Supersymmetric Quantum Field Theories

from

Induced Representations

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I David Hartley hereby certify that this thesis has been composed by myself, that it is a record of my own work, and that it has not been accepted in partial or complete fulfilment of any other degree of professional qualification.

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I hereby certify that the candidate has fulfilled the conditions of the Resolutions and Regulations appropriate to the degree of Ph.D.

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Abstract

This thesis investigates the application of the method of induced representations in supersymmetric quantum field theories.

First, the main features of the theory of induced representations of Lie groups, Lie algebras and Lie superalgebras are presented. The procedure for obtaining irreducible representations in the important case of Lie groups or algebras with an invariant, Abelian subgroup or subalgebra is described.

This procedure is then applied to the Poincaré group for arbitrary dimensions of space-time. The representations obtained are used to construct free quantum fields, without the use of a Lagrangian. The usual characteristics of these fields, such as the field equations, are shown to be consequences of the representation theory.

The induced representation procedure for algebras is demonstrated by the construction of the irreducible supermultiplets for the $N = 1$ Poincaré superalgebra (in those dimensions for which it exists). Again, the construction proceeds from the representations, not from a Lagrangian.

Finally, a mixture of the group and algebra versions of the procedure developed in the preceding parts of the thesis is applied to the inhomogeneous orthosymplectic superalgebra. This superalgebra is relevant to BRST covariant quantisation of gauge fields. A consequence of the systematic application of the representation theory is the derivation of the Parisi-Sourlas mechanism in pseudo-Euclidean space.

Contents

Chapter 1

Introduction

Symmetry principles have played an increasingly central role in modern physics. This is nowhere more evident than in supersymmetric quantum field theories. Here, symmetry has been raised to the level of a first principle, and in some cases been used in the absence of experimental evidence as a justification for various models. Even though no direct experimental evidence for supersymmetry has been found, there is an enormous amount of theoretical physics research devoted to the topic. The reasons for this are more theoretical than physical. Supersymmetry seems to offer the tantalizing prospects of unifying all known particles and interactions (including gravity), and at the same time producing quantum field theories without divergences. Whether or not these hopes can be realised remains to be seen. Certainly, the supersymmetric theories produced to date have not agreed well even qualitatively with experiment, let alone quantitatively. Nevertheless, the lure is strong enough for many people to ignore these difficulties for the present and place their faith in the eventual success of a theory based upon supersymmetry.

The development of supersymmetry in theoretical physics has been recounted many times. Several reviews of the subject and its history can be found in the collection edited by Jacob (1986) (the one by Sohnius deserves especial mention).

Attempts at finding a unified theory of particles and interactions culminated in the celebrated paper of Coleman and Mandula (1967). This paper discusses the possible symmetries of the S-matrix in a class of relativistic quantum field theories. The symmetries considered are all based on Lie groups (so the generators satisfy commutation relations), and only straightforward Poincaré space-time symmetry is allowed (ie. there is no conformal space-time symmetry, for example). Loosely speaking, the conclusion is that all such symmetries must be a direct product of the Poincaré group with an internal symmetry group. So under the assumptions of Coleman and Mandula, it is not possible to have a single symmetry group, the irreducible representations of which describe elementary particles of different spins.

The restriction to particles of a single spin was overcome by the advent of supersymmetry (Berezin and G.I. Kac 1970, Gol'fand and Likhtman 1971, Volkov and Akulov 1973, Wess and Zumino 1974). This supersymmetry includes a set of space-time transformation generators which satisfy *anti*-commutation rules, and so avoids the theorem of Coleman and Mandula.

This development, and especially the renormalisable model of Wess and Zumino, stimulated a surge in interest. Extended supersymmetries (Dondi and Sohnius 1974), with several sets of anticommuting generators were considered, superfields (Salam and Strathdee 1974) invented, along with a whole host of variations on and developments of the theme. These include supersymmetric Yang-Mills theory, supergravity, superstrings and latterly supermembranes.

Supersymmetry is based not on a Lie algebra, but on a Lie superalgebra (also known as a $\mathbb{Z}_2$-graded Lie algebra). A Lie superalgebra can be divided into two subspaces, called even and odd. The Lie product of a Lie algebra becomes a graded Lie product for a Lie superalgebra. For elements which are purely even or odd, the graded Lie product is antisymmetric (like the Lie product) unless both of the arguments belong to the odd subspace, when it is symmetric. Lie superalgebras have been a topic of mathematical study in their own right. The standard review of the subject, containing a classification of all the simple Lie superalgebras, is by V.G. Kac (1977). Other reviews can be found in, for example, Corwin *et al* (1975), Rittenberg (1978), and Scheunert (1979).

Many of the features of Lie algebras carry over, or have a modified counterpart, in Lie superalgebras. One such feature is the method of induced representations. It generalises easily, and in fact turns out to be more practical for Lie superalgebras than it is for Lie algebras (this will become clear in chapters 7 and 8).

The method of induced representations was used by Frobenius (1898) to study the representations of finite groups. Since then it has been generalised to many other structures, including Lie algebras (Harish-Chandra 1951 theorem 4) and rings (Higman 1955). Wigner (1939) constructed all of the unitary irreducible representations of the Poincaré group by induction, and Mackey (see Mackey 1968, for example) showed how useful and pervasive induced representations are in physics.

Some of the early treatments of supersymmetry in physics were rather haphazard, as might well be expected in a new subject. To a large extent, this lack of clarity and rigour has been rectified, but some of the bad habits acquired in the early stages of development seem to persist still today. One of the main aims of this thesis is to demonstrate expressly that a systematic application of well-grounded mathematics to the physics of supersymmetry is possible. To this end, the use of superspace, supergroups and superfields is avoided. This is not meant to imply that these objects are mathematically unsound. Rather, it is as reaction against the somewhat careless manner in which they are often used.

That the features of a physical theory owe much to the representations of the underlying symmetry or supersymmetry is well known. Nevertheless, the full extent of this influence does not seem to be as widely appreciated as it might be. The other main aim of this thesis is to bring out as much as possible the manifestations of the representation theory. This will be done by following the programme of Weinberg (1964*a*,*b*) in constructing quantum field theories with representations right at the starting point.

In accordance with these two aims, the arguments used, and the structure of the whole thesis, tend to flow from mathematics towards physics. A consequence of the second aim is that there are no Lagrangians or principles of least action to be found in this thesis. They are in fact unnecessary – at least in the theory of free fields.

Chapter 2 contains an account of the theory of induced representations for Lie groups and Lie algebras. Particular attention is paid to the two forms of the theory for Lie algebras: induced and produced representations.

Chapter 3 is concerned with the connection between induced representations of Lie groups and produced representations of Lie algebras, and the special (but extremely useful) case when the group or algebra has an Abelian, invariant subgroup or subalgebra. The generalisation of these ideas to Lie superalgebras is also discussed.

Chapter 4 marks the transition to physics by giving a number of examples of the ideas contained in the preceding two chapters. The representations encountered in relativistic classical field theory are shown to be induced representations by explicit construction. The connection between Lie group and Lie algebra representations is illustrated by a simple example.

Chapter 5 generalises Weinberg's construction of free relativistic quantum fields to arbitrary dimensions of space-time. Particular consequences of this construction include some restrictions on the particles which can be described by a given field, and the first-order field equations (such as the Dirac equation) which it must satisfy. The origin of these features in the induced representations is emphasised.

Chapter 6 brings out the points made in chapter 5 with a number of specific examples. These range from the scalar and spinor fields to the antisymmetric, second-rank tensor field. In each case, the particle content is shown to be the one normally expected, and the field equations are derived. For the antisymmetric, second-rank tensor field, the corresponding gauge potential is considered, and compared with the light-cone gauge formalism.

Chapter 7 contains the first substantial appearance of supersymmetry in this thesis. The multiplets of $N = 1$ Poincaré supersymmetry in space-times of four or more dimensions are constructed. These multiplets are well-known, but the approach here differs in that it is a more systematic application of the general theory of chapters 2 and 3 than usual. One result of this chapter which has not been derived in practice, although it is clearly possible to do so in principle, is the explicit set of supertransformation rules for the multiplet.

Chapter 8 concerns a supersymmetric theory based on a Lie superalgebra known as iosp$(d,2\,|\,2)$. This superalgebra is not directly connected with Poincaré supersymmetry and the search for a unified field theory. Rather, it has to do with the quantisation of non-Abelian gauge field theories. By applying the techniques of the previous chapters to this superalgebra, it is shown that the Parisi-Sourlas dimensional reduction mechanism utilised in these theories is a consequence of the produced representation. A final section indicates the directions which further work in this area could take.

Of course, much of this thesis contains familiar ideas simply presented in such a way as to bring out a particular line of development. Material which does not seem to have appeared in the literature before is indicated as it occurs in the text. Specifically, the new material consists of:

(i) an isomorphism between induced Lie group representations and produced Lie algebra representations (a homomorphism from group to algebra has appeared previously) (section 3.1);

(ii) the generalisation to arbitrary space-time dimensions of Weinberg's method of finding field equations (chapter 5);

(iii) the explicit supertransformation rules for $N = 1$ Poincaré supersymmetry in more than four dimensions (chapter 7);
(iv) the construction of the irreducible representation of iosp$(m,n \mid 2)$ (sections 8.1 to 8.3); and
(v) the proof of the Parisi-Sourlas mechanism for pseudo-Euclidean space (section 8.4).

Some remarks about notation and conventions are in order. To avoid monotonous repetition, a few qualifications are omitted, but nevertheless implied. Lie algebras are always taken to be finite-dimensional, and Lie groups to be linear. Subgroups of Lie groups are always understood to be Lie subgroups unless otherwise stated, and the same for Lie algebras and Lie superalgebras. By subspace is always meant a non-trivial subspace. Representations are normally expressed in terms of linear operators (as opposed to matrices), and denoted (Φ,V) where Φ is a homomorphism into the set of linear operators on the carrier space V. Summation over repeated indices is always implied except where expressly indicated; the ranges of values taken by the various index variables are given near the beginning of each chapter or section.

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Chapter 2

Induced representations of Lie groups and Lie algebras

Induced representations and their close counterparts, produced representations, will be used extensively in the physical theories examined in later chapters. For this reason, it seems appropriate to give a description of the main ideas in the general theory first, before dealing with various specific applications. In fact, two chapters will be required to provide such a description. This chapter presents the methods of induction for Lie group representations and induction and production for Lie algebra representations. The following chapter will deal with further developments such as the relationship between group and algebra representations and the extension of the methods to Lie superalgebras. The reason for focussing first on the general theory is that many properties of individual physical theories can be traced back to quite general properties of induced representations. It is worthwhile to appreciate this because it gives both a deeper understanding of the specific physical theory and also an opportunity to find fruitful analogies between theories.

In section 2.1, induced representations for linear Lie groups are introduced with a simple but powerful formulation. An alternative formulation, more complicated but often more useful in physics, is presented in section 2.2. Section 2.3 is concerned with some general properties of these representations. Attention switches to Lie algebras in section 2.4, with a description of the universal enveloping algebra of a Lie algebra. This important concept is required for the construction of produced and induced representations of Lie algebras in sections 2.5 and 2.6. Finally, some general properties of the Lie algebra representations are discussed in section 2.7. Most of the material for this chapter is drawn from standard references (Higman 1955, Mackey 1968, Blattner 1969, Dixmier 1977, Duflo 1972).

2.1 Induced representations of Lie groups

Basically, the aim of the method of induced representations is to construct a representation of a group G by using a representation of a subgroup H of G. Although the method applies more generally, the groups and subgroups considered here will always be Lie groups unless otherwise stated. There are two main formulations or realisations of induced representations: one found commonly in the physics literature, and the other more often found in the mathematics literature. In the physics approach, the representation of H is extended to one of G by using clever combinations of coset representatives for G/H. At first glance, it seems mildly surprising that the result obtained is indeed a representation of G, and remarkable that it is sometimes irreducible. In the mathematics approach, on the other hand, the representation of H is used to restrict a much larger representation of G to an invariant subspace, removing the element of surprise from the procedure. However, the mathematics realisation is more abstract and difficult to relate to physical problems. For these reasons, both realisations will be presented, beginning in this section with the more mathematical, and then in section 2.2 deriving the more physical from this to show the connection between the two.

Let G be a Lie group. There is a standard representation of G which is carried by a set of functions over G. The codomain and character of these functions is not important just here. The set of real-valued differentiable functions or the set of complex-valued square-integrable functions would both be suitable examples. In this representation, an element $x \in G$ is represented by a linear operator $\Phi(x)$ which acts on a function ψ over G according to

$$\{\Phi(x)\psi\}(x') = \psi(x^{-1}x') \qquad \text{for all } x' \in G. \tag{2.1}$$

The braces { } have been put in just to emphasise that $\Phi(x)$ acts on the function ψ to give a new function $\Phi(x)\psi$. It does not act on $\psi(x')$, which is just a value of the function ψ.

To see that Φ is a representation, let x, x' and x'' be arbitrary members of G and ψ be any function over G. Then, using (2.1),

$$\begin{aligned}\{\Phi(x)\{\Phi(x')\psi\}\}(x'') &= \{\Phi(x')\psi\}(x^{-1}x'')\\ &= \psi(x'^{-1}x^{-1}x'')\\ &= \psi((xx')^{-1}x'')\\ &= \{\Phi(xx')\psi\}(x'').\end{aligned}$$

This is true for all $x''\in G$, so it follows that $\Phi(x)\Phi(x') = \Phi(xx')$ for all $x,x'\in G$, and thus the $\Phi(x)$ form a representation. Since the set of functions over G is an infinite-dimensional vector space, the representation is infinite-dimensional. When using the representation (2.1), it will be necessary to use different carrier spaces for particular purposes, such as, for example, the set of complex-valued square-integrable functions for quantum mechanics. Whatever the modification to the carrier space, (2.1) will define the operators of this representation throughout.

Let H be a subgroup of G, and W be a finite-dimensional vector space carrying a representation Γ of H. The representation (2.1) may be modified so that the carrier space is the set of functions from G to W. This representation is reducible, because it is possible to project out an invariant subspace of the new carrier space using the representation (Γ, W) of H as in the following proposition.

Proposition 2.1 Let V be the set of functions ψ from G to W satisfying

$$\psi(xy) = \Gamma(y^{-1})\{\psi(x)\} \qquad \text{for all } x\in G,\ y\in H. \tag{2.2}$$

Then V is an invariant subspace of the carrier space of the representation Φ.

Comment. The braces have been put in again just to emphasise the subject of the operator. In this case, $\Gamma(y^{-1})$ acts on the W-vector $\psi(x)$.

Proof of proposition. Clearly, the condition (2.2) is non-trivial and linear in ψ, so V is a vector space, and a subspace of the space of functions from G to W. That V is invariant (closed under the action of $\Phi(x)$ for all x in G), can be seen by taking any $x, x' \in G$, $y \in H$ and $\psi \in V$, and evaluating

$$\begin{aligned}\{\Phi(x')\psi\}(xy) &= \psi(x'^{-1}x\, y) && \text{by (2.1)}\\ &= \Gamma(y^{-1})\{\, \psi(x'^{-1}x)\,\} && \text{by (2.2)}\\ &= \Gamma(y^{-1})\{\, \{\Phi(x')\psi\}(x)\,\} && \text{by (2.1),}\end{aligned}$$

so that $\Phi(x')\psi$ itself satisfies (2.2) and is thus also in V. □

Comment. The inverse of y rather than, say, y itself is needed in (2.2) for consistency: if $x \in G$, $y, y' \in H$, and $\psi \in V$ then

$$\begin{aligned}\psi(xyy') &= \Gamma(y'^{-1})\{\, \psi(xy)\,\}\\ &= \Gamma(y'^{-1})\{\, \Gamma(y^{-1})\{\psi(x)\}\,\}.\end{aligned}$$

But $yy' \in H$ also, so

$$\begin{aligned}\psi(xyy') &= \Gamma(\, (yy')^{-1})\{\, \psi(x)\,\}\\ &= \Gamma(y'^{-1}y^{-1})\{\, \psi(x)\,\},\end{aligned}$$

which is consistent with $\Gamma(y'^{-1}y^{-1}) = \Gamma(y'^{-1})\Gamma(y^{-1})$. If y^{-1} in (2.2) is replaced with, say, y then this consistency disappears.

From here on, the emphasising braces will be used much more sparingly, but the domains of the various operators should always be borne in mind.

Since V is invariant, the operators $\Phi(x)$ of (2.1) can be restricted to act just on V, completing the construction of the induced representation. No new notation will be needed for the restriction.

Definition 2.2 (Φ, V) defined by (2.1) and (2.2) is the representation of G *induced* from the representation (Γ, W) of $H \subset G$.

Comment. An equivalent representation can be obtained by modifying (2.1) and (2.2) to

$$\Phi(x)\psi(x') = \psi(x'x)$$

and

$$\psi(yx) = \Gamma(y)\psi(x),$$

for all $x \in G$, $y \in H$, but the first definitions given are more convenient for the eventual applications.

2.2 The coset space formulation

As it stands, the realisation of the induced representation in section 2.1 is not very practical, because the functions over the group G are difficult to relate to physical problems. Much more common in physics are functions over the coset space G/H of G with its subgroup H, so the induced representation will be transformed to give a realisation in terms of such functions.

Let G be a Lie group, H a subgroup of G, and K the left coset space G/H (ie. $K = \{ xH : x \in G \}$). There are several standard features of K which are required here. Firstly, there is a canonical projection map $\pi : G \to K$ of G onto K, defined by

$$\pi(x) = xH \qquad \text{for } x \in G,$$

so that

$$\pi(xy) = \pi(x) \qquad \text{for all } x \in G, y \in H.$$

Any $z \in K$ can be written $z = \pi(x)$ for some $x \in G$. The canonical action of $x' \in G$ on z can be written as left multiplication and defined by

$$x'z = \pi(x'x),$$

or

$$x'\pi(x) = \pi(x'x).$$

The choice of x for z does not affect the value of $\pi(x'x)$, so the action is well defined. K contains a point which is stable under the action of all elements of H, namely

$$z_0 = \pi(x_0),$$

where x_0 is the identity element of G. Furthermore, any element x of G satisfying $x\, z_0 = z_0$ must be an element of H. Finally, a set of coset representatives is needed. This is a mapping $B : K \to G$ such that

$$\pi(B(z)) = z \qquad \text{for all } z \in K, \tag{2.3a}$$

or equivalently,

$$B(z)z_0 = z \qquad \text{for all } z \in K. \tag{2.3b}$$

B is also required to vary smoothly over K. The construction of B is (sometimes) a little technical; the details are discussed in the comments after definition 2.5.

With these tools, the induced representation (Φ, V) of (2.1) and (2.2) can be converted to give the desired realisation over K. Let U be the space of functions from K to W. This will be the carrier space of the new realisation. In order to derive the operators for the new realisation, it is necessary to establish a link between V and U. Define two mappings $S: V \to U$ and $T: U \to V$ for $\psi \in V$ and $\phi \in U$ by,

$$\{S\psi\}(z) = \psi(B(z)) \qquad \text{for all } z \in K, \tag{2.4a}$$

and

$$\{T\phi\}(x) = \Gamma(h(x)^{-1})\,\phi(\pi(x)) \qquad \text{for all } x \in G, \tag{2.4b}$$

where

$$h(x) = B(\pi(x))^{-1}x$$

is the unique member of H locating x within its left coset.

Proposition 2.3 The mappings S and T provide an isomorphism between V and U.

Proof. $S\psi$ is clearly a function from K to W, and hence a member of U, but $T\phi$ needs to be checked. If $x \in G$ and $y \in H$, then

$$\begin{aligned} h(xy) &= B(\pi(xy))^{-1}xy \\ &= B(\pi(x))^{-1}xy \\ &= h(x)y. \end{aligned}$$

Hence

$$T\phi(xy) = \Gamma(h(xy)^{-1})\,\phi(\pi(xy))$$

$$= \Gamma(\, y^{-1} h(x)^{-1}\,)\, \phi(\, \pi(x)\,)$$

$$= \Gamma(\, y^{-1}\,)\, T\phi(x),$$

so $T\phi$ is indeed a member of V. For any $\psi \in V$ then, $TS\psi$ is also in V, and for any $x \in G$,

$$\begin{aligned} TS\psi(x) &= \Gamma(\, h(x)^{-1}\,)\, S\psi(\, \pi(x)\,) && \text{by (2.4}b\text{)} \\ &= \Gamma(\, h(x)^{-1}\,)\, \psi(\, B(\pi(x))\,) && \text{by (2.4}a\text{)} \\ &= \psi(\, B(\pi(x))\, h(x)\,) && \text{by (2.2)} \\ &= \psi(x). \end{aligned}$$

Thus, considered as an operator on V, $TS = 1$. Likewise, for any $\phi \in U$, $ST\phi$ is also in U, and for any $z \in K$,

$$\begin{aligned} ST\phi(z) &= T\phi(\, B(z)\,) && \text{by (2.4}a\text{)} \\ &= \phi(z) && \text{by (2.4}b\text{),} \end{aligned}$$

since

$$\pi(\, B(z)\,) = z,$$

and so

$$h\,(\, B(z)\,) = x_0 \qquad \text{(the identity of } G \text{ and } H\text{)}.$$

Thus, considered as an operator on U, $ST = 1$. Since both $TS = 1$ and $ST = 1$, S and T are inverse mappings, and provide an isomorphism between V and U. □

Using this isomorphism, the operators $\Phi(x)$ for $x \in G$ can be converted to give linear operators $\Delta(x)$ on U by setting

$$\Delta(x) = S\Phi(x)T \qquad \text{for all } x \in G, \tag{2.5}$$

This $\Delta(x)$ has, of course, nothing whatsoever to do with a variation, a small change in x, or any such thing. The action of $\Delta(x)$ on $\phi \in U$ then follows by straightforward evaluation.

Proposition 2.4 Acting on $\phi \in U$, the operators $\Delta(x)$ defined by (2.5) give, for all $z \in K$,

$$\Delta(x)\phi(z) = \Gamma(B(z)^{-1} x\, B(x^{-1}z)\,)\phi(x^{-1}z\,). \tag{2.6}$$

Proof. Evaluating $\Delta(x)\phi$ at any point $z \in K$ gives

$$\begin{aligned} \Delta(x)\phi(z) &= \Phi(x)T\phi(B(z)) && \text{by (2.5) and (2.4}a\text{)} \\ &= T\phi(x^{-1}B(z)) && \text{by (2.1)} \\ &= T\phi(B(x^{-1}z)\,B(x^{-1}z)^{-1}\,x^{-1}B(z)), \end{aligned}$$

where an apparently superfluous factor has been introduced, but for a good reason. Consider the following calculation in K,

$$\begin{aligned} B(x^{-1}z)^{-1}\,x^{-1}B(z)\,z_0 &= B(x^{-1}z)^{-1}\,x^{-1}z && \text{by (2.3)} \\ &= z_0, \end{aligned}$$

since

$$B(x^{-1}z)\,z_0 = x^{-1}z \qquad \text{by (2.3).}$$

Thus $B(x^{-1}z)^{-1}\,x^{-1}B(z)$ is an element of H, and (2.2) can be applied since $T\phi \in V$, giving

$$\begin{aligned} \Delta(x)\phi(z) &= \Gamma(B(z)^{-1}x\,B(x^{-1}z))T\phi(B(x^{-1}z)) \\ &= \Gamma(B(z)^{-1}x\,B(x^{-1}z))\phi(x^{-1}z), && \text{using (2.4}b\text{).} \quad \square \end{aligned}$$

Equation (2.6) gives a statement of the operators $\Delta(x)$ purely in terms of U and K, without reference to V or the $\Phi(x)$, and so allows the following alternative definition of an induced representation.

Definition 2.5 Let B provide a set of coset representatives for the left coset space $K = G/H$, U be the space of W-valued functions over K, and the operators $\Delta(x)$ on U be *defined* by (2.6) (ie. not by (2.5) any longer). Then (Δ, U) is the representation of G *induced* from the representation (Γ, W) of $H \subset G$.

Comments. There are several technical points concerning (Δ, U).

(i) By virtue of the relationship (2.5), the representations (Δ, U) and (Φ, V) are equivalent.

(ii) It may not be possible to find a single smooth mapping B from K to G for the coset representatives. Instead, several mappings may be required, each covering different areas of K, with smooth transformations between one mapping and another where they overlap.

(iii) Comment (ii) is sometimes phrased in the language of fibre bundles as follows. The collection (G, K, π) is a fibre bundle. For any open set $L \subset K$, there exists a local cross section $B_L : L \to G$ giving, for each $z \in L$, an element $B_L(z) \in G$ such that

$$\pi(B_L(z)) = z.$$

The local cross sections will be taken as smoothly varying in L. (Comparison with (2.3) shows that *smoothly varying local cross section* is the fibre-bundle phrase for a set of smoothly varying coset representatives for elements of an open set.) Except in the simplest cases, it is not possible to find a smooth global cross section covering the whole of K. Two local cross sections B_L and $B_{L'}$ for open sets $L, L' \subset K$ are said to be *compatible* if they are related on $L \cap L'$ by a function in $C^\infty(G, G)$. An *atlas of local cross sections* is a complete set of mutually compatible local cross sections such that the whole of K is covered. For convenience, the subscript L will be dropped from $B_L(z)$ with the understanding that the implied L is an open neighbourhood of z and the $B_L(z)$ all belong to an atlas of local cross sections.

(iv) Thus, in (2.6), $B(z)$ and $B(x^{-1}z)$ may refer to different, unrelated local cross sections or coset representatives.

(v) Taking $B(z_0) = x_0$ (the identity of G) gives the useful relation for $\phi \in U$, $z \in K$,

$$\phi(z) = \Delta(B(z)^{-1})\,\phi(z_0). \tag{2.7}$$

(vi) Direct verification that (2.6) does indeed define a representation is straightforward, although the outcome is not immediately apparent from inspection. This is what makes the physics realisation surprising.

2.3 Properties of induced Lie group representations

There are several simple general properties of induced representations which will be useful in later applications. Proofs of these properties can be found in standard texts (eg. Mackey 1968).

Let (Φ, V) be the representation of a Lie group G induced from the representation (Γ, W) of the subgroup $H \subset G$ (either definition 2.2 or definition 2.5 will do). It follows fairly quickly from the definitions that (Φ, V) cannot be irreducible unless (Γ, W) is. However, this is only a necessary condition, not a sufficient one. More useful results regarding reducibility are possible for the special case described in section 3.2.

Another important property is that it makes essentially no difference if the induction procedure is carried out in stages through a chain of subgroups. That is, let F be a subgroup of H, and suppose that (Γ, W) is in fact induced from the representation (Δ, U) of F. Then (Φ, V) is equivalent to the representation of G induced *directly* from (Δ, U) of F. This fact becomes very useful in relativistic field theories of massless particles, as will be seen in section 4.3.

An inner product for the induced representation carrier space may be constructed from an inner product in W. Take first the induced representation of definition 2.2. Let μ be the left-invariant Haar measure on G (see eg. Helgason 1978), and let $\langle \cdot , \cdot \rangle_W$ be an inner product in W. Then $\langle \cdot , \cdot \rangle_V$ defined by

$$\langle \phi , \psi \rangle_V = \int_G d\mu(x) \langle \phi(x), \psi(x) \rangle_W , \qquad (2.8a)$$

is an inner product for $\phi, \psi \in V$. (Remember $\phi(x)$ and $\psi(x)$ are vectors in W). For the realisation of definition 2.5, an invariant measure on the coset space K can usually be obtained from μ by projection. (If not, then it can be shown that at least a quasi-invariant measure exists, and the definition (2.6) of the operators can be altered to allow for this (Mackey 1968), but the applications to be considered here will not require this complication). It may be possible to find several alternative measures on K, but as far as the use here is concerned, it makes no difference: μ is essentially unique. Denoting the invariant measure on K by μ also, since no confusion will arise by doing this, the inner product in the coset space realisation is

$$\langle \phi , \psi \rangle_U = \int_K d\mu(z) \langle \phi(z), \psi(z) \rangle_W , \tag{2.8b}$$

for $\phi, \psi \in U$.

Finally, there are some properties concerning unitarity. If $\langle \cdot , \cdot \rangle_W$ is a Hermitean inner product, ie.

$$\langle w , w' \rangle_W = \langle w', w \rangle_W^*$$

for arbitrary w, w' in W, and if μ is a real measure, then the inner products (2.8) are also Hermitean. Further, if $\langle \cdot , \cdot \rangle_W$ is invariant for H, ie.

$$\langle \Gamma(y)w , \Gamma(y)w' \rangle_W = \langle w , w' \rangle_W$$

for all $y \in H$ and $w, w' \in W$, then the inner products (2.8) are invariant for G. Combining these two results, if the representation (Γ, W) of H is unitary under the inner product $\langle \cdot , \cdot \rangle_W$, then the induced representation will also be unitary under the inner product defined by equations (2.8). Such a unitary representation is appropriate, for example, for a classical field theory, as will be discussed in chapter 4.

2.4 The universal enveloping algebra of a Lie algebra

Many of the features of Lie groups correspond closely to features of Lie algebras, and it is often more convenient to use one or the other formalism in solving problems. For this reason alone, it is worth examining the Lie algebraic form of induced representation theory. In addition, the extension from physical theories with ordinary (Lie) symmetries to ones with supersymmetries is very often much more satisfactory in an algebraic rather than a group formalism, and this is particularly so with induced representations (see section 3.4 for further comments on some of the difficulties with induced representations of Lie supergroups). At the heart of both the induction and production procedures is the universal enveloping algebra of the Lie algebra, so this will be examined first.

Let g be a real Lie algebra. The universal enveloping algebra $U(g)$ may be regarded as the real vector space of polynomials (including an identity, or polynomial of order 0) in the elements of g, with the added condition that two polynomials are equal if they may be transformed into each other by use of the Lie products of g as commutation rules (see Dixmier 1977, or Helgason 1978 for a rigorous definition). So, for example, if X, X', X'' are elements of g, and $[X, X'] = X''$, then X; X^2X'; and $3X'' - 2X^4X''X + 6$ are typical elements of $U(g)$, and

$$X^2X' = XX'' + XX'X.$$

Let $X_1,\ldots,X_m$ (where $m = \dim g$) be basis elements of g. According to the Poincaré-Birkhoff-Witt theorem (Birkhoff 1937, Witt 1937), a basis for $U(g)$ is formed by the set of products $X_1^{r_1}X_2^{r_2}\ldots X_m^{r_m}$ where $r_1,\ldots,r_m \in \mathbb{N}$ (the natural numbers 0,1,2,...). That is, the terms of any element of $U(g)$ can be shuffled using the Lie products for g to give a sum of terms in which the basis elements of g are arranged in order. Of course, the actual order is not important: an equally good basis for U(g) is formed by the set of products $X_m^{r_m} X_{m-1}^{r_{m-1}} \ldots X_1^{r_1}$. Such basis elements will be abbreviated to X^r where

$$X^r = X_1^{r_1} X_2^{r_2} \ldots X_m^{r_m}, \tag{2.9}$$

or any other fixed ordering, and $r = (r_1, \ldots, r_m) \in \mathbb{N}^m$.

Let h be a Lie subalgebra of g, and U(h) its universal enveloping algebra. U(g) is a vector space *over* $\mathbb{R}$, which means that the scalars which form part of the definition of the vector space are elements of $\mathbb{R}$. However, it is also possible to regard the elements of U(h) as the "scalars", and consider U(g) as a vector space over U(h) instead of over $\mathbb{R}$. This can be seen as follows.

If $\{X_1, \ldots, X_m\}$ is a basis for g (where $m = \dim g$), then suppose that it is so ordered that $\{X_{n+1}, \ldots, X_m\}$ is a basis for h ($\dim h = m-n$) and $\{X_1, \ldots, X_n\}$ is a basis for a complement h^c of h in g. This complement may not be a Lie algebra, but it is still a real vector space. The basis elements of U(g) can be arranged so that all factors belonging to h appear to the left of all factors belonging to h^c, as for example in $X_m^{r_m} X_{m-1}^{r_{m-1}} \ldots X_1^{r_1}$. Then every element A in U(g) can be written uniquely in the form

$$A = \sum_{r \in \mathbb{N}^n} A_{r_n \ldots r_1} X_n^{r_n} \ldots X_1^{r_1}$$

with the $A_{r_n \ldots r_1}$ all belonging to U(h). In this way, the products $X_n^{r_n} \ldots X_1^{r_1}$ (or equivalently, $X_1^{r_1} \ldots X_n^{r_n}$) form a basis for U($g$) regarded as a vector space over U(h), technically known as a *left U(h)-module*. Exactly the same reasoning allows U(g) to be regarded as a *right* U(h)-module, in which the factors belonging to h appear to the right. The bases for U(g) as left and right U(h)-modules are exactly the same, and the basis elements can be written in abbreviated form as X^r with $r \in \mathbb{N}^n$ instead of $\mathbb{N}^m$. Thus an element A of U(g) considered as a left U(h)-module can be written

$$A = \sum_{r \in \mathbb{N}^n} A_r X^r, \tag{2.10a}$$

with the A_r all in U(h). Alternatively, for U(g) as a right U(h)-module, A can be written

$$A = \sum_{r \in \mathbb{N}^n} X^r A_r, \tag{2.10b}$$

again with the A_r all in U(h).

It is well worth noting that there is a natural (and even unique) extension of any representation of a Lie algebra g to a representation of its universal enveloping algebra U(g). In this extension, the unit element of U(g) is mapped to the identity operator (see eg. Dixmier 1977).

2.5 Produced representations of Lie algebras

Construction of produced representations of Lie algebras proceeds along very similar lines to that of induced representations of Lie groups presented in section 2.1. This is by no means accidental, since the two sorts of representation are in close correspondence, as will be seen in section 3.1. As for Lie groups, there are two formulations. Both will be examined here, the point of this being that the first is easier to understand as a procedure for obtaining representations, and the second is more convenient when it comes to applications.

Let g be a real Lie algebra. The starting point of the construction is again a standard representation of g. The carrier space is the set of linear functions over $U(g)$. Again, the codomain of the functions is unimportant at this stage. It will vary for different applications, but it has no bearing on the definition of the operators or the validity of the representation. Each element X in g is represented by a linear operator $\Phi(X)$ acting on linear functions ψ over $U(g)$ by

$$\{\Phi(X)\psi\}(A) = \psi(AX) \qquad \text{for all } A \in U(g). \qquad (2.11)$$

Braces have been introduced here for emphasis.

To check that Φ defined by (2.11) is a representation, let X and X' be arbitrary elements of g, ψ any linear function over $U(g)$, and A any element of $U(g)$. Then

$$\begin{aligned}\Phi(X)\{\Phi(X')\psi\}(A) &= \Phi(X')\psi(AX) && \text{by (2.11)}\\ &= \psi(AXX'),\end{aligned}$$

so

$$\begin{aligned}\{[\Phi(X),\Phi(X')]\psi\}(A) &= \psi(AXX') - \psi(AX'X)\\ &= \psi(A[X,X'])\\ &= \{\Phi([X,X'])\psi\}(A) && \text{by (2.11).}\end{aligned}$$

Thus, Φ is indeed a representation. Since U(g) is a countably infinite dimensional vector space, the set of linear functions over U(g) must be at least countably infinite in dimension, and so the representation is infinite dimensional. As will be seen in section 3.1, this representation is related to the standard group representation defined by (2.1).

Let h be a subalgebra of g, and W a finite-dimensional vector space carrying a representation Γ of h. The representation (2.11) may be modified so that the carrier space is the set of linear functions from U(g) to W. Just as for the group, this representation is reducible, because it is possible to project out an invariant subspace of the new carrier space using the representation (Γ, W) of h as in the following proposition.

Proposition 2.6 Let V be the set of linear functions ψ from U(g) to W satisfying

$$\psi(BA) = \Gamma(B)\{\psi(A)\} \qquad \text{for all } A \in \mathrm{U}(g),\ B \in \mathrm{U}(h). \tag{2.12}$$

Then V is an invariant subspace of the carrier space of the representation Φ of equation (2.11).

Proof. Clearly, the condition (2.12) is non-trivial and linear in ψ, so V is a vector space, and a subspace of the space of linear functions from U(g) to W. That V is invariant (closed under the action of $\Phi(X)$ for all X in U(g)), can be seen by taking any $X \in g$, $A \in \mathrm{U}(g)$, $B \in \mathrm{U}(h)$ and $\psi \in V$, and evaluating

$$\begin{aligned}
\{\Phi(X)\psi\}(BA) &= \psi(BAX) && \text{by (2.11)}\\
&= \Gamma(B)\{\psi(AX)\} && \text{by (2.12)}\\
&= \Gamma(B)\{\{\Phi(X)\psi\}(A)\} && \text{by (2.11),}
\end{aligned}$$

so that $\Phi(X)\psi$ itself satisfies (2.12) and is thus also in V. □

Comment. Condition (2.12) is the direct analogue of condition (2.2) for the group representations. Again, the particular order BA on the left hand side of (2.12) is required for consistency.

Since V is invariant, the operators $\Phi(X)$ of (2.11) can be restricted to act just on V, completing the construction of the produced representation. No new notation will be needed for the restriction.

Definition 2.7 (Φ, V) defined by (2.11) and (2.12) is the representation of the Lie algebra g *produced* from the representation (Γ, W) of its subalgebra h.

Comments. (i) The term *co-induced* is used by some authors for the produced representation (see eg. Dixmier 1977).
(ii) (Φ, V) is closely related to the abstract realisation (definition 2.2) of an induced representation of a Lie group. The connection will be given in section 3.1.

The space V merits closer examination. As explained earlier, $U(g)$ can be regarded as a left $U(h)$-module. Inspection of equation (2.12) in this light reveals that any $\psi \in V$ is a homomorphic mapping from $U(g)$ to W preserving the $U(h)$-module structure. V is thus denoted $V = \mathrm{Hom}_{U(h)}(U(g), W)$. By virtue of this homomorphic nature, the representation can be put into a form similar in spirit to the coset space realisation (definition 2.5) of the induced Lie group representation.

Let X^r, $r \in \mathbb{N}^n$ be basis elements for $U(g)$ as a left $U(h)$-module, and let k be the real vector space spanned by *real* combinations of the X^r. $U(g)$ is spanned by "$U(h)$-combinations" of the X^r, and $\mathbb{R}$ is a subspace of $U(h)$, so k is just a subspace of $U(g)$. In the special case in which h is an invariant subalgebra of g, h^c is isomorphic to the quotient algebra

g/h. Then k is isomorphic to $\mathrm{U}(g/h)$. But in general, k is not a universal enveloping algebra, just a vector space.

Let U be the space of linear functions from k to W: $U = \mathrm{L}(k,W)$. (This U should not be confused with the universal enveloping algebras, which always appear in plain type with a Lie algebra as argument, eg. $\mathrm{U}(g)$.) Then the produced representation of definition 2.7 can be converted to an equivalent representation carried by U as follows. First establish a link between V and U by defining two mappings $S : V \to U$ and $T : U \to V$ for $\psi \in V$ and $\phi \in U$ by,

$$\{S\psi\}(Z) = \psi(Z) \qquad \text{for all } Z \in k, \tag{2.13a}$$

and

$$\{T\phi\}(A) = \sum_{r \in \mathbb{N}^n} \Gamma(A_r)\,\phi(X^r) \qquad \text{for all } A \in \mathrm{U}(g), \tag{2.13b}$$

where A has been expressed in the X^r basis using (2.10).

Proposition 2.8 S and T of (2.13) are well defined mappings providing an isomorphism between U and V.

Proof. Obviously, $S\psi$ is an element of U. The only condition for $T\phi$ to belong to V which is not immediately apparent is that it must satisfy (2.12). Any product BA with $B \in \mathrm{U}(h)$ and $A \in \mathrm{U}(g)$ has $\mathrm{U}(h)$-components $(BA)_r = B\,(A_r)$ in the X^r basis for $\mathrm{U}(g)$, so

$$\begin{aligned}\{T\phi\}(BA) &= \sum_{r \in \mathbb{N}^n} \Gamma(\,(BA)_r\,)\,\phi(X^r) && \text{by (2.13}b\text{)}\\ &= \sum_{r \in \mathbb{N}^n} \Gamma(B)\Gamma(A_r)\,\phi(X^r)\\ &= \Gamma(B)\{\,\{T\phi\}(A)\,\},\end{aligned}$$

which shows that $T\phi$ does satisfy (2.12). S and T are thus both well defined.

Since the mapping S is nothing but a restriction of the function $\psi \in V$, it is easy to see that every element of U can be obtained in this way. So S is an onto mapping. In addition, if ψ is any element of V, and $S\psi$ its image in U, then further applying T yields

$$\begin{aligned} TS\psi(A) &= \sum_{r \in \mathbb{N}^n} \Gamma(A_r)\, S\psi(X^r) && \text{by (2.13}b\text{)} \\ &= \sum_{r \in \mathbb{N}^n} \Gamma(A_r)\, \psi(X^r) && \text{by (2.13}a\text{)} \\ &= \sum_{r \in \mathbb{N}^n} \psi(A_r X^r) && \text{by (2.12)} \\ &= \psi(A), \qquad \text{for all } A \in \mathrm{U}(g). \end{aligned}$$

So $TS\psi = \psi$ for any $\psi \in V$. If ψ and ψ' are arbitrary elements of V, then

$$\begin{aligned} S\psi = S\psi' &\Rightarrow TS\psi = TS\psi' \\ &\Rightarrow \psi = \psi', \end{aligned}$$

so S is a one-to-one mapping. Since it is also onto, S is therefore an isomorphism, and T is its inverse. □

This isomorphism between V and U establishes an equivalent representation of g on U, with operators $\Delta(X)$ for $X \in g$ defined by

$$\Delta(X) = S\Phi(X)T. \tag{2.14}$$

An explicit formulation of these operators solely in terms of the elements of U is given by the following proposition.

Proposition 2.9 Acting on $\phi \in U$, the operators $\Delta(X)$ defined by (2.14) give, for all $Z \in k$,

$$\Delta(X)\phi(Z) = \sum_{r \in \mathbb{N}^n} \Gamma((ZX)_r)\, \phi(X^r). \tag{2.15}$$

Proof. Evaluating the left hand side of (2.15) using (2.14),

$$\begin{aligned} \Delta(X)\phi(Z) &= S\Phi(X)T\,\phi(Z) \\ &= \Phi(X)T\,\phi(Z) && \text{by (2.13}a\text{)} \end{aligned}$$

$$= T\,\phi(ZX) \qquad \text{by (2.11)}$$
$$= \sum_{r \in \mathbb{N}^n} \Gamma(\,(ZX)_r\,)\,\phi(\,X^r\,) \qquad \text{by (2.13}b\text{).} \qquad \square$$

Equation (2.15) can be used as an alternative definition of the produced representation. This formulation is similar in spirit to the coset space realisation of the induced group representation (definition 2.5), although a direct connection between equations (2.15) and (2.6) is less illuminating than one between the more abstract forms of definitions 2.1 and 2.7. As for the coset space realisation, it is not immediately obvious from equation (2.15) alone that the operators $\Delta\,(X)$ do form a representation. For completeness, the alternative definition is as follows.

Definition 2.10. Let U be the space of W-valued linear functions over k, and the operators $\Delta\,(X)$ on U be *defined* by (2.15) (ie. not by (2.14) any longer). Then $(\Lambda\,,\,U\,)$ is the representation of g *produced* from the representation $(\Gamma\,,\,W\,)$ of $h \subset g$.

2.6 Induced representations of Lie algebras

For the *induced* representation of a Lie algebra g, the construction is very similar in outline to that for the produced representation, so less details will be given here. Let h be a subalgebra of g, and W a finite-dimensional vector space carrying a representation Γ of h. The starting point is the representation of g carried by the universal enveloping algebra U(g) itself. This is simply the natural mapping of g into U(g), with g acting on U(g) by left multiplication. A trivial extension to a representation on the direct product space U(g)$\times W$ can be made by defining operators $\Phi(X)$ for $X \in g$ by

$$\Phi(X)\{ A\times w \} = \{X A\}\times w \qquad \text{for all } A \in \mathrm{U}(g),\ w \in W, \tag{2.16}$$

where braces have been added for emphasis. Φ is obviously a representation of g.

Once again, this initial representation is reducible. Let R be the set comprising elements in U(g)$\times W$ of the form

$$\{AB\}\times w - A \times \{\Gamma(B)w\}, \tag{2.17}$$

with $A \in \mathrm{U}(g)$, $B \in \mathrm{U}(h)$ and $w \in W$. R is a subspace of U(g)$\times W$, so the quotient space (U(g)$\times W$)$/R$ can be formed. This is called the *tensor product space of U(g) and W over U(h),* and denoted $\mathrm{U}(g)\otimes_{\mathrm{U}(h)}W$. Elements of $\mathrm{U}(g)\otimes_{\mathrm{U}(h)}W$ can be written $A\otimes w$ with $A \in \mathrm{U}(g)$ and $w \in W$ by means of the canonical projection of U(g)$\times W$ onto $\mathrm{U}(g)\otimes_{\mathrm{U}(h)}W$ (in which R projects onto the zero vector). Such elements then satisfy

$$\{AB\}\otimes w = A \otimes \{\Gamma(B)w\}, \tag{2.18}$$

for all $A \in \mathrm{U}(g)$, $B \in \mathrm{U}(h)$ and $w \in W$. In fact, an alternative definition of $\mathrm{U}(g)\otimes_{\mathrm{U}(h)}W$ is the set of elements $A\otimes w$ with $A \in \mathrm{U}(g)$ and $w \in W$ together with the identity (2.18).

Since the action (2.16) of Φ is simply left multiplication in the U(g) component, elements of the form (2.17) transform into other elements of the same form. So R is an invariant subspace of U(g)$\times W$ under the action of Φ. Therefore, the operators $\Phi(X)$ for $X \in g$ can be defined to act on a new carrier space $V = \mathrm{U}(g)\otimes_{\mathrm{U}(h)}W$ by "passage to the quotient", which amounts here to nothing more than repeating (2.16) with elements of $\mathrm{U}(g)\otimes_{\mathrm{U}(h)}W$ instead of U(g)$\times W$:

$$\Phi(X)\{A\otimes w\} = \{X A\}\otimes w \qquad \text{for all } A \in \mathrm{U}(g),\ w \in W. \tag{2.19}$$

Just as a check, it is not hard to see that (2.18) will still be satisfied by $\Phi(X)\{AB\otimes w\}$, so $\Phi(X)$ is well defined as an operator on V.

Definition 2.11 (Φ, V) defined by equations (2.19) and (2.18) is the representation of the Lie algebra g *induced* from the representation (Γ, W) of its subalgebra h.

Comment. The induced representation of a Lie algebra is useful in quantum field theory (see section 5.1).

Like the produced representation, the induced representation can be re-formulated by regarding U(g) as a U(h)-module. Let X^r, $r \in \mathbb{N}^n$ be basis elements for U(g) as a *right* U(h)-module, and let k be the real vector space spanned by *real* combinations of the X^r. Define the new carrier space to be $U = k\otimes W$ (a tensor product over ordinary numbers), which has elements of the form $Z\otimes w$ with $Z \in k$ and $w \in W$. Finally, define the operators $\Delta(X)$ for $X \in g$ by

$$\Delta(X)\{Z\otimes w\} = \sum_{r\in\mathbb{N}^n} X^r \otimes \{\Gamma((Z X)_r)w\}, \tag{2.20}$$

for all $Z\otimes w$ in U. The $(Z X)_r$ are the U(h)-components of $Z X$ in U(g) as a right U(h)-module (cf. equation (2.10b)).

By an argument which follows very closely that given for the produced representation, (Φ , V) and (Δ , U) are equivalent representations. So again there is an alternative definition.

Definition 2.12 Let $U = k \otimes W$ and Δ be defined by equation (2.20). Then (Δ , U) is the representation of the Lie algebra g *induced* from the representation (Γ , W) of its subalgebra h.

2.7 Properties of produced and induced Lie algebra representations

Some properties of induced Lie group representations also hold for produced or induced Lie algebra representations (proofs of the following statements may be found in Dixmier 1977). For example, in order that either the produced or induced representation of a Lie algebra be irreducible, it is necessary that the subalgebra representation also be irreducible. Again, as for groups, more useful results on irreducibility are possible for the special case described in section 3.3. The theorem on induction or production by stages also holds: a representation induced or produced in several stages through a chain of subalgebras is equivalent to a representation respectively induced or produced directly from the starting subalgebra. This property becomes particularly useful when it is extended to Lie superalgebras because it allows the representations for the even part of a superalgebra to be constructed first, and then the representations of the whole superalgebra to be considered as multiplets of even subalgebra representations.

Other features of induced Lie group representations do not have counterparts for Lie algebras, or if they do, the counterparts may be obscure. In particular, it is difficult to define a suitable inner product for the carrier space of the Lie algebra representations. The difficulty will be seen more easily after the connection between group and algebra representations has been considered, so this point will not be elaborated on here.

An important feature of the Lie group representations (important for examination of both classical and quantum theories in physics) which does have an algebraic analogue is contragradience. If (Φ , V) is a representation of a Lie algebra g then its *contragradient* or *dual* representation is (Δ , U) where the carrier space U is the dual vector space of V, and the operators $\Delta(X)$ are defined by

$$\{\Delta(X)u\}(v) = -u(\Phi(X)v), \qquad (2.21)$$

for all $X \in g$, $u \in U$ and $v \in V$. The notation $(\Delta , U) = (\Phi , V)^*$ will be used to denote contragradience. (This notation should not lead to confusion with the complex conjugate representation, which would be denoted (Φ^*, V)).

Induced and produced representations are contragradient in the following way. Let g be a Lie algebra, and h a subalgebra of g. If (Φ , V) is the representation of g *induced* from the representation (Γ , W) of h, and (Δ , U) is the representation of g *produced* from its dual $(\Gamma , W)^*$, then $(\Delta , U) = (\Phi , V)^*$ up to equivalence (see Blattner 1969 and Higman 1955 for proofs).

