University of St Andrews

ABSTRACT

The solutions to Fredholm integral equations of the second kind are approximated using power series techniques. On expansion, power series whose coefficients are given in terms of unknown functionals are formed. By truncating these series and using approximations, either in the form of Taylor series or rational approximants, the problem is reduced to a system of equations which is normally solved iteratively. A type of Pade approximant, called an S fraction, and a two point rational approximant, known as an M fraction, are both considered. Special attention is paid to those integral equations arising from second order two point boundary value problems as each series then contains only one unknown functional. The two point approximant is in this case an especially powerful approximation as it ensures automatic satisfaction of the boundary conditions. The methods are illustrated by a series of examples, one of which indicates how the existence and uniqueness of solutions for certain types of boundary value problems could be determined. A comparison of the various approximations is made for each example and it is seen that the two point approximant can provide an accurate result even when only a few terms of the series are considered.

THE SOLUTION OF FREDHOLM INTEGRAL EQUATIONS

USING RATIONAL APPROXIMATION

WITH SPECIAL APPLICATION TO TWO POINT

BOUNDARY VALUE PROBLEMS

by

GILLIAN BROWN

A dissertation submitted for the degree of Master
of Science at the University of St. Andrews.

August 1978

TH 9199

CERTIFICATE

I certify that Gillian Mary Brown has spent four terms

in research work at the Mathematical Institute,

University of St. Andrews, that she has fulfilled the

conditions of Ordinance General No. 12 and Senate

Regulations under Resolution of the University Court,

1974, No. 2, and that she is qualified to submit the

accompanying dissertation in application for the

degree of M.Sc.

DECLARATION

Except where reference is made to the work of others,

the research described in this thesis and the

composition of the thesis are my own work. No part of

the work has been submitted for either professional

qualifications or another degree at this or any other

university.

CONTENTS

ACKNOWLEDGEMENTS

ABSTRACT

The solutions to Fredholm integral equations of the second kind are approximated using power series techniques. On expansion, power series whose coefficients are given in terms of unknown functionals are formed. By truncating these series and using approximations, either in the form of Taylor series or rational approximants, the problem is reduced to a system of equations which is normally solved iteratively. A type of Padé approximant, called an S fraction, and a two point rational approximant, known as an M fraction, are both considered. Special attention is paid to those integral equations arising from second order two point boundary value problems as each series then contains only one unknown functional. The two point approximant is in this case an especially powerful approximation as it ensures automatic satisfaction of the boundary conditions. The methods are illustrated by a series of examples, one of which indicates how the existence and uniqueness of solutions for certain types of boundary value problems could be determined. A comparison of the various approximations is made for each example and it is seen that the two point approximant can provide an accurate result even when only a few terms of the series are considered.

INTRODUCTION

Both integral and differential equations are continually occurring in physical problems and are therefore of great interest to the mathematician. The study of differential equations started soon after the invention of differential and integral calculus in the seventeenth century, to which it is a natural sequel. By the middle of the nineteenth century the analytical theory of differential equations was well developed. It was not however until the turn of the century that Fredholm and Volterra developed their theories of integral equations. Throughout the present century both differential and integral equations have been a source of continuous research, and with the advent of the large modern computer the scope of the many approximate methods of solution has widened considerably.

Since integral and differential equations are closely related they are frequently considered together. Many books such as Davis(1960), Pogorzelski(1966) and Rabenstein(1966) have been written on the analytical methods of solution available. Although the theory of linear operators is well developed, the theory of nonlinear operators is beset by many difficulties. The powerful existence theorems of the linear operators often no longer hold in the nonlinear case and general methods of solution have to give way to a multitude of special techniques. For this reason many methods of solution, both analytical and numerical, involve the process of linearisation.

When considering numerical methods of solution it is useful to classify differential equations into two types. In integral equations we tend to classify equations as being of 'Volterra type' or 'Fredholm type' and analogously we can classify differential equations as being of 'initial value type' or 'boundary value type' depending

on where the boundary conditions are given. One of the many problems
of nonlinear differential equations is that the arbitrary constants
of integration occur nonlinearly; this can especially be a difficulty
with boundary value problems. Fox(1962) considers some of the basic
numerical methods of solution for ordinary differential equations
which do not require extensive computing.

Here we consider both linear and nonlinear Fredholm integral
equations. We look at approximations based on power series expansions
of the solution. The coefficients of these series are given in terms
of unknown functionals. By truncating the series we reduce the
problem to a system of equations which we normally have to solve by
iterative methods.

Particular attention has been paid to those integral equations
arising from second order two point boundary value problems.
Simplification then occurs since each series now contains only one
unknown functional. Examples illustrating both the general method and
the special application to boundary value problems have been given.
Whenever possible a comparison of the approximations with the exact
solution has been made. It will be seen that our methods of approximation
are especially powerful when solving two point boundary value problems.

We consider two types of rational approximant. The first is a
special case of a one point or Padé approximant known as an S fraction.
It is hoped that the S fraction will provide a better one point
approximation than the corresponding truncated Taylor series with
which it is compared. The second is a two point approximant, called an
M fraction, which fits series expansions about both endpoints of the
interval concerned. This approximant has better global convergence
than either of the one point approximations.

CHAPTER 1

THE GENERAL METHOD

We consider the Fredholm integral equation of the second kind,

$$y(x) = f(x) + \lambda \int_0^1 K[x, t, y(t)] \, dt \qquad (1.1)$$

$$0 \le t \le 1 \qquad 0 \le x \le c \qquad c \ge 1$$

where $f(x)$ and $K[x, ty(t)]$ are given functions, and wish to find a solution for $y(x)$. Since a linear transformation of the variables can easily be made, we restrict the problem to the interval $[0,1]$ without loss of generality. We look at two basic types of approximation which fit power series expansions about the endpoints of the interval $[0,1]$. The first is a one point approximation based on expansion about the lower endpoint x=0 of the interval. The second is a two point approximation which fits expansions about both the lower and the upper endpoints.

First we consider the one point approximations about x=0. Normally we assume that the kernel $K[x,t,y(t)]$ of (1.1) is an analytic function of x so that it is possible to expand $K[x,t,y(t)]$ in a Taylor series

$$K[x,t,y(t)] = \sum_{n=0}^{\infty} x^n K_n[t, y(t)] \qquad (1.2)$$

about x=0 where

$$K_n[t, y(t)] = \frac{1}{n!} \frac{d^n K}{dx^n} [0, t, y(t)] \qquad (1.3)$$

This assumption is not however necessary. If the kernel is continuous but has a discontinuous first derivative in x it may be possible to expand the right hand side of (1.1) by splitting the range of

integration and expanding the integral as a whole. This is the case
with the two point boundary value problems described in Chapter 5.
Alternatively the kernel may have an asymptotic expansion as x tends
to zero in which case the $K_n[t,y(t)]$ will no longer be the derivatives
given in (1.3). Similarly we later assume that $f(x)$ is an analytic
function of x and can therefore be expanded in Taylor series although
this is not strictly necessary.

When the kernel can be expanded in the form (1.2), the Fredholm
integral equation may be treated as an equation with degenerate
kernel of infinite rank so that

$$y(x) = f(x) + \lambda \sum_{n=0}^{\infty} x^n \int_0^1 K_n[t, y(t)] \, dt \qquad (1.4)$$

Suppose that we truncate the infinite series (1.4) after N terms and
write

$$y(x) = f(x) + \lambda \sum_{n=0}^{N-1} x^n \int_0^1 K_n[t, y(t)] \, dt + \mathcal{E}(N) \qquad (1.5)$$

where $\mathcal{E}(N)$ is the truncation error. Then if we let $y_N(x)$ be the
truncated approximation to $y(x)$ we have

$$y_N(x) = f(x) + \lambda \sum_{n=0}^{N-1} x^n \int_0^1 K_n[t, y_N(t)] \, dt \qquad (1.6)$$

In the special case when the integral equation is linear and
$K_n[t,y(t)] = K_n(t)y(t)$ we may solve (1.6) exactly by the method of
degenerate kernels to find an approximate solution $y_N(x)$ to (1.1).
This well known method is described at the beginning of Chapter 2.

Usually, however, we are concerned with iterative methods for
solution of (1.6). We assume that the function $f(x)$ can be expanded
in a Taylor series about x=0 of the form

$$f(x) = \sum_{n=0}^{\infty} x^n \frac{f^{(n)}(0)}{n!} \qquad (1.7)$$

so that we can express $y_N(x)$ as

$$y_N(x) = \sum_{n=0}^{N-1} x^n \left\{ \frac{f^{(n)}(0)}{n!} + \lambda \int_0^1 K_n[t, y_N(t)] dt \right\} \qquad (1.8)$$

However since we implicitly assume that $y(x)$ is an analytic function in $[0,1]$ this is equivalent to expanding $y(x)$ directly in a Taylor series about $x=0$ and truncating after N terms. Thus

$$y_N(x) = \sum_{n=0}^{N-1} x^n \frac{y^{(n)}(0)}{n!} \qquad (1.9)$$

On equating coefficients in (1.8) and (1.9) we find that

$$y^{(n)}(0) = f^{(n)}(0) + \lambda \int_0^1 \frac{d^n K}{dx^n}[0, t, y_N(t)] dt \qquad (1.10)$$

$$n = 0, 1, \ldots, N-1$$

where we now define

$$y_N(x) = y_N\left(x, y(0), y'(0), \ldots, y^{(N-1)}(0)\right) \qquad (1.11)$$

Thus (1.10) and (1.11) form a system of N equations for the N unknowns $y(0), y'(0), \ldots, y^{(N-1)}(0)$. Obviously a simple form is when (1.11) defines the truncated Taylor series about $x=0$ given by (1.9). However, in the hope of finding a better approximation, we also consider the case when (1.11) defines a rational approximant to the series (1.9). Both these methods are described in Chapter 2.

The system of equations (1.10) and (1.11) is usually nonlinear and must be solved iteratively. It is useful to define the following N dimensional column vectors

$$\underline{Y} = \left\{ y(0), y'(0), \ldots, y^{(N-1)}(0) \right\}$$

$$\underline{f} = \left\{ f(0), f'(0), \ldots, f^{(N-1)}(0) \right\} \qquad (1.12)$$

$$\underline{K}[t, y_N(t)] = \left\{ K[0, t, y_N(t)], \frac{dK}{dx}[0, t, y_N(t)], \ldots, \frac{d^{N-1}K}{dx^{N-1}}[0, t, y_N(t)] \right\}$$

so that we may write the system of equations in the form

$$Y = f + \lambda \int_0^1 K[t, y_N(t,Y)] dt \qquad (1.13)$$

or $$Y = G(Y) \qquad (1.14)$$

where the function G is defined appropriately. It should be noted that since G involves the evaluation of an integral it may have to be evaluated numerically. The simplest iterative method is the first order scheme

$$Y^{[r]} = G(Y^{[r-1]}) \qquad r = 1, 2, \ldots \qquad (1.15)$$

where a starting vector $Y^{[0]}$ is given. Wherever possible this scheme has been used and throughout this work it will be denoted by the coding (F). Unfortunately this scheme does not always converge and we may have to consider a higher order iterative process. The one used here is Powell's hybrid method for solving nonlinear algebraic equations, Powell(1968). This consists of minimising the sum of squares of the components of $Y - G(Y)$ using a combination of optimisation techniques.

$$Y^{[r]} = \min \sum_{i=0}^{N-1} \left\{ Y_i^{[r-1]} - G_i(Y^{[r-1]}) \right\}^2 \qquad (1.16)$$

Again a starting vector $Y^{[0]}$ must be specified. Whenever this scheme has been used it will be denoted throughout this work by the coding (P).

Thus by using one of the iterative schemes (F) or (P) we may find Y and hence an approximate solution $y_N(x)$, as defined by (1.11), to the original integral equation (1.1).

Unfortunately the error in the one point approximations increases with x for given N, so that near x=1 the approximations may be highly inaccurate. We therefore now consider approximations with better global convergence. The two point approximations fit series expansions

about both the endpoints of the interval $[0,1]$. As before we assume that the kernel of the Fredholm equation is an analytic function of x although this assumption is not strictly necessary. Then we may expand the kernel in Taylor series

$$K[x,t,y(t)] = \sum_{n=0}^{\infty} \frac{x^n}{n!} \frac{d^n K}{d x^n} [0,t,y(t)] \qquad (1.17)$$

and

$$K[x,t,y(t)] = \sum_{n=0}^{\infty} \frac{(x-1)^n}{n!} \frac{d^n K}{d x^n} [1,t,y(t)] \qquad (1.18)$$

about x=0 and x=1 respectively. We also assume that the function $f(x)$ can be expanded in a similar way, i.e.

$$f(x) = \sum_{n=0}^{\infty} x^n \frac{f^{(n)}(0)}{n!} \qquad (1.19)$$

and

$$f(x) = \sum_{n=0}^{\infty} (x-1)^n \frac{f^{(n)}(1)}{n!} \qquad (1.20)$$

So from (1.1) $y(x)$ can now be expressed as the power series

$$y(x) = \sum_{n=0}^{\infty} x^n \left\{ \frac{f^{(n)}(0)}{n!} + \lambda \int_0^1 \frac{1}{n!} \frac{d^n K}{d x^n} [0,t,y(t)] dt \right\} \qquad (1.21)$$

Suppose that we truncate this series after N terms so that

$$y(x) = \sum_{n=0}^{N-1} x^n \left\{ \frac{f^{(n)}(0)}{n!} + \lambda \int_0^1 \frac{1}{n!} \frac{d^n K}{d x^n} [0,t,y(t)] dt \right\} + \varepsilon(N) \qquad (1.22)$$

where $\varepsilon(N)$ is the truncation error. If we let $y_N(x)$ be an approximation to $y(x)$ then we may write

$$y_N(x) = \sum_{n=0}^{N-1} x^n \left\{ \frac{f^{(n)}(0)}{n!} + \lambda \int_0^1 \frac{1}{n!} \frac{d^n K}{d x^n} [0,t,y_N(t)] dt \right\} \qquad (1.23)$$

