The Meaning of Quantifiers and the Epsilon Calculus:

on the Logical Formalization of Dependence Relations

Nicola Bonatti

University of St Andrews

This thesis is submitted in partial fulfilment for the degree of
Master of Philosophy (MPhil)
at the University of St Andrews

September 2020

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Abstract

This work argues that the expressive and inferential powers of quantified logic are not exhausted by the classical quantifiers $\forall$ and $\exists$. Indeed, based on both the Model and Proof theoretic semantics frameworks, the work will highlight two relations of dependence as arising within formulas and inferences which are not correctly represented by the standard interpretation of $\forall$ and $\exists$. Instead, I will claim that the quantifiers 'for all' and 'there exists' should be interpreted as choice functions − according to the ε-operator of the Epsilon Calculus. Finally, I will consider as a case study the Set theory formulated in the Epsilon Calculus so as to consider how the choice functions interpretation of quantifiers affects debates concerning impredicative definitions and the logical/combinatorial view of collections.

Acknowledgements

This work is the product of a year and a half of research, during which I have developed and changed the main ideas motivating the research. I'm sincerely thankful to my supervisor, professor Franz Berto, who has agreed to supervise my work even if it relies outside of his main expertises. His challenging feedbacks helped me to develop the main arguments of this work. Moreover, his warnings about making the philosophical point out of formal results helped me not to get lost into technicalities.

Moreover, I would like to mention Francesca Boccuni, Jack Woods, Leon Horsten, Norbert Gratzl and Tim Button for feedbacks on several parts of this work.

Last but not least, I thank my friends and fellow students Alessandro, Tom and Duccio for precious informal chats on the work. A special mention goes to Antonio for having introduced me to LaTeX, without his help this dissertation would not look so professional!

"My father was a business man and I am a business man. I want philosophy to be business-like, to get something done, to get something settled."

Letter from Wittgenstein to M. O'C. Drury, 1930

For my father, the inherited will.

Contents

Introduction

It is even illuminating that one can derive
$\Phi \to \forall x\varphi(x)$ from $\Phi \to \varphi(a)$ if Φ is an
expression in which 'a' does not occur and if 'a'
stands in $\varphi(a)$ only in the argument places.

(Frege, 1879, p. 21)

What does it mean to say that 'for all x such that ϕ(x)' and 'there exists an x such that $\phi(x)$'? Since Frege (1879) formalization of quantificational logic − which (Van Heijenoort, 1967, p.1) considers "the most important single work ever written in logic" after the Aristotelian syllogistic − three interpretations of universal and existential quantified sentences have been proposed.

According to the first one while universal statements are equivalent to the (perhaps infinite) conjunction of their instances, existential statements are equivalent to the (perhaps infinite) disjunction of their instances. This is the view advocated by the algebraic school of logic, championed by George Boole and Ernst Schröder. Apart from its intuitive appeal, this interpretation soon runs into troubles, as remarked by Sher (2012). I will just sketch two of the main objections: i) there might be not enough constant letters for all the objects in the domain (if, for example, the domain has the size of the real numbers); ii) finitely long formulas are replaced by (potentially) infinitely long ones. That is why I will not discuss this interpretation any further[1].

According to the second interpretation quantifiers are higher-order predicates which make assertions on the extension of the (lower-order) predicates. More precisely, while existential sentences asserts that the extension of the predicate is not empty, universal sentences asserts that the extension of the predicate is the whole domain of discourse. This interpretation was hinted by Frege (1879) himself, but only fully formalized by Mostowski (1957) and Lindström (1966) with the Theory of Generalized Quantifiers (GQ) − which will be extensively discussed in §1. This interpretation is represented by the classical quantifiers $\forall$ ('for all') and $\exists$ ('there exists') adopted in classical predicate logic.

Finally, there is a further interpretation by which quantified sentences are instructions to find witnesses for the predicates. More precisely, while existential sentences assert that an arbitrary individual satisfies the predicate, universal sentences asserts that any individual from the domain satisfies the predicate (or, equivalently, that there is no counterexample to the predicate). This interpretation is formalized by Hilbert and Bernays (1939) ε-operator, which is part of the Epsilon Calculus (EC)[2]. As I will extensively explain in §1, the ε-operator is interpreted in the metalanguage by an (arbitrary) choice function, which allows to explicitly define the classical quantifiers $\forall$ and $\exists$. That is why I will refer to this as the choice functions interpretation of quantifiers, formalized by the ε-operator ('an x such that, if any') of EC.

This work intends to argue for the choice functions interpretation of quantifiers over the higher-order predicates one. In this respect, the present work belongs to the discussion in philosophy of logic concerning the meaning of logical constants. Consider, for example, the disagreement about the inference rules and the truth functions of the classical negation ($\neg$) and implication ($\to$), which have raised alternative formal systems to Classical Logic, such as Intuitionist and Relevant Logics. The present work shares similarities with both of them: on the one hand, as for Intuitionist and

[1] Hintikka (1997) notes that Charles Pierce, although belonging to the algebraic school of logic, advocates an interpretation of the quantifiers 'for all' and 'there exists' closer to the choice functions one, which is discussed below.

[2] A similar interpretation is offered by Hintikka (1996) Game Theoretical Semantics (GTS), which is however adopted for (a version of) the classical quantifiers $\forall$ and $\exists$, rather than the ε-operator. I will carry out a full comparison with Hintikka (1996) GTS in footnotes 11 and 15.

Relevant logics, I will suggest to replace a logical operator − the quantifiers ∀ and ∃ − with the ε-operator. On the other hand, unlike in Intuitionist and Relevant logics, I will justify my claim within First-order Logic (FOL). Indeed, the Epsilon Calculus (EC) with equality is a conservative extension of FOL (Zach, 2017).

Questions about the meaning of logical constants can be addressed according to two formal frameworks, either Model or Proof theoretic semantics. According to the first one, the metalinguistic interpretation of logical constants is hold fixed through models in order to recursively define the truth-conditions of formulas (Zimmermann, 2019). According to the latter, the meaning of logical constant is defined by their Natural Deduction/Sequent Calculus rules of inference (Schroeder-Heister, 2018). In the present work, I will adopt both accounts in order to carry out the argument for the choice functions interpretation of quantifiers (ε-operator) over the higher-order predicates one (∀ and ∃). Moreover, I will evaluate to which extent the ε-operator meets the Model and Proof theoretic conditions of logicality – so as to reject two compelling objections against the choice functions interpretation of quantifiers, which undermines its logicality. That is why, in order to appreciate the discussion below, the reader should be committed to the methodological frameworks of Model and Proof theoretic semantics adopted, respectively, in §Chapters I and II. Indeed, in the following discussion, philosophical considerations on the meaning of quantifiers will go hand-by-hand with Model and Proof theoretic results. This argumentative style is in line with the analytic tradition in philosophy starting with Frege (1879), which attempts to face philosophical problems through logical analysis. However, in the present context, the leading role of mathematical logic for philosophical discussion takes the following form: Model and Proof theory explain how expressions of generality (namely, the quantifiers) are adopted in deductive reasoning, by stressing their expressive and inferential behaviour in formal logic. The focus on both the expressive and inferential power of quantificational logic is motivated by Van Heijenoort (1967) distinction between logic as universal language and as inferential calculus. Finally, the methodological assumption of Model and Proof theoretic semantics strengthens the dissertation conclusion: the quantifiers ∀ and ∃ are choice functions regardless if they are interpreted according to Model or Proof theoretic semantics.

The methodological assumption of Model and Proof theoretic semantics also delimits the scope of the present work: by focusing on the meaning of quantifiers as suggested by their truth-conditions and inferential rules as stated by (classical) First-order Logic, the target of the discussion is the logical meaning of quantifiers. However, this focus leaves out the (broad) literature on the natural language behaviour of quantifiers, as mostly studied by linguistics and computer-science. Nonetheless, in a couple of footnotes below I will point out how the main idea of quantifiers as choice functions − and the formal device of the ε-operator − has been advocated by linguistics and computer scientists.

The leading argument for the ε-operator can be summarized as follow: whether we consider why quantified sentences are true, as defined by Model theory, or how quantified formulas can be deduced from premises, as stated by Proof theory, there are relevant notions of dependence relations which are left out by the interpretation of the classical quantifiers ∀ and ∃. These notions of dependences are of two kinds: the semantic notion of *functional-dependence*, by which the value of a variable is defined by that of another one, and the syntactic notion of *term-dependence*, by which an instantial term can occur in the condition of a subsequent quantified formula. Therefore, this dissertation brings together the (unrelated) works on the Model and Proof theoretic formalization of dependence relations done by, respectively, Hintikka (1996), Väänänen (2007) and Meyer Viol (1995), Alechina and Van Lambalgen (1996). Indeed, I will claim that EC provides several advantages over FOL for the logical formalization of dependences relations. This suggest that the Model and Proof theoretic understanding of quantification relies on the choice functions interpretation of quantifiers, formalized by the ε-operator.

The structure of the work can be summarized as follows. In §Chapter I, I will focus on the model theoretic characterization of quantifiers, explaining how different interpretation of a logical constant can be evaluated based on the recursive truth-conditions that they define. In §1, I will introduce the formal setting characterizing the interpretation of quantifiers as higher-order predicates − as formalized by the classical quantifiers ∀ and ∃ of FOL − and as choice functions − formalized by the ε-operator of EC. Then in §2, in order to compare and evaluate the two opposite interpretations, I will introduce the inferential rule of Functional Instantiation (FI), claiming that is a conservative extension of the model theoretic semantics for the classical quantifiers. More precisely, the syntactic notions of scope nested quantifiers define a semantic relation called *functional dependence*, which is represented by the Skolem functions introduced by FI. Therefore I will take FI as a condition for the correct interpretation of quantifiers, so as to compare FOL and EC. In §2.1 I will carefully explain that FI is invalid in FOL due to the restrictions of the metalanguage − namely, ZFC − by which no global choice functions is definable in FOL. Instead in §2.2, following Leisenring (1969) theorem, I will explain why Skolem functions can be conservatively added to EC. I will conclude that, if the representation of functional-dependence is taken as a condition for the correct interpretation of quantifiers, then the classical quantifiers ∀ and ∃ should be replaced by the ε-operator.

In §Chapter II, I will focus on the proof theoretic characterization of quantifiers, explaining how the Natural Deduction (NK) and Sequent Calculus (SK) rules define, respectively, the sense and meaning of logical constants. More precisely, in §7 I will formulate the proof theoretic semantic thesis within NK, claiming that the understanding of the inferential rules for logical constants is sufficient to grasp also their meaning. Then, in §7.1, I will introduce the NK rules for the classical quantifiers ∀ and ∃, comparing the *indirect* rules of Existential Elimination (∃E) and Universal Introduction (∀I) of Gentzen (1969) with the *direct* ones of Existential Instantiation (EI) and Universal Generalisation (UG) proposed by Quine (1950). I will argue that the restrictions on both direct and indirect rules represent the same (strict partial order) relation of *term-dependence*, by which the instantial variable introduced by an application of either direct or indirect rules might appear in the active assumptions of a later instantial variable, thus defining a relation between the value of the latter with that of the former. Based on this, in §7.2, I will claim that by introducing the ε-operator in the EI rule, term-dependence can be represented at the syntactic level of formulas as (un)nested ε-terms. I will conclude that the understanding of the NK rules of ∀ and ∃ bears on that of the ε-operator. In §8 I will then move to consider the proof theoretic semantics thesis within SK, claiming that the structural and operational rules implicitly define the meaning of logical constants. In §8.1 I will first introduce (among others) the structural rule of Substitution. However, in §8.2, I will point out that the double-line Rules of ∀ and ∃ plus the other structural rules are not sufficient to implicitly define the meaning of those logical constants, because the eigenvariables condition breaks the symmetry between Left and Right rules for the classical quantifiers. That is why, I will propose to extend the structural rule of Substitution with ε-terms so as to reformulate the double-line rules of ∀ and ∃. Based on this, I will prove the symmetry between the Right and Left rules for the classical quantifiers. Therefore, I will conclude that the ε-operator is a meaning constituent of the classical quantifiers ∀ and ∃.

Moreover, the work will consider two possible objections against the logicality of the ε-operator, drawn from model theoretic considerations. Indeed, in the literature there is a common sense of scepticism about the ε-operator, which is considered a mathematical operator rather than a logical constant. Instead, in §3, I will adopt Feferman (2006) objection to the case of EC, which claims that even if the ε-operator is a syntactic first-order operator, is semantically interpreted as higher-order quantification over functions. However, I will resists this objection by claming that it is a *petitio principii:* the ε-operator is semantically higher-order if and only if is interpreted according to ZFC, which is precisely the assumption at issue in the debate over the correct interpretation of quantifiers (as explained above). However, even if the ε-operator is a genuinely first-order operator, it fails the Tarski and Corcoran (1986) and Sher (1991) thesis for logical constants, as explained in §4. I will resist this objection by presenting in §4.1 the revised condition of Woods (2014), by which the notion of choice function interpreting the ε-operator is invariant under permutation of the domain. Moreover, in §4.2, I will support Woods (2014) proposal by claiming that the revised invariance condition is coherent with the assumption of the topic neutrality of logical constants − which is indeed displayed by the arbitrary denotation of the ε-operator.

In the second part of the work, I will adopt EC and the ε-operator in order to shed light over debates in the philosophy of mathematics. Indeed, it should already be appreciated that the ε-operator offers a novel framework by which thinking about quantificational logic. Given that EC is a conservative extension of FOL (more of this in §1), the classical logician should not (in principle) object the adoption of EC, where the quantifiers are interpreted as choice functions. By replacing the classical quantifiers $\forall$ and $\exists$ with the ε-operator, two main differences emerge: syntactically, the ε-operator is a variable-binding operator which forms term from open sentences, like $\varepsilon_x\varphi(x)$. Semantically, the ε-operator is equivalent to a global choice function, such that $f(A) \in A$ for $A \neq \varnothing$. Based on these two features, in the last chapter, I will argue that in the axiomatic Set theory formulated in EC − as formalized by Bourbaki (1968) (called BK) − impredicative definitions are harmless, namely they do not involve paradoxes. Based on this theory, I will consider several consequences for the debate over impredicative definitions and the distinction between logical classes and combinatorial sets. More precisely, §Chapter III is divided into two self-standing parts, which provide case study of the theoretical strengths of EC, concerning: the theory of logical definitions and axiomatic Set theory. I will introduce both in turn.