Finally, it should be noted that even the "more practical" produced and induced Lie algebra representations are very cumbersome. To evaluate completely the operators of the representation requires the calculation of commutators with arbitrary universal enveloping algebra elements. This is an intractable task in all but the simplest cases. So for application to physical theories, the group version of the method of induction is much to be preferred. The exception to this policy is for Lie superalgebras, where the algebraic version works well, at least for the odd part (see chapters 7 and 8).

Chapter 3

Developments of induced representation theory

Before the methods of induction and production presented in chapter 2 can be used in the applications of later chapters, some more basic work is required.

First of all, a link between the Lie group and Lie algebra representations must be established, as this will prove indispensable for supersymmetric field theories. A homomorphic link from group to algebra representations has long been known (Blattner 1969, Duflo 1977), but a connection in the opposite direction does not appear to have been made. Both directions will be considered in this chapter, and in section 3.1 a link from algebra to group representations will be shown to exist for a restricted class of groups (which fortunately includes those used in the following chapters).

For Lie groups or algebras with an invariant subgroup or subalgebra, there is a somewhat surprising, but highly useful construction which gives rise to irreducible induced or produced representations. Even more remarkably, *all* irreducible representations of these groups and algebras are of this construction. Section 3.2 presents the well-known procedure for representations of semi-direct product groups. This is not the most general case possible, but it is a little simpler than more general treatments, and it is also all that will be required subsequently. Section 3.3 presents the corresponding construction and results for Lie algebras.

Finally, in section 3.4, the notions of induced and produced representations are generalised to Lie superalgebras. Since this is mostly straightforward, the discussion is fairly brief, with only those points where differences arise receiving special attention.

3.1 The relationship between group and algebra representations

The task of this section is to set out the relationship between the induced representations of Lie groups and the induced and produced representations of Lie algebras presented in chapter 2. It is necessary to establish this link because it will be vitally important in the construction of supersymmetric field theories in later chapters. Some of the usual presentations of this topic (Blattner 1969, Duflo 1972) are rather densely written, so this section expands and makes more explicit the steps of the argument. In particular, special attention is paid to the link *from* algebra *to* group representations, since this direction is not discussed by Blattner or Duflo.

The procedure will be as follows. First, the more abstract formulations of the induced group and produced algebra representations (definitions 2.2 and 2.7) will be shown to be equivalent. This will be done by showing the equivalence of the larger "standard" representations and then proving that the invariant subspaces for the induced and produced representations correspond to each other under the equivalence mapping. From this, it follows automatically that the "more practical" formulations (definitions 2.5 and 2.10) must also be equivalent, but for convenience, the correspondence will be given explicitly. A few of the proofs in this section are valid only for simply connected Lie groups, but as the groups used later are all of this kind, the restriction is not too severe.

For the rest of this section, G will be a simply connected Lie group, g the real Lie algebra of G, and $\mathrm{U}(g)$ the universal enveloping algebra of g. H will be a Lie subgroup of G, with h and $\mathrm{U}(h)$ the corresponding real Lie algebra and universal enveloping algebra. In addition, W will be a finite-dimensional vector space carrying a representation Γ of H, and hence of h and U(h) as well.

The first task is to examine the "standard" representations. Let $(\Phi\ ,\ C^{\omega}(G,W\ )\ )$ be the "standard" representation of G given by equation (2.1). That is, for x in G and ψ in $C^{\omega}(G,W\ )$, Φ is the operator defined by

$$\{\Phi(x)\psi\}(x') = \psi(x^{-1}x') \qquad \text{for all } x' \in G. \tag{3.1}$$

In the original definitions for the induced representation, the functions from G to W did not need to be analytic (C^{ω}). $C^{\omega}(G,W\ )$ is, however, clearly closed under the action of Φ, so (3.1) still defines a representation. The reason for the restriction will be apparent very soon.

Before any comparison can be made with the Lie algebra representations, $(\Phi\ ,\ C^{\omega}(G,W\ )\ )$ must be converted to a Lie algebra representation as well. Let X be some element of g , and $\exp(\varepsilon X\ )$ (with ε real) an element of the one-parameter subgroup of G generated by X. Note that

$$\Phi(\ e^{\varepsilon X}\ )\psi(x) = \psi(\ e^{-\varepsilon X}\ x\ ).$$

This suggests the natural definition for $(\Phi\ ,\ C^{\omega}(G,W\ )\ )$ as a representation of g: for X in g and ψ in $C^{\omega}(G,W\ )$,

$$\{\Phi(X)\psi\}(x) = \frac{d}{d\varepsilon}\ \psi(\ e^{-\varepsilon X}\ x\ )\Big|_{\varepsilon=0} \qquad \text{for all } x \in G\ . \tag{3.2a}$$

In (3.2a), ε is real. By setting

$$\Phi(1)\psi\ =\ \psi, \tag{3.2b}$$

the representation (3.2) extends naturally to a representation of the universal enveloping algebra U(g) with identity element 1. As a representation of U(g), Φ is exactly the same as the set of differential operators on the carrier space, which is why the carrier space has to be restricted to the set of infinitely differentiable functions. Furthermore, if the inverse mapping is to be defined, these functions must be reconstructed from their derivatives, so they must be analytic.

For the produced Lie algebra representation, the "standard" representation Δ (given in equation (2.11) using the symbol Φ) is carried by the space of linear mappings from $U(g)$ to W, denoted $L(U(g),W)$. The action of $\Delta(X)$ for X in g on a vector ϕ in $L(U(g),W)$ is

$$\{\Delta(X)\phi\}(A) = \phi(AX) \qquad \text{for all } A \in U(g). \tag{3.3a}$$

With

$$\Delta(1)\phi = \phi, \tag{3.3b}$$

Δ becomes a representation of $U(g)$.

The purpose of the next few propositions is to show that $(\Phi, C^{\omega}(G,W))$ and $(\Delta, L(U(g),W))$ are equivalent representations of $U(g)$. To do this, an invertible mapping between the carrier spaces which preserves the structure of the representations will be found. Define first the mapping S from $C^{\omega}(G,W)$ to $L(U(g),W)$ by $\psi \to S\psi$, where

$$\{S\psi\}(A) = \Phi(A)\psi(x_0) \qquad \text{for all } A \in U(g), \tag{3.4}$$

and where x_0 is the identity element of G. It is easy to see that $S\psi$ defined by (3.4) is indeed a W-valued function over $U(g)$, so the mapping S is well defined. Note that S directly links the values at the identities:

$$S\psi(1) = \psi(x_0) \qquad \text{for all } \psi \in C^{\omega}(G,W). \tag{3.5}$$

Proposition 3.1 S defined by equation (3.4) is a homomorphism from $(\Phi, C^{\omega}(G,W))$ to $(\Delta, L(U(g),W))$. That is,

$$\Delta(A)S = S\Phi(A) \qquad \text{for all } A \in U(g). \tag{3.6}$$

Proof. S is clearly a linear mapping. Let X, ψ and A be arbitrary elements of g, $C^{\omega}(G,W)$ and $U(g)$. Then $S\psi \in L(U(g),W)$ and so

$$\begin{aligned}
\Delta(X)\{S\psi\}(A) &= \{S\psi\}(AX) && \text{by (3.3)} \\
&= \Phi(AX)\psi(x_0) && \text{by (3.4)} \\
&= \Phi(A)\{\Phi(X)\psi\}(x_0) \\
&= \{S\Phi(X)\psi\}(A) && \text{by (3.4).}
\end{aligned}$$

Since this holds for all $A \in \mathrm{U}(g)$ and all $\psi \in \mathrm{C}^{\omega}(G,W)$, the operator equation

$$\Delta(X)S = S\Phi(X)$$

holds for all $X \in g$. The desired conclusion follows by natural extension to $\mathrm{U}(g)$. □

To define a mapping in the opposite direction, from $\mathrm{L}(\mathrm{U}(g),W)$ to $\mathrm{C}^{\omega}(G,W)$, the exponential mapping can be used. Recall that for a simply connected, linear Lie group, every representation of the corresponding Lie algebra gives, upon exponentiation, a representation of the group (see Cornwell 1984 vol. II ch. 11 sec. 7). So the representation Δ of g exponentiates to a representation of G, which will be denoted Δ as well.

Define the mapping T from $\mathrm{L}(\mathrm{U}(g),W)$ to $\mathrm{C}^{\omega}(G,W)$ by $\phi \to T\phi$ where

$$\{T\phi\}(x) = \Delta(x^{-1})\phi(1) \qquad \text{for all } x \in G. \tag{3.7}$$

(Recall that 1 is the identity of $\mathrm{U}(g)$). Clearly $T\phi$ is a mapping from G to W. Moreover, since the exponential mapping is analytic $T\phi$ is a function in $\mathrm{C}^{\omega}(G,W)$, so T is well defined. Like S, T directly links the values at the identities:

$$T\phi(x_0) = \phi(1) \qquad \text{for all } \phi \in \mathrm{L}(\mathrm{U}(g),W). \tag{3.8}$$

Proposition 3.2 T defined by (3.7) is a homomorphism from $(\Delta, \mathrm{L}(\mathrm{U}(g),W))$ to $(\Phi, \mathrm{C}^{\omega}(G,W))$. That is,

$$\Phi(A)T = T\Delta(A) \qquad \text{for all } A \in \mathrm{U}(g). \tag{3.9}$$

Proof. It is easiest to show the homomorphism property for Φ and Δ as group representations. Equation (3.9) is then an immediate corollary. From (3.7), T is clearly a linear mapping. Let x, x' and ϕ be arbitrary elements of G and L(U(g),W). Then $T\phi \in C^{\omega}(G,W)$ and so

$$\begin{aligned}
\Phi(x)\{T\phi\}(x') &= \{T\phi\}(x^{-1}x') && \text{by (3.1)}\\
&= \Delta(x'^{-1}x)\phi(1) && \text{by (3.7)}\\
&= \Delta(x'^{-1})\Delta(x)\phi(1) &&\\
&= T\Delta(x)\phi(x') && \text{by (3.7)}
\end{aligned}$$

Since this holds for all x' in G and all ϕ in L(U(g),W), the operator equation

$$\Phi(x)T = T\Delta(x)$$

holds for all x in G. The desired result follows by natural extension. □

Having established a homomorphic mapping in each direction between the two representations (3.2) and (3.3), it remains only to show that they are each others' inverse in order to demonstrate that the carrier spaces are isomorphic (as vector spaces) and the representations thus equivalent. This is the object of the next proposition, after which the main theorem is an easy step.

Proposition 3.3 The mappings S and T of equations (3.4) and (3.7) are inverse mappings of each other. That is,

$$ST = 1 \qquad \text{and} \qquad TS = 1,$$

1 being the identity operator in the appropriate space in each case.

Proof. For any ϕ in L(U(g),W), $ST\phi$ is also in L(U(g),W), and for any A in U(g),

$$\begin{aligned}
\{ST\phi\}(A) &= \Phi(A)\{T\phi\}(x_0) && \text{by (3.4)} \\
&= T\{\Delta(A)\phi\}(x_0) && \text{by (3.9)} \\
&= \Delta(A)\phi(1) && \text{by (3.8)} \\
&= \phi(A) && \text{by (3.3).}
\end{aligned}$$

Since this holds for all A and all ϕ, $ST = 1$ as an operator in L(U(g),W). Likewise, for any ψ in $C^\omega(G,W)$, $TS\psi$ is also in $C^\omega(G,W)$, and for any x in G,

$$\begin{aligned}
\{TS\psi\}(x) &= \Delta(x^{-1})S\psi(1) && \text{by (3.7)} \\
&= S\Phi(x^{-1})\psi(1) && \text{by (3.6)} \\
&= \Phi(x^{-1})\psi(x_0) && \text{by (3.5)} \\
&= \psi(x) && \text{by (3.1).}
\end{aligned}$$

Thus, as an operator on $C^\omega(G,W)$, $TS = 1$. □

Theorem 3.4 Let G be a simply connected Lie group, g its real Lie algebra and U(g) its universal enveloping algebra. Let W be some arbitrary finite-dimensional vector space. The "standard" representations (Φ , $C^\omega(G,W)$) and (Δ , L(U(g),W)) of U(g) given by equations (3.2) and (3.3) are equivalent representations.

Proof. By propositions 3.1, 3.2 and 3.3, the mappings S and $T = S^{-1}$ of equations (3.4) and (3.7) provide an isomorphism between the representations, explicitly demonstrating the equivalence. □

With the "standard" representations shown to be equivalent, most of the work to show the equivalence of the induced group and produced algebra representations is done. All that is left to do is to show that the carrier spaces match. To move from the "standard" representation to the induced Lie group representation needs only the imposition of the condition (2.2) to restrict the carrier space. That is, the representation of G induced from the representation (Γ, W) of the subgroup H of G is carried by the space V of functions ψ in $C^\omega(G,W)$ satisfying

$$\psi(x\,y) = \Gamma(y^{-1})\{\psi(x)\} \qquad \text{for all } x \in G\,,\, y \in H\,. \tag{3.10}$$

The representation (Γ, W) of H gives a representation of its real Lie algebra h in the usual way: for any Y in h (and with ε real),

$$\Gamma(Y) = \frac{d}{d\varepsilon}\,\Gamma(\mathrm{e}^{\varepsilon Y})\Big|_{\varepsilon=0}. \tag{3.11a}$$

With the obvious definition

$$\Gamma(1) = 1, \tag{3.11b}$$

where the right hand side is the identity operator in W, this extends naturally to a representation of the universal enveloping algebra $\mathrm{U}(h)$.

To obtain the produced Lie algebra representation from the "standard" one of (3.3) requires the imposition of condition (2.12). That is, the representation of $\mathrm{U}(g)$ produced from (Γ, W) of $\mathrm{U}(h)$ is carried by the space U of functions ϕ in $\mathrm{L}(\mathrm{U}(g),W)$ satisfying

$$\phi(BA) = \Gamma(B)\{\phi(A)\} \qquad \text{for all } A \in \mathrm{U}(g),\, B \in \mathrm{U}(h). \tag{3.12}$$

Using theorem 3.4, it is not too difficult to show that the restrictions (3.10) and (3.12) are equivalent, so that the induced group and produced algebra representations are equivalent as well.

Theorem 3.5 Let G be a simply connected Lie group and H a simply connected Lie subgroup of G. Let g, h, $\mathrm{U}(g)$ and $\mathrm{U}(h)$ be their Lie algebras and universal enveloping algebras, and (Γ, W) be a finite dimensional representation of H, converted to give a representation of $\mathrm{U}(h)$ as in (3.11). Then the representation (Φ, V) of $\mathrm{U}(g)$ obtained from the induced representation of G and given by (3.2) and (3.10) is equivalent to the produced representation (Δ, U) of $\mathrm{U}(g)$ given by (3.3) and (3.12).

Proof. By theorem 3.4, the "standard" representations (Φ , $C^{\omega}(G, W)$) and (Δ , L(U(g),W)) are equivalent. All that needs to be shown is that the subspaces V and U are mapped into each other by the equivalence transformations S and $T = S^{-1}$ of equations (3.4) and (3.7).

First, consider the image of V under the mapping S. For any ψ in V, let Y and A be arbitrary members of h and U(g). Then

$$\{S\psi\}(YA) = \Phi(YA)\psi(x_0) \qquad \text{by (3.4)}$$

$$= \frac{d}{d\varepsilon}\,\Phi(A)\psi(e^{-\varepsilon Y})\Big|_{\varepsilon=0} \qquad \text{by (3.2)}$$

$$= \frac{d}{d\varepsilon}\,\Gamma(e^{\varepsilon Y})\,\{\Phi(A)\psi(x_0)\}\Big|_{\varepsilon=0} \qquad \text{by (3.10)}$$

$$= \Gamma(Y)\{S\psi\}(A) \qquad \text{by (3.11) and (3.4).}$$

This extends easily to give

$$\{S\psi\}(BA) = \Gamma(B)\{S\psi\}(A)$$

for all B in U(h), so $S\psi$ satisfies the condition (3.12). Thus the image of V under S is contained in U.

It could still be that U is larger than the image of V, so the opposite direction also must be checked. Combining the definition (3.3) with the condition (3.12) readily gives

$$\{\Delta(Y)\phi\}(1) = \Gamma(Y)\{\phi(1)\}$$

for all Y in h and ϕ in U. Using the exponential mapping,

$$\{\Delta(y)\phi\}(1) = \Gamma(y)\{\phi(1)\} \qquad (3.13)$$

for all y in H and ϕ in U. This allows the condition (3.10) to be checked. For any $\phi \in U$, let x and y be arbitrary members of G and H, and consider the image $T\phi$ of ϕ in $C^{\omega}(G, W)$.

$$\{T\phi\}(xy) = \Delta(y^{-1}x^{-1})\phi(1) \qquad \text{by (3.7)}$$

$$= \Delta(y^{-1})\{\Delta(x^{-1})\phi\}(1)$$

$$= \Gamma(y^{-1})\{\Delta(x^{-1})\phi(1)\} \qquad \text{by (3.13)}$$

$$= \Gamma(y^{-1})\{T\phi(x)\} \qquad \text{by (3.7).}$$

So $T\phi$ satisfies the condition (3.13), and hence (3.10), and U is mapped into V by T. Thus U and V are isomorphic and the theorem is proved. □

Comments. (i) Blattner (1969) and Duflo (1972) prove a homomorphic mapping from the group to the algebra representation, which Blattner states is an onto mapping. The opposite direction is not discussed.

(ii) Since the representations are equivalent, it is possible to use the same notation for both without too much risk of confusion. Henceforth, unless careful distinction is required, the representations (Φ, V) and (Δ, U) will be denoted by the same symbols. Which realisation is being used will be made clear, if necessary, by the argument of a function $\psi \in V$. For example, $\Phi(X)\psi(x)$ refers to the induced group representation, and $\Phi(X)\psi(A)$ to the produced algebra representation.

Corollary 3.6 If A' and A'' are two elements of $\mathrm{U}(g)$, then

$$\Phi(A')\psi(x) = \Phi(A'')\psi(x) \qquad \text{for all } x \in G$$

if, and only if

$$\Phi(A')\psi(A) = \Phi(A'')\psi(A) \qquad \text{for all } A \in \mathrm{U}(g).$$

Proof. This follows immediately from theorem 3.5. □

Corollary 3.7 The "more practical" realisations (definitions 2.5 and 2.10) of the induced group and produced algebra representations are equivalent as representations of the universal enveloping algebra.

Proof. Propositions 2.3 and 2.8 show that the "more practical" realisations are equivalent to the abstract ones, so the corollary is a direct consequence of theorem 3.5. □

Although corollary 3.7 suffices for its purpose, it is still instructive to see the connection between the "more practical" realisations written down explicitly. No great detail will be given here, because everything follows straightforwardly from the more abstract form, but some new notation will be introduced for use in later chapters.

Let K be the left coset space G/H, B provide a set of smoothly varying coset representatives for K, and Φ be the induced representation of G carried by $C^{\omega}(K,W)$, defined by (cf. equation (2.6))

$$\Phi(x)\psi(z) = \Gamma(B(z)^{-1}x B(x^{-1}z))\psi(x^{-1}z) \tag{3.14}$$

for all $x \in G$, $\psi \in C^{\omega}(K,W)$, and $z \in K$. To convert (3.14) to a representation of U(g), it is necessary once again to introduce the one-parameter subgroup $\exp(\varepsilon X)$ with $X \in g$ and ε real. For the purpose of clarity, the notation

$$Xz\cdot\partial f(z) = \frac{d}{d\varepsilon} f(\mathrm{e}^{\varepsilon X} z)\Big|_{\varepsilon=0} \tag{3.15}$$

will be adopted, where f is any analytic function over K, the codomain of f being arbitrary. Then (3.14) gives

$$\Phi(X)\psi(z) = \Gamma(B(z)^{-1}\{X - Xz\cdot\partial\}B(z))\psi(z) - Xz\cdot\partial\,\psi(z).$$

As might be expected, this form of the representation can become a little unwieldy, but it is sometimes unavoidable. On these occasions, the abbreviation

$$\Gamma_z(X) = \Gamma(B(z)^{-1}\{X - Xz\cdot\partial\}B(z)) \tag{3.16}$$

is very useful, since it signals that it is not simply Γ being used, but that there is some z dependence as well. By itself, Γ_z does not provide a representation. With this abbreviation,

$$\Phi(X)\psi(z) = \Gamma_z(X)\psi(z) - Xz\cdot\partial\,\psi(z). \tag{3.17}$$

Turning to the produced algebra representation, let X^r, $r \in \mathbb{N}^n$ be basis elements for U(g) as a left U(h)-module ($n = \dim g - \dim h$), and let k be the *real* vector space spanned by the X^r. Let $(\Phi, \mathrm{L}(k,W))$ be the produced representation of g (cf. definition 2.10) defined by

$$\Phi(X)\psi(Z) = \sum_{r \in \mathbb{N}^n} \Gamma(\,(ZX)_r\,)\psi(X^r) \tag{3.18}$$

for all $X \in g$, $\psi \in \mathrm{L}(k,W)$, and $Z \in k$.

By corollary 3.7, the representations (3.17) and (3.18) are equivalent. The isomorphic mapping between the carrier spaces, using the argument of ψ to indicate which is which, is given explicitly by

$$\psi(Z) = \Phi(Z)\psi(z_0) \qquad \text{for all } Z \in k\,, \tag{3.19a}$$

(where z_0 is the identity coset H in G/H) and conversely

$$\psi(z) = \sum_{r \in \mathbb{N}^n} \Phi(\,(B(z)^{-1})_r\,)\psi(X^r) \qquad \text{for all } z \in K. \tag{3.19b}$$

Note that

$$\psi(z_0) = \psi(1),$$

so the values at the identities match, just as for the more abstract versions.

Note. In some places later in this thesis, Lie algebra representations carried by C^∞ functions are used. These functions should be understood to be analytic, C^ω, not simply infinitely differentiable.

3.2 Irreducible representations of semi-direct product groups

In general, induced representations of Lie groups and Lie algebras are reducible. This is hardly surprising, seeing that they originate from the highly reducible "standard" representations. However, if a Lie group or Lie algebra has a semi-direct product structure, then there are very useful consequences for the reducibility of induced representations. In fact, it is then possible to prove that certain induced representations of the group or algebra are always irreducible, and even more importantly, that *all* irreducible representations of the group or algebra are equivalent to induced representations. These properties are quite well known (especially for groups), so only a short description will be given, mainly to establish a pattern to be applied in later chapters. The procedure for groups will be described in this section, followed by the analogous procedure for algebras in the next.

The main results for groups (theorem 3.9) hold for *any* group of type I (see Mackey 1968 for definitions), which includes any connected semisimple Lie group with an invariant Lie subgroup. However, the procedure is easier to follow for a restricted class of semi-direct product groups, which includes those to be studied later, so the general case will not be treated here.

Let G be a Lie group with the semi-direct product structure $G = K \circledS H$, where the invariant subgroup K is Abelian. Any element of G can be written *uniquely* in the form yx where y belongs to K and x to H. A physically important example of such a group is the proper orthochronous Poincaré group of translations and rotations in pseudo-Euclidean space. In this example, K is the group of translations and H

the group of rotations, and any Poincaré transformation can be uniquely decomposed as a rotation followed by a translation.

Neither H nor K gives the irreducible induced representations of G, but instead a specially constructed subgroup is used. Let χ be a one-dimensional (and thus irreducible) representation of K. The *little group for* χ, denoted $H(\chi)$, is defined to be the subgroup of H given by

$$H(\chi) = \{ x \in H : \chi(x y x^{-1}) = \chi(y) \text{ for all } y \in K \}. \qquad (3.20)$$

Similarly, the *stability group for* χ, denoted $S(\chi)$, is defined to be the subgroup of G given by

$$S(\chi) = \{ x \in G : \chi(x y x^{-1}) = \chi(y) \text{ for all } y \in K \}. \qquad (3.21)$$

Obviously, the little group and the stability group for χ are quite similar. In a sense, $H(\chi)$ is the "stability group" for χ in H. $S(\chi)$ differs from $H(\chi)$ in that it includes elements of G not contained in H. In particular, K itself is a subgroup of $S(\chi)$, since it is Abelian. From this it follows that $S(\chi)$ has the semi-direct product structure $S(\chi) = K \circledS H(\chi)$, and that any element of $S(\chi)$ can be uniquely written as yx where y belongs to K and x to $H(\chi)$.

Let Γ be a representation of the little group $H(\chi)$ carried by a finite-dimensional vector space W. This extends naturally to a representation of the stability group $S(\chi)$, denoted $\chi\Gamma$ and carried by W, where

$$\chi\Gamma(y x) = \chi(y)\, \Gamma(x) \qquad \text{for all } y \in K,\ x \in H(\chi). \qquad (3.22)$$

The construction is completed by letting (Φ, V) be the representation of G induced from the representation $(\chi\Gamma, W)$ of the stability group $S(\chi)$. Referring to the definition 2.5 (where it should be noted that the symbols K, Γ, H, x, y etc. had other meanings) the carrier space V is $C^\infty(L, W)$, where L is isomorphic to the left coset space $G/S(\chi)$. Because of the semi-direct product structure of both G and $S(\chi)$, L is also isomorphic to the left coset space $H/H(\chi)$. From this it follows that the coset representatives $B(z)$ in G for $z \in L$ can all be chosen to be elements of H. Assuming that this has been done, the action of the representation Φ can be stated as follows.

Proposition 3.8 Let $y\,x$ with $y \in K$ and $x \in H$ be an arbitrary element of G, and ψ an arbitrary function in $C^\infty(L,W)$. The operator $\Phi(y\,x)$ acts on ψ by

$$\Phi(y\,x)\psi(z) = \chi(B(z)^{-1}y\,B(z))\Gamma(B(z)^{-1}x\,B(x^{-1}z))\psi(x^{-1}z), \quad (3.23)$$

for all $z \in L$.

Proof. From the definition 2.5, the action of Φ is given by

$$\Phi(y\,x)\psi(z) = \chi\Gamma(\,B(z)^{-1}\,y\,x\,B((y\,x)^{-1}z)\,)\,\psi(\,(yx)^{-1}z\,).$$

Since $L = G/S(\chi)$, and $K \subset S(\chi)$, then $y\,z = z$ for all y in K and z in L. Thus,

$$(y\,x\,)^{-1}z \;=\; x^{-1}y^{-1}z \;=\; x^{-1}z\,.$$

Also, the argument of $\chi\Gamma$ can be decomposed into canonical form by

$$B(z)^{-1}y\,x\,B((yx)^{-1}z) \;=\; \{B(z)^{-1}\,y\,B(z)\}\,\{B(z)^{-1}x\,B(x^{-1}z)\}.$$

The first factor in braces { } is an element of K since K is invariant, and the second factor is an element of $H(\chi)$ since the coset representatives have been chosen to be in H. Use of (3.22) then reduces the above action to (3.23). □

Proposition 3.8 completes the construction of the special induced representations of G. Theorem 3.9 summarises the construction and gives the properties which make these special representations so interesting.

Theorem 3.9 Let $G = K \circledS H$ be a Lie group with K being Abelian. Let χ be a one-dimensional representation of K, and $H(\chi)$ and $S(\chi)$ the corresponding little group and stability group defined by equations (3.20) and (3.21). Let (Γ,W) be a finite-dimensional representation of $H(\chi)$ and $(\chi\Gamma,W)$ defined by (3.22) the corresponding representation of $S(\chi)$. Finally, let (Φ,V) be the representation of G induced from the representation $(\chi\Gamma,W)$ of $S(\chi)$.

(*a*) If $(\chi\Gamma,W)$ is irreducible, then (Φ,V) is also irreducible.

(*b*) Conversely, if (Δ,U) is *any* irreducible representation of G, then there exist irreducible representations χ of K and Γ of $H(\chi)$ such that (Δ,U) is equivalent to (Φ,V).

Proof. See Mackey (1968). □

Comments. (i) Theorem 3.9 may be paraphrased as follows. (*a*) All representations of G induced using irreducible representations of K and $H(\chi)$ are themselves irreducible, and (*b*), *all* irreducible representation of G are of this construction.

(ii) Theorem 3.9 also holds if the word "irreducible" is replaced everywhere by "unitary irreducible".

Theorem 3.9(*b*) raises the hope of finding *all* the irreducible representations of G, for if all the irreducible representations χ and Γ can be found, then induction gives the complete set of irreducible representations of G. This procedure is made easier by using the orbit structure of the set of representations χ.

Consider the set of all the irreducible representations of K. Define an equivalence relation, denoted $\sim$, for two representations χ and χ' by $\chi \sim \chi'$ if, and only if, there is some x in G such that

$$\chi'(y) = \chi(x y x^{-1}) \qquad \text{for all } y \in K.$$

This equivalence relation divides the irreducible representations of K into disjoint orbits. It is not hard to prove that, for $\chi \sim \chi'$, $H(\chi)$ and $H(\chi')$

are isomorphic, and that the representations induced from $(\chi\Gamma,W)$ and $(\chi'\Gamma,W)$ are equivalent (see eg. Mackey 1968). So representations of G constructed from the same orbit are equivalent. Conversely, the little groups for χ from different orbits may be quite dissimilar.

Thus inequivalent, irreducible representations of G may be found by splitting the set of irreducible representations of K into orbits, taking one member χ of each orbit, taking inequivalent, irreducible, finite-dimensional representations of $H(\chi)$, and inducing. The problem of finding all the irreducible representations of G becomes that of finding all the irreducible representations of the $H(\chi)$. In the physically important example of the Poincaré group, the latter problem is solvable, and the very satisfying construction of all irreducible representations of the Poincaré group ensues (Wigner 1939).

3.3 Irreducible representations of Lie algebras with an invariant subalgebra

As for the induced representations of semi-direct product groups, the induced representations of Lie algebras containing an invariant subalgebra have some very interesting properties. Almost all of the statements in section 3.2 have counterparts for Lie algebras, *except* that it is more difficult, if not impossible, to set out a programme to construct all of the irreducible representations. For Lie algebras, the general case is no more difficult to treat than the special case corresponding to the semi-direct product groups, so the special case will be given as an example at the end.

Let g be a real Lie algebra with an invariant subalgebra k (k need not be Abelian). Let (χ,U) be a finite-dimensional irreducible representation of k. The *stability algebra for* χ, denoted by $s(\chi)$, is defined as the set of X in g for which there is some linear mapping σ from U to itself such that

$$\chi([X,Y])u = (\sigma\chi(Y) - \chi(Y)\sigma)u \tag{3.24}$$

for all $Y \in k, u \in U$. It is not difficult to see that $s(\chi)$ is indeed a subalgebra of g. The stability algebra can be thought of as the largest subalgebra of g containing k which is representable on U.

Let W be a finite-dimensional vector space carrying a representation $\chi\Gamma$ of $s(\chi)$. (The use of two symbols $\chi\Gamma$ here is just to maintain consistent notation with section 3.2. Γ is not in general a representation by itself because no "little algebra" can be defined for the general case). $\chi\Gamma$ also gives a representation of k by restriction. Assume that, as a representation of k, $\chi\Gamma$ is merely a direct sum of copies of χ. Finally, let (Φ,V) be the representation of g induced from $(\chi\Gamma,W)$ of $s(\chi)$. For this algebraic induced representation construction, there is an analogue of theorem 3.9 for groups.

Theorem 3.10 Let g be a real Lie algebra with an invariant subalgebra k, (χ,U) an absolutely irreducible, finite-dimensional representation of k, and $s(\chi)$ the stability algebra for χ defined by (3.24). Let $(\chi\Gamma,W)$ be a finite-dimensional representation of $s(\chi)$ such that, restricted to k, $\chi\Gamma$ is a direct sum of copies of χ. Finally, let (Φ,V) be the representation of g induced from $(\chi\Gamma,W)$.

(*a*) If $(\chi\Gamma,W)$ is irreducible, then (Φ,V) is also irreducible.

(*b*) Conversely, if (Δ,Z) is any irreducible representation of g, the there exist irreducible representations (χ,U) and $(\chi\Gamma,W)$ such that (Δ,Z) is equivalent to (Φ,V).

Proof. (*a*) see Blattner (1969). (*b*) see Dixmier (1971,1977 sec. 5.4). □

Comment. (χ,U) must be *absolutely irreducible* for the theorem to be valid. This means it must remain irreducible under any extension of the field of scalars for U. This will be true if the scalars form an algebraically closed field, as is the case for the complex numbers, but it may not be true if, for example, U is a real vector space (see eg. Dixmier 1977 for details).

For the special case where k is Abelian, and g is a vector-space direct sum $g = k \oplus h$, where h is a Lie algebra, some slight simplifications occur. First, since any irreducible representation χ of k is one-dimensional, the definition of the stability algebra becomes

$$s(\chi) = \{ X \in g : \chi([X,Y]) = 0 \text{ for all } Y \in k \} \tag{3.25}$$

It is also possible to define a *little algebra for* χ, $h(\chi)$, as the subalgebra of h given by

$$h(\chi) = \{ X \in h : \chi([X,Y]) = 0 \text{ for all } Y \in k \} \tag{3.26}$$

Just as for groups, $s(\chi) = k \oplus h(\chi)$, so the irreducible representation $(\chi\Gamma, W)$ of $s(\chi)$ can be obtained by letting

$$\chi\Gamma(Y) = \chi(Y)\,1 \qquad \text{for } Y \in k, \tag{3.27a}$$

and

$$\chi\Gamma(X) = \Gamma(X) \qquad \text{for } X \in h(\chi), \tag{3.27b}$$

where (Γ, W) is an irreducible representation of $h(\chi)$, and 1 is the identity operator on W. Finally, the basis for U(g) considered as a U($s(\chi)$)-module (used in the induced representation construction) is exactly the same as the basis for U(h) considered as a U($h(\chi)$)-module. These features all have their counterparts for groups, as can be seen from a comparison with section 3.2.

If g is the Lie algebra of a simply connected Lie group G, then they are related by the exponential mapping. As might be expected, the groups $S(\chi)$ and $H(\chi)$ correspond to the Lie algebras $s(\chi)$ and $h(\chi)$ under the same mapping.

Unfortunately, the programme outlined in section 3.2 for constructing all of the irreducible representations of a group does not carry over very well into Lie algebras. There is an extended formulation of the Lie algebraic version of the construction of irreducible induced representations, which makes use of "primitive ideals" of the universal enveloping algebra (Dixmier 1977). These primitive ideals have an orbit structure similar to that for groups, but the formulation makes use of a so-called "adjoint algebraic group" (similar to the adjoint representation of a group carried by its Lie algebra), and so it is not purely algebraic. As will be indicated in the next section, this may have drawbacks when it comes to generalising to Lie superalgebras.

An explicit example of an induced representation for a Lie algebra of the kind described here is given in section 4.4, where produced representations of the Lie algebra iso(1,1) are constructed and compared with those obtained by induction for the proper, orthochronous Poincaré group in two dimensions.

3.4 Induced and produced representations of Lie superalgebras

In this section, the concepts and results presented so far for Lie algebras will be generalised to cover Lie superalgebras as well. Not everything survives the transition, but what does remains virtually unchanged in form. So only a brief outline is needed here, with just the main differences emphasised. Definitions and descriptions of the basic properties of Lie superalgebras will not be repeated here, but are given in standard reviews (eg. Kac 1977, Rittenberg 1978, Scheunert 1979), where most of the material in this section can be found.

For the whole of this section, $g = g_0 \oplus g_1$ will be a Lie superalgebra with g_0 and g_1 the even and odd parts respectively. Subscripts 0 and 1 will denote the even and odd parts of a graded vector space.

First, some general comments about representations are in order. A representation of a Lie superalgebra g is, as usual, a homomorphic mapping of g into a set of linear operators on some vector space V. It is sometimes necessary to distinguish between graded and non-graded representations according to whether the carrier space V is graded or not. In fact, this distinction spreads to many related areas. For example, a graded representation may be *graded-irreducible* (i.e. V has no invariant *graded* subspaces) and yet be reducible, because V might have an invariant *non-graded* subspace. This can have unexpected consequences, such as in the superalgebraic form of Schur's Lemma: Let (Φ,V) be a finite-dimensional, graded-irreducible representation of g. If S is an even linear operator on V (i.e. $SV_i \subset V_i$, $i = 0,1$) that commutes with Φ, then S is proportional to the identity operator, as usual. However, if S is an odd linear operator on V (i.e. $SV_0 \subset V_1$ and

$SV_1 \subset V_0$) that commutes with the even part of the representation and anti-commutes with the odd part, then *either* S is the zero operator, *or* S is proportional to a linear operator I on V with $I^2 = -1$. The operator I exists if, and only if, (Φ,V) is reducible despite being graded-irreducible.

For the most part, this distinction will not be important in the chapters to follow, so the qualifier "graded" will often be omitted. So, for example, "irreducible" will mean "graded-irreducible" unless a distinction is required and the qualifier is re-introduced.

As for Lie algebras, a universal enveloping superalgebra U(g) can be defined for a Lie superalgebra g (cf. section 2.4). The definition is very similar to that for Lie algebras, but it is worth noting that, if Y is some odd element of g, then $Y^2 \in g_0$. This produces a slight change in the Poincaré-Birkhoff-Witt theorem. Let $X_1,\ldots,X_m$ (where $m = \dim g_0$) be basis elements of g_0, and $Y_1,\ldots,Y_n$ (where $n = \dim g_1$) be basis elements of g_1. A basis for U(g) is formed by the set of products $X_1^{r_1} X_2^{r_2} \ldots X_m^{r_m} Y_1^{s_1} Y_2^{s_2} \ldots Y_n^{s_n}$ where $r_1,\ldots,r_m \in \mathbb{N}$ as usual, but $s_1,\ldots,s_n \in \{0,1\}$. Once again, the actual order of the basis elements of g is not important; the odd and even basis elements may even be mingled. Such basis elements for U(g) will be abbreviated (cf. equation (2.9)) by $X^r Y^s$ where

$$X^r Y^s = X_1^{r_1} \ldots X_m^{r_m} Y_1^{s_1} \ldots Y_n^{s_n} , \tag{3.28}$$

or any other fixed ordering, $r = (r_1,\ldots,r_m) \in \mathbb{N}^m$, and $s = (s_1,\ldots,s_n) \in \{0,1\}^n$.

U(g) plays an important part in the representation theory of Lie superalgebras. Let h be a Lie sub-superalgebra of g, and U(h) its universal covering superalgebra. It is possible to regard U(g) as either a right or a left U(h)-module. Suppose that a basis for h in g is the set of elements $\{X_{m-k+1},\ldots,X_m,Y_{n-l+1},\ldots,Y_n\}$. Then a basis for U($g$) as a U($h$)-module (either left or right) is formed by the elements $X^r Y^s$ where $r \in \mathbb{N}^k$ (instead of $\mathbb{N}^m$), and $s \in \{0,1\}^l$ (instead of $\{0,1\}^n$). An important point to note is that if h contains all of the even part g_0 of g, then the U(h)-module is finite-dimensional, since its basis has only 2^l elements. This fact becomes very useful later, because it makes explicit evaluation of some representations feasible, whereas for Lie algebras, this was too complicated (see sections 2.5 and 2.6).

Just as for Lie algebras, there is a natural, unique extension of every representation of g to one of U(g) in which the identity of U(g) is represented by the identity operator.

Induced and produced representations of Lie superalgebras are defined in exactly the same way as for Lie algebras (see sections 2.5 and 2.6 for more details). Let h be a Lie sub-superalgebra of g, and (Γ,W) a finite-dimensional representation of h. For the *induced* representation of g, the carrier space is the tensor product space over U(h), $\mathrm{U}(g)\otimes_{\mathrm{U}(h)}W$, with U($g$) considered as a right U(h)-module. The action of the representation is simply left multiplication on the U(g) component. For the *produced* representation, the carrier space is the space of U(h)-homomorphisms $\mathrm{Hom}_{\mathrm{U}(h)}(\mathrm{U}(g),W)$, with U($g$) considered as a left U(h)-module. The action of the representation is right multiplication on the U(g) argument. Both representations also have alternative formulations in terms of the real vector space k spanned by the basis elements for U(g) as a U(h)-module. So the definitions of induced and produced representations transfer smoothly to Lie superalgebras.

Unfortunately, the same is not true of the properties of these representations. Infinite dimensional representations are more complicated for Lie superalgebras, and it is not always clear if the representations obtained by the process analogous to that in section 3.3 will be irreducible (either graded-irreducible or irreducible at all). However, some comfort may be drawn from checking the Casimir operators on a representation once it is constructed, to get an indication of its status. The Casimir operators for frequently used Lie superalgebras are well established (Jarvis and Green 1979, Jarvis and Murray 1983, Scheunert 1983*b*,*c*).

Not very much mention has yet been made of Lie supergroups. The reason for this is that the connection between produced Lie algebra representations and induced Lie group representations changes drastically for superalgebras and supergroups. Induced representations of Lie supergroups can actually be defined (eg. Williams and Cornwell 1987*a*), but they may correspond to a Lie superalgebra representation in which the odd part is trivial. Conversely, an irreducible produced representation of a Lie superalgebra does give rise to a representation of a corresponding supergroup. But the supergroup representation may not be irreducible. Instead, it may contain several irreducible induced supergroup representations. An example of this state of affairs is given by the important super-Poincaré algebra and super-Poincaré group (Williams and Cornwell 1987*b*). Such problems make it easier to work in an algebraic formulation.

Nevertheless, a link with group representations is still desirable because the algebra representations can be very difficult to calculate. Fortunately, as will be seen in chapters 7 and 8, it is possible to make use of just the correspondence which exists between ordinary Lie group and Lie algebra representations. Manageable superalgebra representations can then be constructed as "supermultiplets" of ordinary Lie group representations. So it will not be necessary to come to grips with the problems of Lie supergroups.

The position has finally been reached, where the general theory of this chapter and the last can be applied to specific physical theories. This will start in the next chapter, with a withdrawal back to ordinary Lie groups and algebras and consideration of second-quantised fields in d-dimensional Minkowski space-time.

Chapter 4

Examples: representations for relativistic classical fields in two or more dimensions

This chapter is the bridge between the mainly mathematical material up until this point and the more physical considerations of the following chapters. Several examples of the method of induced representations will be given, as well as an explicit example of the link between induced group and produced algebra representations. The notation and results of this chapter will be used extensively in the remainder of this thesis.

Section 4.1 contains some preliminary definitions concerning d-dimensional Minkowski space-time and the d-dimensional Poincaré and Lorentz groups for $d \geq 2$. Representations carried by so-called covariant fields over space-time are constructed in section 4.2 by the method of induced representations for groups. This demonstrates in particular the coset space formulation of section 2.2. In section 4.3, the special procedure of section 3.2 for semi-direct products of groups is illustrated by the construction of all *irreducible* representations of the

Poincaré group. This construction was first made by Wigner (1939) for four-dimensional Minkowski space-time, and the d-dimensional case turns out to be a fairly straightforward generalisation. Section 4.4 contains a Lie algebraic derivation of the covariant field representation for two-dimensional Minkowski space-time using the method of production. This is followed by an explicit verification, for this particular case, of the link shown generally in section 3.1 between the induced group and produced algebra representations.

4.1 Minkowski space-time and the Poincaré group in two or more dimensions

Before embarking on the construction of the representations, a few definitions and remarks concerning Minkowski space-time and the Poincaré and Lorentz groups in d dimensions are in order. In this section, and for the rest of this chapter, indices λ, μ, ν, ρ will take values $0, 1, \ldots, d-1$ and summation signs will be omitted, summation over repeated indices being implied unless otherwise indicated.

For d-dimensional Minkowski space-time (with $d \geq 2$), one dimension is time-like and the other $d-1$ are space-like. The metric is pseudo-Euclidean, with metric tensor $g = \mathrm{diag}(1, -1, \ldots, -1)$.

The space-time symmetry group is the d-dimensional Poincaré group, also known as the inhomogeneous Lorentz group. This comprises the set of space-time translations and the d-dimensional Lorentz transformations, which are the transformations leaving the metric tensor invariant. The Lorentz group is isomorphic to the pseudo-orthogonal group $O(d-1,1)$ and the Poincaré group can be denoted $IO(d-1,1)$, where "I" stands for "inhomogeneous". $IO(d-1,1)$ has the semi-direct product structure $IO(d-1,1) = I(d-1,1) \circledS O(d-1,1)$, where $I(d-1,1)$ is the invariant sub-group of translations.

A typical element of $IO(d-1,1)$ can be written $(t\,,\Lambda)$ where $t = (t^0, t^1, \ldots t^{d-1})$ is the d-component real vector describing the translation and Λ is the $d \times d$ matrix of the Lorentz transformation. The product of two elements $(t\,,\Lambda)$ and $(t'\,,\Lambda')$ of $IO(d-1,1)$ is

$$(t\,,\Lambda)\,(t',\Lambda') = (t + \Lambda t', \Lambda\,\Lambda'), \tag{4.1a}$$

the identity element is $(\,0\,,1\,)$, and the inverse of $(t\,,\Lambda)$ is

$$(t\,,\Lambda)^{-1} = (\,-\Lambda^{-1}t\,,\Lambda^{-1}\,). \tag{4.1b}$$

In fact, the set of translations I(d–1,1) is isomorphic to Minkowski space-time itself. This implies that space-time can be regarded as the left coset space IO(d–1,1)/O(d–1,1). The canonical projection map π : IO(d–1,1) $\rightarrow$ IO(d–1,1)/O(d–1,1) is given for $(t\ ,\Lambda)$ in IO(d–1,1) by

$$\pi(t\ ,\Lambda) = t.$$

The canonical action of $(t\ ,\Lambda)$ in IO(d–1,1) on a space-time point x is then

$$(t\ ,\Lambda)\,x = \Lambda\,x + t. \qquad (4.2)$$

In this thesis, consideration of the Lorentz transformations is restricted to those which are proper and orthochronous. That is, the connected component ISO_0(d–1,1) of IO(d–1,1) is selected. Wigner (1939) goes on to examine the other components, and Weinberg (1964a,b) includes them in his construction of quantum fields (to be generalised to d dimensions in chapter 5), but this will not be done here, as the main results follow from the representations of ISO_0(d–1,1). Actually, since spinor representations will be required later, the group which will be used is the universal covering group of ISO_0(d–1,1). This will be denoted P and called, loosely, the "Poincaré group" (rather than the universal covering group of the connected component of the Poincaré group). It should be noted that I(d–1,1) is its own universal covering group, so P has the semi-direct product structure P = I(d–1,1) Ⓢ L where L is the universal covering group of SO_0(d–1,1), or "Lorentz group" for short. A typical member of P will again be written $(t\ ,\Lambda)$, with Λ now in L and equations (4.1) describe the product for P. Minkowski space-time can be regarded as the left coset space P/L, and equation (4.2) again describes the canonical action of P on space-time.

P is a simply connected Lie group, so the results of section 3.1 apply.

The real Lie algebra of P is iso(d–1,1), which has a similar decomposition into iso(d–1,1) = i(d–1,1) ⊕ so(d–1,1) as a vector space direct sum. If $M_{\lambda\mu}$ ($= -M_{\mu\lambda}$) and K_λ are basis elements for iso(d–1,1), then their Lie products are

$$\begin{aligned} [\, M_{\lambda\mu}, M_{\nu\rho} \,] &= g_{\lambda\nu} M_{\mu\rho} - g_{\lambda\rho} M_{\mu\nu} + g_{\mu\rho} M_{\lambda\nu} - g_{\mu\nu} M_{\lambda\rho} \\ [\, M_{\lambda\mu}, K_\nu \,] &= g_{\lambda\nu} K_\mu - g_{\mu\nu} K_\lambda \\ [\, K_\lambda, K_\mu \,] &= 0. \end{aligned} \tag{4.3}$$

$\{M_{\lambda\mu}\}$ is a basis for the Lie algebra so(d–1,1) of the Lorentz group L, and $\{K_\lambda\}$ is a basis for the invariant subalgebra i(d–1,1) corresponding to the translation group.

In a unitary representation of the Poincaré group P, the Poincaré algebra elements will be represented by anti-Hermitean operators. It is often convenient in physical applications to use a set of elements $J_{\lambda\mu}$ ($= - J_{\mu\lambda}$) and P_λ given by

$$\begin{aligned} J_{\lambda\mu} &= -i\,\hbar\, M_{\lambda\mu} \\ P_\lambda &= -i\,\hbar\, K_\lambda \,/ l_0, \end{aligned} \tag{4.4}$$

where l_0 is some arbitrary, non-zero constant with dimensions of length, so that K_λ is dimensionless. Any real linear combinations of these elements will be represented by Hermitean operators. The Lie products (4.3) become

$$\begin{aligned} [\, J_{\lambda\mu}, J_{\nu\rho} \,] &= -i\,\hbar\,(g_{\lambda\nu} J_{\mu\rho} - g_{\lambda\rho} J_{\mu\nu} + g_{\mu\rho} J_{\lambda\nu} - g_{\mu\nu} J_{\lambda\rho}) \\ [\, J_{\lambda\mu}, P_\nu \,] &= -i\,\hbar\,(g_{\lambda\nu} P_\mu - g_{\mu\nu} P_\lambda) \\ [\, P_\lambda, P_\mu \,] &= 0. \end{aligned} \tag{4.5}$$

Strictly speaking, $J_{\lambda\mu}$ and P_λ are basis elements not of the real Lie algebra iso(d–1,1) but of its complexification. However, if only *real* linear combinations of these elements are made, then multiplying the result by i produces an element of iso(d–1,1), which maps to an element of P by exponentiation. $J_{\lambda\mu}$ and P_λ will be called the *generators* of the Lorentz and translation groups.

4.2 "Covariant field" representations

"Covariant field" is a term used for a relativistic field which carries some finite-dimensional representation of the Lorentz group L. So the appropriate induced representations for relativistic fields will be those induced from representations of L. As already noted, Minkowski space-time is identifiable with the left coset space P/L, so the formulation of section 2.2 will be used.

