HARMONIC MEAN ESTIMATES FOR RECAPTURE DEBUGGING

BY

ABDULGHANI A. GHONAIM AL-HARBI

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ABSTRACT

In this thesis we examine the problem of estimating the number of errors (bugs) in a reliability system using the recapture debugging model suggested by Nayak (1988). The reliability system contains a certain number N of errors. Each causes system failures independently of the others . The times between failures for any bug are assumed to be independent exponential random variables with a parameter λ common to all bugs. We assume the system is observed for a fixed length of time.

The maximum likelihood estimate of N was considered by Nayak. We derive the profile likelihood interval for N, and consider as a point estimate the harmonic mean of the endpoints. This estimate was used for the Jelinski-Moranda model by Joe and Reid (1985). The exact probability distribution of the harmonic mean estimator is computed. A generalization of the harmonic mean estimate, called the weighted harmonic mean estimate is proposed as a further improvement. A comparison is drawn between this estimator and the maximum likelihood estimator, using their computed distributions for various values of N.

ABBREVIATIONS & NOTATIONS

ABBREVIATIONS

HME	Harmonic Mean Estimator
J-M	Jelinski-Moranda model
MLE	Maximum Likelihood Estimator
LTP	Lower tail probability
p.d.f.	Probability density function
TAP	Target area probability
UTP	Upper tail probability
WHME	Weighted Harmonic mean Estimator

NOTATIONS

N	Initial number of errors (bugs) contained in the program
T_i	Detection time of the i^{th} error
$t_{(i)}$	i^{th} order statistic of the detection times
τ	Length of time for which the process is observed
R	Number of distinct bugs seen by time τ
$L(N, \lambda)$	Likelihood function for the parameters N, λ
λ	Failure rate for any bug
M_i	Number of repetitions of the i^{th} error before time τ
f(x)	Probability density function of x
p	Probability that a bug has been detected by time τ
	[$p = 1 - \exp(-\lambda\tau)$]
f(x| y)	Conditional density of X given Y
Z	Total number of failures by time τ (Z=R+M)

$L(N)$	Profile Likelihood function of N
H	Harmonic mean
$h_{r,k}$	Defined in section (3.2)
N_1	Lower endpoint of the profile likelihood interval
N_2	Upper endpoint of the profile likelihood interval
c	Level of the profile likelihood interval $(0 < c < 1)$
^, ~, *	Indicate estimators
$\hat{P}_k$	$P[\hat{N} = k]$, $k = 1, 2, ..., \infty$
$\tilde{P}_k$	$P[\tilde{N} = k]$, $k = 1, 2, 3, ...$
H^*	Weighted Harmonic mean
w	Weighting factor $(0 < w < 1)$

CHAPTER 1

INTRODUCTION

1.1. IMPORTANCE OF SOFTWARE RELIABILITY

Today software is beginning to dominate our lives and is clearly crucial to any computer system. Many process control systems, such as nuclear power-plant safety control systems, air traffic control systems and the various space programs, depend on a computer system. Obviously, these must not fail if at all possible, since any failure could be disastrous. It is now a well-noted fact that the software failure is often the main cause of system failures. Hence, the reliability of these systems must be well-regulated in advance of actual use.

There is increasing competition among software producers and customers are increasingly discerning in choosing between software products. Therefore the producers must offer some tangible evidence of software quality to satisfy the customers' needs. An important aspect of software quality is the reliability of a system. In a narrower sense, " reliability is a metric which is the probability of operational success of software" (Shooman, 1983). So software reliability measurement plays an important role in determining the best software.

1. 2. BASIC CONCEPTS

We first define some basic concepts namely software errors, failures and software reliability.

A software error is present when the software does not do what the user reasonably expects it to do (Myers, 1976). A fault can be defined as a program defect or manufacturing imperfection that has caused one or more actual failures, or could potentially cause a failure. Software faults can also be due to poor communication of software requirements between the programmer and the user or to an ignorance of the user requirements. Some errors are hard to find since they have very small failure rates and thus need a long execution time before they cause a failure in the program. Note that throughout the thesis we describe either errors or faults as " bugs ".

Software failures are usually due to the manifestation of design fault and occur during execution of a program with no advance warning. Musa et al. (1987), defined a failure as " a departure of operation from requirements, conveniently described for software in terms of the output state of a run ".

Another term that should be defined is software reliability. It has been defined as "the probability of failure-free operation of a computer program for a specified time in a specified environment" (Musa et al., 1987).

To achieve highly reliable software and avoid the occurrence of the faults in the design the software is put through a process of testing and debugging. The program is executed with the aim of finding any errors. Once a software failure occurs, the program error causing this failure is corrected, and the software exercised again. The process continues until the software is believed to be error-free.

1.3. THE SCOPE OF THE THESIS

One important aspect of software reliability modelling is estimating the number of errors in a program system. There is a lot of work in the literature concerning software reliability models, but Joe (1989) remarks that there is comparatively less work concerned with statistical inference for these models.

In chapter 2 we discuss one of the best known models for analysing software failure data, originally introduced by Jelinski and Moranda (1972). We then consider how this procedure can be improved by the use of recapture debugging, which was proposed by Nayak (1988). We will mainly be concerned with point estimation of the number of errors N in the system.

In chapter three, we derive the joint distribution of the sufficient statistics for the recapture debugging model. This distribution is used later to compute the exact probability distributions of the estimators of the total number of bugs. The maximum likelihood estimator for recapture debugging is derived. Three qualitatively different cases for the profile likelihood function to determine the value of the MLE will be presented.

Harmonic mean estimates, used by Joe and Reid (1985) for the J-M model are applied in chapter four to the recapture debugging model. A generalisation called the weighted harmonic mean estimator is proposed. Some numerical results for the weighted harmonic mean estimator are discussed and its performance compared with that of the maximum likelihood estimator.

The final chapter of the thesis contains conclusions and suggestions for further work.

CHAPTER 2

RECAPTURE DEBUGGING AND THE JELINSKI-MORANDA MODEL

2. 1. THE JELINSKI-MORANDA MODEL (J-M)

Since the early 1970s, a large number of different models have been proposed for estimating software reliability, residual error content and some related parameters, such as the mean time to next failure. One of the earliest and probably still amongst the most commonly used models for describing failures in computer software was proposed by Jelinski & Moranda (1972). Shooman (1972) develops a model similar to J-M model. Musa (1975) was the first to argue that software reliability models should be based on the execution time rather than calender time. Both models are refinements of the model by Jelinski & Moranda.

For situation where the J-M model is applicable, Forman & Singpurwalla (1977) propose an empirical stopping rule for deciding when to stop testing and accept the program. This rule is based upon the comparison of the relative likelihood function for N to the approximate normal relative likelihood for N.

The assumptions of the J-M model are as follows :

1- The program contains N errors (software bugs) at the start of testing. Each occurs independently of the others.

2- Each time a failure occurs, the detected fault is immediately removed and no new faults are introduced during the debugging process.

3- The detection times T_i, $i = 1, 2, \ldots, N$ are exponentially distributed random variables with a common density function given by

$$f(t) = \lambda \exp(-\lambda t) \quad , t \geq 0 , \lambda > 0 \qquad (2.1.1)$$

The same model, although not called by this name, was also studied in the statistical literature. There the interest was in other types of application rather than in software reliability. Relevant references are Blumenthal & Marcus (1975) and Goudie & Goldie (1981).

Inference about N is usually based either on censored sampling, in which the process is observed until some predetermined number r of errors have occurred or on truncated sampling in which observation ends after a fixed length of time τ. Blumenthal & Marcus (1975) concentrate mainly on truncated sampling: Goudie & Goldie (1981) are primarily concerned with censored sampling.

The assumptions of the J-M model have often been criticized. Littlewood & Verrall (1981) attempt to overcome the inability of the J-M model to cope with reliability decay. They argue that this is a serious drawback. Another possible weakness is the assumption that all errors have the same effect on the reliability.

Littlewood (1981) notes that error with the greatest contribution to the overall failure rate will often be discovered first, and so be removed earliest.

Despite these criticisms Meinhold & Singpurwalla (1983) suggest that the J-M model remains central to the topic of software reliability. Langberg and Singpurwalla (1985) provide an alternative motivation for the J-M model. They provide a unification of software reliability models by illustrating that many other well-known models, such as Littlewood & Verrall (1973) and Goel & Okumoto (1979) can be obtained by specifying prior distributions for the parameters of the J-M model. Raftery (1988) also provides estimates based on this model, of the number of faults in a system. His approach is also Bayesian. He suggests that, for the J-M model, point estimation is not very useful, because the posterior distribution of N is often quite diffuse.

2.2. THE LIKELIHOOD FUNCTION FOR THE JELINSKI MORANDA MODEL

Suppose that we use truncated sampling for the J-M model and observe the process until time τ. The probability that the i^{th} bug has been detected by time τ is $1 - \exp(-\lambda \tau)$. We record the successive failure times $t_{(1)} \leq t_{(2)} \leq \ldots \leq t_{(r)} \leq \tau$ where R is a binomial random variable with parameters N and $P = \{1 - \exp(-\lambda \tau)\}$.

The conditional p.d.f. of the time to detect the i^{th} bug, given that it is detected before time τ, is

$$f(t \mid T_i \le \tau) = \frac{\lambda \exp(-\lambda t)}{1-\exp(-\lambda \tau)} \qquad 0 \le T_i \le \tau \qquad (2.2.1)$$

The conditional density of the first r order statistics, given that they come before time τ, is

$$f(t_{(1)}, t_{(2)}, \ldots, t_{(r)} \mid R = r > 0) = r! \prod_{i=1}^{r} \frac{\lambda \exp(-\lambda t_{(i)})}{1-\exp(-\lambda \tau)}$$

$$= r!\,\lambda^r [\exp\{-\lambda \sum_{i=1}^{r} t_{(i)}\}] [1-\exp(-\lambda\tau)]^{-r} \qquad (2.2.2)$$

$$0 \le t_{(1)} \le t_{(2)} \le \ldots \le t_{(r)} \le \tau$$

For $r > 0$ the joint density for r and $t_{(1)}, t_{(2)}, \ldots, t_{(r)}$ is

$$f(r, t_{(1)}, \ldots, t_{(r)}) = f(t_{(1)}, \ldots, t_{(r)} \mid r) \,.\, P(R=r)$$

$$= r!\,\lambda^r \exp(-\lambda \sum_{i=1}^{r} t_{(i)}) [1-\exp(-\lambda\tau)]^{-r}$$

$$\cdot \binom{N}{r} [1-\exp(-\lambda\tau)]^r \exp[-\lambda\tau(N-r)]$$

$$= \frac{N!\,\lambda^r}{(N-r)!} \exp[-\lambda\{(N-r)\tau + \sum_{i=1}^{r} t_{(i)}\}] \qquad (2.2.3)$$

$$r = 1, 2, \ldots, N$$

$$0 \le t_{(1)} \le \ldots \le t_{(r)} \le \tau$$

Note that from the binomial distribution

$$P[R=0] = \exp\{-\lambda\tau N\}$$

Regarding the above density as a function of the parameters gives the joint likelihood for N and λ. For each N, the likelihood function (2.2.3) is maximized by

$$\hat{\lambda} = \frac{r}{\tau(N-r+\beta)}, \text{ where } \beta = \frac{\sum_{i=1}^{r} t_i}{\tau}$$

This expression can be substituted into (2.2.3), giving the profile likelihood function for N, which may be taken as

$$\frac{N(N-1)\ldots(N-r+1)}{(N-r+\beta)^r} \qquad (2.2.4)$$

In order to find the MLE of N, this profile likelihood function is then maximized with respect to N.

Blumenthal & Marcus (1975), Littlewood & Verrall (1981), Goudie & Goldie (1981) and Joe & Reid (1985) have all illustrated that the solution of the ML equations for the J-M model can produce infinite estimates of N. Blumenthal & Marcus (1975) in fact studied two MLE's of N using firstly the unconditional distribution of the observations, and secondly the conditional distribution given r. Both these MLE's share the disadvantage of becoming degenerate with fairly large probability. They preferred a Bayes estimator of N, described in (Watson & Blumenthal, 1980) as a maximum modified likelihood estimator. Littlewood & Verrall (1981) present a simple, necessary and sufficient condition for the MLE to be finite and suggest that this condition should be tested prior to using the J-M model. In Joe & Reid (1985), the authors propose the use of the harmonic mean estimator based on the endpoints of the likelihood interval. The Harmonic mean estimator will be discuss later in the context of the recapture debugging.

For censored sampling, Goudie & Goldie (1981) noted the weakness of the information provided by the data when both N and λ are unknown. Goudie (1990) emphasised that the major weakness of the J-M model is that the information about N can only be partially disentangled from that about λ. The model is slow to distinguish between slow detection of a large number of errors and the relatively faster detection of a smaller number.

2.3. NAYAK's RECAPTURE DEBUGGING MODEL

Nayak (1988) and (1989) presents a procedure called recapture debugging which provides a modification of the J-M model. In his model he also assumes that the software contains N errors at the start of testing. Any detected error is corrected and a counter is inserted in the software to count the number of times that the error would have recurred in the remaining test time. It is assumed that the errors are accessed according to independent homogeneous Poisson processes (IHPP) with a common rate λ. Hence the times $T_{(1)}, T_{(2)}, \ldots, T_{(N)}$ to the first detections of the errors have the same distributions as in the J-M model. Thus we observe these times $t_{(1)}, t_{(2)}, \ldots, t_{(r)}$ for those errors that are detected by time τ, together with the number M_i of times that the i^{th} detected error recurs before time τ ($i = 1, 2, \ldots, r$). Since we detect the i^{th} bug at time $t_{(i)}$, the number M_i of further occurrences has a Poisson distribution with mean $\lambda(\tau - t_{(i)})$. Thus

$$P(M_i = m_i) = \frac{\{\lambda(\tau - t_{(i)})\}^{m_i} \exp\{-\lambda(\tau - t_{(i)})\}}{m_i\,!} \qquad (2.3.1)$$

Since the processes for different bugs have been assumed independent,

$$P[M_{(1)}=m_{(1)},\dots,M_{(r)}=m_{(r)}]=\frac{\lambda^{m}\prod_{i=1}^{r}(\tau-t_{(i)})^{m_i}}{\prod_{i=1}^{r}m_i\,!}\exp[-\lambda(r\tau-\sum_{i=1}^{r}t_{(i)})]$$

(2.3.2)

where $m = \Sigma m_i$.

Recalling that the times to the first detections of the errors have the same distribution as in the J-M model we express the likelihood function in the form

$$L(N,\lambda)=f(r,t_{(1)},\dots,t_{(r)})\,.\,P(m_{(1)},\dots,m_{(r)};\lambda,N)$$

$$=\frac{N!\,\lambda^{r}}{(N-r)!}\exp\{-\lambda[(N-r)\tau+\sum_{i=1}^{r}t_{(i)}]\}\cdot\frac{\lambda^{m}\prod_{i=1}^{r}(\tau-t_{(i)})^{m_i}}{\prod_{i=1}^{r}m_i!}\exp\{-\lambda[r\tau-\sum_{i=1}^{r}t_{(i)}]\}$$

where the first factor comes from (2.2.3). Hence we have

$$L(N,\lambda)=\left[\frac{N!\,\lambda^{m+r}}{(N-r)!}\exp(-N\lambda\tau)\right]\left[\frac{\prod_{i=1}^{r}(\tau-t_{(i)})^{m_i}}{\prod_{i=1}^{r}m_i\,!}\right]$$

(2.3.3)

We can see from the equation above that, if the process is observed for a time τ, the number R of distinct bugs and the total number M of further occurrences of these bugs are jointly sufficient for N and λ.

By using counters to observe the frequency of occurrence of particular bugs, the recapture debugging procedure provides more information about λ. This assists is disentangling the information about N from that about λ. Nayak found both maximum likelihood and method of moments estimators of the parameters of the process. He compared them with the Joe and Reid estimators for the J-M model and found that his estimators were more stable than theirs .

CHAPTER 3

MAXIMUM LIKELIHOOD ESTIMATION

3.1. THE JOINT DISTRIBUTION OF Z AND R

The probability distribution of the maximum likelihood estimator can be computed from the joint distribution of sufficient statistics. Nayak (1988) gave the joint distribution of R and M as

$$P(R=r, M=m) = \binom{N}{r} \frac{(\lambda\tau)^{m+r} \exp(-N\lambda\tau)}{m!} I_{r,m} \qquad (3.1.1)$$

$$r=1, 2, \ldots, N$$
$$m=0, 1, 2, \ldots$$

where

$$I_{r,m} = \int_0^1 \cdots \int_0^1 \left(\sum_{i=1}^{r} u_i\right)^m du_1 \ldots du_r \qquad (r \geq 1)$$

$$= \frac{1}{(r-1)!} \sum_{s=0}^{r-1} \binom{r-1}{s} \frac{1}{m+s+1} \sum_{k=0}^{r-1} (-1)^k \binom{r}{k} (-k)^{r-1-s} \{(r-1)^{m+s+1} - k^{m+s+1}\}$$

Note also that

$$P[R=M=0] = \exp\{-N\lambda\tau\}$$

However, Goudie (1989) offers a simpler expression for the probability function of the sufficient statistics. He draws an analogy between the classical occupancy problem [Feller 1968] and the problem of detection of the bugs using recapture debugging. For classifying failures according to which of the N errors caused them can be seen as equivalent to allocating balls randomly to N different cells. Using the classical occupancy distribution and setting Z = R + M we have (cf. Feller, 1968, p 60)

P [r cells are occupied | z balls have been randomly distributed amongst the N cells]

$$= \frac{\binom{N}{r}}{N^z} \sum_{v=0}^{r} (-1)^v \binom{r}{v} (r-v)^z$$

In the context of recapture debugging, this probability is equivalent to the probability that r distinct bugs have been seen when Z failures have occurred in total.