In §10, I will first consider the theory of logical definitions, where mathematical objects (such as sets or classes) are defined in a first-order theory (usually, axiomatic Set theory) by means of the formula:

$$\forall x((x = b) \leftrightarrow R(x)) \tag{1}$$

Where the condition is expressed by the fact that the *definiendum* b must not occur in the *definiens* $R[x]$. However, this condition is not sufficient: under certain conditions (which will be discussed below) formula (1) gives rise to paradoxes, such as Russell's and Richard's ones. That is why many authors in the literature − mainly Russell (1908) and Poincaré (2012) − blame as the source of the paradoxes a kind of circularity in their defying formula (1), where the mathematical object (set or class) is defined by quantifying over the universe to which it belongs. Russell (1908) and Poincaré (2012) distinguish *predicative* from *impredicative* definitions, disallowing the latter based on their Vicious Circle Principle (VCP). However, it is intuitively clear that the predicativity primarily concerns the scope of the quantifiers in (1). That is why, Hintikka (2012) claims that (im)predicative definitions correspond to different patterns of quantifiers (in)dependence in the defying formula (1). In §10.1, I will support Hintikka (2012) claim by pointing out the difference between Russell (1908) and Poincaré (2012) criteria of predicativity, which can be interpreted as different patterns of quantifiers (in)dependence. Then in §10.2, based on the insights summarized above, I will claim that these patterns of quantifiers (in)dependence can be represented in EC at the syntactic level of (un)nested ε-terms. Finally, in §10.3, I will consider where my proposal leaves the discussion over VCP. More precisely, I will claim that the issue of adopting impredicative definitions primarily rests on semantics, rather than ontological (Gödel, 1944) grounds. Indeed, if impredicative definitions are formulated within EC, then the issue becomes of whether adopting descriptions which might be empty, namely not determining a set. Based on Quine (1985) distinction, I will argue that these descriptions, even if they *specify* impredicatively, they do not *individuate* impredicatively.

In §11, I will move to consider the axiomatic Set theory formulated in EC, as formalized by Bourbaki (1968) (called BK). After introducing the axiomatic setting, I will point out the main differences between BK and ZFC, which concern the Axiom of Foundation and the Axiom of (Global) Choice. Indeed, the ε-operator is equivalent to the Axiom of Global Choice, which asserts the existence of a global choice function defined on the class of all sets. That is why in §11.1, I will argue that BK distinguishes between logical classes and combinatorial sets as follows: while classes are introduced by sound inferences of formulas like (1) − where the relation R is said to be collectivizing in x, denoted as $Coll_xR$ − combinatorial sets are denoted by the ε-operator as $\varepsilon_y\forall x(x \in y) \leftrightarrow R$. I will then compare BK with ZFC and NBG, claiming that the former keeps track of the difference between the logical grammar of classes and the set theoretic operations. Finally, in §11.2 I will consider the BK definition of cardinal numbers as:

$$|t| = \varepsilon_x(x \approx t) \tag{2}$$

Following Leisenring (1969), I will prove that all the relevant properties of cardinal numbers (like Hume Principle) can be deduced from (2). Then, I will compare the BK definition of cardinal numbers with the Frege-Russell and Zermelo-von Neumann proposals, explaining the several advantages of the former over the latter. Based on these features, I will develop an abstractionist account of cardinal numbers for BK which resembles Cantor (1915) one, but is able to resist the objections from Frege (1980). The main idea can be summarized as follows: the ε-term in (2) allows to abstract an arbitrary set which is defined by the only property of being equinumerous to the given set. Such arbitrary set is the cardinal of the given set. In conclusion, given that the BK definition of cardinal numbers is explicit, I will explain why it is not affected by the 'Bad Company' objection moved by Boolos (1987) to the implicit definitions of cardinal numbers (like Hume Principle) adopted by the neologicist program in the philosophy of mathematics (Wright, 1983).

Part I

The Model Theoretic Characterization of Quantifiers

The leading idea of this chapter is that the meaning of quantifiers is not exhausted by their ranging over a set, asserting its non-emptiness or exceptionless – as per the interpretation of the classical quantifiers ∀ and ∃ as higher-order predicates. Instead, quantifiers are terms forming operators which represent the functional dependences among bounded variables – as per the interpretation of quantifiers as choice functions, formalized by the ε-operator. The comparison between these two accounts will bring about a comparison between the metalanguage of FOL – namely First-order Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC)[3] – and the one of EC – namely ZF_{EC}. I will argue that the latter best capture our intuitions about quantification theory, thus advocating for the choice function interpretation of quantifiers.

The paper is organized as follow: in §1, I will first introduce and compare the two interpretations of quantifiers – as higher-order predicates or choice functions. While the former interpretation is formalized according to the metalanguage of FOL – namely ZFC – the latter is formalized by the extensional semantics of the ε-operator, which composes EC. In §2, in order to evaluate the two interpretations of quantifiers, I will introduce the rule of inference of 'Functional Instantiation' (FI), which resembles the procedure of Skolemization for first-order formulas by introducing function symbols which represent the dependences among the values of the quantified variables. I will claim that the introduction of FI is a conservative extension of the model theoretic semantics of first-order quantifiers. On the assumption that model theoretic semantics specifies the meaning of quantifiers, I will take FI as a condition for a proper interpretation of quantifiers.

However, in §3, I will explain that FI cannot be proved in FOL due to the metalanguage of ZFC which asserts the existence of first-order model with non-definable elements. In those models, the value of the function symbol cannot be specified, thus falling to provide an interpretation for it. Instead in §4, following the proof of Leisenring (1969), I will point out that FI is a theorem of EC, and so (Skolem) function symbols can be conservatively added to EC. The extensional semantics of the ε-operator provides an interpretation for the function symbols introduced by FI, thus accounting for representation of dependence relations among quantified variables. Moreover, I will stress the differences between the metalanguage of FOL – namely ZFC – and the one of EC – namely ZF_{EC}, showing that the ε-operator is equivalent to the Axiom of Global Choice.

Finally, in §5 and §6, I will consider two possible objections to the claim that the classical quantifiers ∀ and ∃ should be replaced by the ε-operator. More precisely, in §5 I will consider whether the ε-operator is a semantical first or higher-order quantifier, by adapting an objection of Feferman (2006). I will argue that this objection is question-begging: the ε-operator is covert second-order quantification over function only if it is interpreted according to ZFC – and not to ZF_{EC}. Therefore, this objection presupposes that the right metalanguage is the one of ZFC, which is what is at issue in the debate on whether the quantifiers should be interpreted using the ε-operator. Then, in §6, I will consider whether the ε-operator is a logical constant or not. Indeed, the ε-operator, unlike the classical quantifiers ∀ and ∃ – fails the Model theoretic condition of logicality, as formalized by the Tarski and Corcoran (1986)-Sher (1991) thesis. However, Woods (2014) has proposed a revision of the Tarski-Sher thesis according to which also the ε-operator is a logical constant. I will support Woods (2014) proposal based on two arguments: on the one hand,

[3]Someone could object that not the fully axiomatized ZFC is required to formalize the model theory of FOL. However, as I will extensively explain in §2.1, the Axiom of Choice (and so also full ZFC) is required to prove the theorem at issue in §2.

Woods (2014) account solves the domain-size variance objection to the Tarski-Sher thesis for the classical quantifiers $\forall$ and $\exists$. On the other hand, the arbitrary reference of the ε-operator captures the topic neutrality intuition behind the Tarski-Sher thesis.

Before continuing, I will briefly introduce the methodological framework of model theoretic semantics adopted in this chapter. Model theoretic semantics is a special form of truth-conditional semantics. According to it, the truth-conditions of sentences are defined by models, which are mathematical structures composed by a non-empty set and an interpretation function. Model theory provides the interpretations of the (non-logical) expressions of the object-language and determine (in the metalanguage) the truth-values of its declarative sentences. Therefore, model theoretic semantics formalizes Tarski truth definitions. However, the interpretation of the logical constants is kept fixed thorough the models. That is why the semantic clauses of, for example, object-language sentences containing $\wedge$, involve the conjunction in the truth-conditions defined by the metalanguage. This is the well known '*logocentric predicament*'(Sheffer, 1926, p. 228), according to which "in order to give an account of logic, we must presuppose and employ logic." Therefore, model theoretic semantics does not define the meaning of logical constants − as instead does proof theoretic semantics (see §Chapter II). Nevertheless, there is another feature of model theoretic semantics which is relevant for the comparison of different interpretations of a logical constant. Indeed, the truth-conditions defined by model theoretic semantics are recursive, which thus provide a *compositional* theory of meaning. The compositionality of formal and natural languages have been taken as a condition for their learnability, based on the idea that the meaning of an arbitrarily long sentence must be a function of a finite number of features of that sentence (Davidson, 1965). Therefore, given that different interpretations of a logical constant provide alternative truth-conditions, and given the (rough) connection between recursive truth-conditions and the learnability of a language, I conclude that model theoretic semantics provides two criteria for evaluating alternative interpretations of a logical constant: i) which notions the interpretation adopts in order to (recursively) define the truth-conditions and, ii) how those interpretations represent the truth-conditions of intuitively valid argument. I will adopt both conditions below to compare and evaluate two different interpretations of quantifiers. While (i) will be adopted in §1 in order to introduce the interpretations of quantifiers as either higher-order predicates or choice functions, (ii) will be discussed at the end of §2 to claim that FI is an extension of the standard model theoretic semantics.

1 Two Interpretations of Quantifiers

Since Frege (1879) formalization of quantification theory, the interpretation of quantifiers as higher-order predicates has predominated the logical, philosophical and linguistic discussion Peters and Westerståhl (2006). However, I will introduce an alternative interpretation of quantifiers as choice functions, which tracks back to Hilbert and Bernays (1939). In this paragraph, I will introduce the two interpretations of quantifiers according to their formal systems, laying down several notions which will be relevant for the following discussion.

On the one hand, by higher-order predicates I mean the theory of Generalized Quantifiers (GQ) developed by Mostowski (1957) and Lindström (1966). The idea is that, while an existential sentence $\exists x\varphi(x)$ asserts that the set of individuals satisfying the property φ − denoted by A[4] − is not empty, a universal sentence $\forall x\varphi(x)$ states that A is the entire universe of discourse. In model theoretic terms, the semantic value of the quantifiers $\forall$ and $\exists$ in a model $\mathcal{M}$ for any set A is:

$$\exists_{\mathcal{M}} = A \subseteq \mathcal{M} : A \neq \varnothing$$
$$\forall_{\mathcal{M}} = \mathcal{M}$$

Moreover, the model theoretic versions of truth-conditions for existentially and universally quantified formulas follow quite naturally from the semantic value of the quantifiers:

[4] I will adopt the following notation: capital letters for sets and Greek letters for predicates and formulas.

$$\exists x\varphi(x) \text{ is true in } \mathcal{M} \textit{ iff } \mid \varphi \mid \in \exists_{\mathcal{M}}$$

$$\forall x\varphi(x) \text{ is true in } \mathcal{M} \textit{ iff } \mid \varphi \mid \subseteq \forall_{\mathcal{M}}$$

Even if GQ was introduced to formalize *n*-adic determiners of natural language, the present discussion will consider only the Universal (∀) and Existential (∃) quantifiers[5]. Indeed, my concern regards the logicality of quantification, which is fully captured by the classical quantificational logic. More precisely, Lindström (1966) has proved that standard FOL with the classical quantifiers ∀ and ∃ is the strongest logic that has both the compactness, completeness and Lowenheim-Skolem properties. That is why, any proper extension of First-Order Logic with generalized quantifiers has to detect something non-trivial about the set-theoretic universe. Therefore, I will assume that the interpretation of ∀ and ∃ as higher-order predicates is the one adopted by the model theoretic semantics of FOL.

On the other hand, the choice functions interpretation of quantifiers was first suggested by the seminal paper of (Goldfarb, 1979, p. 357) who tracks back the idea to the works of David Hilbert, Thoralf Skolem and Jacques Herbrand:

> "The connection between quantifiers and choice functions or, more precisely, between quantifier-dependence and choice functions, is the heart of how classical logicians in the twenties viewed the nature of quantification."

The idea is that the truth of an existential sentence such as $\exists x\varphi(x)$ allows us to pick and x such that A. The EC is axiomatized by two axioms schemes − in addition to the standard axioms and deduction rules of FOL − namely:

Ax.1 $\varphi(t) \to \varepsilon_x\varphi(x)$ (Critical Formula)

Ax.2 $\forall x[\varphi(x) \leftrightarrow \psi(x)] \to [\varepsilon_x\varphi(x) \leftrightarrow \varepsilon_x\psi(x)]$ (Extensionality)

The informal interpretation of $\varepsilon_x\varphi(x)$ is "an x such that φ, if any". This rough idea is fully captured by the extensional semantics of the ε-operator, which is composed by:

1. A structure $\mathcal{M}_{EC} =< \mathcal{D}, \Phi, f >$ which consists of a nonempty domain $\mathcal{D} \neq \varnothing$, an interpretation function Φ and an (arbitrary) choice function *f*.
2. An (arbitrary) choice function f such that of any $A \subseteq \mathcal{D}$:

$$f : \mathcal{P}(\mathcal{D}) \to \mathcal{D} = \begin{cases} x \in A & \text{if } A \neq \varnothing, \\ x \in D & \text{if } A = \varnothing. \end{cases}$$

3. An assignment function s: Var $\to \mathcal{D}$. Based on these notions, the valuation of ε-terms is defines as:

$$Val_{\mathcal{D},\Phi,s}(\varepsilon_x A(x)) = \Phi(Val_{\mathcal{D},\Phi,s}(A(x))) = \Phi\{m \in \mathcal{D} : \mathcal{D}, \Phi, s[x/m] \vDash A(x)\}$$

A word of caution about the 'arbitrary' choice function *f*. Indeed, the ε-operator denotes a choice function which picks out a representative object for any non-empty set, and an arbitrary object from the domain $\mathcal{D}$ if the set in question is empty. It should be remarked that any choice function on the power set of the domain $\mathcal{D}$ would do exactly as well as any other as the denotation of the ε-operator. To capture this fact, we focus on ε-invariant sentences:

Def.1 A formula $\varphi \in$ EC will be called ε-invariant *iff* for any model $\mathcal{M}$ of φ and choice functions *f*, g of $\mathcal{M}$, then: $(\mathcal{M}, f) \vDash \varphi \leftrightarrow (\mathcal{M}, g) \vDash \varphi$.

Based on the notion of ε-invariant sentences, it is possible to prove that:

[5] I will consider GQ in the context of EC in §5.

Th.1 If a formula $\varphi \in$ EC is ε-invariant, then there is a first-order (epsilon-free) formula $\delta \in$ FOL; such that for every $f \in \mathcal{M}$: $(\mathcal{M}, f) \vDash \varphi \leftrightarrow \mathcal{M} \vDash \delta$.