Let W be a finite-dimensional vector space carrying a representation Γ of L. The obvious choice for a smoothly varying set of coset representatives B in P for P/L is

$$B(x) = (x, 1) \tag{4.6}$$

for each space-time point x. Let V be the set of W-valued functions over space-time. The operators of the induced representation Φ of P are defined by (cf. equation (2.6))

$$\{\Phi(t, \Lambda)\psi\}(x) = \Gamma(B(x)^{-1}(t, \Lambda)B((t, \Lambda)^{-1}x))\{\psi((t, \Lambda)^{-1}x)\},$$

for all space-time points x where (t, Λ) and ψ are arbitrary elements of P and V. Using (4.1) and (4.6),

$$\begin{aligned} B(x)^{-1}(t, \Lambda)B((t, \Lambda)^{-1}x) &= (-x, 1)(t, \Lambda)(\Lambda^{-1}(x-t), 1) \\ &= (0, \Lambda), \end{aligned}$$

so the action simplifies to

$$\{\Phi(t, \Lambda)\psi\}(x) = \Gamma(\Lambda)\{\psi(\Lambda^{-1}(x-t))\} \tag{4.7}$$

Just by way of clarification, the component form of (4.7) will be given. Let the indices a, b take values $1,\ldots,\dim W$. Any vector ψ in V has $\dim W$ components ψ_a which are real or complex functions over space-time, according as W is a real or complex vector space. The linear operators $\Gamma(\Lambda)$ on W are in one-to-one correspondence with a set of square matrices with components $\Gamma(\Lambda)_{ab}$. The V-vector $\Phi(t\,,\Lambda)\psi$ has component functions $\{\Phi(t\,,\Lambda)\psi\}_a$ given by

$$\{\Phi(t\,,\Lambda)\psi\}_a(x) = \Gamma(\Lambda)_{ab}\,\psi_b(\,\Lambda^{-1}(x-t)\,).$$

An inner product for the carrier space V was given in section 2.3 and defined by equation (2.8) as

$$\langle\phi\,,\psi\rangle_V = \int d^dx\ \langle\,\phi(x)\,,\psi(x)\,\rangle_W\ , \tag{4.8}$$

for $\phi,\psi\in V$. Here, $\langle\,\cdot\,,\cdot\,\rangle_W$ is the (Hermitean) inner product in W, and d^dx is the infinitesimal volume element obtained from the Poincaré invariant measure over space-time.

It is possible to restrict the representation (Φ,V) of P to one carried by a Hilbert space, as follows. Let H be the subspace of V consisting of Borel functions ψ in V satisfying $\langle\psi\,,\psi\rangle_V < \infty$. (A Borel function ψ in V is one for which the set $\psi^{-1}(E)$ is a Borel set in space-time for every Borel set E in W). H is then a Hilbert space if the inner product in H is set equal to that in V. Furthermore, it is not hard to see that H is closed under the action (4.7) of $\Phi(t\,,\Lambda)$ for any $(t\,,\Lambda)$ in P.

Corresponding to the representation (Φ,H) of the Poincaré group P, there is a representation of the Poincaré algebra iso(d–1,1). This representation is carried by the subspace H^∞ of infinitely differentiable functions in H. H^∞ is dense in H. The representation is obtained in the usual way by differentiation. For the generators P_λ and $J_{\lambda\mu}$ of the Poincaré group, and for ψ in H^∞, the representation is

$$\Phi(P_\lambda)\psi(x) = i\hbar\,\frac{\partial}{\partial x^\lambda}\,\psi(x) \tag{4.9a}$$

$$\Phi(J_{\lambda\mu})\psi(x) = i\hbar\left(x_\lambda\frac{\partial}{\partial x^\mu} - x_\mu\frac{\partial}{\partial x^\lambda}\right)\psi(x) + \Gamma(J_{\lambda\mu})\psi(x) \tag{4.9b}$$

In physical theories, P_λ is identified as the momentum and $J_{\lambda\mu}$ as the total angular momentum. The two terms on the right hand side of (4.9*b*) the represent the orbital and spin angular momentum respectively.

The covariant field representation (Φ, H^∞) is neither irreducible nor in general unitary. The reducibility can be seen by evaluating the Casimir operator $\Phi(P^\lambda)\Phi(P_\lambda)$ on any ψ in H^∞:

$$\Phi(P^\lambda)\Phi(P_\lambda)\psi = -\hbar^2 \frac{\partial^2}{\partial x^\lambda \partial x_\lambda}\psi.$$

For an irreducible representation, this should be proportional to ψ, but for a general element of H^∞, this need not be so. Thus, the Klein-Gordon equation

$$\frac{\partial^2}{\partial x^\lambda \partial x_\lambda}\psi = -\left(\frac{mc}{\hbar}\right)^2\psi,$$

where m is a real constant with dimensions of mass and c is the speed of light, can be viewed as a necessary condition for restricting the carrier space to an irreducible representation. This still may not be sufficient to guarantee irreducibility as there may be other, higher order Casimir operators, but it is certainly necessary.

As L is non-compact, it has no finite-dimensional unitary representations (apart from the trivial one). Thus in general (Γ, W) is not unitary, and so neither is (Φ, H^∞).

A few examples will serve to illustrate the induced representations constructed here. These examples will also prove useful in chapter 5 when constructing free quantum fields.

Example 4.1: Scalar field The simplest example of a covariant field, and the only one which carries a unitary representation of the Poincaré group P, is the scalar field. This is obtained by letting the representation (Γ,W) of the Lorentz group L be trivial. It will be convenient to denote the operators of the scalar representation by a superscript 0. The vectors ψ in H are just complex-valued (square-integrable, Borel) functions over space-time. Equation (4.7) simplifies to give

$$\Phi^0(t\ ,\Lambda)\phi(x) = \phi(\Lambda^{-1}(x-t)) \tag{4.10}$$

for any $(t\ ,\Lambda)\in P$, $\phi \in H$ and space-time point x. The inner product (4.8) becomes simply

$$\langle\phi\ ,\phi'\rangle = \int d^dx\ \ \phi(x)^*\phi'(x), \tag{4.11}$$

for $\phi,\phi' \in H$, where * denotes complex conjugation. It is not hard to verify that (Φ^0,H) is unitary with this inner product.

For the generators P_λ and $J_{\lambda\mu}$ of the Poincaré group, and for ϕ in H^∞, the infinitely differentiable subspace of H, the representation is

$$\Phi^0(P_\lambda)\phi(x) \quad = i\hbar \frac{\partial}{\partial x^\lambda}\ \phi(x) \tag{4.12a}$$

$$\Phi^0(J_{\lambda\mu})\phi(x) \quad = i\hbar \left(x_\lambda \frac{\partial}{\partial x^\mu} - x_\mu \frac{\partial}{\partial x^\lambda}\right)\phi(x) \tag{4.12b}$$

From this example, it is apparent that the general representation differs from the scalar one basically in the spin angular momentum part Γ. Consequently, much of the work in constructing a general representation can be simplified by first constructing a scalar one, and many of the features of the general case should already appear in the scalar case. This observation is quite useful, as will be seen, for example, in chapter 8.

Example 4.2: Spinor field When used in a particular (rather than general) sense, the *spinor* representation in d dimensions is defined here to be that corresponding to the γ matrices for so(d–1,1). These are a set of d square matrices γ^{λ} which satisfy

$$\{ \gamma^{\lambda}, \gamma^{\mu} \} = 2 g^{\lambda\mu} 1,$$

where $\{ \cdot , \cdot \}$ is the anti-commutator, and 1 is the identity matrix. The γ matrices have 2^{l} rows and columns, where $l = d/2$ if d is even, and $(d-1)/2$ if d is odd.

The spinor representation of L will be denoted Γ. It is 2^{l}-dimensional, but for $d \geq 4$ and even, it is completely reducible into two distinct representations, each of dimension 2^{l-1}. Nevertheless, it is convenient to persist with the larger representation because it is necessary for wider considerations such as charge conjugation and parity.

More details concerning the γ matrices and the spinor representations in d dimensions are to be found in the appendix.

The generators $J_{\lambda\mu}$ of L are represented by

$$\Gamma(J_{\lambda\mu}) = \tfrac{i\hbar}{4} [\gamma_{\lambda}, \gamma_{\mu}] = \tfrac{i\hbar}{2} \gamma_{\lambda} \gamma_{\mu} \qquad (\lambda \neq \mu) \tag{4.13}$$

and representatives for elements of L are obtained by exponentiation. The elements ψ of H are spinors and have components ψ_{α} ($\alpha = 1,\ldots,2^{l}$) which are complex-valued functions over space-time. The representation of P carried by the spinor field is

$$\Phi(t, \Lambda)\psi(x) = \Gamma(\Lambda)\psi(\Lambda^{-1}(x-t)) \tag{4.14}$$

for $(t, \Lambda) \in P$, $\psi \in H$. For the generators, this gives

$$\Phi(P_{\lambda}) = \Phi^{0}(P_{\lambda})$$

$$\Phi(J_{\lambda\mu}) = \Phi^{0}(J_{\lambda\mu}) + \Gamma(J_{\lambda\mu}),$$

where the Φ^{0} operators on the right hand sides are understood to be those of equation (4.12) extended in the natural and obvious way.

There is no invariant positive-definite inner product for this representation, since Γ is not unitary. However, there is at least an invariant *indefinite* product, given by

$$\langle \psi , \psi' \rangle = \int d^d x \ \psi_\alpha(x)^* \gamma^0{}_{\alpha\beta} \, \psi'_\beta(x), \tag{4.15}$$

where $\psi, \psi' \in H$ and the repeated indices α, β are summed over $1, \ldots, 2^l$.

Example 4.3: Vector field The vector representation of L is the fundamental one formed of d-dimensional square matrices Λ. All of the matrices are real, so it is possible to have a set of real vector fields. In this representation, the matrices of the group and algebra elements are

$$\begin{aligned} \Gamma(\Lambda)^\nu{}_\rho &= \Lambda^\nu{}_\rho, \\ \Gamma(J_{\lambda\mu})^\nu{}_\rho &= i\hbar(\delta^\nu_\lambda g_{\mu\rho} - \delta^\nu_\mu g_{\lambda\rho}). \end{aligned} \tag{4.16}$$

Any real vector $B \in H$ has components B_λ which are real functions over space-time. For $B \in H$ and $(t\,,\Lambda) \in P$,

$$\Phi(t\,,\Lambda)B(x) = \Lambda B(\Lambda^{-1}(x-t)) \tag{4.17}$$

gives the covariant vector field representation. The operators of the scalar representation of iso(d–1,1) can be extended in the obvious way to give

$$\begin{aligned} \Phi(P_\lambda) &= \Phi^0(P_\lambda) \\ \Phi(J_{\lambda\mu}) &= \Phi^0(J_{\lambda\mu}) + \Gamma(J_{\lambda\mu}). \end{aligned}$$

As for spinors, there is no positive definite invariant inner product, just an invariant indefinite one, given by

$$\langle B\,, B' \rangle = \int d^d x \ B_\lambda(x) g^{\lambda\mu} B'_\mu(x), \tag{4.18}$$

where $B, B' \in H$. If complex vector fields are used, $B_\lambda(x)$ should be replaced by $B_\lambda(x)^*$ in (4.18).

Example 4.4: Antisymmetric second rank tensor field This representation is most easily expressed by setting

$$\Gamma(\Lambda) = \Lambda \otimes \Lambda \tag{4.19}$$

and taking vectors F in H to have $d(d-1)/2$ independent component functions $F_{\lambda\mu} = -F_{\mu\lambda}$. It would also be possible to anti-symmetrise the matrices of the representation, but the result is more cumbersome:

$$\Gamma(\Lambda)^{\lambda\mu}{}_{\nu\rho} = \tfrac{1}{2}(\Lambda^{\lambda}{}_{\nu}\Lambda^{\mu}{}_{\rho} - \Lambda^{\lambda}{}_{\rho}\Lambda^{\mu}{}_{\nu}),$$

so (4.19) will be used instead. Then the generators $J_{\lambda\mu}$ are represented by

$$\Gamma(J_{\lambda\mu}) = (J_{\lambda\mu}) \otimes 1 + 1 \otimes (J_{\lambda\mu}), \tag{4.20}$$

where $(J_{\lambda\mu})$ are the generators of the vector representation given in equation (4.16). Like vector fields, antisymmetric tensor fields carry a real representation of L. The induced representation of the Poincaré group is

$$\Phi(t\,,\Lambda)F(x) = \Lambda \otimes \Lambda\ F(\Lambda^{-1}(x-t)) \tag{4.21}$$

for $(t\,,\Lambda) \in P$, $F \in H$. With the appropriate extension,

$$\Phi(P_\lambda) = \Phi^0(P_\lambda)$$

$$\Phi(J_{\lambda\mu}) = \Phi^0(J_{\lambda\mu}) + \Gamma(J_{\lambda\mu}).$$

An (indefinite) invariant inner product for H is given by

$$\langle F\,,F'\rangle = \int d^dx\ F^{\lambda\mu}(x)F'_{\lambda\mu}(x), \tag{4.22}$$

for $F,F' \in H$.

In four dimensions, the antisymmetric tensor representation is not irreducible, but has two parts. These can be extracted from the components of the tensor field as follows. Let $\widetilde{F}$ be the antisymmetric tensor (sometimes called the *dual* of F) given by

$$\widetilde{F}^{\lambda\mu} = \tfrac{1}{2}\,\varepsilon^{\lambda\mu\nu\rho}F_{\nu\rho},$$

where $\varepsilon^{\lambda\mu\nu\rho}$ is the completely antisymmetric tensor in four dimensions, with $\varepsilon^{0123} = 1$. Then the components of the two (pseudo-real) sub-representations are

$$F^{+}{}_{\lambda\mu} = F_{\lambda\mu} + i\widetilde{F}_{\lambda\mu}$$

$$F^{-}{}_{\lambda\mu} = F_{\lambda\mu} - i\widetilde{F}_{\lambda\mu}\,.$$

To conclude this section, the link between these induced representations and classical field theory will be considered. This is most easily seen for a real scalar field over Minkowski space-time. Let O and O' be two observers with inertial frames of reference, and suppose they both observe a real quantity which varies over space and time. Let x_q and x'_q be their respective coordinates for a space-time point q. Each observer can describe the varying values of the real quantity as a function of space and time, but the functional form will differ, giving functions ϕ and ϕ'. At any one space-time point q, both O and O' observe the same value of the quantity, so the functions must satisfy

$$\phi'(x'_q) = \phi(x_q).$$

Since O and O' have inertial frames of reference, they are related by a Poincaré transformation $(t\,,\Lambda)$, so that

$$x'_q = \Lambda x_q + t.$$

Then,

$$\phi'(\Lambda x_q + t) = \phi(x_q).$$

So, for any general coordinate x, the function ϕ' of O' is related to the function ϕ of O by

$$\phi'(x) = \phi(\,\Lambda^{-1}(x-t)\,).$$

Comparison with equation (4.10) shows this is exactly the transformation law of a covariant scalar field, with ϕ' corresponding to $\Phi^0(t\,,\Lambda)\phi$ just as x'_q corresponds to $(t\,,\Lambda)\,x_q$.

Similar arguments apply to more complicated fields. Hence classical fields correspond directly to induced representations of the space-time group, or equivalently, produced representations of the its Lie algebra. It will be seen in chapter 5 that quantised fields transform differently to classical fields, and in fact correspond directly to induced representations of the Lie algebra. This difference is not very important on the whole, but it does generate frequent changes of sign and some differences in interpretation.

4.3 Irreducible representations

By virtue of its semi-direct product nature, *all* of the unitary irreducible representations of the Poincaré group P can be constructed by the method of induced representations (see theorem 3.9). The procedure presented in section 3.2 will be followed here, first finding the orbit structure of the representations of the translation subgroup $\mathrm{I}(d-1,1)$, then the corresponding little groups, and finally arriving at the induced representations.

Simultaneous treatment of all dimensions $d \geq 2$ would obscure the development of the construction, because the two and three dimensional cases have some peculiarities. So $d = 2$ and 3 will be considered later, and $d \geq 4$ will be assumed for the present.

First, the orbit structure of the invariant Abelian subgroup $\mathrm{I}(d-1,1)$ must be determined. There are infinitely many unitary irreducible representations, all one-dimensional. They may be denoted χ_p, where $p = (p^0, p^1, \ldots, p^{d-1})$ is a real d-dimensional vector, and

$$\chi_p(t\,,1\,) = e^{ip\cdot t/\hbar} \tag{4.23}$$

for any $(t\,,1\,)$ in $\mathrm{I}(d-1,1)$. Here, $p \cdot t$ stands for $p_\mu t^\mu$, and $\hbar$ has been introduced so that p has dimensions of momentum, which will be convenient later. Using the equivalence relation ~ discussed after theorem 3.9, the representations χ_p may be divided into disjoint orbits. Under this relation, for two vectors p and q, $\chi_p \sim \chi_q$ if and only if there is some $(t\,,\Lambda)$ in P such that

$$\chi_p(t',1\,) = \chi_q(\,(t,\Lambda)^{-1}\,(t'\,,1\,)\,(t,\Lambda)\,)$$

for all $(t'\,,1\,)$ in $\mathrm{I}(d-1,1)$. Using (4.1) and (4.23), this becomes

$$\chi_p(t'\,,1\,) = \chi_{\Lambda q}(t',1\,),$$

so $\chi_p \sim \chi_q$ if and only if there is some Λ in the Lorentz group L such that $p = \Lambda q$.

Thus, the orbit structure of the representations χ_p can be identified with the orbit structure of the momentum vectors p under the action of L. Two vectors in the same orbit will have the same value of $p \cdot p$ (although the converse is not true: see for example classes (iii) and (iv) below). It is not hard to see that there are six different classes of orbit:

(i) $p \cdot p > 0$ with $p^0 > 0$,

(ii) $p \cdot p > 0$ with $p^0 < 0$,

(iii) $p \cdot p = 0$ with $p^0 > 0$,

(iv) $p \cdot p = 0$ with $p^0 < 0$,

(v) $p \cdot p < 0$ (with p^0 unrestrained), and

(vi) $p = 0$.

There are infinitely many orbits in each of classes (i), (ii) and (v), one for each value of $p \cdot p$. When applied to physical theories, representations from class (i) lead to particles with positive rest mass (recall $g = \mathrm{diag}(1, -1, \ldots, -1)$), class (ii) to negative rest masses, class (iii) to particles moving at the speed of light with positive energy, class (iv) to the same with negative energy, class (v) to particles moving faster than light, and class (vi) to no really good physical interpretation. Of these, only classes (i) and (iii) give physically realistic particles, so the others will not be considered further.

Only one representation is required from each orbit, because the final representations of P are equivalent for all χ_p in the one orbit. This one chosen representation will be denoted χ_k. Let $m > 0$ be a real constant with dimensions of mass, and c be the speed of light. Then

$$k = (mc, 0, \ldots, 0)$$

is a convenient representative for the class (i) orbit with $p \cdot p = m^2c^2$ (it is, of course, the momentum of the rest state of a massive particle, and the orbit is the mass shell). For the (single) orbit of class (iii), a suitable choice is

$$k = (k^0,0,\ldots,0,k^0),$$

with $k^0 > 0$. That is, $k^{d-1} = k^0$, and $k^j = 0$ for $j = 1,\ldots,d-2$. This orbit is the forward light-cone – the mass shell for a massless particle. For the remainder of this section, whenever the letters i,j,k, and l appear as indices, they take values $1,\ldots,d-2$.

Having established the orbits and their representatives χ_k, the next step in the procedure of section 3.2 is to find the little groups, $L(\chi_k)$. The little group for χ_k is the set of $(0,\Lambda)$ in L satisfying

$$\chi_k(\,(0,\Lambda)^{-1}\,(t\,,1\,)\,(0,\Lambda)\,) = \chi_k(t\,,1\,)$$

for all $(t\,,1\,)$ in $\mathrm{I}(d-1,1)$. Using (4.1) and (4.23), this becomes

$$\chi_{\Lambda k}(t\,,1\,) = \chi_k(t\,,1\,),$$

or just

$$\Lambda k = k. \tag{4.24}$$

So the little group $L(\chi_k)$ is the set of all Λ in L which leave k invariant. For convenience, the χ will be omitted, and the little group just labelled $L(k)$.

For $k = (mc,0,\ldots,0)$, the Lorentz transformations leaving k invariant are, by inspection, the spatial rotations. The little group $L(k)$ is thus the universal covering group $\mathrm{Spin}(d-1)$ of $\mathrm{SO}_0(d-1)$. For low dimensions, these groups are isomorphic to some better-known groups:

$$\mathrm{Spin}(3) = \mathrm{SU}(2),$$

$$\mathrm{Spin}(4) = \mathrm{SU}(2) \otimes \mathrm{SU}(2),$$

$$\mathrm{Spin}(5) = \mathrm{Sp}(4),$$

$$\mathrm{Spin}(6) = \mathrm{SU}(4).$$

(where Sp(4) has the real Lie algebra sp(4) with complexification C_2). For all dimensions $d \geq 4$, Spin(d–1) is the simply connected group with Lie algebra so(d–1).

For $k = (k^0, 0, \ldots, 0, k^0)$, the spatial rotations in the k^i components clearly leave k invariant, but there are other possibilities. It is best to go over to the Lie algebra so(d–1,1) of L, and find the little algebra $l(k)$, recalling from section 3.3 that $l(k)$ and $L(k)$ are linked by the exponential mapping. Any Λ in L can be written

$$\Lambda = e^M,$$

where M is an element of so(d–1,1). If Λ is in $L(k)$, then condition (4.24) implies

$$Mk = 0. \tag{4.25}$$

Let

$$M = \xi^{\mu\lambda} M_{\lambda\mu}$$

where $M_{\lambda\mu}$ are the basis elements of so(d–1,1) and $\xi^{\lambda\mu} = -\xi^{\lambda\mu}$. In the fundamental representation carried by Minkowski space-time, the $M_{\lambda\mu}$ are $d \times d$ matrices given by

$$(M_{\lambda\mu})^\nu{}_\rho = -(\delta^\nu_\lambda g_{\mu\rho} - \delta^\nu_\mu g_{\lambda\rho}).$$

Equation (4.25) quickly reduces to

$$\xi^{\lambda 0} - \xi^{\lambda\, d-1} = 0,$$

which places no constraints on the values of ξ^{ij}. Thus a basis for the little algebra $l(k)$ is formed by the elements M_{ij} (which generate the spatial rotations mentioned earlier) and M_{i+} $(= -M_{+i})$ where

$$M_{i+} = (M_{i0} + M_{id-1})/\sqrt{2}. \tag{4.26}$$

The factor of $\sqrt{2}$ has been introduced for later convenience. The M_{ij} form a basis for so(d–2) and together with the M_{i+} they have the Lie products, using (4.3),

$$
\begin{aligned}
[M_{ij}, M_{kl}] &= g_{ik} M_{jl} - g_{il} M_{jk} + g_{jl} M_{ik} - g_{jk} M_{il} \\
[M_{ij}, M_{k+}] &= g_{ik} M_{j+} - g_{jk} M_{i+} \\
[M_{i+}, M_{j+}] &= 0.
\end{aligned}
$$

Comparing these Lie products with those in equation (4.3) for iso(d–1,1), it is apparent that $l(k)$ is isomorphic to iso(d–2).

Thus the little group $L(k)$ for $k = (k^0, 0, \ldots, 0, k^0)$ is the universal covering group I(d–2) Ⓢ Spin(d–2) of $\mathrm{ISO}_0(d-2)$. The latter is also known as the Euclidean group E(d–2). It should be stressed that $L(k)$ is a subgroup of L, so the inhomogeneous part I(d–2) of $L(k)$ is not a subgroup of the inhomogeneous part I(d–1,1) of P. The two have nothing to do with one another.

The final step in the procedure of section 3.2 is the induction itself. This is the same for both k under consideration, so neither will be singled out. Assume, for the moment, that a unitary irreducible representation Γ of $L(k)$ is given, carried by a finite-dimensional inner product space W (the problem of finding (Γ, W) will be considered shortly). The irreducible induced representation Δ of the Poincaré group P is carried by the Hilbert space $H = \mathrm{L}^2(L/L(k), W)$. This consists of square integrable Borel functions from the coset space $L/L(k)$ to W. It is not hard to see that $L/L(k)$ is isomorphic to the orbit of k (ie. the mass shell), so the Hilbert space vectors can be regarded as W-valued functions over p, where p is constrained to lie on the mass shell (ie. $p \cdot p = k \cdot k$ and $p^0 > 0$). The remaining ingredient in the induced representation construction is the set of coset representatives $B(p)$ in L for p in $L/L(k)$. These can be chosen to be smoothly varying over p, and will satisfy

$$B(p)k = p. \tag{4.27}$$

Equation (3.23) can be used to evaluate the induced representation action.

Proposition 4.5 Let ψ be an arbitrary vector in H. The action of the induced representation Δ is given, for $(t\ ,\Lambda)$ in P by

$$\{\Delta(t,\Lambda)\psi\}(p) = e^{ip\cdot t/\hbar}\Gamma(B(p)^{-1}\Lambda B(\Lambda^{-1}p))\{\psi(\Lambda^{-1}p)\} \tag{4.28}$$

for all p on the mass shell.

Proof. From (3.23) (replacing Φ by Δ, yx by (t,Λ), χ by χ_k, and z by p),

$$\{\Delta(t,\Lambda)\psi\}(p) = \chi_k((0,B(p))^{-1}(t\ ,1)(0,B(p)))$$
$$\times\Gamma(B(p)^{-1}\Lambda B(\Lambda^{-1}p))\{\psi(\Lambda^{-1}p)\}.$$

But, using equation (4.1), the argument of χ_k becomes $(B(p)^{-1}t\ ,1\ )$, and then (4.28) follows from equations (4.23) and (4.27). □

Comment. Equation (4.28) does not explicitly refer to k. This has been done because any choice of k in the same orbit will give an equivalent representation. Given a particular set of $B(p)$, the k being used can always be found as $k = B(p)^{-1}p$.

Inequivalent representations of the Poincaré group are obtained first of all by using k from different orbits. Then, for each k, the choice of (Γ,W) can be varied. For $k = (mc,0,\ldots,0)$, the little group $L(k)$ is Spin(d–1), which is simple, compact and simply connected. The finite dimensional unitary irreducible representations of this group are all known. They can be obtained from the irreducible representations of the simple Lie algebra so(d–1) by exponentiation.

For $k = (k^0,0,\ldots,0,k^0)$, the little group $L(k)$ is I(d–2) Ⓢ Spin(d–2). It is non-compact and not simple, but its unitary irreducible representations can all be constructed by repeating the procedure just followed for P. The representations of I(d–2) are characterised by a real

$d-2$ dimensional vector q, and the orbits consist of all q having the same Euclidean norm $|q|$. However, only finite-dimensional representations of $L(k)$ are of interest, which simplifies matters a little. Finite-dimensional representations will arise only when $\mathrm{I}(d-2)$ is represented trivially, that is, only in the orbit with $|q| = 0$. For this case, the little group of Spin(d–2) is in fact the whole of Spin(d–2) itself. So the required representations of $L(k)$ are just the well-known representations of Spin($d-2$) (finite-dimensional, unitary and irreducible) extended trivially to $\mathrm{I}(d-2) \circledS \mathrm{Spin}(d-2)$. This fairly innocuous-looking extension has far-reaching consequences, because it is the source of the first order field equations for massless fields! This will be spelt out in chapter 5.

It should be kept in mind that for $d = 4$, Spin(d–2) is the universal covering group of SO(2), and is Abelian, so all its irreducible representations are one-dimensional. This is why each massless elementary particle in four dimensions has only one value of helicity, unless parity is conserved.

To complete the construction of the unitary irreducible representations of P only requires an inner product in H. This was discussed in section 2.3 and defined by equation (2.8) as

$$\langle \phi, \psi \rangle_H = \int d\mu(p) \langle \phi(p), \psi(p) \rangle_W , \tag{4.29}$$

for $\phi, \psi \in H$. Here, $\langle \cdot , \cdot \rangle_W$ is the (Hermitean) inner product in W, and μ is an L-invariant measure over the mass shell. For both classes of orbit, $k = (mc, 0, \ldots, 0)$ and $k = (k^0, 0, \ldots, 0, k^0)$, p^0 can be set to

$$p^0 = \sqrt{k \cdot k + \mathbf{p}^2}, \tag{4.30a}$$

and $d\mu(p)$ taken as

$$d\mu(p) = \frac{d^{d-1}\mathbf{p}}{2p^0}. \tag{4.30b}$$

where $\mathbf{p}$ is the $(d-1)$-component vector $\mathbf{p} = (p^1,\ldots,p^{d-1})$, and ,

$$\mathbf{p}^2 = \sum_{a=1}^{d-1} p^a p^a.$$

Alternatively, p^0 can be left free, and some generalised functions or distributions used. Then $d\mu(p)$ can be written as

$$d\mu(p) = d^d p\ \delta(p^2 - k^2)\theta(p^0), \tag{4.30c}$$

where $p^2 = p \cdot p$ and $k^2 = k \cdot k$ is m^2c^2 for $k = (mc,0,\ldots,0)$ and 0 for $k = (k^0,0,\ldots,0,k^0)$. (θ is the step function, which, roughly speaking, is zero if its argument is negative and 1 if its argument is positive). In later sections, (4.30c) will usually be adopted, because of its neatness and manifest Lorentz invariance.

As Spin(d–1) and Spin(d–2) are compact Lie groups, their finite-dimensional representations are all unitary (or can be made unitary by a similarity transformation). So $\langle \cdot , \cdot \rangle_W$ can be taken such that Γ is unitary, and hence (Δ,H) is unitary as well.

The construction of the representations of P for $d \geq 4$ is complete. In fact, most of the foregoing discussion applies to the cases $d = 2$ and $d = 3$ as well, with just a few changes in detail.

For $d = 3$, the same six classes of orbit for χ_p arise. They are easy to visualise in $d = 3$: in order (i) to (vi), they are the interiors of the forward and backward light cones, the forward and backward light cones themselves, the space outside the cones, and the origin point. Again, only the same two classes give physically realistic theories. However, the little group $L(k)$ for both $k = (mc,0,0)$ $(mc > 0)$ and $k = (k^0,0,k^0)$ $(k^0 > 0)$ is $\mathbb{R}^+$, the multiplicative group of positive real numbers. This arises in the first class as the universal covering group of SO(2), and in the second as I(1). Equation (4.28) continues to give the action of the induced representation in three dimensions, and (4.29) and (4.30) give the inner product. $\mathbb{R}^+$ has an infinite number of one-dimensional unitary irreducible representations Γ. So the carrier space W of Γ is just $\mathbb{C}$, and the inner product in W is simply

$$\langle \phi(p), \psi(p) \rangle = \phi(p)^* \psi(p)$$

For $d = 2$, orbits (iii), (iv) and (v) each split into two, giving

(iii*a*) $p \cdot p = 0$ with $p^0 > 0, p^1 > 0$,

(iii*b*) $p \cdot p = 0$ with $p^0 > 0, p^1 < 0$,

(iv*a*) $p \cdot p = 0$ with $p^0 < 0, p^1 > 0$,

(iv*b*) $p \cdot p = 0$ with $p^0 < 0, p^1 < 0$,

(v*a*) $p \cdot p < 0$ with $p^1 > 0$, and

(v*b*) $p \cdot p < 0$ with $p^1 < 0$.

This is because in two-dimensional Minkowski space, the four arms of the null lines are all disjoint, as are the two halves of the space-like sector. Nevertheless, only classes (i) (unchanged), (iii*a*) and (iii*b*) are relevant, with suitable representatives being $k = (mc,0)$, $k = (k^0,k^0)$ and $k = (k^0,-k^0)$ respectively ($mc,k^0 > 0$). Each of the corresponding orbits is one-dimensional, since p^0 and p^1 are related through $p \cdot p = k \cdot k$.

Surprisingly, for $d = 2$, the little groups $L(k)$ are all trivial: there are no transformations in L which leave any of the physically realistic k unchanged except for the identity transformation. This comes about because L is only a one-dimensional Lie group, and that degree of freedom is used to move around the orbit. Accordingly, L can be smoothly parameterised by the elements of a momentum orbit.

As the little groups are trivial, the representation (Γ,W) is eliminated from the foregoing procedure, leaving just one irreducible representation of P for each orbit. This representation is carried by $H = \mathrm{L}^2(L\ ,\mathbb{C}\ )$, and (4.28) becomes, for $\psi \in H,\ (t,\Lambda)\in P$,

$$\Delta(t,\Lambda)\psi(p) \;=\; \mathrm{e}^{ip\cdot t/\hbar}\psi(\Lambda^{-1}p) \tag{4.31}$$

for all p in the orbit of k. Equations (4.29) and (4.30) give the inner product for H, with

$$\langle\, \phi(p),\psi(p)\,\rangle = \phi(p)^*\psi(p).$$

Returning to the general dimension $d \geq 2$, the representation of the Lie algebra iso(d–1,1) corresponding to (4.28) or (4.31) can be obtained by differentiation, as detailed near the end of section 3.1. It is carried by the dense subspace H^∞ of H, comprising infinitely differentiable functions in H. For the generators P_λ and $J_{\lambda\mu}$ of the Poincaré group, and for $\psi\in H^\infty$, the representation is (cf. equations (3.16) and (3.17))

$$\Phi(P_\lambda)\psi(p) \;=\; p_\lambda\,\psi(p) \tag{4.32a}$$

$$\Phi(J_{\lambda\mu})\psi(p) \;=\; i\,\hbar\,\left(p_\lambda\frac{\partial}{\partial p}{}^{\mu} - p_\mu\frac{\partial}{\partial p}{}^{\lambda}\right)\psi(p) + \Gamma_p(J_{\lambda\mu})\psi(p), \tag{4.32b}$$

for all p in the orbit of k, where

$$\Gamma_p(J_{\lambda\mu}) \;=\; \Gamma\left(B(p)^{-1}\left(J_{\lambda\mu} + i\,\hbar\,\left(p_\lambda\frac{\partial}{\partial p}{}^{\mu} - p_\mu\frac{\partial}{\partial p}{}^{\lambda}\right)\right)B(p)\right). \tag{4.33}$$

For $d = 2$, the Γ_p term is absent.

It is worth noting that, as for the covariant field representations, the angular momentum splits up into orbital and spin parts.

An explicit comparison of the $d = 2$ representation obtained here with one obtained by the method of produced representations for Lie *algebras* will be made in section 4.4.

To conclude this section, a specific example will be given.

Example 4.6: Scalar representation The simplest irreducible representation of P is found by using the trivial representation of the little group $L(k)$ with $\Gamma(\Lambda) = 1$ for all Λ in $L(k)$. It will be convenient to use the superscript 0 for this representation. Vectors ψ in H are complex-valued functions over the mass shell. Equation (4.28) becomes

$$\Delta^0(t,\Lambda)\psi(p) = e^{ip\cdot t/\hbar}\psi(\Lambda^{-1}p) \tag{4.34}$$

for $(t,\Lambda)\in P, \psi\in H$, and p on the mass shell.

The inner product is

$$\langle\phi,\psi\rangle = \int d^dp\ \delta(p^2-k^2)\theta(p^0)\phi(p)^*\psi(p)$$

for $\phi,\psi\in H$. The generators P_λ and $J_{\lambda\mu}$ are represented by

$$\begin{aligned} \Delta^0(P_\lambda)\psi(p) &= p_\lambda\,\psi(p) \\ \Delta^0(J_{\lambda\mu})\psi(p) &= i\hbar\left(p_\lambda\frac{\partial}{\partial p^\mu} - p_\mu\frac{\partial}{\partial p^\lambda}\right)\psi(p). \end{aligned} \tag{4.35}$$

As for covariant fields, higher spin representations differ from this scalar representation only in the parts involving Γ, and so the scalar representation provides much useful information. Explicit examples of such higher spin representations are not very illuminating, since they require a choice of coset representatives $B(p)$, which is always messy. A few will be examined in chapter 6 in the construction of quantum fields.

4.4 Comparison of group and algebraic construction for two dimensions.

As an example of the equivalence between induced group representations and produced algebra representations, the irreducible produced representations of the Lie algebra iso(1,1) will be constructed and compared with those obtained via induced representations of the Poincaré group in the last section. To construct the produced representation, the procedure of sections 3.3 and 2.5 (particularly the "more practical" formulation of definition 2.10) will be followed. For comparison, g, h and k of section 3.3 are here iso(1,1), so(1,1) and i(1,1) (and k here is a momentum vector).

In most cases, the produced representation of a Lie algebra is too complicated for explicit evaluation, but for iso(1,1), there are only three basis elements. For the sake of comparison, the three generators for the corresponding Poincaré group, P_0, P_1 and $J = J_{01}$ ($= -J_{10}$), will be used. It is convenient to define the combinations

$$P_{\pm} = (P_0 \pm P_1)/\sqrt{2}. \tag{4.36}$$

The only non-zero Lie products are

$$[\, J\,, P_{\pm}\,] = \pm\, i\, \hbar\, P_{\pm}.$$

An irreducible representation χ of the invariant subalgebra i(1,1) is

$$\chi(P_\lambda) = k_\lambda, \tag{4.37}$$

where k is a real two component vector, and λ takes the values 0,1. To obtain representations corresponding to those in the last section, k should be taken as either $(mc,0)$ or (k^0,k^0), with $mc, k^0 > 0$. $k_{\pm}$ can be defined in the same way as $P_{\pm}$.

As

$$\chi(\,[\, J\,, P_{\pm}\,]\,) = \pm\, i\, \hbar\, \chi(P_{\pm}) = \pm\, i\, \hbar\, k_{\pm}\,,$$

which does not vanish for both P_+ and P_-, the stability algebra $s(\chi)$ is just i(1,1) (see equation (3.25) for the definition of $s(\chi)$). Hence the little algebra of so(1,1) is trivial. So there is no need of a representation Γ of the little algebra, as χ by itself provides the necessary representation of $s(\chi)$.

Great simplifications occur for the universal enveloping algebras also. Considered as a U($s(\chi)$)-module, U(iso(1,1)) is just the universal enveloping algebra of the one-dimensional subalgebra, U(so(1,1)). The basis elements for this are the simple powers J^r ($r = 0,1,2,\ldots$) of the generator J. Any element A in U(iso(1,1)) can be written

$$A = \sum_{r=0}^{\infty} A_r J^r \tag{4.38}$$

where all of the A_r are elements of U(i(1,1)). There is no need for the special space called k in section 2.5 (which was defined to be the real or complex vector space spanned by the basis of U(iso(1,1)) as a U(i(1,1))-module), because this is not any different from U(so(1,1)).

The carrier space for the produced representation is the space of complex-valued linear functions over U(so(1,1)), $U = \mathrm{L}(\,\mathrm{U(so(1,1))}\,, \mathbb{C}\,)$. Because the example is so simple, it is possible to evaluate the right multiplication of U(so(1,1)) by iso(1,1). This allows the explicit statement of the action of the produced representation in the following proposition.

Proposition 4.7 For any $\psi \in U$ and $r \in \mathbb{N}$, the produced representation Δ is given by

$$\Delta(J)\psi(J^r) = \psi(J^{r+1}) \tag{4.39a}$$

$$\Delta(P_+)\psi(J^r) = k_+\psi(\,(J - i\hbar)^r\,) \tag{4.39b}$$

$$\Delta(P_-)\psi(J^r) = k_-\psi(\,(J + i\hbar)^r\,) \tag{4.39c}$$

Proof. The action of $X \in$ iso(1,1) on $Z \in$ U(so(1,1)) is given by

$$ZX = \sum_{s=0}^{\infty} (ZX)_s J^s,$$

where the $(ZX)_s$ are all in U(i(1,1)). Equation (2.15) then gives the action of the produced representation as

$$\Delta(X)\psi(Z) = \sum_{s=0}^{\infty} \chi((ZX)_s)\psi(J^s).$$

It suffices to consider only the basis elements $Z = J^r$, since ψ is linear. For $X = J$, clearly,

$$J^r J = J^{r+1},$$

from which (4.39a) follows immediately. For $X = P_{\pm}$, note that

$$\begin{aligned} J P_{\pm} &= P_{\pm} J + [J, P_{\pm}] \\ &= P_{\pm}(J \pm i\hbar) \end{aligned}$$

By induction,

$$J^r P_{\pm} = P_{\pm}(J \pm i\hbar)^r.$$

Recalling (4.37), the required results, (4.39b) and (4.39c) follow immediately. □

Before comparing this produced Lie algebra representation with the induced group representation of the last section, it is worth looking more closely at the Lorentz group and the momentum orbit. In the fundamental two-dimensional matrix representation of so(1,1), J has components

$$J^{\lambda}{}_{\mu} = i\hbar\,(\delta^{\lambda}_{0} g_{1\mu} - \delta^{\lambda}_{1} g_{0\mu}).$$

That is,

$$J = -i\hbar \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

A Lorentz transform $\Lambda = \exp(i\xi J/\hbar)$ is

$$\Lambda = \begin{pmatrix} \cosh\xi & \sinh\xi \\ \sinh\xi & \cosh\xi \end{pmatrix}.$$

When applied to the orbit representative momentum k, this gives

$$p^0 = k^0 \cosh\xi + k^1 \sinh\xi$$
$$p^1 = k^0 \sinh\xi + k^1 \cosh\xi$$

Noting that $p_+ \sim p_0 + p_1 \sim p^0 - p^1$,

$$p_+ = e^{-\xi} k_+$$
$$p_- = e^{\xi} k_-. \tag{4.40}$$

Any momentum p in the orbit of k can thus be written

$$p = \exp(i\xi J/\hbar)k, \qquad \text{with } \xi = -\ln\frac{p_+}{k_+}, \text{ or } \ln\frac{p_-}{k_-} \tag{4.41}$$

(since one of $k_\pm$ might be zero). This shows the parameterisation of the Lorentz group by the momentum orbits for $d = 2$ mentioned in the last section. Also, for any function f over the momentum orbit,

$$\begin{aligned}\frac{d}{d\xi} f(p) &= \frac{dp^0}{d\xi}\frac{\partial}{\partial p^0} f(p) + \frac{dp^1}{d\xi}\frac{\partial}{\partial p^1} f(p)\\ &= \left(p_0\frac{\partial}{\partial p^1} - p_1\frac{\partial}{\partial p^0}\right) f(p)\end{aligned} \tag{4.42}$$

From the last section, the induced representation of the two-dimensional Poincaré group can be differentiated to give a representation of iso(1,1) carried by infinitely differentiable functions of momentum. Denote this representation (Φ,V). Then, for ϕ in V, using (4.32) and (4.36),

$$\Phi(P_\pm)\phi(p) = p_\pm\phi(p) \tag{4.43a}$$
$$\Phi(J)\phi(p) = i\hbar\left(p_0\frac{\partial}{\partial p^1} - p_1\frac{\partial}{\partial p^0}\right)\phi(p). \tag{4.43b}$$

The mapping from the representation (Φ,V) to the produced representation (Δ,U) is very straightforward and not very enlightening. The opposite direction is a little more interesting. All of the details of the calculation will be left in the open, rather than put into the proof of a proposition, because it is these details themselves which constitute the explicit demonstration of equivalence.

For any ψ in U, define a function ψ' in V by

$$\psi'(p) = \sum_{r=0}^{\infty} \frac{(-i\xi/\hbar)^r}{r!}\psi(J^r), \tag{4.44}$$

where $p = \exp(i\xi J/\hbar)k$ (see (4.41)). Clearly, ψ' is an infinitely differentiable function over the orbit of k, so it is in V.

Consider first the representation of P_+. Acting on any arbitrary ψ in U, $\Delta(P_+)$ produces a new function $\Delta(P_+)\psi$ in U, which in turn maps under (4.44) to a function $(\Delta(P_+)\psi)'$ in V given by

$$\begin{aligned}
(\Delta(P_+)\psi)'(p) &= \sum_{r=0}^{\infty} \frac{(-i\xi/\hbar)^r}{r!}\{\Delta(P_+)\psi\}(J^r) && \text{by (4.44)}\\
&= \sum_{r=0}^{\infty} \frac{(-i\xi/\hbar)^r}{r!}k_+\psi((J-i\hbar)^r) && \text{by (4.39}b\text{)}\\
&= k_+\sum_{r=0}^{\infty} \frac{1}{r!}\Big(\Delta((-i\xi/\hbar)(J-i\hbar))\Big)^r\psi(1) \\
& && \text{by (4.39}a\text{)}\\
&= k_+\exp(\Delta(-\xi - i\xi J/\hbar))\psi(1)\\
&= k_+ e^{-\xi}\exp(\Delta(-i\xi J/\hbar))\psi(1)\\
&= p_+\psi'(p), && \text{by (4.40) and (4.44).}
\end{aligned}$$

This should be compared with (4.43a). The calculation for P_- proceeds likewise. For J,

$$\begin{aligned}
(\Delta(J)\psi)'(p) &= \sum_{r=0}^{\infty} \frac{(-i\xi/\hbar)^r}{r!}\{\Delta(J)\psi\}(J^r) && \text{by (4.44)}\\
&= \sum_{r=0}^{\infty} \frac{(-i\xi/\hbar)^r}{r!}\psi(J^{r+1}) && \text{by (4.39}a\text{)}
\end{aligned}$$

$$= \sum_{r=0}^{\infty} i\hbar \frac{d}{d\xi} \frac{(-i\xi / \hbar)^{r+1}}{(r+1)!} \psi (J^{r+1})$$

$$= i\hbar \frac{d}{d\xi} \psi'(p) \qquad \text{by (4.44)}$$

$$= i\hbar \left(p_0 \frac{\partial}{\partial p^1} - p_1 \frac{\partial}{\partial p^0} \right) \psi'(p) \qquad \text{by (4.42),}$$

which corresponds exactly to (4.43*b*). Thus , for any X in iso(1,1),

$$(\Delta(X)\psi)' = \Phi(X)\psi' ,$$

demonstrating the equivalence of the representations (Φ,V) and (Δ,U).

Chapter 5

Free relativistic quantum fields in four or more dimensions

Quantum field theory is a very diverse topic, with many, widely differing approaches. The most studied example is relativistic quantum field theory in four dimensional Minkowski space-time, for which the fields transform according to the Poincaré group of Lorentz transformations and space-time translations.

One approach to this theory, developed by Weinberg (1964*a*, *b*, *c*, 1965*a*, and 1969, but see 1965*b* for a summary), highlights the part played by induced representations of the Poincaré group, P. In this approach, the first step is the construction of free fields. The basic ingredients are momentum space annihilation and creation operators, and space-time field operators. The momentum space operators carry a unitary irreducible representation of P, and the space-time operators carry a "covariant field" representation. Weinberg constructs just the momentum space representation by the method of induction, but, as shown in chapter 4, the "covariant field" representation is also an induced representation. Many of the features of free field theory, such as free field equations and the particle content of a field, follow directly from the interplay between the two representations. In particular, there is no need for a Lagrangian or the principle of least action.

After the construction of free quantum fields, propagators for the fields can be calculated, and a set of Feynman rules derived. An interaction Hamiltonian is required for this last step, but even the choice of this is constrained to a large extent by the representations chosen initially.

For massless particles of spin one or more, the introduction of gauge potentials is required in order to cope with the long-range interactions of electromagnetism and gravitation. Once again, the field equations and transformation properties of the gauge potentials are completely determined by the initial representations. Even gauge transformations and charge conservation are found without the use of Lagrangians.

The object of this chapter is to extend Weinberg's approach to Minkowski space-time of any dimension $d \geq 4$. The appropriate representations have already been constructed in chapter 4. In chapter 4, the representations for two and three dimensions were examined also, but since there were no additional complications, only simplifications, it suffices to describe the construction for four or more dimensions here. Fields for two and three dimensions can be constructed using the same general procedure. A brief account of quantum fields and their transformation laws is given in section 5.1. This leads to the construction of free quantum fields in section 5.2. Finally, section 5.3 examines the consequences of the construction, especially the field equations and particle content of a field.

In this chapter, Minkowski space-time, the Poincaré group P, the Lorentz group L, and so on are all d-dimensional, with $d \geq 4$. The metric tensor for Minkowski space-time is $g = \mathrm{diag}(1,-1,\ldots,-1)$. More details and precise definitions of P and L can be found in section 4.1. Indices λ, μ, ν, ρ will take values $0,1,2,\ldots d-1$, and i, j, k will takes values $1,2,\ldots,d-2$. Summation signs will be omitted, summation over repeated indices being implied unless otherwise indicated.

An immediate extension to the work in this chapter would be to treat interactions and construct Feynman rules as Weinberg does. Only the free fields are considered here, as the main interest still lies ahead, in the construction of supersymmetric multiplets.

The important thing to note here is the spirit of the approach. There are no Lagrangians, and the principle of least action does not appear. Everything is determined by the basic ideas of what a quantum field is, and how it behaves under Poincaré transformations. This brings out most clearly the role of the representation theory.

5.1 Quantum fields, Poincaré transformations and canonical commutation relations

This section sets out some general ideas about quantum fields, particularly concerning the transformation properties of the fields and the canonical commutation relations of annihilation and creation operators. It is not intended as an account of quantum field theory, or even free field theory, since this can be found in the many texts on the subject. The main points to be made here are that quantum fields transform in the opposite sense (ie. contragradiently) to classical fields, and that the canonical commutation relations for a quantum field theory arise from the Hilbert space inner product of the corresponding classical field theory.

It should be stressed that the notion of quantum field used here is very crude. If taken too seriously, it will quickly lead to inconsistencies in interpretation and calculation. Of course, this is a common feature of many approaches to quantum field theory. The particular definitions used here have been chosen because they suit the main purpose, which is to highlight the role of induced representations. Much more refined and sustainable definitions can be found in, for instance, Streater and Wightman (1964) or Bogoliubov *et al* (1975).