Writing $k = r - v$, we thus have

$$P[\,R = r \mid Z = z\,] = \frac{\binom{N}{r}}{N^z} \sum_{k=1}^{r} (-1)^{r-k} \binom{r}{k} k^z \qquad (3.1.2)$$

$$\text{for} \quad \begin{array}{l} z = 1, 2, \ldots \\ r \leq \min(z, N) \end{array}$$

We know for each bug that the number of occurrences by time τ has a Poisson distribution with mean $\lambda\tau$ and the total number of occurrences Z has therefore a Poisson distribution with mean $N\lambda\tau$.

Thus

$$P[Z=z] = \frac{(N\lambda\tau)^z}{z!}\exp(-N\lambda\tau), \quad z=0,1,2,\ldots \qquad (3.1.3)$$

Hence, from (3.1.2) and (3.1.3), we obtain

$$P[R=r,Z=z] = \begin{cases} \exp\{-N\lambda\tau\}, & r=z=0 \\ \dfrac{\binom{N}{r}(\lambda\tau)^z}{z!}\exp(-N\lambda\tau)\displaystyle\sum_{k=1}^{r}(-1)^{r-k}\binom{r}{k}k^z, & \begin{array}{l} z=1,2,\ldots \\ r=1,2,\ldots,\min(z,N) \end{array} \end{cases} \qquad (3.1.4)$$

Figure 3.1 shows the sample space of the sufficient statistics (Z,R).

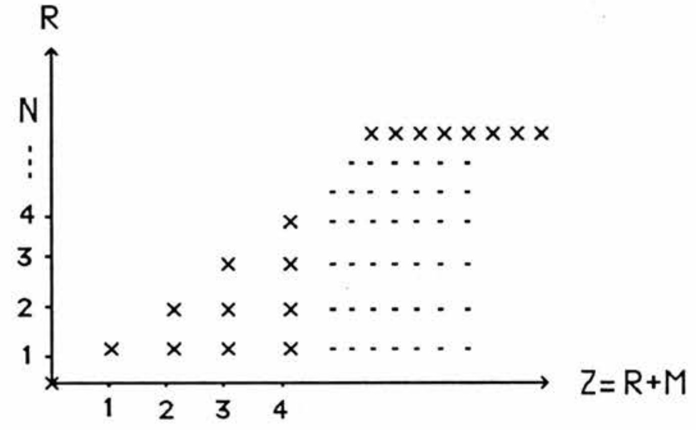


Figure 3.1

The sample space of the sufficient statistics (Z, R).

3.2. THE MAXIMUM LIKELIHOOD ESTIMATE

In the special case where we have seen no observations, and therefore have $r = 0$, it is clear from (3.1.4) that as we are assuming $\lambda > 0$, the MLE of N is zero. From now on we therefore assume that $r > 0$ or equivalently that $z > 0$. Consider now the logarithm of equation (2.3.3), which is

$$\text{Log}\, L(N, \lambda) = \text{Log}\, N! + (m + r)\,\text{Log}\, \lambda - N\lambda\tau + \sum_{i=1}^{r} m_i\, \text{Log}\,(\tau - t_{(i)})$$

$$- \text{Log}\,(N - r)! - \sum_{i=1}^{r} \text{Log}\, m_i! \qquad (3.2.1)$$

Taking the partial derivative with respect to λ and setting equal to zero, we obtain

$$\frac{\partial \log L}{\partial \lambda} = \frac{m + r}{\lambda} - N\tau$$

Solving this equation for λ, the estimated value of λ is

$$\hat{\lambda} = \frac{m + r}{N\tau}$$

This can be substituted into (2.3.3), giving the profile likelihood function for N, which may be taken as

$$L(N) = \frac{N(N-1)\ldots(N-r+1)}{N^{r+m}} \qquad (3.2.2)$$

We note that for r ≥1

$$L(k) > L(k-1)$$

$$\Leftrightarrow \left(\frac{k-1}{k}\right)^{r+m} > \frac{k-r}{k} = 1 - \frac{r}{k}$$

$$\Leftrightarrow \quad m < h_{r,k} \qquad (3.2.3)$$

where $$h_{r,k} = \frac{\operatorname{Log}(1-\frac{r}{k})}{\operatorname{Log}(1-\frac{1}{k})} - r \qquad (k > r)$$

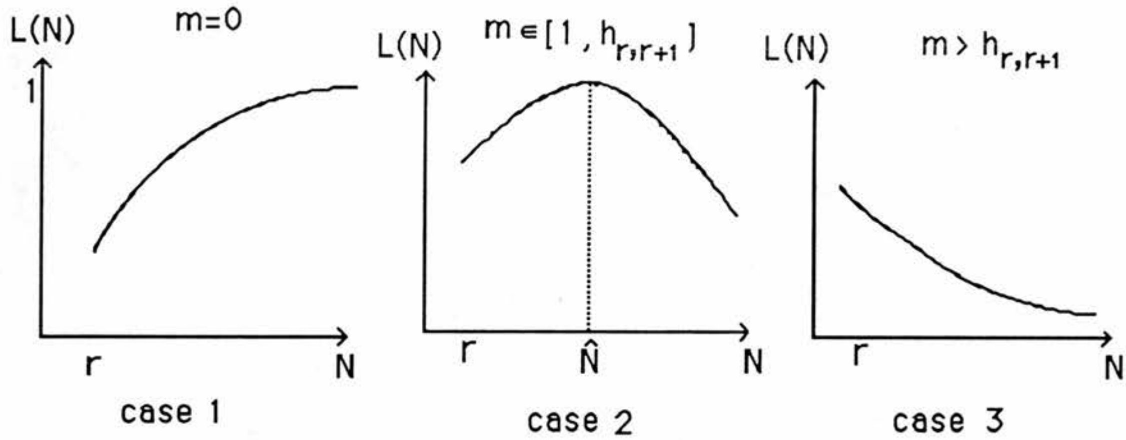
Similarly,

$$L(k+1) < L(k)$$

$$\Leftrightarrow m > h_{r,k+1} \qquad (3.2.4)$$

It will be convenient also to put that $h_{r,r} = \infty$. For each fixed r, it can be shown that $h_{r,k}$ is a decreasing function of k.

For given values of m and r, the profile likelihood function (3.2.2) can be computed and graphed. Three qualitatively different cases are shown in Figure 3.2(a-c). For convenience the likelihood function is drawn in continuous curve. We now discuss these three cases in turn.



a b c

Figure 3.2 (a - c)

The three possible different types of the profile likelihood function.

Case 1 : For m = 0 and $r \geq 1$, the profile likelihood is known to be an increasing function of N. Taking the logarithm of the profile likelihood function (3.2.3) gives

$$\operatorname{Log} L(N) = \sum_{i=1}^{r} \operatorname{Log}(N-i+1) - r \operatorname{Log}(N)$$

$$\frac{\partial \operatorname{Log} L(N)}{\partial N} = \sum_{i=1}^{r} \frac{1}{N-i+1} - \frac{r}{N}$$

$$\frac{\partial \operatorname{Log} L(N)}{\partial N} = r \left[\frac{1}{H} - \frac{1}{N} \right]$$

where H is the harmonic mean of N, N -1, . . ., N - r + 1. Since H < N and $r \geq 1$, we conclude that

$$\frac{\partial \operatorname{Log} L(N)}{\partial N} > 0$$

Hence for $r \geq 1$ and $m = 0$, $L(N)$ is increasing in N and thus the MLE, which we denote by $\hat{N}$, equals infinity.

Case 2 : If $m \in [1, h_{r,\ r+1}]$, where $h_{r,\ r+1} > 1$, then $L(N)$ is unimodal over the integers r, $r+1$, ... In fact, it follows from (3.2.3) and (3.2.4), that if $h_{r,\ k+1} < m < h_{r,\ k}$, the MLE of N is k.

Case 3 : If $m > h_{r,\ r+1}$ and $r \geq 1$ then $L(N)$ is decreasing in N and $\hat{N} = r$.

3.3 COMPUTING THE DISTRIBUTION OF THE MLE

A program has been written for computing the probability distribution of the MLE as follows :

Since we are assuming that $Z > 0$, we need the joint distribution of R and Z conditional on $Z > 0$, which can be found from equation (3.1.4.). The conditional probability that $Z = z$ and $R = r$ can be computed taking increasing values of r for each fixed z, and successively larger values of z. The computation can cease when the sum of these probabilities is sufficiently close to 1 i.e. (0.99999). For each point (r, z) we can find the value of the MLE. (Note that this value is infinity when $m = 0$). The distribution of the MLE then follows by aggregating the probabilities as follows :

$$P[\hat{N} = k] = \sum_{(z,r):\hat{N}=k} P(Z = z, R = r \mid z > 0)$$

$$k = 1, 2, \ldots, \infty$$

CHAPTER 4

HARMONIC MEAN ESTIMATES

4.1 THE JOE AND REID ESTIMATOR

4.1.1 DEFINITION OF THE ESTIMATOR

Figure 4.1 shows that, for a given level $c \in (0,1)$, there are three types of profile likelihood interval for N. For the J-M model, Joe and Reid (1985) proposed estimating N using the harmonic mean (HM) of the endpoints of this interval.

In case 1, L(N) is increasing and tends to one as N tends to infinity, so the profile likelihood interval is $[N_1, \infty)$, where

$N_1 = \text{Inf} \{ N \geq r : L(N) > c \}$.

The HM of N_1 and ∞ is

$$H = 2 N_1$$

In case 2, the profile likelihood interval is $[N_1, N_2]$, where

$N_1 = \text{Inf} \{ r \leq N \leq \hat{N} : L(N) > c\, L(\hat{N}) \}$

$N_2 = \sup \{ N \geq \hat{N} : L(N) > c\, L(\hat{N}) \}$.

The HM of N_1 and N_2 is

$$H = [\frac{1}{2} (\frac{1}{N_1} + \frac{1}{N_2})]^{-1} .$$

In case 3, the profile likelihood interval is $[\, r\ ,\ N_2\,]$, where

$$N_2 = \sup\{\, N \geq r : L(N) > c\,L(\hat{N})\,\}\ .$$

The HN of r and N_2 is

$$H = \left[\frac{1}{2}\left(\frac{1}{r}+\frac{1}{N_2}\right)\right]^{-1}$$

The harmonic mean estimator (HME), which we denote by $\tilde{N}$, can be defined as the closest integer to H .

For a chosen value of c, the probability distribution of $\tilde{N}$ can also be computed from the joint distribution of Z and R given Z>0. The method for computing the probability distribution of $\hat{N}$ can again be used. The only difference here is that $\tilde{N}$ cannot be infinite.

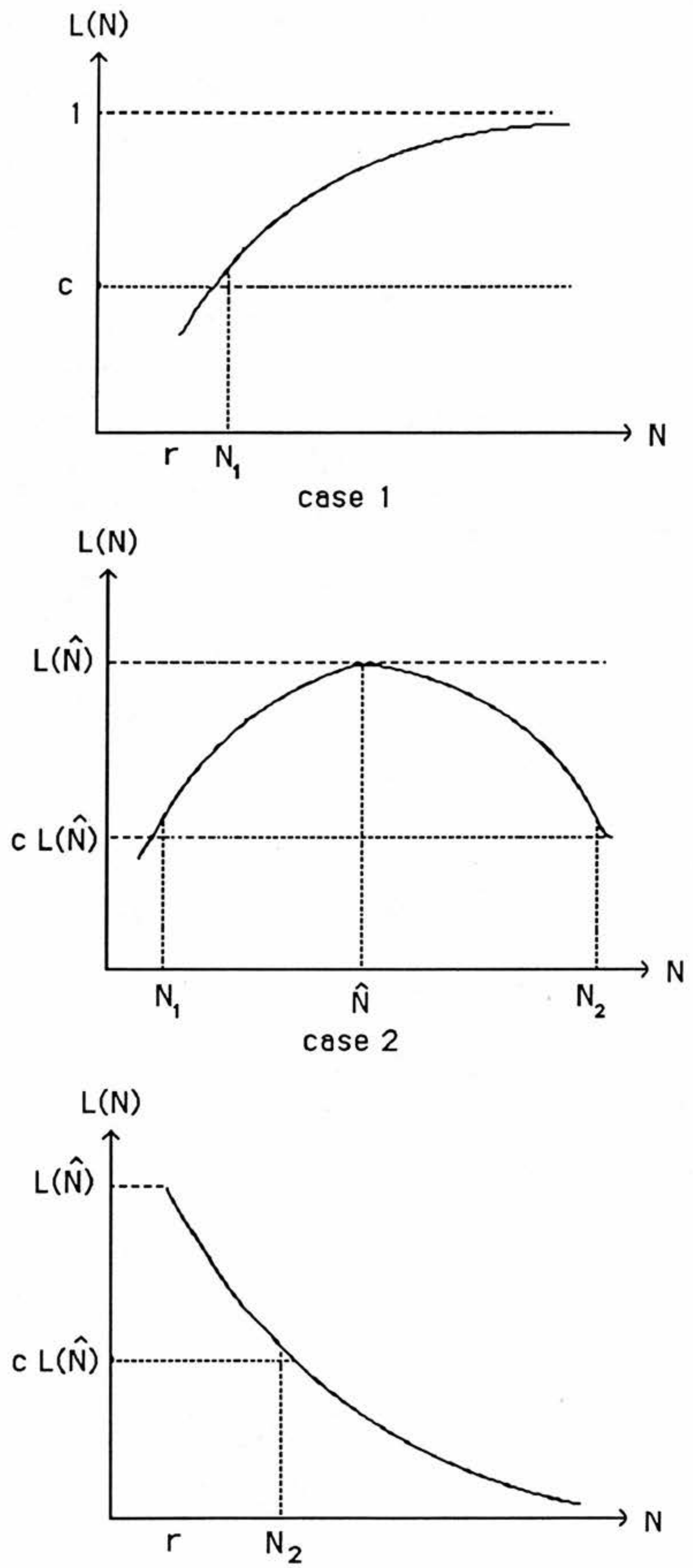


Figure 4-1

The three possible different types of the profile likelihood

interval for N.

In table 1 we give the exact probability distribution of the maximum likelihood and the harmonic mean estimators using the values N =10, $\lambda = 1$ and τ = 2.3, 1.4, 0.69 and 0.29 which are considered by Nayak (1988). We also put c = 0.5, the value recommended by Joe & Reid (1985) for the J-M model. This table shows that the probability that $\hat{N} = \infty$ is very high when τ is small. Figures 4.2 & 4.3 show that for small τ (0.29) the probability distributions of $\hat{N}$ and $\tilde{N}$ are irregular, whereas, when τ is large (2.3) they are fairly smooth. We should mention that, the probability that $\hat{N} = \infty$ is shown on these Figures, so that the exhibited probabilities sum to one. Note that when τ = 0.29 the harmonic mean estimator has much higher probability of being even than odd.

Table 1

The probability distributions of $\hat{N}$ and $\tilde{N}$ (N = 10 , λ = 1).

k	$\hat{p}_k$	$\tilde{p}_k$	$\hat{p}_k$	$\tilde{p}_k$	$\hat{p}_k$	$\tilde{p}_k$	$\hat{p}_k$	$\tilde{p}_k$
	τ =2.3		τ=1.4		τ=0.69		τ=0.29	
1	0000	.0000	.0000	.0000	.0031	.0031	.0270	.0270
2	.0000	.0000	.0003	.0002	.0232	.0153	.0762	.1812
3	.0000	.0000	.0020	.0014	.0379	.0291	.0215	.0675
4	.0001	.0001	.0067	.0048	.0165	.0291	.0007	.0180
5	.0013	.0010	.0274	.0115	.0808	.0124	.0777	.0006
6	.0091	.0081	.0513	.0415	.0730	.0607	.0162	.2240
7	.0440	.0313	.0945	.0388	.0757	.0226	.0024	.0005
8	.1439	.1058	.1550	.0839	.1600	.1075	.0588	.0899
9	.2886	.2342	.1480	.1092	.0495	.0436	.0009	.0023
10	.2965	.2779	.1724	.1376	.0202	.0289	.0001	.0002
11	.1635	.1818	.1131	.1378	.0568	.0737	.0029	.0085
12	.0333	.0949	.0893	.1279	.0320	.0983	.0003	.1714
13	.0136	.0274	.0383	.0703	.0699	.0711	.0218	.0502
14	.0021	.0160	.0061	.0312	.0001	.0007	.0000	.0000
15	.0018	.0125	.0452	.0552	.0384	.0589	.0007	.0029
16	.0010	.0030	.0045	.0433	.0001	.0254	.0000	.0002
17	.0000	.0021	.0000	.0061	.0000	.0001	.0000	.0000
18	.0008	.0000	.0109	.0000	.0022	.0000	.0000	.0000
19	.0002	.0011	.0145	.0138	.0566	.0017	.0064	.0000
20≥	.0003	.0029	.0152	.0854	.0315	.3181	.0014	.1555
∞	.0000		.0053		.1907		.6848	

$\hat{p}_k$, $\tilde{p}_k$ respective probabilities that $\hat{N}$, $\tilde{N}$ take the value k.