Proof. (Blass and Gurevich, 2000, p. 1272) □

That is why EC is conservative extension of FOL − this result is also known as the Second Epsilon Theorem (Zach, 2017). Moreover, **Th.1** implies that the ε-invariant formulas are the formulas in which the ε-operator is used solely for quantification. Indeed, the universal $\forall$ and existential $\exists$ quantifiers can be explicitly defined in terms of the ε-operator as:

$$\exists x\varphi(x) \equiv \varphi(\varepsilon_x \varphi(x))$$

$$\forall x\varphi(x) \equiv \varphi(\varepsilon_x \neg\varphi(x))$$

Egli et al. (1995) give a clear justification of these equivalences:

> "If there is an x that has the property A, then there is also a description $\varepsilon_x A$ that satisfies A. To look at it the other way around: if there is a description $\varepsilon_x A$ that satisfies A, then there must be as well an x that satisfies A. The intuitions behind the second Hilbert rule are a bit more difficult: $\neg A$ denotes the complement of the set A. By $\varepsilon_x \neg A$ this complement is assigned one element, which either is an element of $\neg A$, if $\neg A$ is not empty, or is an arbitrary element from the domain, if $\neg A$ is empty. Precisely the latter, however, has to be the case, as $\varepsilon_x \neg A$ is an element of A. Therefore, $\neg A$ must be empty, and thus A denotes generality or: $\forall xA$. The second Hilbert rule can be derived from the first by replacing A by $\neg A$ and invoking contraposition."

Given that EC is conservative over FOL and given the equivalence of $\forall$, $\exists$ and the ε-operator, I consider the comparison between FOL and EC carried out in §2.1 and §2.2 as a comparison between the two different accounts of quantification.

2 Functional Instantiation

One of the improvements of Frege (1879) logic over Aristotelian syllogistic − in which no more than one quantifier occurs in each statement − is that a quantifier may fall into the scope of another, expressing a dependence relation between the values of the variables. I will argue that in order to better represent the semantic dependences among quantified variables, the rule of Functional Instantiation (FI) should be considered as part of the model theoretic semantics for quantifiers, regardless the interpretation as higher-order predicates or choice function. According to the rule of 'Functional Instantiation' (FI) − first introduced by Hintikka (1998) − for every formula in prenex normal form[6]:

$$\frac{\forall x_1 ... x_n \exists y \varphi(x_1 ... x_n, y)}{\forall x_1 ... x_n \varphi(x_1 ... x_n, f(x_1 ... x_n))} \ \textbf{(FI)}$$

Where f is a new n-ary function symbol added into the language[7]. Given that FI resembles the Skolemization procedure for first-order formulas − as it will be further explained in §4 − I

[6]A formula of FOL is in prenex normal form if and only if it is formed by a quantifier prefix followed by a quantifier-free formula. It is a well-known meta theorem that any FOL formula is equivalent to one in prenex normal form. The same definition and metatheorem hold in EC − see Leisenring (1969) p.54].

[7]It is interesting to note that, if f is a 0-ary function, then f becomes an Henkin constant − which have been used by Henkin (1949) for the completeness proof of FOL. Moreover, in this case, FI becomes similar to the Natural Deduction rule of Existential Elimination (EE). However, there is an important difference: while FI represent only the *functional-dependence* within formulas, EE is restricted by the *term-dependence* arising between the inferential steps of a deduction. I will extensively discuss this latter notion of dependence relation in §Chapter II.

will refer to the function symbol $f(x_1...x_n)$ as the Skolem function[8]. The idea behind FI is that the premise is true if there is a way of choosing, given any individual x, an individual y such that $\varphi(x, y)$. Moreover, in order to express that the choice of y depends on x, the new function f is introduced in the conclusion. Then, $f(x)$ denotes an appropriate y[9].

In this paragraph, I will justify the introduction of FI by arguing that is a *conservative extension* of the model theoretic semantics for first-order quantifiers. The introduction of FI is *conservative* because of the syntactic notions of scope and nested quantifiers defines the semantic relation of dependence among the values of quantified variables. Moreover, this dependence relation is functional: the value of a variable is totally determined by the value of another variable − as will be showed by several examples. The introduction of FI is an *extension* of the model theoretic semantics for first-order quantifiers because the functional dependences are left implicit by the truth conditions of quantifiers interpreted as higher-order predicates. Instead FI, by introducing functional symbols such as $f(x_1...x_n)$, names the functional dependence between the values of x and y. What I mean here by saying that FI is an extension of the truth-conditions of the classical quantifiers will be clear in §2.2, when I will prove − following Leisenring (1969) result − that the function symbols introduced by FI can be conservatively added to EC.

On the assumption that model theoretic semantics attempts to represent the truth-conditions of intuitively valid argument (see §Chapter I), I will conclude that FI is a necessary condition for the proper interpretation of quantifiers. Indeed, as Haack (1978) remarks, many attempts to change classical logic − such as FOL where the quantifiers are interpreted as higher-order predicates − come from worries about the adequate representation of various kind of informal argument − such as (FI). That is why, in §2.1 and §2.2, I will evaluate the theoremacity of FI in both FOL and EC so as to compare the interpretations of quantifiers as higher-order predicates and as choice functions.

First of all, I will explain how the semantic notion of dependence among the values of quantified variables is determined by two syntactic notions: the ones of scope and nested quantifiers. Indeed, adopting Krynicki (1995) notation, a quantifier prefix[10] is a triple $Q = (A_Q, B_Q, D_Q)$, where A_Q and E_Q are disjoint finite sets of variables called respectively universal and existential variables of Q, and D_Q is a relation between universal and existential variables of Q such as $(D_Q \subseteq A_Q \times E_Q)$, called the dependence relation of Q. If $< x, y > \in D_Q$ then the existential variable y depends on the universal variable x in Q. Then,

Def.2 We say that a quantifier prefix $Q = (A_Q, B_Q, D_Q)$ is linear if there is a linear ordering $<$ on $A_Q \cup E_Q$ such that $(Q, <)$ is:

i) irreflexive, no variable depends from itself;

ii) asymmetric, for no $< x, y > \in D_Q$, if x depends on y, then y depends on x;

iii) transitive, if x depends on y and y depends on z, then x depends on z;

iv) connected, for all $x, y \in (A_Q \times E_Q)$, then either x depends on y, or y depends on x.

However, $(Q, <)$ is a syntactical relation which concerns nested quantified variables, and not their values. Yet, the truth-conditions for quantified formulas are recursively defined on the complexity of formulas (namely, from the outside to the inside of the formula) − thus according to the linear

[8]Given that the semantic interpretation of quantifiers is here at issue, I will consider the rule of inference FI from a model theoretic point of view, namely as: $\forall x_1...x_n \exists y \varphi(x_1...x_n, y) \rightarrow \forall x_1...x_n \varphi(x_1...x_n, f(x_1...x_n))$

[9]Someone could object that FI considers only one possible pattern of quantifiers interaction, namely the one where the existential quantifier is in the scope of the universal ones. However, it is possible to formulate an equivalent version of FI* where the universal quantifier is in the scope of the universal one is replaced by a n-ary function symbols $g(x_1...x_n)$ known as Herbrand functions. In footnote n.X, I will explain why FI* does not challenge the argument below.

[10]Quantifiers and quantifier prefix are different concepts; for instance '$\forall$' is the quantifier, but '$\forall x$' is the quantifier prefix. While quantifiers are considered as both semantical and syntactical concepts, quantifier prefixes are syntactical concepts, which are interpreted semantically by quantifiers.

order $(Q, <)$. That is why in FOL, the syntactic notions of scope and nested quantifiers determine a semantic relation of dependence. This is shown by the well-known fact that reversing the order of existential and universal quantifiers produce a non-equivalent formula. It should be remarked that I am here focusing only on the dependences relations among the values of quantified variables, and not on their independences. Indeed, since the work of Henkin (1961), it is well known that by weakening the condition (iv), we obtain a partial order over the set of quantified variables, which is formalized by Henkin's branching quantifiers. As explained by Hintikka (1996), branching quantifiers allow to represent the independence between the values of variables, arguing that independence relation is a common phenomena in both formal and informal languages. However, the commitment to branched quantifiers is a highly debated issue (Neale, 1990)[11]. Instead, a weaker claim is here at issue: the semantic notion of dependence among quantified variables follows straightforwardly from the syntactic notions of scope and nested quantification. In this sense, the semantic notion of dependence is *conservative* over the standard model theoretic semantics for quantifiers.

However, I have to explain in more details what the semantic notion of dependence amounts to and how it is related to FI. Consider a formula like $\forall x \exists y \varphi(x, y)$: saying that $\exists y$ depends on $\forall x$ means that the witness of $\exists y$ may vary with the value interpreting $\forall x$. Therefore, there are two relevant aspects of the notion of dependence: i) the co-variance between the values of $\forall x$ and $\exists y$; ii) the value of $\exists y$ is totally determined by the ones of $\forall x$. That is why the semantic notion of dependence is *functional dependence*: given the value of a variable is it possible to define the value of another variable Väänänen (2007). Moreover, it is clear why FI captures both aspects of the notion of functional dependence: on the one hand, the introduction of the Skolem function $f(x_1...x_n)$ names the dependence relation between $\forall x$ and $\exists y$. On the other hand, by replacing $\exists y$ with $f(x_1...x_n)$, FI asserts that the value of the former is completely specified by the latter. The idea that the function symbols introduced by FI are adopted to represent the dependences between the values of bonded variables is well explained by Shoenfield (1967) p.55] example:

> "There is a type of reasoning frequently used in mathematics which we have not yet considered. To illustrate it, suppose that we are discussing natural numbers and have proved that for every x, there is a prime y such that $y > x$. In the course of a later proof we might say: let y be a prime such that $y > x$. We would then have to keep in mind through the rest of the proof that y depends upon x. If we wished to indicate this by the notation, we would say instead: for each x, let $f(x)$ be a prime greater than x. Of course, f would be a new symbol which does not appear in the result we are trying to prove."

I will now explain why FI is an *extension* of the model theoretic semantics for first-order quantifiers interpreted as higher-order predicates. I will argue that the Skolem functions introduced by FI allows representing the − intuitive − truth conditions for nested quantified formulas, which are left implicit by their standard truth conditions. Indeed, as remarked in §1, the GQ account of quantifiers as higher-order predicates defines also their truth conditions. In the case of $\forall$ and $\exists$, their truth conditions only concern the cardinality of the quantified sets, namely their non-emptiness or exceptionless. However, as I will point out, the GQ truth conditions do not work well in the case of nested quantified formula, where the value of $\exists y$ is picked out from the model according to the

[11] Given the huge literature raise dy Hintikka (1996), I will briefly summarize the similarities and differences between mine and his works. With Hintikka, I think that the (logical) expressive power of quantifiers relies on their interaction, which gives rise to different notions of functional dependence. Moreover, unlike FOL, both Epsilon Calculus and Hintikka's logic with branching quantifiers can express the independence between the values of the variables − see §10.2. However, EC has an advantage over Hintikka's logic with branching quantifiers: while the latter is expressively equivalent to ΣSOL (and so not closed under classical negation), the former is instead a conservative extension of FOL, which thus validate all of the classical theorems. That is why, in order to represent functional dependence, then ε-operator is a better choice than the branching quantifiers.

ones of $\forall x$. This intuitive idea is fully captured by FI, which introduces a Skolem function to name the specific dependence relation. Consider the following example:

1. Everyone loves someone.
   a) There is a woman that every man loves.
   b) Every man loves a different woman.

These sentences are formalized as:

a*) $\exists x Woman(x) \wedge \forall y(Man(y) \rightarrow Loves(y,x))$

b*) $\forall x Man(x) \rightarrow \exists y(Woman(y) \wedge Loves(x,y))$

However, formula (b*) does not correctly represent the meaning of sentence (b). Formula (b*) is true *iff* the set of all men is contained in the (non-empty) intersection of the set Woman(y) with the binary relation Loves (x,y) of man loving a woman. Instead, (b) says that every man has his own lover. Instead, (b) says that every man has his own lover. That is why the FI formulation of (b) introduces a function symbol $f(x)$ which picks a man and returns the man's lover, as:

$$\forall x Man(x) \rightarrow [Woman(f(x)) \wedge Loves(x, f(x))] \tag{3}$$

There are two versions of the claim that FI is an extension of the model theoretic semantics for the classical quantifiers $\forall$ and $\exists$. According to the strong one, adding (Skolem) function symbols is conservative in EC but not in FOL, and so EC is an extension of FOL $-$ as I will extensively explain in §2.1 and §2.2. According to the weaker claim, in GQ theory there are relational (or polyadic) quantifiers that cannot be expressed by any linear iteration of monadic quantifiers (Peters and Westerståhl, 2006). I will here explain this latter point. A generalized quantifier Q of type $< n_1, ...n_k >$ is said to be *monadic* if it bounds n variables in k formulas $-$ and *relational* otherwise. Obviously, the classical quantifiers $\forall$ and $\exists$ are monadic. Then, for example, a relational quantifier Q of type <2> can be defined by *iteration* of the classical ones, as $\forall x \exists y$. Nevertheless, the relational quantifier contained in sentence (b) $-$ which is interpreted as 'every x has its own y' $-$ cannot be defined by any linear order of monadic generalized quantifiers (Sher, 2012). This limitative result for the expressive power of GQ theory is known in the literature as the 'Frege boundary' (Keenan, 1992). Instead, FI introduces a functional symbol in order to name the dependence relation between the value of x and y, as showed in (3). That is why, FI is an extension of the model theoretic semantics of the classical quantifiers[12].

I will conclude by considering two further examples, drawn from analysis and group theory. Indeed, introducing function symbols to name the dependence relation detected in the axioms of a theory is a common practice among mathematicians. First of all, consider the formal definition of continuity and uniform continuity at a value x for a function $f : R \rightarrow R$ on the real numbers, respectively:

1. $\forall x \forall \epsilon \exists \delta \forall y |x - y| < \delta \rightarrow |f(x) - f(y)| < \epsilon$
2. $\forall \epsilon \exists \delta \forall x \forall y |x - y| < \delta \rightarrow |f(x) - f(y)| < \epsilon$

For students of analysis, it is helpful to realize that these statements assert the existence of a function (called a modulus of continuity) that specifies δ in terms of ϵ Wolf (2005). If f is

[12] It is interesting to note that many properties of relational generalized quantifiers $-$ such as branching and resumption $-$ are connected to natural language phenomena such as indefinite descriptions and anaphoric pronouns (Peters and Westerståhl, 2006). Given the limitative result of GQ theory concerning relational generalized quantifiers, many linguistics like Egli et al. (1995) and Meyer Viol (1995) have adopted the ε-operator in order to formalize those natural language phenomena. However, these works concerns mainly linguistics and thus fall outside the scope of the present dissertation.

continuous on its entire domain, then δ becomes a function of x and ϵ − namely (1). If f is uniformly continuous, then δ is a function of ϵ alone − as (2). Therefore, the definitions of continuity and uniform continuity correspond to different patterns of quantifiers dependence. Another example comes from one of the axioms of group theory, namely:

$$\forall x \exists x^{-1}(x \circ x^{-1} = e)$$

Where x^{-1} is called the inverse element. However, x^{-1} is a symbol introduced in the language of group theory in order to name the function *f(x)* which maps each x to its inverse. Therefore, as long as a specified model is considered, introducing function symbols to name specific dependence relations is a *conservative extensions* over the model theoretic semantics of quantified formulas. That is why, FI is considered as a *desideratum* of the correct interpretation of quantifiers. However, I have argue that FI is a valid rule, so the conclusion must be true in all models in which the premise is. Therefore, the comparison between the theoremacity of FI in both FOL and EC, carried out in §2.1 and §2.2, will be considered as a comparison between the two interpretations of quantifiers.