Roughly speaking, a relativistic field ψ is just a function over d-dimensional Minkowski space-time. For a relativistic *quantum* fields, the value $\psi(x)$ of the function at any space-time point x is a linear operator on a Fock space F. The Fock space, at its simplest, is generated by the Hilbert space H of a classical field theory (for convenience, the wave-function of quantum mechanics will be termed a classical field). F may be written as an infinite direct sum of tensor products of H

$$F = H_0 \oplus H \oplus H \otimes H \oplus H \otimes H \otimes H \oplus \ldots$$

where H_0 is a one-dimensional Hilbert space ($\mathbb{C}$, for example). The tensor products may also be endowed with some symmetry properties.

In general, a quantum field ψ may be more than just a single function: it may be a set of functions $\psi_1,\ldots,\psi_n$ for some positive integer n.

If H carries a unitary representation Δ of the Poincaré group P, then Fock space itself is a Hilbert space, carrying a unitary representation of P. The representation carried by F will be of the form

$$\Phi = 1 \oplus \Delta \oplus \Delta \otimes \Delta \oplus \Delta \otimes \Delta \otimes \Delta \oplus \ldots$$

where 1 is the trivial representation, and any symmetry structure in the tensor products follows that of F. The inner product in F is also inherited from H in a natural manner.

For quantum fields, the behaviour under Poincaré transformations is the opposite of that for classical fields. The next few paragraphs demonstrate this behaviour for a covariant quantum field, although the same argument can be applied to other types of field, such as momentum space fields carrying irreducible representations. The presentation here (which follows that of Bjorken and Drell (1965)) should be understood not as a general proof, but more as a demonstration for particular kinds of quantum fields. Nonetheless, the conclusion is actually general, and can be shown to be so by more careful (and more involved) arguments.

Let W be a finite-dimensional vector space carrying a representation Γ of the Lorentz group L. Take indices a,b,c,d to range over the values

1,2,...,dim W, and sum over repeated indices, as usual. Let ψ_a be a set of relativistic quantum fields. Given two vectors Ψ and Ξ in F, it is possible to define a set of functions $f^{\Psi\Xi}{}_a$ over space-time by

$$f^{\Psi\Xi}{}_a(x) = \langle \Psi , \psi_a(x)\Xi\rangle, \tag{5.1}$$

for each space-time point x, where $\langle \cdot , \cdot \rangle$ is the inner product in F. The $f^{\Psi\Xi}{}_a(x)$ can be regarded as components of a vector $f^{\Psi\Xi}(x)$ in W. $f^{\Psi\Xi}$ is then a W-valued function over space-time with component functions $f^{\Psi\Xi}{}_a$.

It is possible to generate a representation U of P from $f^{\Psi\Xi}$ by defining the operator $U(t,\Lambda)$ for each (t,Λ) in P by

$$U(t,\Lambda) f^{\Psi\Xi} = f^{\Phi(t,\Lambda)\Psi\ \Phi(t,\Lambda)\Xi}. \tag{5.2}$$

Since Φ is a representation, it follows that the U provides a homomorphic mapping from P. The carrier space of the representation U is just the set of W-valued functions $f^{\Psi\Xi}$ for all Ψ and Ξ in F.

According to the principles of quantum field theory, $f^{\Psi\Xi}{}_a(x)$ should be a measurable quantity for each space-time point x (at least, if the field operators are self-adjoint). This implies that $f^{\Psi\Xi}$ is a *classical* covariant field over space-time, and so U should be an induced representation of P. Since W carries a representation Γ of L, the appropriate type of representation is the one induced from Γ. This type of representation was discussed in section 4.2. From equation (4.7), U will have the form

$$\{ U(t,\Lambda)f^{\Psi\Xi} \}(x) = \Gamma(\Lambda) \{ f^{\Psi\Xi}(\Lambda^{-1}(x-t)) \},$$

or, in components,

$$\{ U(t,\Lambda)f^{\Psi\Xi} \}_a(x) = \Gamma(\Lambda)_{ab}\, f^{\Psi\Xi}{}_b(\Lambda^{-1}(x-t)), \tag{5.3}$$

for all (t,Λ) in P, and each space-time point x.

Combining (5.2) and (5.3), and using the definition (5.1) of $f^{\Psi\Xi}$ gives

$$\langle \Phi(t,\Lambda)\Psi, \psi_a(x)\Phi(t,\Lambda)\Xi\rangle = \Gamma(\Lambda)_{ab}\langle \Psi, \psi_b(\Lambda^{-1}(x-t))\Xi\rangle.$$

Since Φ is unitary, the left-hand side of the last expression is equal to

$$\langle \Psi, \Phi(t,\Lambda)^{-1}\psi_a(x)\Phi(t,\Lambda)\Xi\rangle.$$

This identity holds for any Ψ and Ξ in F, so it follows (at least weakly) that

$$\Phi(t,\Lambda)^{-1}\psi_a(x)\Phi(t,\Lambda) = \Gamma(\Lambda)_{ab}\psi_b(\Lambda^{-1}(x-t)).$$

Under the transformation $\Psi \to \Phi(t,\Lambda)\Psi$, the operators $\psi_a(x)$ transform to $\Phi(t,\Lambda)\psi_a(x)\Phi(t,\Lambda)^{-1}$, which, by the above reasoning, is given by

$$\Phi(t,\Lambda)\psi_a(x)\Phi(t,\Lambda)^{-1} = \Gamma(\Lambda^{-1})_{ab}\psi_b(\Lambda x + t). \tag{5.4}$$

for all (t,Λ) in P, and each space-time point x. Comparison with the transformations of equation (4.7) shows that the quantum field operators transform in the opposite sense to the classical fields.

There are several ways to arrive at the transformation rule (5.4). One is to specify some field operators concretely in terms of annihilation and creation actions in F (eg. Streater and Wightman 1964 p.103). Explicit calculation then verifies equation (5.4). This approach is reminiscent of Mackey's systems of imprimitivity (Mackey 1968), where projection-valued measures are considered instead of operator valued fields.

Note that there is another important difference between the quantum and classical fields. The representation on classical fields transforms one field into another: one function into another function. For quantum fields, on the other hand, (5.4) involves just a product of operators. The representation does not transform one quantum field into another quantum field, but rather transforms the field *value* at one space-time point into a field value at another point. There is in fact only one quantum field involved (counting a set of quantum fields for a W-vector as a single entity).

If the quantum fields ψ_a are infinitely differentiable functions over space-time, then corresponding to (5.4), a representation of the Lie algebra iso(d–1,1) can be determined. With the usual generators P_λ and $J_{\lambda\mu}$ of the Poincaré group (defined in equation (4.4) and having the Lie products (4.5)), equation (5.4) yields

$$[\Phi(P_\lambda),\psi_a(x)] = -i\hbar \frac{\partial}{\partial x^\lambda} \psi_a(x) \tag{5.5a}$$

$$[\Phi(J_{\lambda\mu}),\psi_a(x)] = -i\hbar \left(x_\lambda \frac{\partial}{\partial x^\mu} - x_\mu \frac{\partial}{\partial x^\lambda}\right)\psi_a(x) - \Gamma(J_{\lambda\mu})_{ab}\psi_b(x) \tag{5.5b}$$

Just as for classical fields the right-hand side of (5.5a) represents the momentum, and the terms on the right-hand side of (5.5b) represent the orbital and spin angular momentum respectively.

Equations (5.5) for covariant quantum fields should be compared with equations (4.9) for covariant classical fields to see that the signs on the right-hand sides have reversed.

From the reversal of sense between equations (4.7) and (5.4), or the corresponding sign changes between equations (4.9) and (5.5), it is apparent that the representation carried by quantum fields is contragradient to that carried by classical fields. The classical field representation is an induced representation of the Poincaré group P, which means that the corresponding representation of the Poincaré algebra iso(d–1,1) is equivalent to a produced representation, as shown in section 3.1. It was noted in section 2.7 that induced and produced representations of Lie algebras are contragradient. The contragradient quantum field representation (5.5) is thus equivalent to an induced representation of iso(d–1,1). This point should be emphasised: the *produced* representations of the Poincaré algebra are appropriate for *classical* fields, and the *induced* representations are appropriate for *quantum* fields.

As mentioned earlier, these considerations apply to other types of field just as well as to covariant fields over space-time. Suppose the function $f^{\Psi\Xi}{}_a$ used earlier were defined for a quantum field ϕ_a over momentum space and vectors Ψ and Ξ in the Fock space F by

$$f^{\Psi\Xi}{}_a(p) = \langle\Psi, \phi_a(p)\Xi\rangle, \tag{5.6}$$

Equation (5.2) can be re-interpreted to define a representation U of P for this new $f^{\Psi\Xi}$. Suppose that this representation is unitary and irreducible. These representations were all constructed in section 4.3. The momentum p must be restricted to the orbit or mass shell with $p{\cdot}p = m^2c^2$ and $p^0 > 0$, where m is a positive or zero constant with units of mass and c is the speed of light. In addition, $f^{\Psi\Xi}(p)$ must be a vector in a finite-dimensional space W carrying a representation Γ of the appropriate little group. So the indices a,b,c,d take values 1,2,...,dim W. For (t,Λ) in P, the representation is given by (4.28) as

$$U(t,\Lambda)f^{\Psi\Xi}(p) = e^{ip\cdot t/\hbar}\Gamma(\,B(p)^{-1}\Lambda\, B(\Lambda^{-1}p)\,)f^{\Psi\Xi}(\Lambda^{-1}p), \tag{5.7}$$

for all p on the mass shell. In (5.7), the $B(p)$ form a set of smoothly varying set of Lorentz transformations such that, for some fixed momentum k (on the mass shell),

$$B(p)k = p$$

for all p on the mass shell.

As before, the combination of (5.6), (5.7) and (5.2) implies a transformation law for the operators $\phi_a(p)$, which is

$$\Phi(t,\Lambda)\phi_a(p)\Phi(t,\Lambda)^{-1} = e^{-i\Lambda p\cdot t/\hbar}\Gamma(B(\Lambda p)^{-1}\Lambda B(p))^{-1}{}_{ab}\phi_b(\Lambda p). \tag{5.8}$$

for all (t,Λ) in P and all p on the mass shell.

This transformation law is again contragradient to the classical one in equation (4.28), although this time, the reversal is a little more complicated. The corresponding Lie algebra representation is

$$[\Phi(P_\lambda),\phi_a(p)] = -p_\lambda\,\phi_a(p) \tag{5.9a}$$

$$[\Phi(J_{\lambda\mu}),\phi_a(p)] = -i\hbar\left(p_\lambda\frac{\partial}{\partial p}{}^{\mu} - p_\mu\frac{\partial}{\partial p}{}^{\lambda}\right)\phi_a(p) - \Gamma_p(J_{\lambda\mu})_{ab}\phi_b(p), \tag{5.9b}$$

for all p on the mass shell, where $\Gamma_p(J_{\lambda\mu})$ is given in equation (4.33) as

$$\Gamma_p(J_{\lambda\mu}) = \Gamma\left(B(p)^{-1}\left(J_{\lambda\mu} + i\hbar\left(p_\lambda\frac{\partial}{\partial p}{}^{\mu} - p_\mu\frac{\partial}{\partial p}{}^{\lambda}\right)\right)B(p)\right). \tag{5.10}$$

Once again, the changes of sign between equations (5.9) and their classical counterparts (4.32) demonstrate that the quantum and classical fields carry contragradient representations.

To conclude this section, it will be demonstrated that the canonical commutation relations for the creation and annihilation operators of a quantum field theory arise from the Hilbert space inner product of the corresponding classical field theory.

Let (Δ,H) be a unitary, irreducible representation of the Poincaré group P. H is a Hilbert space containing functions over momenta lying on a mass shell. Let k be a fixed momentum on the mass shell, and (Γ,W) be the unitary, irreducible representation of the little group $L(k)$ used in inducing (Δ,H) (see chapter 4 for details). The functions in H can be interpreted as the fields of a classical field theory. Let (Φ,F) be the Fock space representation of P built from (Δ,H) as outlined at the beginning of this section.

Up until this point, the exact nature of the quantum fields discussed has not been specified: they have just been any (infinitely differentiable) operator-valued functions. There has also been no connection made between the representation carried by Fock space and the transformation rules of the quantum field: for example, covariant fields (which carry a reducible representation) can be defined on a Fock space built from an irreducible Hilbert space representation.

Now consider a set of momentum space annihilation and creation operators $a_b(p)$ and $a_b(p)^\dagger$ where p takes all values on the mass shell, and b ranges over $1,2,\ldots,\dim W$ (the adjoint is defined with respect to the Fock space inner product). Suppose the annihilation operators transform like ϕ in (5.8), using the same representation (Γ,W) as the functions in H.

The definition of these annihilation and creation operators is fraught with difficulties, and requires some comment. For example, let Ω be the "vacuum state" vector in F, which is a vector of unit norm lying entirely in the H_0 subspace of F. The annihilation operator $a_b(p)$ yields the null vector when acting on Ω, while the creation operator $a_b(p)^\dagger$ is usually thought of as acting on Ω to produce a vector in the "single-particle" subspace of F containing just H. This single-particle vector is an eigenfunction of the $\Phi(P_\lambda)$ operators with eigenvalues p_λ. Unfortunately, such an eigenfunction ψ must satisfy

$$\Phi(P_\lambda)\psi(p') = p_\lambda \psi(p')$$

for all p' on the mass shell. Using (5.5a) on the left hand side, this yields

$$(p'_\lambda - p_\lambda)\psi(p') = 0$$

for all p' on the mass shell. This equation cannot be satisfied by any function in H except the zero function. It is non-trivially satisfied only by generalised functions such as the delta function

$$\psi(p') = \delta^d(p' - p).$$

But these functions have infinite norm, and so do not belong to H.

Hence the creation operators are not actually operators on F, since their action yields objects outside of F. The same is true for the annihilation operators. For this reason, it is normal to regard the annihilation and creation operators as operator-valued generalised functions and to "smear" them with suitable functions as operator-valued distributions. Nevertheless, it is possible to carry on as if $a_b(p)$ and $a_b(p)^\dagger$ were Fock space operators, with the knowledge that it is always possible to smear them with suitable functions in order to check results more rigorously.

With these comments in mind, it will come as no surprise to find generalised functions in the canonical commutation relations.

Suppose the inner product in W corresponds to a metric tensor with components η^{ab}. That is, for $u,v \in W$,

$$\langle u , v \rangle_W = u_a{}^* \eta^{ab} v_b \,. \tag{5.11}$$

Since (Γ,W) is unitary, the inner product must be Hermitean, so $(\eta^{ab})^* = \eta^{ba}$.

For any arbitrary vector ψ in H, define the Fock space operator $\tilde{\psi}$ by

$$\tilde{\psi} = \int d\mu(p)\, \psi_a(p)^* \eta^{ab}\, a_b(p) \;, \tag{5.12a}$$

where μ is the real invariant measure on the mass shell (see equation (4.30) in section 4.3). The F-adjoint operator

$$\tilde{\psi}^\dagger = \int d\mu(p)\, \psi_a(p) \eta^{ba}\, a_b(p)^\dagger \;, \tag{5.12b}$$

acts on the vacuum state vector Ω in F to "create" the single-particle state which is the natural embedding of $\psi \in H$ in F. $\tilde{\psi}$ itself annihilates the vacuum state vector.

For two vectors ψ and ϕ in H, the Fock space inner product of $\tilde{\psi}^\dagger\Omega$ and $\tilde{\phi}^\dagger\Omega$ is just

$$\begin{aligned}\langle\, \tilde{\psi}^\dagger\Omega\, , \tilde{\phi}^\dagger\Omega\,\rangle &= \langle\, \psi\, , \phi\,\rangle_H \\ &= \int d\mu(p)\; \psi_a(p)^*\eta^{ab}\; \phi_b(p)\ ,\end{aligned}$$

by equations (4.29) and (5.11), where the integration is over the whole mass shell. The left hand side of this relation can be rewritten as $\langle\, \Omega\, , \tilde{\psi}\tilde{\phi}^\dagger\Omega\,\rangle$, which, since $\tilde{\psi}\Omega = 0$, can also be written $\langle\, \Omega\, , [\, \tilde{\psi}, \tilde{\phi}^\dagger\,]\Omega\,\rangle$. Here, $[\,\cdot\, , \cdot\,]$ indicates either commutator or anti-commutator, whichever is suitable. Expanding this using (5.12) gives

$$\begin{aligned}&\langle\, \Omega\, ,\int\!\!\int d\mu(p)d\mu(p')\psi_a(p)^*\eta^{ab}\phi_c(p')\eta^{dc}\; [\, a_b(p)\, , a_d(p')^\dagger\,]\Omega\,\rangle \\ &\qquad = \int d\mu(p)\; \psi_a(p)^*\eta^{ab}\; \phi_b(p).\end{aligned}$$

Taking $\langle\, \Omega\, , \Omega\,\rangle = 1$, this requirement is satisfied if

$$\int d\mu(p')\phi_c(p')\eta^{dc}\; [\, a_b(p)\, , a_d(p')^\dagger\,] = \phi_b(p). \tag{5.13}$$

It is convenient to use the version (4.30b) of $d\mu(p)$, for which

$$d\mu(p) = \frac{d^{\,d-1}\mathbf{p}}{2p^0}$$

(recall that $\mathbf{p}$ is the $(d-1)$-dimensional spatial part of p, and that $p^0 = \sqrt{m^2c^2 + \mathbf{p}^2}$). If η_{ab} are the components of the inverse of the metric tensor, so that

$$\eta_{ab}\,\eta^{bc} = \eta^{cb}\,\eta_{ba} = \delta^{\,b}_{\,a}\,,$$

then (5.13) is satisfied by taking

$$[\, a_b(p)\, , a_c(p')^\dagger\,] = 2\eta_{bc}\, p^0\delta^{\,d-1}(\, \mathbf{p} - \mathbf{p'}\,). \tag{5.14}$$

Equation (5.14) gives the canonical commutation relations for the annihilation and creation operators of a quantised W-vector field. As has been demonstrated, these relations can be derived from the inner product of H, which is the Hilbert space of the classical W-vector field theory. The derivation just presented will prove useful in finding commutation rules in a supermultiplet in chapter 8.

5.2 Construction of free quantum fields

Covariant quantum fields are commonly used to describe elementary particles in physics. As mentioned in section 4.2, the covariant field representations of the Poincaré group P are reducible. On the other hand, *elementary* particles should, almost by definition, correspond to irreducible representations of P. So the construction of free quantum fields can be viewed as a matter of combining the covariant and irreducible representations of P in some way.

Weinberg's approach is to take momentum space annihilation and creation operators carrying an irreducible representation. These operators are interpreted as creating and annihilating momentum eigenstates of an elementary particle. The covariant field operator for the same elementary particle should then be some linear combination of these annihilation and creation operators.

Following this procedure, let $a_\alpha(p)$ and $a_\alpha(p)^\dagger$ be annihilation and creation operators respectively for some elementary particle of mass $m \geq 0$ (the massive and massless cases will be treated together as far as possible). The momentum p takes values on the mass shell characterised by $p \cdot p = m^2c^2$ and $p^0 > 0$, where c is the speed of light. Suppose that the annihilation operators carry a unitary irreducible representation of P similar to that of equation (5.8). This representation is obtained using a finite-dimensional representation (Γ,W) of the little group $L(k)$ of the Lorentz group L, where k is some fixed momentum on the mass shell. (Recall that in section 4.3, $L(k)$ was found to be $\mathrm{Spin}(d-1)$ for $m > 0$ and $\mathrm{I}(d-2) \circledS \mathrm{Spin}(d-2)$ for $m = 0$). The indices α,β,γ will take values $1,2,\ldots,\dim W$ for the whole of this section.

For brevity, the little group element in (5.8) will be written as

$$R_p(\Lambda) = B(\Lambda p)^{-1} \Lambda B(p) \tag{5.15}$$

for all Λ in L and all p on the mass shell.

The representation carried by the annihilation operators then reads

$$\Phi(t,\Lambda) a_\alpha(p) \Phi(t,\Lambda)^{-1} = e^{-i\Lambda p\cdot t/\hbar} \Gamma(R_p(\Lambda)^{-1})_{\alpha\beta}\, a_\beta(\Lambda p), \tag{5.16a}$$

for all (t,Λ) in P and all p on the mass shell. Recall that $\Phi(t,\Lambda)$ is a unitary operator and $\Gamma(R_p(\Lambda)^{-1})_{\alpha\beta}$ is a number. Taking adjoints on each side yields the transformation rule for the creation operators:

$$\Phi(t,\Lambda) a_\alpha(p)^\dagger \Phi(t,\Lambda)^{-1} = e^{i\Lambda p\cdot t/\hbar} \Gamma^*(R_p(\Lambda)^{-1})_{\alpha\beta}\, a_\beta(\Lambda p)^\dagger. \tag{5.16b}$$

In (5.16b), Γ^* is the "complex conjugate" of the representation Γ, defined by

$$\Gamma^*(R)_{\alpha\beta} = (\Gamma(R)_{\alpha\beta})^* \qquad \text{for all } R \text{ in } L(k).$$

As the next step in the field construction, let the set of fields ψ_a form a covariant V-vector quantum field over Minkowski space-time, where V is a finite-dimensional space carrying a representation Δ of the Lorentz group L. The indices a,b will take values $1,2,\ldots,\dim V$ for the rest of this section. The vector field ψ has component fields ψ_a which transform according to (5.4) by

$$\Phi(t,\Lambda)\psi_a(x)\Phi(t,\Lambda)^{-1} = \Delta(\Lambda^{-1})_{ab}\psi_b(\Lambda x + t). \tag{5.17}$$

The main task is to express the covariant field as a linear combination of the annihilation and creation operators in a manner consistent with the transformation rules (5.16) and (5.17). The next proposition shows that there is very little room for choice in this linear combination, and provides the starting point for determining the field equations.

Proposition 5.1 Let ψ be a covariant quantum field over space-time constructed from momentum space annihilation and creation operators $a(p)$ and $a(p)^\dagger$. Suppose the covariant field transforms by (5.17) and the momentum space operators by (5.16) under the Fock space representation Φ of the Poincaré group. Then the operators $\psi_a(x)$ are given by

$$\begin{aligned}\psi_a(x) &= N\int d^dp\ \delta(p^2-m^2c^2)\theta(p^0)\Delta(B(p))_{ab} \\ &\qquad\times\{\mathrm{e}^{-ip\cdot x/\hbar}\,u_{b\alpha}\,a_\alpha(p)+\mathrm{e}^{ip\cdot x/\hbar}\,v_{b\alpha}\,a_\alpha(p)^\dagger\}\end{aligned} \tag{5.18}$$

where N is a normalisation constant to be fixed in individual cases (see section 5.5), and the $\dim V\times\dim W$ matrices u and v satisfy

$$\Delta(R)u = u\,\Gamma(R) \tag{5.19a}$$

$$\Delta(R)v = v\,\Gamma^*(R) \tag{5.19b}$$

for all R in the little group $L(k)$.

Proof. First take the operator $\psi_a(x)$ to be a linear combination of the annihilation and creation operators $a_\alpha(p)$ and $a_\alpha(p)^\dagger$ over all α and all p on the mass shell. The most general combination possible is

$$\begin{aligned}\psi_a(x) &= N\int d^dp\ \delta(p^2-m^2c^2)\theta(p^0) \\ &\qquad\times\{u(x,p)_{a\beta}\,a_\beta(p)+v(x,p)_{a\beta}\,a_\beta(p)^\dagger\}\end{aligned} \tag{5.20}$$

where the integration is over the whole of the mass shell, and $d^dp\ \delta(p^2-m^2c^2)\theta(p^0)$ is the infinitesimal invariant measure. The coefficients $u(x,p)_{a\beta}$ and $v(x,p)_{a\beta}$ are components of complex $\dim V\times\dim W$ matrices. They will be essentially fixed by applying Poincaré transformations to both sides and using both (5.16) and (5.17).

First consider a translation $(t, 1)$ alone. Applying $\Phi(t, 1)$ to the covariant field operator gives

$$\begin{aligned}\Phi(t,1)\psi_a(x)\Phi(t,1)^{-1} &= \psi_a(x+t). \qquad\qquad \text{by (5.17)}\\ &= N\int d^dp\ \delta(p^2-m^2c^2)\theta(p^0)\\ &\qquad \times\ \{u(x+t,p)_{\alpha\beta}\, a_\beta(p) + v(x+t,p)_{\alpha\beta}\, a_\beta(p)^\dagger\},\end{aligned}$$

by (5.20). On the other hand, using (5.20), the operators representing the translation can be applied instead to the creation and annihilation operators:

$$\begin{aligned}&\Phi(t,1)\psi_a(x)\Phi(t,1)^{-1}\\ &\qquad = N\int d^dp\ \delta(p^2-m^2c^2)\theta(p^0)\\ &\qquad\qquad \times\{u(x,p)_{\alpha\beta}\,\Phi(t,1)a_\beta(p)\Phi(t,1)^{-1}\\ &\qquad\qquad\quad + v(x,p)_{\alpha\beta}\,\Phi(t,1)a_\beta(p)^\dagger\Phi(t,1)^{-1}\}\\ &\qquad = N\int d^dp\ \delta(p^2-m^2c^2)\theta(p^0)\\ &\qquad\qquad \times\{u(x,p)_{\alpha\beta}\ \mathrm{e}^{-ip\cdot t/\hbar}\ a_\beta(p)\\ &\qquad\qquad\quad + v(x,p)_{\alpha\beta}\ \mathrm{e}^{ip\cdot t/\hbar}\ a_\beta(p)^\dagger\} \qquad\qquad \text{by (5.16).}\end{aligned}$$

Comparing the two integrands,

$$u(x+t,p)_{\alpha\beta} = u(x,p)_{\alpha\beta}\,\mathrm{e}^{-ip\cdot t/\hbar}$$

and

$$v(x+t,p)_{\alpha\beta} = v(x,p)_{\alpha\beta}\ \mathrm{e}^{ip\cdot t/\hbar}\ .$$

From this it follows that

$$u(x,p)_{\alpha\beta} = u(0,p)_{\alpha\beta}\ \mathrm{e}^{-ip\cdot x/\hbar}$$

and

$$v(x,p)_{\alpha\beta} = v(0,p)_{\alpha\beta}\ \mathrm{e}^{ip\cdot x/\hbar}\ ,$$

so, defining

$$u(p)_{\alpha\beta} = u(0,p)_{\alpha\beta}$$

and

$$v(p)_{\alpha\beta} = v(0,p)_{\alpha\beta}\ ,$$

equation (5.20) becomes

$$\psi_a(x) = N \int d^dp\ \delta(p^2 - m^2c^2)\theta(p^0) \times \{ e^{-ip\cdot x/\hbar}\, u(p)_{a\beta}\, a_\beta(p) + e^{ip\cdot x/\hbar}\, v(p)_{a\beta}\, a_\beta(p)^\dagger \} \tag{5.21}$$

Next consider a Lorentz transformation $(0,\Lambda)$ by itself. Applying $\Phi(0,\Lambda)$ to $\psi_a(x)$ and using (5.17) and (5.21),

$$\begin{aligned}
\Phi(0,\Lambda)\psi_a(x)\Phi(0,\Lambda)^{-1} &= N \int d^dp\ \delta(p^2 - m^2c^2)\theta(p^0)\Delta(\Lambda^{-1})_{ab} \\
&\quad \times \{ e^{-ip\cdot\Lambda x/\hbar}\, u(p)_{b\alpha}\, a_\alpha(p) + e^{ip\cdot\Lambda x/\hbar}\, v(p)_{b\alpha}\, a_\alpha(p)^\dagger \} \\
&= N \int d^dp\ \delta(p^2 - m^2c^2)\theta(p^0)\Delta(\Lambda^{-1})_{ab} \\
&\quad \times \{ e^{-ip\cdot x/\hbar}\, u(\Lambda p)_{b\alpha}\, a_\alpha(\Lambda p) + e^{ip\cdot x/\hbar}\, v(\Lambda p)_{b\alpha}\, a_\alpha(\Lambda p)^\dagger \},
\end{aligned}$$

using the Lorentz invariance of the measure to transfer the Lorentz transformation from the x to the p. On the other hand, applying $\Phi(0,\Lambda)$ to the annihilation and creation operators by (5.16), and using (5.21) gives

$$\begin{aligned}
\Phi(0,\Lambda)\psi_a(x)\Phi(0,\Lambda)^{-1} &= N \int d^dp\ \delta(p^2 - m^2c^2)\theta(p^0) \\
&\quad \times \{ e^{-ip\cdot x/\hbar}\, u(p)_{a\beta}\, \Gamma(R_p(\Lambda)^{-1})_{\beta\gamma}\, a_\gamma(\Lambda p) \\
&\quad\quad + e^{ip\cdot x/\hbar}\, v(p)_{a\beta}\, \Gamma^*(R_p(\Lambda)^{-1})_{\beta\gamma}\, a_\gamma(\Lambda p)^\dagger \}.
\end{aligned}$$

Comparing the last two equations, the integrands must be equal, so that

$$\Delta(\Lambda^{-1})_{ab}\, u(\Lambda p)_{b\gamma} = u(p)_{a\beta}\, \Gamma(R_p(\Lambda)^{-1})_{\beta\gamma}$$

and

$$\Delta(\Lambda^{-1})_{ab}\, v(\Lambda p)_{b\gamma} = v(p)_{a\beta}\, \Gamma^*(R_p(\Lambda)^{-1})_{\beta\gamma}.$$

Writing these as matrix equations, and multiplying on the left by $\Delta(\Lambda)$ and on the right by $\Gamma(R_p(\Lambda))$ and $\Gamma^*(R_p(\Lambda))$ as appropriate gives

$$u(\Lambda p)\Gamma(R_p(\Lambda)) = \Delta(\Lambda)u(p) \tag{5.22a}$$

$$v(\Lambda p)\Gamma^*(R_p(\Lambda)) = \Delta(\Lambda)v(p) \tag{5.22b}$$

Two special cases are important. First, take $p = k$ (the fixed momentum) and $\Lambda = B(q)$, where q is some arbitrary momentum on the mass shell. Then $\Lambda p = q$. The little group element $R_p(\Lambda) = R_k(B(q))$ can be evaluated by (5.15), and gives (recall $B(k) = 1$),

$$\begin{aligned} R_k(B(q)) &= B(B(q)k)^{-1}B(q)\,1 \\ &= 1. \end{aligned}$$

Thus equations (5.22) yield

$$u(q) = \Delta(B(q))u(k) \tag{5.23a}$$

$$v(q) = \Delta(B(q))v(k) \tag{5.23b}$$

for all mass shell momenta q. So, knowing the coset representatives $B(q)$, the matrices $u(q)$ and $v(q)$ are all fixed by the values at $q = k$. Define the constant $\dim V \times \dim W$ matrices u and v by

$$\begin{aligned} u_{\alpha\beta} &= u(k)_{\alpha\beta} \\ v_{\alpha\beta} &= v(k)_{\alpha\beta}. \end{aligned}$$

Equation (5.18) in the proposition then follows directly from (5.21) and (5.23).

For the second special case, take $p = k$ and $\Lambda = R$, where R is an arbitrary element of the little group $L(k)$. Then $\Lambda p = k$, and $R_p(\Lambda)$ is given by (5.15) as

$$\begin{aligned} R_p(\Lambda) = R_k(R) &= B(Rk)^{-1}RB(k) \\ &= R. \end{aligned}$$

Equations (5.22) then yield equation (5.19) directly. □

Comment. There is no need for the "creation" part of the field ψ_a to be the adjoint of the "annihilation" part – unless the field describes a particle which is its own anti-particle. In general, the creation operator

$a_\alpha(p)^\dagger$ for the particle can be replaced in (5.18) by the creation operator $b_\alpha(p)^\dagger$ for the anti-particle.

Equation (5.18) is the familiar expression for a covariant field as a Fourier transform of creation and annihilation operators for momentum eigenstates. It is worth emphasising how much of the expression has been closely governed by the choice of transformation laws: the mass shell itself, and the exponential and matrix parts of the coefficients all owe their form to the representations. Note also that the relations (5.19) essentially fix the matrices u and v. This will be seen more clearly in the next section when the field equations for the covariant field ψ are extracted from these relations. Some examples of the field construction will be given in the next chapter.

5.3 Particle content and field equations

Equation (5.18) gave the expansion of a covariant quantum field in terms of annihilation and creation operators. Associated with the covariant field, there is a representation Δ of the Lorentz group L (see equation (5.17)), and associated with the annihilation and creation operators, there are representations Γ and Γ^* of the little group. Perhaps not surprisingly, it is not possible to construct a covariant field for an arbitrary Δ from annihilation and creation operators for an arbitrary Γ. Only certain combinations of Δ and Γ are allowed. Equations (5.19), as well as fixing the coefficient matrices u and v of the field expansion, also provide a compatibility criterion for Δ and Γ. In other words, equations (5.19) place restrictions on the possible elementary particle content of a covariant quantum field.

For massless fields, it was stated in section 4.3 that the representation Γ was trivial on the inhomogeneous part of the little group. When this fact is used in equations (5.19), the results are the first order field equations for massless fields.

These two consequences of equations (5.19) are the main subjects of this section. The conventions for the indices in this section are as follows: λ,μ,ν,ρ range over $0,1,\ldots,d-1$; i,j,k over $1, 2, \ldots, d-2$; a,b,c,d over $1, 2, \ldots, \dim V$; and $\alpha,\beta,\gamma,\delta$ over $1, 2, \ldots, \dim W$, where V and W are the finite-dimensional carrier spaces of the representations Δ and Γ.

Before considering the particle content and massless field equations, there is another field equation which can be traced back to the representation theory. As was pointed out in section 4.2, the Klein-Gordon equation is a necessary condition for a covariant field to be in an irreducible representation. The covariant quantum field of (5.18) has been constructed from an irreducible representation, so it should be possible to check that the Klein-Gordon equation is satisfied. Applying the operator $\Phi(P_\lambda)$ representing the translation generator to the covariant field operator $\psi_a(x)$ twice, using equation (5.5*a*),

$$[\Phi(P^\lambda), [\Phi(P_\lambda), \psi_a(x)]] = -\hbar^2 \frac{\partial^2}{\partial x^\lambda \partial x_\lambda} \psi_a(x).$$

Evaluating the right-hand side using the expansion (5.18),

$$\begin{aligned} -\hbar^2 \frac{\partial^2}{\partial x^\lambda \partial x_\lambda} \psi_a(x) \\ &= -\hbar^2 N \int d^d p \ \delta(p^2 - m^2c^2)\theta(p^0)\Delta(B(p))_{ab} \\ &\quad \times \frac{\partial^2}{\partial x^\lambda \partial x_\lambda} \{ e^{-ip\cdot x/\hbar} u_{b\alpha} a_\alpha(p) + e^{ip\cdot x/\hbar} v_{b\alpha} a_\alpha(p)^\dagger \} \\ &= -\hbar^2 N \int d^d p \ \delta(p^2 - m^2c^2)\theta(p^0)\Delta(B(p))_{ab} \\ &\quad \times \frac{-p^\lambda p_\lambda}{\hbar^2} \{ e^{-ip\cdot x/\hbar} u_{b\alpha} a_\alpha(p) + e^{ip\cdot x/\hbar} v_{b\alpha} a_\alpha(p)^\dagger \} \\ &= m^2c^2 \psi_a(x), \end{aligned}$$

since p is restricted to the mass shell, where $p^2 = m^2c^2$. Thus the Klein-Gordon equation is automatically satisfied by the covariant quantum field constructed in section 5.2, as expected.

Note that each component ψ_a of ψ individually satisfies the Klein-Gordon equation. This is because each of them individually carries a one-dimensional irreducible representation of the space-time translation group.

Returning to the investigation of equations (5.19), it will be convenient to treat massive and massless fields separately. The particle content of a field is easiest to find for massive fields, while the extra field equations arise for massless fields.

In section 4.3, the little group $L(k)$ for $k = (mc,0,\ldots,0)$ (a convenient orbit representative for the mass shell of $m > 0$) was shown to be Spin(d–1). This group is simple, compact and simply connected, so its finite-dimensional representations are easily obtained from those of the corresponding complex simple Lie algebra. For even dimensions $d = 2l$ ($l \geq 2$), the complex simple Lie algebra is B_{l-1}, and for odd dimensions $d = 2l + 1$ ($l \geq 2$), it is D_l. The unitary irreducible representations of B_l and D_l can be labelled by a collection of l natural numbers $n = \{n_1,\ldots,n_l\}$ (see, for example, Cornwell 1984 vol. II). The same labelling will be used for the corresponding representations of Spin(d–1) as well.

Spin(d–1) is a subgroup of the Lorentz group (or, more fully, of the universal covering group L of the connected component of the Lorentz group O(d–1,1)). So the representation Δ of L associated with the covariant quantum field ψ provides a representation of Spin(d–1) as well. Generally, Δ will be a reducible representation of Spin(d–1), and , as Spin(d–1) is compact, it will be completely reducible. That is, as a representation of Spin(d–1), Δ is equivalent to a direct sum

$$\Gamma^{\,n} \oplus \Gamma^{\,n'} \oplus \ldots \oplus \Gamma^{\,n''},$$

where $\{\Gamma^n, \Gamma^{n'}, \ldots, \Gamma^{n''}\}$ is a set of irreducible representations labelled by n,n',n'' in $\mathbb{N}^l$. This is known as the branching rule for decomposing Δ on Spin(d–1). For all R in Spin(d–1) then, $\Delta(R)$ can be written

$$\Delta(R) = S\,(\Gamma^n(R) \oplus \Gamma^{n'}(R) \oplus \ldots \oplus \Gamma^{n''}(R))S^{-1}, \tag{5.24}$$

where S is a $\dim V \times \dim V$ matrix similarity transformation.

Using this decomposition in equation (5.19a), and recalling that Γ is the particular irreducible representation of Spin(d–1) for the elementary particles described by the field,

$$(\Gamma^n(R) \oplus \Gamma^{n'}(R) \oplus \ldots \oplus \Gamma^{n''}(R))S^{-1}u = S^{-1}u\,\Gamma(R) \tag{5.25}$$

for all R in Spin(d–1). $S^{-1}u$ is a dim $V \times$ dim W matrix which can be divided by horizontal partitions corresponding to the direct sum components into a stack of smaller matrices with dim W columns. Denote the smaller matrix corresponding to Γ^n by $(S^{-1}u)^n$ and so on. Pictorially,

$$\begin{bmatrix} \\ S^{-1}u \\ \\ \end{bmatrix} = \begin{bmatrix} (S^{-1}u)^n \\ (S^{-1}u)^{n'} \\ \vdots \\ (S^{-1}u)^{n''} \end{bmatrix}.$$

Equation (5.25) decomposes to give a set of matrix equations: for all R in Spin(d–1),

$$\begin{aligned} \Gamma^n(R)(S^{-1}u)^n &= (S^{-1}u)^n\,\Gamma(R) \\ \Gamma^{n'}(R)(S^{-1}u)^{n'} &= (S^{-1}u)^{n'}\,\Gamma(R) \\ &\vdots \\ \Gamma^{n''}(R)(S^{-1}u)^{n''} &= (S^{-1}u)^{n''}\,\Gamma(R). \end{aligned}$$

Since all of the representations in these equations are irreducible, Schur's lemma applies. Each of the matrices $(S^{-1}u)^n$, ..., $(S^{-1}u)^{n''}$ must be either zero, or else square (dim $W \times$ dim W) and non-singular. In the second case, Γ and the corresponding representation, Γ^m say, from the direct sum are equivalent. A similarity transformation can then be performed on Γ so that they are identical. $(S^{-1}u)^m$ is then a constant multiple of the dim $W \times$ dim W identity matrix. This requirement is most easily met by taking

$$u_{\alpha\beta} = (S)^m{}_{\alpha\beta}\,, \tag{5.26}$$

where $(S)^m$ is the appropriate submatrix of S partitioned vertically in the same fashion as $S^{-1}u$.

Equation (5.26) virtually fixes the coefficient matrix u of the expansion (5.18). A possible complication arises if the representation Γ

appears more than once in the decomposition, but it creates no real difficulty, None of the examples examined later will have this feature, except for the massless spinor field, but even in that example, the difficulty is avoided. Once conventions about the exact form of the representations have been established, the only remaining freedom in u is a multiplicative constant.

In exactly the same manner, the coefficient matrix v for the creation operators in the field expansion (5.18) can be found from (5.19*b*). If Γ^* is transformed so that it is identical with one of the representations, $\Gamma^{m'}$ say, from the direct sum (5.24), then v can be taken as

$$v_{\alpha\beta} = (S)^{m'}{}_{\alpha\beta} , \tag{5.27}$$

which is again fixed to within a constant.

These constants of proportionality for u and v need not be the same. Using arguments arising from crossing symmetry and the spin-statistics connection (see Weinberg 1964*a*,*b*), it is possible to fix the *relative* constant between u and v up to a phase factor.

Of course, Γ and Γ^* may be equivalent representations. This occurs if Γ is potentially real or pseudo-real (see, for example, Cornwell 1984 vol. I p.127). Suppose that this is the case. If the representation corresponding to Γ appears only once in the branching rule for Δ, then this is also the representation corresponding to Γ^*. If Γ is pseudo-real, it will not be possible to transform *both* Γ and Γ^* simultaneously to be identical to the same representation in the branching rule, so one of (5.26) or (5.27) must be changed. If Γ is either potentially real or pseudo-real, then there exists a non-singular, unitary matrix Z such that, for all R in Spin(d–1),

$$\Gamma^*(R) = Z^{-1}\Gamma(R)Z,$$

and

$$Z^*Z = \pm 1$$

(+1 for potentially real Γ and –1 for pseudo-real Γ). Replacing Γ^* in (5.19*b*) gives

$$\Delta(R)v = v\, Z^{-1}\Gamma(R)Z,$$

for all R in Spin(d–1). If (5.19*a*) is satisfied by u, then (5.19*b*) can be satisfied by taking $v = u\, Z$. Thus, if Γ is potentially real or pseudo-real and u is fixed by (5.26), then (5.27) should be replaced by

$$v_{\alpha\beta} = (S)^{m'}{}_{\alpha\gamma}\, Z_{\gamma\beta}. \tag{5.28}$$

However, Γ may also be essentially complex (the only remaining possibility), in which case Γ and Γ^* are inequivalent. This occurs if Γ corresponds to the representation $\{n_1,\ldots,n_l\}$ of D_l, where $l \geq 3$ and odd, and $n_{l-1} \neq n_l$. An important example of this is when Γ is one of the "spinor" representations $\{0, \ldots, 0, 0, 1\}$ or $\{0, \ldots, 0, 1, 0\}$ for $d = 7, 11, 15, \ldots$. In these cases, v must be fixed separately from u by (5.27).

Besides fixing the coefficient matrices u and v, the above discussion has an important converse implication. In order for (5.19) to be satisfied, the representation Γ must be equivalent to *at least one* of the representations $\Gamma^{n}, \Gamma^{n'}, \ldots, \Gamma^{n''}$ in the branching rule for Δ. That is, Δ must contain Γ at least once for the field construction (5.18) to be possible at all. If Γ is essentially complex, then Δ must contain Γ^* separately as well as Γ. Fortunately, this last requirement is met in all cases of interest.

So, not all combinations of Δ and Γ are possible. A covariant quantum field transforming by Δ can describe only those elementary particles transforming by the representations contained in the branching rule for Δ on Spin(d–1). This is the promised restriction on the particle content of a field.

For fields describing massless particles, there is a very similar property, and an important difference. In section 4.3, the little group $L(k)$ for $k = (k^0,0,\ldots,0,k^0)$ with $k^0 > 0$ (a convenient orbit representative for the forward light cone) was found to be isomorphic to I(d–2) Ⓢ Spin(d–2). It should be emphasised that I(d–2) here consists of Lorentz transformations, and is not a subgroup of the space-time translation group. The only finite-dimensional, unitary, irreducible representations of $L(k)$ are those of the compact part Spin(d–2) extended to $L(k)$ by representing the inhomogeneous part I(d–2) trivially.

The finite-dimensional, irreducible representations of Spin(d–2) arise, as for the massive fields, from the representations $n = \{n_1,\ldots,n_l\} \in \mathbb{N}^l$ of the simple, complex Lie algebras B_l and D_l. In this case, for even dimensions $d = 2l$ ($l \geq 3$), the appropriate algebra is D_{l-1}, and for odd dimensions $d = 2l + 1$ ($l \geq 2$), it is B_{l-1}. For $d = 4$, the Lie algebra of Spin(d–2) is just one-dimensional, and the representations are the usual

$$\Gamma^n(\theta) = e^{in\theta/2},$$

where θ parameterises Spin(2), and n takes any integer value.

Essentially complex representations occur for $d = 4$ and $n \neq 0$, and for D_l with odd $l \geq 3$ and $n_{l-1} \neq n_l$. The most interesting examples of this are the "spinor" representations $\{ 0, \ldots, 0, 0, 1 \}$ or $\{ 0, \ldots, 0, 1, 0 \}$ for $d = 8, 12, 16, \ldots$ (note that these differ from the corresponding dimensions for massive fields).

As usual, let Δ be the covariant field's representation and Γ the irreducible representation of $L(k)$ carried by the annihilation operators.

Equations (5.19) can each be broken into two parts. Take (5.19*a*) first. For all R in the Spin(d–2) part of $L(k)$, the coefficient matrix u of the annihilation operators must satisfy

$$\Delta(R)u = u\,\Gamma(R). \tag{5.29a}$$

For all T in the inhomogeneous part I(d–2) of $L(k)$, $\Gamma(T) = 1$, so (5.19*a*) gives

$$\Delta(T)u = u. \tag{5.29b}$$

Similarly, equation (5.19*b*) for the coefficient matrix v of the creation operators yields

$$\Delta(R)v = v\,\Gamma^*(R) \tag{5.30a}$$

for all R in Spin(d–2), and

$$\Delta(T)v = v \tag{5.30b}$$

for all T in I(d–2).

Equations (5.29*a*) and (5.30*a*) can be treated in exactly the same way as (5.19*a*) and (5.19*b*) were for massive fields. Δ reduces completely on Spin(d–2), and the representation Γ (and Γ^* as well, if Γ is essentially complex) must be contained in the branching rule if (5.29*a*) and (5.30*a*) are to be satisfied. The coefficient matrices u and v are then given by the equivalents of equations (5.26), (5.27) and (5.28).

However, equations (5.29*b*) and (5.30*b*) must be satisfied too, and this places extra restrictions on which representations in the branching rule are suitable. In fact, the possible particle content of a massless field is even more restricted than that of a similar massive field. The most outstanding example of this is the covariant vector field. The massive field may be constructed from annihilation and creation operators transforming by a (d–1)-dimensional vector representation of Spin(d–1) or by a trivial scalar representation. As will be seen in chapter 6, the massless vector field may describe only scalar particles, which, at first sight, is at odds with the use of the vector gauge potential to describe photons. However, the gauge potential is not a true covariant field (see chapter 6), so there is no conflict.

Equations (5.29*b*) and (5.30*b*) are most conveniently examined in the corresponding algebraic form. Since they are the same, only (5.29*b*) need be considered. Recall from section 4.3 (equation (4.26)) that the generators of the inhomogeneous part of $L(k)$ are

$$J_{+i} = (J_{0i} + J_{d-1\,i})/\sqrt{2},$$

and note that, since $k = (k^0,0,\ldots,0,k^0)$, J_{+i} can be re-expressed as

$$J_{+i} = k^{\lambda}J_{\lambda i}/k^0\sqrt{2}. \tag{5.31}$$

It is easy to check that (5.31) in fact defines the generators of the inhomogeneous part of $L(k)$ for any choice of k on the forward light cone, not just the one taken here.

Upon differentiation at the origin of $\mathrm{I}(d-2)$, equation (5.29*b*) becomes

$$\Delta(J_{+i})u = 0, \tag{5.32a}$$

which, by (5.31) is

$$k^{\lambda}\Delta(J_{\lambda i})u = 0. \tag{5.32b}$$

Before using (5.32) to derive the field equations, it is necessary to establish a preliminary result.

Proposition 5.2 If the coefficient matrix u satisfies (5.29*a*), then the columns of u are eigenvectors of $\Delta(J_{0d-1})$, all having the same eigenvalue. That is,

$$\Delta(J_{0d-1})u = i\hbar Q\, u, \tag{5.33}$$

where Q is a constant ($i\hbar$ has been extracted for later convenience).

Proof. In algebraic form, (5.29*a*) is

$$\Delta(J_{ij})u = u\,\Gamma(J_{ij}).$$

As $[J_{ij}, J_{0d-1}] = 0$, it follows that

$$\Delta(J_{ij})\Delta(J_{0d-1})u = \Delta(J_{0d-1})u\,\Gamma(J_{ij}).$$

Thus $\{\Delta(J_{0d-1})u\}$ satisfies (5.29*a*) as well as u. But, as shown earlier, (5.29*a*) fixes u to within a constant factor, so $\{\Delta(J_{0d-1})u\}$ must be proportional to u. □

Comment. If the branching rule for Δ contains Γ more than once, then (5.33) may not hold, since (5.29*a*) no longer fixes u to within a constant. However, for the main fields of interest, this complication does not arise.

It should be possible to determine the constant Q in (5.33) using the weights of Δ and the highest weight of Γ. Indeed, this should lead to a generalisation of Weinberg's result for 4 dimensions, which is that massless particles of helicity j ($2j$ an integer) can be described only by covariant fields with Lorentz group representations (s,s') satisfying $s - s' = j$. Nevertheless, in the examples to be considered, it is straightforward enough to find Q explicitly.

