0/PROTON**2*1.0/GRAV**(3.0/2.0)*Z(I)**2*1.0/ Y(I)*YP(I)*UTH
     3**(2.0/3.0)
      BMASS(I)=AMASS(I)/1.985D33
      PGDG=UTH*UPH
      AMVAR=Z(I)**2*(1.0/Y(I))*UTH**(2.0/3.0)*YP(I)
      WRITE(6,3)Z(I),Y(I),YP(I),YPP(I),PGDG,AMVAR
      YP(I)=CYP(I)
   88 CONTINUE
   99 CONTINUE
      STOP
      END
```

APPENDIX III

Subroutine for the computation of the Fermi Dirac integrals using the approximation formulae by W. J. Cody and H. C. Thacher, Jr., for $x\,(= \ln \lambda(\zeta)\cdot\Lambda_0)$:

$$-\infty < x \leq 1$$
$$1 \leq x \leq 4$$
$$4 \leq x < \infty$$

and for each order $k = -\frac{1}{2}\ ,\ \frac{1}{2}\ ,\ \frac{3}{2}$

where $$F_k(x) = \int_0^\infty \frac{t^k}{e^{t-x}+1}\,dt$$.

```
      SUBROUTINE FCIC(L,X,L)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION GAM(3)
      DIMENSION CNL(5,3),CNM(5,3),CNH(5,3),CDL(5,3),CDM(5,3),CDH(5,3)
      DIMENSION CNL1(5),CDL1(5),CNM1(5),CDM1(5),CNH1(5),CDH1(5)
      DIMENSION CNL2(5),CDL2(5),CNM2(5),CDM2(5),CNH2(5),CDH2(5)
      DIMENSION CNL3(5),CDL3(5),CNM3(5),CDM3(5),CNH3(5),CDH3(5)
      EQUIVALENCE (CNL(1,1),CNL1(1)),(CNL(1,2),CNL2(1)),
     1(CNL(1,3),CNL3(1)),(CDL(1,1),CDL1(1)),(CDL(1,2),CDL2(1)),
     2(CDL(1,3),CDL3(1))
      EQUIVALENCE (CNM(1,1),CNM1(1)),(CNM(1,2),CNM2(1)),
     1(CNM(1,3),CNM3(1)),(CDM(1,1),CDM1(1)),(CDM(1,2),CDM2(1)),
     2(CDM(1,3),CDM3(1))
      EQUIVALENCE (CNH(1,1),CNH1(1)),(CNH(1,2),CNH2(1)),
     1(CNH(1,3),CNH3(1)),(CDH(1,1),CDH1(1)),(CDH(1,2),CDH2(1)),
     2(CDH(1,3),CDH3(1))
      DATA GAM/1.00,0.50,0.75/
      DATA PIE/3.141592653589793/
      DATA CNL1/
     1-1.253314128820E 00,-1.723663557701E 00,-6.559045729258E-01,
     2-6.342283197682E-02,-1.488383106116E-05/
      DATA CDL1/
     1 1.000000000000E 00, 2.191780925980E 00, 1.605815955406E 00,
     2 4.443669527481E-01, 3.624232288112E-02/
      DATA CNM1/
     1 1.073812769400E 00, 5.600330366000E 00, 3.688221127000E 00,
     2 1.174339281600E 00, 2.364193552700E-01/
      DATA CDM1/
     1 1.000000000000E 00, 4.603184066700E 00, 4.307591067400E-01,
     2 4.215113214500E-01, 1.183260160100E-02/
      DATA CNH1/
     1-8.222559330000E-01,-3.620369345000E+01,-3.015385410000E+03,
     2-7.049871579000E+04,-5.698145924000E+04/
      DATA CDH1/
     1 1.000000000000E 00, 3.935689841000E+01, 3.568756266000E+03,
     2 4.181893625000E+04, 3.385138907000E+05/
      DATA CNL2/
     1-3.133285305570E-01,-4.161873852293E-01,-1.502208400588E-01,
     2-1.339579375173E-02,-1.513350700138E-05/
```

```
      DATA CDL2/
     1 1.CCCCCOOOCCCCE CC, 1.872608675SC2E CC, 1.145204446578E CO,
     2 2.570225587573E-01, 1.639902543568E-02/
      DATA CNM2/
     1 6.7817662666CCE-01, 6.331240179100E-01, 2.94479651772CE-01,