As with the one point approximations we assume that $y(x)$ can be expanded directly in a Taylor series about x=0. We can therefore equate coefficients in (1.23) to give as before

$$y^{(n)}(0) = f^{(n)}(0) + \lambda \int_0^1 \frac{d^n K}{d x^n} [0,t,y_N(t)] dt \qquad (1.24)$$

$$n = 0, 1, \ldots, N-1$$

This procedure may be repeated using power series expansions about x=1. Suppose we truncate these series after M terms to form an approximation $y_M(x)$ to $y(x)$. Then we may derive a system of M equations equivalent to (1.24) of the form

$$y^{(n)}(1) = f^{(n)}(1) + \lambda \int_0^1 \frac{d^n K}{dx^n} [1, t, y_M(t)] dt \tag{1.25}$$

$$n = 0, 1, \ldots, M-1$$

We are now in a position to define the following set of equations from (1.24) and (1.25)

$$y^{(n)}(0) = f^{(n)}(0) + \lambda \int_0^1 \frac{d^n K}{dx^n} [0, t, y_{NM}(t)] dt$$

$$n = 0, 1, \ldots, N-1 \tag{1.26}$$

$$y^{(n)}(1) = f^{(n)}(1) + \lambda \int_0^1 \frac{d^n K}{dx^n} [1, t, y_{NM}(t)] dt$$

$$n = 0, 1, \ldots, M-1$$

where $\quad y_{NM}(x) = y_{NM}\left(x, y(0), y'(0), \ldots, y^{(N-1)}(0), y(1), y'(1), \ldots, y^{(M-1)}(1)\right) \tag{1.27}$

Thus we have a set of M+N equations for the M+N unknowns $y(0), y'(0), \ldots \ldots, y^{(N-1)}(0), y(1), y'(1), \ldots \ldots, y^{(M-1)}(1)$. The approximation $y_{NM}(x)$ that we use here is a two point rational approximant which is described in detail in Chapter 3. The system of equations (1.26) and (1.27) are nonlinear and must therefore be solved iteratively. If we define N+M dimensional column vectors

$$\underline{Y} = \left\{ y(0), y'(0), \ldots, y^{(N-1)}(0), y(1), y'(1), \ldots, y^{(M-1)}(1) \right\}$$

$$\underline{f} = \left\{ f(0), f'(0), \ldots, f^{(N-1)}(0), f(1), f'(1), \ldots, f^{(M-1)}(1) \right\} \tag{1.28}$$

$$\underline{K}[t, y_{NM}(t)] = \left\{ K[0, t, y_{NM}(t)], \frac{dK}{dx}[0, t, y_{NM}(t)], \ldots, \frac{d^{N-1}K}{dx^{N-1}}[0, t, y_{NM}(t)], \right.$$

$$\left. K[1, t, y_{NM}(t)], \frac{dK}{dx}[1, t, y_{NM}(t)], \ldots, \frac{d^{M-1}K}{dx^{M-1}}[1, t, y_{NM}(t)] \right\}$$

we can write the above system of equations in the form

$$\underline{Y} = \underline{f} + \lambda \int_0^1 \underline{K} [t, y_{NM}(t, \underline{Y})] \, dt \qquad (1.29)$$

or $\qquad \underline{Y} = \underline{G}(\underline{Y}) \qquad (1.30)$

where the function $\underline{G}$ is defined appropriately. As previously the function $\underline{G}$ will usually have to be evaluated numerically since an integral is involved. Thus we have reduced the two point approximation to a form similar to that of the one point approximation in (1.14). Again we solve for $\underline{Y}$ iteratively using either the first order iterative scheme (F) or Powell's method (P) as already described. Having solved for $\underline{Y}$ we have an approximate solution $y_{NM}(x)$, as defined by (1.27), to the original integral equation (1.1).

CHAPTER 2

ONE POINT APPROXIMATIONS

In the previous chapter we outlined some basic methods, involving power series expansions, used to find an approximate solution to a Fredholm integral equation of the second kind. Here we consider in detail one point approximations based on a power series expansion of the integral equation about x=0. Three different one point approximations are discussed.

a) Approximation by the Method of Degenerate Kernels

The solution of linear Fredholm integral equations by the method of degenerate kernels is well known and included in most of the basic textbooks on integral equations, for instance see Chambers(1976), Pogorzelski(1966) or Mikhlin(1964). For completeness we include the method in the form used here to find an approximate solution to a linear Fredholm integral equation of the second kind. So let us consider the linear integral equation

$$y(x) = f(x) + \lambda \int_0^1 K(x,t)\, y(t)\, dt \qquad (2.1)$$

As in Chapter 1, we assume that the kernel can be expanded in a power series about x=0

$$K(x,t) = \sum_{n=0}^{\infty} x^n K_n(t) \qquad (2.2)$$

so that (2.1) becomes an integral equation with a degenerate kernel of infinite rank

$$y(x) = f(x) + \lambda \sum_{n=0}^{\infty} x^n \int_0^1 K_n(t)\, y(t)\, dt \qquad (2.3)$$

If we truncate the infinite series (2.2) after N terms we may find an

approximation $y_N(x)$ to $y(x)$ satisfying

$$y_N(x) = f(x) + \lambda \sum_{n=0}^{N-1} x^n \int_0^1 K_n(t) y_N(t) dt \qquad (2.4)$$

Equation (2.4) can now be solved exactly to find $y(x)$ by the method of degenerate kernels. Suppose we let

$$I_n(N) = \int_0^1 K_n(t) y_N(t) dt \qquad (2.5)$$

$$n = 0, 1, \ldots, N-1$$

where of course, for a fixed value of N, the $I_n(N)$ are constants. Multiplying equation (2.4) throughout by $K_m(x)$ and integrating between x=0 and x=1 gives us the linear system of equations

$$I_m(N) = F_m + \lambda \sum_{n=0}^{N-1} J_{mn} I_n(N) \qquad (2.6)$$

$$m = 0, 1, \ldots, N-1$$

where $F_m = \int_0^1 K_m(x) f(x) dx$ and $J_{mn} = \int_0^1 K_m(x) x^n dx$

assuming that all the integrals exist. Here we solve the linear set of equations (2.6) for the $I_m(N)$ by using a FORTRAN library subroutine due to Wilkinson and Reinsch(1971). Given a set of linear equations in the form $A\underline{x} = \underline{b}$, the routine uses Crout's method with partial pivoting to decompose the matrix A into upper and lower triangular matrices. An approximation to $\underline{x}$ is found by back-substitution and this is then updated until full machine accuracy is obtained. Having found the $I_m(N)$ we can write down an approximation to $y(x)$ for a given value of N,

$$y_N(x) = f(x) + \lambda \sum_{n=0}^{N-1} x^n I_n(N) \qquad (2.7)$$

Since the kernel has been expanded in a series about x=0 it is

obvious that $y_N(x)$ will be a closer approximation to $y(x)$ near $x=0$ than near $x=1$. Unfortunately, for small N, $y_N(x)$ is frequently a very poor approximation to $y(x)$. For this reason we later consider a rational approximant as an alternative approximation to $y(x)$.

b) Iterating the Taylor series

When the integral equation is nonlinear we can no longer use the method of degenerate kernels to find an approximate solution $y_N(x)$. Instead we must use an iterative scheme. So consider the nonlinear equation of the second kind

$$y(x) = f(x) + \lambda \int_0^1 K[x,t,y(t)]\,dt \qquad (2.8)$$

As seen in Chapter 1, if both $K[x,t,y(t)]$ and $f(x)$ can be expanded in power series about $x=0$, as given by (1.2) and (1.7) respectively, then we may write

$$y(x) = \sum_{n=0}^{\infty} x^n \left\{ \frac{f^{(n)}(0)}{n!} + \lambda \int_0^1 K_n[t,y(t)]\,dt \right\} \qquad (2.9)$$

On truncating the infinite power series (2.9) we can find an approximation $y_N(x)$ to $y(x)$ satisfying

$$y_N(x) = \sum_{n=0}^{N-1} x^n \left\{ \frac{f^{(n)}(0)}{n!} + \lambda \int_0^1 K_n[t,y_N(t)]\,dt \right\} \qquad (2.10)$$

If we assume that $y(x)$ can be expanded directly in a Taylor series about $x=0$, then on truncation after N terms we have

$$y_N(x) = \sum_{n=0}^{N-1} x^n \frac{y^{(n)}(0)}{n!} \qquad (2.11)$$

Thus as shown in Chapter 1 we can equate coefficients in (2.10) and (2.11) to form the system of equations

$$y^{(n)}(0) = f^{(n)}(0) + \lambda \int_0^1 \frac{d^n K}{dx^n}[0,t,y_N(t)]\,dt \qquad (2.12)$$

$$n = 0, 1, \ldots, N-1$$

where we have here assumed that the kernel has been expanded in a Taylor series about x=0 and that the $K_n[t,y(t)]$ are therefore of the form given in (1.3). We define $y_N(x)$ to be the truncated Taylor series about x=0 as given by (2.11). This system of equations can be solved for the $y^{(n)}(0)$ by one of the iterative schemes (F) or (P) and we can therefore find a one point approximation $y_N(x)$ to $y(x)$.

As in the case of approximation by the method of degenerate kernels, it is obvious that $y_N(x)$, being a truncated power series about x=0, will be a closer approximation to $y(x)$ near x=0 than near x=1. However we find that for small N, $y_N(x)$ may be a very poor approximation to $y(x)$. For this reason we consider a Padé approximant to $y(x)$ as an alternative one point approximation.

It should be noted that when the function $f(x)$ is already in the form of a truncated power series, solution of the linear Fredholm equation by iterating the Taylor series is directly equivalent to solution by the method of degenerate kernels, for a fixed value of N. Thus in the linear examples described in Chapters 4 and 6, it has been found necessary to use only one of the methods described above.

It is also useful to consider separately the first order scheme (F) for finding an approximation $y_N(x)$ to $y(x)$ by iterating the Taylor series about x=0. This scheme may be written in the form

$$y_N^{[r+1]}(x) = \sum_{n=0}^{N-1} x^n \left\{ \frac{f^{(n)}(0)}{n!} + \lambda \int_0^1 K_n[t, y_N^{[r]}(t)] dt \right\} \quad (2.13)$$

where the rth iterate of $y_N(x)$ is denoted by $y_N^{[r]}(x)$ and an appropriate starting function $y_N^{[o]}(x)$ is given. This is obviously closely related to the method of successive approximations, described in detail in such books as Davis(1960) and Pogorzelski(1966). It is therefore possible to use the theory of the method of successive approximations to find a bound for λ inside which the first order scheme (2.13) will converge.

Outside this bound a higher order iterative scheme normally has to be used.

Starting Vectors for the Iterative Schemes

It is important to find suitable starting vectors for the iterative schemes. For many systems of equations the first order iterative scheme (F) may fail to converge at all; however when convergence of the first order scheme does occur it will usually do so whatever starting vector is taken. For strongly nonlinear equations most iterative methods fail to converge unless we have a good starting vector and Powell's method is no exception. In general there is no systematic way of finding a suitable starting vector but we outline some of the possibilities here.

Clearly the starting function $y_N^{[o]}(x)$ must be in the form of a truncated power series about $x=0$. The simplest starting function to take is

$$y_N^{[o]}(x) = \sum_{n=0}^{N-1} x^n \frac{f^{(n)}(0)}{n!} \qquad (2.14)$$

or in the vector notation of (1.12) $\underline{Y}^{[o]} = \underline{f}$. Alternatively it may be possible to assume a solution of the form

$$y_R(x) = C_0 + C_1 x + \ldots + C_{R-1} x^{R-1} \qquad (2.15)$$

for some small R (i.e. R = 2 or 3) and by substitution in the integral equation to solve analytically for the c_i's. We then take $y_N^{[o]}(x) = y_R(x)$.

Thirdly we can try and linearise the original integral equation (2.8) and solve using the method of degenerate kernels. We can then use the 'linearised' solution as a starting vector for the nonlinear equation.

c) Iterating the Padé Approximant

It is well known that rational functions known as Padé approximants

$$P_{NM}(z) = \frac{\sum_{s=0}^{M} \alpha_s z^s}{\sum_{t=0}^{N} \beta_t z^t} \qquad (2.16)$$

can be constructed so that the expansion of $P_{NM}(z)$ about z=0 agrees with a given number of terms of the power series expansion of the function

$$G(z) = \sum_{n=0}^{\infty} a_n z^n \qquad (a_0 \neq 0) \qquad (2.17)$$

The classical theory of Padé approximants is described in Wall(1948); for more recent theory see, for instance, Gragg(1972). Many different Padé approximants could be used but here we consider only one. The particular Padé approximant used can be expressed in terms of continued fractions called S fractions which correspond to P_{NN} and P_{NN+1}, N = 0,1,..... The Nth convergent of this fraction can be written

$$S_N(z) = \frac{c_0}{1} + \frac{c_1 z}{1} + \frac{c_2 z}{1} + \ldots + \frac{c_{N-1} z}{1} \qquad (2.18)$$

where the c_n are independent of N and are chosen so that the fraction, when expanded out, fits N terms of (2.17). Thus

$$c_n = c_n(a_0, a_1, \ldots, a_n) \qquad (2.19)$$
$$n = 0, 1, \ldots, N-1.$$

The algorithm used to find the $\{c_n\}$ is called the corresponding sequence (CS) algorithm, due to Murphy and O'Donohoe(1977).

So consider Fredholm's integral equation of the second kind and,

as previously, expand the right hand side in a Taylor series about
x=0

$$y(x) = \sum_{n=0}^{\infty} x^n \left\{ \frac{f^{(n)}(0)}{n!} + \lambda \int_0^1 K_n[t, y(t)] \, dt \right\} \qquad (2.20)$$

We can now form the S fraction $y_N(x)$ corresponding to this infinite
series. Since we implicitly assume that $y(x)$ can be expanded in a
Taylor series about x=0

$$y(x) = \sum_{n=0}^{\infty} x^n \frac{y^{(n)}(0)}{n!} \qquad (2.21)$$

we can equate coefficients for the first N terms in (2.20) and (2.21).
Then

$$y^{(n)}(0) = f^{(n)}(0) + \lambda \int_0^1 \frac{d^n K}{dx^n} [0, t, y_N(t)] \, dt \qquad (2.22)$$
$$n = 0, 1, \ldots, N-1$$

where we have assumed that the kernel has been expanded in a Taylor
series about x=0 and that the $K_n[t, y(t)]$ are therefore of the form
given in (1.3). $y_N(x)$ is the S fraction to the series (2.20) and hence
to (2.21). From (2.18) and (2.19) it is obvious that

$$y_N(x) = \frac{y(0)}{1} + \frac{C_1[y(0), y'(0)]x}{1} + \ldots + \frac{C_{N-1}[y(0), \ldots, y^{(N-1)}(0)]x}{1} \qquad (2.23)$$
$$= y_N(x, y(0), y'(0), \ldots, y^{(N-1)}(0))$$

as required. We can thus solve the system of N equations (2.22) by one
of the iterative schemes described in Chapter 1. It is hoped that this
approximant will provide a better approximation than either the method
of degenerate kernels or the iterated Taylor series. Obviously these
methods of approximation could be repeated using other Padé approximants
and different results might be obtained.