2.1 Functional Instantiation in First Order Logic

Yet FI is not a sound rule of inference for FOL, because it is not truth-preserving. Indeed, the logical equivalence between a first-order formula and its Skolem Normal Form (SNF) cannot be proved within FOL. Namely, the following inference is invalid[13]:

$$\forall x \exists y \varphi(x, y) \nvDash \forall x \varphi(x, f(y)) \tag{4}$$

The aim of this paragraph is to highlight the reason why FI is not a theorem of FOL, namely, how to obtain a counterexample to it. In order to do so, I will develop the comparison between FI and the process of Skolemization for first-order formulas, according to which each $\exists$ is eliminated and the corresponding variable is substituted with a n-ary Skolem function. That is why, I will refer to the consequent of FI as the SNF of the antecedent. Moreover, given that the process of Skolemization is a metatheorem of FOL, I will consider FI within the metalanguage of FOL − namely, ZFC

In proof theoretic terms, FI replaces each existential quantifier in a formula with a function symbol (a Skolem function) which is supposed to denote an arbitrary relation on the domain. Yet, in model theoretic terms, the Skolem function is a term which must be interpreted and so the Skolem function has to define a functional relation over the domain. Here the problems come: in ZFC not all sets (or relations) that are proved to exist can also be defined. Due to this fact, there are first-order model where no Skolem function is definable, thus producing a counter model to FI. The connection between the existence of non-definable sets and the failure of FI in FOL is proved by the fact that if FI is formulated in full SOL − where the Skolem functions are existentially quantified − then it is possible to prove the logical equivalence of first-order formulas and their (second-order) SNF. Therefore, as long as first-order models are considered, ZFC cannot prove FI because it is not able to provide an interpretation for all the Skolem functions − whose existence can nonetheless be proved in second-order models. Given that FI is a condition for a correct interpretation of quantifiers, I will conclude that this argument undermines the interpretation of quantifiers as higher-order predicates − which is formalized according to ZFC, as explained in §2.

Considered as an inferential rule, FI by itself only replace each existential quantifier in a formula with an $n-$ary Skolem function. Given a language $\mathcal{L}$ − namely FOL added with non-logical symbols − we call $\mathcal{L}^{Sk}$ the Skolem expansion of $\mathcal{L}$ with the (Skolem) function symbols. Moreover, a theory $\mathcal{T}$ − namely a set of formulas of $\mathcal{L}$ − has *built-in* Skolem functions if for every formula $\varphi(x_1 ... x_n)$, $\mathcal{T}$ has a n-ary function symbol f such that:

[13] Instead, the converse formula $\forall x \varphi(x, f(y)) \vDash \forall x \exists y \varphi(x, y)$ is a theorem of FOL − see van Dalen (1980).

$$\mathcal{T} \vdash \forall x_1...x_n(\exists y\varphi(x_1...x_n, y) \to \varphi(x_1...x_n, f(x_1...x_n,))) \tag{5}$$

Those formulas are the axioms of the Skolem theory $\mathcal{T}^{Sk}$. Then it is possible to prove that:

Th.2 Let $\mathcal{T}$ be a theory in $\mathcal{L}$. Then there are an expansion $\mathcal{L}^{Sk}$ of $\mathcal{L}$ and an extension $\mathcal{T}^{Sk}$ of $\mathcal{T}$ ($\mathcal{T}^{Sk}$ a theory in $\mathcal{L}^{Sk}$) such that $\mathcal{T}^{Sk}$ has *built-in* Skolem functions.

Proof. (Chang and Keisler, 1990, p. 165) □

It should be remarked that *built-in* Skolem functions are already part of the language $\mathcal{L}$ of $\mathcal{T}$. In other words, **Th.2** states that there are enough function symbols in the language to witness all existential quantifiers. However, FI does not tell how to eliminate Skolem functions symbol from a formula and it does not tell anything about how $\varphi(x_1...x_n, y)$ and $f(x_1...x_n)$ are related. That is why we have to consider the interpretation of the Skolem function in an extended model − and here the problems come. Indeed:

Th.3 Each $\mathcal{M} \in \text{Mod}(\mathcal{T})$ has a Skolem expansion $\mathcal{M}^{Sk} \in \text{Mod}(\mathcal{T}^{Sk})$.

Proof. (van Dalen, 1980, p. 137) □

The proof of **Th.3** essentially relies on the Axiom of Choice (AC). More precisely, given the formula $\forall x\exists y\varphi(x, y)$, AC is adopted to 'choose' an appropriate y as value for $f(x)$, thus defining the Skolem function $f(x)$. The role played by AC in the proof of **Th.3** will be clear at the end of the paragraph. By now, let put AC aside in order to focus on the semantic step required to pass from $\mathcal{M}$ to $\mathcal{M}^{SK}$. This step is the one from:

for each $a \in \mathcal{M}$ there is some $b \in \mathcal{M}$ such that $\mathcal{M} \vDash_{(x/a,y/b)} \varphi(x, y)$

to saying that

for each $a \in \mathcal{M}$ we define $f(a)$ to be one of these corresponding b.

The Skolem function $f(x)$ is a function $f : \mathcal{M} \to \mathcal{M}$. In $\mathcal{M}^{Sk}$ the interpretation function Φ^{Sk} has to assign a n-ary functional relation over the domain to the n-ary Skolem function symbol $f(x)$, in order to evaluate the whole formula $\forall x\varphi(x, f(x))$. This is why the Skolem function symbol must define a relation on the domain. Yet, the problem of defining a Skolem function is strictly connected to the problem of 'finding' a value for it. If there are several suitable y for a given x, we have not given a rule for choosing a particular one of these y and this means one can dispute whether we have really defined the function $f(x)$. This is remarked by the fact that in the case of $\forall x\exists!y\varphi(x, y)$ the Skolem function for φ is uniquely determined and its proof does not require AC (van Dalen, 1980, p.138).

Def.3 We say that a theory $\mathcal{T}$ has *definable* Skolem functions $f_\varphi(x)$ if for any formula $\varphi(w, v)$ there is a formula $\psi(w, v)$ such that:

i) $\mathcal{T} \vdash \forall w(\exists v\varphi(w, v) \to \exists u(\psi(w, u) \wedge \varphi(w, u)))$,

ii) $\mathcal{T} \vdash \forall w\forall u\forall v((\varphi(w, u) \wedge \varphi(w, v)) \to u = v)$.

Those are the defining conditions for a Skolem function: while (i) is called the existence condition, (ii) is called the uniqueness condition (Shoenfield, 1967, p.59). These conditions explain how the Skolem function $f(x)$ and the binary relation $\varphi(x, y)$ are connected. If these conditions are met, then is possible to introduce the defying axiom for the Skolem function $f_\varphi(x)$:

$$y = f_\varphi(x) \leftrightarrow \psi(x, y) \tag{6}$$

Which provides the value picked out by the Skolem function $f_\varphi(x)$. Before considering some examples, it is important to remark that the notion of *definable* Skolem function is not primarily model theoretic, but set theoretic. Indeed, a model is composed by a non-empty domain − that is a non-empty set − and so the problem of *definable* Skolem functions correspond to the problem of which relations are definable in the domain. In ZFC, the notion of definable function is stated as follow by (Fraenkel et al., 1973, p.70):

> "A definable operation (function) on $x_1...x_n$ is given by a condition $B(y)$, without parameters other than $x_1...x_n$, such that one can prove in *ZFC* that for any given $x_1...x_n$ there is exactly one y which fulfils the condition $B(y)$; this set y depends, in general, on $x_1...x_n$ and is taken to be the value of the operation for $x_1...x_n$."

This remark the connection between *definable* Skolem functions and their values. Consider two examples. Let $\mathcal{T}$ be $\mathcal{N}$, namely the theory of natural numbers defined by the Peano axioms, which has *definable* Skolem functions for every formula. To prove it, we use the fact that every non-empty subset of $\mathcal{N}$ has a smallest element. So for a formula $\varphi(x, y)$ we can define a Skolem function by the formula $\psi(x, z)$ given by:

$$(\forall z \neg\varphi(z, y) \wedge x = 0 \vee (\varphi(x, y) \wedge \forall z(z < x \rightarrow \neg\varphi(z, y))) \tag{7}$$

Then $\mathcal{N}$ has *definable* Skolem functions:

$$\mathcal{N} \vdash \forall y(\exists x\psi(x, y) \wedge (\exists x\varphi(x, y) \rightarrow \forall x(\psi(x, y) \rightarrow \varphi(x, y)))) \tag{8}$$

Indeed, (8) meets both the existence and uniqueness conditions for definable Skolem functions. But we are not so lucky in all first-order models − and that is why FI is invalid in FOL. If the model does not have the right sort of structure, there might be no way to define the Skolem function. Consider for example $\mathcal{R}$, which is the theory of real numbers. Subsets of $\mathcal{R}$ do not in general contain a least element. This is a consequence of a more general result, namely:

Th.4 In ZFC there is no definable well-ordering of the set of all real numbers.

Proof: Feferman (1965).

Due to **Th.4**, there are first-order model with non-empty sets of which no element can be defined. But, as remarked above, a definable element is required to meet the uniqueness condition (ii) for *definable* Skolem functions. Without definable values, there are no *definable* Skolem functions. And without *definable* Skolem functions, the interpretation function Φ^{Sk} of the first-order model $\mathcal{M}^{Sk}$ fails to assign a n-ary functional relation to the n-ary Skolem function symbol introduced by FI.

At this point, someone could take this result as a valid reason to drop FI, arguing that, from a model theoretic perspective, there are no such things as the semantic values of the Skolem functions symbols introduced by FI. Yet this is not the case: as soon as we formulate FI in SOL (FI_{SOL}) − where the Skolem functions are existentially quantified − then the following is a logical consequence over full models (Wolf, 2005):

$$\forall x\exists y\varphi(x, y) \vDash \exists f\forall x\varphi(x, f(x)) \tag{9}$$

Why so? Because the very Skolem functions which cannot be defined in first-order model due to the ZFC assumption about the existence of non-empty sets with non-definable elements, they nonetheless exist by AC. However, the existence of non-definable sets and relations (such as Skolem functions) cannot be captured by first-order models which quantify over individuals, but only by second-order models which quantify also over subsets and relations of the domain. Moreover, this explains why AC is required by the proof of **Th.3**: in order to expand a model with Skolem functions, AC is required to guarantee the existence of such functions, even if they defying axioms cannot be specified.

In conclusion, I will sum up the results concerning the (in)validity of FI in FOL and SOL, explaining why it undermines the interpretation of quantifiers as higher-order predicates. Once established that any theory formulated in FOL can be expanded with *built-in* Skolem function symbols − as for **Th.2** − we have considered the interpretation of those symbols. Given that we are considering first-order language, the semantic value of a n-ary Skolem function is a n-ary functional relation over the domain. Yet, Skolem function symbols define relations on the domain *iff* the conditions of existence (i) and uniqueness (ii) are satisfied. However, according to ZFC, there are models where there exist non-definable relations. While this fact cannot be captured by first-order models − which explain the invalidity of FI in FOL − it can be captured by (full) second-order models − explaining the validity of FI_{SOL} in SOL.

It should be remarked that the existence of non-definable sets in ZFC is a well-known fact − which tracks back to the debate over AC Moore (2012). However, less attention has been focused on how this set theoretic assumption limits the expressive power of quantification theory in FOL. Indeed, as remarked in §2, the Skolem functions introduced by FI works as arbitrary names for the dependences relations among quantified variables detected in the formulas. Nonetheless, if first-order models of ZFC are considered, then there are relations in the models that cannot be specified by any Skolem functions. This highlights how poorly ZFC formalize the inferential patterns concerning functional dependencies − which have been argued in §2 to be a *conservative extension* of the model theoretic semantics for quantifiers. That is why the invalidity of FI in FOL undermines the interpretation of quantifiers as higher-order predicates.

2.2 Functional Instantiation in Epsilon Calculus

However, EC provides better results concerning the validity of FI. In this paragraph, I will first present the logical equivalence of first-order formulas and their SNF in EC − following the proof of Leisenring (1969). Then, I will explain why it is possible to expand every EC model with *definable* Skolem function. Moreover, I will stress that Leisenring (1969) proof does not require the Axiom of Choice (AC). Indeed, I will explain why the formulation of ZF set theory according to EC (ZF_{EC}) is consistent with the negation of AC. Moreover, I will highlight a stronger result: the ε-operator is equivalent to a global-choice operator, which existence is implied by the Axiom of Global Choice (AGC) of the von Neumann-Bernays-Gödel set theory (NBG). This last result is remarkable for the dialectic of the work: indeed, it is well known that no global-choice operator is available in ZFC, pain the inconsistency Fraenkel et al. (1973). But so, the metalanguage of FOL, namely ZFC, does not only make invalid FI − thus limiting the expressive power of quantification theory in FOL − but it also rules out the very interpretation of quantifiers as choice functions, formalized by the ε-operator. First of all, it is possible to prove that adding function symbols is conservative over EC:

Th.5 Let EC* be the expansion of EC with the n-ary function symbol f. Then, for any set of formulas X and any formula C:

$X, \forall x \exists y \varphi(x,y) \vdash_{EC} C$ *iff* $X, \forall x \varphi(x, f(x)) \vdash_{EC^*} C$

Proof. (Leisenring, 1969, pp. 55-56) □

This is the result we were looking for: every EC formula with nested quantifiers can be turned into its SNF − as for FI. I will first explain how each EC model $\mathcal{M}_{EC} :< \mathcal{M}, \Phi, f >$ can be expanded to one $\mathcal{M}^{Sk}_{EC}$ with *definable* Skolem function[14]. This will highlight that the invariant choice function f adopted by the extensional semantics of the ε-operator allows to represent the inferential patterns of functional dependences asserted by FI. The idea is to identify each ε-term $\varepsilon_y \varphi(x,y)$ with its Skolem function symbol $f_\varphi(x)$, because an ε-term with free variables represent

[14]In footnote n.X I have explained that FI could be replaced by FI*, which introduces Herbrand functions. However, FI* would not challenge the argument discussed above because (Leisenring, 1969, p. 55) proves that also Herbrand functions can be conservatively added to EC. That is why, the same argument for FI works as well as for FI*.

a function of those variables, which picks out a suitable value for x given any value of those free variables. That is why,

Def.4 We define a mapping $(.)^* : \mathcal{L}_{EC} \to \mathcal{L}_{EC}^{Sk}$ between the language of EC and its Skolem expansion $\mathcal{L}_{EC}^{Sk}$, which sends each ε-term to its built-in Skolem function:
$(\varepsilon_y \varphi(x,y))^* \to f(x)$.