Proposition 5.3 Let ψ be a covariant field transforming by the representation Δ of L, and constructed, as in (5.18), from elementary particle operators for massless particles transforming by the representation Γ of $L(k)$. Then ψ satisfies the first-order field equations

$$\Delta(J_{\lambda\mu})_{ab}\partial^{\mu}\psi_b - i\hbar Q\,\partial_{\lambda}\psi_a = 0, \tag{5.34}$$

where Q is a constant, fixed by (5.33), and $\partial_{\lambda} = \partial/\partial x^{\lambda}$.

Proof. Consider the left-hand side of (5.32*b*) with the index i replaced consecutively by 0 and d–1:

$$\begin{aligned} k^{\lambda}\Delta(J_{\lambda 0})u &= k^{d-1}\Delta(J_{d-1\,0})u, \\ &= -k_0\Delta(J_{0\,d-1})u, \end{aligned}$$

$$\begin{aligned} k^{\lambda}\Delta(J_{\lambda\, d-1})u &= k^{0}\Delta(J_{0\, d-1})u\,, \\ &= -k_{d-1}\Delta(J_{0\, d-1})u\,. \end{aligned}$$

As $k_i = 0$, (5.32) and the last two equations can be combined as

$$k^{\lambda}\Delta(J_{\lambda\mu})u \quad = -k_{\mu}\,\Delta(J_{0\, d-1})u\,.$$

By (5.33), then,

$$k^{\lambda}\Delta(J_{\lambda\mu})u + i\hbar\, Q\, k_{\mu}\, u = 0. \tag{5.35}$$

This equation needs to be generalised to any light-cone momentum p.

Using the Lie products (4.5), it is possible to find the adjoint representation of L carried by the $J_{\lambda\mu}$:

$$\Lambda\, J_{\lambda\mu}\,\Lambda^{-1} = \Lambda^{-1}{}_{\lambda}{}^{\nu}\,\Lambda^{-1}{}_{\mu}{}^{\rho}\,J_{\nu\rho}$$

for all Λ in L. In particular, for the coset representatives $B(p)$, this implies that

$$B(p)^{\lambda}{}_{\nu}\,\Delta(J_{\lambda\mu})\Delta(B(p) = B(p)_{\mu}{}^{\rho}\,\Delta(B(p))\Delta(J_{\nu\rho}). \tag{5.36}$$

Using (5.23) for the definition of $u(p)$,

$$\begin{aligned} p^{\lambda}\Delta(J_{\lambda\mu})u(p) &= B(p)^{\lambda}{}_{\nu}\,k^{\nu}\,\Delta(J_{\lambda\mu})\Delta(B(p))u \\ &= k^{\nu}B(p)_{\mu}{}^{\rho}\,\Delta(B(p))\Delta(J_{\nu\rho})u && \text{by (5.36)} \\ &= B(p)_{\mu}{}^{\rho}\,\Delta(B(p))\,\{-i\hbar\, Q\, k_{\rho}\}u && \text{by (5.35)} \\ &= -i\hbar\, Q\, p_{\mu}\, u(p) && \text{by (5.23).} \end{aligned}$$

So the generalisation of (5.35) is very similar in form:

$$p^{\lambda}\Delta(J_{\lambda\mu})u(p) + i\hbar\, Q\, p_{\mu}\, u(p) = 0 \tag{5.37}$$

for all light-cone momenta p, where Q is fixed by (5.33). Exactly the same equation holds for $v(p)$ as well.

From the expansion (5.18) with $m = 0$, the derivative of ψ_a is

$$\begin{aligned} \frac{\partial}{\partial x^{\lambda}}\psi_a(x) &= N\int d^dp\;\delta(p^2)\theta(p^0)\Delta(B(p))_{ab} \\ &\quad\times\frac{\partial}{\partial x^{\lambda}}\{\,e^{-ip\cdot x/\hbar}\,u_{b\alpha}\,a_{\alpha}(p) + e^{ip\cdot x/\hbar}\,v_{b\alpha}\,a_{\alpha}(p)^{\dagger}\,\} \end{aligned}$$

$$= N \int d^d p \; \delta(p^2)\theta(p^0) \frac{-ip_\lambda}{\hbar} \{ e^{-ip\cdot x/\hbar} u(p)_{b\alpha} a_\alpha(p) + e^{ip\cdot x/\hbar} v(p)_{b\alpha} a_\alpha(p)^\dagger \}.$$

Thus (5.37) gives the field equation

$$\Delta (J_{\lambda\mu})_{ab} \partial^\lambda \psi_b + i\hbar Q\, \partial_\mu \psi_a = 0,$$

from which (5.34) follows by simple rearrangement. □

Equation (5.34) gives the well-known first order field equations (Dirac equation, Maxwell's equations etc) for free fields, as will be seen in the examples in the next chapter. In addition it imposes further constraints on the possible particle content of a covariant field. This too will be apparent from the examples in the next chapter.

Chapter 6

Examples: free relativistic quantum fields in four or more dimensions

In this chapter, some specific examples of the general construction of chapter 5 will be given. Sections 6.1 to 6.4 describe in detail the construction of the scalar, spinor, vector and antisymmetric second-rank tensor fields. The expansion of these fields as Fourier transforms of the corresponding annihilation and creation operators is explicitly derived. All of the examples here are for massless particles, since these have the added interest of first-order field equations. These are derived, and turn out to be reassuringly familiar. In section 6.5, the gauge potential for the second-rank tensor field is considered. Interestingly, Lorentz transformations are found to induce a gauge transformation in the gauge potential. A comparison between this construction and the light-cone gauge formalism is made in section 6.6.

In this chapter, as usual, Minkowski space-time, the Poincaré group P, the Lorentz group L, and so on are all d-dimensional, with $d \geq 4$. The metric tensor for Minkowski space-time is

$g = \mathrm{diag}(1,-1,\ldots,-1)$. More details and precise definitions of P and L can be found in section 4.1. Indices λ, μ, ν, ρ will take values $0,1,\ldots d-1$, and i, j, k will take values $1,2,\ldots,d-2$. Summation signs will be omitted, summation over repeated indices being implied unless otherwise indicated.

6.1 The scalar field

The covariant scalar field representation was given in example 4.1 in section 4.2 for a classical relativistic field. A quantised scalar field transforms using the trivial representation of L: $\Delta(\Lambda) = 1$ for all Λ in L. It thus has a single component ϕ and transforms under the Fock space representation Φ of P by a rule contragradient to (4.10), namely

$$\Phi(t,\Lambda)\phi(x)\Phi(t,\Lambda)^{-1} = \phi(\Lambda x + t)$$

for all (t,Λ) in P. On the little group $L(k)$, Δ is of course trivial, so the only possible representation Γ of $L(k)$ for the annihilation and creation operators is the trivial one. The corresponding irreducible scalar representation of P was given for classical fields in example 4.6. Let the annihilation and creation operators be $a(p)$ and $b(p)^\dagger$ with p on the forward light cone. (Choosing a and b independent will give a field which is not self-adjoint). Equations (5.16) for the transformation of these operators become

$$\Phi(t,\Lambda)a(p)\Phi(t,\Lambda)^{-1} = e^{-i\Lambda p\cdot t/\hbar}a(\Lambda p),$$

$$\Phi(t,\Lambda)b(p)^\dagger\Phi(t,\Lambda)^{-1} = e^{i\Lambda p\cdot t/\hbar}\, b(\Lambda p)^\dagger.$$

for all (t,Λ) in P and all p on the forward light cone.

Since Δ and Γ are trivial, so are the coefficient matrices u and v of the field expansion (5.18). Thus the expansion of the scalar field in terms of elementary particle operators is simply

$$\phi(x) = N\int d^d p\; \delta(p^2)\theta(p^0)\{\, e^{-ip\cdot x/\hbar}\, a(p) + e^{ip\cdot x/\hbar}\, b(p)^\dagger\,\} \tag{6.1}$$

For the scalar field, there is no field equation other than the Klein-Gordon equation.

Conventionally, the dimensions of the scalar field are taken to be $[\phi(x)] = [\text{length}]^{-(d-1)/2}$ (this gives $\phi^\dagger\phi$ the right dimensions for a probability density). From the canonical commutation relations (5.14), the annihilation and creation operators have dimensions [momentum] $^{-(d-2)/2}$. The measure in (6.1) has dimensions [momentum] $^{d-2}$. This means that the normalisation constant N must have dimensions $[N]$ = [length] $^{-(d-1)/2}$ [momentum] $^{-(d-2)/2}$. These units can be supplied by the combination of constants $(k_0\, \hbar^{-(d-1)})^{1/2}$, since $\hbar$ has dimensions [length] [momentum], and k_0 is an arbitrary but fixed momentum. It is also convenient to introduce a factor of $(2\pi)^{-(d-1)/2}$ since (6.1) is a Fourier transform. Thus (6.1) becomes finally

$$\phi(x) = \frac{k_0^{1/2}}{(2\pi \hbar)^{(d-1)/2}} \int d^d p\; \delta(p^2)\theta(p^0)$$
$$\times \{ e^{-ip\cdot x/\hbar}\, a(p) + e^{ip\cdot x/\hbar}\, b(p)^\dagger \}$$

6.2 The spinor field

As for the classical field in example 4.2, the covariant quantum spinor field transforms by the representation of the d-dimensional Lorentz group corresponding to the γ matrices for so(d–1,1). The calculations here are simplified by a particular construction of these γ matrices, but the results will be valid generally, upon application of the appropriate similarity transformations.

The γ matrices have 2^l rows and columns, where $l = d/2$ if d is even, and $(d-1)/2$ if d is odd. Indices a,b,c will therefore take values $1,2,\ldots,2^l$ in this section.

Let $\psi_a(x)$ be the quantum field operators for the covariant spinor field at the space-time point x. The transformation law is

$$\Phi(t,\Lambda)\psi_a(x)\Phi(t,\Lambda)^{-1} = \Delta(\Lambda^{-1})_{ab}\psi_b(\Lambda x + t)$$

for all (t,Λ) in P, where Φ is the Fock space representation of P, and Δ is the spinor representation of L.

The generators $J_{\lambda\mu}$ ($= -J_{\mu\lambda}$) of L are represented by

$$\Delta(J_{\lambda\mu}) = \tfrac{i\hbar}{4}[\gamma_\lambda, \gamma_\mu] = \tfrac{i\hbar}{2}\gamma_\lambda\gamma_\mu \qquad (\lambda \neq \mu). \tag{6.3}$$

As shown in the appendix, the 2^l-dimensional γ matrices can be constructed from the 2^{l-1}-dimensional τ matrices (which are the "γ" matrices for so(d–2) with a negative-definite metric tensor). The construction is

$$\gamma_0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} \tau_i & 0 \\ 0 & -\tau_i \end{pmatrix}, \quad \gamma_{d-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{6.4}$$

where $i = 1,\ldots,d-2$ and 1 is the 2^{l-1}-dimensional identity matrix. With this construction, the generators of the little group $L(k)$ for massless particles are

$$\Delta(J_{ij}) = \frac{i\hbar}{4}[\gamma_i, \gamma_j]$$

$$= \frac{i\hbar}{4}\begin{pmatrix} [\tau_i, \tau_j] & 0 \\ 0 & [\tau_i, \tau_j] \end{pmatrix}$$

$$= \begin{pmatrix} \Gamma(J_{ij}) & 0 \\ 0 & \Gamma(J_{ij}) \end{pmatrix}$$

and

$$\Delta(J_{+i}) = \frac{-i\hbar}{2}\gamma_i(\gamma_0 + \gamma_{d-1}).$$

Here

$$\Gamma(J_{ij}) = \frac{i\hbar}{4}[\tau_i, \tau_j] \tag{6.5}$$

are the generators of a spinor representation of Spin(d–2). Thus, on Spin(d–2), the representation Δ of L decomposes to $\Gamma \oplus \Gamma$.

For the coefficient matrices u and v in the expansion of ψ in terms of creation and annihilation operators, the restrictions $\Delta(J_{+i})u = \Delta(J_{+i})v = 0$ due to equations (5.29b) and (5.30b) become

$$\gamma_i(\gamma_0 + \gamma_{d-1})u = \gamma_i(\gamma_0 + \gamma_{d-1})v = 0.$$

As γ_i is invertible,

$$(\gamma_0 + \gamma_{d-1})u = (\gamma_0 + \gamma_{d-1})v = 0. \tag{6.6}$$

In block form, from (6.4),

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}u = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}v = 0.$$

Thus u and v have the block form

$$u = \begin{pmatrix} u' \\ 0 \end{pmatrix}, \quad v = \begin{pmatrix} v' \\ 0 \end{pmatrix}$$

where u' and v' have 2^{l-1} rows. This has the effect of eliminating from consideration one of the two Γ in the decomposition of Δ, thus avoiding the possible complications mentioned in section 5.3.

Unfortunately, the spinor representations Γ of Spin(d–2) are not all irreducible, so there must be further restraints. It is best to deal with the odd dimensions and even dimensions separately.

First, let d be odd, with $d \geq 5$. In this case, Γ corresponds to the representation $\{0, \ldots, 0, 1\}$ of B_{l-1} (recall $l = (d-1)/2$), and is irreducible. Since this is the only irreducible representation in the branching rule for Δ, the annihilation operators must transform by Γ.

Let $a_\alpha(p)$ and $b_\alpha(p)^\dagger$ be annihilation and creation operators, where p takes values on the forward light cone, and indices α,β range over $1, 2, \ldots, 2^{l-1}$.

It is now possible to fix the coefficient matrices completely. Equation (5.29a) becomes, in block form

$$\begin{pmatrix} \Gamma(R) & 0 \\ 0 & \Gamma(R) \end{pmatrix} \begin{pmatrix} u' \\ 0 \end{pmatrix} = \begin{pmatrix} u' \\ 0 \end{pmatrix} \Gamma(R),$$

for all R in Spin(d–2). By Schur's lemma, this leaves u' proportional to the identity matrix. The creation operator coefficient v' is easily found from u'. The representations of B_{l-1} are all either potentially real or pseudo-real. In either case, as shown in the appendix, the complex conjugate representation Γ^* of Spin(d–2) is given by

$$\Gamma^*(R) = Z^{-1}\Gamma(R)Z \tag{6.7}$$

or all R in Spin(d–2), where Z is the charge conjugation matrix for the τ matrices. So,

$$\begin{aligned} \Delta(R)u\,Z &= u\,\Gamma(R)Z \\ &= u\,Z\,\Gamma^*(R), \end{aligned}$$

and equation (5.30a) is satisfied by taking $v = u\,Z$, that is, $v' = Z$.

The normalisation of the spinor field is the same as that of the scalar field. So for $d \geq 5$ and odd, the spinor field is given in terms of annihilation and creation operators by

$$\psi_a(x) = \frac{k_0^{\,1/2}}{(2\pi\hbar)^{(d-1)/2}} \int d^d p \; \delta(p^2)\theta(p^0)\Delta(B(p))_{ab}\, u_{b\alpha} \times \{ e^{-ip\cdot x/\hbar} a_\alpha(p) + e^{ip\cdot x/\hbar} Z_{\alpha\beta} b_\beta(p)^\dagger \}$$

where

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ i.e. } u_{b\alpha} = \begin{cases} \delta_{b\alpha} & \text{for } b = 1, \ldots, 2^{l-1} \\ 0 & \text{for } b = 2^{l-1}+1, \ldots, 2^l. \end{cases}$$

If the γ matrices are modified by a similarity transformation, then u is also modified by the same transformation.

For d even, there is a spare γ matrix and a spare τ matrix, corresponding to γ_5 in four dimensions. These spare matrices can be used to decompose the spinor representations into two "Weyl spinor" representations. It is convenient to persevere with the larger representation for the covariant field, but irreducible representations are needed for the elementary particle annihilation and creation operators.

For $d \geq 6$, the representation Γ of Spin($d-2$) generated from the γ matrices corresponds to the representation $\{0, \ldots, 0, 1\} \oplus \{0, \ldots, 1, 0\}$ of D_{l-1} (where $l = d/2$). Denote $\{0, \ldots, 0, 1\}$ by Γ^1, and $\{0, \ldots, 1, 0\}$ by Γ^2. Both of these representations are 2^{l-2}-dimensional and irreducible. In four dimensions also, the representation Γ splits into two representations of so(2), usually labelled $\Gamma^{1/2}$ and $\Gamma^{-1/2}$. For the sake of uniformity, these will be labelled Γ^1 and Γ^2 here. The decomposition of Δ on Spin($d-2$) contains Γ^1 and Γ^2 twice each. One copy of each is eliminated by the conditions (5.29b) and (5.30b). So there may be two kinds of spinor field, one for elementary particle operators transforming by Γ^1 and one for Γ^2.

Let $a_\alpha^\sigma(p)$ and $b_\alpha^\sigma(p)^\dagger$ be the annihilation and creation operators for Γ^σ, with $\sigma = 1, 2,\ \alpha = 1, 2, \ldots, 2^{l-2}$, and p on the forward light cone.

Taking the coefficient matrices

$$u^\sigma = \begin{pmatrix} u^{\sigma\prime} \\ 0 \end{pmatrix}, \quad v^\sigma = \begin{pmatrix} v^{\sigma\prime} \\ 0 \end{pmatrix}$$

for the different spinors, the $2^{l-1} \times 2^{l-2}$ matrices $u^{\sigma\prime}$ and $v^{\sigma\prime}$ must satisfy (from (5.29*a*) and (5.30*a*))

$$\begin{aligned} \Gamma(R)u^{\sigma\prime} &= u^{\sigma\prime}\,\Gamma^\sigma(R) \\ \Gamma(R)v^{\sigma\prime} &= v^{\sigma\prime}\,\Gamma^{\sigma}{}^*(R) \end{aligned}$$

for all R in Spin(d–2) and $\sigma = 1, 2$. Since $\Gamma = \Gamma^1 \oplus \Gamma^2$, Schur's lemma leads straight away to

$$u^{1'} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u^{2'} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(or a constant multiple of this), where 1 is the $2^{l-2} \times 2^{l-2}$ identity matrix.

The representations Γ^1 and Γ^2 are essentially complex for d = 4, 8, 12, ... In these dimensions, Γ^1 is isomorphic to Γ^{2*}. This means that there is only one spinor field in these dimensions, instead of two. Of course, there is still the adjoint field describing the anti-particle, but for d = 6, 10, 14, ..., there exist different spinor fields whose particles are totally unrelated. In all cases (Γ^σ potentially real, pseudo-real or essentially complex), the charge conjugation matrix of the τ matrices can be used to find v^σ, because it applies to the larger representation Γ. Given that

$$\Gamma(R)u^{\sigma\prime} = u^{\sigma\prime}\,\Gamma^\sigma(R)$$

for all R in Spin(d–2), it follows that

$$\Gamma^*(R)u^{\sigma'} \quad = u^{\sigma'}\Gamma^{\sigma}{}^*(R),$$

since $u^{\sigma'}$ is a constant multiple of a real matrix. Using (6.7) gives

$$\Gamma(R)Z\,u^{\sigma'} \quad = Z\,u^{\sigma'}\Gamma^{\sigma}{}^*(R),$$

which is just the requirement for $v^{\sigma'}$. So $v^{\sigma'}$ can be taken as $v^{\sigma'} = Z\,u^{\sigma'}$ (note that there is a reversal of order from the corresponding equation for odd dimensions d).

So, for $d \geq 4$ and even, there are two kinds of spinor field ψ_a^{σ} (or essentially just one if $d = 4, 8, 12, \ldots$), given in terms of annihilation and creation operators by

$$\begin{aligned}\psi_a^{\sigma}(x) = \frac{k_0^{\,1/2}}{(2\pi\hbar)^{(d-1)/2}}\int d^dp\;\delta(p^2)\theta(p^0)\Delta(B(p))_{ab} \\ \times \{\, e^{-ip\cdot x/\hbar}\,u_{b\;\alpha}^{\sigma}\,a_{\alpha}(p) \\ +\, e^{ip\cdot x/\hbar}\,Z'_{bc}\,u_{c\;\alpha}^{\sigma}\,b_{\alpha}(p)^{\dagger}\,\}\end{aligned}$$

where the $2^l \times 2^{l-2}$ coefficient matrices are

$$u^1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix},\quad u^2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

and the $2^l \times 2^l$ matrix Z' is built from the $2^{l-1} \times 2^{l-1}$ charge conjugation matrix Z:

$$Z' = \begin{pmatrix}Z & 0\\ 0 & 0\end{pmatrix}$$

Again, if the γ matrices undergo a similarity transformation, then so must u and v as well.

Finally, the constant Q in the field equation (5.34) must be determined. This was given by (5.33) in

$$\Delta(J_{0d-1})u \quad = i\hbar Q\;u.$$

Using the definition (6.3) of the representation Δ this becomes

$$\gamma_0 \gamma_{d-1} u = 2Q\, u.$$

From (6.6),

$$\gamma_{d-1} u = -\gamma_0 u,$$

and since $(\gamma_0)^2 = \frac{1}{2}\{\gamma_0, \gamma_0\} = g_{00} = 1$, it follows that

$$\gamma_0 \gamma_{d-1} u = -u,$$

and thus $Q = -\frac{1}{2}$.

Let ψ denote the spinor field for any dimension d (either one if there are two). Then, using (6.3), the field equation (5.34) is

$$\tfrac{i\hbar}{4} [\gamma_\lambda, \gamma_\mu]_{ab} \partial^\mu \psi_b + \tfrac{i\hbar}{2} \partial_\lambda \psi_a = 0.$$

where $\partial_\lambda = \partial / \partial x^\lambda$. Removing a factor of $\frac{i\hbar}{2}$, pre-multiplying by γ^λ and contracting over λ using

$$[\gamma_\lambda, \gamma_\mu] = 2(\gamma_\lambda \gamma_\mu - g_{\lambda\mu}),$$

gives

$$\gamma^\lambda{}_{ab}\, \partial_\lambda \psi_b = 0.$$

This last is the Dirac equation for massless particles in d dimensions. It is, nevertheless, encouraging to see it arise as a necessary consequence of the most straightforward field construction from first principles, and not as an initial postulate or from a Lagrangian.

6.3 The vector field

The main reason for including this section is that it contains something of a surprise. The massless vector field turns out to be more interesting for what it isn't than for what it is. It isn't the field for photons. This unusual statement results from the considerations about particle content in the last chapter, as will be seen here.

The massless, covariant vector field transforms by the d-dimensional fundamental representation of L. That is, $\Delta(\Lambda) = \Lambda$. So let $A_\lambda(x)$ be the quantum field operators for the vector field at the point x in space-time. Under the Fock space representation Φ of P, the vector field transforms by

$$\Phi(t,\Lambda)A_\lambda(x)\Phi(t,\Lambda)^{-1} = \Lambda^\mu{}_\lambda A_\mu(\Lambda x + t),$$

for all (t,Λ) in P. The generators of L are represented by

$$\Delta(J_{\lambda\mu})^\nu{}_\rho = i\hbar(\delta^\nu_\lambda g_{\mu\rho} - \delta^\nu_\mu g_{\lambda\rho}).$$

The first task is to find the matrix solution u to the equation $\Delta(J_{+i})u = 0$. Recall from (5.32) that this is equivalent to

$$k^\lambda \Delta(J_{\lambda i})u = 0.$$

For a single column w of u, this is

$$\begin{aligned} k^\lambda \Delta(J_{\lambda i})^\nu{}_\rho w^\rho &= i\hbar k^\lambda(\delta^\nu_\lambda g_{i\rho} - \delta^\nu_i g_{\lambda\rho})w^\rho \\ &= i\hbar\,(k^\nu w_i - \delta^\nu_i k^\lambda w_\lambda) \\ &= 0, \end{aligned}$$

for all i and all ν if, and only if, $w_i = 0$ and $k \cdot w = 0$. The only solutions to this have w proportional to k. Thus u can be taken as the $d \times 1$ column matrix $u = k / k_0$ (the factor of k_0 is introduced to make u dimensionless).

Next, the representation Γ for the elementary particles must be found, using equation (5.29a):

$$\Delta(R)u = u\,\Gamma(R)$$

for all R in Spin(d–2). In its decomposition on Spin(d–2), Δ contains the (d–2)-dimensional representation and two trivial representations, accounting for one dimension each. Since $R\,k = k$ for all R in Spin(d–2), it follows that Γ is one of the trivial representations $\Gamma(R) = 1$.

This is important, because it implies that the covariant vector field can describe only scalar, that is, spinless particles. In particular, a *covariant vector field cannot describe photons* (which transform by the vector representation of Spin(d–2)). Of course, the vector potential in electromagnetism does just that: describes photons. But there is no contradiction, because the vector potential is not a true vector, as will be seen in section 6.5. Let the annihilation and creation operators for these particles be $a(p)$ and $b(p)^\dagger$, with p on the forward light cone.

As Γ is trivial, Γ^* is as well, so the coefficient matrix v for the creation operators can be taken to be the same as u.

Finally, note that

$$\Delta(B(p))u = B(p)k/k_0 = p/k_0.$$

This completes the list of items in the field construction (5.18).

In terms of annihilation and creation operators, the covariant vector field is given by

$$A_\lambda(x) = \frac{k_0^{\;1/2}}{(2\pi\hbar)^{(d-1)/2}}\int d^dp\;\delta(p^2)\theta(p^0)p_\lambda \times \{\,e^{-ip\cdot x/\hbar}\,a(p) + e^{ip\cdot x/\hbar}\,b(p)^\dagger\,\}$$

A slightly different normalisation has been used here, with the extra k_0^{-1} disappearing. This is just to make it more obvious that

$$A_\lambda(x) = i\hbar\,\partial_\lambda\,\phi(x),$$

where ϕ is a scalar field (cf. (6.2)). The field equation for A (which is clearly $\partial_\lambda A^\lambda = 0$) can be obtained in the same manner as that for the spinor in the last section. It will not be derived here, since it is not very interesting.

6.4 The antisymmetric, second-rank tensor field

The classical covariant, antisymmetric, second-rank tensor field was examined in example 4.4. The corresponding quantum field is as usual quite similar. Let $F_{\lambda\mu}(x) = -F_{\mu\lambda}(x)$ be field operators for the quantum field at a space-time point x. The covariant representation of L may be taken as $\Delta(\Lambda) = \Lambda \wedge \Lambda$, where $\wedge$ is the anti-symmetric tensor product. Under the Fock space representation Φ of P,

$$\Phi(t,\Lambda)F_{\lambda\mu}(x)\Phi(t,\Lambda)^{-1} = \Lambda^{\nu}{}_{\lambda}\Lambda^{\rho}{}_{\mu}F_{\nu\rho}(\Lambda x + t),$$

for all (t,Λ) in P. The generators of L are represented by

$$\Delta(J_{\lambda\mu}) = J_{\lambda\mu} \otimes 1 + 1 \otimes J_{\lambda\mu}, \tag{6.8}$$

where $J_{\lambda\mu}$ here denotes the fundamental d-dimensional matrix representation

$$(J_{\lambda\mu})^{\nu}{}_{\rho} = i\hbar\,(\delta^{\nu}_{\lambda}g_{\mu\rho} - \delta^{\nu}_{\mu}g_{\lambda\rho}),$$

and 1 is the d-dimensional identity matrix.

The task is to find the coefficient matrix u satisfying $\Delta(J_{+i})u = 0$, and the corresponding irreducible representation Γ of Spin(d–2) such that

$$\Delta(R)u = u\,\Gamma(R)$$

for all R in Spin(d–2). This is a fairly long, detailed process, but the result is unavoidable.

Let w be a single column of u. The rows of w (and u) are labelled by antisymmetric indices just as for F : $w^{\lambda\mu} = -w^{\mu\lambda}$. The column w itself must satisfy

$$k^{\lambda}\Delta(J_{\lambda i})w = 0,$$

which, by direct application of the above definitions gives

$$g_{i\rho}(k^{\mu}w^{\rho\nu} - k^{\nu}w^{\rho\mu}) - k_{\rho}(\delta^{\mu}_{i}w^{\rho\nu} - \delta^{\nu}_{i}w^{\rho\mu}) = 0. \tag{6.9}$$

This must be satisfied for all $\mu,\nu = 0, 1, \ldots, d{-}1$ and all $i = 1, \ldots, d{-}2$. Take $\mu = 0$ and $\nu = d{-}1$. The second term in (6.9) disappears, leaving essentially

$$w^{i\,0} = w^{i\,d-1}.$$

With $\mu = 0$ and $\nu = j$, all that is left after a little cancellation is

$$w^{i\,j} = \delta^{i\,j}\, w^{0d-1}.$$

But the left-hand side of this is antisymmetric, while the right-hand side is symmetric, so both sides must equal zero separately.

In summary, w has

$$w^{\lambda\mu} = -w^{\mu\lambda},$$

$$w^{i\,j} = w^{0d-1} = 0,$$

$$w^{i\,0} = w^{i\,d-1}.$$

This leaves just $d{-}2$ independent elements in w. Thus u can have $d{-}2$ independent columns, which can be conveniently labelled by the indices i,j etc.. By inspection, the matrix u can be taken as

$$u^{\lambda\mu}{}_i = (k^\lambda\, \delta\, {}^{\mu}_{i} - k^\mu\, \delta\, {}^{\lambda}_{i})/\, k_0 . \tag{6.10}$$

This is then the largest possible solution of $\Delta\,(J_{+i})u = 0$. (It is also dimensionless).

Turning to the compact part of the little group, if R is a Lorentz transformation belonging to Spin($d{-}2$), then R has the d-dimensional matrix form

$$R = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & & & & 0 \\ : & & R' & & : \\ 0 & & & & 0 \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix}, \tag{6.11}$$

where R' is the $(d{-}2)$-dimensional square matrix with elements

$$R'^{\,i}{}_j = R^i{}_j \,.$$

In particular, $R^{\,0}{}_i = R^{\,d-1}{}_i = 0.$

Using the matrix u of (6.10), $\Delta(R)u$ becomes

$$\begin{aligned}\Delta(R)^{\lambda\mu}{}_{\nu\rho}\, u^{\nu\rho}{}_i &= R^{\lambda}{}_{\nu} R^{\mu}{}_{\rho}\, u^{\nu\rho}{}_i \\ &= (R^{\lambda}{}_{\nu} k^{\nu} R^{\mu}{}_{\rho} \delta^{\rho}_{i} - R^{\mu}{}_{\rho} k^{\rho} R^{\lambda}{}_{\nu} \delta^{\nu}_{i})/\, k_0 \\ &= (k^{\lambda} R^{\mu}{}_{i} - k^{\mu} R^{\lambda}{}_{i})/\, k_0 \\ &= u^{\lambda\mu}{}_{j}\, R^{j}{}_{i}\,.\end{aligned}$$

Comparing this with the requirement

$$\Delta(R)u = u\,\Gamma(R),$$

the irreducible representation Γ is clearly the fundamental $(d-2)$-dimensional representation of Spin$(d-2)$.

Since Γ is real, the coefficient matrix v for the creation operators may be set identical to u, with the annihilation and creation operators transforming by the same representation Γ.

Let $a^i(p)$ and $b^i(p)^\dagger$ be the annihilation and creation operators for the antisymmetric tensor field, with p on the forward light cone. Under the Fock space representation $\Phi(t,\Lambda)$ of (t,Λ) in P, these operators transform by

$$\Phi(t,\Lambda)a^i(p)\Phi(t,\Lambda)^{-1} = \mathrm{e}^{-i\Lambda p\cdot t/\hbar} R_p(\Lambda)_j{}^i\, a^j(\Lambda p), \tag{6.13a}$$

$$\Phi(t,\Lambda)b^i(p)^\dagger\Phi(t,\Lambda)^{-1} = \mathrm{e}^{i\Lambda p\cdot t/\hbar} R_p(\Lambda)_j{}^i\, b^j(\Lambda p)^\dagger, \tag{6.13b}$$

where $R_p(\Lambda)$ is the little group element defined in (5.15).

It is useful to note that for the coset representatives $B(p)$, $\Delta(B(p))u$ is given by

$$\begin{aligned}&B(p)^{\lambda}{}_{\nu} B(p)^{\mu}{}_{\rho}(k^{\nu}\delta^{\rho}_{i} - k^{\rho}\delta^{\nu}_{i})/\, k_0 \\ &\quad= (p^{\lambda} B(p)^{\mu}{}_{i} - p^{\mu} B(p)^{\lambda}{}_{i})/\, k_0.\end{aligned}$$

It is also useful if the same normalisation is adopted here as for the vector field.

The antisymmetric second-rank tensor field can thus be expanded in terms of its momentum-space annihilation and creation operators as

$$F^{\lambda\mu}(x) = \frac{k_0^{1/2}}{(2\pi\hbar)^{(d-1)/2}} \int d^dp \ \delta(p^2)\theta(p^0)(p^\lambda B(p)^\mu{}_j - p^\mu B(p)^\lambda{}_j)$$

$$\times \{ e^{-ip\cdot x/\hbar} a^j(p) + e^{ip\cdot x/\hbar} b^j(p)^\dagger \} \qquad (6.12)$$

This is very similar to the usual four-dimensional field, and suggests the gauge potential immediately, but that topic will be left until the next section. Note that $B(p)^\lambda{}_i$ is the d-dimensional generalisation of the polarisation vectors in four dimensions.

In $d = 4$, it is possible to construct two separate fields, one for each helicity. The two parts of the covariant field were separated in example 4.4. For $d \geq 5$, there is only one field, and it combines all of the possible "helicities" at once. This difference comes about because, for $d = 4$, Spin(d–2) is the universal covering group of SO(2), which has essentially complex representations. The two helicities correspond to different, inequivalent representations. For $d \geq 5$, the d–2 "helicities" correspond to the rows of a single, real representation, so there is only one field. In other words, there would be no polarised light in a Minkowski space-time of five or more dimensions! This kind of difference between four and higher dimensions arises for all except scalar fields and the spinor fields already mentioned in section 6.2.

Finally, the first-order field equations (5.34) for $F^{\lambda\mu}$ can be found. The constant Q in (5.34) is fixed by (5.33) from

$$\Delta(J_{0d-1})u = i\hbar Q u .$$

Using (6.10) and a little algebra, the value of Q is found to be 1. The first-order field equations are then

$$\Delta(J_{\lambda\mu})^{\nu\rho}{}_{\nu'\rho'} \partial^\mu F^{\nu'\rho'} - i\hbar \partial_\lambda F^{\nu\rho} = 0,$$

where the indices ν',ρ' range over $0, 1, \ldots, d{-}1$. Using (6.9) quickly transforms the field equations to

$$\partial_\lambda F^{\lambda\mu} = 0,$$

and

$$\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0,$$

which are precisely the d-dimensional generalisation of Maxwell's equations for free electromagnetic fields.

It is straightforward to repeat the constructions just performed, but for larger representations. The generalisations of the Rarita-Schwinger field strength $F^{\lambda\mu\alpha}$ for spin $3/2$ and the fourth rank traceless tensor $R^{\lambda\mu\nu\rho}$ for spin 2 contain no new surprises. The procedure is a little laborious, but it is gratifying to see the expected results at the end, without having done any guess-work. This is also encouraging for the application of the method to the less well-established supersymmetric field theories starting in the next chapter.

6.5 Gauge potentials

Just as in four dimensions, it is possible in d dimensions to express the covariant fields for higher spin particles as certain derivatives of gauge potential fields. This is most easily seen for the antisymmetric, second-rank tensor fields of section 6.4. Gauge potentials for other fields differ from this mainly in complexity.

Let $F^{\lambda\mu}(x)$ be the quantum field operators for the antisymmetric, second-rank tensor field, and let $a^i(p)$ and $b^i(p)^\dagger$ be the corresponding annihilation and creation operators. For simplicity, consider just self-adjoint fields $F^{\lambda\mu}$, for which $b^i(p)^\dagger = a^i(p)^\dagger$.

From the expansion (6.12) of the field, it is immediately apparent that $F^{\lambda\mu}$ can be written as

$$F^{\lambda\mu} = \partial^\lambda A^\mu - \partial^\mu A^\lambda, \tag{6.14}$$

where

$$A^\lambda(x) = \frac{i\,\hbar k_0^{\;1/2}}{(2\pi\hbar)^{(d-1)/2}} \int d^dp\; \delta(p^2)\theta(p^0)B(p)^\lambda{}_i \times \{ e^{-ip\cdot x/\hbar}\, a^i(p) - e^{ip\cdot x/\hbar}\, a^i(p)^\dagger \} \tag{6.15}$$

A is the gauge potential for F, or the vector potential. Equation (6.15) does not give the only solution to (6.14). There is, of course, the possibility of gauge transformations. If any set of quantum fields Θ^λ is added to A^λ, then the result still yields $F^{\lambda\mu}$ provided

$$\partial^\lambda\Theta^\mu - \partial^\mu\Theta^\lambda = 0.$$

For example, Θ could be a covariant vector field as in section 6.3, where

$$\Theta^\lambda = \partial^\lambda\phi \tag{6.16}$$

and ϕ is a scalar field.

Since A is a d-dimensional vector, and there are only $d-2$ independent annihilation and creation operators, there must be two conditions on A to fix the gauge. The exact form of A in (6.15) depends on the coset representatives $B(p)$, so the choice of gauge finally rests in a choice of coset representatives. Nevertheless, one of the gauge fixing conditions is already determined by the general form of (6.15). As

$$p_\lambda B(p)^\lambda{}_i = B(p)^{-1}{}_i{}^\lambda p_\lambda = k_i = 0,$$

for all p on the forward light cone, it follows that

$$\partial_\lambda A^\lambda = 0. \tag{6.17}$$

A in (6.15) is thus in the Lorentz gauge. A gauge transformation of the form (6.16) will maintain condition (6.17), and amounts to just a change of coset representatives.

The gauge potential A is not a covariant vector field. Under Poincaré transformations, A transforms in a way which combines the normal behaviour of a vector field with an additional gauge transformation. This is explained in the following proposition.

Proposition 6.1 Under the Fock space representation $\Phi(t,\Lambda)$ of (t,Λ) in P, the vector potential A of (6.15) transforms by

$$\Phi(t,\Lambda)A^\lambda(x)\Phi(t,\Lambda)^{-1} = \Lambda_\nu{}^\lambda A'^\nu(\Lambda x + t), \tag{6.18}$$

where A' is a different vector potential, related to A by a gauge transformation

$$A'^\lambda(x) = A^\lambda(x) + \partial^\lambda \phi_\Lambda(x), \tag{6.19}$$

where $\phi_\Lambda(x)$ is given by

$$\phi_\Lambda(x) = \frac{\hbar^2 k_0{}^{1/2}}{(2\pi\hbar)^{(d-1)/2}} \int d^dp\ \delta(p^2)\theta(p^0)R_p(\Lambda^{-1})^0{}_i \times \{ e^{-ip\cdot x/\hbar} a^i(p) + e^{ip\cdot x/\hbar} a^i(p)^\dagger \}. \tag{6.20}$$

Proof. The best way to proceed is to expand $A^\lambda(x)$ by (6.15) and then use the transformation rules (6.13) of the creation and annihilation operators. With $R_p(\Lambda)$ (the little group element) defined by (5.15),

$$\begin{aligned}
&\Phi(t,\Lambda)A^\lambda(x)\Phi(t,\Lambda)^{-1}\\
&= \frac{i\hbar k_0^{1/2}}{(2\pi\hbar)^{(d-1)/2}}\int d^dp\,\delta(p^2)\theta(p^0)B(p)^\lambda{}_i\\
&\qquad\times\{e^{-ip\cdot x/\hbar}e^{-i\Lambda p\cdot t/\hbar}R_p(\Lambda)_j{}^i a^j(\Lambda p)\\
&\qquad\qquad - e^{ip\cdot x/\hbar}e^{i\Lambda p\cdot t/\hbar}R_p(\Lambda)_j{}^i a^j(\Lambda p)^\dagger\}\\
&= \frac{i\hbar k_0^{1/2}}{(2\pi\hbar)^{(d-1)/2}}\int d^dp\,\delta(p^2)\theta(p^0)B(p)^\lambda{}_i R_p(\Lambda)_j{}^i\\
&\qquad\times\{e^{-i\Lambda p\cdot(\Lambda x+t)/\hbar}a^j(\Lambda p) - e^{i\Lambda p\cdot(\Lambda x+t)/\hbar}a^j(\Lambda p)^\dagger\}
\end{aligned}$$

Note that the summation in the product of the B and R matrices is only over $i = 1, \ldots, d-2$. This can be written

$$B(p)^\lambda{}_i R_p(\Lambda)_j{}^i = B(p)^\lambda{}_\mu \delta_i^\mu \delta^i{}_\nu R_p(\Lambda)_j{}^\nu,$$

and the two Kronecker deltas expressed as

$$\delta_i^\mu \delta^i_\nu = \delta^\mu_\nu - \delta^\mu_0 \delta^0_\nu - \delta^\mu_{d-1}\delta^{d-1}_\nu .$$

With a little rearrangement, this is

$$\delta_i^\mu \delta^i_\nu = \delta^\mu_\nu - (k^\mu \delta^0_\nu - \delta^\mu_{d-1} k_\nu)/k_0.$$

The product of the B and R matrices becomes

$$\begin{aligned}
B(p)^\lambda{}_i R_p(\Lambda)_j{}^i &= B(p)^\lambda{}_\mu R_p(\Lambda)_j{}^\mu\\
&\qquad - B(p)^\lambda{}_\mu (k^\mu \delta^0_\nu - \delta^\mu_{d-1}k_\nu)R_p(\Lambda)_j{}^\nu/k_0.
\end{aligned}$$

Expanding $R_p(\Lambda)$ by (5.15), the first term in this expression is

$$\begin{aligned}
B(p)^\lambda{}_\mu R_p(\Lambda)_j{}^\mu &= B(p)^\lambda{}_\mu (B(p)^{-1}\Lambda^{-1}B(\Lambda p))^\mu{}_j\\
&= (\Lambda^{-1})^\lambda{}_\mu B(\Lambda p)^\mu{}_j .
\end{aligned}$$

The second term is

$$B(p)^\lambda{}_\mu (k^\mu \delta^0_\nu - \delta^\mu_{d-1}k_\nu)R_p(\Lambda)_j{}^\nu/k_0 = p^\lambda R_p(\Lambda)_j{}^0/k_0,$$

since

$$B(p)^\lambda{}_\mu k^\mu = p^\lambda$$

and

$$\begin{aligned} k_\nu R_p(\Lambda)_j{}^\nu &= k_j \qquad \text{(since } R_p(\Lambda) \text{ is in the little group)} \\ &= 0. \end{aligned}$$

Putting all of this back into the main calculation,

$$\begin{aligned} &\Phi(t,\Lambda)A^\lambda(x)\Phi(t,\Lambda)^{-1} \\ &= \frac{i\hbar k_0^{1/2}}{(2\pi\hbar)^{(d-1)/2}} \int d^dp\, \delta(p^2)\theta(p^0) \\ &\qquad \times \{ (\Lambda^{-1})^\lambda{}_\mu B(\Lambda p)^\mu{}_j - p^\lambda R_p(\Lambda)_j{}^0 / k_0 \} \\ &\qquad \times \{ e^{-i\Lambda p\cdot(\Lambda x+t)/\hbar} a^j(\Lambda p) - e^{i\Lambda p\cdot(\Lambda x+t)/\hbar} a^j(\Lambda p)^\dagger \} \\ &= \frac{i\hbar k_0^{1/2}}{(2\pi\hbar)^{(d-1)/2}} \int d^dp\, \delta(p^2)\theta(p^0)(\Lambda^{-1})^\lambda{}_\mu \\ &\qquad \times \{ B(p)^\mu{}_j - p^\mu R_p(\Lambda^{-1})^0{}_j / k_0 \} \\ &\qquad \times \{ e^{-ip\cdot(\Lambda x+t)/\hbar} a^j(p) - e^{ip\cdot(\Lambda x+t)/\hbar} a^j(p)^\dagger \} \end{aligned}$$

using (5.15) and the Lorentz invariance of the measure. This yields the required

$$\Phi(t,\Lambda)A^\lambda(x)\Phi(t,\Lambda)^{-1} = \Lambda_\nu{}^\lambda A'^\nu(\Lambda x + t),$$

and A' is seen to be given by

$$\begin{aligned} A'^\lambda(x) = \frac{i\hbar k_0^{1/2}}{(2\pi\hbar)^{(d-1)/2}} &\int d^dp\, \delta(p^2)\theta(p^0) \\ &\times \{ B(p)^\lambda{}_j - p^\lambda R_p(\Lambda^{-1})^0{}_j / k_0 \} \\ &\times \{ e^{-ip\cdot x/\hbar} a^j(p) - e^{ip\cdot x/\hbar} a^j(p)^\dagger \}. \end{aligned}$$

The first term on the second line of this expression ($B(p)^\lambda{}_j$) gives rise to $A^\lambda(x)$ (cf. 6.15)). The factor p^λ in the second term means that the result can be written as a derivative $\partial^\lambda \phi_\Lambda(x)$. Equations (6.19) and (6.20) then follow immediately. □

Comments. Note that the gauge transformation depends only on the Lorentz transformation Λ, and not on the translation t. In a sense the gauge transformation is *induced* by the Lorentz transformation. The new gauge field A' is still in the Lorentz gauge. The second gauge condition for A will not be satisfied by A', but rather by $\Lambda^{-1}A'$.

In the formalism presented here, once the coset representatives $B(p)$ have been chosen, the gauge is fixed, and the vector potential does not transform as a covariant field. This can be useful, as it can help to determine the form of the interaction Hamiltonian (Weinberg 1965a). However, it is often convenient to have a covariant vector potential. An auxiliary condition is then required to specify which Fock space vectors correspond to "physical" states. Such a formalism will be considered in chapter 8, again using the theory of induced representations, but this time for a supersymmetric field theory.

6.6 The light-cone gauge formalism

The light-cone gauge formalism first appeared under the name of the "infinite-momentum frame" for electrodynamics (Kogut and Soper 1970, Chang *et al* 1973). For the vector potential A of the electromagnetic field, the gauge conditions in the light-cone gauge are

$$\partial_\lambda A^\lambda = 0,$$

and

$$A^0(x) + A^{d-1}(x) = 0. \tag{6.21}$$

The advantages of using the light-cone gauge rather than some other gauge do not really start to appear until non-Abelian gauge theories are considered, but then considerable simplifications in the quantisation of these theories occur. In fact, the light-cone gauge was for some time the only gauge in which supersymmetric gauge theories, supergravity and superstrings could be quantised, or in which S-matrix calculations for supersymmetric theories could be performed. Some of these uses of the light-cone gauge can be found in Brink, Lindgren and Nilssen (1983), Mandelstam (1983), Brink, Green and Schwarz (1983), Bengtsson (1983), and Green and Schwarz (1984).

The vector potential described in the last section is not in the light-cone gauge. Nevertheless, it does share some features of the light-cone gauge theory: it contains no auxiliary degrees of freedom (there are just $d-2$ annihilation and creation operators), and hence the quantised theory is just as straightforward as the classical theory.

It is worthwhile to introduce the coordinate system of the light-cone gauge formalism, and to re-examine the vector potential of the last section.

Let v be an arbitrary d-dimensional vector with components $v^0, \ldots, v^{d-1}$. v can be expressed in terms of the $(d-2)$-dimensional vector $\mathbf{v}$ with components v^i ($i = 1, 2, \ldots, d-2$, as usual) and two *light-cone* components

$$v^{\pm} = (v^0 \pm v^{d-1})/\sqrt{2},$$

or

$$v_{\pm} = (v_0 \pm v_{d-1})/\sqrt{2}.$$

(Note that $v^+ = v_-$ and $v^- = v_+$).

These definitions match the ones used in chapter 4, and are particularly convenient for the choice $k = (k^0, 0, \ldots, 0, k^0)$ of standard light-cone momentum. They can be generalised easily to accommodate any other choice of k.

When using light-cone coordinates, it is convenient to let indices λ, μ, ν, ρ take values $1, 2, \ldots, d-2, +, -$, and indices i, j, k, l values $1, 2, \ldots, d-2$. The metric tensor for Minkowski space-time then has components

$$\begin{aligned} g_{ij} &= -\delta_{ij} \\ g_{\pm i} &= g_{i\pm} = 0 \\ g_{+-} &= g_{-+} = 1 \\ g_{++} &= g_{--} = 0, \end{aligned}$$

and $g^{\lambda\mu} = g_{\lambda\mu}$. An inner product of two d-dimensional vectors p and x becomes

$$p \cdot x = p^{\lambda} x_{\lambda} = -p^i x^i + p^+ x^- + p^- x^+.$$

As

$$dv^+ dv^- = dv^0 dv^{d-1},$$

the invariant volume element on the forward light cone just becomes

$$d\mu(p) = d^d p \, \delta(p^2) \theta(p^+ + p^-),$$

with

$$d^dp \;=\; d^{d-2}\mathbf{p}dp^+dp^-.$$

In fact, all formulae containing only covariant vectors and tensors remain valid when the components are re-interpreted in terms of light-cone coordinates. Only those formulae containing non-covariant parts or specific components of a vector (such as in the θ function above) need attention, and they are quickly converted.