     2 8.01320711419CE-02, 1.339182129400E-02/
      DATA CDM2/
     1 1.0CCCCCCCCCCCE CC, 1.4374C4CC3970E-01, 7.086621484500E-02,
     2 2.345794947350E-03,-1.294499288350E-05/
      DATA CNH2/
     1 8.2244997626CCE-01, 2.CC463C3393CCE+01, 1.8268093446CCE+03,
     2 1.2226530374CCE+04, 1.404075009200E+05/
      DATA CDH2/
     1 1.0CCCCCCCCCCCE CC, 2.34862C765900E+01, 2.2C1348374300E+03,
     2 1.144267359600E+04, 1.65847159CCC0E+05/
      DATA CNL3/
     1-2.349963985406E-01,-2.927373637547E-01,-9.883097588738E-02,
     2-8.251386379551E-03,-1.874384153223E-05/
      DATA CDL3/
     1 1.CCCCCCOCCCCCE CC, 1.608597109146E CO, 8.275289530880E-01,
     2 1.52232238285CE-01, 7.69512C475C64E-03/
      DATA CNM3/
     1 1.15302134C2COE CC, 1.059155897200E CO, 4.689880309500E-01,
     2 1.18829087840CE-01, 1.94387557870CE-02/
      DATA CDM3/
     1 1.CCCCCCCCCCCCE CC, 3.734895384100E-02, 2.3248458137CCE-02,
     2-1.37667708740CE-03, 4.64663927810CE-05/
      DATA CNH3/
     1 2.46740023684CE CC, 2.191675823680E+02, 1.238293790750E+04,
     2 2.2C667724968CE+C5, 8.494429200340E+05/
      DATA CDH3/
     1 1.CCCCCC000000E 00, 8.911251406190E+01, 5.045756696670E+03,
     2 9.09C759463C4CE+04, 3.89960915641OE+05/
      FN=0.0
      FD=0.0
      N=(L+3)/2
      IF(X-1.C)1,4,4
 4    IF(X-4.C)2,2,3
 1    CCNTINUE
```

```
      EX=DEXP(X)
      DC 1C M=1,5
      K=6-M
      FN=EX*FN+CNL(K,N)
      FD=EX*FD+CDL(K,N)
1C    CONTINUE
      DD=FN/FD
      Y=EX*(GAM(N)*DSQRT(PIE)+EX*DD)
      GC TC 5
2     CONTINUE
      DC 2C M=1,5
      K=6-M
      FN=X*FN+CNM(K,N)
      FD=X*FD+CDM(K,N)
2C    CCNTINUE
      DD=FN/FD
      Y=DD
      GO TO 5
3     CCNTINUE
      C=2*N-1
      PX=1.C/X/X
      SX=DSQRT(X)
      DC 3C M=1,5
      K=6-M
      FN=PX*FN+CNH(K,N)
      FD=PX*FD+CDH(K,N)
3C    CCNTINUE
      DD=FN/FD
      Y=SX**C*(2.C/C+PX*DD)
      GC TC 5
    5 U=Y/GAM(N)/DSQRT(PIE)
      RETURN
      END
```

APPENDIX IV

REGIONS OF DEGENERACY OF THE ELECTRON GAS ON THE ($\log\rho - \log T$) PLANE

The various regions of degeneracy of the electron gas on the ($\log \rho$, $\log T$) plane can be shortly discussed in the following:

(D) Complete Degeneracy

In a completely degenerate gas, the density is high enough so that all the available electron states having energies less than some maximum energy are filled. The occupation index for the Fermi gas is $\left[\exp(\alpha + \theta E) + 1\right]^{-1}$ so that the maximum density of electrons in phase space is

$$n_e(p)\,dp = \frac{2}{h^3}\,4\pi p^2\,dp$$

or the total number of density of electrons in a completely degenerate electron gas is

$$n_e = \frac{8\pi}{3h^3}\,p_0^3$$

where p_0 = maximum momentum of the nonrelativistic electrons.