However, as for FOL, also in EC the problems concern the interpretation of the Skolem function symbols. The idea behind **Th.5** is that the function symbol *f(x)* is added into the language of EC to name the ε-term $\varepsilon_y \varphi(x,y)$ – which refers to an arbitrary value of y. Then, the invariant choice function f of $\mathcal{M}_{EC}$ can be adopted to evaluate the Skolem function symbol according to its ε-formula, as:

$$Val^{\mathcal{M},f,s}(\varepsilon_y \varphi(x,y)) = Val^{\mathcal{M}_{EC}^{Sk},f,s} f_{\varphi}(x) \tag{10}$$

Moreover, the Skolem function defines a relation on the domain. Indeed, the invariant choice function f interpreting the ε-operator is extensional, namely if the ε-terms $\varepsilon_y \varphi(x,y)$ and $\varepsilon_y \psi(x,y)$ are co-extensional, then the invariant choice function f of assigns the same value to both terms – see the **Ax.2** in §1. Therefore, the Uniqueness condition (ii) in §2.1 can be formulated in EC as:

$$\varepsilon_u \varphi(w,u) \wedge \varepsilon_v \varphi(w,v) \to u = v \tag{11}$$

Therefore, in EC, the arbitrary individual picked out by the extensional semantics of the ε-operator allows defining the value of the Skolem function, proving that every EC model can be expanded with Skolem functions[15].

However, someone could argue that the extensional semantics for the ε-operator implicitly assumes AC. As noted by Carnap (1961) and Leisenring (1969), the ε-operator and AC share many similarities: unlike the other axioms of ZFC, AC merely asserts the existence of a selection set y for a given set x without actually specifying the members of y. Similarly, the ε-operator refers to a choice function for any subsets x of the domain without specifying which member of x is being selected. Nonetheless, in ZF$_{EC}$ it can be proved that:

Th.6 If the Axiom of Replacement is restricted to ε-free formulas, then ZF_{EC} is consistent with the negation of the Axiom of Choice.

Proof. (Fraenkel et al., 1973, p. 73) □

Indeed, it is important to remark the difference between AC and the ε-operator: while the former asserts that the values of the choice function for each non-empty set form a further set – the choice set – the latter does not imply so. Therefore, the question of the derivability of AC in EC depends on the Axiom of Replacement, which asserts that for every given definable function with domain a set A, there is a set whose elements are all the values of the function. Indeed, if ε-terms like $\varepsilon_u(u \in w)$ are allowed among the definable function of the Axiom of Replacement, then it is clear that $\varepsilon_u(u \in w)$ forms a choice set – as for AC.

Instead, by comparing ZF$_{EC}$ and ZFC, is it possible to highlight that the ε-operator has greater inferential power than AC. This was first pointed out by Wang (1957), pp.66-67]:

[15] As remarked in the §Introduction, Hintikka (1996) Game Theoretical Semantics (GTS) is one of the sources of the idea that quantifiers are devices adopted to witness the predicates in quantified formulas. Moreover, as explained in §2, Hintikka (1996) claims that the classical quantifiers ∀ and ∃ should be replaced by branching quantifiers, interpreted according to GTS. In a nutshell, the main idea of GTS is that the truth-conditions for quantified formulas are given by their Skolem and Herbrand functions, which in GTS correspond to the strategies of verifying and falsifying formulas. In footnote n.X, I have already pointed out the advantages of the ε-operator over branching quantifiers. Then, given that in EC Skolem (and Herbrand) functions can be conservatively added, it would be interesting to investigate whether the ε-operator can be interpreted according to GTS. Nonetheless, this topic would lead us to far away from the present comparison with the classical quantifiers. That is why I leave this topic to a further work.

"There are also cases where, although the ε-rule yield the desired result, the axiom of choice would not. For example, in the Zermelo theory we can infer '$\forall xR(x, \varepsilon_y Rxy)$' from '$\forall x\exists yR(x, y)$' by the ε-rule, but we cannot infer '$\exists f\forall xR(x, f(x))$' from '$\forall x\exists yR(x, y)$' by the axiom of choice, on account of the absence of a universal set in Zermelo's theory."

Indeed, the ε-operator is equivalent to a global-choice operator, which existence is asserted by the Axiom of Global Choice (AGC) of the von Neumann-Bernays-Gödel set theory (*NBG*). Indeed, both operators allow choosing an arbitrary element from each non-empty set. This can be checked by comparing the ε-axioms presented in §1 with the ones for the global choice operator σ introduced by Bernays (1991):

Ax.1*$a \in C \rightarrow \sigma(C) \in C$

Ax.2*$(A \leftrightarrow B) \rightarrow \sigma(A) = \sigma(B)$

While **Ax.1*** asserts that for every non-empty set C, the value of $\sigma(C)$ is a member of C, **Ax.2*** states that if two sets A and B are co-extensional, then the values of $\sigma(A)$ and $\sigma(B)$ is the same individual. Then, it is clear that **Ax.1*** is equivalent to the ε-axiom of Critical Formulas and **Ax.2*** to the one of Extensionality. Someone could object that this result undermines the argument for the choice function interpretation of quantifiers − as formalized by EC: given that the assumption of AC is already highly controversial Moore (2012), then an interpretation of quantifiers which is equivalent to the stronger AGC must be rejected. Nevertheless, as soon as someone endorses the argument for the logicality of FI proposed in §2, then (s)he is also committed to the assumption of either AC or AGC in order to prove FI, respectively, in SOL and EC.

The equivalence between the ε-axioms and AGC is remarkable for the present discussion because in ZFC no global choice operator is available (Fraenkel et al., 1973, p. 71). This follows from **Th.4**, namely from the existence of non-empty sets with no definable elements − such as the non-empty set $\mathcal{R}$ of all well-orderings of the real numbers. If a global choice operator like the ε-operator was available in ZFC, then one could prove the existence of a definable element of $\mathcal{R}$, namely $\varepsilon_x\mathcal{R}(x)$. Therefore, the metalanguage of FOL, namely ZFC, does not only make invalid FI − thus limiting the expressive power of quantification theory in FOL − but it also rules out the very interpretation of quantifiers as choice function, formalized by the ε-operator.

3 What Kind of Logic is the Epsilon Calculus?

Due to **Th.1**, every EC formula is equivalent to a FOL formula. But this result depends on the Compactness theorem and therefore on the presence of infinite structures Blass and Gurevich (2000). Otto (2000) found a nonelementary property of finite structures which is expressible by a sentence of EC, but not by any first-order sentences. This raises the suspect that EC is actually more expressible than FOL − but to which extent? Notice that EC formulas are evaluated at structures equipped with an invariant choice function − as explained in §2. Then, it is possible to prove that:

Th.7 Every EC formula is equivalent to some Σ-SOL formula.

Proof. (Blass and Gurevich, 2000, p. 1274) □

Where Σ-SOL is the existential fragment of SOL. However, this last result could cast doubts about whether the formulation of the quantifiers $\forall$ and $\exists$ according to the ε-operator is actually first-order. The idea would be that the ε-operator, even if it takes first-order variables just like $\forall$ and $\exists$, it could be contextually eliminated by an higher-order quantification over functions, as: $\exists f[\varphi(f(\varphi(x)))]$. Then, the ε-operator would be second-order quantification first-order-ified. This is precisely Feferman (2006) objection against the branched quantifiers adopted by Hintikka

(1996) Independence Friendly Logic. Given that FOL expanded with branching quantifiers has the expressive power of Σ-SOL Hintikka (1996) and given that EC is at least expressible as Σ-SOL, then Feferman (2006) objection extends also to EC, where the quantifiers are interpreted according to the ε-operator. I will adapt and expand Feferman (2006) argument.

The ε-operator can be considered in the framework of GQ, presented in §2. In this framework, it is possible to distinguish between syntactically first-order quantifiers − which bound first-order variables − and semantically higher-order quantifiers − which are interpreted by quantification over higher-order functions. However, the extensional semantics for the ε-operator presented in §1 requires to existentially quantify over an (invariant) choice function $f : \mathcal{P}(\mathcal{M}) \to \mathcal{M}$. Therefore in GQ, the ε-operator, even if syntactically first-order, is semantically higher order.

Nevertheless, I will claim that this objection is a *petitio principii*: the ε-operator is covert second-order quantification over choice functions only if it is interpreted according to *ZFC* − and not ZF_{Ec}. Moreover, I will resist Feferman (2006) objection by pointing out that if the ε-operator is added to the theory of GQ, then the class of definable generalized quantifier can be expanded − as proved by Caicedo (1991). Therefore, there is no common background framework − either *ZFC* or the theory of GQ − according to which determine whether the ε-operator is a semantical higher-order quantifier or not.

First of all, it should be remarked that Feferman (2006) distinction between syntactical and semantical first-order quantifiers correspond to the distinction between expressibility and definability of generalized quantifiers in FOL. Indeed, there are generalized quantifiers − such as '*more than a half of*' − which even if they are first-order expressible, in the sense that there is a formula which correctly represent their truth-conditions, they are not first-order definable, because they are interpreted by higher-order quantification over arbitrary functions − as explained by Iacona (2018). Based on this distinction and Caicedo (1991) result, I will argue that there are generalized quantifiers that even if are not definable in (an extension of) FOL, they are definable adopting the resources of the ε-operator. Indeed, consider the generalized quantifier:

$$Q_1 x, y(\varphi(x, y)) = \quad \varphi \textit{ defines an equivalence relation having uncountably many equivalence classes.}$$

Keisler (1970) proves that Q_1 is not definable even in $\mathcal{L}_\omega$ (Q_2), where $\mathcal{L}_\omega$ is a first-order logic with infinitary language and (Q_2) is the generalized quantifier "there are uncountably many...". However, Caicedo (1991) has proved that Q_1 can be defined in $\mathcal{L}_\omega$ (Q_2) enriched with the ε-operator as:

$$eq(\varphi(x, y)) \wedge Q_2 y(y = \varepsilon_x \varphi(x, y))$$

Where $eq(\varphi)$ says that $\varphi(x, y)$ defines an equivalence relation, because in such case $Q_2 y(y = \varepsilon_x \varphi(x, y))$ asserts that there is an uncountable set of representatives for the relation. This example proves that by adding the ε-operator to the theory of GQ, the class of definable generalized quantifier is expanded. Therefore, there is no way of rejecting the ε-operator as higher-order quantification over functions without being already committed to ZFC or to the standard GQ theory. However, as I have argued in §2.1, ZFC limits the expressive power of quantification theory within FOL, thus undermining Feferman (2006) objection.

4 Is the ε-operator a (Model theoretic) Logical Constant?

In the previous paragraphs, I have pointed out that the comparison between FOL and EC concerns two different interpretations of quantifiers: as higher-order predicates or choice functions. To compare these two interpretations, I proposed a rule of inference FI for nested quantifiers which introduces functional symbols to name the *dependence relations* between the values of variables.

Moreover, I have argued that the representation of functional dependence is a *conservative extension* of the model theoretic semantics for quantifiers and so it should be considered as a *desiderata* for the correct interpretation of quantifiers. While FI can be proved only in SOL on the assumption of AC, it is instead a theorem of EC − where (Skolem) function symbols can be conservatively added. Therefore, I have argued that the classical quantifiers ∀ and ∃ should be replaced with the ε-operator.

Yet, there is a serious concern which triggers my entire revisionist argument for the classical quantifiers of FOL. The argument of the previous paragraph for the derivability of FI in EC is a model theoretic one: it proves that FI is a logical consequence of the EC axioms. However, the model theoretic definition of logical consequence depends on the distinction between the logical and non-logical vocabulary. Therefore, the argument of paragraph §2 relies on the assumption that the ε-operator is a logical constant. Consider the Tarski and Corcoran (1986)-Sher (1991) thesis, which is the well-known condition to determine if an expression is a logical constant or not. Then, while according to the Tarski-Sher thesis the classical quantifiers ∀ and ∃ are logical constants, the ε-operator is not. This objection would undermine the logicality of the whole EC, suggesting that the ε-operator is rather a mathematical concept. That is why I will first present the Tarski-Sher thesis, explaining why the ε-operator fails it. Then §4.1, following Woods (2014), I will weaken the Tarski-Sher thesis so as to consider the ε-operator as a logical constant. Finally, in §4.2, I will argue that Woods's solution is not *ad hoc* because it improves the debate over the logicality of quantifiers: on the one hand it explains the meaning invariance of the classical quantifiers ∀ and ∃, on the other hand, it follows the intuition on topic neutrality behind the Tarski-Sher thesis.

Tarski and Corcoran (1986) initially considered the logicality of notions, which are the extensional denotation of logical constant on every domain. More precisely, given a domain D it is possible to construct a type-hierarchy over $\mathcal{D}$ as follow:

- d and t are type-symbols;
- d denotes $\mathcal{D}$;
- t denotes $\{T, F\}$;
- If $S_1, ..., S_n$, and S are type-symbols, $(S_1, S_2, ...Sn \to S)$ is a type symbol;
- $(S_1, ...Sn \to S)$ denotes the set of functions from the Cartesian product of $S_1, ..., S_n$, to S.

Then, a notion θ is a function from every domain $\mathcal{D}$ to the members $\theta^{\mathcal{D}}$ of the type-hierarchy over $\mathcal{D}$. In this framework it is possible to formalize the generalized account of quantifiers presented in §2, according to which quantifiers are higher-order predicates that yield truth values when applied to subsets of the domain, or $((d \to t) \to t)$. Sher (1991) expands Tarski account by stating a further condition for logical expressions. That is why Tarski-Sher thesis can be spelt out into two conditions:

(TS1) A notion θ is logical if is invariant under bijections.

(TS2) An expression ϵ in a language L is logical if it denotes a logical notion.

I will refer to the conjunction of these two theses as 'TS'. Then it is clear why the ε-operator fails TS1 and consequently TS2: any expression denoting a function of type $((d \to t) \to d)$ − namely any variable-binding operator − is not isomorphism-invariant on every domain containing more than one object because we can simply permute its range for a counterexample. Yet, as I have argued in the previous paragraph, the EC captures our intuitions on the validity of FI, suggesting also the logicality of the ε-operator. That is why Woods (2014) has weakened the Tarski-Sher thesis so as to extend the definition of logical constant to the ε-operator.