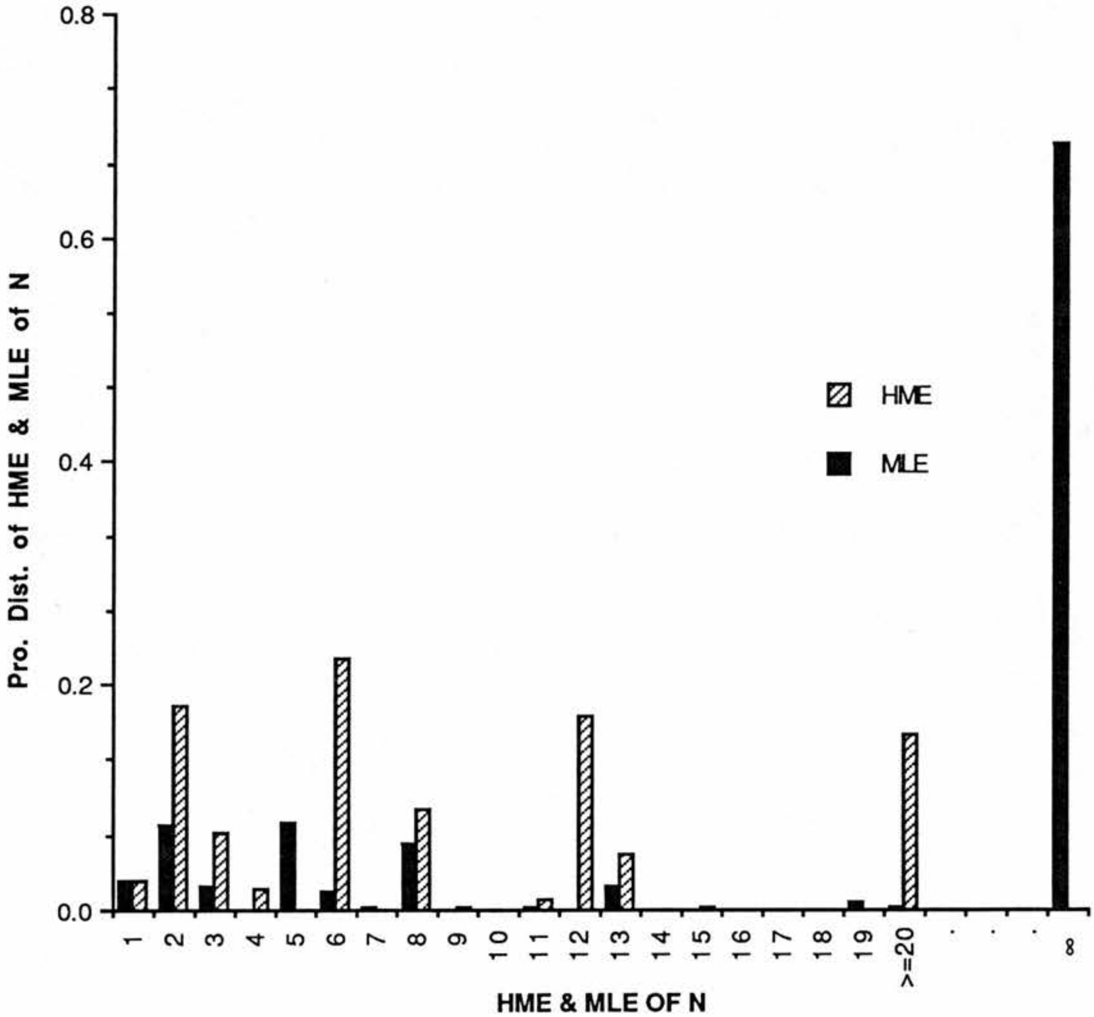


Figure 4.2

The probability distributions of the HME and the MLE for

N=10, τ =0.29 and c=0.5.

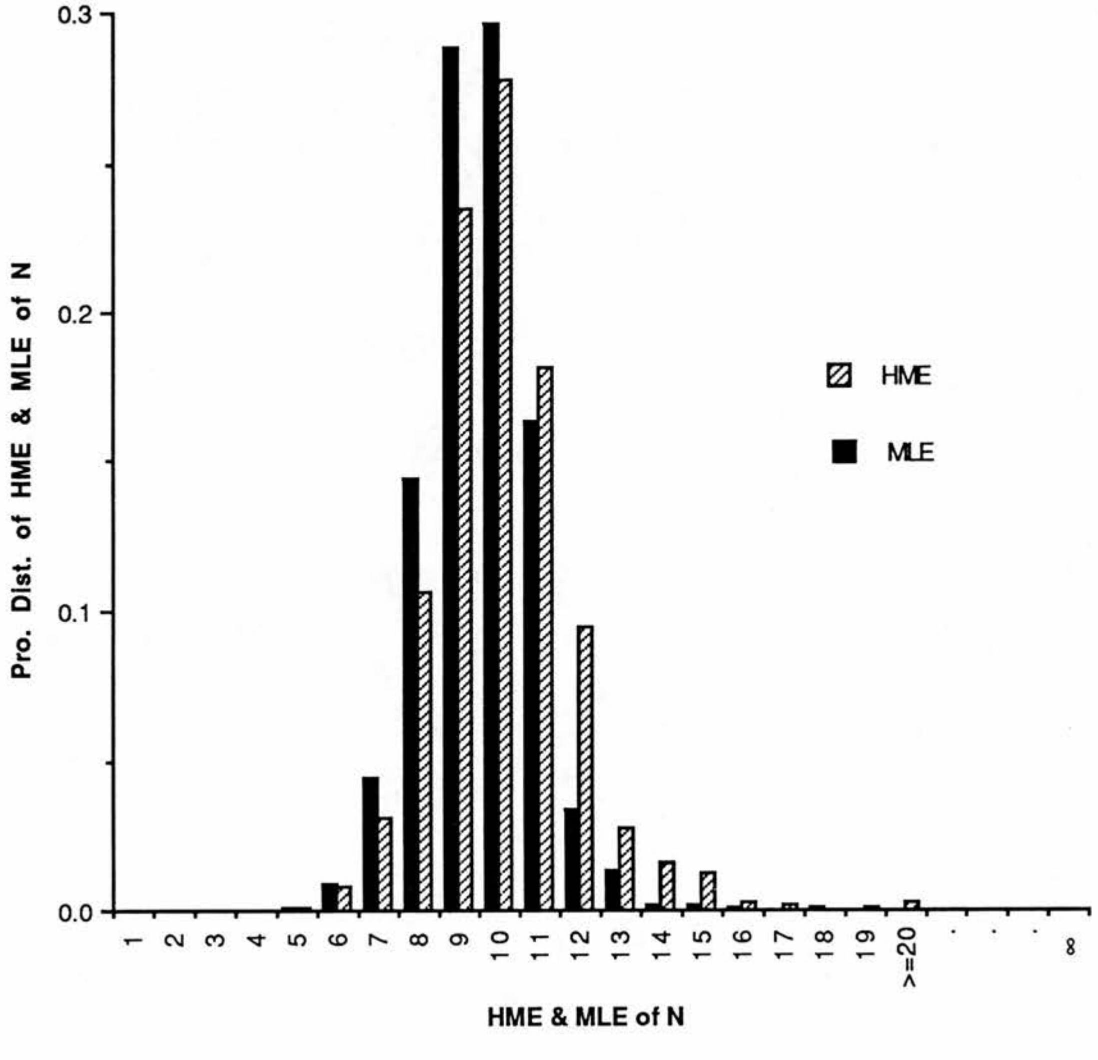


Figure 4.3

The probability distributions of the HME and the MLE for

N =10 , τ = 2.3 and c= 0.5.

4.1.2 CHOICE OF THE LEVEL C OF THE PROFILE LIKELIHOOD INTERVAL

It will be useful to define the criteria which are used in this section to summarise the distribution of $\tilde{N}$.

1. THE MEDIAN

The smallest number k satisfying $P[\tilde{N} \leq k] \geq 0.5$.

2. THE TARGET AREA PROBABILITY [TAP]

The probability of the estimator being within ±1 of the true value i.e. $P[N - 1 \leq \tilde{N} \leq N + 1]$.

3. THE LOWER TAIL PROBABILITY [LTP]

The probability that the estimator is less than or equal to the true value divided by two i.e. $P[\tilde{N} \leq N / 2]$.

4. THE UPPER TAIL PROBABILITY [UTP]

The probability that the estimator is greater than or equal to the true value multiplied by two i.e. $P[\tilde{N} \geq 2N]$.

In tables 2-5, we give summary statistics for the HME for N = 5, 10, 15 and 20 and varying values of p and c (p = 0.1, 0.2, . . . , 0.9 and c = 0.1, 0.2, . . ., 0.9).

THE MEDIAN

It can easily be seen from these tables that the median has got the best results (in terms of closeness to N) when p is 0.9. Also as p decreases the median stays equal to the true value for longest when c is large. However, for the large values of N when c =0.90, the median tends to be too large as p gets closer to 0.2.

For $p = 0.1$ there is a general tendency to badly underestimate N except for $c = 0.90$ when $N = 20$. This effect persists for $p = 0.2$ when c is small.

TAP

By looking at the TAP, we can see that for $p \in [0.8, 0.9]$, the maximum value of this probability occurs when $c = 0.7$, 0.8 or 0.9 In some cases these values of c are poor for small values of p.

LTP

For $p = 0.1$, 0.2, the LTPs are considerably better for large values of c than for small values of c except when $N = 5$. For $p = 0.8$, 0.9, we can see that all the LTPs are small. However, for $N = 5$ small values of c perform rather better.

UTP

For small values of p ($p \leq 0.3$), small values of c perform better than large values of c. For $p \geq 0.5$, large values of c are preferable when $N = 15$ or 20, whereas, for small values of N, the position is less clear-cut.

CONCLUSION :

It is clear that there is quite a bit of instability in the sizes of the above tabulated probabilities. Also, no one value of c is consistently better than the others. Nonetheless, the above argument leads us to prefer large values of c i.e. 0.8, 0.9 since these values perform better for large value of p. However, it should be noted that for small values of p, these values of c give large UTP combined with a median which is often a serious underestimate of N when $p = 0.1$.

Table 2

Summary statistics for the HME (N = 5, λ =1).

THE MEDIAN

P	VALUES OF C 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	2	2	2	2	2	2	2	2	2
0.2	2	2	2	2	2	2	2	2	2
0.3	4	4	3	3	3	3	3	2	2
0.4	4	4	4	4	6	6	7	6	6
0.5	5	4	4	4	6	6	7	6	6
0.6	6	5	5	5	6	6	7	6	6
0.7	6	6	6	6	6	6	5	5	5
0.8	6	6	6	5	5	5	5	5	5
0.9	6	5	5	5	5	5	5	5	5

TAP

P	VALUES OF C 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	.1940	.1940	.1603	.1603	.1603	.1603	.0002	.0027	.0027
0.2	.3549	.3549	.2465	.2466	.2462	.2462	.0035	.0183	.0189
0.3	.4590	.4589	.2750	.2763	.2732	.2732	.0162	.0507	.0552
0.4	.4987	.4981	.2707	.2771	.2659	.2659	.0459	.0958	.1120
0.5	.4864	.4837	.2611	.2818	.2550	.2550	.1000	.1499	.1887
0.6	.4528	.4460	.2728	.3196	.2736	.2736	.1868	.2235	.2898
0.7	.4407	.4327	.3376	.4120	.3564	.3564	.3211	.3506	.4310
0.8	.4990	.5113	.5050	.5796	.5379	.5379	.5296	.5719	.6322
0.9	.7081	.7592	.8034	.8320	.8208	.8208	.8203	.8582	.8744

LTP

P	VALUES OF C 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	.8011	.8011	.8011	.8022	.8022	.8022	.8022	.8190	.8190
0.2	.6093	.6100	.6100	.6170	.6170	.6170	.6170	.6712	.6712
0.3	.4333	.4362	.4362	.4553	.4553	.4553	.4553	.5470	.5470
0.4	.2826	.2899	.2899	.3234	.3234	.3234	.3234	.4358	.4358
0.5	.1651	.1780	.1780	.2215	.2215	.2215	.2215	.3289	.3289
0.6	.0847	.1014	.1014	.1439	.1439	.1439	.1439	.2235	.2235
0.7	.0380	.0534	.0534	.0832	.0832	.0832	.0832	.1257	.1257
0.8	.0148	.0235	.0235	.0360	.0360	.0360	.0360	.0493	.0493
0.9	.0035	.0052	.0052	.0068	.0068	.0068	.0068	.0080	.0080

UTP

P	VALUES OF C 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	.0038	.0011	.0011	.0180	.0180	.0180	.0180	.1780	.1780
0.2	.0280	.0092	.0092	.0634	.0634	.0634	.0634	.3060	.3060
0.3	.0844	.0309	.0309	.1224	.1224	.1224	.1224	.3794	.3789
0.4	.1708	.0685	.0685	.1797	.1797	.1797	.1797	.3997	.3971
0.5	.2670	.1166	.1166	.2186	.2186	.2186	.2186	.3736	.3654
0.6	.3358	.1600	.1600	.2236	.2236	.2236	.2236	.3104	.2917
0.7	.3325	.1753	.1753	.1842	.1842	.1842	.1842	.2195	.1897
0.8	.2300	.1376	.1376	.1059	.1059	.1059	.1059	.1142	.0843
0.9	.0687	.0484	.0484	.0249	.0249	.0249	.0249	.0254	.0140

Table 3

Summary statistics for the HME (N =10, λ =1).

THE MEDIAN

P	VALUES OF C								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	2	2	2	2	2	2	2	2	2
0.2	4	4	4	4	6	6	8	12	20
0.3	6	6	8	8	8	7	8	12	17
0.4	10	11	10	10	12	12	11	12	13
0.5	14	13	13	12	12	12	11	11	10
0.6	14	13	13	12	12	12	11	11	10
0.7	14	13	12	12	11	11	11	10	10
0.8	13	12	12	11	11	10	10	10	10
0.9	11	11	10	10	10	10	10	10	10

TAP

P	VALUES OF C								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	.0261	.0122	.0122	.0751	.0001	.0001	.0030	.0030	.0030
0.2	.1238	.0611	.0611	.1610	.0034	.0030	.0310	.0310	.0309
0.3	.2186	.1191	.1191	.1656	.0250	.0220	.0926	.0928	.0919
0.4	.2454	.1571	.1563	.1314	.0780	.0716	.1605	.1642	.1572
0.5	.2241	.1800	.1748	.1334	.1470	.1450	.2112	.2282	.2048
0.6	.2110	.2134	.1959	.1938	.2143	.2204	.2512	.2925	.2545
0.7	.2409	.2740	.2441	.2847	.3085	.3023	.3118	.3709	.3512
0.8	.3189	.3719	.2580	.4088	.4645	.4368	.4324	.4948	.5129
0.9	.5344	.6145	.6125	.6551	.6948	.7055	.7055	.7409	.7494

LTP

P	VALUES OF C								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	.8933	.8933	.8933	.8933	.6259	.6259	.6260	.6260	.6260
0.2	.6505	.6507	.6507	.6507	.3812	.3812	.3823	.3823	.3823
0.3	.3968	.3984	.3984	.3984	.2320	.2321	.2396	.2399	.2399
0.4	.2086	.2146	.2147	.2149	.1434	.1441	.1634	.1661	.1661
0.5	.0969	.1081	.1084	.1094	.0883	.0915	.1178	.1267	.1267
0.6	.0400	.0517	.0529	.0561	.0521	.0596	.0795	.0950	.0950
0.7	.0140	.0211	.0235	.0283	.0279	.0364	.0440	.0573	.0573
0.8	.0034	.0061	.0079	.0106	.0106	.0142	.0152	.0195	.0195
0.9	.0004	.0007	.0009	.0011	.0011	.0013	.0013	.0014	.0014

UTP

P	VALUES OF C								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	.0005	.0007	.0024	.0024	.0163	.0163	.0910	.0910	.3584
0.2	.0124	.0155	.0323	.0323	.0949	.0949	.2458	.2458	.5153
0.3	.0605	.0731	.1154	.1154	.2142	.2140	.3325	.3325	.4989
0.4	.1426	.1653	.2186	.2186	.3054	.3034	.3215	.3215	.3929
0.5	.2099	.2326	.2720	.2720	.3176	.3081	.3511	.3511	.2720
0.6	.2136	.2258	.2429	.2429	.2513	.2287	.1625	.1625	.1655
0.7	.1503	.1502	.1540	.1540	.1428	.1148	.0806	.0806	.0785
0.8	.0613	.0547	.0549	.0549	.0447	.0297	.0230	.0230	.0205
0.9	.0061	.0040	.0040	.0040	.0029	.0014	.0013	.0013	.0009

Table 4

Summary statistics for the HME (N =15, λ =1).

THE MEDIAN

P	VALUES OF C								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	4	4	4	4	6	6	8	11	10
0.2	6	6	8	10	12	12	11	12	20
0.3	14	13	14	16	15	15	18	17	20
0.4	19	18	20	20	20	20	18	18	17
0.5	23	22	21	20	20	19	18	17	16
0.6	21	20	20	19	19	18	17	17	16
0.7	19	18	18	17	17	16	16	15	15
0.8	18	17	17	16	16	16	15	15	15
0.9	16	16	16	15	15	15	15	15	15

TAP

P	VALUES OF C								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	.0197	.0193	.0528	.0437	.0001	.1381	.0001	.0001	.0027
0.2	.1272	.1158	.1786	.1294	.0060	.1903	.0060	.0060	.0340
0.3	.2083	.1615	.1830	.1461	.0401	.1387	.0401	.0104	.0792
0.4	.1904	.1180	.1105	.1510	.1012	.1307	.1004	.0484	.0991
0.5	.1524	.1026	.1048	.1837	.1536	.1657	.1513	.1098	.1339
0.6	.1441	.1508	.1770	.2206	.1945	.2132	.1995	.1698	.2031
0.7	.1719	.2305	.2683	.2554	.2601	.2721	.2848	.2403	.2873
0.8	.2721	.3357	.3555	.3469	.3808	.3960	.4081	.3708	.4199
0.9	.4606	.5368	.5589	.5766	.5834	.6177	.6248	.6442	.6619

LTP

P	VALUES OF C								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	.9082	.9083	.7704	.7704	.7704	.7932	.4910	.4910	.4910
0.2	.5740	.5763	.3920	.3923	.3923	.4673	.2767	.2767	.2788
0.3	.2438	.2518	.1533	.1564	.1564	.2383	.1746	.1755	.1872
0.4	.0756	.0857	.0572	.0649	.0649	.1130	.1001	.1048	.1253
0.5	.0218	.0286	.0239	.0318	.0318	.0505	.0494	.0582	.0734
0.6	.0075	.0107	.0103	.0140	.0140	.0207	.0220	.0286	.0336
0.7	.0022	.0032	.0033	.0044	.0044	.0067	.0081	.0099	.0105
0.8	.0003	.0004	.0005	.0008	.0008	.0011	.0015	.0016	.0016
0.9	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

UTP

P	VALUES OF C								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	.0007	.0009	.0027	.0027	.0128	.0123	.0559	.1938	.1938
0.2	.0242	.0307	.0526	.0526	.1132	.0981	.2215	.4057	.4057
0.3	.1208	.1433	.1839	.1839	.2661	.2135	.3189	.4144	.4143
0.4	.2448	.2721	.2822	.2822	.3261	.2619	.3057	.3186	.3170
0.5	.2843	.2967	.2640	.2640	.2607	.2249	.2345	.2082	.2007
0.6	.2214	.2128	.1734	.1734	.1517	.1394	.1405	.1117	.0986
0.7	.1162	.0961	.0740	.0740	.0630	.0547	.0547	.0409	.0329
0.8	.0283	.0200	.0137	.0137	.0126	.0084	.0084	.0062	.0051
0.9	.0007	.0005	.0002	.0002	.0002	.0001	.0001	.0001	.0001

Table 5

Summary statistics for the HME (N = 20, λ =1).