In light-cone coordinates, the vector potential (6.15) is

$$A^{\lambda}(x) \;=\; \frac{i\hbar k_-^{1/2}}{(2\pi\hbar)^{(d-1)/2}}\int d^dp\;\delta(p^2)\theta(p^+ + p^-)B(p)^{\lambda}{}_i \times \{\, e^{-ip\cdot x/\hbar}\,a^i(p) \;-\; e^{ip\cdot x/\hbar}\,a^i(p)^{\dagger}\,\} \tag{6.22}$$

(a factor of $\sqrt{2}$ in the normalisation has been removed for convenience). The light-cone gauge condition is now given by $A^+(x) = 0$. To satisfy this condition requires a gauge transformation on the vector potential of (6.22). Let

$$A'^{\lambda}(x) \;=\; \frac{i\hbar k_-^{1/2}}{(2\pi\hbar)^{(d-1)/2}}\int d^dp\;\delta(p^2)\theta(p^+ + p^-)B'(p)^{\lambda}{}_i \times \{\, e^{-ip\cdot x/\hbar}\,a^i(p) \;-\; e^{ip\cdot x/\hbar}\,a^i(p)^{\dagger}\,\} \tag{6.23}$$

and suppose that

$$\partial^{\lambda}A'^{\mu} \;-\; \partial^{\mu}A'^{\lambda} \;=\; \partial^{\lambda}A^{\mu} \;-\; \partial^{\mu}A^{\lambda},$$

so that they give the same tensor field, but that

$$A'^{+}(x) = 0.$$

In terms of the coefficient matrices $B(p)$ and $B'(p)$, the first conditions become

$$p^{\lambda}B'(p)^{\mu}{}_i \;-\; p^{\mu}B'(p)^{\lambda}{}_i \;=\; p^{\lambda}B(p)^{\mu}{}_i \;-\; p^{\mu}B(p)^{\lambda}{}_i$$

and the second

$$B'(p)^{+}{}_i = 0$$

for all p on the forward light cone. These conditions can be satisfied by taking

$$B'(p)^{\lambda}{}_{i} = B(p)^{\lambda}{}_{i} - B(p)^{+}{}_{i}\, p^{\lambda}/p^{+}.$$

A' is then the vector potential in the light-cone gauge. It is not hard to verify that the $B'(p)$ cannot form a set of coset representatives for Minkowski space-time considered as P/L (because of the non-linearity in the second term), and that the light-cone gauge potential is thus not of the construction in section 6.5.

Chapter 7

Poincaré supersymmetry in four or more dimensions

Poincaré supersymmetry is the name given to a symmetry based on a Lie superalgebra whose even part is the Poincaré algebra (and possibly an internal symmetry algebra) and whose odd part carries the spinor representation of the Lorentz algebra. Roughly speaking, if the spinor representation is the smallest available then the supersymmetry is called *unextended* and labelled $N = 1$. If the odd part of the superalgebra carries N copies of the smallest representation, then the supersymmetry is called *extended*. The reason for considering dimensions of space-time greater than four is, briefly, the hope of obtaining reasonable four-dimensional extended supersymmetric theories. This chapter applies the method of induction to find irreducible representations, suitable for a quantum field theory, of the $N = 1$ Poincaré superalgebra in four or more dimensions. These take the form of multiplets (also called supermultiplets) of elementary particle annihilation and creation operators. Only multiplets of massless particles are considered here; repetition of the procedure for massive particles presents no real difficulty.

Of course, Poincaré supersymmetry has been an area of intense research for about fifteen years, and the results of this chapter have been found before. In particular, the enumeration of all the irreducible supermultiplets for $N \geq 1$ has been given in Yang and Wybourne (1986) and Strathdee (1987). The approach taken here differs from the usual treatments in that it adheres more rigidly to the method of induced representations in constructing the supermultiplets. This keeps the mathematical and "physical" aspects of the models further apart, and avoids giving semi-physical justifications for what are really mathematical results (see especially section 7.2). Another advantage of the present approach is that it yields explicit expressions for the supertransformation laws within a supermultiplet without the need for any of the guess-work or trial-and-error methods of the usual treatments (for example the "Noether method" (Freedman *et al* 1976)).

Section 7.1 gives a description of the Lie superalgebra of Poincaré supersymmetry, and shows that $N = 1$ supersymmetry does not exist in space-times of dimension 5,6 or 7 modulo 8. This fact, although known, does not seem to be widely appreciated, and in the past, the possibility of $N = 1$ Poincaré superalgebras for $d = 0,1$ mod 8 seems to have been overlooked (eg. van Holten and van Proeyen 1982, Cremmer 1982, Nahm 1982). Section 7.2 applies the method of induction to find the irreducible representations of the $N = 1$ Poincaré superalgebra. An uncommon feature here is the use of a closed stability superalgebra; the usual treatments make some use of a "little superalgebra", which is either not closed, or not a sub-superalgebra of the Poincaré superalgebra (more on this matter will be found in section 7.3). The problem arises because the Poincaré superalgebra does not have the same semi-direct product structure appropriate to the "little algebra" method, so section 7.2 uses the more general method described in section 3.3. The stability superalgebra used here can also be found in Freedman (1979). Section 7.3 is devoted to the construction of all the possible finite-dimensional irreducible representations of the stability superalgebra, and section 7.4 concludes by interpreting these in terms of the particle content of supermultiplets and giving the explicit form of the supertransformation rules.

The notation used in this chapter is explained as it is introduced. Square brackets $[\cdot,\cdot]$ will denote a graded Lie product. In representations, this is a commutator unless both arguments are odd, when it is an anti-commutator. Products with mixed arguments (linear combinations of even and odd elements) must be broken into a sum of products with graded arguments. Unless explicitly stated, the term "Poincaré superalgebra" will refer to the unextended ($N = 1$) Poincaré superalgebra.

7.1 The $N = 1$ Poincaré superalgebra in four or more dimensions

In this section, the superalgebra for the simplest Poincaré supersymmetry (ie. $N = 1$) will be presented. This will also establish some of the notation and conventions to be used throughout this chapter. Examining the structure constants of the superalgebra reveals an important result: there is *no* $N = 1$ Poincaré superalgebra in space-times of dimension $d = 5,6$ or 7 modulo 8.

As usual, indices λ, μ, ν, ρ will take values $0,1,\ldots d{-}1$, and i, j, k will take values $1,2,\ldots,d{-}2$. "Spinor" representations of the d-dimensional Lorentz group will be taken to mean those constructed from the corresponding γ matrices. The calculations here are simplified by a particular construction of these γ matrices, but the results will be valid generally, upon application of the appropriate similarity transformations. The γ matrices have 2^l rows and columns, where $l = d/2$ if d is even, and $(d-1)/2$ if d is odd. Indices a,b,c will take values $1, 2, \ldots, 2^l$. Summation signs will be omitted, summation over repeated indices being implied unless otherwise indicated. To make equations easier to read, "spinor" indices will be written as superscripts.

Let $J_{\lambda\mu}$ $(= - J_{\mu\lambda})$ and P_λ be the generators of the Poincaré group and Q^a the generators of a set of supertransformations. These elements form the basis of a Lie superalgebra called the Poincaré superalgebra with graded Lie products

$$
\begin{aligned}
[J_{\lambda\mu}, J_{\nu\rho}] &= -i\hbar\,(g_{\lambda\nu} J_{\mu\rho} - g_{\lambda\rho} J_{\mu\nu} + g_{\mu\rho} J_{\lambda\nu} - g_{\mu\nu} J_{\lambda\rho}) \\
[J_{\lambda\mu}, P_\nu] &= -i\hbar\,(g_{\lambda\nu} P_\mu - g_{\mu\nu} P_\lambda) \\
[P_\lambda, P_\mu] &= 0 \\
[J_{\lambda\mu}, Q^a] &= \tfrac{-i\hbar}{2}\,(\gamma_\lambda \gamma_\mu)^{ab} Q^b \qquad (\lambda \neq \mu) \\
[P_\lambda, Q^a] &= 0 \\
[Q^a, Q^b] &= (\gamma^\lambda C)^{ab} P_\lambda ,
\end{aligned}
\tag{7.1}
$$

where C is the charge conjugation matrix for the γ matrices (see the appendix for details).

Several features of these graded Lie products merit some comment. Firstly, since P_λ and Q^a commute, it is clear that $P^\lambda P_\lambda$ is a Casimir operator for the superalgebra. Thus any irreducible supermultiplet contains particles with the same mass. From the last graded product, it is apparent that the Q^a have dimensions [momentum] $^{-1/2}$. As usual, the odd part of the superalgebra carries a representation of the even part. In this case, it is the spinor representation

$$D(J_{\lambda\mu}) = \tfrac{i\hbar}{2}(\gamma_\lambda \gamma_\mu) \qquad (\lambda \neq \mu) \tag{7.2}$$

of so(d–1,1) extended trivially on the P_λ. That is,

$$[J_{\lambda\mu}, Q^a] = -D(J_{\lambda\mu})^{ab} Q^b. \tag{7.1}$$

(This equation will be treated as part of the graded Lie product set (7.1)). For an extended Poincaré supersymmetry, the representation D would be a direct sum of several copies of the spinor representation. There would also possibly be other basis elements generating automorphisms among the Q^a carrying different spinor representations.

Most importantly, the structure constants in the graded Lie products (7.1) are not real. For physical applications, it is necessary that at least the even part of the superalgebra is a real Lie algebra.

Consequently, the whole superalgebra must be real. It is not always possible to satisfy this stipulation.

For the even part, the elements

$$M_{\lambda\mu} = \frac{i}{\hbar} J_{\lambda\mu}$$
$$K_{\lambda} = \frac{i\zeta}{\hbar} P_{\lambda}$$

form a dimensionless basis for the real Lie algebra iso(d–1,1) (ζ is an arbitrary non-zero constant with dimensions of length). The Q^a need to be modified too, the most general possibility being to take a new set of basis elements Q'^a given by

$$Q'^a = S^{ab} Q^a$$

where S is some non-singular 2^l-dimensional square matrix. This amounts to performing a similarity transformation (using S) on the spinor representation D. To have a real superalgebra, the structure constants must all be real, so the question is whether there exists a similarity transformation S such that D becomes a real representation of so(d–1,1). That is, is D a potentially real representation?

For odd dimensions d, the spinor representations are potentially real if $d = 1$ or 3 mod 8, and pseudo-real if $d = 5$ or 7 mod 8. For even d, the spinor representation D reduces to two inequivalent "Weyl" spinor representations. If $d = 2$ mod 8, these Weyl representations are both potentially real, and hence, so is D. If $d = 0$ or 4 mod 8, the Weyl spinor representations are mutually conjugate: the complex conjugate representation of one is equal (not just equivalent) to the other. The total representation D is thus potentially real. But if $d = 6$ mod 8, the Weyl representations are both pseudo-real, so there is no way to make D real.

To summarise, the real (unextended) Poincaré superalgebra *does not exist* in dimensions d = 5,6 or 7 mod 8. Extended real Poincaré superalgebras do exist for these dimensions, but only for N even. Attention will henceforth be confined to the other dimensions $d = 0,1,2,3,4 \bmod 8$ (and $d > 4$). In each of these cases, it is possible to adopt a set of conventions for the γ matrices and the charge conjugation matrix C whereby the last graded Lie product in (7.1) has real structure constants as well in the transformed basis.

As mentioned above, when $d = 2 \bmod 8$, the spinor representation can be decomposed into two independent real representations (called the Majorana-Weyl representations). In this case, it is possible to have a smaller supersymmetry algebra than that of (7.1) by using just those odd generators corresponding to one or the other of the two parts of D. In order to treat all cases together, the algebra of (7.1) will be used for the moment, even though this is an extended ($N = 2$) algebra for $d = 2 \bmod 8$. The elimination of the extra degrees of freedom for this special case will be made later at an appropriate place.

7.2 Irreducible representations of the $N = 1$ Poincaré superalgebra

One crucial observation to be made from the graded Lie products (7.1) is that the Poincaré superalgebra contains an Abelian, invariant subalgebra i(d–1,1) spanned by the generators P_λ. This allows the procedures of section 3.3 to come into effect, yielding irreducible representations by the methods of production and induction. In this section, the general construction for certain irreducible induced representations, appropriate for supersymmetric, massless quantum field theories, will be given.

The first thing to note is that, without P_λ, the remaining generators $J_{\lambda\mu}$ and Q^a do *not* form a superalgebra by themselves, because the product of two Q's gives a combination of P's. The Poincaré superalgebra does not have the same nice "semi-direct product" structure as the Poincaré (Lie) algebra, the even part. Therefore, it is not possible to define a "little superalgebra" (denoted $h(\chi)$ in section 3.3). Nonetheless, it is still possible to find a stability superalgebra (denoted $s(\chi)$ in section 3.3) and use the construction of theorem 3.10, generalised to superalgebras, to find irreducible representations.

Let χ be the one-dimensional, absolutely irreducible representation of i(d–1,1)

$$\chi(P_\lambda) = k_\lambda$$

where $k = (k^0,0,\ldots,0,k^0)$ ($k^0 > 0$). As seen in chapter 4, this χ is appropriate for massless particles. From equation (3.25), the stability superalgebra is defined to be the set of all Poincaré superalgebra elements X satisfying

$$\chi([X, P_\lambda]) = 0.$$

By inspection, this includes P_λ and Q^a . From section 4.3, it also includes J_{ij} and J_{+i}. By the graded Lie products (7.1), the set $\{P_\lambda, Q^a, J_{ij}, J_{+i}\}$ does indeed span a closed superalgebra. So this is the basis for $s(\chi)$.

The next requirement in the procedure of theorem 3.10 is an irreducible representation Γ of $s(\chi)$ carried by a finite-dimensional vector space W. (Γ was called $\chi\Gamma$ in section 3.3). A complete construction of the possible (Γ, W) will be left until the next section, and just enough details given here to enable the induced representations of the Poincaré superalgebra to be found. The indices σ, τ will be used for the matrix elements of Γ, taking values 1,2,...,dim W (where dim W also will be determined in the next section). Another notation (given shortly) will also prove useful.

Restricted to i(d–1,1), Γ must be a direct sum of copies of χ. This gives straight away

$$\Gamma(P_\lambda) = k_\lambda\, 1, \tag{7.3}$$

where 1 is the identity operator in W. Considered just as a representation of the iso(d–2) algebra spanned by $\{J_{ij}, J_{+i}\}$, Γ must be completely reducible to a direct sum of irreducible representations. As explained in section 4.3, these irreducible representations must represent J_{+i} trivially. Thus

$$\Gamma(J_{+i}) = 0. \tag{7.4}$$

On the compact part so(d–2), the basis of W can be arranged so that

$$\Gamma(J_{ij}) = \sum_{q\in I} \oplus \Gamma^q(J_{ij}) \tag{7.5}$$

where I is a set listing the labels q of the irreducible representations Γ^q in the decomposition, and $\sum\oplus$ denotes a direct sum. This gives an alternative notation, $\Gamma(X)^{q\,q'}_{\alpha\,\alpha'}$ for the matrix elements of $\Gamma(X)$ ($X \in s(\chi)$), where q,q' take values in I, α takes values 1,2,...,dim Γ^q, and α' takes values 1,2,...,dim $\Gamma^{q'}$. Clearly

$$\sum_{q \in I} \dim \Gamma^q = \dim W.$$

This is just enough information to be able to give a general expression for the induced representation, but it is also already possible to get some hint as to the form of $\Gamma(Q^a)$ as well. Consider the Lie product

$$[\, J_{+i} \,, Q^a \,] = -D(J_{+i})^{ab} Q^b.$$

In the representation Γ, the left-hand side is zero because $\Gamma(J_{+i})$ vanishes. This leaves

$$D(J_{+i})^{ab} \Gamma(Q^b) = 0.$$

Comparison with section 6.2 shows that this is very similar to the equation satisfied by the coefficient matrix u in the spinor field expansion. So something of the form

$$\Gamma(Q^a) \sim u^a \otimes \text{something else}$$

might be expected. Further examination of this subject will be left to the next section.

Turning to the induction of the representation, the only remaining basis elements in the Poincaré superalgebra are $J_{0d\,-1}$ and J_{-i}. Using the "more practical" formulation of definition 2.12 in section 2.6, the basis for the carrier space of the induced representation is the set of elements

$$J_{0d-1}{}^{n_0} J_{-1}{}^{n_1} \dots J_{-d-2}{}^{n_{d-2}} \otimes e_\sigma \tag{7.6}$$

with $n_0, \dots, n_{d-2} \in \mathbb{N}$ and $\{e_\sigma\}$ a basis for W.

Fortunately, it is not necessary to deal with this basis. As stated in chapter 2, it is possible to perform the induction in stages. The obvious first step is to induce a representation of the Poincaré algebra iso(d–1,1). The carrier space is exactly the same as that for the final representation.

Let Φ be the induced representation of the Poincaré algebra iso(d–1,1) carried by the vector space with basis given by (7.6). According to theorem 3.5, the corresponding *produced* representation would be equivalent to an induced representation of the Poincaré group, P. As discussed in chapter 5, the same holds for this *induced* representation Φ and a representation of P carried by quantum field operators. This allows the (intractable) basis (7.6) to be abandoned.

Equations (7.3), (7.4) and (7.5) give rise to an induced representation of P which is a direct sum of irreducible induced representations of P, each of which corresponds to a massless elementary particle. Let $a_\sigma(p)$ be a set of annihilation operators with p on the forward light cone. The representation Φ of P induced from (Γ,W) is given by (cf. equation 5.16)

$$\Phi(t,\Lambda)\, a_\sigma(p)\Phi(t,\Lambda)^{-1} = e^{-i\Lambda p\cdot t/\hbar}\Gamma(R_p(\Lambda)^{-1})_{\sigma\tau}\, a_\tau(\Lambda p), \tag{7.7a}$$

for all (t,Λ) in P. Alternatively, the annihilation operators can be labelled $a\,{}^q_\alpha(p)$, in which case, (7.7a) becomes (using (7.5))

$$\Phi(t,\Lambda) a\,{}^q_\alpha(p)\Phi(t,\Lambda)^{-1} = e^{-i\Lambda p\cdot t/\hbar}\Gamma^q(R_p(\Lambda)^{-1})_{\alpha\beta}\, a\,{}^q_\beta(\Lambda p), \tag{7.7b}$$

for all (t,Λ) in P (there is, of course, no summation over the q in Γ). The little group element $R_p(\Lambda)$ is, from equation (5.15),

$$R_p(\Lambda) = B(\Lambda p)^{-1}\Lambda\, B(p), \tag{7.8}$$

where the $B(p)$ are the coset representatives for p on the forward light cone, with

$$B(p)k = p. \tag{7.9}$$

Equations (7.7) can be differentiated to find the Lie algebra representation, but it is much more convenient to stay with the group version.

As the last step, the representation of the Poincaré superalgebra must be induced from Φ. Since the carrier space for the final representation has the same basis, it will be carried by the same annihilation operators. In fact, the same symbol Φ can be used without any risk of confusion. A preliminary result must be established first.

Proposition 7.1 For *any* representation Φ of the Poincaré superalgebra (and its natural extension to incorporate the Lorentz group L),

$$\Phi(0,\Lambda)^{-1}\Phi(Q^a)\Phi(0,\Lambda) = D(\Lambda)^{ab}\Phi(Q^b), \tag{7.10}$$

for all Λ in L.

Proof. In the representation Φ, the Lie product of $J_{\lambda\mu}$ and Q^a is, by (7.1),

$$[\,\Phi(J_{\lambda\mu})\,,\,\Phi(Q^a)\,] = -D(J_{\lambda\mu})^{ab}\Phi(Q^b).$$

The result follows immediately by exponentiation. □

From section 3.1, the bases of the group and algebra forms of the induced representation match at the identities of the coset spaces. That is, $a_\sigma(k)$ and $1 \otimes e_\sigma$ have the same kind of transformation. Since

$$\Phi(Q^a)(1 \otimes e_\sigma) = 1 \otimes \Gamma(Q^a)_{\tau\sigma} e_\tau ,$$

the transformation for $a_\sigma(k)$ is

$$[\Phi(Q^a)\,,\,a_\sigma(k)\,] = \Gamma(Q^a)_{\tau\sigma}\, a_\tau(k). \tag{7.11}$$

The brackets $[\,\cdot\,,\cdot\,]$ denote a graded commutator, the grading of the annihilation and creation operators being inherited from the (graded) Fock space.

From (7.8), (7.9) and (7.7*a*),

$$\Phi(0,B(p))a_\sigma(k)\Phi(0,B(p))^{-1} = a_\sigma(p). \tag{7.12}$$

So

$$[\Phi(Q^a)\,,\,a_\sigma(p)\,]$$
$$= [\Phi(Q^a)\,,\,\Phi(0,B(p))\,a_\sigma(k)\,\Phi(0,B(p))^{-1}\,] \qquad \text{by (7.12)}$$

$$= \Phi(0,B(p))\,[\Phi(0,B(p))^{-1}\Phi(Q^a)\,\Phi(0,B(p))\,,\,a_\sigma(k)\,]\Phi(0,B(p))^{-1}$$

$$= D(B(p))^{ab}\,\Phi(0,B(p))\,[\Phi(Q^b)\,,\,a_\sigma(k)\,]\Phi(0,B(p))^{-1} \qquad \text{by (7.10)}$$

$$= D(B(p\,))^{ab}\,\Gamma(Q^b)_{\tau\sigma}\;a_\tau(p) \qquad \text{by (7.11) and (7.12)}$$

The second-last step is equivalent to shuffling Q^a through the monomial in the basis element (7.6), but it is much simpler to do in the group version, as here.

That completes the induction of the Poincaré superalgebra representation. The results of this section are summarised in the following theorem.

Theorem 7.2 Let χ be the one-dimensional representation

$$\chi(P_\lambda) = k_\lambda$$

with $k = (k^0,0,\ldots,0,k^0)$ $(k^0 > 0)$. The stability superalgebra $s(\chi)$ is spanned by $\{P_\lambda, Q^a, J_{ij}, J_{+i}\}$. Let Γ be an irreducible representation of $s(\chi)$, carried by a finite-dimensional vector space W, such that

$$\Gamma(P_\lambda) = k_\lambda 1.$$

Then

(*a*) $\quad \Gamma(J_{+i}) = 0.$

(*b*) The basis of W can be arranged so that

$$\Gamma(J_{ij}) = \sum_{q\in I}\oplus\Gamma^q(J_{ij})$$

where the Γ^q are irreducible representations of so(d–2). Let $a_\sigma(p)$ be a set of annihilation operators carrying Γ, and Φ be the representation of the Poincaré superalgebra induced from (Γ,W). Extending Φ to include the Poincaré group P, its action is

$$\Phi(t,\Lambda)\,a_\sigma(p)\Phi(t,\Lambda)^{-1} = e^{-i\Lambda p\cdot t/\hbar}\,\Gamma(R_p(\Lambda)^{-1})_{\sigma\tau}\,a_\tau(\Lambda\,p),$$

$$[\Phi(Q^a)\,,\,a_\sigma(p)\,] = D(B(p))^{ab}\,\Gamma(Q^b)_{\tau\sigma}\;a_\tau(p), \qquad (7.13)$$

for all (t,Λ) in P, where

$$R_p(\Lambda) = B(\Lambda p)^{-1} \Lambda B(p),$$

and

$$B(p)k = p.$$

Comment. The set of annihilation operators $a_\sigma(p)$ (or better, $a^{\,q}_{\alpha}(p)$) form a supermultiplet. Equation (7.13) gives the supertransformation laws within the supermultiplet. Just which particles belong to the supermultiplet, and precisely what the supertransformation laws are rests entirely on the choice of Γ. The alternative notation for equations (7.13) is

$$\begin{aligned} \Phi(t,\Lambda)\, a^{\,q}_{\alpha}(p) \Phi(t,\Lambda)^{-1} &= e^{-i\Lambda p\cdot t/\hbar} \Gamma^q(R_p(\Lambda)^{-1})_{\alpha\beta}\, a^{\,q}_{\beta}(\Lambda p), \\ [\Phi(Q^a), a^{\,q}_{\alpha}(p)] &= \sum_{q' \in I} D(B(p))^{ab}\, \Gamma(Q^b)^{q'q}_{\alpha'\alpha}\; a^{\,q'}_{\alpha'}(p). \end{aligned} \tag{7.14}$$

7.3 Finite-dimensional, irreducible representations of the stability superalgebra

To complete the determination of the supermultiplet and transformation laws of theorem 7.2, it is necessary to fix the irreducible representation Γ of the stability superalgebra $s(\chi)$. This can be done in three steps. First, the odd generators can be divided into two sets, which can be treated separately. Secondly, it simplifies matters greatly to use a certain basis for $s(\chi)$ with real structure constants. Finally, it is possible to decouple the odd part of $s(\chi)$ from the even part, yielding a simple but general construction for all of the finite-dimensional, irreducible representations.

As discussed in section 7.2, the stability superalgebra $s(\chi)$ is spanned by the set $\{P_\lambda, Q^a, J_{ij}, J_{+i}\}$. The representation Γ of P_λ and J_{+i} was already found to be

$$\Gamma(P_\lambda) = k_\lambda 1,$$
$$\Gamma(J_{+i}) = 0.$$

So only J_{ij} and Q^a remain to be considered. $\Gamma(J_{ij})$ was already found to have the form

$$\Gamma(J_{ij}) = \sum_{q\in I} \oplus \Gamma^q(J_{ij}),$$

but the set I of irreducible so($d-2$) representations is yet to be determined.

From (7.1), the relevant graded Lie products are

$$\begin{aligned}
[J_{ij}, Q^a] &= -D(J_{ij})^{ab} Q^b \\
[Q^a, Q^b] &= (\gamma^\lambda C)^{ab} P_\lambda . \qquad (7.15)
\end{aligned}$$

For consistency with the spinor fields of section 6.2, the construction of the γ matrices given in the appendix will be used. A Majorana representation here would perhaps be more convenient (it would lead more quickly to (7.26). The final results are valid for any representation, upon application of the appropriate similarity transformation. Take

$$\gamma_0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} \tau_i & 0 \\ 0 & -\tau_i \end{pmatrix}, \quad \gamma_{d-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
$$C = \begin{pmatrix} 0 & \varepsilon {}^*Z \\ \varepsilon Z & 0 \end{pmatrix}, \tag{7.16}$$

where τ_i are the "γ" matrices for so(d–2) (with negative-definite metric tensor), Z is a unitary charge conjugation matrix for the τ matrices and the factor ε depends on the dimension d of space-time:

$$\varepsilon = \begin{cases} 1 & \text{for } d = 0{,}1 \bmod 4 \\ i & \text{for } d = 2{,}3 \bmod 4. \end{cases}$$

In the spinor representation D of so(d–1,1), the generators J_{ij} of Spin(d–2) decompose to give

$$D(J_{ij}) = \begin{pmatrix} \Delta(J_{ij}) & 0 \\ 0 & \Delta(J_{ij}) \end{pmatrix}, \tag{7.17}$$

where

$$\Delta(J_{ij}) = \tfrac{i\hbar}{4}\,[\tau_i, \tau_j] \tag{7.18}$$

is the spinor representation of Spin(d–2).

Let the matrix elements of Δ be labelled by indices r,s taking values $1,2,\ldots,2^{l-1}$ (where $l = d/2$ if d is even, and $(d-1)/2$ if d is odd), and let

$$z = 2^{l-1}. \tag{7.19}$$

The decomposition (7.17) of D suggests splitting the odd generators into two sets Q^r_+ and Q^r_- defined by

$$Q_+^r = Q^r,$$
$$Q_-^r = Q^{r+z} \qquad r = 1,2,\ldots,z \tag{7.20}$$

(the + and – labels here are two arbitrary designations).

Proposition 7.3 In terms of the $Q_\pm^r$ of (7.20),

(a) the graded Lie product with the J_{ij} becomes

$$[\, J_{ij}\, , Q_\pm^r\,] = -\Delta\,(J_{ij})^{rs} Q_\pm^s\,, \tag{7.21}$$

(b) in the representation Γ of $s(\chi)$,

$$[\,\Gamma(Q_+^r)\,,\Gamma(Q_+^s)\,] = -2\,\varepsilon\, k^0 Z^{rs} \tag{7.22}$$
$$[\,\Gamma(Q_+^r)\,,\Gamma(Q_-^s)\,] = 0$$
$$[\,\Gamma(Q_-^r)\,,\Gamma(Q_-^s)\,] = 0.$$

Proof. (a) follows by inspection from (7.15) and (7.17).

(b) In the representation Γ, (7.15) gives

$$\begin{aligned}[\,\Gamma(Q^a)\,,\Gamma(Q^b)\,] &= (\gamma^\lambda C\,)^{ab}\Gamma(P_\lambda) \\ &= k^0(\,(\gamma_0 + \gamma_{d-1})C\,)^{ab} \qquad \text{by (7.3)}\end{aligned}$$

In block matrix form, the right-hand side of this expression is, by (7.16)

$$\begin{aligned}k^0\,(\gamma_0 + \gamma_{d-1})C &= k^0\left(\begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & \varepsilon\,{}^*Z \\ \varepsilon Z & 0 \end{pmatrix}\right) \\ &= -2\,\varepsilon\, k^0\begin{pmatrix} Z & 0 \\ 0 & 0 \end{pmatrix}\end{aligned}$$

In block form, $\Gamma(Q^a)$ is

$$\Gamma(Q) = \begin{pmatrix} \Gamma(Q_+) \\ \Gamma(Q_-) \end{pmatrix}$$

yielding the desired result. □

Comment. The representatives $\Gamma(J_{ij})$, $\Gamma(J_{i\,+})$, $\Gamma(Q_\pm^r)$ and the matrix $k^\lambda(\gamma_\lambda C\,)$ form what is often called the "little superalgebra". This is not the same as the little algebra defined in section 3.3: it only exists (or closes) in the representation Γ, whereas the little algebra of section 3.3 is a subalgebra of the main algebra.

Proposition 7.3 shows that, as far as the representation Γ of $s(\chi)$ is concerned, the generators $Q^{\,r}_{+}$ and $Q^{\,s}_{-}$ decouple to become two independent sets. Furthermore, from part (*b*), bearing in mind that $[\,\cdot\,,\cdot\,]$ here is an anticommutator of matrices,

$$\Gamma(Q^{\,r}_{-})^2 = 0.$$

By Engel's theorem, any finite-dimensional representation Γ of $Q^{\,r}_{-}$ can thus be made to be super-upper-triangular (ie. with zeros on the diagonal). So the only irreducible representation is

$$\Gamma(Q^{\,r}_{-}) = 0. \tag{7.23}$$

It is worth pausing here to note the origin of (7.23). In most treatments of Poincaré supersymmetry, some sort of semi-physical justification is given for this step. For example, it is argued that the application of $Q^{\,r}_{-}$ to physical states would give states of zero norm, which are not physically interesting, so it is best to set the action of $Q^{\,r}_{-}$ to zero. By keeping the mathematical and physical aspects further apart, as in the presentation here, it can be seen that no physical reasoning is required: (7.23) is a necessary condition for an irreducible representation of $s(\chi)$, regardless of any physical interpretation.

Turning to the second step mentioned at the start of this section, a transformation of basis for $s(\chi)$ will simplify the determination of the remaining elements $\Gamma(J_{ij})$ and $\Gamma(Q^{\,r}_{+})$ of the representation.

With a suitable choice of conventions for the τ matrices, the representation Δ of (7.18) is unitary (see appendix for details). In all dimensions $d = 0,1,2,3,4 \bmod 8$, Δ is a potentially real representation of Spin(d–2). So there exists a unitary 2^{l-1}-dimensional square matrix S such that

$$\Delta^R(J_{ij}) = S^{-1}\Delta(J_{ij})S \tag{7.24}$$

defines a real, orthogonal representation Δ^R of Spin(d–2). (Note: this is not the same S as in section 7.1).

Proposition 7.4 The charge conjugation matrix Z for the τ matrices can be taken as

$$Z = SS^{\mathrm{T}}$$

(where $^{\mathrm{T}}$ stands for transpose).

Proof. As shown in the appendix, Z must satisfy

$$Z^{\dagger}Z = 1$$

and

$$Z^{-1}\Delta(R)Z = \Delta^*(R)$$

for all R in Spin(d–2). For the first of these,

$$(SS^{\mathrm{T}})^{\dagger}SS^{\mathrm{T}} = (S^{\mathrm{T}})^{\dagger}S^{\dagger}SS^{\mathrm{T}} = 1,$$

as required. For the second,

$$\begin{aligned} \Delta^*(R) &= (S\Delta^R(R)S^{\dagger})^* && \text{since } S \text{ is unitary} \\ &= S^*\Delta^R(R)S^{\mathrm{T}} && \text{since } \Delta^R \text{ is real} \\ &= S^*S^{-1}\Delta(R)SS^{\mathrm{T}} \\ &= (SS^{\mathrm{T}})^{-1}\Delta(R)SS^{\mathrm{T}}, \end{aligned}$$

from which the result follows. □

Using the matrix S, the basis for $s(\chi)$ can now be transformed to make the structure constants real. However, only the J_{ij} and Q_+^r graded Lie products are needed.

Proposition 7.5 A dimensionless basis for the real Lie superalgebra $s(\chi)$ is given by

$$K_\lambda = \tfrac{i\zeta}{\hbar} P_\lambda$$
$$M_{ij} = \tfrac{i}{\hbar} J_{ij}$$
$$M_{+i} = \tfrac{i}{\hbar} J_{+i}$$
$$\Theta^r_\pm = \eta\, (S^{-1})^{rs} Q^s_\pm,$$

where ζ is an arbitrary but fixed length, and η is a constant such that

$$\eta^2 = \frac{-1}{\varepsilon k_0}.$$

In this basis,

$$[\, M_{ij}\, , \Theta^r_\pm\,] = -\Delta^R (M_{ij})^{rs} \Theta^s_\pm\, , \tag{7.25}$$

$$[\, \Gamma(\Theta^r_+)\, , \Gamma\, (\Theta^s_+)\,] = 2\, \delta^{rs}\, 1. \tag{7.26}$$

Proof. The dimensionless basis elements for the real Lie algebra iso(d–1,1) were discussed in section 4.1. What is new here is the $\Theta^r_\pm$. By inspection, $\Theta^r_\pm$ is dimensionless, and (7.25) follows easily from (7.24) and (7.21). Finally,

$$\begin{aligned}
[\Gamma(\Theta^r_+), \Gamma(\Theta^s_+)] &= \eta^2 (S^{-1})^{rr'} (S^{-1})^{ss'} [\Gamma(Q^{r'}_+), \Gamma(Q^{s'}_+)] \\
&= \frac{-1}{\varepsilon k_0} (S^{-1})^{rr'}\, (-2\varepsilon k^0) Z^{r's'} (S^{-1})^{ss'} \\
&= 2(S^{-1} Z\, (S^{\mathrm{T}})^{-1})^{rs} \\
&= 2\, \delta^{rs}\, 1. \qquad \text{by proposition 7.4.} \qquad \square
\end{aligned}$$

At this point, it is worth considering other constructions of the γ matrices and making a few remarks about the particular case of $d = 2$ mod 8.

Note that, since $\Gamma(Q^r_-) = 0$, the complete set of odd generators can be written as

$$\Gamma(Q^a) = \delta^{ar} \Gamma(Q^r_+).$$

The factor δ^{ar} in this expression matches the coefficient matrix u which was used in the construction of the spinor field in section 6.2 (and which arose in a similar manner). So

$$\begin{aligned} \Gamma(Q^a) &= u^{ar}\Gamma(Q^r_+) \\ &= \eta^{-1}u^{ar}\, S^{rs}\,\Gamma(\Theta^s_+). \end{aligned} \tag{7.27}$$

Recall that for $d = 2 \bmod 8$, the representation Δ of Spin($d-2$) decomposes into the direct sum of two potentially real representations. This means that one or the other of the corresponding subsets of the $\Gamma(Q^a)$ can be set to zero. To express this, the symbol Δ can be re-interpreted in $d = 2 \bmod 8$ as one of the potentially real, irreducible representations and the indices r,s as ranging over $1,2,\ldots,2^{l-2}$. The matrix S can also be adjusted accordingly.

Let n be 2^{l-2} for $d = 2 \bmod 8$, and 2^{l-1} otherwise. For the rest of this chapter, Δ will denote an n-dimensional, potentially real, unitary, spinor representation of Spin(d–2) (for $d = 2 \bmod 8$ there are two possibilities). Δ^R will be an equivalent real, orthogonal representation and S the unitary equivalence matrix. The indices r,s will range over $1,2,\ldots,n$. The basis elements Θ^r_+ will be taken such that the graded Lie products (7.25) and (7.26) hold.

Other constructions of the γ matrices will differ from the one used here by a similarity transformation. This same similarity transformation will also apply to the Q^a, and can be absorbed into the spinor coefficient matrix u. This matrix is then defined by the two conditions

$$\begin{aligned} D(J_{+i})^{ab}u^{br} &= 0, \\ D(J_{ij})^{ab}u^{bs} &= u^{ar}\,\Delta(J_{ij})^{rs}, \end{aligned} \tag{7.28}$$

and equation (7.27) gives the $\Gamma(Q^a)$ in terms of the non-trivial ones.

Equation (7.26) suggests straight away that the $\Gamma(\Theta^r_+)$ are the generators of a Clifford algebra. However, before making use of this, it is worth making one final adjustment which will decouple the odd and even elements M_{ij} and Θ^r_+. Recall that the finite-dimensional carrier space for Γ is W, and that the indices σ,τ,υ take values $1,2,\ldots\dim W$.

Proposition 7.6 Define the $\dim W \times \dim W$ matrices $\Gamma(L_{ij}) = -\Gamma(L_{ji})$ by

$$\Gamma(L_{ij}) = \Gamma(M_{ij}) - \tfrac{1}{4}\,\Gamma(\Theta^r_+)\Delta^R(M_{ij})^{rs}\,\Gamma(\Theta^s_+), \tag{7.29}$$

or, in component form,

$$\Gamma(L_{ij})_{\sigma\tau} = \Gamma(M_{ij})_{\sigma\tau} - \tfrac{1}{4}\,\Gamma(\Theta^r_+)_{\sigma\upsilon}\,\Delta^R(M_{ij})^{rs}\,\Gamma(\Theta^s_+)_{\upsilon\tau}$$

(note the summation over the υ). These matrices, together with the $\Gamma(\Theta^r_+)$ and the unit matrix 1, form the basis of a real Lie superalgebra with graded Lie products

$$\begin{aligned}
[\Gamma(L_{ij}),\Gamma(L_{kl})] &= -\delta_{ik}\,\Gamma(L_{jl}) + \delta_{il}\,\Gamma(L_{jk}) - \delta_{jl}\,\Gamma(L_{ik}) + \delta_{jk}\,\Gamma(L_{il}) \\
[\Gamma(L_{ij}),1] &= 0 \\
[\Gamma(L_{ij}),\Gamma(\Theta^r_+)] &= 0 \\
[1,\Gamma(\Theta^r_+)] &= 0 \\
[\Gamma(\Theta^r_+),\Gamma(\Theta^s_+)] &= 2\,\delta^{rs}\,1.
\end{aligned} \tag{7.30}$$

Proof. Only the first and middle products need to be verified, the last comes from proposition 7.5. The proof is straightforward, but long. The middle relation will be verified first. Note that

$$[\Gamma(M_{ij}),\Gamma(\Theta^r_+)] = -\Delta^R(M_{ij})^{rs}\Gamma(\Theta^s_+) \qquad \text{by (7.25)},$$

and

$$\begin{aligned}
&[\Gamma(\Theta^{r'}_+)\Delta^R(M_{ij})^{r's'}\,\Gamma(\Theta^{s'}_+),\Gamma(\Theta^r_+)] \\
&\quad = \Delta^R(M_{ij})^{r's'}\{\Gamma(\Theta^{r'}_+)[\Gamma(\Theta^{s'}_+),\Gamma(\Theta^r_+)] \\
&\qquad\qquad\qquad - [\Gamma(\Theta^{r'}_+),\Gamma(\Theta^r_+)]\Gamma(\Theta^{s'}_+)\} \\
&\quad = \Delta^R(M_{ij})^{r's'}\{\,2\delta^{rs'}\Gamma(\Theta^{r'}_+) - 2\delta^{rr'}\Gamma(\Theta^{s'}_+)\} \\
&\quad = -4\Delta^R(M_{ij})^{rs'}\Gamma(\Theta^{s'}_+),
\end{aligned}$$

using

$$\Delta^R(M_{ij})^{\mathrm{T}} = -\Delta^R(M_{ij}) \qquad (\Delta^R \text{ is orthogonal}).$$

Combining these two expressions with the definition (7.29) (note the factor $\tfrac{1}{4}$) gives the desired result (the middle relation).

Turning to the first relation, the result just obtained can be used to give

$$[\Gamma(L_{ij}), \Gamma(\Theta_+^r)\Delta^R(M_{kl})^{rs}\Gamma(\Theta_+^s)] = 0. \tag{7.31}$$

The remaining expression needed is

$$\begin{aligned}
&[\Gamma(M_{ij}), \Gamma(\Theta_+^r)\Delta^R(M_{kl})^{rs}\Gamma(\Theta_+^s)] \\
&\quad = \Delta^R(M_{kl})^{rs}\{\Gamma(\Theta_+^r)[\Gamma(M_{ij}), \Gamma(\Theta_+^s)] \\
&\qquad\qquad + [\Gamma(M_{ij}), \Gamma(\Theta_+^r)]\Gamma(\Theta_+^s)\} \\
&\quad = \Delta^R(M_{kl})^{rs}\{-\Gamma(\Theta_+^r)\Delta^R(M_{ij})^{ss'}\Gamma(\Theta_+^{s'}) \\
&\qquad\qquad - \Delta^R(M_{ij})^{rr'}\Gamma(\Theta_+^{r'})\Gamma(\Theta_+^s)\} \\
&\quad = \Gamma(\Theta_+^r)\Delta^R([M_{ij}, M_{kl}])^{rs}\Gamma(\Theta_+^s).
\end{aligned} \tag{7.32}$$

This allows the verification of the first relation:

$$\begin{aligned}
&[\Gamma(L_{ij}), \Gamma(L_{kl})] = [\Gamma(L_{ij}), \Gamma(M_{kl})] \qquad \text{by (7.29) and (7.31)} \\
&\quad = [\Gamma(M_{ij}), \Gamma(M_{kl})] - \tfrac{1}{4}[\Gamma(M_{ij}), \Gamma(\Theta_+^r)\Delta^R(M_{kl})^{rs}\Gamma(\Theta_+^s)] \\
&\quad = \Gamma([M_{ij}, M_{kl}]) - \tfrac{1}{4}\Gamma(\Theta_+^r)\Delta^R([M_{ij}, M_{kl}])^{rs}\Gamma(\Theta_+^s)
\end{aligned}$$

by (7.32). The desired relations for $\Gamma(L_{ij})$ the follow directly from the Lie products for M_{ij}. □

Comment. $\Gamma(L_{ij})$ is often called the *superspin* .

Equations (7.30) (and especially the middle one) show that the Lie superalgebra spanned by $\Gamma(L_{ij})$, $\Gamma(\Theta_+^r)$ and 1 is a direct sum of an even Lie algebra and a superalgebra whose even part is one-dimensional and central. So the irreducible representations of the whole superalgebra can be constructed from irreducible representations of these sub-superalgebras.

From the first relation in equations (7.30), the $\Gamma(L_{ij})$ span an so(d–2) Lie algebra. The finite-dimensional irreducible representations of this are well known. Let $F_{ij} = -F_{ji}$ be a set of $m \times m$ matrices providing an irreducible representation of so(d–2).

From the last relation in equations (7.30), the $\Gamma\ (\Theta^r_+)$ generate the Clifford algebra associated with so(n) (recall $r = 1,2,\ldots,n$). Since n is even, there is only one finite-dimensional irreducible representation of the Clifford algebra (up to equivalence) — the "γ" matrices, which are $2^{n/2}$-dimensional. Let G^r be a set of $2^{n/2} \times 2^{n/2}$ matrices satisfying

$$\{ G^r, G^s \} = 2\, \delta^{rs}\, 1,$$

where $\{\cdot\,,\cdot\}$ is the matrix anticommutator and 1 is the identity matrix.

These matrices then combine to give the irreducible representation of the larger superalgebra:

$$\Gamma\ (L_{ij}) = F_{ij} \otimes 1,$$
$$\Gamma\ (\Theta^r_+) = 1 \otimes G^r.$$

The size $\dim W$ of the representation Γ is thus

$$\dim W = 2^{n/2} m\,. \tag{7.33}$$

This leads back to the irreducible representation of the M_{ij} by (7.29)

$$\Gamma\ (M_{ij}) = F_{ij} \otimes 1 + \tfrac{1}{4}\, 1 \otimes G^r \Delta^R (M_{ij})^{rs}\, G^s. \tag{7.34}$$

This completes the construction of the irreducible representations of the stability superalgebra $s(\chi)$. The results of this section are summarised in the following theorem.

Theorem 7.7 Let P_λ, Q^a, J_{ij}, J_{+i} be the generators spanning the stability superalgebra $s(\chi)$ of section 7.2. Let n be 2^{l-2} for $d = 2$ mod 8, and 2^{l-1} otherwise, and let Δ denote an n-dimensional, potentially real, unitary, spinor representation of so(d–2). Let u be the $2^l \times n$ matrix defined by

$$D\, (J_{+i})^{ab} u^{br} = 0,$$
$$D\, (J_{ij})^{ab} u^{bs} = u^{ar}\, \Delta\, (J_{ij})^{rs}$$

(where D is the spinor representation of so(d–1,1)), and S be a unitary $n \times n$ matrix transforming Δ to a real representation,

$$\Delta^R(J_{ij}) = S^{-1}\Delta(J_{ij})S .$$

Finally, let the constant η satisfy

$$\eta^2 = \frac{-1}{\varepsilon k_0} ,$$

where

$$\varepsilon = \begin{cases} 1 & \text{for } d = 0{,}1 \bmod 4 \\ i & \text{for } d = 2{,}3 \bmod 4. \end{cases}$$

Then the finite-dimensional irreducible representations (Γ, W) of $s(\chi)$ are all of the form

$$\begin{aligned} \Gamma(P_\lambda) &= k_\lambda 1, \\ \Gamma(J_{+i}) &= 0, \\ \Gamma(J_{ij}) &= -i\hbar F_{ij} \otimes 1 + \tfrac{1}{4} 1 \otimes G^r \Delta^R(J_{ij})^{rs} G^s \\ \Gamma(Q^a) &= \eta^{-1} u^{ar} S^{rs} 1 \otimes G^r, \end{aligned} \tag{7.35}$$

where the F_{ij} matrices provide an $m \times m$ irreducible representation of so$(d-2)$ and the G^r are a set of $2^{n/2} \times 2^{n/2}$ generators of a Clifford algebra, satisfying

$$\{G^r, G^s\} = 2\delta^{rs} 1.$$

This representation is $2^{n/2}m$ -dimensional.

Comments. (i) The form of $\Gamma(Q^a)$ ($u^a \otimes$ something else) anticipated in section 7.2 has been realised here.

(ii) It will generally be necessary to perform a similarity transformation on Γ to get it into the form used in section 7.2, where $\Gamma(J_{ij})$ was a direct sum of irreducible representations.

(iii) It is also possible to construct the representation of the Q^a by organising them into a set of Fermi annihilation and creation operators. Since this approach is also commonly used to construct the γ matrices themselves, it is clear that the two methods are equivalent.

7.4 Particle supermultiplets and transformation rules

Sections 7.2 and 7.3, and particularly theorems 7.2 and 7.7, provide a complete construction of the massless, irreducible, $N = 1$ Poincaré supermultiplets. However, in the form given, the particle content of a supermultiplet and the supersymmetric transformation laws are not readily seen. The purpose of this section is to clarify these two matters.

Finding the particle content of the supermultiplet amounts to finding the decomposition of the matrices $\Gamma(J_{ij})$ of theorem 7.7 into irreducible so(d–2) representations. It is simplest to consider just the smallest possible supermultiplet first. This minimal supermultiplet is the one obtained by taking the trivial so(d–2) representation $F_{ij} = 0$ in theorem 7.7. The particle content of the larger supermultiplets can be found by taking direct products of the minimal supermultiplet with non-trivial F_{ij}.

Assume then, that $F_{ij} = 0$. The decomposition of $\Gamma(J_{ij})$ is assisted by two observations. The first is that equation (7.34) with $F_{ij} = 0$,

$$\Gamma(M_{ij}) = \tfrac{1}{4} G^{r} \Delta^{R} (M_{ij})^{rs} G^{s} \tag{7.36}$$

specifies an embedding of so(d–2) in the Lie algebra so(n) (recall n is the dimension of Δ^{R}). Let the so(n) basis elements be denoted H^{rs} ($= -H^{sr}$). Then the so(d–2) basis elements in the embedding are

$$M_{ij} = -\tfrac{1}{2} \Delta^{R} (M_{ij})^{rs} H^{rs} . \tag{7.37}$$

In this embedding, the spinor representation of so(n) (generated by the G matrices) branches to the representation Γ of so(d–2). This is borne out by examining the right-hand side of (7.37) in the spinor representation:

$$
\begin{aligned}
&-\tfrac{1}{2}\Delta^R(M_{ij})^{rs} \times -\tfrac{1}{4}[G^r, G^s] \\
&\quad = \tfrac{1}{4}\Delta^R(M_{ij})^{rs} G^r G^s && \text{since } \Delta^R(M_{ij})^{\mathrm{T}} = -\Delta^R(M_{ij}) \\
&\quad = \Gamma(M_{ij}) && \text{by (7.36).}
\end{aligned}
$$

The second observation is that in the embedding (7.37), the vector representation of so(n) branches to the spinor representation Δ^R of so(d–2). This is also easy to verify. The vector representation of so(n) is

$$(H^{rs})^{tu} = -\delta^{rt}\delta^{su} + \delta^{ru}\delta^{st} \tag{7.38}$$

(cf. (4.16) for example). In this representation, the right-hand side of (7.37) becomes

$$\tfrac{1}{2}\Delta^R(M_{ij})^{rs}(\delta^{rt}\delta^{su} - \delta^{ru}\delta^{st}) = \Delta^R(M_{ij})^{tu}.$$

Reversing the order of these two observations gives a straightforward means of finding the decomposition of $\Gamma(J_{ij})$. First find an embedding of so(d–2) in so(n) such that the vector representation of so(n) branches to Δ^R. Then find the branching rule for the spinor representation of so(n) with the same embedding. The result is the same as the decomposition of $\Gamma(J_{ij})$.