4.1 The ε-operator as a Generalized Notion

As explained above, TS1 concerns the logicality of notions, namely functions from a domain $\mathcal{D}$ to some elements of the type hierarchy over $\mathcal{D}$. However, even if most expressions have a definite denotation on every domain, this is not the case for the ε-operator. Indeed, the ε-operator denotes a choice function which picks out a 'representative' object for any non-empty set, and an arbitrary object from the domain $\mathcal{D}$ if the set in question is empty. It should be remarked that any choice function on the power set of the domain $\mathcal{D}$ would do exactly as well as any other as the denotation of the ε-operator. This idea is well expressed by Miraglia (1996):

> "[...] the property that f has of being a choice function − the property by which $f(A) \in A$, for any subset A of the universe − is preserved under an arbitrary transformation of the domain. In other words, the result of applying a transformation of the type under consideration to any choice function is still a choice function."

Therefore, the ε-operator should not be considered as a logical constant denoting exactly one such function. Indeed, as stated by **Df.1**, we are focusing on ε-invariant sentences, namely those that can be interpreted by any choice functions on $\mathcal{P}(\mathcal{D})$. In order to accommodate the indefinite denotation of the ε-operator − which determines a range of admissible choice functions − Woods (2014) proposes to replace the definition of notion used by TS1 which that of 'generalized notion': a generalized notion is a function from every domain $\mathcal{D}$ to a set of objects of the same type in the type-hierarchy over $\mathcal{D}$. While Tarski-Sher *notions* determine unique logical operations, Woods *generalized notions* specify sets of them. Then − following Woods (2014) − it is possible to reformulate TS1 and TS2 as follow:

(WI1) A generalized notion θ^* is logical if and only if the set of admissible members of the type-hierarchy over $\mathcal{D}$ that θ^* denotes on $\mathcal{D}$ is isomorphism invariant.

(WI2) An expression ϵ^* in a language $\mathcal{L}$ is logical if ϵ^* denotes a logical generalized notion.

Following Woods (2014), I will refer to the conjunction of these two notions as 'Weak Invariance' (WI). Then the generalized notion denoted by the ε-operator is isomorphism invariant. Therefore, Woods (2014) proves that the ε-operator is a logical constant.

4.2 Two Arguments for Generalized Notions

Yet, someone could argue that this is an *ad hoc* solution: we have to modify the initial definition of 'notion' so as to extend the Tarski-Sher thesis to the ε-operator. But this is not the case: as I will point out, if we move to generalized notions, then it is possible to solve an objection to the Tarski-Sher thesis about the logicality of the classical quantifiers. The objection to the Tarski-Sher thesis that I want to consider is the concern of McGee (1996) and Feferman (1999) about the constant interpretation of quantifiers[16] across domains of different sizes. Let Q be the notion denoted by the universal quantifier $\forall$, then according to TS1, the invariance under bijection between two domains $\mathcal{D}$ and $\mathcal{D}'$ of the same size forces Q to be the same notion on $\mathcal{D}$ and $\mathcal{D}'$. However, if $\mathcal{D}$ and $\mathcal{D}'$ do not have the same size, the same logical symbol $\forall$ may be interpreted completely differently on $\mathcal{D}$ and $\mathcal{D}'$ according to TS1. Bonnay (2006) proposes the example of a strange quantifier Q_7 that is interpreted as the universal quantifier on finite domains of size less than 7 and like the existential quantifier on all other domains and which nonetheless passes the Tarski-Sher test. This is a compelling objection, as (Feferman, 1999, p. 38) points out:

> "It seems to me there is a sense in which the usual operations of the first-order predicate calculus have the same meaning independent of the domain of individuals over which they are applied. This characteristic is not captured by invariance under bijections."

[16]The objection concerns any generalized quantifiers, but I will focus only on the classical $\forall$ and $\exists$.

Instead, consider the denotation of classical quantifiers as generalized notions. What a domain determines for the denotation of $\forall$ and $\exists$ can be seen as a range of admissible functions, but in this case, there is always only one. More precisely, let θ^D be the image of the generalized notion θ on a domain D, then:

- $\forall^{\mathcal{D}}$ is the singleton containing the function from the domain $\mathcal{D}$ to the singleton of the domain.
- $\exists^{\mathcal{D}}$ is the singleton containing the function from the domain $\mathcal{D}$ to the set of the nonempty subsets of $\mathcal{D}$.

Woods (2014) calls definite generalized notions those that denote a singleton on every domain − like $\forall$ and $\exists$ − and indefinite generalized notions otherwise − like the ε-operator. Then the definite generalized notion denoted by $\forall$ and $\exists$ are logical: there is a bijection ζ from $\mathcal{D}$ to $\mathcal{D}'$ which maps the singleton $\forall^{\mathcal{D}}$ with the singleton $\forall^{\mathcal{D}}$', and similarly for $\exists$. Moreover, the isomorphism ζ takes admissible functions to admissible, though not necessarily identical, functions. Indeed the size of $\mathcal{D}$ and $\mathcal{D}'$ could be different and so also the function of $\forall^{\mathcal{D}}$ and $\forall^{\mathcal{D}'}$'. That is why WI1 is independent from the size of the domains: the isomorphism is established between the range of admissible functions, and not among the objects denoted on every domain. It follows that for WI2 the domain variance objection to TS2 does not arise: $\forall$ and $\exists$ denote on every domain a specific function, regardless of the cardinality of the domain. Since Woods (2014) proposal allows to better understand the logicality of both the ε-operator and the quantifiers $\forall$ and $\exists$, I conclude that it is a reasonable correction of the Tarski-Sher thesis.

There is a further reason to consider WI1 as a valid condition for logical constant: it captures the idea of topic neutrality which also justifies TS. I will first present the idea of topic neutrality, explaining why it justifies TS; then I will point out that the same idea is captured by the ε-operator if considered as a logical notion according to WI1. The idea that the formality of logic concern its topic neutrality can be summarized as follow : given that logic − namely the study of valid inferences − can be applied to any subjects, then the formal notions adopted by logic have to be insensitive to the features of the objects considered. The idea − which is also shared by Sher (1991) and McGee (1996) − traces back to the work of (Mostowski, 1957, p. 17) on generalized quantifiers, according to which logic "should not allow us to distinguish between different elements of [the domain]". Then it is clear why TS determines only those expressions that are topic neutral: if logic deals with formal notions that can be applied to any objects, then replacing an object with another do not affect the formal notion. Formally, this means that the meaning of logical notions is invariant over arbitrary bijections of the domain, namely TS.

Above I have explained why the ε-operator fails the TS condition. Then it is also clear why it could be argued that the ε-operator is sensitive to the identity condition and features of objects of the domain. Indeed, the ε-operator is a variable-binding operator which forms terms from open sentences. Then the ε-operator seems to work like a name, which depends on the object named. However, this is just part of the story about the ε-operator: as I have explained above, the ε-operator is an indefinite description operator. More precisely, in the case of the ε-operator, the domain does not determine a single object as its denotation, but rather a range of candidate objects of the particular semantical type. Then there are two relevant aspects about the semantics of the ε-operator: on the one hand, it is a referential expression which denotes an object of the domain; on the other hand, its denotation is indefinite, namely, it refers to an arbitrary object of the domain. That is why − in order to extend the definition of logical constant to the ε-operator − the notions of TS1 (namely functions from a domain D to members of the type-hierarchy over D) have to be replaced with the generalized notions of WI1 (which are functions from a domain D to a set of objects of the same type in the type-hierarchy over D). I want to stress that the indefiniteness of the denotation of the ε-operator − formalized as a generalized notion − meet the condition of topic neutrality. Consider $\varepsilon_x A$ that refers to a member of A: if A is not a singleton, then there is no way to specify which object is chosen; moreover, if A is empty, then the ε-operator refers to an arbitrary object of the domain. The arbitrary denotation of the ε-operator does not distinguish among the features or identity conditions of the objects of the domain. Given that the arbitrary denotation of the ε-operator formalized by WI is what we wanted the notion of invariance of TS to capture in the first place, then it is reasonable to treat WI as valid condition for logical constants.

5 Summary

In conclusion, I will sum up the main arguments of the present discussion:

1. The rule of Functional Instantiation (FI) is a conservative extension of the model theoretic semantics for quantifiers, and so it is a *desiderata* for the correct interpretation of quantifiers. Indeed, the syntactic notions of scope and nested quantification define a semantic relation of dependence among the values of the quantified variables. This dependence relation is functional, because: i) the value of a variable may vary according to the one of another variable; ii) the value of a variable is completely determined by the one of another variable. That is why FI introduces function symbols (Skolem functions) which work as arbitrary names for the dependence relations. Moreover, once interpreted, these function symbols capture the intuitive idea according to which the truth of a nested quantified formula allows to pick out an individual from the model.

2. Nonetheless, FI cannot be proved in FOL due to the metalanguage of *ZFC*, and so the interpretation of quantifiers as higher-order predicates is undermined. Given that we are considering first-order language, the semantic value of a n-ary Skolem function is a n-ary relation over the domain. However, for a Skolem function to define a relation over the domain, the conditions of existence and uniqueness must be satisfied. The uniqueness condition requires that the value of the Skolem function must be definable. Yet, the metalanguage of *ZFC* implies the existence of first-order models with no definable elements ‚Äì such as R. So, there are models where no Skolem function can be defined, thus providing a counterexample to FI. Nevertheless, the existence of the very Skolem functions which cannot be defined in first-order model is guaranteed by the Axiom of Choice (AC). That is why in SOL it is possible to prove the logical equivalence over full models of second-order formulas and their Skolem Normal Form (where the Skolem function is existentially quantified).

3. Instead, FI can be proved in EC thanks to the extensional semantics of the ε-operator which provides an interpretation for the Skolem functions introduced by FI, thus advocating for the interpretation of quantifiers as choice function. Indeed, in EC, the arbitrary individual picked out by the extensional semantics of the ε-operator allows to define the value of the Skolem function, proving that every EC model can be expanded with Skolem functions. Moreover, unlike for SOL, the EC proof of FI does not require AC. Indeed, the ε-operator is equivalent to the Axiom of Global Choice. This fact remarks the distinctions between the metalanguage of FOL − namely *ZFC* − and the one of EC − namely ZF_{EC}.

4. Rejecting the ε-operator by claiming that is covert second-order quantification over functions is a *petitio principii*, which presuppose that the ε-operator is interpreted according to *ZFC* − and not ZF_{EC}. This is remarked by the fact that if the theory of GQ is applied to EC, then the class of definable generalized quantifier is expanded. Therefore, there is no common background framework − either *ZFC* or the theory of GQ − according to which determine whether the ε-operator is a semantical higher-order quantifier or not.

5. As long as TS is a valid condition for logical expressions, so also WI is. Indeed, the classical definition of ‘notion’ would rule out the ε-operator from the set of logical constants. However, the arbitrary reference of the ε-operator is in accordance with the intuition of topic neutrality which justifies TS itself: to pick an arbitrary object means that any object is a well suited

candidate of reference. That is why the ε-operator does not discriminate among the objects of the domain. Therefore, the ε-operator has to be formalized as a generalized notion, namely a function from the domain to a set of admissible choice functions on the power set of the domain. Moreover, the definition of generalized notion allows to solve the domain-size objection to TS for the classical quantifiers $\forall$ and $\exists$. Given that WI better accounts for the meaning of both the ε-operator and the classical quantifiers $\forall$ and $\exists$, then it is a valid correction of TS.

Part II

The Proof Theoretic Characterization of Quantifiers

Investigations over the meaning of logical constants can be conducted also according to a proof theoretic point of view, by which the understanding of the inferential rules for a logical constant are sufficient to grasp also its meaning. This is the core idea of proof theoretic semantics, which will be adopted in this chapter as a methodological framework in order to highlight the notions involved by the inferential rules of the classical quantifiers $\forall$ and $\exists$. Indeed, this chapter has a twofold aim: on the one hand, I will adopt both Natural Deduction and Sequent Calculus in order to highlight two different aspects − respectively, *term-dependencies* and *arbitrariness* − which are part of the inferential rules for the classical quantifiers $\forall$ and $\exists$. On the other hand, I will explain how these notions are better represented within EC, claiming that the ε-operator is part of the proof theory related to $\forall$ and $\exists$. Therefore, the main thesis of this chapter can be roughly summarized as follow: the understanding of the inferential rules for the classical quantifiers $\forall$ and $\exists$ bears on the understanding of the inferential rules of the ε-operator. It is worth mentioning that (von Plato, 2014, p. 417) traces back the link between the provability of quantified formulas − like $\forall x\varphi(x)$ or $\exists x\varphi(x)$ − and that of arbitrary instances − such as $\varepsilon_x\varphi(x)$ − to Frege (1879), but only fully formalized by Gentzen (1969):

> "[...] because Frege explained generality in the first place by stating that each its instance is a 'fact', and just later added the 'illuminating' observation by which generality can be inferred from an arbitrary instance. It took over fifty years to arrive at a perfect understanding of the quantifiers, in the form of autonomous, purely formulated rules of inference for the universal and existential quantifiers in the work of Gentzen."

The chapter is organized as follows: in §6, I will first introduce the methodological framework of proof theoretic semantics, according to the main claims shared in the literature. In §7, I will present and discuss the Natural Deduction rules for the classical quantifiers. I will argue that the distinction between direct and indirect rules of inference for $\forall$ and $\exists$ relies on how the relation among instantial terms (called 'term dependence') is represented. Moreover, I will claim that, while (in)direct rules represent the order relation at the semantic level of, respectively, variables and subordinate proofs values, the Natural Deduction system extended with the ε-operator (NK_ε) instantiate term-dependence at the syntactic level of subordinate ε-terms.

In §8 I will introduce the SK double-lines rules for $\forall$ and $\exists$, with the structural rule of substitution, explaining how they (implicitly) define the meaning of the classical quantifiers. I will distinguish a weaker and stronger version of this claim so as to argue that, in both cases, the ε-operator is a structural constituent of the classical quantifiers.

6 Proof Theoretic Semantics

According to proof theoretic semantics, the meaning of a logical constant is defined by its inferential rules, as stated by either Natural Deduction or Sequent Calculus. For this chapter, I will assume that the reader is familiar with the systems of Natural Deduction (NK) and Sequent Calculus (SK) for Classical Logic − but check the note below. In this section I will introduce only the main features common to most accounts of proof theoretic semantics − for an overview see

Schroeder-Heister (2018). The proof theoretic semantics claim was originally advocated by Gentzen (1969), according to which, while introduction/left rules are a kind of definition of a constant, elimination/right rules are the consequences of these definitions, in the sense that their application is a kind of inversion of introduction/left rules. However, only Prawitz (1965) fully formalized this idea by interpreting the Normalization theorem for (a restricted version of) NK − which is equivalent to the Cut Elimination theorem for SK − as a meaning constituent condition for logical constant.