THE MEDIAN

P	VALUES OF C								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	4	4	4	4	6	6	8	12	20
0.2	10	12	14	13	13	14	20	30	31
0.3	19	24	23	24	23	26	27	25	23
0.4	29	27	29	28	27	26	25	24	23
0.5	29	27	26	26	25	25	24	23	22
0.6	27	25	24	24	23	22	22	21	20
0.7	25	24	23	23	22	22	21	21	20
0.8	23	23	22	22	21	21	21	20	20
0.9	22	21	21	21	20	20	20	20	20

TAP

P	VALUES OF C								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	.0004	.0000	.0279	.0000	.0901	.0075	.1846	.0000	.2919
0.2	.0155	.0001	.1014	.0014	.2052	.0650	.1558	.0006	.1109
0.3	.0273	.0047	.0913	.0199	.1833	.1164	.0816	.0127	.0320
0.4	.0778	.0283	.0912	.0716	.1529	.1183	.0789	.0616	.0640
0.5	.0863	.0700	.1251	.1254	.1542	.1141	.1143	.1291	.1336
0.6	.1092	.1185	.1576	.1711	.1750	.1504	.1624	.1917	.1942
0.7	.1396	.1857	.2108	.2345	.2056	.2139	.2357	.2499	.2499
0.8	.2126	.2799	.3043	.3145	.3122	.3379	.3717	.3650	.3692
0.9	.3843	.4651	.5313	.5297	.5325	.5566	.5602	.5934	.6135

LTP

P	VALUES OF C								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	.9334	.8509	.8509	.8509	.6664	.6664	.6674	.3755	.3930
0.2	.5636	.4270	.4276	.4276	.2798	.2799	.2965	.1895	.2525
0.3	.1972	.1452	.1499	.1499	.1086	.1103	.1406	.1559	.1878
0.4	.0480	.0463	.0541	.0545	.0490	.0549	.0721	.0910	.1196
0.5	.0121	.0166	.0215	.0229	.0226	.0288	.0331	.0435	.0637
0.6	.0033	.0052	.0068	.0082	.0085	.0111	.0126	.0161	.0250
0.7	.0006	.0010	.0014	.0018	.0021	.0028	.0035	.0040	.0057
0.8	.0000	.0001	.0001	.0002	.0002	.0003	.0004	.0004	.0005
0.9	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

UTP

P	VALUES OF C								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	.0007	.0010	.0026	.0026	.0099	.0372	.0372	.1198	.3044
0.2	.0340	.0443	.0692	.0692	.1250	.2051	.2051	.3452	.4930
0.3	.1673	.1872	.2325	.2325	.2963	.3021	.3021	.3647	.4058
0.4	.3005	.2806	.3036	.3036	.3285	.2672	.2663	.2783	.2808
0.5	.2954	.2431	.2361	.2354	.2371	.1785	.1726	.1729	.1629
0.6	.1832	.1467	.1288	.1250	.1181	.0926	.0813	.0785	.0681
0.7	.0650	.0541	.0437	.0388	.0335	.0289	.0223	.0195	.0169
0.8	.0080	.0072	.0048	.0038	.0033	.0028	.0018	.0015	.0013
0.9	.0001	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

4.2. THE WEIGHTED HARMONIC MEAN ESTIMATOR

4.2.1 DEFINITION AND INITIAL INVESTIGATION

How might the HME be improved ?

A generalisation of the HM would be to use the weighted harmonic mean (WHM) given by

$$H^* = [w(\frac{1}{N_1}) + (1-w)(\frac{1}{N_2})]^{-1}$$

where w is a weighting factor ($0 < w < 1$). The weighted harmonic mean estimator of N (WHME) is N^* which is defined as the closest integer to H^*. Putting $w = 0.5$ gives the Joe & Reid estimator and changing the values of the weight will substantially affect the distribution of the HME. This can be seen quite clearly from tables 6-9 when we choose pairs of values of c and w, where c= 0.25, 0.5, 0.75, w= 0.25, 0.5, 0.75 and the values of p and N are the same as previously.

We assess this estimator using the corresponding criteria to those used in the last subsection for the Joe & Reid estimator.

THE MEDIAN

We can see from these tables that for $p \geq 0.5$ the best results for the median occur when (c, w) = (0.75, 0.75). But when p =0.1, 0.2, for these values of c and w the WHME badly underestimates N. It can be also observed that for (c, w) = (0.25, 0.25) the estimator performs better for small p, but then, it overestimates N when p is large. It is when p is large that the most information is available and we would wish the estimator to do well in this situation.

TAP

For small values of p, the small values of w give the best results when N = 5, but no value of (c, w) gives consistently good results for the other values of N. When p = 0.9 the largest and smallest TAPs consistently occur for (0.75, 0.75) and (0.25, 0.25) respectively. For moderate and large values of p, (c, w)= (0.25, 0.25) performs relatively poorly when N ≥ 10, but no value of (c, w) consistently performs better than the others.

LTP

The probabilities of LT for small values of w are consistently better than for large values of w and the differences are large when p is small (especially when N = 5). For large p however, the LTPs are generally small for all (c, w).

UTP

Here large values of w are consistently better than small values of w for all values of N. Overall the best performances occur when (c, w) = (0.25, 0.75) and (0.75, 0.75) for small and large values of p respectively.

CONCLUSION

It is again hard to establish the best value of (c, w) for all values of N and p. However, the above discussion leads us to prefer the value of (c, w) to be (.75, .75), since this value gives better results for large p. We should however mention that for small p this value gives poor LTPs for all values of N.

Table 6

Summary statistics for the WHME (N=5, λ=1).

THE MEDIAN

P	VALUES OF (C,W)								
	(.25,.25)	(0.5,.25)	(.75,.25)	(.25,0.5)	(0.5,0.5)	(.75,0.5)	(.25,.75)	(0.5,.75)	(.75,.75)
0.1	4	4	4	2	2	2	1	1	1
0.2	4	4	4	2	2	2	1	1	1
0.3	6	4	4	3	3	3	2	2	2
0.4	8	10	7	4	6	7	2	4	5
0.5	8	10	7	4	6	7	3	4	5
0.6	8	10	7	5	6	7	5	4	5
0.7	8	7	6	6	6	5	5	6	5
0.8	6	6	5	6	5	5	5	5	5
0.9	6	5	5	5	5	5	5	5	5

TAP

P	VALUES OF (C,W)								
	(.25,.25)	(0.5,.25)	(.75,.25)	(.25,0.5)	(0.5,0.5)	(.75,0.5)	(.25,.75)	(0.5,.75)	(.75,.75)
0.1	.7778	.7767	.7599	.1603	.1603	.0002	.0198	.1627	.1628
0.2	.6089	.6018	.5473	.2466	.2462	.0035	.0758	.2609	.2616
0.3	.4897	.4705	.3764	.2756	.2732	.0162	.1569	.3076	.3122
0.4	.4152	.3811	.2612	.2735	.2659	.0463	.2444	.3153	.3320
0.5	.3792	.3331	.2118	.2700	.2550	.1027	.3191	.3021	.3436
0.6	.3763	.3270	.2342	.2929	.2736	.1976	.3758	.2995	.3767
0.7	.4074	.3696	.3358	.3696	.3564	.3509	.4404	.3561	.4663
0.8	.4925	.4922	.5322	.5371	.5379	.5829	.5752	.5269	.6405
0.9	.7081	.7576	.8204	.8157	.8208	.8621	.8257	.8169	.8749

LTP

P	VALUES OF (C,W)								
	(.25,.25)	(0.5,.25)	(.75,.25)	(.25,0.5)	(0.5.0.5)	(.75,0.5)	(.25,.75)	(0.5,.75)	(.75,.75)
0.1	.0415	.0415	.0425	.8011	.8022	.8022	.9622	.8190	.8190
0.2	.0655	.0662	.0732	.6100	.6170	.6170	.8597	.6712	.6712
0.3	.0727	.0759	.0950	.4362	.4553	.4553	.7123	.5470	.5470
0.4	.0659	.0745	.1080	.2899	.3234	.3234	.5434	.4358	.4358
0.5	.0502	.0662	.1097	.1780	.2215	.2215	.3764	.3289	.3289
0.6	.0320	.0540	.0965	.1014	.1439	.1439	.2308	.2235	.2235
0.7	.0169	.0387	.0686	.0534	.0832	.0832	.1185	.1257	.1257
0.8	.0074	.0209	.0334	.0235	.0360	.0360	.0443	.0493	.0493
0.9	.0021	.0050	.0067	.0052	.0068	.0068	.0073	.0080	.0080

UTP

P	VALUES OF (C,W)								
	(.25,.25)	(0.5,.25)	(.75,.25)	(.25,0.5)	(0.5.0.5)	(.75,0.5)	(.25,.75)	(0.5,.75)	(.75,.75)
0.1	.0207	.1807	.1780	.0011	.0180	.0180	.0002	.0011	.0178
0.2	.0821	.3248	.3060	.0092	.0634	.0634	.0032	.0092	.0606
0.3	.1761	.4331	.3794	.0309	.1224	.1224	.0144	.0302	.1102
0.4	.2837	.5032	.3997	.0685	.1797	.1797	.0386	.0647	.1478
0.5	.3772	.5295	.3736	.1166	.2186	.2186	.0740	.1030	.1588
0.6	.4262	.5022	.3104	.1600	.2236	.2236	.1075	.1253	.1388
0.7	.4047	.4103	.2195	.1753	.1842	.1842	.1161	.1119	.0928
0.8	.2967	.2517	.1142	.1376	.1059	.1059	.0818	.0627	.0414
0.9	.1118	.0705	.0254	.0484	.0249	.0249	.0223	.0123	.0070

Table 7

Summary Statistics for the WHME (N=10, λ=1).

THE MEDIAN

P	VALUES OF (C,W)								
	(.25,.25)	(0.5,.25)	(.75,.25)	(.25,0.5)	(0.5,0.5)	(.75,0.5)	(.25,.75)	(0.5,.75)	(.75,.75)
0.1	4	4	4	2	2	2	1	1	1
0.2	8	12	16	4	6	8	2	4	5
0.3	14	12	16	8	8	8	5	7	6
0.4	16	18	16	11	12	11	8	9	9
0.5	18	18	14	13	12	11	10	10	10
0.6	17	14	11	13	12	11	11	10	10
0.7	15	13	11	13	11	10	11	10	10
0.8	13	11	11	12	11	10	11	10	10
0.9	11	11	10	10	10	10	10	10	10

TAP

P	VALUES OF (C,W)								
	(.25,.25)	(0.5,.25)	(.75,.25)	(.25,0.5)	(0.5,0.5)	(.75,0.5)	(.25,.75)	(0.5,.75)	(.75,.75)
0.1	.0000	.0122	.0001	.0122	.0001	.0030	.0030	.0030	.0030
0.2	.0001	.0611	.0030	.0611	.0034	.0310	.0310	.0310	.0309
0.3	.0019	.1191	.0220	.1191	.0250	.0928	.0925	.0928	.0918
0.4	.0122	.1563	.0719	.1517	.0780	.1642	.1608	.1642	.1566
0.5	.0428	.1748	.1483	.1798	.1470	.2280	.2140	.2280	.1998
0.6	.0994	.1966	.2394	.2112	.2143	.2901	.2678	.2901	.2322
0.7	.1713	.2514	.3585	.2673	.3085	.3559	.3631	.3559	.2983
0.8	.2651	.3938	.5145	.3749	.4645	.4510	.4985	.4510	.4558
0.9	.5531	.6651	.7191	.6281	.6948	.7065	.6688	.7065	.7377

LTP

P	VALUES OF (C,W)								
	(.25,.25)	(0.5,.25)	(.75,.25)	(.25,0.5)	(0.5,0.5)	(.75,0.5)	(.25,.75)	(0.5,.75)	(.75,.75)
0.1	.5976	.6259	.6259	.8933	.6259	.6260	.9685	.8933	.8934
0.2	.3109	.3812	.3812	.6507	.3812	.3823	.8111	.6507	.6519
0.3	.1459	.2320	.2320	.3984	.2320	.2399	.5567	.3985	.4063
0.4	.0690	.1432	.1434	.2146	.1434	.1661	.3122	.2156	.2375
0.5	.0376	.0869	.0883	.1081	.0883	.1267	.1485	.1127	.1478
0.6	.0222	.0477	.0521	.0517	.0521	.0950	.0658	.0636	.0990
0.7	.0106	.0207	.0279	.0211	.0279	.0573	.0395	.0368	.0577
0.8	.0027	.0061	.0106	.0061	.0106	.0195	.0107	.0142	.0195
0.9	.0002	.0007	.0011	.0007	.0011	.0014	.0011	.0013	.0014

UTP

P	VALUES OF (C,W)								
	(.25,.25)	(0.5,.25)	(.75,.25)	(.25,0.5)	(0.5,0.5)	(.75,0.5)	(.25,.75)	(0.5,.75)	(.75,.75)
0.1	.0192	.0914	.0914	.0007	.0163	.0910	.0001	.0020	.0020
0.2	.1236	.2553	.2553	.0155	.0949	.2458	.0029	.0228	.0228
0.3	.2933	.3725	.3723	.0731	.2142	.3325	.0205	.0754	.0754
0.4	.4311	.4029	.4007	.1653	.3054	.3215	.0612	.1372	.1372
0.5	.4701	.3584	.3472	.2326	.3176	.2511	.1024	.1646	.1646
0.6	.4113	.2681	.2384	.2258	.2513	.1625	.1072	.1373	.1373
0.7	.2797	.1578	.1161	.1502	.1428	.0806	.0692	.0767	.0767
0.8	.1147	.0551	.0298	.0547	.0447	.0230	.0221	.0228	.0228
0.9	.0104	.0040	.0014	.0040	.0029	.0013	.0013	.0013	.0013

Table 8

Summary statistics for the WHME (N=15, λ=1).

THE MEDIAN

P	VALUES OF (C,W)								
	(.25,.25)	(0.5,.25)	(.75,.25)	(.25,0.5)	(0.5,0.5)	(.75,0.5)	(.25,.75)	(0.5,.75)	(.75,.75)
0.1	8	12	14	4	6	8	2	4	5
0.2	16	18	16	8	12	11	5	8	9
0.3	24	24	22	13	15	18	10	13	16
0.4	28	26	22	18	20	18	14	16	16
0.5	28	23	19	21	20	17	17	17	16
0.6	24	20	18	20	19	17	17	17	16
0.7	20	18	16	18	17	16	16	16	15
0.8	18	17	16	17	16	15	16	15	15
0.9	17	16	15	16	15	15	15	15	15

TAP

P	VALUES OF (C,W)								
	(.25,.25)	(0.5,.25)	(.75,.25)	(.25,0.5)	(0.5,0.5)	(.75,0.5)	(.25,.75)	(0.5,.75)	(.75,.75)
0.1	.1598	.0004	.3115	.0092	.0001	.0001	.0000	.0027	.1406
0.2	.2462	.0102	.2517	.0552	.0060	.0060	.0006	.0344	.2184
0.3	.1552	.0387	.1791	.0788	.0401	.0401	.0105	.0824	.1777
0.4	.0753	.0667	.1588	.0690	.1012	.1004	.0504	.0986	.1274
0.5	.0762	.0952	.1691	.0901	.1536	.1513	.1243	.1057	.1356
0.6	.1119	.1404	.2032	.1572	.1945	.1993	.2153	.1591	.1875
0.7	.1619	.2122	.2915	.2400	.2601	.2825	.3181	.2722	.2584
0.8	.2305	.3524	.4407	.3290	.3808	.4022	.4466	.4084	.4077
0.9	.4293	.6121	.6127	.5300	.5834	.6461	.5900	.6394	.6617

LTP

P	VALUES OF (C,W)								
	(.25,.25)	(0.5,.25)	(.75,.25)	(.25,0.5)	(0.5,0.5)	(.75,0.5)	(.25,.75)	(0.5,.75)	(.75,.75)
0.1	.4681	.4681	.4692	.7704	.7704	.4910	.9312	.7932	.7932
0.2	.1990	.2013	.2149	.3920	.3923	.2767	.6517	.4673	.4695
0.3	.0815	.0896	.1217	.1533	.1564	.1746	.3366	.2383	.2509
0.4	.0341	.0443	.0767	.0572	.0649	.1001	.1400	.1130	.1382
0.5	.0152	.0224	.0402	.0239	.0318	.0494	.0512	.0509	.0750
0.6	.0062	.0102	.0158	.0103	.0140	.0220	.0168	.0221	.0337
0.7	.0016	.0032	.0045	.0033	.0044	.0081	.0046	.0081	.0105
0.8	.0002	.0004	.0008	.0005	.0008	.0015	.0008	.0015	.0016
0.9	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

UTP

P	VALUES OF (C,W)								
	(.25,.25)	(0.5,.25)	(.75,.25)	(.25,0.5)	(0.5,0.5)	(.75,0.5)	(.25,.75)	(0.5,.75)	(.75,.75)
0.1	.0155	.0564	.1944	.0009	.0128	.0559	.0001	.0021	.0558
0.2	.1497	.2372	.4210	.0307	.1132	.2215	.0084	.0374	.2163
0.3	.3693	.3806	.4700	.1433	.2661	.3184	.0561	.1278	.2877
0.4	.4882	.4068	.3983	.2709	.3261	.3054	.1279	.1972	.2394
0.5	.4517	.3379	.2750	.2889	.2607	.2320	.1472	.1793	.1594
0.6	.3175	.2190	.1531	.1943	.1517	.1328	.1012	.1082	.0862
0.7	.1535	.0965	.0631	.0791	.0630	.0467	.0402	.0408	.0315
0.8	.0373	.0200	.0126	.0147	.0126	.0065	.0061	.0062	.0051
0.9	.0012	.0005	.0002	.0003	.0002	.0001	.0001	.0001	.0001

Table 9

Summary statistics for the WHME (N=20, λ=1).