The actual methods of finding these branching rules vary from case to case. They can be found in Yang and Wybourne (1986) and Strathdee (1987), where the multiplets for extended Poincaré supersymmetries also are given. The results for dimensions 8,9,10,11 are summarised in the following table. Multiplets containing representations of so(d–2) higher than the second-rank tensor are excluded (these would correspond to particles of spin > 2 in four dimensions; this limitation also excludes dimensions $d > 11$).

d	n	dim$F$	dim$W$	tensorial	spinorial
8	8	1	16	1 + 6 + 1	4 + 4*
		6	96	6 + 1 + 15 + 20 + 6	4* + 20' + 20'* + 4
9	8	1	16	7 + 1	8
		7	112	27 + 21 + 7 + 1	8 + 56
10	8	1	16	8	8'
		8	128	35 + 28 + 1	8' + 56
11	16	1	256	84 + 44	128

Table 7.8 Massless supermultiplets for d = 8 to 11

The first column in this table shows the dimension of space-time. The second shows the number of non-trivial odd generators (alternatively the size of the smallest real spinor representation of so(d–2)). The third column shows the size of the so(d–2) representatives F_{ij} in theorem 7.7 – the minimal supermultiplets arise when F_{ij} is trivial. The fourth column shows the size of the supermultiplet, given by

$$\dim W \;=\; 2^{n/2} \dim F\,.$$

Finally, the fifth and sixth columns show the decomposition of $\Gamma(J_{ij})$ into irreducible representations. These irreducible representations are labelled by their size, with a dash being used to distinguish spinor representations from tensor representations of the same size, and a star indicating a complex-conjugate representation.

As mentioned earlier, the non-minimal supermultiplets are obtained simply using the Clebsch-Gordan series for so(d–2). For example, in the second line of table 7.8,

$$\begin{aligned} 6\times(1+6+1) &= 6\times 1 \;+\; 6\times 6 \;+\; 6\times 1 \\ &= 6 \;+\; (1 \;+\; 15 \;+\; 20) \;+\; 6. \end{aligned}$$

For $d = 4$, the J_{ij} span so(2), so the F_{ij} are all one-dimensional. The value of n is 2, so all $d = 4$ supermultiplets are 2-dimensional, containing just a spin j particle and a spin $j + \frac{1}{2}$ particle ($j = 0, \pm\frac{1}{2}, \pm 1, \ldots$). If parity must be conserved alone, it is necessary to include complex conjugate representations, doubling the size of the supermultiplet.

One point which can be seen quickly is that the supermultiplets contain equal numbers of Bose and Fermi degrees of freedom. Since n (which is either 2^{l-1} or 2^{l-2}) is even, the spinor representation of so(n) reduces to two Weyl representations (still of so(n)) of equal size. The two subspaces are projected out using the matrices

$$\Pi_{\pm} = (1 \pm G')/2,$$

where

$$G' = G^1 G^2 \ldots G^n$$

is the analogue of γ^5 in 4 dimensions. It is not hard to see that

$$G' G^r = -G^r G' ,$$

so that

$$\Pi_+ G^r = G^r \Pi_-$$
$$\Pi_- G^r = G^r \Pi_+ ,$$

for any r in 1,2,...,n.

Thus the action of one G^r matrix on a vector in the "+" Weyl subspace yields a vector in the "–" subspace, and vice versa. This line of argument, coupled with the fact that the G^r carry a spinor representation of so(d–2), shows that one Weyl subspace will contain only tensorial representations of so(d–2), and the other only spinorial representations. Hence there are equal numbers of Bose and Fermi degrees of freedom, as is borne out in table 7.8.

The second matter requiring clarification is that of the supersymmetric transformation laws within a multiplet. These are essentially fixed in theorem 7.7 by choosing the matrices G^r. But it is desirable to perform a similarity transformation on the $\Gamma(J_{ij})$ to bring them into a direct sum form (equation (7.5)), and this same similarity transformation will affect the $\Gamma(Q^a)$. The problem, then, is to express the matrix elements of $\Gamma(Q^a)$ in the form $\Gamma(Q^a)^{q\,q'}_{\alpha\,\alpha'}$ where $q,q' \in I$, the set of so(d–2) representations in the supermultiplet, and α takes values $1,2,\ldots,\dim \Gamma^q$ etc.

To do this, recall from equation (7.21) in section 7.3 that the Q^r_+ (the "non-trivial" Q^a) and the J_{ij} have the graded Lie product

$$[J_{ij}, Q^r_+] = -\Delta(J_{ij})^{rs} Q^s_+ .$$

This holds in the representation Γ as well. Exponentiating to the universal covering group Spin(d–2) gives

$$\Gamma(R)\Gamma(Q^r_+)\Gamma(R^{-1}) = \Delta(R^{-1})^{rs}\,\Gamma(Q^s_+),$$

and hence

$$\Delta(R)^{rs}\,\Gamma(R)\Gamma(Q^s_+) = \Gamma(Q^r_+)\Gamma(R) \tag{7.39}$$

for all R in Spin(d–2).

With $\Gamma(R)$ a direct sum of irreducible representations $\Gamma^q(R)$, it is possible to partition $\Gamma(Q^r_+)$ in the same way. For an individual partition matrix $\Gamma(Q^r_+)^{qq'}$ where $q,q' \in I$, equation (7.39) becomes

$$\Delta(R)^{rs}\,\Gamma^q(R)\Gamma(Q^s_+)^{qq'} = \Gamma(Q^r_+)^{qq'}\Gamma^{q'}(R)$$

for all R in Spin(d–2). (There is no implicit sum over the repeated labels q and q'). The individual partition matrices $\Gamma(Q^r_+)^{qq'}$ are thus irreducible tensor operators. (This is immediately so for $d = 1,2,3$ mod 8, where Δ is irreducible. For $d = 0,4$ mod 8, it is necessary first to adjust the basis for Δ so that it decomposes into the two irreducible complex conjugate parts. Any such transformation will be absorbed in the spinor coefficient matrix u (see equations (7.28))). Application of the Wigner-Eckart theorem yields

$$\Gamma\,(Q^{\,r}_{\,+})^{q\,q'}_{\alpha\,\alpha'} = \left(\begin{smallmatrix}\Delta & q\\ r & \alpha\end{smallmatrix}\,\middle|\,\begin{smallmatrix}q'\\ \alpha'\end{smallmatrix}\right),$$

where the right-hand side is an so(d–2) Clebsch-Gordan coefficient. Using equation (7.27) gives

$$\Gamma\,(Q^{\,a})^{q\,q'}_{\alpha\,\alpha'} = u^{\,ar}\left(\begin{smallmatrix}\Delta & q\\ r & \alpha\end{smallmatrix}\,\middle|\,\begin{smallmatrix}q'\\ \alpha'\end{smallmatrix}\right), \tag{7.40}$$

which is the desired result.

It is finally possible to give an explicit expression for all the transformation rules within a Poincaré supermultiplet. Define $u(p)^{ar}$ as in chapter 5 for free quantum fields by

$$u(p)^{ar} = D\,(B(p)\,)^{ab}u^{br}.$$

Inserting equation (7.40) into the transformations (7.14) at the end of section 7.2 gives, for all (t,Λ) in P,

$$\Phi\,(t,\Lambda)\,a^{\,q}_{\,\alpha}(p)\Phi\,(t,\Lambda)^{-1} = \mathrm{e}^{-i\Lambda p\cdot t/\hbar}\Gamma^{q}\,(R_p(\Lambda\,)^{-1}\,)_{\alpha\beta}\,a^{\,q}_{\,\beta}(\Lambda\,p),$$

$$[\,\Phi(Q^{\,a})\,,a^{\,q}_{\,\alpha}(p)\,] = \sum_{q'\in I} u\,(p)^{ar}\left(\begin{smallmatrix}\Delta & q\\ r & \alpha\end{smallmatrix}\,\middle|\,\begin{smallmatrix}q'\\ \alpha'\end{smallmatrix}\right)a^{\,q'}_{\,\alpha'}(p). \tag{7.41}$$

Equations (7.41) are an explicit expression of the transformation and supertransformation rules of an unextended Poincaré supermultiplet in d = 0,1,2,3,4 mod 8 dimensions. Everything is expressed in terms which can be calculated or found from tables: the Clebsch-Gordan coefficients, the index set (table 7.8), and the spinor matrix function (section 6.2). The only awkward object is the matrix function $R_p(\Lambda\,)$, but this is a necessary complication of using irreducible representations.

A final comment. The supertransformation rules in (7.41) look very straightforward, almost as if they could have been guessed without so much work – $Q^{\,a}$ is a spinor, so the spinor matrix function $u(p)$ is needed, and the Clebsch-Gordan coefficients arise because the left-hand side transforms like a direct product of the spinor representation Δ and the Γ^q representation. And indeed, the precise evaluation of the coefficients has required only the last few pages. The bulk of the work in sections 7.3 and 7.4 has been directed at finding the particle content of a supermultiplet. This is the more difficult problem.

Chapter 8

Supersymmetry with the Lie superalgebra iosp($m,n|2$)

Although Poincaré supersymmetry is the supersymmetric theory most studied (and indeed, the word "supersymmetry" is often used just to mean Poincaré supersymmetry), there are many other Lie superalgebras which give rise to interesting physical theories. One such is the inhomogeneous, orthosymplectic Lie superalgebra iosp($m,n|2$). This chapter applies the method of produced representations to find irreducible representations of this superalgebra, and studies some of their properties.

In section 8.1, the superalgebra is defined and described. The method of produced representations is particularly easy to apply, and this is done for representations corresponding to massless particles. Unfortunately, the most straightforward form of these representations is impractical for interpretation as a field theory, so in section 8.2, a conversion is made to a more useful form. This is done for the simplest possible irreducible representation, yielding the massless scalar supermultiplet. In section 8.3, an inner product for this supermultiplet is developed, and this is used in section 8.4 to prove the validity of the Parisi-Sourlas mechanism directly in pseudo-Euclidean space, something which has not previously been done. Finally, section 8.5 concludes with a description of BRST theories. There have been a number of suggestions that the supersymmetry underlying BRST theories is iosp(d, $2 \mid 2$). Section 8.5 discusses this idea, and indicates a line of further investigation.

8.1 The iosp(m,n |2) superalgebra and its irreducible representations

This section sets out the definition of the inhomogeneous, orthosymplectic Lie superalgebra, and establishes notation and conventions for the remainder of the chapter. Since this superalgebra has a semi-direct product structure, with an Abelian, invariant sub-superalgebra, the methods of induction and production can be used to find irreducible representations. This is just a straightforward application of the procedures described in chapter 3. Unfortunately, the resulting representations do not lend themselves easily to physical interpretation, so some modification is required, but this task is left for the next section.

As might be inferred from the name, iosp(m,n | 2) contains both a pseudo-Euclidean part and a symplectic part. In this chapter, the (m,n)-dimensional pseudo-Euclidean space has $m,n > 1$ in order to allow a (1,1)-dimensional reduction in section 8.4. A "light-cone" coordinate system (cf. section 6.6) is used, with indices a,b,c,d taking the values $1,\ldots,m+n-2, +, -$, and indices λ,μ,ν,ρ the values $1,\ldots,m+n-2$. The reasons behind this notation will be clear in sections 8.4 and 8.5. For the symplectic space, indices $\alpha,\beta,\gamma,\delta$ are used, taking the values 1,2. As before, square brackets $[\,\cdot\,,\cdot\,]$ denote graded Lie products.

The metric tensor g for the pseudo-Euclidean space has the components $g^{ab} = g_{ab}$ with

$$g_{\lambda\mu} = \mathrm{diag}(-1,\ldots,-1,1,\ldots,1) \quad \text{(with -1 occuring $m-1$ times and 1 $n-1$ times)}$$

$$g_{-+} = g_{+-} = 1$$

$$g_{\pm\lambda} = g_{\lambda\pm} = 0$$

$$g_{++} = g_{--} = 0.$$

For the symplectic space, the (Hermitean) metric tensor Ω has $\Omega^{\alpha\beta} = \Omega_{\alpha\beta}$ with

$$\Omega_{\alpha\beta} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

The inhomogeneous orthosymplectic superalgebra iosp(m,n | 2) is a direct generalisation of the Lie algebra iso(m,n). It can be defined in terms of the graded Lie products of its basis elements. Consider a set of elements denoted J_{ab} ($= -J_{ba}$), $K_{\alpha\beta}$ ($= K_{\beta\alpha}$), $L_{a\beta}$, P_a, and Q_α. These span a vector space of dimension $\{ \frac{1}{2}(m+n)(m+n-1) + 3(m+n) + 5 \}$. Define a grading on this space such that J_{ab}, $K_{\alpha\beta}$ and P_a are even basis elements, and $L_{a\beta}$ and Q_α are odd. Equipped with the following graded Lie products this graded vector space becomes the complexification of the Lie superalgebra iosp(m,n | 2):

$$\begin{aligned}
[J_{ab}, J_{cd}] &= -i\hbar\,(g_{ac}J_{bd} - g_{bc}J_{ad} + g_{bd}J_{ac} - g_{ad}J_{bc}),\\
[K_{\alpha\beta}, K_{\gamma\delta}] &= -\hbar\,(\Omega_{\alpha\gamma}K_{\beta\delta} + \Omega_{\beta\gamma}K_{\alpha\delta} + \Omega_{\beta\delta}K_{\alpha\gamma} + \Omega_{\alpha\delta}K_{\beta\gamma}),\\
[J_{ab}, K_{\gamma\delta}] &= 0,\\
[J_{ab}, L_{c\delta}] &= -i\hbar\,(g_{ac}L_{b\delta} - g_{bc}L_{a\delta}),\\
[K_{\alpha\beta}, L_{c\delta}] &= -\hbar\,(\Omega_{\alpha\delta}L_{c\beta} + \Omega_{\beta\delta}L_{c\alpha}),\\
[L_{a\beta}, L_{c\delta}] &= i\hbar\,(\Omega_{\beta\delta}J_{ac} - ig_{ac}K_{\beta\delta}),\\
[J_{ab}, P_c] &= -i\hbar\,(g_{ac}P_b - g_{bc}P_a),\\
[K_{\alpha\beta}, Q_\gamma] &= -\hbar\,(\Omega_{\alpha\gamma}Q_\beta + \Omega_{\beta\gamma}Q_\alpha),\\
[L_{a\beta}, P_c] &= i\hbar\, g_{ac}Q_\beta\\
[L_{a\beta}, Q_\gamma] &= i\hbar\,\Omega_{\beta\gamma}P_a\\
[J_{ab}, Q_\gamma] &= [K_{\alpha\beta}, P_c] = 0,\\
[P_a, P_b] &= [P_a, Q_\beta] = [Q_\alpha, Q_\beta] = 0.
\end{aligned} \tag{8.1}$$

By inspection, the set $\{ \frac{i}{\hbar}J_{ab}, \frac{i}{\hbar}K_{\alpha\beta}, \frac{e^{i\pi/4}}{\hbar}L_{a\beta}, \frac{i\zeta}{\hbar}P_a, \frac{e^{i\pi/4}\zeta}{\hbar}Q_\alpha \}$ (where ζ is an arbitrary, fixed, non-zero length) forms a dimensionless basis for the *real* Lie superalgebra iosp(m,n | 2). The elements used in

(8.1) are more convenient because they may be represented by pseudo-Hermitean operators (ie. the even elements by Hermitean operators and the odd elements by anti-Hermitean operators).

In order to have some feel for the nature of iosp(m,n | 2), it can be regarded as a Poincaré algebra for a superspace with (m,n) even dimensions and 2 odd dimensions. The basis elements J_{ab} generate rotations in the even subspace, the $K_{\alpha\beta}$ rotations in the odd subspace, and the $L_{a\beta}$ super-rotations between even and odd subspaces. The inhomogeneous elements P_a and Q_α generate translations and super-translations.

Together, the elements J_{ab}, $K_{\alpha\beta}$ and $L_{a\beta}$ span the homogeneous part osp(m,n | 2) of the superalgebra. An alternative definition of this orthosymplectic superalgebra, which shows more clearly that it is a generalisation of the pseudo-orthogonal algebra, is as follows. An orthosymplectic metric tensor can be defined as a square supermatrix G (with $m+n$ even rows and 2 odd rows) having g_{ab} and $\Omega_{\alpha\beta}$ in the diagonal partitions and zero elsewhere. The orthosymplectic superalgebra osp(m,n | 2) is then the set of real square supermatrices M (with $m+n$ even rows and 2 odd rows) satisfying

$$M^{\mathrm{ST}}G + (-1)^{|M|}G\,M = 0,$$

where $^{\mathrm{ST}}$ denotes the supertranspose and $|M|$ the degree of M. This is clearly an extension of the similar formula for so(m,n).

The inhomogeneous elements P_a and Q_α span an Abelian, invariant sub-superalgebra, here denoted i(m,n | 2). Thus iosp(m,n | 2) has a "semi-direct product" structure which can be exploited to find irreducible representations by the methods of production or induction, as in chapters 2 and 3. Although the induced representations are the ones which are appropriate for the second quantised theories sought, the produced representations are more convenient for calculating the inner products which become the norms of states in the second quantised theory. So the procedures of sections 2.5, 2.6 and 3.3 will be used to construct the irreducible produced representations. The corresponding induced representations would differ only by a few signs. For comparison with section 3.3, the algebras denoted g, h, and k there are iosp($m,n \mid 2$), osp($m,n \mid 2$) and i($m,n \mid 2$) here.

Let χ be the one-dimensional, absolutely irreducible, representation of i($m,n \mid 2$) given by

$$\chi(P_-) = 2k_0,$$

$$\chi(P_+) = \chi(P_\lambda) = \chi(Q_\alpha) = 0,$$

where k_0 is an arbitrary, constant, non-zero momentum. Note that

$$\chi(P{\cdot}P + Q{\cdot}Q) = 0,$$

so the eventual representations will be a generalisation of the massless particle representations of section 4.3.

Proposition 8.1 The little superalgebra (called $h(\chi)$ in section 3.3) of the homogeneous sub-superalgebra osp($m,n \mid 2$) is isomorphic to iosp($m-1,n-1 \mid 2$) with basis elements $J_{\lambda\mu}$, $K_{\alpha\beta}$, $L_{\lambda\beta}$, $J_{+\lambda}$ and $L_{+\alpha}$, the last two sets forming a basis for the inhomogeneous part.

Proof. The little superalgebra for χ consists of all elements X of osp($m,n \mid 2$) for which

$$\chi([X, P_a]) = \chi([X, Q_\alpha]) = 0. \tag{8.2}$$

Since the Lie product of J_{ab} with Q_α vanishes, it follows that the little superalgebra includes the iso($m-1,n-1$) algebra spanned by $J_{\lambda\mu}$ and $J_{+\lambda}$ (the demonstration of this is the same as that given in section 4.3

for the little algebra for massless particles). Inspection of equations (8.1) quickly shows that $X = K_{\alpha\beta}$, $L_{\lambda\beta}$ and $L_{+\alpha}$ also satisfy (8.2). For example,

$$\chi([L_{+\alpha}, Q_\beta]) = \chi(i\hbar\Omega_{\alpha\beta}P_+) = 0.$$

Inspection of equations (8.1) also quickly shows that none of the remaining elements, $J_{-\lambda}$, J_{+-} and $L_{-\alpha}$, nor any linear combination of them satisfies (8.2). The elements $J_{\lambda\mu}$, $K_{\alpha\beta}$ and $L_{\lambda\beta}$ obviously span an osp(m–1,n–1 | 2) subalgebra of osp($m,n \mid 2$). All that is needed is the graded Lie products involving the $J_{+\lambda}$ and $L_{+\alpha}$, and these follow straightforwardly from (8.1), giving

$$\begin{aligned}
[J_{\lambda\mu}, J_{+\nu}] &= -i\hbar(g_{\lambda\nu}J_{+\mu} - g_{\mu\nu}J_{+\lambda}),\\
[K_{\alpha\beta}, L_{+\gamma}] &= -\hbar(\Omega_{\alpha\gamma}L_{+\beta} + \Omega_{\beta\gamma}L_{+\alpha}),\\
[L_{\lambda\beta}, J_{+\mu}] &= i\hbar g_{\lambda\mu}L_{+\beta}\\
[L_{\lambda\beta}, L_{+\gamma}] &= i\hbar\Omega_{\beta\gamma}J_{+\lambda}\\
[J_{\lambda\mu}, L_{+\gamma}] &= [K_{\alpha\beta}, J_{+\nu}] = 0,\\
[J_{+\lambda}, J_{+\mu}] &= [J_{+\lambda}, Q_\beta] = [Q_\alpha, Q_\beta] = 0.
\end{aligned}$$

Comparison of the products with those in (8.1) shows that the $J_{+\lambda}$ and $L_{+\alpha}$ do indeed span the inhomogeneous part of an iosp(m–1,n–1 | 2) superalgebra. □

Comment. This little superalgebra is clearly just an extension of the little algebra for massless particle representations of the Poincaré group. As in that case it should be stressed that the inhomogeneous part of the little superalgebra is *not* contained in the inhomogeneous part of the original superalgebra.

Irreducible representations of this little superalgebra can be obtained by repeated induction or production, with finite dimensional representations arising when the inhomogeneous part is represented trivially. Assume that this has been carried out, and that Γ is such an irreducible representation of iosp(m–1,n–1 | 2), carried by a finite-dimensional vector space W.

The stability superalgebra $s(\chi)$ is just the vector-space direct sum of the little superalgebra iosp(m–1,n–1 | 2) and the inhomogeneous part i(m,n | 2) of the original superalgebra. Let $\chi\Gamma$ denote the irreducible representation of $s(\chi)$ which is equal to χ for elements of i(m,n | 2) and Γ for elements of iosp(m–1,n–1 | 2). Since χ is one-dimensional, the carrier space for $\chi\Gamma$ is still W.

The elements $J_{-\lambda}$, J_{+-} and $L_{-\alpha}$ form a basis for the remainder of iosp(m,n | 2) (as a vector space only), and thus generate a basis for the universal enveloping superalgebra U(iosp(m,n | 2)) regarded as a left - module (see section 2.4, and, for example, Kac 1977). These generated basis elements are of the form

$$\prod_{\lambda} J_{-\lambda}^{\,r_\lambda} J_{+-}^{\,r_0} L_{-1}^{\,s_1} L_{-2}^{\,s_2} ,$$

where $r_0, r_\lambda \in \mathbb{N}$ and $s_1, s_2 \in \{0,1\}$. For the sake of clarity, these will be abbreviated to

$$J^{\,r} L^{\,s} = \prod_{\lambda} J_{-\lambda}^{\,r_\lambda} J_{+-}^{\,r_0} L_{-1}^{\,s_1} L_{-2}^{\,s_2} , \tag{8.3}$$

with $r \in \mathbb{N}^{m+n-2}$ and $s \in \{0,1\} \times \{0,1\}$. The order of the iosp($m$,$n$ | 2) factors in these monomials is not important, but it is convenient to use the order (8.3) in the produced representation.

With these definitions, the produced representation (Φ, V) of iosp(m,n | 2) can be determined. The construction of definition 2.7 in section 2.5 will be used. The carrier space of the produced representation is the space of $s(\chi)$-homomorphisms

$$V = \mathrm{Hom}_{s(\chi)}(\, \mathrm{U}(\, \mathrm{iosp}(m,n \,|\, 2)) \,, W \,).$$

This is the set of linear mappings ϕ from U(iosp(m,n | 2)) to W satisfying

$$\phi\,(Y A\,) = \chi\Gamma(Y\,)\, \phi\,(A\,) \tag{8.4}$$

for all $Y \in s(\chi)$ and $A \in$ U(iosp($m,n \mid 2$)). Each element X of iosp($m,n \mid 2$) is represented by a linear operator $\Phi(X)$ which acts on $\phi \in V$ to give $\Phi(X)\phi \in V$ defined by

$$\Phi(X)\phi(A) = \phi(AX). \tag{8.5}$$

As was seen in section 2.5, to characterise a function ϕ in V, it suffices to specify ϕ on the elements $J^r L^s$. That is, to find $\Phi(X)\phi$, it is enough to evaluate $\phi(J^r L^s X)$ for all $r \in \mathbb{N}^{m+n-2}$ and $s \in \{0,1\} \times \{0,1\}$. In principle, it is simple to do this – just use the graded Lie products (8.1) to reduce the term $J^r L^s X$ to a sum of terms of the form $B J^r L^s$ with $B \in$ U($s(\chi)$), and then use (8.4) – but in practice, this is too cumbersome to be useful. Besides being impractical, this formulation bears little resemblance to field theory. So the representation must be modified to an equivalent form carried by functions over coordinate or momentum space.

8.2 The irreducible scalar multiplet for iosp(m,n | 2)

An alternative, more practical realisation of the representations of iosp(m,n | 2) is required before explicit evaluation of the linear operators can take place. To have a field theory over momentum space, the irreducible representations of the iso(m,n) subalgebra should be involved. Unlike the Poincaré superalgebra example examined in chapter 7, it is not possible to simply perform the induction in stages, using iso(m,n) as an intermediate algebra. This is because for iosp(m,n | 2), the stability superalgebra $s(\chi)$ is not a subalgebra of iso(m,n). Still, it is possible to convert the iso(m,n) part of the representations of the last section into the equivalent Poincaré group representations, as will be seen here. In this and the next two sections, only the simplest representations, the scalar multiplets, will be considered, as these suffice to demonstrate the construction and the dimensional reduction. Higher order representations will again be examined in the final section.

Consider the produced representation (Φ,V). From equation (8.5), it is apparent that for any ϕ in V,

$$\phi(J^r L^s) = \Phi(L^s)\phi(J^r).$$

Since $s \in \{0,1\} \times \{0,1\}$, which has four elements, it follows that ϕ can be characterised completely by evaluating the four functions ϕ, $\Phi(L_{-1})\phi$, $\Phi(L_{-2})\phi$ and $\Phi(L_{-1}L_{-2})\phi$ on the J^r alone.

Comparison with section 4.3 shows that the J^r for $r \in \mathbb{N}^{m+n-2}$ are precisely the elements involved in a "massless" produced representation of the (m,n)-dimensional Poincaré algebra iso(m,n). Thus each of the four functions above, restricted to the space spanned by the J^r, carries a produced representation of the Poincaré algebra.

This will be the representation produced from the obvious restrictions of χ and Γ. As discussed in chapter 3, the produced representation is equivalent to one derived from an induced representation of the corresponding simply connected Lie group, in this case the universal covering group of $ISO_0(m,n)$.

Such an equivalence provides a way around the difficulty mentioned in the last section. The group representation is much easier to evaluate than the algebra one. So the four functions above can be replaced by an equivalent set of four functions carrying the group representation and the superalgebra representation then reconstructed from the latter four functions.

Induction of the representation of the Poincaré group proceeds along exactly the same lines as in chapter 4. Only a brief outline will be given here.

For the same χ as before (or rather, its exponentiation), the little group is isomorphic to the covering group of $ISO_0(m-1,n-1)$. A representation of this is then provided by exponentiating (Γ,W). The induced group representation is carried by analytic$W$-valued functions over the coset space $SO_0(m,n)/\ ISO_0(m-1,n-1)$. This coset space is isomorphic to the null surface in (m,n)-dimensional pseudo-Euclidean space, which can be parameterised by $m+n$ coordinates p^a satisfying $p\cdot p\ =0\ \ (p\neq 0)$. The main difference from the previous case is that when m and n are both greater than one, the null surface does not split into two independent halves with $p^0>0$ and $p^0<0$: there exist both pure space and pure time rotations.

Denote the induced representation of the Poincaré group by (Φ_0,V_0), where V_0 is the space of analytic W-valued functions over the null surface. For $\phi_0 \in V_0$, the generators of the group representation are (cf. equations (4.32))

$$\Phi_0(P_a)\,\phi_0(p) = p_a \phi_0(p),$$
$$\Phi_0(J_{ab})\phi_0(p) = i\hbar\left(p_a \frac{\partial}{\partial p^b} - p_b \frac{\partial}{\partial p^a}\right)\phi_0(p) + \Gamma_p(J_{ab})\phi_0(p),$$

for all p on the null surface, where

$$\Gamma_p(J_{ab}) = \Gamma_0\left(B(p)^{-1}\left(J_{ab} + i\hbar\left(p_a \frac{\partial}{\partial p^b} - p_b \frac{\partial}{\partial p^a}\right)\right)B(p)\right).$$

The $B(p)$ are a set of smoothly varying coset representatives for $SO_0(m,n)/ISO_0(m-1,n-1)$, satisfying

$$B(p)k = p,$$

where k is the fixed momentum on the null surface given by

$$\chi(P_a) = k_a.$$

Differing notation to distinguish the representation just given and the produced algebra representation is not required, since which one is being used can be seen from the argument of the carrier space function. The correspondence between the two is given (see section 3.1) for $\phi_0 \in V_0$ by

$$\phi_0(k) = \phi_0(1),$$

where 1 is the unit element of the enveloping algebra.

Returning to the superalgebra, it is convenient to introduce the notation

$$\phi(p,\alpha) = \{\Phi(L_{-\alpha})\phi\}(p), \tag{8.6}$$
$$\phi(p,\alpha\beta) = \{\Phi(L_{-\alpha}L_{-\beta})\phi\}(p) = -\phi(p,\beta\alpha).$$

Any function $\phi \in V$ is completely determined by evaluating $\phi(p)$, $\phi(p,\alpha)$, and $\phi(p,\alpha\beta)$ for $\alpha,\beta = 1,2$, and all points p on the null surface. So, for each X in iosp$(m,n\,|\,2)$ and $\phi \in V$, the complete specification of $\Phi(X)\phi$ (which is also a member of V) requires the calculation of $\{\Phi(X)\phi\}(p)$, $\{\Phi(X)\phi\}(p,\alpha)$, and $\{\Phi(X)\phi\}(p,\alpha\beta)$.

Note that the representation Φ_0 acts on functions over the null surface. Accordingly, operators like $\Phi_0(J_{ab})$ will be defined to act on functions like $\phi(p,\alpha)$ as if they were functions of p alone. That is,

$$\Phi_0(J_{ab})\phi(p,\alpha) = \Phi_0(J_{ab})\{\Phi(L_{-\alpha})\phi\}(p), \tag{8.7}$$

as opposed to $\Phi(L_{-\alpha})\{\Phi_0(J_{ab})\phi\}(p)$.

With these definitions, it is finally possible to evaluate the action of the operators of the produced representation from their definition (8.5). From here on, consideration will be restricted to the scalar multiplet, which is the representation in which Γ is trivial. That is,

$$\Gamma(J_{\lambda\mu}) = \Gamma(K_{\alpha\beta}) = \Gamma(L_{\lambda\alpha}) = \Gamma(J_{+\lambda}) = \Gamma(L_{+\alpha}) = 0. \tag{8.8}$$

Theorem 8.2 The operators of the produced representation (Φ,V) (with $\Gamma = 0$) of iosp$(m,n\,|\,2)$ defined in section 8.1, are given explicitly in the realisation of this section by

$$\begin{aligned}
\Phi(J_{\lambda\mu})\phi(p) &= \Phi_0(J_{\lambda\mu})\phi(p),\\
\Phi(J_{\lambda\mu})\phi(p,\alpha) &= \Phi_0(J_{\lambda\mu})\phi(p,\alpha),\\
\Phi(J_{\lambda\mu})\phi(p,\alpha\beta) &= \Phi_0(J_{\lambda\mu})\phi(p,\alpha\beta);
\end{aligned}$$

$$\begin{aligned}
\Phi(J_{-\lambda})\phi(p) &= \Phi_0(J_{-\lambda})\phi(p),\\
\Phi(J_{-\lambda})\phi(p,\alpha) &= \Phi_0(J_{-\lambda})\phi(p,\alpha),\\
\Phi(J_{-\lambda})\phi(p,\alpha\beta) &= \Phi_0(J_{-\lambda})\phi(p,\alpha\beta);
\end{aligned}$$

$$\Phi(J_{+\lambda})\phi(p) = \Phi_0(J_{+\lambda})\phi(p),$$
$$\Phi(J_{+\lambda})\phi(p,\alpha) = \Phi_0(J_{+\lambda})\phi(p,\alpha) + i\hbar\frac{p_\lambda}{p_-}\phi(p,\alpha),$$
$$\Phi(J_{+\lambda})\phi(p,\alpha\beta) = \Phi_0(J_{+\lambda})\phi(p,\alpha\beta) + 2i\hbar\frac{p_\lambda}{p_-}\phi(p,\alpha\beta) + (i\hbar)^2\Omega_{\alpha\beta}\Phi_0(J_{-\lambda})\phi(p);$$

$$\Phi(J_{+-})\phi(p) = \Phi_0(J_{+-})\phi(p),$$
$$\Phi(J_{+-})\phi(p,\alpha) = \Phi_0(J_{+-})\phi(p,\alpha) + i\hbar\,\phi(p,\alpha),$$
$$\Phi(J_{+-})\phi(p,\alpha\beta) = \Phi_0(J_{+-})\phi(p,\alpha\beta) + 2i\hbar\phi(p,\alpha\beta);$$

$$\Phi(K_{\alpha\beta})\phi(p) = 0,$$
$$\Phi(K_{\alpha\beta})\phi(p,\gamma) = \hbar\,(\Omega_{\alpha\gamma}\phi(p,\beta) + \Omega_{\beta\gamma}\phi(p,\alpha)),$$
$$\Phi(K_{\alpha\beta})\phi(p,\gamma\delta) = 0;$$

$$\Phi(P_\lambda)\phi(p) = p_\lambda\,\phi(p),$$
$$\Phi(P_\lambda)\phi(p,\alpha) = p_\lambda\,\phi(p,\alpha),$$
$$\Phi(P_\lambda)\phi(p,\alpha\beta) = p_\lambda\,\phi(p,\alpha\beta);$$

$$\Phi(P_-)\phi(p) = p_-\,\phi(p),$$
$$\Phi(P_-)\phi(p,\alpha) = p_-\,\phi(p,\alpha),$$
$$\Phi(P_-)\phi(p,\alpha\beta) = p_-\,\phi(p,\alpha\beta);$$

$$\Phi(P_+)\phi(p) = p_+\,\phi(p),$$
$$\Phi(P_+)\phi(p,\alpha) = p_+\,\phi(p,\alpha),$$
$$\Phi(P_+)\phi(p,\alpha\beta) = p_+\,\phi(p,\alpha\beta) + (i\hbar)^2\Omega_{\alpha\beta}p_-\,\phi(p);$$

$$\Phi(L_{\lambda\alpha})\phi(p) = \frac{p_\lambda}{p_-}\phi(p,\alpha),$$
$$\Phi(L_{\lambda\alpha})\phi(p,\beta) = -\frac{p_\lambda}{p_-}\phi(p,\alpha\beta) - i\hbar\,\Omega_{\alpha\beta}\Phi_0(J_{-\lambda})\phi(p),$$
$$\Phi(L_{\lambda\alpha})\phi(p,\beta\gamma) = -i\hbar\,\Omega_{\beta\gamma}\Phi_0(J_{-\lambda})\phi(p,\alpha);$$

$$\Phi(L_{-\alpha})\phi(p) = \phi(p,\alpha),$$
$$\Phi(L_{-\alpha})\phi(p,\beta) = -\phi(p,\alpha\beta),$$
$$\Phi(L_{-\alpha})\phi(p,\beta\gamma) = 0;$$

$$\Phi(L_{+\alpha})\phi(p) = \frac{p_+}{p_-}\phi(p,\alpha),$$
$$\Phi(L_{+\alpha})\phi(p,\beta) = -\frac{p_+}{p_-}\phi(p,\alpha\beta) + i\hbar\,\Omega_{\alpha\beta}\,\Phi_0(J_{+-})\phi(p),$$
$$\Phi(L_{+\alpha})\phi(p,\beta\gamma) = i\hbar\,\Omega_{\beta\gamma}\,\Phi_0(J_{+-})\phi(p,\alpha) + 2(i\hbar)^2\Omega_{\beta\gamma}\,\phi(p,\alpha);$$

$$\Phi(Q_\alpha)\phi(p) = 0,$$
$$\Phi(Q_\alpha)\phi(p,\beta) = -i\hbar\,\Omega_{\alpha\beta}p_-\phi(p),$$
$$\Phi(Q_\alpha)\phi(p,\beta\gamma) = -i\hbar\,\Omega_{\beta\gamma}p_-\phi(p,\alpha);$$

Proof. Most of these transformations can be verified by straightforward application of the construction (8.4) and (8.5), and the graded Lie products (8.1). The whole proof is extremely long and repetitive, so just a few key examples will be shown here; the remaining relations can be demonstrated in the same manner.

First take the $\Phi(J_{\lambda\mu})$ operator. Restricted to the null surface, ϕ carries the representation Φ_0, so

$$\Phi(J_{\lambda\mu})\phi(p) = \Phi_0(J_{\lambda\mu})\phi(p).$$

(The same is true for all other iso(m,n) elements). To evaluate $\Phi(J_{\lambda\mu})\phi$ at (p,α) just requires the definition (8.6) and the product $[J_{\lambda\mu}, L_{-\alpha}] = 0$:

$$\begin{aligned}\{\Phi(J_{\lambda\mu})\phi\}(p,\alpha) &= \{\Phi(L_{-\alpha})\Phi(J_{\lambda\mu})\phi\}(p)\\ &= \Phi(J_{\lambda\mu})\Phi(L_{-\alpha})\phi(p) + \Phi([L_{-\alpha}, J_{\lambda\mu}])\phi(p)\\ &= \Phi_0(J_{\lambda\mu})\phi(p,\alpha) \qquad \text{by (8.7).}\end{aligned}$$

Likewise

$$
\begin{aligned}
\{\Phi(J_{\lambda\mu})\phi\}(p,\alpha\beta) &= \{\Phi(L_{-\beta})\Phi(J_{\lambda\mu})\phi\}(p,\alpha) \\
&= \Phi(J_{\lambda\mu})\Phi(L_{-\beta})\phi(p,\alpha) \\
&= \Phi_0(J_{\lambda\mu})\phi(p,\alpha\beta).
\end{aligned}
$$

The relations for $J_{-\lambda}$, P_λ and P_- are derived in exactly the same way. The $K_{\alpha\beta}$ rule is equally straightforward.

Next consider $\Phi(J_{+-})$. Immediately,

$$\Phi(J_{+-})\phi(p) = \Phi_0(J_{+-})\phi(p),$$

and then, since $[L_{-\alpha}, J_{+-}] = i\hbar L_{-\alpha}$,

$$
\begin{aligned}
\{\Phi(J_{+-})\phi\}(p,\alpha) &= \{\Phi(L_{-\alpha})\Phi(J_{+-})\phi\}(p) \\
&= \Phi(J_{+-})\Phi(L_{-\alpha})\phi(p) + i\hbar\,\Phi(L_{-\alpha})\phi(p) \\
&= \Phi_0(J_{\lambda\mu})\phi(p,\alpha) + i\hbar\,\phi(p,\alpha).
\end{aligned}
$$

Simply repeating this gives

$$\Phi(J_{+-})\phi(p,\alpha\beta) = \Phi_0(J_{+-})\phi(p,\alpha\beta) + 2i\hbar\phi(p,\alpha\beta).$$

The evaluation of $\Phi(L_{\lambda\alpha})\phi(p)$ and $\Phi(L_{+\alpha})\phi(p)$ is not so straightforward, but the two are very similar, so just $\Phi(L_{\lambda\alpha})\phi(p)$ will be shown here. Consider the action of $L_{\lambda\alpha}$ on the element J^r. Using the graded Lie products (8.1), it is easy to see that

$$J_{+-}{}^r L_{\lambda\alpha} = L_{\lambda\alpha} J_{+-}{}^r$$

for any $r \in \mathbb{N}$. Similarly,

$$J_{-\mu} L_{\lambda\alpha} = L_{\lambda\alpha} J_{-\mu} - i\hbar g_{\lambda\mu} L_{-\alpha},$$

and

$$J_{-\mu}{}^r L_{-\alpha} = L_{-\alpha} J_{-\mu}{}^r,$$

for any $r \in \mathbb{N}$. From these it follows that, using the abbreviation (8.3),

$$J^r L_{\lambda\alpha} = L_{\lambda\alpha} J^r + J' L_{-\alpha}$$

for all $r \in \mathbb{N}^{m+n}$, where J' is some linear combination of the $J^{r'}$ ($r' \in \mathbb{N}^{m+n}$) dependent upon r and λ. Observing that $L_{\lambda\alpha}$ is an so(m–1,n–1) vector, just as P_λ is, it follows that

$$J^r P_\lambda = P_\lambda J^r + J' P_-$$

with exactly the same J'. Hence, since $[L_{-\alpha}, P_-] = 0$,

$$J^r L_{\lambda\alpha} P_- - L_{\lambda\alpha} J^r P_- = J^r P_\lambda L_{-\alpha} - P_\lambda J^r L_{-\alpha},$$

so that, using (8.4), (8.5) and (8.8),

$$\Phi(L_{\lambda\alpha} P_-)\phi(J^r) = \Phi(P_\lambda L_{-\alpha})\phi(J^r).$$

Finally, using $[L_{\lambda\alpha}, P_-] = 0$,

$$\Phi(L_{\lambda\alpha})\phi(p) = \frac{p_\lambda}{p_-}\phi(p,\alpha).$$

Since $[L_{-\alpha}, J_{+\lambda}] = i\hbar L_{\lambda\alpha}$, this can then be used to evaluate the rule for $J_{+\lambda}$.

The final non-trivial calculation is for Q_α. Since $[J^r, Q_\alpha] = 0$ (for $r \in \mathbb{N}^{m+n}$) and $\chi(Q_\alpha) = 0$, it quickly follows that

$$\Phi(Q_\alpha)\phi(p) = 0.$$

Repeating the same procedure as before (with a sign change because the graded Lie product is symmetric here),

$$\begin{aligned}\{\Phi(Q_\alpha)\phi\}(p,\beta) &= \{\Phi(L_{-\beta})\Phi(Q_\alpha)\phi\}(p) \\ &= -\Phi(Q_\alpha)\Phi(L_{-\beta})\phi(p) + \Phi([L_{-\beta}, Q_\alpha])\phi(p) \\ &= 0 + i\hbar\,\Omega_{\beta\alpha}\Phi(P_-)\phi(p) \qquad \text{by (8.1)} \\ &= -i\hbar\,\Omega_{\alpha\beta}\, p_-\,\phi(p).\end{aligned}$$

Similarly,

$$\begin{aligned}\{\Phi(Q_\alpha)\phi\}(p,\beta\gamma) &= -\Phi(Q_\alpha)\{\Phi(L_{-\gamma})\phi\}(p,\beta) \\ &\qquad + i\hbar\,\Omega_{\gamma\alpha}\Phi(P_-)\phi(p,\beta) \\ &= i\hbar\,\Omega_{\alpha\beta}\, p_-\,\phi(p,\gamma) - i\hbar\,\Omega_{\alpha\gamma}\, p_-\,\phi(p,\beta) \\ &= -i\hbar\,\Omega_{\beta\gamma}\, p_-\,\phi(p,\alpha).\end{aligned}$$

The last step can be verified by substituting actual values for α,β and γ. The Q_{α} rule then allows the P_{+} rule to be calculated (since $[L_{-\alpha}, P_{+}] = i\hbar Q_{\alpha}$). □

Comments. To demonstrate the irreducibility of this representation, consider the second order Casimir operator $P{\cdot}P + Q{\cdot}Q$. Acting on $\phi \in V$, $\Phi(P{\cdot}P + Q{\cdot}Q)$ clearly vanishes at (p) and (p,α). For $(p,\gamma\delta)$,

$$
\begin{aligned}
&\Phi(P^{a}P_{a} + Q^{\alpha}Q_{\alpha})\phi(p,\gamma\delta) \\
&\quad = \Phi(P^{\lambda}P_{\lambda} + 2P_{+}P_{-} + \Omega^{\alpha\beta}Q_{\beta}Q_{\alpha})\phi(p,\gamma\delta) \\
&\quad = p^{\lambda}p_{\lambda}\phi(p,\gamma\delta) + 2p_{+}p_{-}\phi(p,\gamma\delta) + 2(i\hbar)^{2}\Omega_{\gamma\delta}p_{-}^{\ 2}\phi(p) \\
&\qquad\qquad + \Omega^{\alpha\beta}(-i\hbar)\Omega_{\gamma\delta}p_{-}(-i\hbar)\Omega_{\alpha\beta}p_{-}\phi(p) \\
&\quad = 0, \qquad \text{since } \Omega^{\alpha\beta}\Omega_{\alpha\beta} = -2.
\end{aligned}
$$

Thus

$$\Phi(P^{a}P_{a} + Q^{\alpha}Q_{\alpha}) = 0,$$

which is evidence for an irreducible representation. Note also that the iso($m-1,n-1$) subalgebra is represented "covariantly".

8.3 An inner product for the irreducible scalar multiplet

Having established an irreducible representation of iosp$(m,n\,|\,2)$, an inner product is required under which the operators will be pseudo-Hermitean. This is necessary so that inner products in the supermultiplet are iosp-invariant. Again, the correspondence between Lie group and Lie algebra representations provides part of the answer. Since this section just extends part of the last, the notation and terminology of section 8.2 are used.

An invariant measure for the null surface is $d^{m+n}p\ \delta(p^2)$, so the operators $\Phi_0(J_{ab})$ and $\Phi_0(P_a)$ are Hermitean under the inner product $\langle\cdot,\cdot\rangle_0$ given by

$$\langle\phi_0,\psi_0\rangle_0 = \int d^{m+n}p\ \delta(p^2)\,\phi_0(p)^*\psi_0(p)$$

for $\phi_0,\psi_0 \in V_0$.

What is required is some extension of this to Φ and V. Unfortunately, the remainder of the domain of functions in V is not a Lie group coset space like the null space, so a naive extension is not possible. However, all that remains to be incorporated is the extra four-dimensional space with basis elements L^s, $s \in \{0,1\}\times\{0,1\}$, so it is not too difficult to find a promising candidate. The simplest possibility will be tried first. It will fail, but only just, and it will lead directly to a satisfactory solution.

Proposition 8.3 Under the inner product $\langle\cdot,\cdot\rangle_1$ for V defined by

$$\langle\phi,\psi\rangle_1 = \int d^{m+n}p\ \delta(p^2)\Omega^{\alpha\beta}\{\phi(p,\alpha\beta)^*\psi(p)-\phi(p)^*\psi(p,\alpha\beta) - \phi(p,\alpha)^*\psi(p,\beta)+\phi(p,\beta)^*\psi(p,\alpha)\}. \tag{8.9}$$

for $\phi,\psi \in V$ the operators $\Phi(X)$ of the irreducible representation of theorem 8.2 are pseudo-Hermitean, *except* for $X = J_{+\lambda}$, J_{+-} and $L_{+\alpha}$.

For these, the combinations $J_{+\lambda} - i\hbar P_\lambda P_-^{-1}$, $J_{+-} - i\hbar P_- P_-^{-1}$ and $L_{+\alpha} - i\hbar Q_\alpha P_-^{-1}$ are represented by pseudo-Hermitean operators.

Proof. Clearly the operators $\Phi(J_{\lambda\mu})$, $\Phi(J_{-\lambda})$, $\Phi(P_\lambda)$ and $\Phi(P_-)$ are Hermitean under $\langle\cdot,\cdot\rangle_1$, since as far as these operators are concerned (see theorem 8.2), each of the four parts is just like $\langle\cdot,\cdot\rangle_0$. Because of the contraction over the symplectic space indices, $\Phi(K_{\alpha\beta})$ is also Hermitean.

For $\Phi(P_+)$, inspection of theorem 8.2 shows that the only terms causing any difficulty are those with arguments $(p,\alpha\beta)$. Thus

$$\begin{aligned}
&\langle\Phi(P_+)\phi,\psi\rangle_1 - \langle\phi,\Phi(P_+)\psi\rangle_1 \\
&\quad= \int d^{m+n}p\;\delta(p^2)\Omega^{\alpha\beta}\{\Phi(P_+)\phi(p,\alpha\beta)^*\psi(p) \\
&\qquad\qquad + \phi(p)^*\Phi(P_+)\psi(p,\alpha\beta)\} \\
&\quad= \int d^{m+n}p\;\delta(p^2)\Omega^{\alpha\beta}\{(i\hbar)^2(-\Omega_{\alpha\beta})p_-\phi(p)^*\psi(p) \\
&\qquad\qquad + \phi(p)^*(i\hbar)^2(\Omega_{\alpha\beta})p_-\psi(p)\} \\
&\quad= 0,
\end{aligned}$$

and $\Phi(P_+)$ is Hermitean.

It is equally straightforward to show that the operators $\Phi(L_{\lambda\alpha})$, $\Phi(L_{-\alpha})$ and $\Phi(Q_\alpha)$ are anti-Hermitean under $\langle\cdot,\cdot\rangle_1$. For example,

$$\begin{aligned}
&\langle\Phi(L_{-\alpha})\phi,\psi\rangle_1 + \langle\phi,\Phi(L_{-\alpha})\psi\rangle_1 \\
&= \int d^{m+n}p\;\delta(p^2)\Omega^{\beta\gamma} \\
&\qquad\{-\phi(p,\alpha)^*\psi(p,\beta\gamma) + \phi(p,\alpha\beta)^*\psi(p,\gamma) - \phi(p,\alpha\gamma)^*\psi(p,\beta)\} \\
&+ \int d^{m+n}p\;\delta(p^2)\Omega^{\beta\gamma} \\
&\qquad\{\phi(p,\beta\gamma)^*\psi(p,\alpha) + \phi(p,\beta)^*\psi(p,\alpha\gamma) - \phi(p,\gamma)^*\psi(p,\alpha\beta)\} \\
&= \int d^{m+n}p\;\delta(p^2)\Omega^{\beta\gamma}\{-\phi(p,\alpha)^*\psi(p,\beta\gamma) - \phi(p,\beta\gamma)^*\psi(p,\alpha)\} \\
&+ \int d^{m+n}p\;\delta(p^2)\Omega^{\beta\gamma}\{\phi(p,\beta\gamma)^*\psi(p,\alpha) + \phi(p,\alpha)^*\psi(p,\beta\gamma)\} \\
&= 0,
\end{aligned}$$

as required.