Clearly, proof theoretic semantics is built upon proof theory, which is a branch of mathematical logic studying the deductive aspect of formal systems, such as how a conclusion Ψ follows from the premise Φ in $\Phi \vdash \Psi$. Indeed, the main idea of proof theoretic semantics − as opposed to the model theoretic semantics introduced in §Chapter I − is to adopt the notion of proof, and not that of truth, as the main source to determine the meaning of a logical expression. That is why proof theoretic semantics is mainly motivated by epistemological considerations, as opposed to the metaphysical ones of model theoretic semantics, which has to account for the denotations of expressions. Indeed, proof theoretic semantics has been advocated by the anti-realism standpoint, most notably Dummett (1991). Finally, proof theoretic semantics is part of a broader program for the analysis of natural language semantics − called *inferentialism* − according to which the meaning of an expression is determined only by its interaction with other expressions (Brandom, 2009).

However, the following discussion diverge from the standard accounts of proof theoretic semantics in a relevant aspect. While from a proof theoretic standpoint there are important differences between how NK and SK represent derivability relations, proof theoretic semantics considers NK and SK on a par for defining the meaning of logical constants. Instead, I will claim that there are relevant differences if the proof theoretic semantics thesis is realized by either NK or SK. Indeed, especially in the case of $\forall$ and $\exists$, NK and SK offer different prospectives over their inferential rules, respectively: by allowing the discharge of assumptions, NK provides a dynamic view over quantified proofs (I will come back on this in §7.2). Indeed, in §7.1, I will adopt NK in order to explain how, in a proof containing nested quantifiers, the values satisfying the quantified predicates are chosen. Instead, in SK, through an explicit statement of the derivability relations, the structural rules characterizing a logical system are brought to surface (I will come back to this in §8). That is why, in §8.2, I will adopt SK in order to claim that the ε-operator is a meaning constituent of the quantifiers $\forall$ and $\exists$. Given that the following sections highly rely on NK and SK, I leave to, respectively, §7 and §8 for a clear explanation of how the definitions of logical constants are carried out according to these systems.

Finally, it should be remarked that proof theoretic semantics is naturally tied to the constructivist understanding of logical operators and, thus, to Intuitionistic logic. Indeed, the elimination rules of Natural Deduction are straightforwardly formulated in Intuitionistic Logic, and only translated into Classical Logic by means of additional rules for the negation ($\neg$). However, the further assumption of Intuitionistic logic cannot be made in the present context. Indeed, the Intuitionistic interpretation of $\exists$ is closely related to the (standard) interpretation of the ε-operator: as for the former a proof of $\exists x\varphi(x)$ means that an object t such that $\varphi(x)$ is true can be constructed, so for the latter $\varepsilon_x\varphi(x)$ provides with an arbitrary instance t such that $t \in A$. That is why, I will stick to Classical logic so as to argue that the (classical) interpretation of $\exists$ relies on the (constructivist) interpretation of the ε-operator [17].

[17] However, the Natural Deduction and Sequent Calculus rules for Intuitionistic EC have been extensively studied in the literature, see Mints (2012) for a survey.

7 Natural Deduction and Term-dependence

NK is composed by introduction and elimination rules for each logical constant. In the next section I will introduce the NK rules for the classical quantifiers ∀ and ∃, distinguishing between the direct and indirect case. In this section, I will first spell out in more details the proof theoretic semantics thesis as applied to NK and then explain how I will adopt this claim in order to compare ∀, ∃ and the ε-operator. Indeed, the claim that NK rules for classical quantifiers define the meaning of ∀ and ∃ can be formulated as follows. Given that Natural Deduction rules attempt to mirror the human logical reasoning, then understanding the rules for classical quantifiers is a sufficient condition to grasp also the meaning of ∀ and ∃. This idea has been proposed in order to account for the universal applicability of logic: anyone who is capable of articulate thought or reasoning at all should be able to understand these inference rules, and should therefore be in a position to grasp the meaning of the logical constant (Peacocke, 1988).

In §X, I will adopt this claim in order to argue that the understanding of the inferential rules for the classical quantifiers ∀ and ∃ bears on the understanding of the inferential rule for the ε-operator. More precisely, I will first argue that both direct and indirect quantifiers rules represent the order relation – called *term-dependecies* – among the instantial terms introduced within an inference. Then, I will point out that the same order relation is instantieded in NK_ε at the syntactic level of nested ε-terms. I will conclude that, if understanding the inferential rules for logical constants is sufficient to grasp also their meaning, and if the relation of terms-dependence is instatieted by both NK_ε and NK (in)direct rules, then the notion of choice process is part of the meaning of both the classical quantifiers ∀, ∃ and the ε-operator.

7.1 Direct and Indirect Rules for Quantifiers

In this section, I will first introduce the NK rules for the quantifiers ∀ and ∃, distinguishing between the direct and indirect case. Then, I will argue that this distinction relies on a logical matter which affect the soundness proof of NK systems, rather than a notational issue. Finally, I will sum up the main accounts that have been proposed in the literature to justify either direct or indirect rules. While the introduction and elimination rules of the other logical connectives follow straightforwardly from their intended meaning, the rules of ∀ and ∃ are more tricky. As remarked by (Anellis, 1991, p.145):

> "It is clear that the difficulties involved in developing a set of quantification rules for systems of natural deduction which are at once both simple, i.e. 'natural', and sound required a significant effort of roughly half a century, by many of the better logicians and textbook writers."

Indeed, systems of NK are of two kinds: those who adopt subordinate proofs for the rules of existential elimination and universal introduction, and those who do not[18]. The former is the NK system introduced by Gentzen (1969). The distinctive feature of the system, as opposed to the one considered below, is that there is no rule for inferring an instance of an existential statement from the existential statement itself. Instead, the rule ∃E allows the inference from the existential statement only when the subordinate proof from an instance (what is usually called a 'typical disjunct') is available. I will refer to these as *indirect* NK rules:

[18]More precisely, there are hybrid NK systems which adopts subordinate proof for the universal introduction rule only, as for Copi (1954).

$\vdots$	$\vdots$	
1	$a \mid \varphi(x/a)$	
$\vdots$	$\mid \vdots$	
n	$\mid \varphi(a)$	
$n+1$	$\forall x\varphi(x)$	$\forall$I, 1, n

Where the variable a does not occur outside the subproof where it is introduced.

1	$\exists x\varphi(x)$	
2	$a \mid \varphi(x/a)$	
$\vdots$	$\mid \vdots$	
n	$\mid \Psi$	
$n+1$	Ψ	$\exists$E, 1, 2, n

Where the variable a does not occur outside the subproof where it is introduced.

Instead, the NK system of Quine (1950) permits to infer an arbitrary instance directly from the existential statement $-$ and the other way around for the introduction of the existential quantifiers. I will refer to these as *direct* NK rules:

1	$\varphi(a)$	
2	$\forall x\varphi(x)$	UG, 1

Where the variable $a \in V$.

1	$\exists x\varphi(x)$	
2	$\varphi(a)$	EI, 1

Where the variable $a \in V$.

However, direct rules have further restrictions. Indeed, $a \in Var$ *if and only if:*

- *Flagging:* No variables is adopted twice, namely for two applications of EI and UG.
- *Ordering:* The variable of each step must be alphabetically later than all free ones in the formula.
- *Local Restriction:* The variable a does not occur in $\varphi(x)$ in any applications of the rules EI and UG.

Therefore, both Gentzen (1969) and Quine (1950) adopt the syntactic category of variables for, respectively $\forall$I, $\exists$E and UG, EI (Pelletier, 1999). However, they diverge on how to account the arbitrariness of the variable a: while according to *direct* rules the variables have to be alphabetically ordered, for *indirect* rules the variable can occur in the subordinate proof only. In order to avoid confusion, I will refer to these variables as 'instantial terms', leaving the context to clarify if I am referring to *direct* or *indirect* rules (or both).

From a philosophical standpoint, the use of subordinate proofs has been justified according to the role played by supposition for logical reasoning. For example, the supposition of φ itself is taken to be equivalent to the assertion of $\varphi \to \varphi$. However, there is an important logical difference between rules adopting subordinate proofs and those who do not, as remarked by Prawitz (1965) distinction between *proper* and *improper* rules of inference. *Proper* inferences are ones that are valid in the straightforward manner: if the premises are true then the conclusion must also be true. Instead, *improper* inferences are valid if the conclusion has been inferred in a specific way. Take for example $\to I$, which allows to conclude $\varphi \to \psi$ in the case in which ψ has been inferred from φ. It is not possible to regard this inference as proper, since in the transition to $\varphi \to \psi$ the assumption φ has been discharged. Thus the conclusion has not been directly inferred from the assumptions. Rather, the conclusion $\varphi \to \psi$ is inferred from the fact that ψ follows from φ. Prawitz

(1965), following Gentzen (1969), considers as improper rules of inference all of those that imply discharging of assumptions, such as $\rightarrow$I, $\vee$E, $\exists$E and $\forall$I.

Yet, on the assumption of the $\rightarrow$I, the use of subordinate proof for the propositional rules of $\vee$E pins down to an aesthetic issue (Pelletier, 1999). Indeed, $\vee$E can be reformulated with the separation of cases, namely from $\varphi \vee \psi$ and $\varphi \rightarrow \theta$ and $\psi \rightarrow \theta$, conclude θ, which does not adopt subordinate proof. Instead, the difference between *direct* and *indirect* rules relies on a logical issue, namely whether to require that each line of a proof is a semantic consequence of all the assumptions active at that inference point. Consider NK systems with the EI rule: even if $\exists x \varphi(x)$ were semantically implied by whatever active assumptions there are, it is not true that $\varphi(a)$ will be implied by those same assumptions, since the the restriction on EI requires the variable a to be fresh, namely not already introduced in the inference. This obviously affects the soundness proof for NK systems with direct rules. Quine (1950) strategy is to prove the soundness of the orthodox system − namely, Gentzen (1969) NK system with indirect rules − and then showing how soundness for the orthodox system transfers to soundness on the given system. Instead, in NK systems with the $\exists E$, the subordinate proof with the instantial term becomes a further assumption and the restriction on the variables that can occur in Ψ guarantee that this formulas does not semantically depend upon the instantial term. Yet, NK systems with indirect rules has to face a different issue: if the instantial terms adopted in the subordinate proofs are variables, as explained above, then it must be explained how to semantically evaluate them.

Therefore, the opposition between direct and indirect rules is not a notational matter − i.e. which set of rules best represents ordinary reasoning − but it is instead a logical issue that, on the further assumption of proof theoretic semantics of §7, concerns the meaning of the quantifiers $\forall$ and $\exists$. Following Fine (1985), the dichotomy between *direct* and *indirect* rules of inference can be divided into two sub-questions[19]: the first one is a *descriptive* one. It is the question of what we are actually doing when we adopt (in)direct rules of inference. The second is a *normative* one. It is the question of what justifies us in doing whatever it is we are doing when we adopt (in)direct rules of inference.

I will adopt this difference to briefly summarize the literature on the debate between direct and indirect rules. Concerning direct rules, Fine (1985) theory of arbitrary objects introduces a formal semantics for FOL based on an expansion of the domain with a set of arbitrary objects, ordered by a relation of dependence, according to which the proof of soundness for NK systems with EI and UG rules follows straightforwardly. Indeed, the main idea of Fine (1985) theory is that the instantial terms adopted by direct rules are actually names which refer to a specific category of objects, namely arbitrary objects. Therefore, according to Fine (1985), the restrictions on EI and UG are justified by how the dependence relation among arbitrary objects is characterized. Concerning indirect rules, there are different accounts for each questions. DeVidi and Korté (2014), following the work of King (1991), reformulates the rules of $\exists$E and $\forall$I so as that the instantial variables adopted in the subordinate proofs are implicitly bound by either an existential or universal quantifier. Therefore, DeVidi and Korté (2014) and King (1991) answer the descriptive question concerning $\exists$E and $\forall$I by stressing the semantic and syntactic difference between singular terms and expressions of generality. Instead, Martino (2001) and Breckenridge and Magidor (2012) introduce the notion of arbitrary reference, according to which a term can refer to a specific object even if in principle it is impossible to single out which object is denoted. Then, by considering the variables in $\exists$E and $\forall$I as actual names, the subordinate proofs are justified by the epistemic ignorance involved by the notion of arbitrary reference.

[19] It should be remarked that Fine (1985) applies the distinction between descriptive and normative questions to direct rules only.

7.2 Term-dependence and the ε-operator

Instead, I will claim that what is at issue in the comparison between EI, UG and ∃E, ∀I is the representation of the order-relation among the instantial terms introduced in a quantified proof, called *terms-dependence* – which has been discussed only by Meyer Viol (1995). The common ground of term-dependence justifies the adoption of NK with either direct or indirect rules, thus answering Fine (1985) normative question. Moreover, I will argue that, while direct rules *explicitly* represent term-dependences through indices on the (instantiated) variables, indirect rules *implicitly* represent term-dependences as subordinate proofs. The different representations of term-dependence explain the restrictions on both direct and indirect rules, thus answering Fine (1985) descriptive question. Finally, I will consider NK_ε, namely NK expanded with the ε-operator. Unlike for NK with either direct or indirect rules, I will show that in NK_ε the relation of term-dependence is represented at the syntactic level of nested ε-terms. On the assumption that the Natural Deduction rules mirror the effective logical reasoning, then the representation of the choice process is part of the understanding of quantified proofs.

The first hypothesis is that both the direct and indirect quantifiers rules represent the choice process of instantial terms adopted in the inference, called *term-dependence*. A choice process is composed by two parts: the choices of the instantial terms and the order of such choices. The choices of instantial terms can be better understood if the NK rules are interpreted as conditions for the assertion and denial of statement containing logical constants. The rough idea is that, if understanding the NK rules for a logical constant, like $\wedge$, is sufficient to grasp also its meaning, then asserting Φ, asserting Ψ and denying $\Phi \wedge \Psi$ involves a clash. Most approaches to proof-theoretic semantics consider introduction rules as governing assertions and elimination rules as governing denials – even if no relevant assumptions hang on this. Then, consider the rules ∃E/EI and ∀I/UG discussed so far. On the one hand, asserting $\forall x \varphi(x)$ would clash if and only if $\varphi(a)$ is denied – for an instantial variable a. On the other hand, denying $\exists x \varphi(x)$ would clash if and only if $\varphi(a)$ is asserted – for an instantial variable a. Obviously, the instantial variable a must be selected according to the restrictions on either direct or indirect rules. That is why, choosing the adequate instantial variables is a key component for the applications of ∃E/EI and ∀I/UG, interpreted as governing the assertions and denials of, respectively, universal and existential sentences.