THE MEDIAN

P	VALUES OF (C,W)								
	(.25,.25)	(0.5,.25)	(.75,.25)	(.25,0.5)	(0.5,0.5)	(.75,0.5)	(.25,.75)	(0.5,.75)	(.75,.75)
0.1	8	12	16	4	6	8	2	4	5
0.2	24	24	43	12	13	24	8	10	16
0.3	36	40	32	23	23	26	17	20	22
0.4	42	34	27	26	27	24	22	22	22
0.5	34	29	26	26	25	24	22	22	21
0.6	29	26	23	25	23	21	22	21	20
0.7	26	24	22	24	22	21	22	21	20
0.8	24	22	21	22	21	20	21	20	20
0.9	22	21	20	21	20	20	20	20	20

TAP

P	VALUES OF (C,W)								
	(.25,.25)	(0.5,.25)	(.75,.25)	(.25,0.5)	(0.5,0.5)	(.75,0.5)	(.25,.75)	(0.5,.75)	(.75,.75)
0.1	.0000	.0004	.0001	.0000	.0901	.0001	.0000	.0000	.0000
0.2	.0003	.0133	.0092	.0002	.2052	.0085	.0006	.0006	.0043
0.3	.0077	.0432	.0527	.0053	.1833	.0500	.0128	.0127	.0407
0.4	.0386	.0595	.0931	.0334	.1529	.1085	.0636	.0616	.1087
0.5	.0837	.0793	.1101	.0785	.1542	.1629	.1359	.1291	.1621
0.6	.1198	.1187	.1537	.1210	.1750	.2050	.1859	.1921	.2145
0.7	.1424	.1800	.2322	.1895	.2056	.2608	.2241	.2824	.2699
0.8	.1882	.2765	.3598	.3050	.3122	.3665	.3625	.3903	.3724
0.9	.3646	.5383	.5532	.5015	.5325	.5883	.5438	.5838	.6135

LTP

P	VALUES OF (C,W)								
	(.25,.25)	(0.5,.25)	(.75,.25)	(.25,0.5)	(0.5,0.5)	(.75,0.5)	(.25,.75)	(0.5,.75)	(.75,.75)
0.1	.6351	.3453	.3745	.8509	.6664	.6675	.9520	.6894	.6849
0.2	.2111	.1194	.1696	.4270	.2798	.2998	.6469	.5101	.3623
0.3	.0527	.0620	.0910	.1452	.1086	.1559	.2892	.2418	.2011
0.4	.0158	.0326	.0531	.0463	.0490	.0910	.1020	.1076	.1072
0.5	.0058	.0134	.0285	.0166	.0226	.0435	.0352	.0421	.0532
0.6	.0019	.0047	.0105	.0053	.0085	.0161	.0106	.0137	.0224
0.7	.0005	.0012	.0022	.0011	.0021	.0040	.0020	.0035	.0055
0.8	.0001	.0001	.0002	.0001	.0002	.0004	.0002	.0004	.0005
0.9	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

UTP

P	VALUES OF (C,W)								
	(.25,.25)	(0.5,.25)	(.75,.25)	(.25,0.5)	(0.5,0.5)	(.75,0.5)	(.25,.75)	(0.5,.75)	(.75,.75)
0.1	.0122	.1227	.3049	.0010	.0099	.1198	.0001	.0020	.0371
0.2	.1677	.4039	.5129	.0443	.1250	.3452	.0126	.0493	.1970
0.3	.4170	.5176	.4717	.1871	.2963	.3647	.0826	.1669	.2612
0.4	.5038	.4570	.3659	.2779	.3285	.2783	.1597	.2213	.3039
0.5	.4106	.3233	.2409	.2306	.2371	.1736	.1481	.1654	.1204
0.6	.2430	.1721	.1182	.1302	.1181	.0813	.0789	.0781	.0515
0.7	.0957	.0570	.0334	.0487	.0335	.0223	.0222	.0195	.0129
0.8	.0152	.0072	.0033	.0070	.0033	.0018	.0018	.0015	.0011
0.9	.0001	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

4.2. 2 CHOOSING APPROPRIATE VALUES OF C AND W

In the previous subsection we considered several values of (c, w) and said that, although the position was not completely clear-cut, we preferred both c and w to be large. So in this subsection, as shown in tables 10-13, we will choose the pairs of values (c, w) to be { (0.75, 0.75), (0.8, 0.8), (0.9,0.8), (0.8, 0.9), (0.9, 0.9) }. The values of p and N are again the same as before.

THE MEDIAN

As we expected, the median is best when p is large. In fact when $p \in [\, 0.7,\ 0.9\,]$ the median equals the true value of N for all the cases shown in the tables. For $p \in [0.4, 0.6]$ there is no major difference between the median and N. For p = 0.1, 0.2 there is a general tendency to badly underestimate N except for large c when p = 0.2 and N = 10, 20. This leads us to prefer c = 0.9 with results when N = 10, 15 suggesting the choice (c, w) = (0.9, 0.9).

TAP

When N = 5, 15 the values (c, w) = (0.75, 0.75), (0.8, 0.9) perform well particularly for small p. However, when N = 10, (c, w) = (0.9, 0.9) is clearly better for small p. When N = 20, the TAPs are less dependent on the value of (c, w) and no one of these five values is better than the others for all p.

LTP

For N = 5, all values of (c, w) under consideration give the same LTPs. For N = 10, 15, 20 we can see that as p gets large all LTPs are small and there is little difference between them. For these values of N, c = 0.9 performs consistently well for small p in comparison with the other values of c.

UTP

Again all values of (c, w) perform well when p is large. When N = 5, 10, 15 the smaller values of c are better than c = 0.9 when p is small. However, when N = 20 none of the other four values is better than (0.9, 0.9) for all values of p.

CONCLUSION

As we expected based on the results in the previous section, large values of c and w give generally satisfactory results. However, from the above discussion we can say that there is no value of (c, w) that is consistently superior to any of the other values. Nonetheless, the results for the median and the LTP lead us to recommend (c, w) = (0.9, 0.9) while noting that this value is less good as regards the UTP for small N.

Table 10

Summary statistics for the WHME for chosen

large c and w when N =5.

THE MEDIAN

P	VALUES OF (C,W)				
	(.75,.75)	(0.8,0.8)	(0.9,0.8)	(0.8,0.9)	(0.9,0.9)
0.1	1	1	1	1	1
0.2	1	1	1	1	1
0.3	2	2	2	2	2
0.4	5	5	5	5	5
0.5	5	5	5	5	5
0.6	5	5	5	5	5
0.7	5	5	5	5	5
0.8	5	5	5	5	5
0.9	5	5	5	5	5

TAP

P	VALUES OF (C,W)				
	(.75,.75)	(0.8,0.8)	(0.9,0.8)	(0.8,0.9)	(0.9,0.9)
0.1	.1628	.0027	.0027	.1628	.0027
0.2	.2616	.0189	.0189	.2616	.0189
0.3	.3122	.0552	.0552	.3122	.0552
0.4	.3320	.1120	.1120	.3320	.1120
0.5	.3436	.1887	.1887	.3436	.1887
0.6	.3767	.2898	.2898	.3767	.2898
0.7	.4663	.4310	.4310	.4663	.4310
0.8	.6405	.6322	.6322	.6405	.6322
0.9	.8749	.8744	.8744	.8749	.8744

LTP

P	VALUES OF (C,W)				
	(.75,.75)	(0.8,0.8)	(0.9,0.8)	(0.8,0.9)	(0.9,0.9)
0.1	.8190	.8190	.8190	.8190	.8190
0.2	.6712	.6712	.6712	.6712	.6712
0.3	.5470	.5470	.5470	.5470	.5470
0.4	.4358	.4358	.4358	.4358	.4358
0.5	.3289	.3289	.3289	.3289	.3289
0.6	.2235	.2235	.2235	.2235	.2235
0.7	.1257	.1257	.1257	.1257	.1257
0.8	.0493	.0493	.0493	.0493	.0493
0.9	.0080	.0080	.0080	.0080	.0080

UTP

P	VALUES OF (C,W)				
	(.75,.75)	(0.8,0.8)	(0.9,0.8)	(0.8,0.9)	(0.9,0.9)
0.1	.0178	.0178	.1778	.0178	.1778
0.2	.0606	.0606	.3033	.0606	.3033
0.3	.1102	.1102	.3672	.1102	.3672
0.4	.1478	.1478	.3678	.1478	.3678
0.5	.1588	.1588	.3137	.1588	.3137
0.6	.1388	.1388	.2249	.1388	.2249
0.7	.0928	.0928	.1281	.0928	.1281
0.8	.0414	.0414	.0497	.0414	.0497
0.9	.0070	.0070	.0075	.0070	.0075

Table11

Summary statistics for the WHME for chosen

large c and w when N = 10.

THE MEDIAN

P	VALUES OF (C,W)				
	(.75,.75)	(0.8,0.8)	(0.9,0.8)	(0.8,0.9)	(0.9,0.9)
0.1	1	1	1	1	1
0.2	5	7	12	5	11
0.3	6	7	12	6	11
0.4	9	9	12	8	11
0.5	10	9	9	9	9
0.6	10	9	9	9	9
0.7	10	10	10	10	10
0.8	10	10	10	10	10
0.9	10	10	10	10	10

TAP

P	VALUES OF (C,W)				
	(.75,.75)	(0.8,0.8)	(0.9,0.8)	(0.8,0.9)	(0.9,0.9)
0.1	.0030	.0030	.0030	.0000	.2674
0.2	.0309	.0309	.0281	.0009	.2705
0.3	.0918	.0919	.0739	.0111	.1775
0.4	.1566	.1572	.1108	.0501	.1215
0.5	.1998	.2048	.1417	.1279	.1491
0.6	.2322	.2545	.2068	.2303	.2342
0.7	.2983	.3512	.3330	.3527	.3531
0.8	.4558	.5129	.5105	.5146	.5146
0.9	.7377	.7494	.7493	.7494	.7494

LTP

P	VALUES OF (C,W)				
	(.75,.75)	(0.8,0.8)	(0.9,0.8)	(0.8,0.9)	(0.9,0.9)
0.1	.8934	.6378	.6378	.9052	.6378
0.2	.6519	.4360	.4360	.7056	.4360
0.3	.4063	.3246	.3246	.4910	.3246
0.4	.2375	.2406	.2406	.3120	.2406
0.5	.1478	.1673	.1673	.1884	.1673
0.6	.0990	.1083	.1083	.1123	.1083
0.7	.0577	.0596	.0596	.0600	.0596
0.8	.0195	.0196	.0196	.0196	.0196
0.9	.0014	.0014	.0014	.0014	.0014

UTP

P	VALUES OF (C,W)				
	(.75,.75)	(0.8,0.8)	(0.9,0.8)	(0.8,0.9)	(0.9,0.9)
0.1	.0020	.0158	.0910	.0158	.0910
0.2	.0228	.0854	.2458	.0854	.2458
0.3	.0754	.1742	.3324	.1742	.3321
0.4	.1372	.2238	.3211	.2238	.3184
0.5	.1646	.2098	.2489	.2098	.2395
0.6	.1373	.1457	.1553	.1457	.1389
0.7	.0767	.0676	.0688	.0676	.0547
0.8	.0228	.0152	.0152	.0152	.0108
0.9	.0013	.0005	.0005	.0005	.0004

Table 12

Summary statistics for the WHME for chosen

large c and w when N = 15.

THE MEDIAN

P	VALUES OF (C,W)				
	(.75,.75)	(0.8,0.8)	(0.9,0.8)	(0.8,0.9)	(0.9,0.9)
0.1	5	7	9	5	8
0.2	9	9	12	8	11
0.3	16	16	16	15	15
0.4	16	16	16	16	15
0.5	16	16	16	15	15
0.6	16	16	16	15	15
0.7	15	15	15	15	15
0.8	15	15	15	15	15
0.9	15	15	15	15	15

TAP

P	VALUES OF (C,W)				
	(.75,.75)	(0.8,0.8)	(0.9,0.8)	(0.8,0.9)	(0.9,0.9)
0.1	.1406	.0027	.0027	.1407	.0000
0.2	.2184	.0361	.0361	.2204	.0023
0.3	.1777	.0976	.0974	.1959	.0237
0.4	.1274	.1451	.1426	.1711	.0798
0.5	.1356	.1805	.1693	.1739	.1439
0.6	.1875	.2242	.2004	.2008	.1957
0.7	.2584	.2995	.2749	.2749	.2745
0.8	.4077	.4417	.4343	.4343	.4343
0.9	.6617	.6655	.6654	.6654	.6654

LTP

P	VALUES OF (C,W)				
	(.75,.75)	(0.8,0.8)	(0.9,0.8)	(0.8,0.9)	(0.9,0.9)
0.1	.7932	.7932	.4910	.7932	.4910
0.2	.4695	.4695	.2788	.4695	.2788
0.3	.2509	.2509	.1872	.2509	.1872
0.4	.1382	.1382	.1253	.1382	.1253
0.5	.0750	.0750	.0734	.0750	.0734
0.6	.0337	.0337	.0336	.0337	.0336
0.7	.0105	.0105	.0105	.0105	.0105
0.8	.0016	.0016	.0016	.0016	.0016
0.9	.0000	.0000	.0000	.0000	.0000

UTP

P	VALUES OF (C,W)				
	(.75,.75)	(0.8,0.8)	(0.9,0.8)	(0.8,0.9)	(0.9,0.9)
0.1	.0558	.0558	.1938	.0558	.1938
0.2	.2163	.2163	.4007	.2163	.4007
0.3	.2877	.2877	.3861	.2877	.3861
0.4	.2394	.2394	.2679	.2394	.2679
0.5	.1594	.1594	.1635	.1589	.1635
0.6	.0862	.0856	.0838	.0834	.0838
0.7	.0315	.0300	.0270	.0270	.0270
0.8	.0051	.0041	.0031	.0031	.0031
0.9	.0001	.0000	.0000	.0000	.0000

Table 13

Summary statistics for the WHME for chosen

large c and w when N = 20.

THE MEDIAN

P	VALUES OF (C,W)				
	(.75,.75)	(0.8,0.8)	(0.9,0.8)	(0.8,0.9)	(0.9,0.9)
0.1	5	7	12	5	11
0.2	16	18	27	16	26
0.3	22	21	20	20	20
0.4	22	21	20	20	20
0.5	21	21	20	20	19
0.6	20	20	20	19	19
0.7	20	20	20	20	20
0.8	20	20	20	20	20
0.9	20	20	20	20	20

TAP

P	VALUES OF (C,W)				
	(.75,.75)	(0.8,0.8)	(0.9,0.8)	(0.8,0.9)	(0.9,0.9)
0.1	.0000	.0023	.0023	.0023	.0023
0.2	.0043	.0417	.0417	.0416	.0411
0.3	.0407	.1088	.1088	.1060	.0976
0.4	.1087	.1472	.1461	.1256	.1046
0.5	.1621	.1677	.1602	.1222	.1186
0.6	.2145	.1997	.1936	.1759	.1875
0.7	.2699	.2502	.2757	.2753	.2787
0.8	.3724	.3692	.4028	.4029	.4030
0.9	.6135	.6135	.6154	.6154	.6154

LTP

P	VALUES OF (C,W)				
	(.75,.75)	(0.8,0.8)	(0.9,0.8)	(0.8,0.9)	(0.9,0.9)
0.1	.6849	.6849	.3930	.6849	.3930
0.2	.3623	.3628	.2525	.3628	.2525
0.3	.2011	.2071	.1878	.2071	.1878
0.4	.1072	.1214	.1196	.1214	.1196
0.5	.0532	.0638	.0637	.0638	.0637
0.6	.0224	.0250	.0250	.0250	.0250
0.7	.0055	.0057	.0057	.0057	.0057
0.8	.0005	.0005	.0005	.0005	.0005
0.9	.0000	.0000	.0000	.0000	.0000

UTP

P	VALUES OF (C,W)				
	(.75,.75)	(0.8,0.8)	(0.9,0.8)	(0.8,0.9)	(0.9,0.9)
0.1	.0371	.0371	.1197	.0371	.0371
0.2	.1970	.1970	.3371	.1970	.1970
0.3	.2612	.2612	.3238	.2612	.2612
0.4	.3039	.2039	.2160	.2039	.2039
0.5	.1204	.1204	.1214	.1204	.1204
0.6	.0515	.0515	.0513	.0512	.0512
0.7	.0129	.0129	.0124	.0123	.0123
0.8	.0011	.0011	.0009	.0009	.0009
0.9	.0000	.0000	.0000	.0000	.0000

4.3 COMPARISON BETWEEN THE WHME AND THE MLE

In the last subsection we have argued that (c, w) = (0.9, 0.9) gives the most suitable estimator. Using table 14 and Figures 4.4-4.15 we now compare this WHME with the MLE. The differences in the performances are as follows :

1. The MLE is infinite with substantial probability and indeed even the median of the MLE is infinite when p is small. The WHME cannot be infinite and this is more realistic since the number of errors in software cannot be infinite. Moreover, even when the median of the MLE is finite there is no case in table 14 where this median is closer to N than the WHME.