Starting Vectors for the Iterative Schemes

Similar starting vectors to those used when iterating the Taylor series can be applied. The simplest starting function to take is the S fraction of $f(x)$, or in the vector notation of (1.12) to let $\underline{y}^{[0]} = \underline{f}$. Other starting vectors may be found by taking the S fraction of those starting functions used in Section b).

CHAPTER 3

TWO POINT APPROXIMATIONS

Since one point approximations about $x=0$ are liable to be inaccurate for given N near $x=1$, we now consider a rational approximation with better global convergence, which fits power series expansions about both $x=0$ and $x=1$. First we outline the basic idea upon which the theory of two point rational approximants is based.

Suppose that a function can be expanded about $z=0$ in the series

$$\sum_{n=0}^{\infty} c_n z^n \tag{3.1}$$

and about $z=\infty$ in the series

$$\sum_{n=1}^{\infty} \frac{b_n}{z^n} \tag{3.2}$$

where (3.1) and (3.2) may be either asymptotic or convergent expansions and it is required that c_0 and b_1 be non zero. Rational functions in the form of continued fractions can now be constructed which agree, when expanded out, with n terms of (3.1) and m terms of (3.2) simultaneously. Here the particular rational function we consider is such that the mth convergent agrees with m terms of each series. Murphy(1966) showed that the required fraction is of the form

$$M_m(z) = \frac{c_0}{1+d_1 z} + \frac{n_2 z}{1+d_2 z} + \dots + \frac{n_m z}{1+d_m z} \tag{3.3}$$

This continued fraction is called an M fraction and is unique, McCabe(1975). The coefficients n_i and d_i are independent of m and have been calculated here using the corresponding sequence (CS) algorithm, Murphy and O'Donohoe(1977), although other methods are

available. Clearly we have

$$n_i = n_i(c_0, c_1, \ldots, c_{i-1}, b_1, b_2, \ldots, b_i)$$

$$i = 2, \ldots, m \tag{3.4}$$

and

$$d_i = d_i(c_0, c_1, \ldots, c_{i-1}, b_1, b_2, \ldots, b_i)$$

$$i = 1, \ldots, m$$

It should be noted that for certain problems either c_0 in (3.1) or b_1 in (3.2) may be zero. It is simple to safeguard this from happening by adding an arbitrary function which can later be subtracted.

Now consider the Fredholm integral equation of the second kind

$$y(x) = f(x) + \lambda \int_0^1 K[x, t, y(t)] \, dt \tag{3.5}$$

In order to form a two point approximant to $y(x)$ we need to expand $y(x)$ in the form of the series (3.1) and (3.2). Suppose that we expand (3.5) in a Taylor series about x=0 so that

$$y(x) = \sum_{n=0}^{\infty} x^n \left\{ \frac{f^{(n)}(0)}{n!} + \lambda \int_0^1 K_n[t, y(t)] \, dt \right\} \tag{3.6}$$

where

$$K[x, t, y(t)] = \sum_{n=0}^{\infty} x^n K_n[t, y(t)] = \sum_{n=0}^{\infty} \frac{x^n}{n!} \frac{d^n K[0, t, y(t)]}{dx^n}$$

and

$$f(x) = \sum_{n=0}^{\infty} x^n \frac{f^{(n)}(0)}{n!}$$

Suppose similarly that we can expand (3.5) in a Taylor series about x=1 so that

$$y(x) = \sum_{n=0}^{\infty} (x-1)^n \left\{ \frac{f^{(n)}(1)}{n!} + \lambda \int_0^1 K_n^*[t, y(t)] \, dt \right\} \tag{3.7}$$

where

$$K[x, t, y(t)] = \sum_{n=0}^{\infty} (x-1)^n K_n^*[t, y(t)] = \sum_{n=0}^{\infty} \frac{(x-1)^n}{n!} \frac{d^n K[1, t, y(t)]}{dx^n}$$

and

$$f(x) = \sum_{n=0}^{\infty} (x-1)^n \frac{f^{(n)}(1)}{n!}$$

As shown in Chapter 1 this is equivalent to assuming expansions of $y(x)$ about $x=0$ and $x=1$ in the forms

$$y(x) = \sum_{n=0}^{\infty} x^n \frac{y^{(n)}(0)}{n!} \tag{3.8}$$

and

$$y(x) = \sum_{n=0}^{\infty} (x-1)^n \frac{y^{(n)}(1)}{n!} \tag{3.9}$$

Now by the simple change of variable

$$z = \frac{x}{1-x} \qquad \text{or} \qquad x = \frac{z}{1+z} \tag{3.10}$$

we can transform (3.6) - (3.9) into expansions about $z=0$ and $z=\infty$ to give

$$y(z) = \sum_{n=0}^{\infty} z^n c_n \qquad \text{and} \qquad y(z) = \sum_{n=0}^{\infty} \frac{b_n}{z^n} \tag{3.11, 3.12}$$

respectively. The c_n and b_n are expressed in terms of the original coefficients of the series, namely

$$\left.\begin{aligned} c_0 &= y(0) \\ c_n &= \sum_{i=1}^{n} \binom{n-1}{i-1} (-1)^{n-i} \frac{y^{(n)}(0)}{n!} \qquad n=1,2,.. \\ b_0 &= y(1) \\ b_n &= \sum_{i=1}^{n} \binom{n-1}{i-1} (-1)^n \frac{y^{(n)}(1)}{n!} \qquad n=1,2,.. \end{aligned}\right\} \tag{3.13}$$

If we now put

$$Y(z) = y(z) - y(1) \tag{3.14}$$

we may approximate $Y(z)$ by the M fraction

$$M_N(z) = M_N(z, y(0), y'(0), ..., y^{(N-1)}(0), y(1), y'(1), ..., y^{(N)}(1)) \tag{3.15}$$

which agrees with N terms of the series (3.11) and N+1 terms of the series (3.12). Thus in the notation of Chapter 1 we can now form a two point approximation to $y(x)$

$$y_{NM}(x) = y(1) + M_N\left(\frac{x}{1-x}\right) \tag{3.16}$$

where $M = N+1$, and hence can define the relationship (1.27)

$$y_{NN+1}(x) = y_{NN+1}\left(x, y(0), y'(0), \ldots, y^{(N-1)}(0), y(1), y'(1), \ldots, y^{(N)}(1)\right) \tag{3.17}$$

Thus on equating coefficients of the series (3.6) and (3.7) with those of the series (3.8) and (3.9) we may derive the system of 2N+1 nonlinear equations

$$y^{(n)}(0) = f^{(n)}(0) + \lambda \int_0^1 \frac{d^n K}{dx^n}[0, t, y_{NN+1}(t)]\,dt$$

$$n = 0, 1, \ldots, N-1$$

$$y^{(n)}(1) = f^{(n)}(1) + \lambda \int_0^1 \frac{d^n K}{dx^n}[1, t, y_{NN+1}(t)]\,dt \tag{3.18}$$

$$n = 0, 1, \ldots, N$$

where $y_{NN+1}(x)$ is as defined above. The system of equations can be solved by one of the iterative schemes (F) or (P), described in Chapter 1, to give an approximate solution to (3.5). Although we only consider the two point approximant described in detail here, obvious extensions to the methods could be made to deal with other fractions. For instance in certain problems it might be advisable to consider a two point approximant using more terms of one series than of the other.

Starting Vectors for the Iterative Schemes

The simplest starting function is to take the appropriate two point approximant to the function $f(x)$ or in the vector notation of (1.28) to let $\underline{Y}^{[0]} = \underline{f}$. Alternatively it may be possible to make use of one point approximations. Suppose that for some small N a one point approximation about $x=0$ has been found. Then if we can repeat the general procedure to find a one point approximation about $x=1$ we are now in the position of being able to calculate a two point approximant to the two series. This may then be used as a starting vector.

CHAPTER 4

APPLICATION OF THE GENERAL METHOD

TO AN EXAMPLE

Computational Procedure

Computing has been done throughout with DOUBLE PRECISION accuracy
on the University of St Andrews IBM 360/44 computer. The programs
have been written in FORTRAN as a series of subroutines so that
they may readily be altered to solve different problems. For each
example results were computed for various numbers of terms in the
series (usually $N = 4,6,8,10$ and 12), although only a few of these
results have been given here due to lack of space.

Details of the various methods of numerical integration used
and the starting vectors for the iterative schemes have been
discussed individually for each example. Since convergence of the
first order iterative scheme (F) may be slow, Aitken's acceleration
formula was used. Powell's method (P) was used with maximum internal
steplength 0.01 and a convergence criteria for minimising the sum of
squares (1.16) of 10^{-16}. Checks were performed using results from the
the first order iterative scheme (F) and comparing them with similar
results from Powell's method (P). In all cases agreement was found
to at least seven decimal figures.

The subroutines used to form the S fraction were tested using
two examples. The coefficients of the S fractions corresponding to
the functions $(1 + x)^{-\frac{1}{2}} + (1 + 2x)^{-\frac{1}{3}}$ and $(1 + x)^{\frac{1}{2}}$ are given by
Drew and Murphy(1977) and Wall(1948) respectively. For both examples
the subroutines were found to be working correctly.

Example demonstrating the General Method

As an illustration of the general method we consider the linear integral equation

$$y(x) = 1 + \lambda \int_0^1 e^{xt} \, y(t) dt \qquad (4.1)$$

where λ is a constant, and $0 \le x \le 1$. Here

$$K[x, t, y(t)] = K(x, t) y(t) = e^{xt} y(t) \qquad (4.2)$$

and is an analytic function of x in $[0, 1]$. Following the general method we can expand $y(x)$ about x=0 and x=1 as

$$y(x) = 1 + \lambda \sum_{n=0}^{\infty} x^n \int_0^1 \frac{t^n}{n!} y(t) \, dt \qquad (4.3)$$

and

$$y(x) = 1 + \lambda \sum_{n=0}^{\infty} (x-1)^n \int_0^1 e^t \frac{t^n}{n!} y(t) dt \qquad (4.4)$$

Truncation of (4.3) after N terms gives

$$y_N(x) = 1 + \lambda \sum_{n=0}^{N-1} x^n \int_0^1 \frac{t^n}{n!} y_N(t) dt \qquad (4.5)$$

The Method of Degenerate Kernels

Since the integral equation is linear we can solve for $y_N(x)$ by the method of degenerate kernels. Suppose we let

$$I_n(N) = \int_0^1 \frac{t^n}{n!} y_N(t) \, dt \qquad (4.6)$$

$$n = 0, 1, \ldots, N-1$$

Then we may derive a system of N linear equations of the form (2.6) in the normal way

$$I_m(N) = \frac{1}{m!(m+1)} + \lambda \sum_{n=0}^{N-1} \frac{I_n(N)}{m!(m+n+1)} \tag{4.7}$$

$$m = 0, 1, \ldots, N-1$$

Solution of these equations for the $I_m(N)$ provides an approximation $y_N(x)$ to $y(x)$. As N increases it is found that the $I_n(N)$ tend to a fixed value I_n as would be expected.

Iterated One Point Approximations

In this example $y_N(x)$ is already in the form of a truncated power series about x=0 from (4.3). If we expand $y(x)$ directly in a Taylor series about x=0 then we can, on equating coefficients as described in Chapter 1, derive the system of N equations

$$\left. \begin{array}{l} y(0) = 1 + \lambda \int_0^1 y_N(t)\, dt \\[12pt] y^{(s)}(0) = \lambda \int_0^1 t^s\, y_N(t)\, dt \qquad s = 1, \ldots, N-1 \end{array} \right\} \tag{4.8}$$

where by definition $y_N(x) = y_N(x, y(0), y'(0), \ldots, y^{(N-1)}(0))$. These equations are solved using either the first order iterative scheme (F) or Powell's method (P).

When $y_N(x)$ defines the truncated Taylor series of $y(x)$ about x=0 we find that, for this example, solving the system of equations (4.8) by iterative methods is directly equivalent to solution by the method of degenerate kernels for a fixed value of N. In both these cases the integrals have been evaluated analytically. Thus no results from iterating the Taylor series have been included in Tables 1 - 3.

It can readily be shown that when the method of successive approximations is applied to the original integral equation (4.1) convergence occurs when $\lambda < \frac{1}{e}$. It would therefore be expected that our

first order scheme (F) for iterating the Taylor series should also converge for these values of λ , and this is indeed found to be the case.

When $y_N(x)$ defines the S fraction to $y(x)$ it is found necessary to evaluate the integrals at each iteration numerically. Here we used Simpson's rule with step size h=0.05. This is undoubtedly a source of some inaccuracy in the results. It was found that decreasing the step size to h=0.01 did improve the accuracy of the solution slightly but also greatly increased the computing time used.

Starting vectors are of course necessary when using the iterative schemes (F) and (P). When iterating either the Taylor series or the S fraction by the first order scheme (F) a starting function $y_N^{[o]}(x) = 1$ may be taken. When the scheme (F) failed to converge, Powell's method (P) was used. Somewhat surprisingly, it was found that for all values of λ tested Powell's method converged when the same starting function $y_N^{[o]}(x) = 1$ was taken.

Iterated Two Point Approximant

Following the general theory of Chapter 3 we derive the system of 2N+1 equations

$$
\left.\begin{aligned}
y(0) &= 1 + \lambda \int_0^1 y_{NN+1}(t)\,dt \\
y^{(s)}(0) &= \lambda \int_0^1 t^s\, y_{NN+1}(t)\,dt \qquad s=1,\ldots,N-1 \\
y(1) &= 1 + \lambda \int_0^1 e^t\, y_{NN+1}(t)\,dt \\
y^{(s)}(1) &= \lambda \int_0^1 e^t\, t^s\, y_{NN+1}(t)\,dt \qquad s=1,\ldots,N
\end{aligned}\right\} \tag{4.9}
$$

where $y_{NN+1}(x) = y_{NN+1}(x, y(0), y'(0), \ldots, y^{(N-1)}(0), y(1), y'(1), \ldots, y^{(N)}(1))$

defines the two point rational approximant. Again these equations
must be solved iteratively using either the first order scheme (F)
or Powell's method (P).