More problematic are the operators $\Phi(J_{+\lambda})$, $\Phi(J_{+-})$ and $\Phi(L_{+\alpha})$. For example, consider $\Phi(J_{+-})$. The $\Phi_0(J_{+-})$ part is Hermitean, because it just acts on the null surface parts of the functions, leaving

$$
\begin{aligned}
&\langle \Phi(J_{+-})\phi,\psi\rangle_1 - \langle \phi,\Phi(J_{+-})\psi\rangle_1 \\
&= \int d^{m+n}p\;\delta(p^2)\Omega^{\alpha\beta}\{(-2i\hbar)\phi(p,\alpha\beta)^*\psi(p) \\
&\qquad\qquad -(-i\hbar)\phi(p,\alpha)^*\psi(p,\beta)+(-i\hbar)\phi(p,\beta)^*\psi(p,\alpha)\} \\
&- \int d^{m+n}p\;\delta(p^2)\Omega^{\beta\gamma}\{\phi(p)^*(2i\hbar)\psi(p,\alpha\beta) \\
&\qquad\qquad -\phi(p,\alpha)^*(i\hbar)\psi(p,\beta)-\phi(p,\beta)^*(i\hbar)\psi(p,\alpha)\} \\
&= -2i\hbar\;\langle\phi,\psi\rangle_1 .
\end{aligned}
$$

Similar calculations lead to

$$
\begin{aligned}
&\langle \Phi(J_{+\lambda})\phi,\psi\rangle_1 - \langle \phi,\Phi(J_{+\lambda})\psi\rangle_1 \\
&= -2i\hbar\int d^{m+n}p\;\delta(p^2)\Omega^{\alpha\beta}\frac{p_+}{p_-}\{\phi(p,\alpha\beta)^*\psi(p)-\phi(p)^*\psi(p,\alpha\beta) \\
&\qquad\qquad -\phi(p,\alpha)^*\psi(p,\beta)+\phi(p,\beta)^*\psi(p,\alpha)\} \\
&= -2i\hbar\;\langle\phi,\Phi(P_+)\Phi(P_-)^{-1}\psi\rangle_1 ,
\end{aligned}
$$

and likewise

$$
\begin{aligned}
&\langle \Phi(L_{+\alpha})\phi,\psi\rangle_1 + \langle \phi,\Phi(L_{+\alpha})\psi\rangle_1 \\
&\qquad\qquad = -2i\hbar\;\langle\phi,\Phi(Q_\alpha)\Phi(P_-)^{-1}\psi\rangle_1 .
\end{aligned}
$$

Straightforward rearrangement then shows that the combinations stated in the proposition are indeed pseudo-Hermitean under $\langle\cdot,\cdot\rangle_1$. □

The irregular operators under $\langle\cdot,\cdot\rangle_1$ are those, and only those, from the homogeneous part of the superalgebra with a + index. These are also the only operators which have a non-vanishing graded Lie product with P_-. This observation leads quickly to a satisfactory inner product for V.

Theorem 8.4 Under the inner product $\langle\,\cdot\,,\cdot\,\rangle$ for V defined by

$$\langle \phi ,\psi \rangle = \int d^{m+n}p\ \delta(p^2)\Omega^{\alpha\beta}\frac{1}{p_-^2}\{\phi(p,\alpha\beta)^*\psi(p) - \phi(p)^*\psi(p,\alpha\beta)$$

$$- \phi(p,\alpha)^*\psi(p,\beta) + \phi(p,\beta)^*\psi(p,\alpha)\}. \quad (8.10)$$

for $\phi,\psi \in V$, the operators of the irreducible produced representation of iosp$(m,n\,|\,2)$ are pseudo-Hermitean.

Proof. The inner product (8.10) is related to (8.9) by

$$\langle \phi,\psi \rangle = \langle \phi, \Phi(P_-)^{-2}\psi \rangle_1,$$

so the Hermiticity of $\Phi(X)$ where $[X, P_-] = 0$ is unaffected. For $J_{+\lambda}$,

$$[P_-^{-2}, J_{+\lambda}] = -2i\hbar P_\lambda P_-^{-3}.$$

So, setting $\psi' = \Phi(P_-)^{-2}\psi$,

$$\begin{aligned}
&\langle \Phi(J_{+\lambda})\phi,\psi \rangle - \langle \phi,\Phi(J_{+\lambda})\psi \rangle \\
&\quad= \langle \Phi(J_{+\lambda})\phi,\Phi(P_-)^{-2}\psi \rangle_1 - \langle \phi,\Phi(P_-)^{-2}\Phi(J_{+\lambda})\psi \rangle_1 \\
&\quad= \langle \Phi(J_{+\lambda})\phi,\psi' \rangle_1 - \langle \phi,\Phi(J_{+\lambda})\psi' \rangle_1 \\
&\qquad\qquad + \langle \phi, 2i\hbar\,\Phi(P_\lambda)\Phi(P_-)^{-1}\psi' \rangle_1 \\
&\quad= \langle \{\Phi(J_{+\lambda}) - i\hbar\,\Phi(P_\lambda)\Phi(P_-)^{-1}\}\phi,\psi' \rangle_1 \\
&\qquad\qquad - \langle \phi, \{\Phi(J_{+\lambda}) - i\hbar\,\Phi(P_\lambda)\Phi(P_-)^{-1}\}\psi' \rangle_1 \\
&\quad= 0,
\end{aligned}$$

by proposition 8.3. Thus, $\Phi(J_{+\lambda})$ is Hermitean under $\langle\,\cdot\,,\cdot\,\rangle$. Similar reasoning applies to $\Phi(J_{+-})$ and $\Phi(L_{+\alpha})$. □

It is worth noting that the inner product $\langle\,\cdot\,,\cdot\,\rangle$ is indefinite. This is normally undesirable, but for the Parisi-Sourlas dimensional reduction and the application to BRST theories outlined in section 8.5, it is essential.

8.4 The Parisi-Sourlas dimensional reduction mechanism in pseudo-Euclidean space.

The argument of Parisi and Sourlas (Parisi and Sourlas 1979) demonstrates that a $(d+2)$-dimensional scalar field coupled to an external random source is equivalent to a free d-dimensional scalar field. The original applications of the argument were to a spin system in a random magnetic field but quantum field theory formulations are equally possible. In the original version of the argument, ghost fields are introduced in order to calculate Green's functions for the $(d+2)$-dimensional field coupled to the source, and it is a supersymmetric invariance of the resulting Lagrangian which is responsible for the equivalence with the d-dimensional system. These supersymmetry transformations, together with the spatial rotational symmetry, the symplectic symmetry of the ghost fields and the translational and supertranslational invariance of the Green's functions, form the inhomogeneous orthosymplectic Lie superalgebra iosp$(d+2\,|\,2)$. The original scalar field, the ghost fields and the random source form an iosp$(d+2\,|\,2)$ supermultiplet.

Most treatments of the Parisi-Sourlas argument (eg. Parisi and Sourlas 1979, Cardy 1983, Klein and Perez 1983, McClain *et al* 1983, Klein *et al* 1984) introduce a superspace formalism, and show that a Berezin integral over $(d+2\,|\,2)$ superspace with an iosp-invariant integrand is equal to a d-dimensional integral over ordinary space with a similar integrand. This result is the key to showing that superfields which are invariant under an osp$(2\,|\,2)$ sub-superalgebra have Green's functions identical to those of an ordinary scalar field in d dimensions. An extension to these arguments has also been made by considering an osp(d+2l | 2l) superalgebra and achieving a 2l-dimensional reduction (Ishizuka and Kikuchi 1986).

The treatments mentioned are all for Euclidean field theories, so a Wick rotation is necessary before the procedure can be applied to relativistic field theories. Recently, there has been interest in the use of iosp(d,2 | 2) superfields to covariantly quantise d-dimensional gauge and string theories in the BRST formalism (see section 8.5 for more on this subject). These approaches make use of the Parisi-Sourlas mechanism. Unfortunately, the usual Wick rotation turning one space-like and one time-like direction into two space-like ones relies on the positive-definiteness of the inner product for the rest of the space. When there is more than one time-like direction originally, the validity of the Wick rotation is unclear. So a version of the Parisi-Sourlas argument directly in pseudo-Euclidean space is desirable.

In this section, a new argument is given that proves the key step of the Parisi-Sourlas mechanism for a (1,1)-dimensional reduction from an (m,n)-dimensional pseudo-Euclidean theory to an (m−1,n−1)-dimensional theory. Using the iosp(m,n | 2) scalar supermultiplet developed in the last three sections, the Wick rotation procedure is avoided entirely. Due to the approach taken in this thesis, the argument also uses no supergroups, superspace coordinates or superfields.

As mentioned above, the key part of the dimensional reduction argument of Parisi and Sourlas consists of showing that the Green's functions of the (d+2)-dimensional supersymmetric field theory (when the fields themselves are invariant under the osp(2 | 2) sub-superalgebra) are equal to those for a d-dimensional Euclidean field theory. The corresponding pseudo-Euclidean dimensional reduction can be demonstrated for a free scalar field by examining the inner product in the iosp($m,n\,|\,2$) scalar supermultiplet. The dimensional reduction in terms of Green's functions for a quantised field theory then follows as a consequence.

Some distributions, or generalised functions will be used to analyse the inner product, so it is first necessary to examine these distributions and their properties. The distributions will be expressed as the limiting functions of some family of ordinary functions.

Proposition 8.5 For real variables t and ε, introduce the function

$$g_\varepsilon(t) = \frac{1}{(t^2+\varepsilon^2)^{1/2}} .$$

This has the properties

(i) $\displaystyle \lim_{\varepsilon\to 0} \frac{d}{dt}\{t\, g_\varepsilon(t)\} = 2\delta(t),$

(ii) $\displaystyle \lim_{\varepsilon\to 0} \frac{d}{dt}\{t\, \frac{d}{dt}\{t\, g_\varepsilon(t)\}\} = 0.$

Proof. (i) A quick way to establish this is to use the following theorem (Jones 1966 p. 71 ex. 26). If f is a real function of t such that

$$\int_{-\infty}^{\infty} f(t)dt = 1,$$

and

$$\int_{-\infty}^{\infty} \frac{|f(t)|}{(1+t^2)^N}\, dt < \infty$$

for some $N \geq 0$, then

$$\delta(t) = \lim_{\eta\to\infty} \eta\, f(\eta t).$$

In the present case,

$$\int_{-\infty}^{\infty} \frac{1}{(1+t^2)^{3/2}}\,dt = \int_{\pi/2}^{\pi/2} (\cos^2\theta)^{3/2} \frac{1}{\cos^2\theta}\,d\theta$$

$$= \int_{\pi/2}^{\pi/2} |\cos\theta|\,d\theta$$

$$= 2.$$

So, take

$$f(t) = \frac{1/2}{(1+t^2)^{3/2}}.$$

The second condition is already satisfied for $N = 0$ by the first, since $|f(t)| = f(t)$. Thus

$$2\delta(t) = \lim_{\eta\to\infty} \frac{\eta}{(1+\eta^2 t^2)^{3/2}}$$

$$= \lim_{\varepsilon\to 0} \frac{\varepsilon^2}{(\varepsilon^2+t^2)^{3/2}}, \qquad \text{where } \varepsilon = 1/\eta.$$

(The two-sided limit is justified because only ε^2 appears). The desired result follows, since

$$\frac{d}{dt}\{t\, g_\varepsilon(t)\} = \frac{d}{dt}\frac{t}{(t^2+\varepsilon^2)^{1/2}}$$
$$= \frac{\varepsilon^2}{(t^2+\varepsilon^2)^{3/2}}.$$

(ii) can be seen immediately by direct evaluation:

$$t\,\frac{d}{dt}\{t\, g_\varepsilon(t)\} = \frac{\varepsilon^2 t}{(t^2+\varepsilon^2)^{3/2}},$$

which is zero at $t = 0$ ($\varepsilon \neq 0$), and vanishes as $\varepsilon \to 0$ for any other value of t. So this function tends to the zero distribution as $\varepsilon \to 0$. By theorem 2.4-3 of Zemanian (1965), the derivative of the function then tends to the derivative of the zero distribution, which gives the required result. □

Proposition 8.6 Let $\varepsilon, \eta \neq 0$ be real numbers, and introduce the two functions

$$G_{\varepsilon\eta}(p) = \frac{1}{16\hbar^2}\left\{\frac{d}{dp_-}\{p_- g_\eta(p_-)\}\, g_\varepsilon(p_+) - \frac{d}{dp_+}\{p_+ g_\varepsilon(p_+)\}\, g_\eta(p_-)\right\}$$

and

$$F_{\varepsilon\eta}(p) = \frac{1}{p_+ p_- + \varepsilon^2} + G_{\varepsilon\eta}(p).$$

$F_{\varepsilon\eta}$ has the following properties:

(i) $$\lim_{\varepsilon,\eta\to 0} p_+ p_- F_{\varepsilon\eta}(p) = 1, \tag{8.11}$$

(ii) $$\lim_{\varepsilon,\eta\to 0} \Phi_0(J_{-+})F_{\varepsilon\eta}(p) = \frac{i}{2\hbar}\,\delta(p_+)\delta(p_-). \tag{8.12}$$

Proof. (i) Since

$$\begin{aligned}\lim_{\varepsilon\to 0} t\,\frac{d}{dt}\{t\, g_\varepsilon(t)\} &= 2t\,\delta(t) \qquad \text{by proposition 8.5}\\ &= 0,\end{aligned}$$

it follows that

$$\lim_{\varepsilon,\eta\to 0} p_+ p_- G_{\varepsilon\eta}(p) = 0,$$

which leaves the required result.

(ii) Observe that

$$\begin{aligned}\Phi_0(J_{+-}) &= i\hbar\left(p_+\frac{\partial}{\partial p^-} - p_-\frac{\partial}{\partial p^+}\right)\\ &= i\hbar\left(p_+\frac{\partial}{\partial p_+} - p_-\frac{\partial}{\partial p_-}\right)\end{aligned}$$

so that

$$\Phi_0(J_{+-})p_+p_- = 0$$

and hence

$$\Phi_0(J_{+-})\frac{1}{p_+p_- + \varepsilon^2} = 0.$$

So the first term in $F_{\varepsilon\eta}(p)$ can be ignored. This leaves

$$\begin{aligned}\Phi_0(J_{-+})F_{\varepsilon\eta}(p) &= i\hbar\left(p_+\frac{\partial}{\partial p_+} - p_-\frac{\partial}{\partial p_-}\right)G_{\varepsilon\eta}(p)\\ &= i\hbar\left(\frac{\partial}{\partial p_+}\{p_+G_{\varepsilon\eta}(p)\} - \frac{\partial}{\partial p_-}\{p_-G_{\varepsilon\eta}(p)\}\right).\end{aligned} \tag{8.13}$$

Consider the first term on the right-hand side.

$$\frac{\partial}{\partial p_+}\{p_+ G_{\varepsilon\eta}(p)\} = \frac{1}{16\hbar^2}\left\{\frac{d}{dp_-}\{p_- g_\eta(p_-)\}\frac{d}{dp_+}\{p_+ g_\varepsilon(p_+)\} - \frac{d}{dp_+}\{p_+\frac{d}{dp_+}\{p_+ g_\varepsilon(p_+)\}\} g_\eta(p_-)\right\}$$

By proposition 8.5, in the limit as $\varepsilon \to 0$, the first term yields a delta function in p_+ while the second term vanishes. Taking the limit as $\eta \to 0$ as well,

$$\lim_{\varepsilon,\eta\to 0}\frac{\partial}{\partial p_+}\{p_+ G_{\varepsilon\eta}(p)\} = \frac{1}{4\hbar^2}\delta(p_+)\delta(p_-).$$

By the same argument,

$$\lim_{\varepsilon,\eta\to 0}\frac{\partial}{\partial p_-}\{p_- G_{\varepsilon\eta}(p)\} = \frac{-1}{4\hbar^2}\delta(p_+)\delta(p_-).$$

Combining these according to (8.13) gives the required result. □

Armed with the function $F_{\varepsilon\eta}$, it is possible to prove the key component of the Parisi-Sourlas mechanism in pseudo-Euclidean space.

Theorem 8.7 Write $p = (\mathbf{p}, p_+, p_-)$ for p on the null surface in pseudo-Euclidean space. For any $\phi, \psi \in V$ satisfying

$$\Phi(L_{+\alpha})\phi = \Phi(L_{+\alpha})\psi = 0, \tag{8.14}$$

the inner product (8.10) in theorem 8.4 reduces to

$$\langle\phi,\psi\rangle = \int d^{m+n-2}\mathbf{p}\,\delta(\mathbf{p}^2)\,\phi(\mathbf{p},0,0)^*\,\psi(\mathbf{p},0,0).$$

Proof. If $\Phi(L_{+\alpha})\phi = 0$ then, from theorem 8.2,

$$\frac{p_+}{p_-}\phi(p,\alpha) = 0, \tag{8.15}$$

$$\frac{p_+}{p_-}\phi(p,\alpha\beta) = i\hbar\,\Omega_{\alpha\beta}\Phi_0(J_{+-})\phi(p) \tag{8.16}$$

Since $\phi(p,\alpha)$ is an infinitely differentiable function of p (for α fixed), the only solution to (8.15) is

$$\phi(p,\alpha) = 0.$$

The same argument applies to ψ, leaving the inner product as

$$\langle \phi ,\psi \rangle = \int d^{m+n}p\ \delta(p^2)\Omega^{\alpha\beta}\frac{1}{p_-^{\ 2}}\{\phi(p,\alpha\beta)^*\psi(p) - \phi(p)^*\psi(p,\alpha\beta)\}$$

(8.17)

Introducing $p_+ p_- F_{\varepsilon\eta}$ into the inner product by (8.11),

$$\langle \phi ,\psi \rangle = \lim_{\varepsilon,\eta \to 0} \langle \phi , p_+ p_- F_{\varepsilon\eta}\,\psi \rangle.$$

Evaluating the right hand side, making use of (8.17)

$$\begin{aligned}
&\langle \phi , p_+ p_- F_{\varepsilon\eta}\,\psi \rangle \\
&= \int d^{m+n}p\ \delta(p^2)\Omega^{\alpha\beta}F_{\varepsilon\eta}(p) \\
&\qquad \times \{\frac{p_+}{p_-}\phi(p,\alpha\beta)^*\psi(p) - \phi(p)^*\frac{p_+}{p_-}\psi(p,\alpha\beta)\} \\
&= \int d^{m+n}p\ \delta(p^2)\Omega^{\alpha\beta}F_{\varepsilon\eta}(p) \\
&\qquad \times \{-i\hbar\,\Omega_{\alpha\beta}\,\Phi_0(J_{+-})\phi(p)^*\psi(p) \\
&\qquad\qquad - \phi(p)^* i\hbar\,\Omega_{\alpha\beta}\,\Phi_0(J_{+-})\psi(p)\} \qquad \text{by (8.16)} \\
&= 2i\hbar \int d^{m+n}p\ \delta(p^2)F_{\varepsilon\eta}(p)\Phi_0(J_{+-})\{\phi(p)^*\psi(p)\} \\
&= -2i\hbar \int d^{m+n}p\ \delta(p^2)\phi(p)^*\psi(p)\Phi_0(J_{+-})F_{\varepsilon\eta}(p),
\end{aligned}$$

using integration by parts. Taking the limit as $\varepsilon,\eta \to 0$, this becomes, by (8.12),

$$\int d^{m+n}p\ \delta(p^2)\delta(p_+)\delta(p_-)\phi(p)^*\psi(p).$$

Integrating out p_+ and p_- gives the desired result. □

Comments. The required reduction of one space and one time dimension has been achieved. The final inner product takes no account of the value of the fields on anything but ordinary $(m-1,n-1)$-dimensional space. What remains is the standard inner product for the momentum representation of an ordinary classical massless scalar field carrying a unitary irreducible representation of iso$(m-1,n-1)$. The condition (8.14) is, incidentally, less restrictive than the osp(2 | 2) invariance used in the usual treatments.

The dimensional reduction for the Green's functions of quantised fields follows by virtue of the direct correspondence between canonical commutation relations for quantised fields and the inner product for classical fields discussed in section 5.1. Coordinate space Green's functions can be obtained by Fourier transformation.

8.5 Further work: BRST theories and iosp(d, 2 | 2)

In this section, some of the basic ideas of the Becchi-Rouet-Stora-Tyutin (BRST) approach to covariant quantisation of non-Abelian gauge theories are described, and a way suggested in which a BRST theory could arise from an iosp(d,2 | 2) field theory. The material presented in this section is not so much a completed work as an indication of a line of further research. Hence a more descriptive tone will be adopted.

BRST quantisation can be regarded as a generalisation to non-Abelian gauge theories of the Gupta-Bleuler method for quantum electrodynamics. In that method, the vector potential is quantised as if all of its components were independent degrees of freedom. The price to be paid for the contradiction with the result of section 6.5 (that there are 2 *dependent* degrees of freedom, so the gauge potential cannot transform as a vector) is that the single-particle Hilbert space must have an indefinite metric tensor. A "physical" Hilbert space can be recovered by means of a projection operator constructed from a gauge-fixing condition. Alternatively, the projection operator can be constructed from an auxiliary field, whose (algebraic) equation of motion corresponds to some gauge-fixing condition. The physical Hilbert space has a definite norm, and contains the correct number of degrees of freedom.

To generalise this method to non-Abelian gauge theories, besides the auxiliary gauge-fixing fields (there are two here), it is necessary to add pairs of scalar Fermi fields. These fields must also have an indefinite metric tensor, in order to avoid contradiction with the spin-statistics theorem. They are known as Faddeev-Popov ghosts (Faddeev and Popov 1967). It was recognised by Becchi, Rouet and Stora (1976), and independently by Tyutin (1975), that the resulting system (covariant gauge potentials, auxiliary fields and ghost fields) possessed supersymmetry. The generators of the supersymmetric BRST transformations are similar to the projection operator of the Gupta-Bleuler method; they determine a physical Hilbert space. A slight difference is that the physical Hilbert space this time is the cohomology space of the BRST operators: first the projection is made, and then any of the remaining states which differ only by a BRST transformation are counted as equivalent.

Several attempts have been made to give the BRST transformations a geometrical interpretation (eg. Bonora *et al* 1982, Baulieu and Thierry-Mieg 1982), and to unify the BRST supersymmetry with the Poincaré symmetry in a single superalgebra or supergroup (eg. Delbourgo and Jarvis 1982, Bowick and Gürsey 1986). The most recent approach has used the iosp(d,2 | 2) superalgebra, and appears to be quite promising.

The iosp(d,2 | 2)-based approach has been demonstrated already (Siegel and Zwiebach 1987*a*, Neveu and West 1986), and indeed extended from non-Abelian gauge theories to include strings and superstrings (Siegel and Zwiebach 1987*b*, Siegel 1987, Neveu and West 1987, Aratyn *et al* 1987*a*,*b*). However, the treatments mentioned all take a different viewpoint to the one taken in this thesis: they use a light-cone gauge formalism in superspace, and frequently not enough detail about the origins of some of the transformations is given. So an approach, such as the one here, based on the representations of the underlying superalgebra might shed some more light on the subject. For example, it would eliminate the need for the light-cone gauge formalism. The results already obtained would also be reinforced by deriving them without the use of superspace and the accompanying Berezin integration. Such an approach has already proved successful in confirming the Parisi-Sourlas mechanism in the last section.

To answer the question of why the iosp(d,2 | 2) superalgebra should be appropriate for BRST theories, it is necessary to describe them in a little more detail, and then reduce the transformations involved to the barest essentials.

Consider a relativistic, BRST-quantised, non-Abelian gauge theory in d -dimensional space-time ($d \geq 4$). (For this section, the indices λ,μ,ν will take the values $0,1,\ldots,d-1$, and the indices α,β,γ the values 1,2). As just described, the elements of the theory are a vector field $A_\lambda(x)$, two gauge fixing auxiliary fields $B_1(x)$ and $B_2(x)$, and two Faddeev-Popov ghost fields $C_1(x)$ and $C_2(x)$. All the fields are massless and the field operators are Hermitean. Each of these fields (ie. each component shown) is actually a vector carrying the adjoint representation of the Lie algebra of the gauge group, but for the sake of readability, this will not be shown explicitly.

There are some constraints relating the gauge fixing fields to the others, namely

$$B_1(x) = -\frac{\partial}{\partial x^\lambda} A^\lambda(x)$$

and

$$B_1(x) + B_2(x) = i f\,(C_1(x) \times C_2(x))$$

where the cross product is the one induced on the adjoint representation by the Lie product of the gauge group, and f is the coupling constant matrix. The relations given here and the transformations to follow are taken from Nishijima (1984) with modifications to the notation.

Under Poincaré transformations, the vector field transforms as a true Lorentz vector (unlike the gauge potential of section 6.5), and the other fields are scalars. Despite being Lorentz scalars, the ghost fields are required to satisfy Fermi statistics, and so must be odd operators. As a consequence, ghost particle states have negative norm.

As well as relativistic invariance, a BRST theory possesses an invariance under the supersymmetric BRST transformations. These are generated by two odd operators $\Phi(Q^B_\alpha)$, which, in this case, have the following graded commutators with the field operators.

$$
\begin{aligned}
[\Phi(Q^B_\alpha), A_\lambda(x)] &= -i\hbar D_\lambda C_\alpha(x) \\
[\Phi(Q^B_\alpha), B_\alpha(x)] &= 0 \\
[\Phi(Q^B_\alpha), C_\beta(x)] &= \hbar B_\alpha(x) \qquad\qquad (\alpha \neq \beta) \\
[\Phi(Q^B_\alpha), C_\alpha(x)] &= \tfrac{1}{2} i\hbar f\,(C_\alpha(x) \times C_\alpha(x)),
\end{aligned}
$$

where there is no summation over the repeated α here. The covariant derivative D_λ is defined by

$$D_\lambda C_\alpha(x) = \frac{\partial}{\partial x}{}_\lambda C_\alpha(x) + f\,(A_\lambda(x) \times C_\alpha(x)).$$

It is obviously undesirable to have states of negative norm, but the BRST transformations provide a way out of this. A "physical" state space (with a positive definite norm) is generated by the intersection of the cohomology spaces of the two BRST operators. This will be made clearer later.

For present purposes, it is not necessary to persist with the complexities of non-Abelian gauge theories. The ideas to be demonstrated can be seen with just an Abelian (say U(1)) gauge theory. In this case, all the cross-product terms in the previous equations disappear. A single gauge-fixing field,

$$B(x) = B_1(x) = -B_2(x)$$

remains, given by

$$B(x) = -\frac{\partial}{\partial x}_{\lambda} A^{\lambda}(x).$$

The BRST transformations simplify to

$$\begin{aligned} [\Phi(Q^B_\alpha), A_\lambda(x)] &= -i\hbar \frac{\partial}{\partial x}_{\lambda} C_\alpha(x) \\ [\Phi(Q^B_\alpha), B(x)] &= 0 \\ [\Phi(Q^B_\alpha), C_\beta(x)] &= -i\hbar\, \Omega_{\alpha\beta} B(x), \end{aligned} \tag{8.18}$$

where

$$\Omega_{\alpha\beta} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

The appearance of the symplectic metric tensor Ω suggests immediately that the ghost fields could carry a representation of the symplectic group Sp(2). This hints at a resolution of the conflict between the Lorentz scalar nature of the ghost fields, and their Fermi statistics by means of an internal Sp(2) symmetry.

To find the complete underlying symmetry and its irreducible representation, the fields should be expanded in terms of annihilation and creation operators. In the BRST theory, the vector field has d independent degrees of freedom and the ghost fields have one each, so the expansions are straightforward:

$$A_\lambda(x) = \frac{1}{(2\pi\hbar)^{(d-1)/2}} \int d^dp\, \delta(p^2)\theta(p^0) \times \{ e^{-ip\cdot x/\hbar} a_\lambda(p) + e^{ip\cdot x/\hbar} a_\lambda(p)^\dagger \}$$

$$C_\alpha(x) = \frac{1}{(2\pi\hbar)^{(d-1)/2}} \int d^dp\, \delta(p^2)\theta(p^0) \times \{ e^{-ip\cdot x/\hbar} c_\alpha(p) + e^{ip\cdot x/\hbar} c_\alpha(p)^\dagger \},$$

where $a_\lambda(p)$ and $c_\alpha(p)$ are the annihilation operators. The gauge-fixing field is just

$$B(x) = \frac{i/\hbar}{(2\pi\hbar)^{(d-1)/2}}\int d^dp\,\delta(p^2)\theta(p^0)p^\lambda \times \{\mathrm{e}^{-ip\cdot x/\hbar}a_\lambda(p) - \mathrm{e}^{ip\cdot x/\hbar}a_\lambda(p)^\dagger\}.$$

Note that the vector field is not constructed from scalar annihilation and creation operators as in section 6.3. This was found to be necessary there, because the operators were required to carry an *irreducible* representation of the Poincaré group. Since the complete symmetry of the current theory must include the BRST transformations, the operators for the vector field need not carry an irreducible representation of the Poincaré group alone. Instead, they should form part of an irreducible representation of the complete symmetry.

Unfortunately, the vector representation of the Lorentz group is not a unitary one, unless the carrier space has an indefinite metric. This yields the canonical commutation relations

$$[a_\lambda(p), a_\mu(p)^\dagger] = -2g_{\lambda\mu}p^0\delta^{d-1}(\mathbf{p}-\mathbf{p}') \tag{8.19a}$$

($\mathbf{p}$ here is the $(d-1)$-dimensional spatial part of p). Likewise the Sp(2) representation carried by the ghost fields can be unitary only if

$$\{c_\alpha(p), c_\beta(p)^\dagger\} = 2\Omega_{\alpha\beta}p^0\delta^{d-1}(\mathbf{p}-\mathbf{p}'). \tag{8.19b}$$

In terms of the annihilation operators, the BRST transformations (8.18) are

$$\begin{aligned} [\Phi(Q^B_\alpha), a_\lambda(p)] &= -p_\lambda c_\alpha(p) \\ [\Phi(Q^B_\alpha), c_\beta(p)] &= \Omega_{\alpha\beta}p^\lambda a_\lambda(p). \end{aligned} \tag{8.20}$$

Incidentally, it is easy to see how the BRST selection mechanism works at this point. Take a particular momentum, say $p^{0} = p^{d-1}$ and $p^{i} = 0$ for $i = 1,\ldots,d-2$. The transformation rules are then

$$
\begin{aligned}
[\Phi(Q^{B}_{\alpha}), a_0(p)] &= -p_0 c_\alpha(p) \\
[\Phi(Q^{B}_{\alpha}), a_i(p)] &= 0 \\
[\Phi(Q^{B}_{\alpha}), a_{d-1}(p)] &= p_0 c_\alpha(p) \\
[\Phi(Q^{B}_{\alpha}), c_\beta(p)] &= \Omega_{\alpha\beta}\, p_0\, (a_0(p) + a_{d-1}(p)).
\end{aligned}
$$

The generators in the kernel of both $\Phi(Q^{B}_{\alpha})$ operators are the $a_i(p)$ and the combination $(a_0(p) + a_{d-1}(p))$. Of these, $(a_0(p) + a_{d-1}(p))$ is also in the image of both operators. So the intersection of the cohomology spaces of the $\Phi(Q^{B}_{\alpha})$ corresponds to the space spanned by the $d-2$ operators $a_i(p)$. These are precisely the "physical" transverse degrees of freedom.

Returning to the annihilation operators for general light-cone momentum p, the complete symmetry of the theory begins to take shape. The operators $a_\lambda(p)$ carry a vector representation of iso($d-1$,1). The ghost operators $c_\alpha(p)$ carry a vector representation of sp(2). If the familiar notation P_λ, $J_{\lambda\mu}$, $K_{\alpha\beta}$ is used for the generators of translations, Lorentz transformations and symplectic rotations, then

$$
\begin{aligned}
[\Phi(P_\lambda), a_\mu(p)] &= -p_\lambda a_\mu(p) \\
[\Phi(J_{\lambda\mu}), a_\nu(p)] &= -i\hbar\, (g_{\lambda\nu} a_\mu(p) - g_{\mu\nu} a_\lambda(p)) \\
[\Phi(K_{\alpha\beta}), a_\lambda(p)] &= 0 \\
[\Phi(P_\lambda), c_\alpha(p)] &= -p_\lambda c_\alpha(p) \\
[\Phi(J_{\lambda\mu}), c_\alpha(p)] &= 0 \\
[\Phi(K_{\alpha\beta}), c_\gamma(p)] &= -\hbar\, (\Omega_{\alpha\gamma} c_\beta(p) + \Omega_{\beta\gamma} c_\alpha(p)).
\end{aligned}
$$

Define some additional operators $\Phi(L_{\lambda\alpha})$ by

$$
\begin{aligned}
[\Phi(L_{\lambda\alpha}), a_\mu(p)] &= -i\hbar\, g_{\lambda\mu} c_\alpha(p) \\
[\Phi(L_{\lambda\alpha}), c_\beta(p)] &= -i\hbar\, \Omega_{\alpha\beta} a_\lambda(p).
\end{aligned} \tag{8.21}
$$

Together with the $\Phi(J_{\lambda\mu})$ and $\Phi(K_{\alpha\beta})$, the $\Phi(L_{\lambda\alpha})$ form a vector representation of the superalgebra osp(d–1,1 | 2) (cf. the graded Lie products (8.1), and especially the products with P_λ and Q_α, to see the transformation rules for vectors). Comparing the $\Phi(L_{\lambda\alpha})$ transformation (8.21) with the BRST transformations (8.18) shows that the BRST operators are *not* part of this superalgebra.

Before proceeding to the final stage of the argument, it is worth turning aside one possible objection. The step to osp(d–1,1 | 2) just made might seem a little contrived. Of course, given a graded vector space it is possible to make it carry any equally-sized representation of any superalgebra desired. So what must be asked is whether the physical theory has the corresponding supersymmetry. This means asking whether the representation is unitary. In this case, the answer is yes. The canonical commutation relations (8.19), with the reasonable addition

$$[a_\lambda(p), c_\alpha(p)^\dagger] = 0,$$

show that the carrier space metric incorporates the osp(d–1,1 | 2) metric tensor (denoted G at the start of section 8.1). This implies the unitarity of the $\Phi(L_{\lambda\alpha})$ operators. So the step to osp(d–1,1 | 2) is justified.

The final problem is how to include the BRST operators. A very little algebra shows that

$$
\begin{aligned}
[\Phi(Q^{B}_{\alpha}),\Phi(Q^{B}_{\beta})] &= 0 \\
[\Phi(J_{\lambda\mu}),\Phi(Q^{B}_{\alpha})] &= 0 \qquad\qquad (8.22)\\
[\Phi(K_{\alpha\beta}),\Phi(Q^{B}_{\gamma})] &= -\hbar\,(\Omega_{\alpha\gamma}\Phi(Q^{B}_{\beta}) + \Omega_{\beta\gamma}\Phi(Q^{B}_{\alpha})).
\end{aligned}
$$

The key observation to make is that osp(d–1,1 | 2) is the homogeneous part of the little superalgebra for a "massless" irreducible representation of iosp(d,2 | 2) (see section 8.1). In the larger superalgebra, there are six odd generators remaining: $L_{\pm\alpha}$ and Q_{α}. From the graded Lie products (8.1), any of these would satisfy the relations (8.22). So it seems worthwhile to construct the iosp(d,2 | 2) representation induced from the vector representation of osp(d–1,1 | 2). This is effectively what has been done in the equivalent light-cone gauge formalism (Siegel and Zwiebach 1987a, Neveu and West 1986). The odd generators $L_{+\alpha}$ are found to fulfill the role of the BRST operators. So the approach suggested should be successful.

Unfortunately, the iosp(d,2 | 2) representation, which is similar to the scalar one in section 8.3, has two extra momentum coordinates, $p_{\pm}$, and four times as many particles in the multiplet as required. However, as pointed out by Neveu and West, and also Spiegelglas (1988), in the "physical" sector annihilated by $\Phi(Q^{B}_{\alpha}) = \Phi(L_{+\alpha})$, the Parisi-Sourlas mechanism operates to remove these surplus degrees of freedom.

So the course of investigation indicated is as follows. First construct the irreducible representation of iosp(d , 2 | 2) induced from the vector representation of osp(d–1,1 | 2). Then examine the cohomology of the $\Phi(L_{+\alpha})$ operators to see if it does indeed contain only "physical" degrees of freedom. Next, the "covariant" iosp(d,2 | 2) field multiplet over space-time should be constructed. The interaction of the covariant and irreducible representations should provide some interesting results, just as it provided the field equations for relativistic fields in section 5.3. Finally, extensions could be made of these ideas back to non-Abelian gauge theories, and then on to strings, superstrings and so on.

Appendix

Spinor representations and γ matrices in d-dimensional Minkowski space-time

The purpose of this appendix is to give some more details about the spinor representations of the Lorentz group in d-dimensional Minkowski space-time. In particular, the γ matrices underlying these representations are described, together with some of their properties. The special realisation of the γ matrices used in chapters 4, 6 and 7 is derived.

The classic paper on spinor representations in space-times of arbitrary dimension is by Brauer and Weyl (1935). This paper generalises the ideas developed by Dirac (1927) in his work on the relativistic electron in four dimensions. The key element is an associative algebra of a kind discovered by Clifford (1878), and now called a Clifford algebra.

There are several different sets of notation used for Clifford algebras and spinor representations. The one adopted here is of the kind usually found in the physics literature. Although Clifford algebras exist over complex fields or real fields of arbitrary signature, attention will be restricted here to the Clifford algebra over a real d-dimensional space of signature $(d-1,1)$: Minkowski space-time. The indices λ,μ will take values $0,1,\ldots,d-1$ and i,j values $1,2,\ldots,d-2$. The Minkowski space-time metric tensor is $g = \mathrm{diag}(1,-1,\ldots,-1)$.

The Clifford algebra is based upon a set of d generators γ_λ and an identity 1. To construct the Clifford algebra basis, all possible tensor products of the generators are formed, and then the anticommutation rule

$$\{\gamma_\lambda, \gamma_\mu\} = 2g_{\lambda\mu}1 \qquad \text{(A.1)}$$

is applied. Since the anticommutator removes all symmetric parts of any tensor product, this gives a total of 2^d independent basis elements.

The Clifford algebra can be represented by $2^l \times 2^l$ matrices, where l is $d/2$ for d even and $(d-1)/2$ for d odd. For d even, this representation is unique up to similarity transformations. For d odd, there are two inequivalent representations (see Brauer and Weyl for details). In the matrix representation, there exists a unitary $2^l \times 2^l$ matrix C, called the charge conjugation matrix, which has the property

$$C^{-1}\gamma_\lambda C = \pm\gamma_\lambda{}^{\mathrm{T}},$$

where the sign $\pm$ depends upon d as detailed below, and $^{\mathrm{T}}$ stands for transpose.

It is not hard to see that the matrix representation for d even is just that for $d+1$ (which is odd) with one generator matrix removed. So it suffices to consider just the odd dimensions at first. The matrices can be constructed easily, using an induction process, as in the following theorem.

Theorem A.1 Let $d = 2l + 1$ be odd, and τ_i be a set of $d-2$ $2^{l-1} \times 2^{l-1}$ matrices satisfying

$$\{\tau_i, \tau_j\} = -2\delta_{ij} 1 \tag{A.2}$$

where 1 is the unit matrix. Let Z be a unitary $2^{l-1} \times 2^{l-1}$ matrix satisfying

$$Z^{-1}\tau_i Z = -\varepsilon^2 \tau_i^{\mathrm{T}} \tag{A.3}$$

where ε is a complex number satisfying

$$\varepsilon^*\varepsilon = 1; \qquad (\varepsilon^*)^2 = \varepsilon^2. \tag{A.4}$$

(i) The matrices

$$\gamma_0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \ \gamma_i = \begin{pmatrix} \tau_i & 0 \\ 0 & -\tau_i \end{pmatrix}, \ \gamma_{d-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{A.5}$$

provide a matrix representation of the Clifford algebra for d-dimensional Minkowski space-time, with

$$\{\gamma_\lambda, \gamma_\mu\} = 2g_{\lambda\mu} 1.$$

(ii) Furthermore, the matrix

$$C = \begin{pmatrix} 0 & \varepsilon^* Z \\ \varepsilon Z & 0 \end{pmatrix}$$

is unitary, and satisfies

$$C^{-1}\gamma_\lambda C = \varepsilon^2 \gamma_\lambda^{\mathrm{T}}. \tag{A.6}$$

Proof. (i) By inspection,

$$(\gamma_0)^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$(\gamma_i)^2 = \begin{pmatrix} \tau_i^2 & 0 \\ 0 & \tau_i^2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$(\gamma_{d-1})^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

Also,

$$\gamma_0\gamma_i = \begin{pmatrix} 0 & \tau_i \\ -\tau_i & 0 \end{pmatrix} = -\gamma_i\gamma_0,$$

$$\gamma_0\gamma_{d-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -\gamma_{d-1}\gamma_0,$$

and, for $i \neq j$,

$$\gamma_i\gamma_j = \begin{pmatrix} \tau_i\tau_j & 0 \\ 0 & \tau_i\tau_j \end{pmatrix} = \begin{pmatrix} -\tau_j\tau_i & 0 \\ 0 & -\tau_j\tau_i \end{pmatrix} = -\gamma_j\gamma_i.$$

Taken together, this set of equations demonstrates the desired result.

(ii) By direct evaluation,

$$C^\dagger C = \begin{pmatrix} 0 & \varepsilon^* Z^\dagger \\ \varepsilon Z^\dagger & 0 \end{pmatrix}\begin{pmatrix} 0 & \varepsilon^* Z \\ \varepsilon Z & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

since $Z^\dagger Z = 1$ and $\varepsilon^*\varepsilon = 1$. So C is unitary. Further,

$$C^{-1}\gamma_0 C = \begin{pmatrix} 0 & \varepsilon^* Z^{-1} \\ \varepsilon Z^{-1} & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \varepsilon^* Z \\ \varepsilon Z & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -(\varepsilon^*)^2 1 \\ -\varepsilon^2 1 & 0 \end{pmatrix}$$

$$= \varepsilon^2\gamma_0{}^{\mathrm{T}} \qquad \text{since } (\varepsilon^*)^2 = \varepsilon^2.$$

$$C^{-1}\gamma_i C = \begin{pmatrix} 0 & \varepsilon^* Z^{-1} \\ \varepsilon Z^{-1} & 0 \end{pmatrix}\begin{pmatrix} \tau_i & 0 \\ 0 & -\tau_i \end{pmatrix}\begin{pmatrix} 0 & \varepsilon^* Z \\ \varepsilon Z & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\varepsilon^*\varepsilon Z^{-1}\tau_i Z & 0 \\ 0 & \varepsilon^*\varepsilon Z^{-1}\tau_i Z \end{pmatrix}$$

$$= \begin{pmatrix} \varepsilon^2\tau_i{}^{\mathrm{T}} & 0 \\ 0 & -\varepsilon^2\tau_i{}^{\mathrm{T}} \end{pmatrix}$$

$$= \varepsilon^2\gamma_i{}^{\mathrm{T}}.$$

$$C^{-1}\gamma_{d-1}C = \begin{pmatrix} 0 & \varepsilon^* Z^{-1} \\ \varepsilon Z^{-1} & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \varepsilon^* Z \\ \varepsilon Z & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -(\varepsilon^*)^2 1 \\ \varepsilon^2 1 & 0 \end{pmatrix}$$

$$= \varepsilon^2 \gamma_{d-1}{}^{\mathrm{T}}.$$

So

$$C^{-1}\gamma_\lambda C = \varepsilon^2 \gamma_\lambda{}^{\mathrm{T}}$$

as required. □

If an additional matrix γ_d is defined by

$$\gamma_d = i\,\gamma_{d-1}, \tag{A.7}$$

then

$$(\gamma_d)^2 = -1,$$

so the d matrices $\gamma_1, \ldots, \gamma_d$ form a set of "τ" matrices for the construction of the next set of "γ" matrices. Thus, the γ matrices for any odd dimension d, or equivalently, for any integer l, can be constructed progressively starting from the 1×1 τ matrix for $l = 1$:

$$\tau_1 = i. \tag{A.8}$$

(The other possibility, $\tau_1 = -i$ gives the alternative representations for d odd mentioned earlier).

For $l = 1$, Z is a 1×1 matrix containing any arbitrary complex number of modulus 1 (so that Z is unitary). Consequently,

$$Z^{-1}\tau_1 Z = \tau_1.$$

Hence (cf. (A.3)),

$$\varepsilon^2 = -1.$$

So for $l = 1$, ε can be taken as either $\pm i$ (both of these satisfy the conditions (A.4) on ε). The choice of sign is arbitrary, since it can be absorbed into the choice of the argument of Z.

Comparing the charge conjugation equations (A.3) and (A.6) for the τ and γ matrices, there is a sign change. So the value of ε^2 will alternate as l is increased, with

$$\varepsilon^2 = (-1)^l. \tag{A.9}$$

For each l, both solutions to this satisfy the conditions (A.4), and the choice of sign can be absorbed into Z. A suitable convention to adopt is

$$\varepsilon = \begin{cases} 1 & l \text{ even} \\ i & l \text{ odd.} \end{cases}$$

When d is even, the γ matrices can be obtained by just discarding one of the "space-like" γ matrices for $d+1$ (ie. not γ_0). Which one is discarded is unimportant, because all realisations for d even are equivalent. A convenient choice in the present realisation is to discard one of the γ_i matrices.

For d even, it is possible to define a second charge conjugation matrix using the surplus γ matrix. Denote the surplus γ matrix by $\overline{\gamma}$. This satisfies

$$\{\overline{\gamma}, \gamma_\lambda\} = 0, \qquad \overline{\gamma}^2 = -1. \tag{A.10}$$

Setting

$$C' = \overline{\gamma} C,$$

the new charge conjugation matrix gives

$$\begin{aligned} C'^{-1} \gamma_\lambda C' &= C^{-1} \overline{\gamma}^{-1} \gamma_\lambda \overline{\gamma} C \\ &= -C^{-1} \overline{\gamma}^{-1} \overline{\gamma} \gamma_\lambda C \\ &= -C^{-1} \gamma_\lambda C. \end{aligned}$$

The second charge conjugation matrix produces the same result as the first, but with a different sign. So for d even, it is possible to choose a charge conjugation matrix of either sign. But for d odd, the sign is fixed according to (A.9).

Turning to the spinor representation, it is straightforward (although laborious) to establish that the matrices $\Gamma\ (M_{\lambda\mu})$ defined by

$$\Gamma\ (M_{\lambda\mu}) = \tfrac{-1}{4}[\gamma_\lambda\ ,\gamma_\mu\ ] \tag{A.11}$$

provide a representation of the Lie algebra so(d–1,1) of the Lorentz group. This representation is called *the* spinor representation of so(d–1,1). The term "spinor representation" is also used in a general sense for any "double-valued" representation of the Lorentz group (ie. a representation of the universal covering group of $SO_0(d–1,1)$ which is not a representation of $SO_0(d–1,1)$ itself). But when it is used in a particular sense, (A.11) is meant.

Likewise, the matrices

$$\Delta(M_{ij}\ ) = \tfrac{-1}{4}[\tau_i\ ,\tau_j\ ] \tag{A.12}$$

form the spinor representation of so(d–2). In the construction of the γ matrices presented above, the matrices $\Delta\ (M_{ij}\ )$ are all anti-Hermitean. This can be seen by first showing by induction that the τ_i matrices are anti-Hermitean. Assume that for some l , the τ_i matrices are anti-Hermitean. Inspection of the construction (A.5) shows that γ_0 is Hermitean, while $\gamma_1,...,\gamma_{d-1}$ are anti-Hermitean. So the matrix $\gamma_d = i\ \gamma_0$ is anti-Hermitean, and the next set $\gamma_1,...,\gamma_d$ of "τ" matrices is anti-Hermitean as well. Obviously, for $l = 1$, the matrix τ_1 of (A.8) is anti-Hermitean. Hence the τ matrices are anti-Hermitean for all l . An immediate consequence of this is that

$$\begin{aligned}\Delta(M_{ij}\ )^\dagger &= \tfrac{-1}{4}[\tau_j{}^\dagger,\tau_i{}^\dagger] \\ &= -\Delta(M_{ij}\ ).\end{aligned} \tag{A.13}$$

Thus the corresponding spinor representation of the Spin($d-2$) or $SO_0(d-2)$ is unitary, as required in section 7.3.

The charge conjugation matrix Z for the τ_i matrices can be used to obtain the complex conjugate representation Δ^*. The complex conjugate representation is defined at the level of the Lie group:

$$\Delta^*(R) = \{\Delta(R)\}^*$$

for all R in Spin($d-2$). For the Lie algebra, this gives

$$\Delta^*(M_{ij}) = \{\Delta(M_{ij})\}^*$$

(although the same is not true for the generators J_{ij} of angular momentum, since they incorporate a factor of i). Since the $\Delta(M_{ij})$ are anti-Hermitean, it follows that

$$\begin{aligned}
\Delta^*(M_{ij}) &= \{\Delta(M_{ij})^\dagger\}^{\mathrm{T}} \\
&= -\Delta(M_{ij})^{\mathrm{T}} && \text{by (A.13)} \\
&= \tfrac{1}{4}[\tau_j{}^{\mathrm{T}}, \tau_i{}^{\mathrm{T}}] && \text{by (A.12)} \\
&= \tfrac{1}{4}[Z^{-1}\tau_j Z, Z^{-1}\tau_i Z] && \text{by (A.3), and since } \varepsilon^4 = 1 \\
&= Z^{-1}\Delta(M_{ij})Z .
\end{aligned}$$

Likewise, for the group,

$$\Delta^*(R) = Z^{-1}\Delta(R)Z ,$$

as required in section 6.2.

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