2. When N = 10 the TAPs for the WHME are better than those for the MLE for $p \leq 0.7$. For N = 5, 15 or 20, all the TAPs displayed are the same for both the WHME and the MLE.

3. For large values of p, the LTPs for the WHM and ML estimators are small and approximately equal. But, when p is small, these probabilities for the WHME are very high, especially for small values of N. This indicates that the WHME tends to underestimate N for these values.

4. The UTPs of the MLE are very high when p is small, which means that the MLE tends to overestimate N in this case. The corresponding UTPs of the WHME are considerably smaller than those of the MLE.

CONCLUSION

From the above discussion, the WHME for the chosen values of (c, w) seems to be the preferred choice as a point estimator. This estimator avoids the possibility of infinite estimates. In the cases considered, the median of the WHME is at least as good as that of the MLE. Consideration of the LTPs and the UTPs shows that the distribution of the WHME has more weight in the lower tail than that of the MLE. Thus the WHME has better UTPs, although it does badly underestimate N when p is small. In terms of the TAPs, there is little difference between the two estimators.

Table 14

Summary statistics for comparing the chosen

WHME with the MLE .

The median

P	N=5		N=10		N=15		N=20	
	WHME	MLE	WHME	MLE	WHME	MLE	WHME	MLE
0.1	1	∞	1	∞	8	∞	11	∞
0.2	1	∞	11	∞	11	∞	26	∞
0.3	2	∞	11	∞	15	25	20	24
0.4	5	∞	11	13	15	15	20	19
0.5	5	6	9	9	15	15	19	19
0.6	5	5	9	9	15	15	19	19
0.7	5	5	10	10	15	15	20	20
0.8	5	5	10	10	15	15	20	20
0.9	5	5	10	10	15	15	20	20

TAP

P	N=5		N=10		N=15		N=20	
	WHME	MLE	WHME	MLE	WHME	MLE	WHME	MLE
0.1	.0027	.0027	.2674	.0000	.0000	.0000	.0023	.0023
0.2	.0189	.0189	.2705	.0009	.0023	.0023	.0411	.0411
0.3	.0552	.0552	.1775	.0111	.0237	.0237	.0976	.0976
0.4	.1120	.1120	.1215	.0501	.0798	.0798	.1046	.1046
0.5	.1887	.1887	.1491	.1279	.1439	.1439	.1186	.1186
0.6	.2898	.2898	.2342	.2303	.1957	.1957	.1875	.1875
0.7	.4310	.4310	.3531	.3527	.2745	.2745	.2787	.2787
0.8	.6322	.6322	.5146	.5146	.4343	.4343	.4030	.4030
0.9	.8744	.8744	.7494	.7494	.6654	.6654	.6154	.6154

LTP

P	N=5		N=10		N=15		N=20	
	WHME	MLE	WHME	MLE	WHME	MLE	WHME	MLE
0.1	.8190	.0594	.6378	.0738	.4910	.0813	.3930	.1013
0.2	.6712	.1274	.4360	.1676	.2788	.1567	.2525	.2004
0.3	.5470	.1867	.3246	.2209	.1872	.1617	.1878	.1821
0.4	.4358	.2204	.2406	.2095	.1253	.1217	.1196	.1192
0.5	.3289	.2171	.1673	.1605	.0734	.0731	.0637	.0637
0.6	.2235	.1761	.1083	.1074	.0336	.0336	.0250	.0250
0.7	.1257	.1111	.0596	.0595	.0105	.0105	.0057	.0057
0.8	.0493	.0468	.0196	.0196	.0016	.0016	.0005	.0005
0.9	.0080	.0079	.0014	.0014	.0000	.0000	.0000	.0000

UTP

P	N=5		N=10		N=15		N=20	
	WHME	MLE	WHME	MLE	WHME	MLE	WHME	MLE
0.1	.1778	.9375	.0910	.9224	.1938	.9057	.0371	.8878
0.2	.3033	.8471	.2458	.7816	.4007	.7134	.1970	.6472
0.3	.3672	.7275	.3321	.5887	.3861	.4746	.2612	.3901
0.4	.3678	.5831	.3184	.3861	.2679	.2786	.2039	.2237
0.5	.3137	.4255	.2395	.2200	.1635	.1507	.1204	.1217
0.6	.2249	.2723	.1389	.1080	.0838	.0693	.0512	.0505
0.7	.1281	.1428	.0547	.0415	.0270	.0218	.0123	.0113
0.8	.0497	.0523	.0108	.0090	.0031	.0027	.0009	.0007
0.9	.0075	.0076	.0004	.0004	.0000	.0000	.0000	.0000

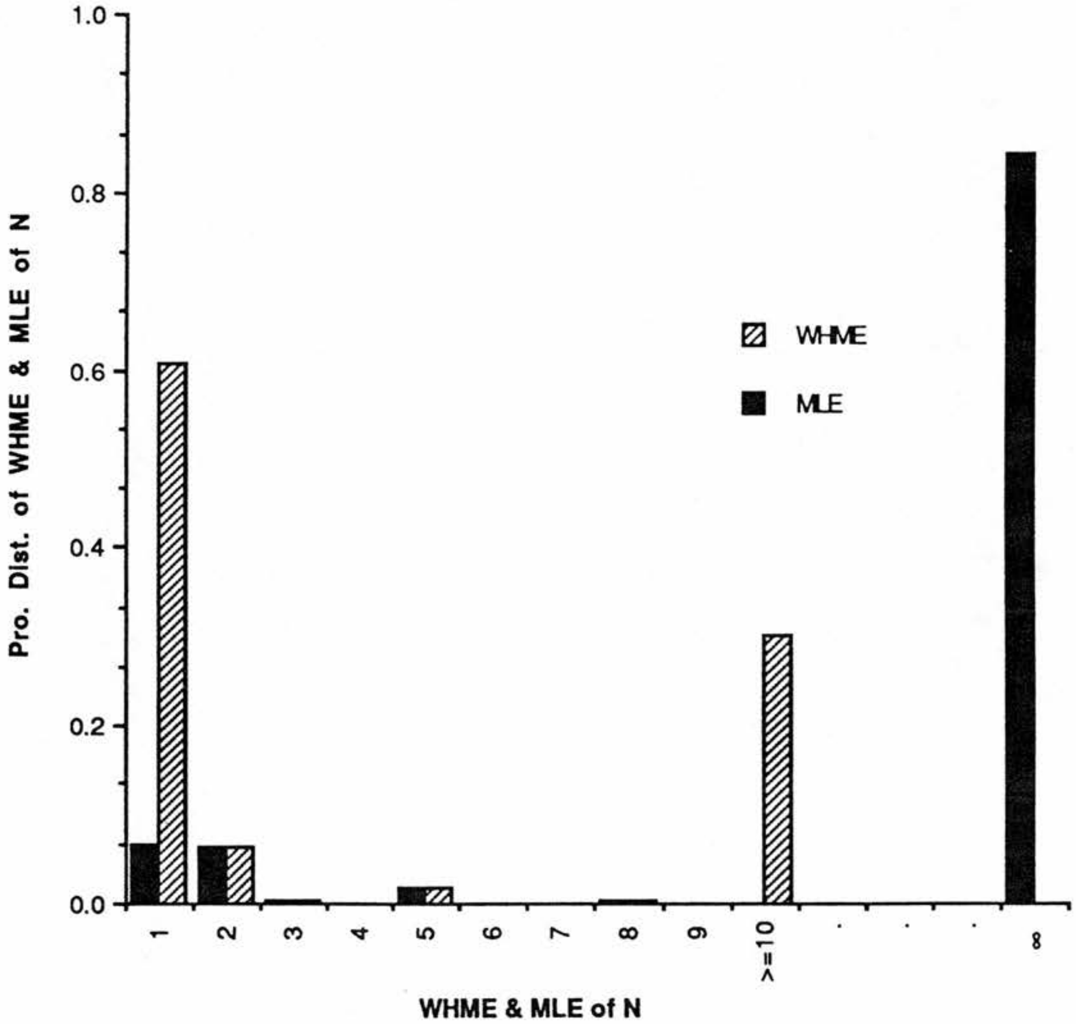


Figure 4.4

Comparison of the WHME and the MLE [with (c,w)=(0.9,0.9)]

when N=5, p=0.2

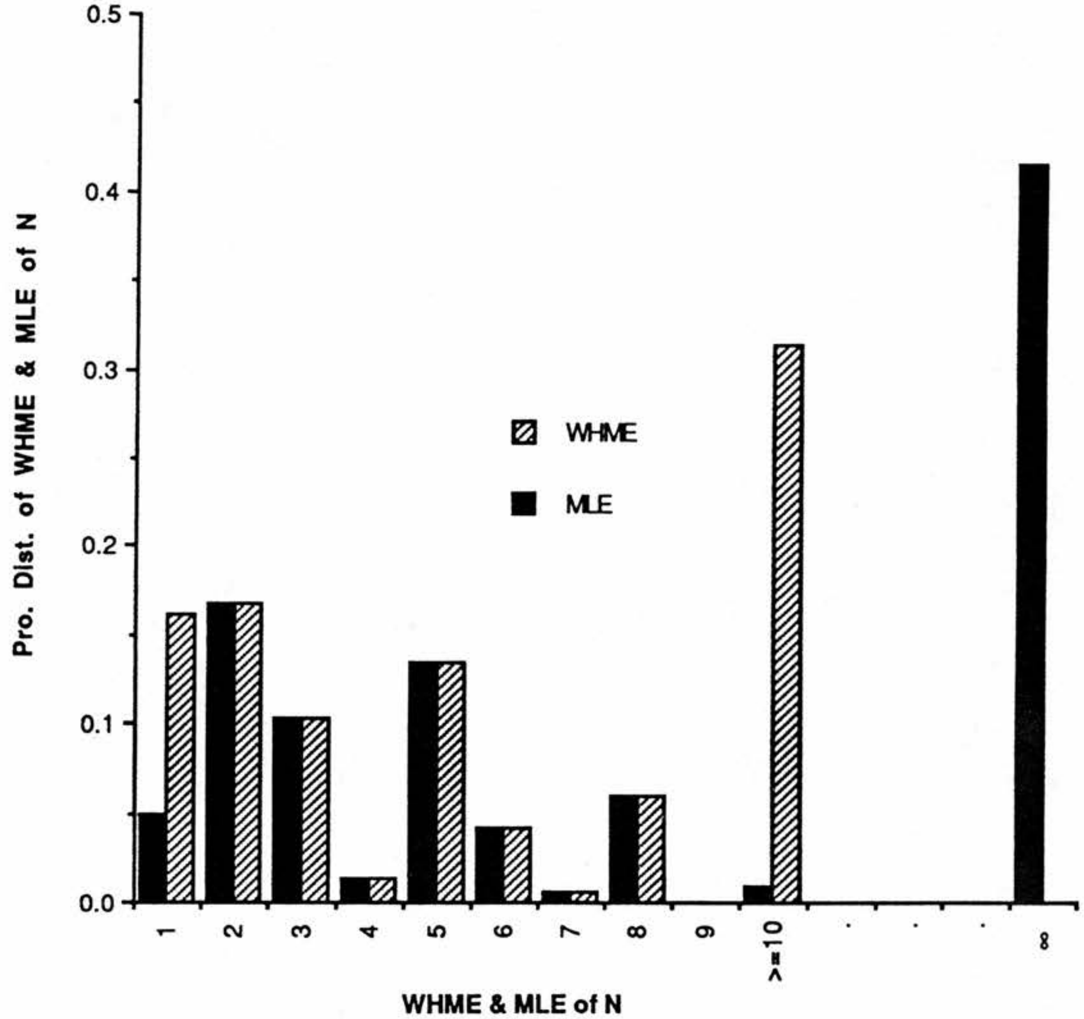


Figure 4.5

Comparison of the WHME and the MLE [with (c,w)=(0.9,0.9)]

when N=5, p=0.5

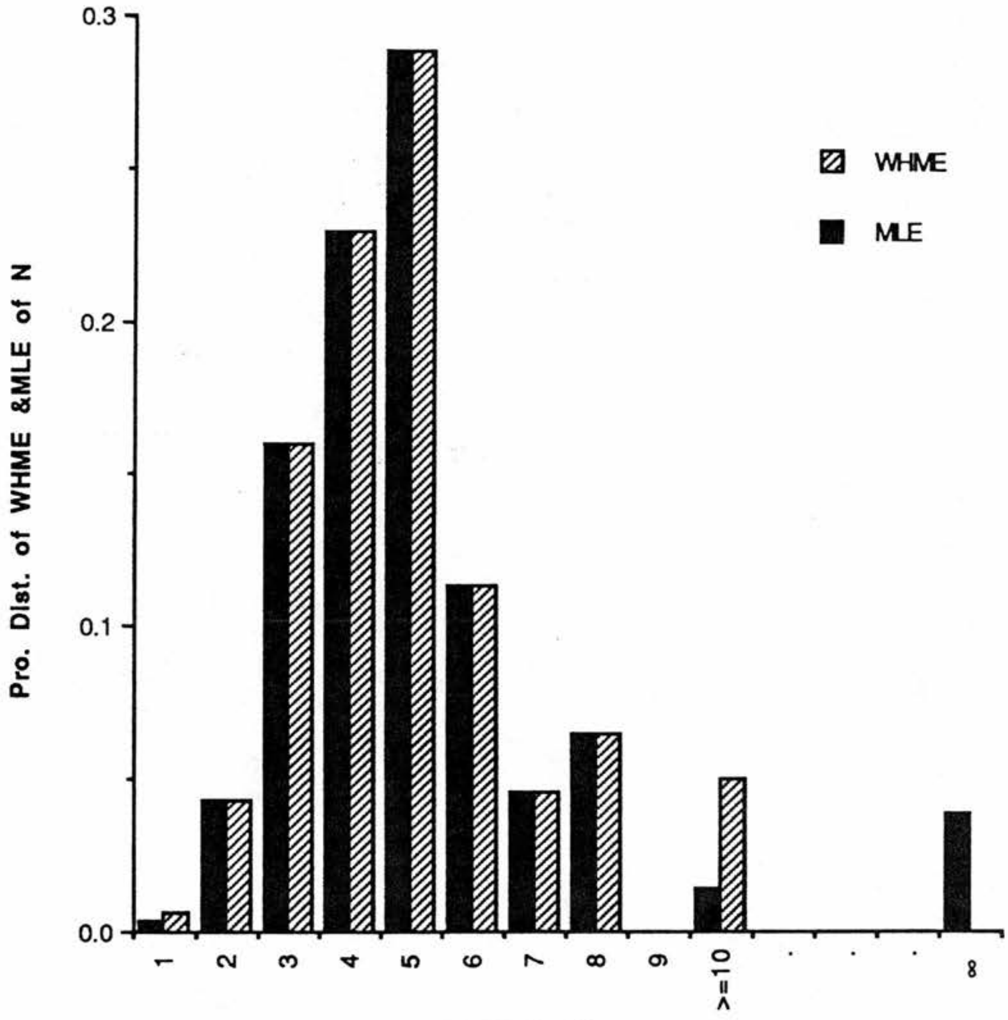


Figure 4.6

Comparison of the WHME and the MLE [with (c,w)=(0.9,0.9)]