As in the case of the S fraction, all the integrals have to be
evaluated numerically at each iteration and we again use Simpson's
rule with step size h=0.05. As before, this is a source of some
inaccuracy in the results. When the step size was decreased to h=0.01
improved accuracy was obtained but the computing time was increased
by an unreasonable amount.

Again starting vectors had to be found for the iterative
schemes and for the first order scheme (F) the starting function
$y_{N,N+1}^{[c]}(x) = 1$ was used. When the scheme (F) failed to converge Powell's
method (P) was used. However, for some of the values of λ tested,
Powell's method with the same starting function $y_{N,N+1}^{[c]}(x) = 1$ failed to
converge. Better starting vectors were found using the method of
degenerate kernels to find a series approximation to $y(x)$ about $x=0$.
The method was repeated to give a series approximation about $x=1$.
Thus a two point approximant $y_{N,N+1}(x)$ can be formed for some small N
and used as a starting function. Fortunately this process involves
little extra work.

The Exact Solution

Unfortunately this integral equation has, as far as the author knows,
no exact solution. For comparison we give an 'exact' solution which has
been computed using both the method of degenerate kernels and the two
point rational approximant. For large N (here we have considered
N=12 and N=14) we can derive series approximations to $y(x)$ about both
$x=0$ and $x=1$ by the method of degenerate kernels. We hope that for
this value of N the coefficients of the series $I_n(N)$ have converged

with sufficient accuracy to a fixed value I_n. We then form the two point approximant to the two series and use this as an 'exact' solution. In all cases this 'exact' solution appears to have converged to at least eight decimal figures.

Results

We now show some of the results computed for various values of λ. Since Simpson's rule with step size h=0.05 is a source of some inaccuracy, we have also shown the results using the smaller step size h=0.01. These figures are given in brackets for the appropriate methods. In Table 1 we have shown the results when $\lambda = \frac{1}{2}$. In Tables 2 and 3 we consider the results when $\lambda = 1$ and $\lambda = 4$ respectively. In all cases four terms of the series about x=0 and five terms of the series about x=1 have been taken. (F) and (P) have been used to denote which iterative scheme was used in each of the iterative methods.

Comments on these Results

In each of the Tables 1 - 3 we have considered a different value of λ and shown the resulting approximations when N=4. As would be expected the iterated two point rational approximant provides a far better approximation to y(x) than either of the one point approximations. The iterated two point approximant has in all cases converged to the exact solution to at least five decimal places. In fact it can be seen by altering the step size in Simpson's rule from h=0.05 to h=0.01 that the error in the sixth decimal place is due to the integration routine. With step size h=0.01 the iterated two point approximant converged to six decimal figures in all the cases tested.

Both the one point approximations give a rough approximation

to $y(x)$ when just four terms of the series about $x=0$ are considered. Unfortunately the iterated S fraction does not always seem to provide the improved approximation that was hoped; when $\lambda = \frac{1}{2}$ it is a poorer approximation than the Taylor series found by the method of degenerate kernels. When N was increased it was found that both the one point approximations quickly converged to the exact solution.

TABLE 1

Fredholm equation $\quad y(x) = 1 + \lambda \int_0^1 e^{xt} y(t) dt$

$$\lambda = \frac{1}{2}$$

y(x)	x = 0	x = 0.2	x = 0.4	x = 0.6	x = 0.8	x = 1
Method of degenerate kernels	2.511305	2.683571	2.881314	3.107910	3.366740	3.661181
Iterated S fraction (P)	2.528357 (2.528356)	2.703048 (2.703048)	2.904653 (2.904653)	3.139388 (3.139388)	3.415856 (3.415855)	3.746081 (3.746080)
Iterated two point approximant (F)	2.517003 (2.517003)	2.690078 (2.690078)	2.889104 (2.889104)	3.118432 (3.118432)	3.383189 (3.383189)	3.689423 (3.689422)
'Exact' solution	2.517003	2.690078	2.889104	3.118432	3.383189	3.689422

N = 4 - y(x) expanded to 4 terms about x=0

y(x) expanded to 5 terms about x=1

TABLE 2

Fredholm equation $\quad y(x) = 1 + \lambda \int_0^1 e^{xt} y(t)\, dt$

$$\lambda = 1$$

y(x)	x = 0	x = 0.2	x = 0.4	x = 0.6	x = 0.8	x = 1
Method of degenerate kernels	-1.745405	-2.080580	-2.466892	-2.911227	-3.420474	-4.001519
Iterated S fraction (P)	-1.722774 (-1.722775)	-2.055796 (-2.055798)	-2.440352 (-2.440354)	-2.885142 (-2.885143)	-3.400254 (-3.400255)	-3.997243 (-3.997244)
Iterated two point approximant (P)	-1.721342 (-1.721344)	-2.054228 (-2.054230)	-2.438654 (-2.438656)	-2.883421 (-2.883423)	-3.398921 (-3.398923)	-3.997430 (-3.997431)
'Exact' solution	-1.721344	-2.054230	-2.438656	-2.883423	-3.398923	-3.997431

$N = 4$ — $y(x)$ expanded to 4 terms about $x=0$

$\qquad\quad\;\;$ $y(x)$ expanded to 5 terms about $x=1$

TABLE 3

Fredholm equation $y(x) = 1 + \lambda \int_0^1 e^{xt} y(t) dt$

$$\lambda = 4$$

y(x)	x = 0	x = 0.2	x = 0.4	x = 0.6	x = 0.8	x = 1
Method of degenerate kernels	0.304383	0.149233	-0.034262	-0.250230	-0.502798	-0.796094
Iterated S fraction (P)	0.314621 (0.314619)	0.158584 (0.158583)	-0.026578 (-0.026579)	-0.246356 (-0.246356)	-0.507290 (-0.507289)	-0.817131 (-0.817128)
Iterated two point approximant (P)	0.314340 (0.314338)	0.158327 (0.158325)	-0.026788 (-0.026789)	-0.246453 (-0.246453)	-0.507155 (-0.507154)	-0.816623 (-0.816620)
'Exact' solution	0.314338	0.158325	-0.026789	-0.246453	-0.507154	-0.816620

N = 4 - y(x) expanded to 4 terms about x=0

y(x) expanded to 5 terms about x=1

CHAPTER 5

TWO POINT BOUNDARY VALUE PROBLEMS

In this chapter we apply the general method to second order two point
boundary value problems. As is well known this type of problem has an
integral equation formulation. It will be seen that our methods of
approximation are particularly suited to dealing with equations of
this kind. This is for two main reasons. The first is that each of the
Taylor series expanded at the boundaries now contains only one unknown
derivative. This means that the systems of equations reduce to a
single equation for each unknown derivative, and hence only one, or at
most two, integrals have to be evaluated at each iteration. Secondly
we find that when the method of iterating the two point approximant
is applied to these examples, the boundary conditions are automatically
satisfied.

So let us consider second order nonlinear differential equations
for $y(x)$ in the general form

$$\frac{d^2 y}{dx^2} + \lambda f(x,y) = 0 \qquad (0 \leq x \leq 1)$$

(5.1)

subject to the boundary conditions

$$y(0) = a \qquad y(1) = b$$

(5.2)

The restriction of the problem to the interval $[0,1]$ involves no loss
of generality. Also, although we specifically consider boundary
conditions of the form (5.2), it is possible to have mixed boundary
conditions involving derivatives.

Much work has been done on problems of this kind. As Scott(1975)
states, "the numerical solution of nonlinear two point boundary value

problems can be a challenging proposition. The basic existence and
uniqueness theory is not as developed as for initial value problems or
linear boundary value problems". Keller(1975) has made a survey on
some of the recent literature concerning numerical methods of solution
for two point boundary value problems.

Here we need to write the differential equation (5.1) in integral
equation form where, of course, the boundary conditions are automatically
incorporated. The corresponding integral equation is

$$y(x) = a + (b-a)x + \lambda \int_0^1 K(x,t) f[t,y(t)] dt \qquad (5.3)$$

where
$$K(x,t) = t(1-x) \qquad 0 \le t \le x$$
$$= x(1-t) \qquad x \le t \le 1 \qquad (5.4)$$

This is a Fredholm integral equation of the second kind and is in the
form (1.1). Suppose we let

$$y(x) = a + (b-a)x + \lambda I(x) \qquad (5.5)$$

Then on expanding $I(x)$ in a Taylor series about $x=0$ we have

$$y(x) = a + (b-a)x + \lambda \sum_{n=0}^{\infty} x^n \frac{I^{(n)}(0)}{n!} \qquad (5.6)$$

with

$$I(0) = 0$$

$$I'(0) = \int_0^1 (1-t) f[t,y(t)] dt$$

$$I''(0) = -f[0,y(0)] = -f[0,a] \qquad (5.7)$$

$$I^{(s)}(0) = -\left[\frac{d^{s-2}}{dx^{s-2}} f[x,y(x)] \right]_{x=0} \qquad s=3,4,\ldots$$

The $I^{(s)}(0)$ are total derivatives of the function $f[x,y(x)]$ evaluated

at x=0 and can therefore be expressed in terms of the derivatives of
y(x) evaluated at x=0

$$I^{(s)}(0) = I_s \left[y(0), y'(0), \ldots, y^{(s-2)}(0) \right] \qquad (5.8)$$

$$s = 3, 4, \ldots$$

However since we implicitly assume that y(x) can be expanded directly
in a Taylor series about x=0 we can put

$$y(x) = \sum_{n=0}^{\infty} x^n \frac{y^{(n)}(0)}{n!} \qquad (5.9)$$

Suppose we now truncate the infinite series (5.6) and (5.9) after N
terms to find an approximation $y_N(x)$ to y(x). Then on equating
coefficients in the two series we can derive the system of N equations

$$y(0) = a$$

$$y'(0) = b - a + \lambda \int_0^1 (1-t) f[t, y_N(t)] dt$$

$$y''(0) = -\lambda f[0, a] \qquad (5.10)$$

$$y^{(s)}(0) = \lambda I_s \left[y(0), y'(0), \ldots, y^{(s-2)}(0) \right]$$

$$s = 3, 4, \ldots, N-1$$

where, as previously, we define the one point approximation to be

$$y_N(x) = y_N \left(x, y(0), y'(0), \ldots, y^{(N-1)}(0) \right) \qquad (5.11)$$

This approximation may be the truncated Taylor series (5.9) or the S
fraction corresponding to that series, both of which are described in
Chapter 2. The system of equations (5.10) is equivalent to the equations
(1.10) that we derived in Chapter 1, and may be solved similarly using
either the first order iterative scheme (F) or Powell's method (P).
Some simplification is however possible since it is clear that we have

in fact only one unknown, namely $y'(0)$. All the other derivatives of $y(x)$ evaluated at $x=0$ are either known or can be expressed in terms of $y'(0)$. Thus we may write

$$y^{(s)}(0) = \lambda I^{(s)}(0) = \lambda I_s [y'(0)] \tag{5.12}$$

$$s = 3, 4, \ldots, N-1$$

We are therefore in effect solving a single equation for $y'(0)$ which may be written

$$y'(0) = b - a + \lambda \int_0^1 (1-t) f [t, y_N (t, y'(0))] dt \tag{5.13}$$

where now $y_N(x) = y_N(x, y'(0))$. It is necessary to find a starting value $y'(0)$ for both the iterative schemes (F) and (P). Possible choices are discussed at the end of this chapter. It should be noted that we could also solve equation (5.13) directly by such means as the bisection method or the method of false position, in which case we would need to find an interval encompassing the root $y'(0)$.

Although we have discussed here iterative schemes for finding $y'(0)$ and hence $y_N(x)$, it is possible in the linear case $f[x, y(x)] = f(x)y(x)$ to solve exactly for $y_N(x)$ by the method of degenerate kernels. Since we have only one unknown functional $y'(0)$ we may derive a single linear equation for $y'(0)$. This is directly equivalent to iterating the Taylor series about $x=0$.

Thus by similar methods to those used previously we can find a one point approximation $y_N(x)$ to $y(x)$. This approximation will now be exact at $x=0$ but, as before, the error increases with x so that it may be inaccurate near $x=1$. We therefore now consider a two point approximation which satisfies the boundary conditions at both $x=0$ and $x=1$.

Suppose we now repeat the procedure of expanding (5.5) using power series expansions about x=1. We truncate these series after M terms to form an approximation $y_M(x)$ to $y(x)$. Then we may derive a system of M equations, similar to (5.10), of the form

$$
\left.
\begin{aligned}
y(1) &= b \\[4pt]
y'(1) &= b - a - \lambda \int_0^1 t\, f\,[t, y_M(t)]\, dt \\[4pt]
y''(1) &= - \lambda f\,[1, b] \\[4pt]
y^{(s)}(1) &= \lambda I^{(s)}(1) = \lambda I_s^*\,[y(1), y'(1), \ldots, y^{(s-2)}(1)] \\[4pt]
&\qquad\qquad s = 3, 4, \ldots, M-1
\end{aligned}
\right\}
\qquad (5.14)
$$

It therefore becomes clear that we have again only one unknown, namely **y'(1)**. All the other derivatives of $y(x)$ evaluated at x=1 are either known or can be expressed in terms of y'(1). Thus

$$
y^{(s)}(1) = \lambda I^{(s)}(1) = \lambda I_s^*\,[y'(1)] \qquad (5.15)
$$
$$
s = 3, 4, \ldots, M-1
$$

We are now in a position to define a set of N+M equations from (5.10) and (5.15) of the form

$$
\left.
\begin{aligned}
&y(0) = a &\qquad& y''(0) = -\lambda f\,[0, a] \\[4pt]
&y'(0) = b - a + \lambda \int_0^1 (1-t)\, f\,[t, y_{NM}(t)]\, dt \\[4pt]
&y^{(s)}(0) = \lambda I_s\,[y'(0)] &\qquad& s = 3, \ldots, N-1 \\[6pt]
&y(1) = b &\qquad& y''(1) = -\lambda f\,[1, b] \\[4pt]
&y'(1) = b - a - \lambda \int_0^1 t\, f\,[t, y_{NM}(t)]\, dt \\[4pt]
&y^{(s)}(1) = \lambda I_s^*\,[y'(1)] &\qquad& s = 3, \ldots, M-1
\end{aligned}
\right\}
\qquad (5.16)
$$

where, as before, we can define a two point approximation $y_{NM}(x)$ to be

$$y_{NM}(x) = y_{NM}(x, y(0), y'(0), \ldots, y^{(K-1)}(0), y(1), y'(1), \ldots, y^{(M-1)}(1))$$ (5.17)

We again take this to be the two point rational approximant described in Chapter 3. However it is obvious that we have only two unknowns, $y'(0)$ and $y'(1)$, and thus

$$y_{NM}(x) = y_{NM}(x, y'(0), y'(1))$$ (5.18)

The system of N+M equations (5.16) may therefore be reduced to a pair of nonlinear equations

$$\left. \begin{array}{l} y'(0) = b - a + \lambda \int_0^1 (1-t) f\,[t, y_{NM}(t, y'(0), y'(1))]\,dt \\[2mm] y'(1) = b - a - \lambda \int_0^1 t\,f\,[t, y_{NM}(t, y'(0), y'(1))]\,dt \end{array} \right\}$$ (5.19)

We solve these equations by either the first order iterative scheme (F) or by Powell's method (P) as described in Chapter 1. We need a starting vector $\underline{Y}$ where $\underline{Y} = \{y'(0), y'(1)\}$ and possible candidates are discussed at the end of this chapter. Thus having found $y'(0)$ and $y'(1)$ we now know the two point approximant $y_{NM}(x)$. Since $y(0)$ and $y(1)$ are given by the boundary conditions, this approximant is exact at both $x=0$ and $x=1$. For this reason iteration of the two point approximant seems a very suitable method for finding approximate solutions to second order boundary value problems.