The electron pressure is given by

$$P_e = \frac{8\pi}{15mh^3}\,p_0^5 \quad \text{or} \quad P_e = \frac{h^2}{20m}\left(\frac{3}{\pi}\right)^{2/3} N_0^{5/3}\left(\frac{\rho}{\mu_e}\right)^{5/3} \qquad (1)$$

or $$P_e = 1.004 \times 10^{13}\left(\frac{\rho}{\mu_e}\right)^{5/3} \text{ dynes/cm}^2$$

This equation shows that the nonrelativistic electron pressure varies as the $5/3$ power of the density.

We may define an approximate boundary line in the ($\log \rho$, $\log T$) plane, dividing it into regions of nondegenerate and degenerate gas by the condition:

$$\frac{N_0 k}{\mu_e}\,\rho T = \frac{h^2}{20m}\left(\frac{3}{\pi}\right)^{2/3} N_0^{5/3}\left(\frac{\rho}{\mu_e}\right)^{5/3} \qquad (2)$$

or, numerically this equation shows that the completely degenerate electron pressure exceeds the nondegenerate electron pressure when

$$\frac{\rho}{\mu_e} > 2.4\times10^{-8}\ T^{3/2} \quad g/cm^3 \tag{3}$$

(2) Completely relativistic degeneracy

$$P_e = \frac{2\pi c}{3h^3}\left(\frac{3h^3}{8\pi}\right)^{4/3} n_e^{4/3} = \frac{2\pi c}{3h^3}\left(\frac{3h^3 N_0}{8\pi}\right)^{4/3}\left(\frac{\rho}{\mu_e}\right)^{4/3} \tag{4}$$

with $p_0 c = 2m_0c^2 = hc\left(\frac{3}{8\pi} n_e\right)^{1/3} = 6.12\times10^{-11}\ n_e^{1/3} = 5.15\times10^{-3}\left(\frac{\rho}{\mu_e}\right)^{1/3}$ MeV.

for $$p_0 c = 1\ MeV \Rightarrow \frac{\rho}{\mu_e} = 7.3\times10^{6}\ g/cm^3 \tag{5}$$

Densities must exceed 10^6 g/cm^3 for a degenerate gas to be relativistic, for which the degeneracy will be essentially complete unless $T > 10^9$ K (equ. 3)

(3) Partially relativistic Degenerate

$$P_e = \frac{\pi m^4 c^5}{3h^3} f(x) = A f(x) \tag{6}$$

$$\frac{\rho}{\mu_e} = \frac{8\pi m^3 c^3}{3h^3 N_0} x^3 = \frac{B}{\mu_e} x^3 \tag{7}$$

$x = p_F/mc$, $p_F = \left(\frac{3h^3}{8\pi} n_e\right)^{1/3}$, $f(x) = x(x^2+1)^{1/2}(2x^3-3) + 3\ln(\sqrt{1+x^2}+x)$

(4) Non relativistic partial degeneracy

$$P_e = \rho \frac{kT}{\mu_e H} \frac{U_{3/2}}{U_{1/2}} \tag{8}$$

$$\frac{\rho}{\mu_e} = \frac{2}{h^3}(2\pi m)^{3/2}(kT)^{3/2}\frac{1}{N_0} U_{1/2} \tag{9}$$

$\Rightarrow \log\left(\frac{\rho}{\mu_e} T^{-3/2}\right) = \log U_{1/2} - 8.044$ this equation relates $\log(\rho/\mu_e T^{-3/2})$ to the degenerate parameter α.