when N=5, p=0.8

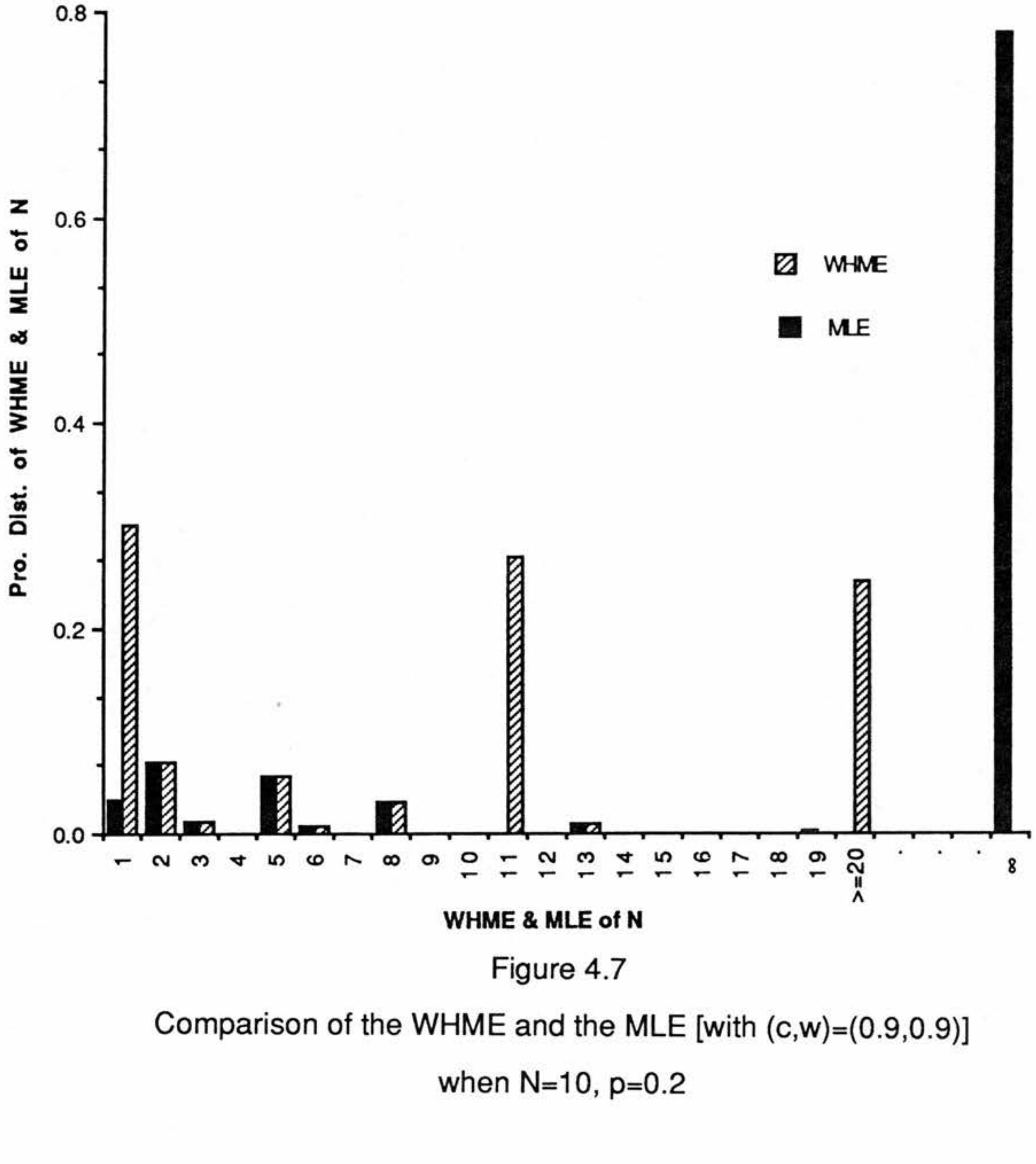


Figure 4.7

Comparison of the WHME and the MLE [with (c,w)=(0.9,0.9)]

when N=10, p=0.2

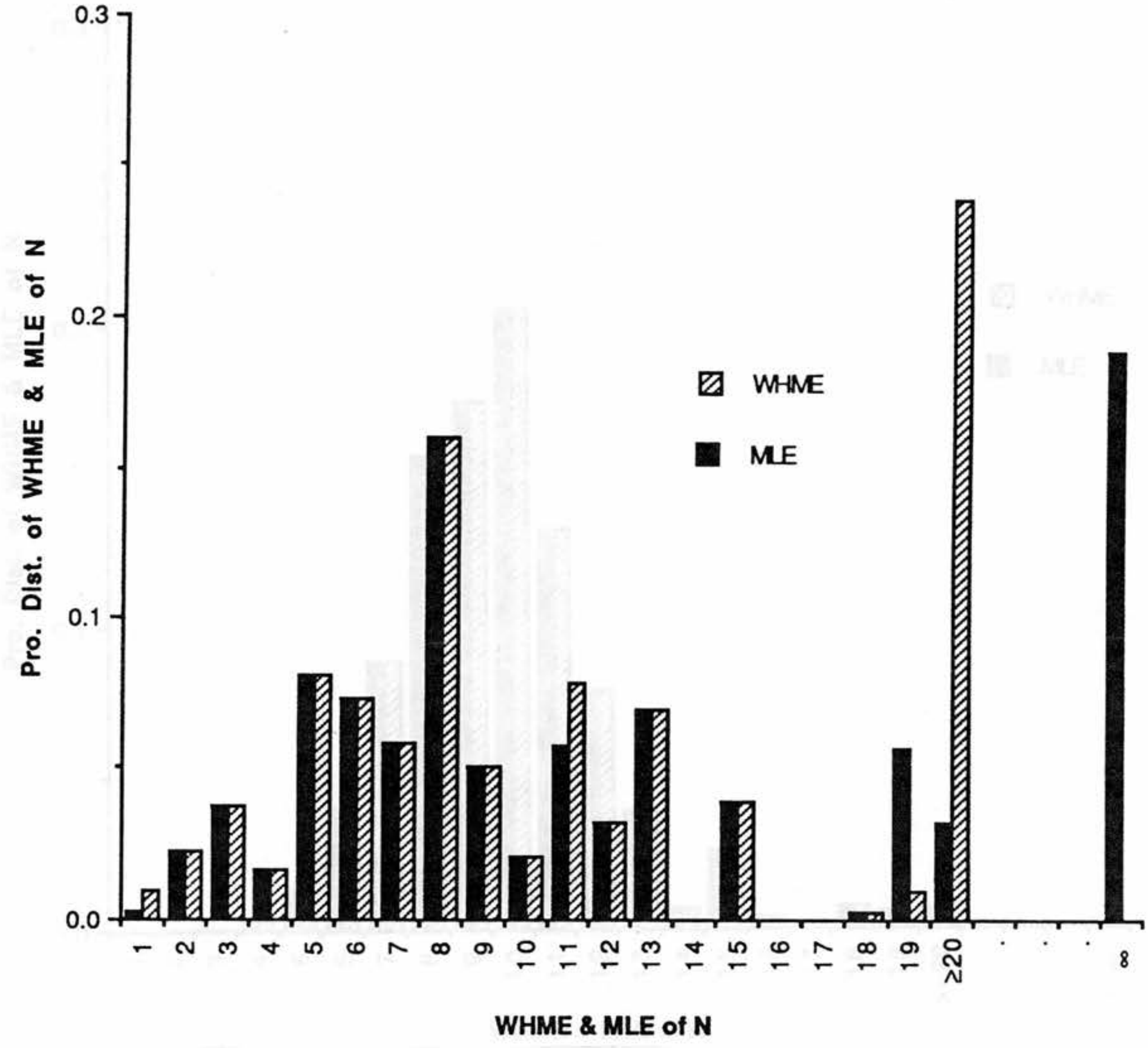


Figure 4.8

Comparison of the WHME and the MLE [with (c,w)=(0.9,0.9)]

when N=10, p=0.5

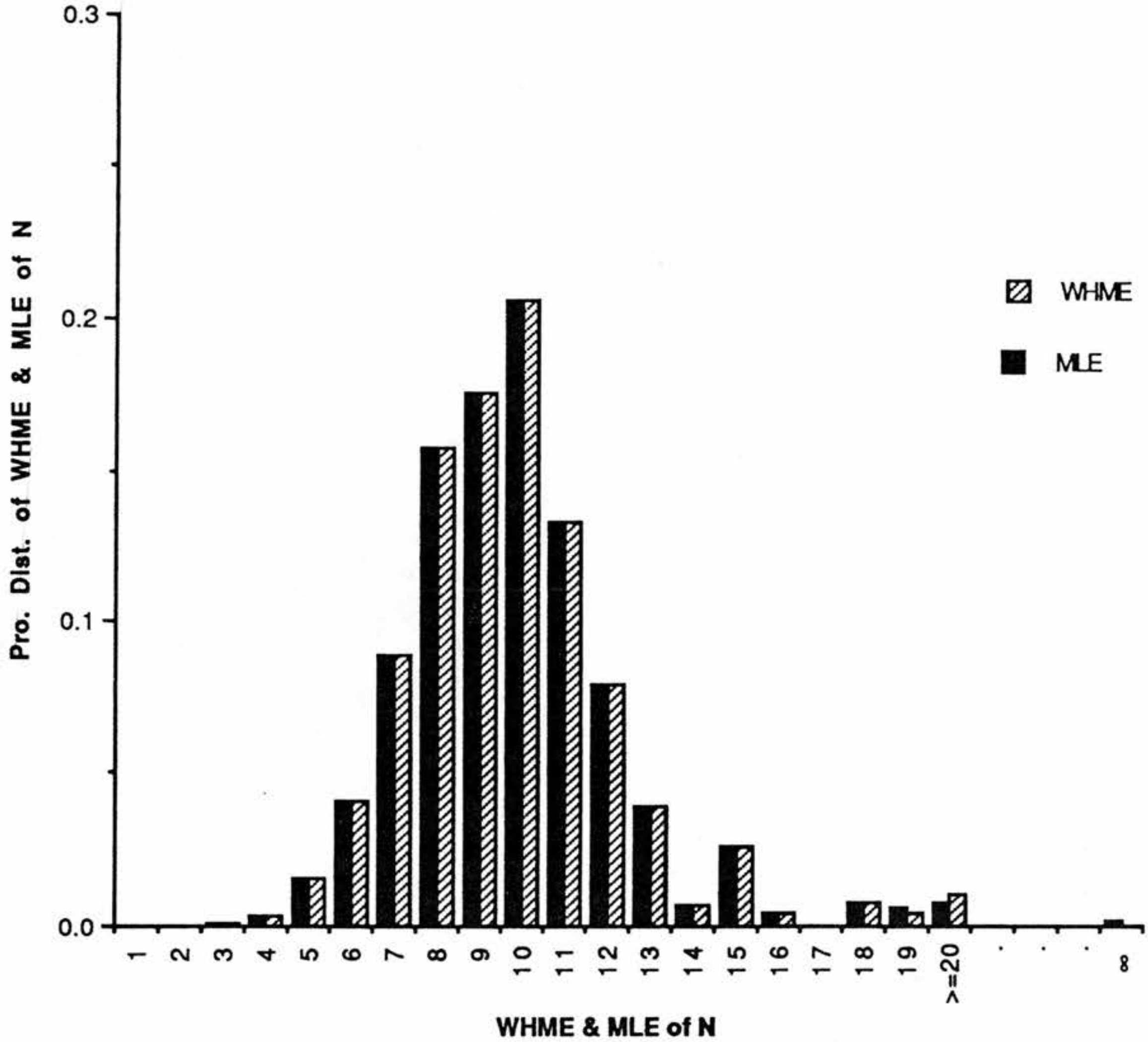


Figure 4.9

Comparison of the WHME and the MLE [with (c,w)=(0.9,0.9)]

when N=10, p=0.8

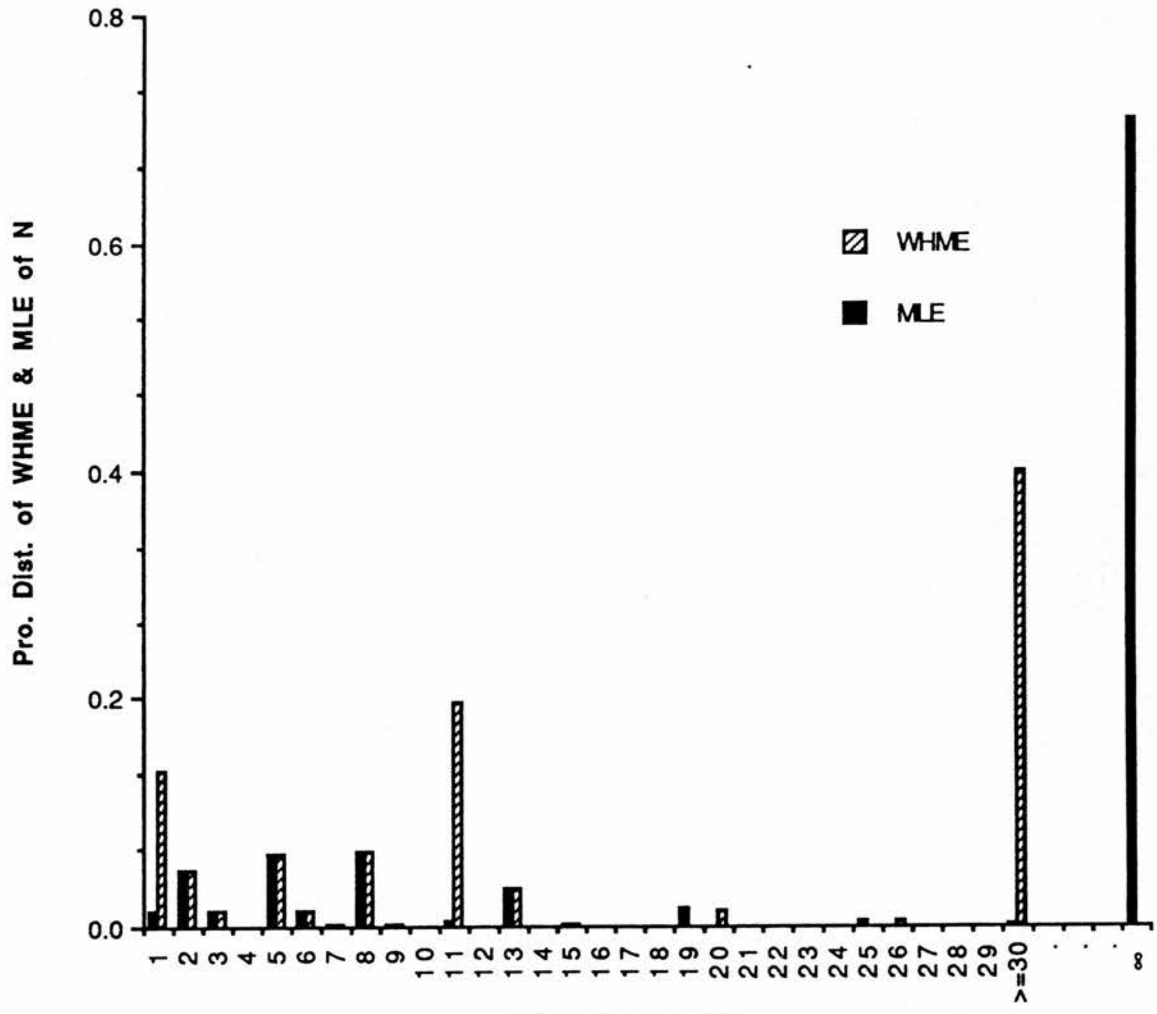


Figure 4.10

Comparison of the WHME and the MLE [with (c,w)=(0.9,0.9)]

when N=15, p=0.2

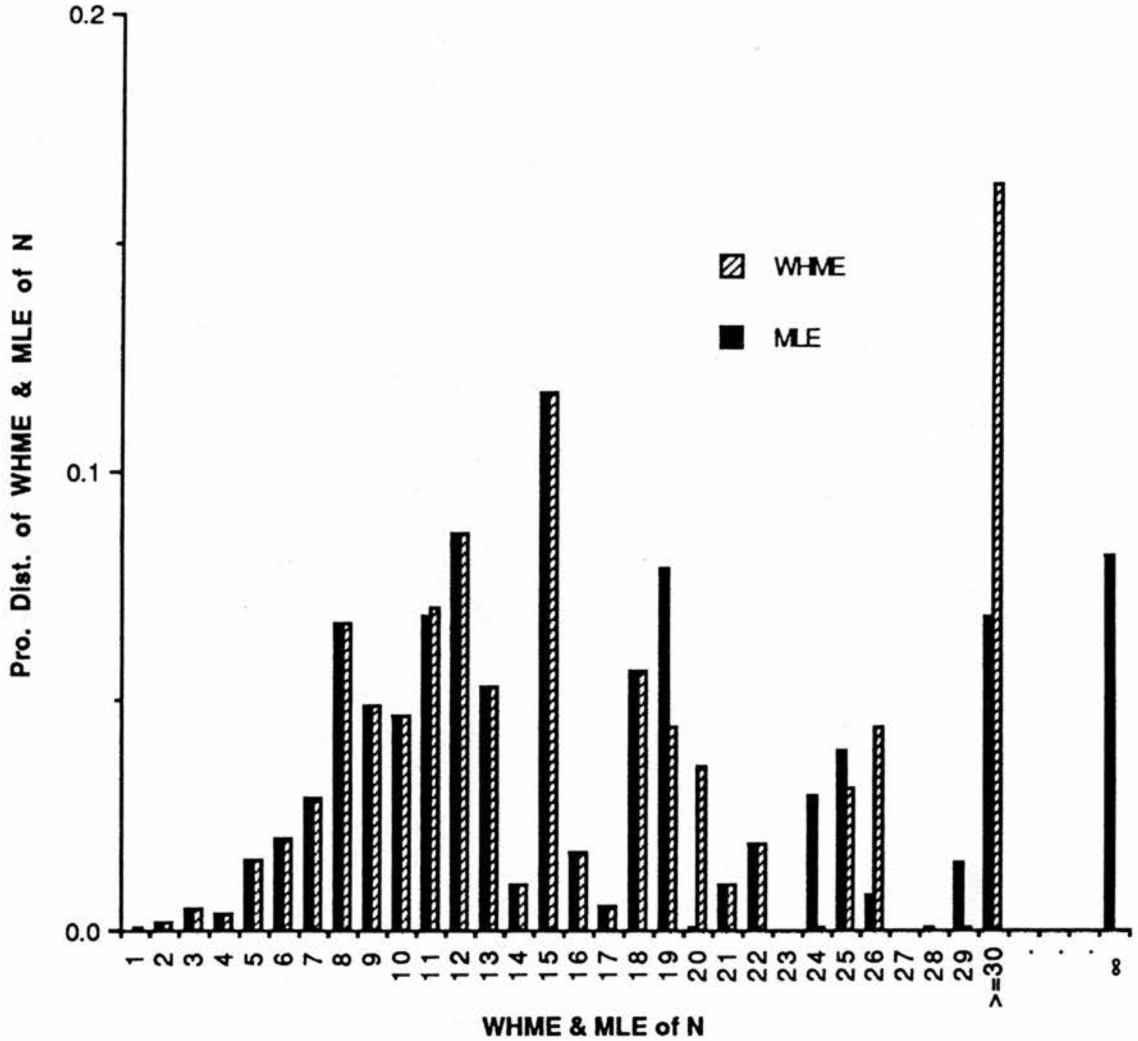


Figure 4.11

Comparison of the WHME and the MLE [with (c,w)=(0.9,0.9)]