The Special Case $f(x,y) = f(y)$

Further simplification is possible when we consider equations of the form

$$\frac{d^2 y}{dx^2} + \lambda f(y) = 0$$ (5.20)

subject to boundary conditions (5.2). For then we have

$$\frac{1}{2} \left(\frac{dy}{dx}\right)^2 = - \lambda \int^y f(y) \, dy + \text{constant} \qquad (5.21)$$

and hence can define a relationship between y'(0) and y'(1)

$$\frac{1}{2} [y'(0)]^2 + \lambda \int_0^a f(y) \, dy = \frac{1}{2} [y'(1)]^2 + \lambda \int_0^b f(y) \, dy \qquad (5.22)$$

In general we must consider taking both the positive and negative
square roots in (5.22). However we often have enough knowledge of the
physical properties of the problem to make this choice obvious. Thus
we can reduce (5.19) to a single nonlinear equation for y'(0).

Starting Vectors for the Various Iterative Schemes

We need to find appropriate starting values for y'(0) and y'(1) in the
iterative schemes (F) and (P). For one point approximations we need to
determine a suitable starting value for y'(0). The simplest choice is
to take y'(0) = b-a and substitute this into the right hand side of
(5.13). Alternatively we can try and linearise the original differential
equation. The resulting integral equation may, if it cannot be solved
analytically, then be solved by the method of degenerate kernels to
give an approximate solution $y_N(x)$. This result can then be used to
give a starting value y'(0) for our nonlinear iterative schemes.

The most systematic way of finding a starting value y'(0) for
the one point approximations is to consider the single equation

$$y'(0) - (b-a) - \lambda \int_0^1 (1-t) f[t, y_N(t, y'(0))] \, dt = 0 \qquad (5.23)$$

By determining where the left hand side changes sign for various
values of N we can find an interval in which there is a root y'(0).
The midpoint of this interval can then be used as a starting value

y'(0) in the iterative schemes (F) and (P). In practice however it is useful to have some idea beforehand of the region in which y'(0) lies; this can often be obtained from a study of the original differential equation.

For two point approximations we need to find suitable starting values for both y'(0) and y'(1). The simplest choice is to take y'(0) = b−a = y'(1) and substitute this into the right hand side of (5.19). Alternatively we can make use of one point approximations. If we can derive these about both x=0 and x=1 then they may be used to provide starting values y'(0) and y'(1) when iterating the two point approximant. It should be noted that this procedure produces little extra work since both the series expansions about x=0 and x=1 are needed to calculate the two point approximant.

CHAPTER 6

BOUNDARY VALUE PROBLEMS: EXAMPLES

In this chapter we consider three differential equations of boundary value type. The first is a simple linear equation while both the second and third examples are nonlinear. We shall show that one of the nonlinear problems has two solutions and that the other, a well known test problem, has inherent difficulties that have to be overcome. The same computational procedure as that described at the start of Chapter 4 has been used. For each example a comparison of the various approximations has been made with the exact solution.

Example 1 - An Introductory Boundary Value Problem

Consider the linear second order differential equation

$$\frac{d^2 y}{d x^2} + \lambda^2 y = 0 \tag{6.1a}$$

subject to the boundary conditions

$$y(0) = a = 2 \qquad y(1) = b = 3 \tag{6.1b}$$

Since this equation has a simple closed form solution it makes a useful first test example. In integral equation form the problem becomes

$$y(x) = a + (b-a)x + \lambda^2 I(x) \tag{6.1c}$$

where

$$I(x) = \int_0^x x(1-t) y(t)dt + \int_x^1 (1-x) t y(t) dt \tag{6.1d}$$

On expanding $I(x)$ in a Taylor series about $x=0$ we can immediately write

$$y(x) = a + \left[b-a + \lambda^2 \int_0^1 (1-t) y(t) dt\right] x + \left[-\lambda^2 y(0)\right] \frac{x^2}{2!} + .. \quad (6.1e)$$

$$... + \left[-\lambda^2 y^{(s-2)}(0)\right] \frac{x^s}{s!} + ...$$

Similarly on expanding $I(x)$ in a Taylor series about $x=1$ we have

$$y(x) = b + \left[b-a - \lambda^2 \int_0^1 t \, y(t) \, dt\right](x-1) + \left[-\lambda^2 y(1)\right] \frac{(x-1)^2}{2!} + .. \quad (6.1f)$$

$$... + \left[-\lambda^2 y^{(s-2)}(1)\right] \frac{(x-1)^s}{s!} + ...$$

First we consider the various one point approximations as applied to this example. On truncating the infinite power series (6.1e) after N terms we can find an approximation $y_N(x)$ satisfying

$$y_N(x) = a + \left[b-a + \lambda^2 \int_0^1 (1-t) y_N(t) dt\right] x + \left[-\lambda^2 y(0)\right] \frac{x^2}{2!} + .. \quad (6.1g)$$

$$... + \left[-\lambda^2 y^{(N-3)}(0)\right] \frac{x^{N-1}}{(N-1)!}$$

On equating coefficients with those of the truncated Taylor series of $y(x)$ expanded directly about $x=0$ we have the relations

$$y(0) = a$$

$$y'(0) = b - a + \lambda \int_0^1 (1-t) y_N(t) dt$$

$$y''(0) = -\lambda^2 y(0) = -\lambda^2 a \qquad\qquad (6.1h)$$

$$y^{(s)}(0) = -\lambda^2 y^{(s-2)}(0) \qquad\qquad s = 3, 4, .., N-1$$

involving one unknown, namely $y'(0)$.

The Method of Degenerate Kernels

Since we are dealing with a linear equation we can solve for $y_N(x)$ using the method of degenerate kernels. As we have only one functional, namely $\int_0^1 (1-t)y_N(t)dt$, we may derive a single linear equation by multiplying (6.1g) throughout by $(1-x)$ and integrating between $x=0$ and $x=1$. On rearranging this equation becomes

$$y'(0) = b - a + \frac{a}{2} + \lambda^2 \sum_{s=1}^{N-1} \frac{y^{(s)}(0)}{(s+1)(s+2)s!} \qquad (6.1j)$$

where

$$y^{(s)}(0) = (-1)^{s/2} \lambda^s a \qquad s \text{ even}$$

$$y^{(s)}(0) = (-1)^{(s-1)/2} \lambda^{s-1} y'(0) \qquad s \text{ odd}$$

and may be solved directly for $y'(0)$. Having found $y'(0)$ we can thus determine an approximation to $y(x)$ in the form of a truncated Taylor series about $x=0$.

As N increases the coefficient $y'(0)$ of the Taylor series converges. The rate of convergence is obviously dependent on the value of λ. When $y'(0)$ is large the method is slow to converge. In Table 4 we consider the coefficient $y'(0)$ for various values of N and λ. It should be noted that for small N $y'(0)$ may be highly inaccurate (as when $\lambda = 3$) and in these cases $y_N(x)$ will be virtually useless as an approximation to $y(x)$.

Iterated One Point Approximations

Suppose we now define $y_N(x) = y_N(x,y(0),y'(0),\ldots,y^{(n+1)}(0))$ to be either the truncated Taylor series about $x=0$ or the S fraction corresponding to that series. We can then solve the system of N equations (6.1h) by either the first order iterative scheme (F) or

TABLE 4

$$y'' + \lambda^2 y = 0 \qquad y(0) = 2 \qquad y(1) = 3$$

The value of y'(0) as found by approximation using the

method of degenerate kernels

y'(0)	N = 4	N = 6	N = 8	N = 10	N = 12	Exact
λ = 1	2.277227	2.281066	2.280999	2.281000	2.281000	2.281000
λ = 2	7.858143	8.468531	8.427463	8.429180	8.429130	8.429131
λ = 3	18.571429	173.764706	102.184332	106.096452	105.856809	105.867022
λ = 4	-9.285714	-20.357798	-22.931770	-22.804088	-22.758531	-22.765714

Powell's method (P). Since alternate coefficients for this example are constant, considerable simplification can be made.

Clearly the system of equations (6.1h) can be reduced to a single equation for $y'(0)$. When $y_N(x)$ defines the Taylor series about $x=0$ this equation is of course the linear equation (6.1j) found by the method of degenerate kernels. Since solution by substitution and by iteration yield the same result, only those using the method of degenerate kernels have been listed.

The method of successive approximations converges for $\lambda < 2$ when applied to the original integral equation (6.1c). We would therefore expect the first order scheme (F) for iterating the Taylor series to converge for these values of λ. This is confirmed by the results.

When $y_N(x)$ defines the S fraction corresponding to $y(x)$, the functional $\int_0^1 (1-t)y_N(t)dt$ has had to be integrated numerically at each iteration. Throughout this example Simpson's rule with step size $h=0.01$ has been used.

A starting value for $y'(0)$ and hence $y_N(x)$ is of course necessary for both the iterative schemes (F) and (P). When iterating either the Taylor series or the S fraction by the first order scheme (F), a starting function $y_N^{[c]}(x) = a + (b-a)x$ was taken. When the iterative scheme (F) failed to converge, Powell's method (P) was used and an appropriate starting value for $y'(0)$ was obtained using the method of degenerate kernels.

Iterated Two Point Approximant

If we truncate the power series (6.1e) and (6.1f) after N and N+1 terms respectively, then we arrive at the following system of

equations

$$y(0) = a \qquad\qquad y''(0) = -\lambda^2 a$$

$$y'(0) = b - a + \lambda^2 \int_0^1 (1-t)\, y_{NN+1}(t)\, dt$$

$$y^{(s)}(0) = -\lambda^2 y^{(s-2)}(0) \qquad\qquad s = 3, \dots, N-1 \qquad (6.1k)$$

$$y(1) = b \qquad\qquad y''(1) = -\lambda^2 b$$

$$y'(1) = b - a - \lambda^2 \int_0^1 t\, y_{NN+1}(t)\, dt$$

$$y^{(s)}(1) = -\lambda^2 y^{(s-2)}(1) \qquad\qquad s = 3, \dots, N$$

where we define $y_{NN+1}(x) = y_{NN+1}(x, y(0), \dots, y^{(N)}(0), y(1), \dots, y^{(N)}(1))$ to be
the two point rational approximant described in Chapter 3. It can
easily be seen that this system of equations contains only two
unknowns, namely $y'(0)$ and $y'(1)$. We can therefore write the two point
approximant as $y_{NN+1}(x) = y_{NN+1}(x, y'(0), y'(1))$ and reduce the 2N+1
equations given by (6.1k) to the pair of equations

$$y'(0) = b - a + \lambda^2 \int_0^1 (1-t)\, y_{NN+1}(t, y'(0), y'(1))\, dt$$

$$y'(1) = b - a - \lambda^2 \int_0^1 t\, y_{NN+1}(t, y'(0), y'(1))\, dt \qquad (6.1l)$$

These two equations are solved iteratively for $y'(0)$ and $y'(1)$ by one
of the iterative schemes (F) or (P). As in the case of the S fraction
the two integrals are evaluated numerically at each iteration using
Simpson's rule with step size h=0.01.

Starting values for both $y'(0)$ and $y'(1)$ have to be found when
using the iterative schemes. Taking $y_{NN+1}^{[0]}(x) = a + (b-a)x$ provides an
adequate starting function for the first order scheme (F). When this

failed to converge Powell's method (P) was used. Suitable starting values for Powell's method were found by using the method of degenerate kernels to provide series approximations about both x=0 and x=1.

The Exact Solution

The boundary value problem (6.1a) with (6.1b) has the simple exact solution

$$y(x) = a \cos \lambda x + \frac{(b - a \cos \lambda) \sin \lambda x}{\sin \lambda} \qquad (6.1m)$$

This solution can be expanded in Taylor series about x=0 and x=1. The coefficients of the series can then be compared with those calculated from the various approximate solutions. Obviously the coefficients can be large when $\sin \lambda$ is small and for these values of λ (e.g. $\lambda = 3$) the approximations are usually slow to converge. The size of the coefficients also tend to increase with λ.

Results

We now show some of the results computed for various values of λ. In Tables 5 and 6 we consider the results when $\lambda = 1$ and $\lambda = 2$ respectively. In both these tables results have been shown for N = 4, i.e. using four terms of the series about x=0 and five terms of the series about x=1. In Tables 7 and 8 results with $\lambda = 3$ and $\lambda = 4$ have been shown. Since the one point approximations are not very good it has been felt necessary to show only the results from the method of degenerate kernels. In these tables results are shown both when N = 4 and N = 8. As before (F) and (P) have been used to denote which iterative scheme was applied.