(5) Extremely relativistic partial degeneracy

$$P_e = n_e kT \frac{(1/3)F_3(\eta)}{F_2(\eta)} = \frac{R}{\mu_e} \rho T \frac{(1/3)F_3(\eta)}{F_2(\eta)} \qquad (10)$$

$$\frac{\rho}{\mu_e} = \frac{16\pi (kT)^3}{N_0 h^3 c^3} \frac{1}{2} F_2(\eta) \qquad (11)$$

where $$F_k(\eta) = \int_0^\infty \frac{x^k\,dx}{\exp(-\eta+x)+1}$$ (from J.P.Cox p. 850)

$(-\eta = \alpha = -\log A)$.

(6) Partially relativistic Partially Degenerate

$$P_e = \frac{16\pi\sqrt{2}}{3} \frac{m^4c^5}{h^3} \beta^{5/2}\left[F_{3/2}(\eta,\beta) + (1/2)\beta F_{5/2}(\eta,\beta)\right] \qquad (12)$$

$$= 16\sqrt{2}\, A \beta^{5/2}\left[F_{3/2}(\eta,\beta) + (1/2)\beta F_{5/2}(\eta,\beta)\right]$$

$$\frac{\rho}{\mu_e} = 3\sqrt{2}\,(B/\mu_e)\,\beta^{3/2}\left[F_{1/2}(\eta,\beta) + \beta F_{5/2}(\eta,\beta)\right] \qquad (13)$$

where $$F_k(\eta,\beta) = \int_0^\infty \frac{x^k (1+\frac{1}{2}\beta x)^{1/2}}{e^{-\eta+x}+1}\,dx \quad , \quad \beta = kT/mc^2$$

The diagram below illustrates the various domains of degeneracy as they are estimated by the above relations.

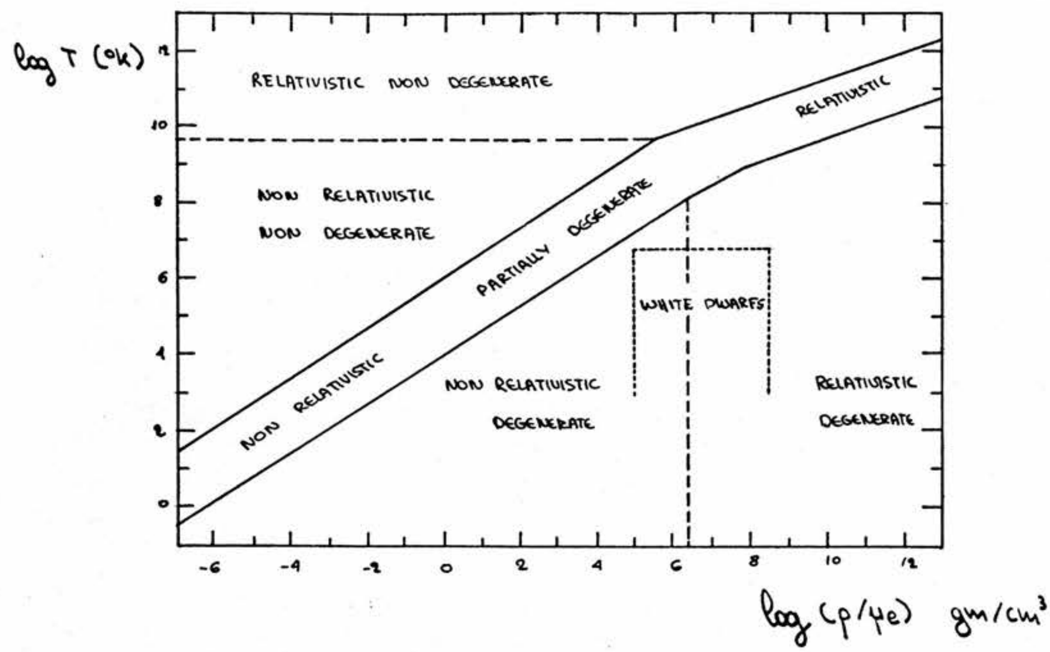


Fig. Regions on the (ρ-T) plane where degeneracy and relativistic effects are shown (from J. Cox p. 847)

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