when N=15, p=0.5

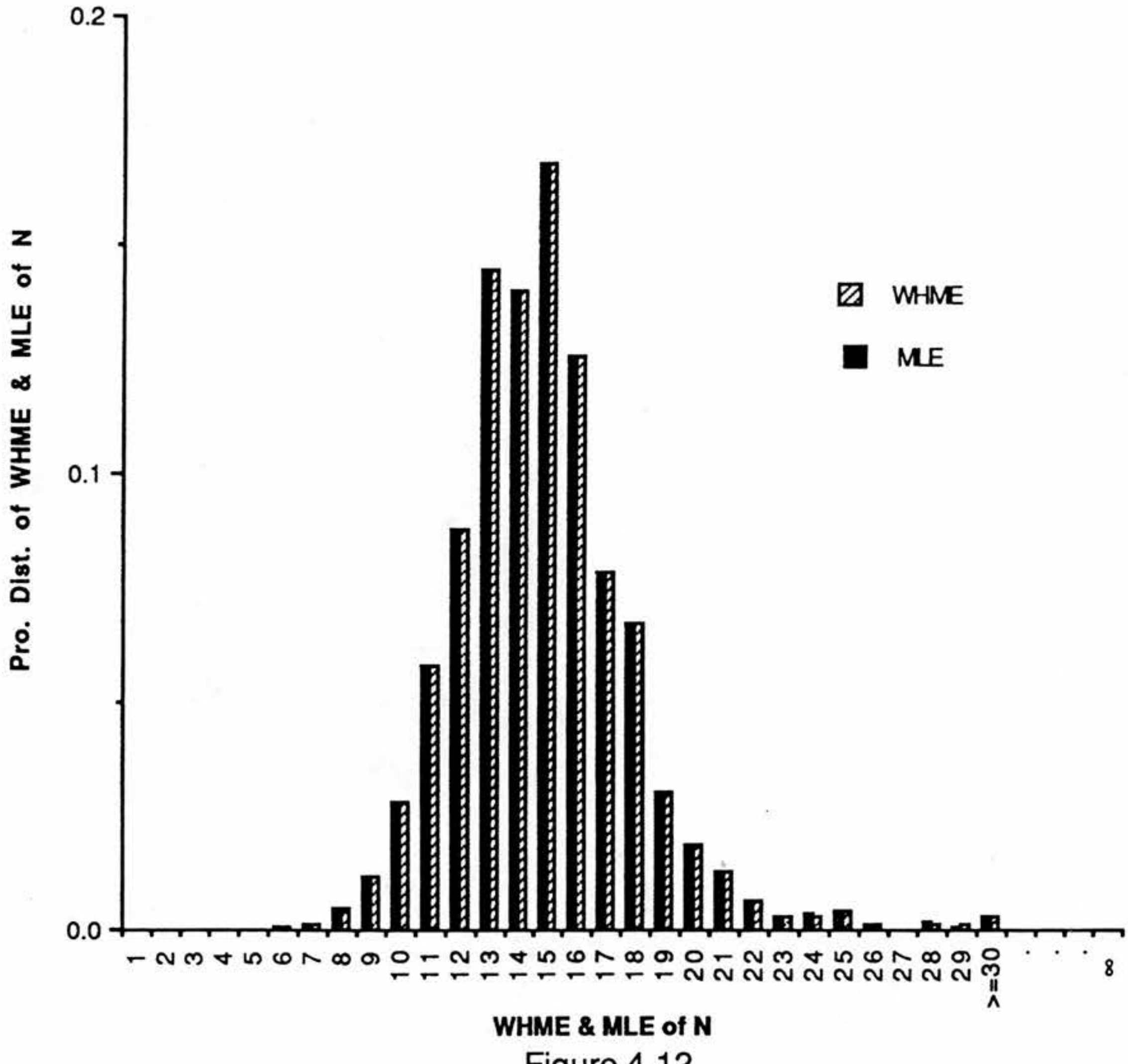


Figure 4.12

Comparison of the WHME and the MLE [with (c,w)=(0.9,0.9)]

when N=15, p=0.8

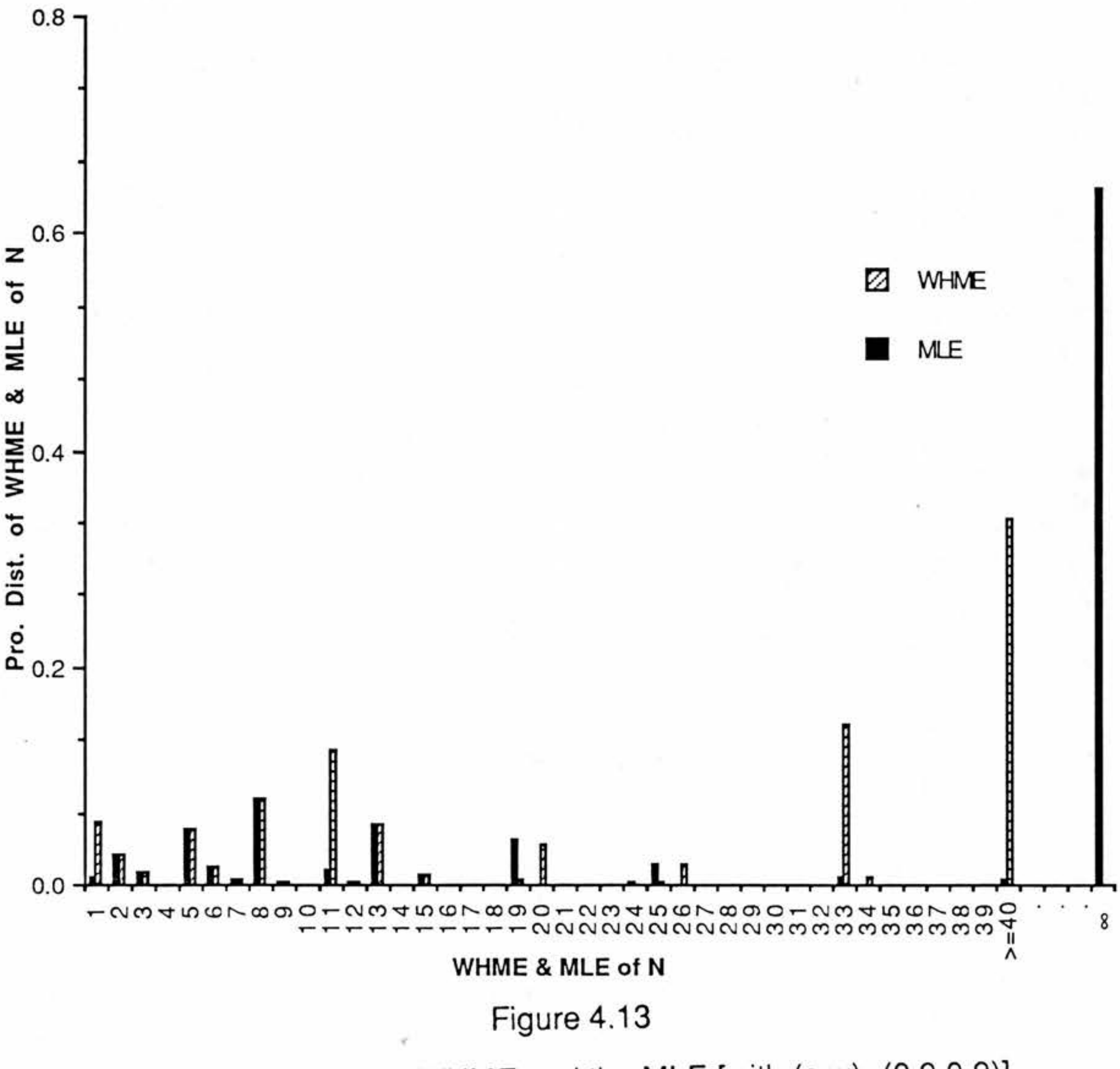


Figure 4.13

Comparison of the WHME and the MLE [with (c,w)=(0.9,0.9)]

when N=20, p=0.2

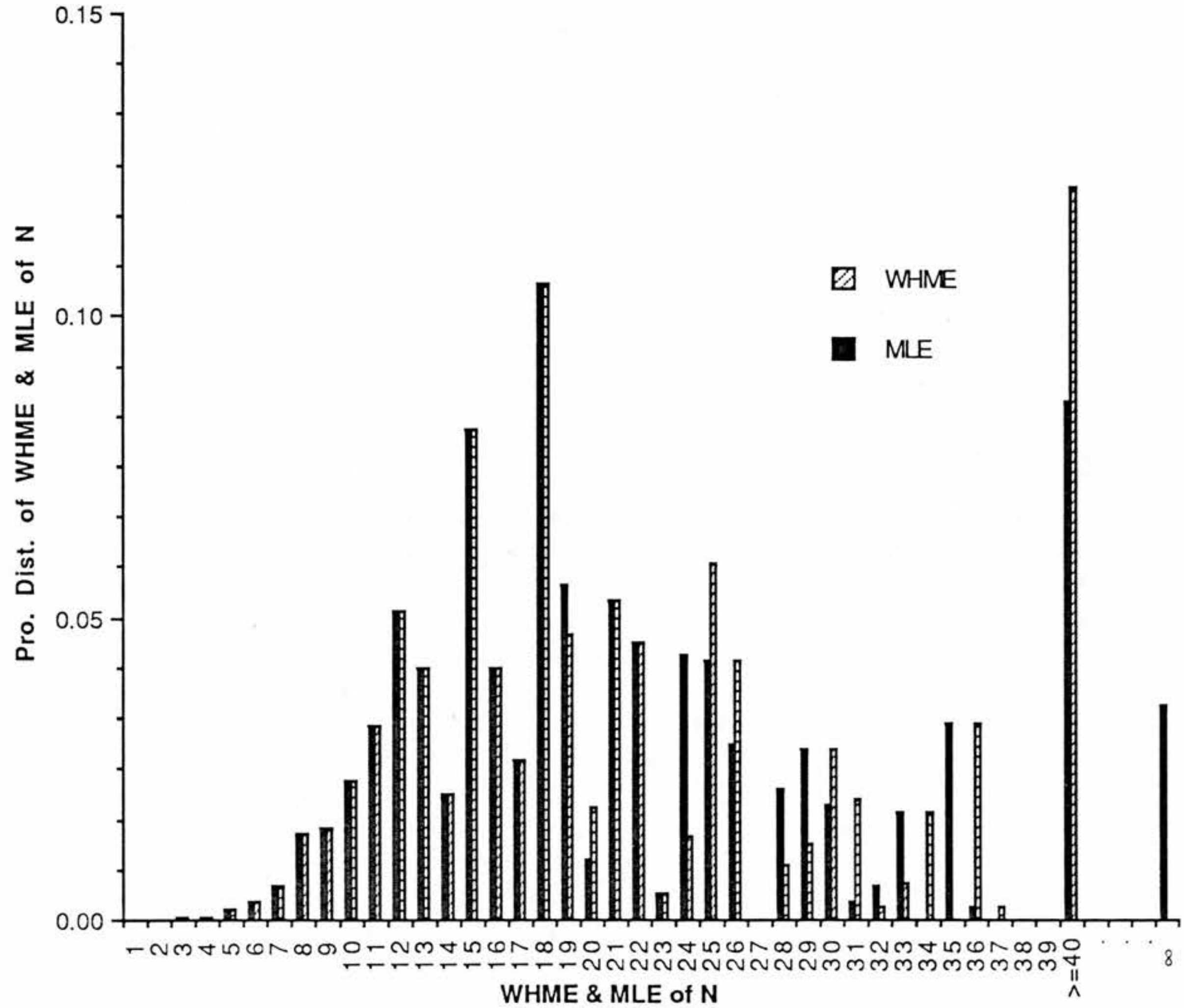


Figure 4.14

Comparison of the WHME and the MLE [with (c,w)=(0.9,0.9)] when N=20, p=0.5

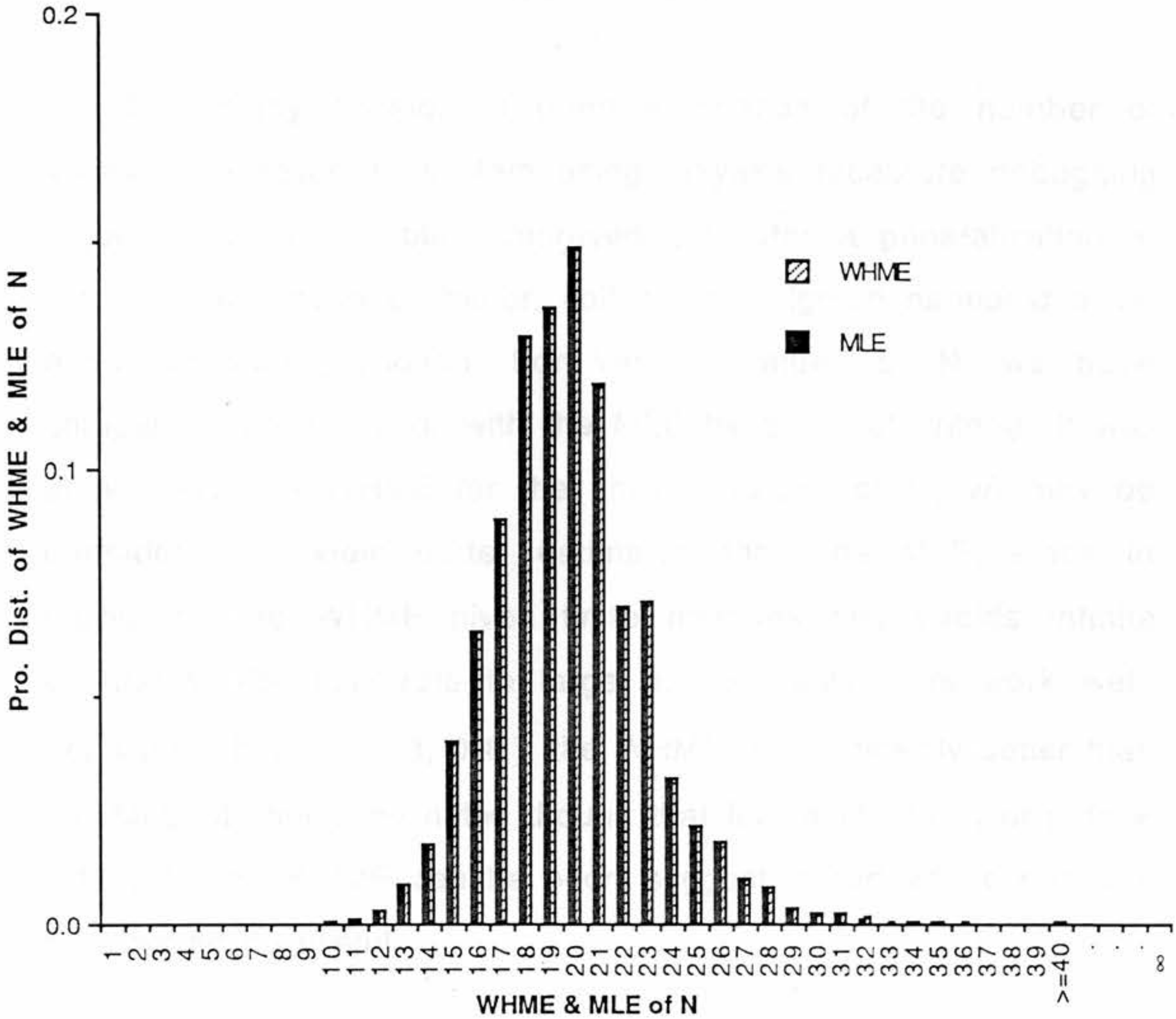


Figure 4.15

Comparison of the WHME and the MLE [with (c,w)=(0.9,0.9)]

when N=20, p=0.8

CHAPTER 5

CONCLUSION

This study considered point estimation of the number of errors in a reliability system using Nayak's recapture debugging model. In order to obtain improved estimates a generalization of the harmonic mean estimator, called the weighted harmonic mean estimator was proposed. For various values of N, we have compared this estimator with the MLE by a set of criteria. It was shown that the WHME for the chosen values of (c, w) may be considered to yield better estimates than the MLE, since in particular, the WHME gives finite medians and avoids infinite estimates. For moderate to large p, both estimators work well. For values of $p \in [\,0.3,\ 0.4\,]$, the WHME is significantly better than the MLE. It should be noted though that for small values of p ($p = 0.1, 0.2$), the WHME can be poor, suggesting further work in this area would be useful.

The weighted harmonic mean estimator, which we have proposed in this thesis, is only one of the possible generalizations of the harmonic mean estimator. A different generalization suggested by Joe and Reid (1985) for the J-M model is to use the closest integer to

$$b^{-1}\left[\frac{1}{2}\, b(N_1) + \frac{1}{2}\, b(N_2)\right] \quad ,$$

where b is any strictly monotone function from [1, ∞] to [0, 1]. There are many possible such functions and this approach could usefully be investigated for the recapture debugging model.

We have studied three different possible cases for the profile likelihood interval for N using the recapture debugging model. For the J-M model, Joe & Reid (1985) also investigated Improving the interval estimate by choosing three different values of c, depending on the shape of the profile likelihood function. Again a similar approach could be tried here and its effect on the point estimate examined.

We also have seen that the harmonic mean estimates avoid the maximum likelihood estimates' problem of taking the value infinity. A different way round this difficulty was given by Blumenthal and Marcus (1975) for the J-M model. They proposed a type of Bayes estimator called the maximum modified likelihood estimator. The use of Bayesian techniques for recapture debugging is a further possibility.

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