Comments on these results

In each of the Tables $5 - 8$ a different value of λ has been considered and the resulting approximations shown for a fixed value of N. In every case the iterated two point approximant is seen to be much better than either of the one point approximations, and has the advantage of automatically satisfying both boundary conditions $(6.1b)$ to the original differential equation. Unfortunately in many cases the iterated S fraction was found to be a poorer approximation to $y(x)$ than the truncated Taylor series. As a check the exact solution to the problem $(6.1m)$ was expanded in a series about x=0 and the S fraction formed. When $\lambda = 1$ and $\lambda = 2$ it was confirmed that for small N the S fraction was a worse approximation to $y(x)$ than the truncated Taylor series. When $\lambda = 3$ the S fraction was a slightly better approximation than the Taylor series but since both results were highly inaccurate for small N, no real improvement was gained.

TABLE 5

$$y'' + \lambda^2 y = 0 \qquad y(0) = 2 \quad y(1) = 3$$

$$\lambda = 1$$

y(x)	x = 0	x = 0.2	x = 0.4	x = 0.6	x = 0.8	x = 1
Method of degenerate kernels	2.0	2.412409	2.726601	2.924356	2.987459	2.897690
Iterated S fraction (F)	2.0	2.415461	2.740122	2.973804	3.124839	3.206180
Iterated two point approximant (F)	2.0	2.413298	2.730385	2.938621	3.029703	3.0
Exact solution	2.0	2.413298	2.730385	2.938621	3.029703	3.0

N = 4   -   y(x) expanded to 4 terms about x=0

y(x) expanded to 5 terms about x=1

TABLE 6

$$y'' + \lambda^2 y = 0 \qquad y(0) = 2 \qquad y(1) = 3$$

$$\lambda = 2$$

y(x)	x = 0	x = 0.2	x = 0.4	x = 0.6	x = 0.8	x = 1
Method of degenerate kernels	2.0	3.369524	4.167619	4.142857	3.043810	0.619048
Iterated S fraction (P)	2.0	3.663726	4.831301	5.501895	5.783821	5.805791
Iterated two point approximant (P)	2.0	3.483352	4.416759	4.652856	4.154370	3.0
Exact solution	2.0	3.483351	4.416758	4.652855	4.154370	3.0

N = 4 — y(x) expanded to 4 terms about x=0

y(x) expanded to 5 terms about x=1

TABLE 7

$$y'' + \lambda^2 y = 0 \qquad y(0) = 2 \quad y(1) = 3$$

$$\lambda = 3$$

y(x)	x = 0	x = 0.2	x = 0.4	x = 0.6	x = 0.8	x = 1	
Method of degenerate kernels	2.0	5.131429	6.205714	3.885714	-3.165714	-16.285714	N = 4
Iterated two point approximant (P)	2.0	21.573041	33.609514	33.905742	22.358311	3.0	
Exact solution	2.0	21.576344	33.615450	33.911712	22.361638	3.0	
Iterated two point approximant (P)	2.0	21.576345	33.615452	33.911714	22.361639	3.0	
Method of degenerate kernels	2.0	20.883207	32.470625	32.692896	21.245757	0.827024	N = 8

TABLE 8

$$y'' + \lambda^2 y = 0 \qquad y(0) = 2 \qquad y(1) = 3$$

$$\lambda = 4$$

y(x)	x = 0	x = 0.2	x = 0.4	x = 0.6	x = 0.8	x = 1	
Method of degenerate kernels	2.0	-0.299048	-2.689524	-3.982857	-2.990476	1.476190	N = 4
Iterated two point approximant (P)	2.0	-2.693619	-5.750823	-5.323286	-1.669161	3.0	
Exact solution	2.0	-2.689367	-5.747401	-5.319138	-1.664357	3.0	
Iterated two point approximant	2.0	-2.689367	-5.747400	-5.319138	-1.664357	3.0	N = 8
Method of degenerate kernels	2.0	-2.719154	-5.789907	-5.358804	-1.642477	3.890676	

Example 2 - <u>A Nonlinear Boundary Value Problem with two Solutions</u>

The second problem considered is the nonlinear second order differential equation

$$\frac{d^2 y}{dx^2} + \lambda y^2 = 0 \qquad \lambda > 0 \qquad (6.2a)$$

subject to the boundary conditions

$$y(0) = a = 1 \qquad y(1) = b = 2 \qquad (6.2b)$$

where λ is a positive constant.

For this example we immediately encounter problems of existence and uniqueness. Unfortunately neither the existence theorem for second order differential equations by Keller(1966) nor the existence theorems of Bailey, Shampine and Waltman(1966) are applicable in this case. Birn, Goldstein and Schindler(1978) consider the existence theorems available for solving the nonlinear Dirichlet problem

$$-\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) A = f(\lambda, A, x, y) \qquad \lambda > 0$$

in the region Λ, where the value of A on the boundary of Λ is given. It can be shown by applying this theory to our example that there will be a solution for $\lambda < \lambda^*$, where λ^* is a real positive number, and no solution when $\lambda > \lambda^*$. Bandle(1975) gives bounds for λ^* by considering upper and lower solutions of the Dirichlet problem but these turn out to be of little practical use. Birn et al. also state that if $\lambda < \lambda^*$ the existence of a second independent solution is almost guaranteed.

First we apply our methods to this example; later the analytic

solution is considered in some detail. Writing (6.2a) in integral equation form we have

$$y(x) = a + (b-a)x + \lambda \int_0^x x(1-t)y^2(t)dt + \lambda \int_x^1 (1-x)ty^2(t)dt \quad (6.2c)$$

or
$$y(x) = a + (b-a)x + \lambda I(x)$$

On expanding $I(x)$ in a Taylor series about $x=0$ we can write

$$y(x) = a + (b-a)x + \lambda \sum_{n=0}^{\infty} x^n \frac{I^{(n)}(0)}{n!} \quad (6.2d)$$

where we can show that

$$\left.\begin{array}{l}
I(0) = 0 \\[2mm]
I'(0) = \int_0^1 (1-t)y^2(t)dt \\[2mm]
I''(0) = -y^2(0) = -a^2 \\[2mm]
I^{(s)}(0) = -\left[\dfrac{d^{s-2}}{dx^{s-2}}y^2(x)\right]_{x=0} = -\sum_{k=0}^{s-2}\binom{s-2}{k}y^{(s-k-2)}(0)\,y^{(k)}(0) \\[4mm]
\qquad\qquad s = 3, 4, \ldots
\end{array}\right\} \quad (6.2e)$$

Similarly by expressing $I(x)$ as a Taylor series about $x=1$ we can write $y(x)$ in the form of the power series

$$y(x) = b + (b-a)(x-1) + \lambda \sum_{n=0}^{\infty}(x-1)^n \frac{I^{(n)}(1)}{n!} \quad (6.2f)$$

with
$$\left.\begin{array}{l}
I(1) = 0 \\[2mm]
I'(1) = -\int_0^1 t\,y^2(t)dt \\[2mm]
I''(1) = -y^2(1) = -b^2 \\[2mm]
I^{(s)}(1) = -\left[\dfrac{d^{s-2}}{dx^{s-2}}y^2(x)\right]_{x=1} = -\sum_{k=0}^{s-2}\binom{s-2}{k}y^{(s-k-2)}(1)\,y^{(k)}(1) \\[4mm]
\qquad\qquad s = 3, 4, \ldots
\end{array}\right\} \quad (6.2g)$$

Iterated One Point Approximations

If we truncate the infinite power series expansion (6.2d) after N
terms then we have

$$
\left.
\begin{array}{l}
y(0) = a \qquad\qquad\qquad\qquad y''(0) = -\lambda a^2 \\[2ex]
y'(0) = b - a + \lambda \int_0^1 (1-t) y_N^2(t)\, dt \\[2ex]
y^{(s)}(0) = -\lambda \displaystyle\sum_{k=0}^{S-2} \binom{s-2}{k} y^{(s-k-2)}(0)\, y^{(k)}(0) \\[3ex]
\qquad\qquad\qquad\qquad S = 3, 4, \ldots, N-1
\end{array}
\right\}
\qquad (6.2h)
$$

where $y_N(x) = y_N(x, y(0), y'(0), \ldots, y^{(N-1)}(0))$ can be either the
truncated Taylor series of $y(x)$ about $x=0$ or the S fraction
corresponding to that series, both of which are described in Chapter 2.

The system of N equations (6.2h) can be solved in the normal
way by one of the iterative schemes (F) or (P). However the system
essentially contains only one unknown, viz. $y'(0)$, and hence we can
write

$$
y'(0) - \left[b - a + \lambda \int_0^1 (1-t)\, y_N^2(t, y'(0))\, dt \right] = 0
\qquad (6.2j)
$$

When $y(x)$ defines the S fraction corresponding to $y(x)$ it is
found necessary to evaluate the integral numerically at each iteration.
Once again Simpson's rule with step size $h=0.01$ was used. Integrals
involving the Taylor series were still evaluated analytically. With
the starting function $y_N^{[0]}(x) = a + (b-a)x$ it was found that both
the truncated Taylor series and the iterated S fraction quickly
converged for $\lambda = \frac{1}{2}$ and $\lambda = 1$. Neither approximation converged when
$\lambda = 2$.

A second solution was sought for $\lambda = \frac{1}{2}$ and $\lambda = 1$ using Powell's
method (P). Clearly this required a different starting vector. If we
let $y_N(x)$ define the truncated Taylor series and consider intervals

in which the left hand side of (6.2j) changes sign for various values of N, then it can clearly be seen that the equation has two roots $y'(0)$ when $\lambda = \frac{1}{2}$ and $\lambda = 1$. By taking the midpoint of the appropriate interval as a starting value, second solutions can then be obtained. However both the truncated Taylor series and the iterated S fraction were found to be poor approximations for small N.

Iterated Two Point Approximant

Suppose we truncate the infinite power series expansions (6.2d) and (6.2f) after N and N+1 terms respectively to derive the system of 2N+1 equations

$$y(0) = a \qquad\qquad y''(0) = -\lambda a^2$$

$$y'(0) = b - a + \lambda \int_0^1 (1-t)\, y^2_{NN+1}(t)\,dt$$

$$y^{(s)}(0) = -\lambda \sum_{k=0}^{s-2} \binom{s-2}{k} y^{(s-k-2)}(0)\, y^{(k)}(0) \qquad s = 3,\ldots, N-1$$

$$y(1) = b \qquad\qquad y''(1) = -\lambda^2 b \qquad\qquad (6.2k)$$

$$y'(1) = b - a - \lambda \int_0^1 t\, y^2_{N N+1}(t)\,dt$$

$$y^{(s)}(1) = -\lambda \sum_{k=0}^{s-2} \binom{s-2}{k} y^{(s-k-2)}(1)\, y^{(k)}(1) \qquad s = 3,\ldots, N$$

We define $y_{NN+1}(x) = y_{NN+1}(x, y(0), y'(0),\ldots, y(0), y(1), y'(1),\ldots, y(1))$ to be the two point rational approximant described in Chapter 3. Clearly the system of equations (6.2k) contains only two unknowns, $y'(0)$ and $y'(1)$, and thus may be reduced to the pair of nonlinear equations

$$y'(0) = b - a + \lambda \int_0^1 (1-t)\, y^2_{NN+1}(t, y'(0), y'(1))\,dt$$

$$y'(1) = b - a - \lambda \int_0^1 t\, y^2_{NN+1}(t, y'(0), y'(1))\,dt \qquad (6.2l)$$

FIGURE 1

$$\frac{dy}{dx} = \pm \sqrt{\frac{2\lambda}{3}} \left(c'^3 - y^3 \right)^{1/2}$$

Since $\frac{d^2y}{dx^2}$ is always negative, $y(x)$ can never have a minimum value. It has a maximum value when $y(x) = c'$. From Figure 1 it is clear that there are two possibilities:

(I) $y(x)$ is strictly increasing in the interval $[0,1]$ so that $y'(0)$ and $y'(1)$ are both positive.

(II) $y(x)$ has a maximum in the interval $[0,1]$ so that $y'(0) > 0$ and $y'(1) < 0$.

Now suppose that $y(x)$ has its maximum at a point x_m. Then we know that $y(x_m) = c'$. We can therefore rewrite (6.2n) in the following forms

$$(I) \qquad \frac{dy}{dx} = +\sqrt{\frac{2\lambda}{3}} \left(c'^3 - y^3\right)^{\frac{1}{2}} \qquad\qquad 0 \leq x \leq 1$$

$$(II) \qquad \frac{dy}{dx} = +\sqrt{\frac{2\lambda}{3}} \left(c'^3 - y^3\right)^{\frac{1}{2}} \qquad\qquad 0 \leq x \leq x_m$$

$$ \qquad\qquad = -\sqrt{\frac{2\lambda}{3}} \left(c'^3 - y^3\right)^{\frac{1}{2}} \qquad\qquad x_m \leq x \leq 1$$

(I) and (II) can be solved in terms of elliptic integrals of the first
kind (see Abramowitz and Stegun(1970)), viz.

$$F(\emptyset, k) = \sqrt[4]{3} \int_{-\infty}^{z} \frac{dz}{(1 - z^3)^{\frac{1}{2}}}$$

where $\qquad \cos\emptyset = \dfrac{z - 1 + \sqrt{3}}{z - 1 - \sqrt{3}} \qquad$ and $\qquad k = \sin\dfrac{5\pi}{12}$

For instance in case (I) the general solution is of the form

$$\sqrt{\frac{2\lambda}{3}}\, x + c = \frac{1}{c'^{\frac{1}{2}}} \frac{1}{\sqrt[4]{3}} F(\emptyset, k) \qquad\qquad (6.2o)$$

where c is the second constant of integration. The constants c and c'
are found from the boundary conditions (6.2b); on elimination of c
we find

$$\frac{1}{\sqrt[4]{3}\, c'^{\frac{1}{2}}} \left[F\left(\cos^{-1}\left\{\frac{b - c' + \sqrt{3}c'}{b - c' - \sqrt{3}c'}\right\}, k\right) - F\left(\cos^{-1}\left\{\frac{a - c' + \sqrt{3}c'}{a - c' - \sqrt{3}c'}\right\}, k\right) \right] = \sqrt{\frac{2\lambda}{3}} \qquad (6.2p)$$

Similarly in case (II) the general solution can be written

$$\sqrt{\frac{2\lambda}{3}}\, x + c = \frac{F(\emptyset, k) - F(\pi, k)}{\sqrt[4]{3}\, c'^{\frac{1}{2}}} \qquad\qquad 0 \leq x \leq x_m$$

$$\left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\}\; (6.2q)$$

$$\sqrt{\frac{2\lambda}{3}}\, x + c = \frac{-F(\emptyset, k) + F(\pi, k)}{\sqrt[4]{3}\, c'^{\frac{1}{2}}} \qquad\qquad x_m \leq x \leq 1$$

where the second constant of integration $c = -\sqrt{\frac{2\lambda}{3}}\, x_n$. Again the constants c and c′ are found from the boundary conditions (6.2b) so that on elimination of c we have

$$\frac{1}{\sqrt[4]{3}\, c'^{1/2}}\left[2\, F(\pi, k) - F\left(\cos^{-1}\left\{\frac{b-c'+\sqrt{3}c'}{b-c'-\sqrt{3}c'}\right\}, k\right) - F\left(\cos^{-1}\left\{\frac{a-c'+\sqrt{3}c'}{a-c'-\sqrt{3}c'}\right\}, k\right)\right] = \sqrt{\frac{2\lambda}{3}} \quad (6.2r)$$

The nonlinear equations (6.2p) and (6.2r) were solved for c′ using the bisection method. The elliptic functions were evaluated using a FORTRAN subroutine based on numerical methods of Bulirsch(1965). The structure of these equations show that for $\lambda > \lambda^* \simeq 1.6$ there is no solution while for λ less than this critical value there are two solutions. See Figure 2.
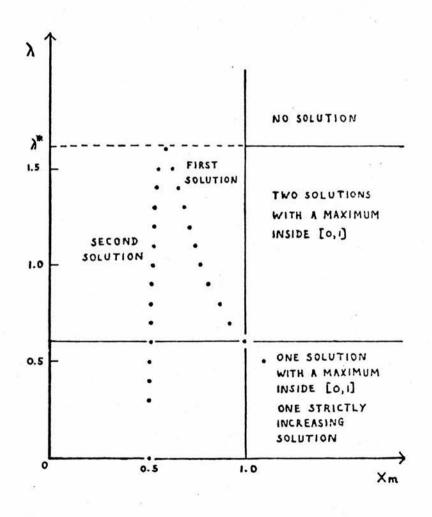
Results

In Tables 9 and 10 we compare the results obtained using rational and Taylor series approximations with the exact solution for $\lambda = \frac{1}{2}$ and $\lambda = 1$. In both tables we have used four terms of the power series for $y(x)$ about x=0 and five terms of the series about x=1. The second solutions are shown in Table 11 where we have here only used the method involving two point approximants. (F), (P) or (B) have been used to denote which iterative scheme was applied.

Comments on these results

Tables 9 and 10 list the approximations to the first solution of (6.2a) for $\lambda = \frac{1}{2}$ and $\lambda = 1$. The iterated two point approximant provides the most accurate approximation to $y(x)$ for small N and has the advantage of automatically satisfying the boundary conditions (6.2b). All the approximations converge quickly to the first solution with increasing N. In both the tables the iterated Taylor series is

FIGURE 2

$$y'' + \lambda y^2 = 0 \qquad y(0)=1 \qquad y(1)=2$$

$$y(x_m) = \max_{0 \le x \le 1} y(x)$$

a closer approximation to $y(x)$ than the iterated S fraction. However this does not always remain the case when N is increased.

Table 11 shows that the iterated two point approximant quickly converges to the second solution for $\lambda = \frac{1}{2}$ and $\lambda = 1$ when using the bisection rule (B). Both the one point approximations were very poor for small N and, as before, neither type was a consistently superior approximation to $y(x)$.

It should be noted that our process is in a sense a more automatic procedure than the calculation of the exact solution. It is also important to note that our approximations failed to converge for values of λ for which there is no solution. It seems very possible that these methods could be useful in determining the existence and uniqueness of solutions to boundary value problems of this type.

63 -

TABLE 9

First solution of $y'' + \lambda y^2 = 0$ $\quad y(0) = 1 \quad y'(1) = 2$

$$\lambda = \frac{1}{2}$$

y(x)	x = 0	x = 0.2	x = 0.4	x = 0.6	x = 0.8	x = 1
Iterated Taylor series (F)	1.0	1.296630	1.560912	1.780500	1.943045	2.036199
Iterated S fraction (F)	1.0	1.298051	1.565311	1.794125	1.980637	2.124543
Iterated two point approximant (F)	1.0	1.295737	1.557779	1.771346	1.922417	2.0
Exact solution	1.0	1.295737	1.557778	1.771346	1.922416	2.0

N = 4 - y(x) expanded to 4 terms about x=0

y(x) expanded to 5 terms about x=1

TABLE 10

First solution of $\quad y'' + \lambda y^2 = 0 \qquad y(0) = 1 \quad y(1) = 2$

$$\lambda = 1$$

y(x)	x = 0	x = 0.2	x = 0.4	x = 0.6	x = 0.8	x = 1
Iterated Taylor series (F)	1.0	1.456967	1.835262	2.096210	2.201139	2.111376
Iterated S fraction (F)	1.0	1.481770	1.890099	2.206014	2.426632	2.561083
Iterated two point approximant	1.0	1.445745	1.807567	2.039662	2.107430	2.0
Exact solution	1.0	1.445749	1.807581	2.039673	2.107433	2.0

N = 4   -   y(x) expanded to 4 terms about x=0

y(x) expanded to 5 terms about x=1

TABLE 11

Second solutions of $y'' + \lambda y^2 = 0$ $y(0) = 1$ $y(1) = 2$

$$\lambda = \tfrac{1}{2}$$

y(x)	x = 0	x = 0.2	x = 0.4	x = 0.6	x = 0.8	x = 1	
Iterated two point approximant (B)	1.0	11.991309	19.927769	20.318354	12.886868	2.0	N = 4
Exact solution	1.0	12.039583	19.967596	20.353277	12.932844	2.0	
Iterated two point approximant (B)	1.0	12.039583	19.967596	20.353277	12.932844	2.0	N = 8

$$\lambda = 1$$

y(x)	x = 0	x = 0.2	x = 0.4	x = 0.6	x = 0.8	x = 1	
Iterated two point approximant (B)	1.0	5.406015	8.586999	8.950732	6.270721	2.0	N = 4
Exact solution	1.0	5.416550	8.597157	8.958129	6.279931	2.0	
Iterated two point approximant (B)	1.0	5.416550	8.597157	8.958129	6.279931	2.0	N = 8

Example 3 - Troesch's Equation : A Well Known Test Problem

The final example is a nonlinear boundary value problem which has
been used extensively as a test problem. This equation, first
investigated by Troesch(1960) and often referred to as Troesch's
equation, is given by

$$\frac{d^2 y}{dx^2} = \lambda \sinh \lambda y \qquad \lambda > 0 \qquad (6.3a)$$

and is solved subject to the boundary conditions

$$y(0) = 0 \qquad y(1) = 1 \qquad (6.3b)$$

Application of the existence theorem for second order differential
equations by Keller(1966) shows that the problem has a unique
solution for all $\lambda > 0$. Difficulties however occur when trying to
solve this problem numerically. If $y'(0)$ is slightly greater than
its real value, the solution has a singularity in the interval $[0,1]$.
(Roberts and Shipman(1972)). Also when λ is greater than four,
Roberts and Shipman(1976) show that if $y'(1)$ is only slightly
greater than its real value, there are discontinuous solutions to
the problem which can cause overflow in the interval $[0,1]$. The
severity of these difficulties increases with λ .

Here we obtain a solution to the problem for small values of
λ. Writing (6.3a) in integral equation form we have

$$y(x) = x + \lambda \int_0^x x(t-1)\sinh \lambda y(t)\, dt + \lambda \int_x^1 (x-1)t \sinh \lambda y(t)\, dt \qquad (6.3c)$$

or

$$y(x) = x + \lambda I(x)$$

If we expand $I(x)$ in a Taylor series about $x=0$ then we have

$$y(x) = x + \lambda \sum_{n=0}^{\infty} x^n \frac{I^{(n)}(0)}{n!} \qquad (6.3d)$$

where

$$
\left.
\begin{aligned}
I(0) &= 0 \\[2mm]
I'(0) &= \int_0^1 (t-1) \sinh \lambda y(t)\, dt \\[2mm]
I''(0) &= \sinh \lambda y(0) = 0 \\[2mm]
I^{(s)}(0) &= \left[\frac{d^{s-2}}{dx^{s-2}} \sinh \lambda y(x) \right]_{x=0} \qquad s = 3, 4, \ldots
\end{aligned}
\right\} \qquad (6.3e)
$$

Similarly if we expand $I(x)$ in a Taylor series about $x=1$ then we can write

$$y(x) = 1 + (x-1) + \sum_{n=0}^{\infty} (x-1)^n \frac{I^{(n)}(1)}{n!} \qquad (6.3f)$$

with

$$
\left.
\begin{aligned}
I(1) &= 0 \\[2mm]
I'(1) &= \int_0^1 t \sinh \lambda y(t)\, dt \\[2mm]
I''(1) &= \sinh \lambda y(1) = \sinh \lambda \\[2mm]
I^{(s)}(1) &= \left[\frac{d^{s-2}}{dx^{s-2}} \sinh \lambda y(x) \right]_{x=1} \qquad s = 3, 4, \ldots
\end{aligned}
\right\} \qquad (6.3g)
$$

Recurrence relationships can be derived for the $I^{(s)}(0)$ and $I^{(s)}(1)$ since

$$
\frac{d^{p+1}}{dx^{p+1}} \sinh \lambda y(x) = \lambda \sum_{R=0}^{P} \binom{P}{R} y^{(p-R+1)}(x) \left\{ \cosh \lambda y(x) \Big|_{k=0} \right.
$$

$$(6.3h)$$

$$
\left. + \lambda \sum_{j=0}^{k-1} \binom{k-1}{j} y^{(k-j)}(x) \frac{d^j}{dx^j} \sinh \lambda y(x) \right\}
$$

$$k = 1, 2, \ldots$$

$$p = 0, 1, \ldots$$

Alternate coefficients of the series expansion about x=0 are zero for this example. The coefficients of the series expansion about x=1 are all positive and become very large as λ increases.

Iterated One Point Approximations

On truncating the infinite series (6.3d) after N terms we may derive the system of N equations

$$
\left.
\begin{aligned}
y(0) &= 0 \qquad\qquad\qquad y''(0) = 0 \\[2mm]
y'(0) &= 1 + \lambda \int_0^1 (t-1) \sinh \lambda\, y_N(t)\, dt \\[2mm]
y^{(s)}(0) &= \lambda \left[\frac{d^{s-2}}{dx^{s-2}} \sinh \lambda\, y(x) \right]_{x=0} \qquad s = 3, 4, \ldots, N-1
\end{aligned}
\right]
\qquad (6.3k)
$$

where $y_N(x) = y_N(x, y(0), y'(0), \ldots, y^{(N-1)}(0))$ can be either the truncated Taylor series of $y(x)$ about x=0 or the S fraction corresponding to that series. The equations are solved using Powell's method (P). However the system essentially contains only one unknown, namely $y'(0)$, and hence we can write

$$
y'(0) = 1 + \lambda \int_0^1 (t-1) \sinh \lambda\, y_N(t, y'(0), y'(1))\, dt
\qquad (6.3L)
$$

where now the one point approximation $y_N(x) = y_N(x, y'(0), y'(1))$.

For both the one point approximations the integral is evaluated numerically at each iteration using Simpson's rule with step size h=0.01. We consider approximations when $\lambda = 1$, 2 and 3; for these values of λ a starting function $y_N^{(0)}(x) = x$ is adequate. However, even with these small values of λ, overflow occurs for certain values of N.

Iterated Two Point Approximant

Now suppose we truncate the infinite series (6.3d) and (6.3f) after N and N+1 terms respectively. Then we may write

$$
\left.
\begin{aligned}
y(0) &= 0 \qquad\qquad\qquad\qquad y''(0) = 0 \\[6pt]
y'(0) &= 1 + \lambda \int_0^1 (t-1)\sinh \lambda y_{NN+1}(t)\, dt \\[6pt]
y^{(s)}(0) &= \lambda \left[\frac{d^{s-2}}{dx^{s-2}} \sinh \lambda y(x)\right]_{x=0} \qquad s=3,4,\dots, N-1 \\[6pt]
y(1) &= 1 \qquad\qquad\qquad\qquad y''(1) = \lambda \sinh \lambda \\[6pt]
y'(1) &= 1 + \lambda \int_0^1 t \sinh \lambda y_{NN+1}(t)\, dt \\[6pt]
y^{(s)}(1) &= \lambda \left[\frac{d^{s-2}}{dx^{s-2}} \sinh \lambda y(x)\right]_{x=1} \qquad s=3,4,\dots, N
\end{aligned}
\right\} \quad (6.3m)
$$

where $y_{NN+1}(x) = y_{NN+1}(x, y(0), y'(0), \dots, y^{(N+1)}(0), y(1), y'(1), \dots, y^{(N)}(1))$ is the two point rational approximant described in Chapter 3. Since this system of 2N+1 equations essentially contains only two unknowns, $y'(0)$ and $y'(1)$, it may be reduced to the pair of nonlinear equations

$$
\left.
\begin{aligned}
y'(0) &= 1 + \lambda \int_0^1 (t-1)\sinh \lambda\, y_{NN+1}(t, y'(0), y'(1))\, dt \\[6pt]
y'(1) &= 1 + \lambda \int_0^1 t \sinh \lambda\, y_{NN+1}(t, y'(0), y'(1))\, dt
\end{aligned}
\right\} \quad (6.3n)
$$

where the two point approximant $y_{NN+1}(x) = y_{NN+1}(x, y'(0), y'(1))$. These equations are solved using Powell's method (P).

Again the integrals were evaluated numerically and, in general, Simpson's rule with step size h=0.01 was used. This was however thought to be a possible source of inaccuracy so that when $\lambda = 5$ a higher order integration routine was used. The rule applied was Newton-Cotes 6 point formula (see Abramowitz and Stegun(1970))

with step size h=0.01. This appears to have improved the accuracy of the results but is thought to be still a source of error due to the large size of the derivatives of $\sinh \lambda y(x)$ when evaluated near x=1.

When λ = 1, 2 and 3 a starting function $y_{N\,N+1}^{[o]}(x)$ = x can be taken. When λ = 4 Powell's method (P) fails to converge with this starting function and a more accurate one has to be found. This was done by considering the simple analysis of the problem by Troesch(1976). He shows that $y(x) < \sinh \lambda x / \sinh \lambda$ (0< x <1) and that $0 < y'(0) < \lambda/\sinh \lambda$. Hence when λ = 4 and 5 we take a starting value $y'(0)$ = $\lambda/\sinh \lambda$. A corresponding starting value for $y'(1)$ can be found by using the relationship (6.3o) derived subsequently. However even with these starting values overflow occurs for most values of N when λ =5.

Since equation (6.3a) does not explicitly contain terms in x we can perform a first integration to give

$$\left[y'(0) \right]^2 = \left[y'(1) \right]^2 - 2\cosh \lambda \qquad (6.3o)$$

where it can be shown that both $y'(0)$ and $y'(1)$ are positive. Thus the two equations (6.3n) may be further reduced to a single equation for $y'(0)$. We solve this equation using the bisection rule (B) with λ = 5. The integral is again evaluated using Newton-Cotes 6 point rule. Unfortunately overflow still occurs for most values of N.

The Exact Solution

A closed form of the solution to Troesch's problem has been given by Roberts and Shipman(1976) in terms of Jacobian elliptic functions. Here we have taken numerical values for $y'(0)$ and $y'(1)$ from this solution and used them to form the two point rational approximant

for various values of N. The results determined for large N (here
we have taken N = 12 or 14) have been quoted as the 'exact'
solution.

Results

In each of Tables 12 - 15 we show results for different values of λ.
When λ = 1, 2 and 3 both the one and two point approximations are
shown using Powell's method (P) and taking four terms of the series
about x=0 and five terms of the series about x=1. When λ = 4 and 5
we just consider the two point approximant. For λ = 4 results
obtained using Powell's method (P) are given taking N = 4 and N = 8.
For λ = 5 we consider the two point approximant obtained by
Powell's method (P) compared with that obtained by the bisection
rule (B). Again we have taken N = 4 since both methods fail to
converge for higher values of N.

Comments on these results

This is a difficult example and it is probably unfair to expect our
methods of approximation to perform well for even quite small values
of λ. The coefficients of the series about x=0 become very small as
λ increases and correspondingly the coefficients of the series about
x=1 become very large. Even with a small step size the truncation
error in the integration routines may be relatively large, and is
therefore a source of error in the results. Accurate starting values
are needed to prevent overflow from occurring.

When λ = 1, 2 and 3 the approximations are fairly well
behaved although they become increasingly slow to converge with
larger λ . In the results shown the iterated Taylor series is a

better approximation to $y(x)$ than the iterated S fraction. However when a larger number of terms in the series is taken (i.e. N is increased) the iterated S fraction becomes the better approximation. Overflow occurred for some values of N when iterating the S fraction for $\lambda = 2$ and $\lambda = 3$.

When $\lambda = 4$ we have shown the resulting approximations when iterating the two point approximant by Powell's method (P). It is noticeable that these approximations are appreciably slower to converge than the approximations for smaller values of λ . When $\lambda = 5$ overflow occurred for most values of N when iterating the two point approximant by Powell's method (P). It is a drawback of Powell's method that, apart from choosing an accurate starting vector and taking a small maximum internal steplength (here we have always taken this to be 0.01), it is difficult to restrict the problem to an area in which overflow cannot take place. We really need to solve the pair of equations (6.3n) subject to constraints on the derivatives. It was hoped that this could be done by solving the single equation for $y'(0)$ using the bisection rule (B). However the interval in which the solution lay was found to be so restricted that the method was impractical and overflow again resulted for most values of N. It is interesting to note that the results obtained by this method when $N = 4$ were poorer than those obtained by Powell's method (P).

TABLE 12

$$y'' = \lambda \sinh \lambda y \qquad y(0) = 0 \quad y(1) = 1$$

$$\lambda = 1$$

y(x)	x = 0	x = 0.2	x = 0.4	x = 0.6	x = 0.8	x = 1
Iterated Taylor series (P)	0.0	0.170234	0.347233	0.537760	0.748580	0.986457
Iterated S fraction (P)	0.0	0.170269	0.346929	0.536307	0.745390	0.982711
Iterated two point approximant (P)	0.0	0.170171	0.347223	0.538534	0.752608	1.0
Exact solution	0.0	0.170171	0.347223	0.538534	0.752608	1.0

N = 4 - y(x) expanded to 4 terms about x=0

y(x) expanded to 5 terms about x=1

TABLE 13

$$y'' = \lambda \sinh \lambda y \qquad y(0) = 0 \qquad y(1) = 1$$

$$\lambda = 2$$

y(x)	x = 0	x = 0.2	x = 0.4	x = 0.6	x = 0.8	x = 1
Iterated Taylor series (P)	0.0	0.108405	0.233704	0.392792	0.602563	0.879911
Iterated S fraction (P)	0.0	0.111613	0.235306	0.380354	0.557148	0.780369
Iterated two point approximant (P)	0.0	0.106518	0.230532	0.393577	0.628467	1.0
Exact solution	0.0	0.106519	0.230522	0.393553	0.628465	1.0

N = 4 — y(x) expanded to 4 terms about x=0

y(x) expanded to 5 terms about x=1

TABLE 14

$$y'' = \lambda \sinh \lambda y \qquad y(0) = 0 \quad y(1) = 1$$

$$\lambda = 3$$

y(x)	x = 0	x = 0.2	x = 0.4	x = 0.6	x = 0.8	x = 1
Iterated Taylor series (P)	0.0	0.060354	0.141207	0.263054	0.446395	0.711727
Iterated S fraction (P)	0.0	0.071901	0.152612	0.245752	0.354922	0.484853
Iterated two point approximant (P)	0.0	0.054186	0.128937	0.253154	0.483229	1.0
Exact solution	0.0	0.054248	0.128777	0.252747	0.483138	1.0

N = 4 - y(x) expanded to 4 terms about x=0

y(x) expanded to 5 terms about x=1

TABLE 15

$$y'' = \lambda \sinh \lambda y \qquad y(0) = 0 \qquad y(1) = 1$$

$\lambda = 4$

y(x)	x = 0	x = 0.2	x = 0.4	x = 0.6	x = 0.8	x = 1	
Iterated two point approximant (P)	0.0	0.024164	0.066922	0.156605	0.357568	1.0	N = 4
Exact solution	0.0	0.024841	0.066504	0.153923	0.356280	1.0	
Iterated two point approximant (P)	0.0	0.024847	0.066517	0.153927	0.356248	1.0	N = 8

$\lambda = 5$

y(x)	x = 0	x = 0.2	x = 0.4	x = 0.6	x = 0.8	x = 1	
Iterated two point approximant (P)	0.0	0.007944	0.032316	0.100349	0.265554	1.0	N = 4
Iterated two point approximant (B)	0.0	0.005926	0.029380	(..)3648	0.273335	1.0	
Exact solution	0.0	0.010753	0.033200	0.092045	0.258217	1.0	

CONCLUSIONS

We have considered Fredholm integral equations of the second kind
and found approximations to the solution using power series
expansions. In particular we have considered using both a Padé
approximant and a two point rational approximant as approximations.
The methods of approximation have been tested by an example which is
described in Chapter 4. As Walsh(1977) states, "if we can obtain
numerical results for a given problem which appear to converge, and
to behave consistently with variations in the parameters, we can be
fairly sure that they represent an analytical solution, even without
a rigorous proof".

Altnough the methods of approximation have been aimed at
solving integral equations, special attention has been paid to two
point boundary value problems. In general, the solution of integral
equations by our methods involves the evaluation of several integrals
at each iteration. This is usually time consuming and must be
considered as a possible source of error when the integrals have to
be evaluated numerically. However the results given in Chapter 4
show that reasonable approximations can be obtained when only a few
terms of the power series expansions are considered. Simplification
occurs when considering two point boundary value problems as the
solution of these differential equations by our method involves the
evaluation of, at most, only two integrals at each iteration.

An important advantage of our methods, when using the two
point rational approximant to solve boundary value problems, is that
the boundary conditions are automatically satisfied. Three second
order differential equations of boundary value type have been

discussed as test examples in Chapter 6. It can be seen that for many problems where calculation of an exact solution is difficult, our approximations provide a reasonable alternative method of solution.

Let us now consider the one point approximations based on power series expansions about the lower endpoint of the interval $[0,1]$. We have used both the truncated Taylor series and the S fraction as one point approximations. Both approximations tend to be slow to converge to the exact solution and are inaccurate for small values of N near the upper endpoint of the interval $[0,1]$. However they have the advantage of being easy to use and provide good starting vectors for other iterative schemes, such as those involving the more accurate two point approximations.

Unfortunately the iterated S fraction does not provide the close approximation to the solution that was hoped. In many of the results the S fraction was a worse approximation than the truncated Taylor series. This was particularly true when N was small and only a few terms of the power series were taken (as was the case in the Tables shown here). A possible reason for this Padé approximant behaving so poorly can be gained from Kershaw(1977). He states that "for a wide class of functions the error in the best polynomial approximation is little worse than that of the error in the comparable rational function", and gives references for further details. It should be stressed, however, that other Padé approximants might yield much better results.

Secondly we consider the two point approximations based on power series expansions about both the lower and the upper endpoints of the interval $[0,1]$. The approximation used here is a two point

rational approximant which fits N terms of the power series expansion
about x=0 and N+1 terms of the power series expansion about x=1.
This approximant converged very quickly to the exact solution for
nearly all the examples tested. Even for small N it is a surprisingly
good approximation. (See Tables of results).

On occasion the methods have failed to converge for certain
values of the parameter λ and number of terms N. This was
particularly the case when using Powell's method (P). It was
difficult to know whether this was due to breakdown of Powell's
method, possibly due to inaccurate evaluation of the integrals, or
whether this was inherent in the approximations themselves. In at
least one example breakdown of the method occurred because the
denominator of the rational approximant was close to zero. However
failure to converge only occurred rarely, normally for isolated values
of N, so cannot be considered a serious drawback.

Special mention should be made of the way multiple solutions
can be found for nonlinear two point boundary value problems as shown
in the second example. Here it can be clearly seen that the method of
iterated two point approximants provides us with two solutions to the
problem. In the third example we see that our methods of approximation,
not surprisingly, cannot cope with the inherent difficulties in the
numerical solution of the problem. It is possible that better
results could be obtained by using a two point approximant dependent
on a large number of terms from the series expansion about x=0 and a
small number of terms from the series expansion about x=1.

Obvious extensions could be made to the work done here,
covering both higher order differential equations and systems of
equations. Other two point boundary value problems with more general

boundary conditions could readily be considered. As suggested above, the two point approximant could be modified for certain examples so that unequal weight was placed on the two series expansions. Other modifications could be made to deal with singularities, possibly involving asymptotic expansions.

We have shown that our methods of approximation using power series expansions can provide accurate approximations to both Fredholm integral equations and boundary value problems. These approximations have the advantage that they are functions of x and hence can be calculated at any point in the interval. In particular we show that the use of two point rational approximants can provide a powerful method of obtaining solutions to two point boundary value problems. This is especially valuable when dealing with those nonlinear problems for which no general methods of solution exist.

REFERENCES

Abramowitz,M. and Stegun,I.A. (1970) "Handbook of Mathematical
        Functions", Dover Publications (7th Ed.), New York.

Bailey,P., Shampine,L.F. and Waltman,P. (1966) Bull.Amer.Math.Soc.
        72,96-98.

Bandle,C. (1975)   Arch.Rational Mech.Anal.,58,219-238.

Birn,J., Goldstein,H. and Schindler,K. (1978) submitted to Solar
        Physics.

Bulirsch,R. (1965) Numerische Mathematik,7,78-90.

Chambers,Ll.G. (1976) "Integral Equations: A Short Course",
        International Textbook Company Ltd., London.

Davis,H.T. (1962)   "Introduction to Nonlinear Differential and
        Integral Equations", Dover Publications, New York.

Drew,D.M. and Murphy,J.A. (1977) J.Inst.Maths.Applics.,19,169-185.

Fox,L. (1962)   "Numerical Solution of Ordinary and Partial
        Differential Equations", Pergamon Press, Oxford.

Gragg,W.B. (1972)   S.I.A.M. Review,14,1-62.

Keller,H.B. (1966) Bull.Amer.Math.Soc.,72,728-731.

Keller,H.B. (1975) in "Numerical Solutions of Boundary Value Problems
        for Ordinary Differential Equations", Ed. A.K. Aziz,
        Academic Press, New York.

Kershaw,D. (1977)   in "State of the Art in Numerical Analysis",
        Ed. D. Jacobs, Academic Press, New York.

McCabe,J.H. (1975) J.Inst.Maths.Applics.,15,363-372.

Mikhlin,S.G. (1964) "Integral Equations", Pergamon Press.

Murphy,J.A. (1966)   Internal research project, Brunel University.

Murphy,J.A. and O'Donohoe,M.R. (1977) Z.Angew.Math.Phys.,$\underline{28}$,1121-1131.

Pogorzelski,W. (1966) "Integral Equations and their Applications",
        Vol. 1, Pergamon Press.

Powell,M.J.D. (1968) Harwell Report AERE-R5947 H.M.S.O.

Rabenstein,A.L. (1966) "Introduction to Ordinary Differential
        Equations", Academic Press.

Roberts,S.M. and Shipman,J.S. (1972) J.Comp.Phys.,$\underline{10}$,232-241.

Roberts,S.M. and Shipman,J.S. (1976) J.Comp.Phys.,$\underline{21}$,279-290.

Scott,M.R. (1975)   in "Numerical Solutions of Boundary Value
        Problems for Ordinary Differential Equations",
        Ed. A.K. Aziz, Academic Press, New York.

Troesch,B.A. (1960)   "Intrinsic Difficulties in the Numerical
        Solution of a Boundary Value Problem", Internal report
        NN-142 TRW,Inc., Redondo Beach, Cal.

Troesch,B.A. (1976)   J.Comp.Phys.,$\underline{21}$,279-290.

Wall,H.S. (1948)   "Analytic Theory of Continued Fractions",
        Van Nostrand, New York.

Walsh,J. (1977)   in "State of the Art of Numerical Analysis",
        Ed. D. Jacobs, Academic Press, New York.

Wilkinson,J.H. and Reinsch,C. (1971) "Handbook for Automatic
        Computation, Vol. II, Linear Algebra", p.93-110,
        Springer-Verlag.