For what concerns the order of the choice process, it could be useful to compare term-dependences with the functional-dependences discussed in §2: while the latter are defined by nested quantified formulas like $\forall x \exists y \varphi(x, y)$, the former arise also with quantifiers combination such as $\exists x \exists y \varphi(x, y)$, where the value adopted to witness the first existential quantifier occurs in the condition of the subsequent choice. It should be remarked that this latter dependence relation cannot be accounted in standard model theoretic semantics, given that the quantifier prefix $\exists x \exists y$ is equivalent to the $\exists y \exists x$ one[20]. That is why I will adopt the $\exists x \exists y$ example in order to compare and evaluate how $\exists E$ and EI represent term-dependences:

1	$\exists x \exists y \varphi(x, y)$	
2	$\exists y \varphi(a, y)$	EI for a, 1
3	$\varphi(a, b)$	EI for b, 2
4	$\exists x \varphi(x, b)$	∃I, 3
5	$\exists y \exists x \varphi(x, y)$	∃I, 4

Instead, in NK systems with indirect rules, the permutation of existential quantifiers is proved

[20]However, from the model theoretic standpoint, the relation between quantification and choice process has been deeply investigated since the works on Game-Theoretical Semantics, see Hintikka (1996). Instead – as far as I know – the relation between the two has been poorly investigated from the proof theoretic point of view, with the exceptions of Meyer Viol (1995) and Alechina and Van Lambalgen (1996).

as follows:

1	$\exists x \exists y \varphi(x, y)$	
2	a $\exists y \varphi(a, y)$	
3	b $\varphi(a, b)$	
4	$\exists x \varphi(x, b)$	∃I, 3
5	$\exists y \exists x \varphi(x, y)$	∃I, 4
6	$\exists y \exists x \varphi(x, y)$	∃E, 2, 3–5
7	$\exists y \exists x \varphi(x, y)$	∃E, 1, 2–6

It should be now clear why NK is the most suitable framework according to which studying the dependence relations as arising within proofs. Indeed, NK has two distinctive features: i) it allows to discharge assumptions previously made; ii) it represents only the active assumptions for each inferential step. While feature (i) indicates the inferential step at which the choice of the instantial term is made, feature (ii) help to keep track of the order among those choices. That is why NK offers a *dynamic* framework for studying logical inferences, as mentioned in §6. Instead, both (i) and (ii) do not hold for SK: the choice of instantial terms is represented only by the substitution boxes − more on this in §8.1 − and the order relation is lost given that SK shows all the open assumptions at each inferential line. However, based on feature (i), someone could object that in NK dependence relations do not arise only with the instantial terms of quantifiers rules, but also from the formulas adopted as assumptions, as for the rules of →I and ∨E. However, Meyer Viol (1995) has remarked that in Classical Logic, but not in Intuitionistic Logic, the order of the discharged assumptions is inessential. Given the present focus on the rules of inference of Classical Logic, I conclude that term-dependence concerns only the quantifiers rules.

Finally, someone could wonder whether and how the NK rules for ∀ affects the choice process. On the one hand, the ∀E rule is obviously committed to the choice of a specific value which satisfies the quantified predicate. However, given that the predicate is universally quantified, the choice of the term is free from dependence relations, given that any values of it would satisfies the predicate. Indeed, also the restrictions on ∀I and UG guarantee that the value chosen to infer the universal formula does not depend on any previous choices, asserting that the instantial term is fresh for the deduction.

I will now further characterize the choice process sketched above. First of all, I will prove that the relations of term-dependence instantiated by direct and indirect rules are equivalent to a strict-partial order. In the case of direct rules, this can be easily checked given that Quine (1950) *Ordering* restriction introduced in §7.1 imposes the alphabetic order over the set Var of instantial terms. I will refer to the order relation of direct rules as '$\prec$'. Instead, for indirect rules, further definitions are required in order to recover the order relation among subordinate proofs. Following Meyer Viol (1995) notation:

Def.5 The suppositions introduced by ∃E and ∀I are identified by the quadruple: $\langle \epsilon, x, \varphi, a \rangle$. Where:

- ϵ is the kind of rule applied. While the ϵ-operator is adopted for $\exists E$, the τ-operator is adopted for $\forall I$.
- φ is the condition the (value of) istantial term has to satisfy which respect to the variable x in φ.
- x is the variable of which all occurrences in φ must be replaced by the istantial term.
- a is the instantial term chosen for the application of either $\exists E$ or $\forall I$.

Let the choice tuple $\langle \epsilon_x \varphi(x), a \rangle$ be short-hand for $\langle \epsilon, x, \varphi, a \rangle$. Then, it is possible to define an order relation among the suppositions introduced by ∃E and ∀I. It should be remarked that this order is displayed in the Fitch (1952) style proofs by the subsequent supposition's bars and their scope. More precisely:

Def.6 We say that the choice tuple $\langle \epsilon_x \varphi(x), a \rangle$ immediately depends on the tuple $\langle \epsilon_x \psi(x), b \rangle$, notation $\langle \epsilon_x \varphi(x), a \rangle \ll \langle \epsilon_x \psi(x), b \rangle$, if b occurs in φ. We say that choice tuple $\langle \epsilon_x \varphi(x), a \rangle$ depends on tuple $\langle \epsilon_x \psi(x), b \rangle$, notation $\langle \epsilon_x \varphi(x), a \rangle < \langle \epsilon_x \psi(x), b \rangle$, if there is a finite sequence of immediate dependence steps connecting $\langle \epsilon_x \varphi(x), a \rangle$ to $\langle \epsilon_x \psi(x), b \rangle$.

So the relation of dependence $<$ is the ancestral relation of the relation of immediate dependence $\ll$. Then, based on the restrictions on EI, UG and ∃E, ∀I, their respective relations $\prec$ and $<$ are equivalent to a strict partial order.

Th.8 The relations $\prec$ and $<$ are strict partial orders.

Proof. Consider first $\prec$. Let $a, b, c \in Q$, where Q is the set of variables introduced by applications of either EI or UG. Then, based on the restrictions of EI and UG: i) Given *Ordering*, if $a \prec b$ and $b \prec c$, then $a \prec c$. So, $\prec$ is transitive. ii) Given *Ordering*, pseudo-variables are adopted in their alphabetic order, so if $a \prec b$, then for no b, $b \prec a$. So, $\prec$ is antysimmetric. iii) Given *Flagging*, for no a, $a \prec a$. So, $\prec$ is irreflixive. Therefore, $\prec$ is a strict partial order.

Consider then $<$. Let $\langle \epsilon_x \varphi(x), a \rangle$, $\langle \epsilon_y \psi(y), b \rangle$ and $\langle \epsilon_z \delta(z), c \rangle$ the suppositions introduced by ∃E applications (the proof can be easily extended to ∀I). Then, based on the restrictions on ∃E and Def.6: i) If $\langle \epsilon_x \varphi(x), a \rangle \ll \langle \epsilon_y \psi(y), b \rangle$ and $\langle \epsilon_y \psi(y), b \rangle \ll \langle \epsilon_z \delta(z), c \rangle$, then $\langle \epsilon_x \varphi(x), a \rangle < \langle \epsilon_z \delta(z), c \rangle$ ($<$ is the ancestral relation of $\ll$). So, $<$ is transitive. ii) Given the restriction on ∃E, no instantial term can be adopted twice, then for no a, $\langle \epsilon_x \varphi(x), a \rangle < \langle \epsilon_x \varphi(x), a \rangle$. So, $<$ is irreflixive. iii) Given the restriction on ∃E, no instantial term can occur outside the scope of a supposition, then, if $\langle \epsilon_x \varphi(x), a \rangle < \langle \epsilon_y \psi(y), b \rangle$ and $a \neq b$, then $\neg(\langle \epsilon_y \psi(y), b \rangle < \langle \epsilon_x \varphi(x), a \rangle)$. So, $<$ is antysimmetric. Therefore, $<$ is a strict partial order. □

Therefore, both *direct* and *indirect* rules instantiate the same order relation, called *term-dependence.* Moreover, the antisymmetric condition, which is met by the restrictions on both direct and indirect rules, guarantees that the instantial term introduced is fresh, namely it does not occur in previous inferential step − this condition will be further discussed in §8.2.

However, it should be remarked that term-dependence is represented at the semantic level by both direct and indirect rules. More precisely, term-dependence is defined *explicitly* for EI, UG on the values of the variables and *implicitly* for ∃E, ∀I on the validity of subordinate proofs. However, I am here stretching the explicit-implicit distinction beyond the standard theory of definitions. What I mean by that is as follows: while each inferential step in a deduction correspond to an assertion, subordinate proofs correspond to hypotheses. Therefore, while for direct rules term-dependence is expressed by the assertive force of inferential steps, for indirect rules term-dependence is represented by the hypothetical force of subordinate proofs. However, in both cases, term-dependence is formalized at the semantic level of, respectively, the values of the variables and the validity of subordinate proofs. This point should be further explained because proof theory is generally considered as free from notions like semantic values. Indeed, I have explained above that quantifiers rules − as stated by either direct or indirect form − comes with restrictions that instantiate the same order relation, called term-dependence. That is why I will first consider the restrictions on direct rules, explaining why they explicitly express term-dependence at the semantic level of the values of instantial variables. This follows straightforward from the fact that Quine (1950) *Ordering* restriction could be reformulated as:

- *Numbering:* It is possible to order the distinct instantial variables in such a way $a_1, a_2, ..., a_n$ that a_j does not immediately depend upon a_i for $1 \leqslant i \leqslant j \leqslant n$.

The two conditions are equivalent, but what the latter highlights is that the restrictions on direct rules are made to order the values of the instantial variables, rather than the instantial variables themself. In this sense, direct rules explicitly express term-dependence at the semantic level.

For what concerns indirect rules, I have explained above that the term-dependence relation is instantieted by subordinate proofs, which are individuated as, for example, $\langle \epsilon_x \varphi(x), a \rangle$. Then it follows quite clearly that the restrictions on indirect rules implicitly represent term-dependence based on the validity of subordinate proofs. Indeed, subordinate proofs are hypothetical: in the case of ∃E and ∀I, they assume that the instantial variable has been deducted in the correct way stated by the restrictions, so as to infer the conclusion. In this sense indirect rules implicitly represent the term-dependence relation. But if so, then it is also clear why indirect rules represent term-dependence at the semantic level: only if the subordinate proof $\langle \epsilon_x \varphi(x), a \rangle$ is valid, then it counts as a step in the term-dependence relation. Therefore, both direct an indirect rules define term-dependence at the semantic level. However, this conclusion is quite at odd with the syntactic character prompted by the proof theoretic semantics thesis of NK. That is why, I will consider below whether the ε-operator provides a better result.

Indeed, as explained above, the choice process formalized by the relation of *terms-dependence* is mainly related to the NK rules for the existential quantifier − as stated by either ∃E or EI. Given the definitional equivalence between the ε-opertaor and the ∃ quantifier, it is reasonable to question whether the inferential rule for the ε-operator instantiate the same order relation. That is why, I will consider the system NK_ε, which is obtained by NK by replacing the ∃E rule with:

1	$\exists x \varphi(x)$	
2	$\varphi(\varepsilon_x \varphi(x))$	EI_ε, 1

Moreover, NK_ε adopts the UG rule with the following restriction: *the instantial term 'a' must not occur in the conclusion of that inference or in any assumption on which the conclusion depends.* Based on this, I will argue that in NK_ε, the relation of *terms-dependence* is instantieted at the syntactic level of subordinate ε-terms. Indeed, the ε-operator is a term forming operator which names the instantial variables adopted by the quantifiers rules. Therefore, the order of the ε-terms in a formula represents the order of the subsequent choices. Consider the aforementioned example about the permutation of existential quantifiers:

1	$\exists x \exists y \varphi(x, y)$	
2	$\exists y \varphi(\varepsilon_x \exists y \varphi(x, y), y)$	EI_ε, 1
3	$\varphi(\varepsilon_x \exists y \varphi(x, y), \varepsilon_y \varphi(\varepsilon_x \exists y \varphi(x, y), y))$	EI_ε, 2
4	$\exists x \varphi(x, \varepsilon_y \varphi(\varepsilon_x \exists y \varphi(x, y), y))$	∃I, 3
5	$\exists y \exists x \varphi(x, y)$	∃I, 4

Given the equivalence between *direct* and *indirect* rules proved above, I will support my claim by showing how the restrictions on *direct* rules (which concern the semantic values of variables) can be formulated in NK_ε at the syntactic level of formulas. The rough idea is to consider the instantial variables of direct rules as short-hand notations for ε-terms, namely: $a \prec Var = \varepsilon_x \varphi(x)$. Then, Quine (1950) restrictions on direct rules can be formulated in NK_ε as:

- $(\textit{Flagging})_{NK_\varepsilon}$: One and the same ε-term should not compare in two different applications of both EI_ε and UG. This is guaranteed by the restriction of UG in NK_ε mentioned above.
- $(\textit{Local Restriction})_{NK_\varepsilon}$: in EI_ε, the ε-term in the conclusion should replace all the occurrences of the bound quantified variable in the premise. This is guaranteed by the rule for renaiming bound variables, which in NK_ε is formulated as:

⋮	⋮
n	$(\varepsilon_x \varphi(x)) = (\varepsilon_y \varphi(y/x))$

- $(Ordering)_{NK_\varepsilon}$: The ε-terms adopted by EI_ε should be ordered according to a strict partial order. This is guaranteed by the antysymmetry of the 'proper subterm of' relation among ε-terms. For example, $b \prec a$ is formalized in NK_ε at the syntactic level of the bare formula as:

$$\underbrace{\varepsilon_y \varphi(\underbrace{\varepsilon_x \exists y \varphi(x, y)}_{\text{a}}, y))}_{\text{b}}$$

Th.8 proved that the relation $\prec$ characterized by the restrictions on direct rules is a strict partial order. Given that the same restrictions can be formalized in NK_ε at the syntactic level of nested ε-terms, thus also EI_ε (and UG) instantiate the relation of *terms-dependence.* It should be remarked that the NK_ε formulation of the restrictions on *direct* rules has been independently proposed by Wang (1955) and Hazen (1987). However, both authors advocate the NK_ε formulation for pedagogical reasons, such as the better perspicuity compared to Quine (1950) formulation. Instead, I have argued that NK_ε instantiates at the syntactic level the same order relation (terms-dependence) which is instead represented at the semantic level by both direct and indirect rules in NK. Therefore, if understanding the inferential rules for logical constants is sufficient to grasp also their meaning, and if the relation of terms-dependence is instatieted by both NK_ε and NK (in)direct rules, then the notion of choice process is part of the meaning of both the classical quantifiers $\forall$, $\exists$ and the ε-operator. A similar point is made by Tennant (1980), who proves that the NK rules of $\forall$ and $\exists$ can be derived from the ε-operator ones. I take this as further evidence supporting the claim the understanding of the inferential rules for the classical quantifiers $\forall$ and $\exists$ bears on the understanding of the inferential rule of the ε-operator.

8 Sequent Calculus and Fresh-variables

Like NK, SK distinguishes between Right and Left rules for each logical constant. That is why, the SK Left and Right rules correspond to, respectively, the NK rules of Introduction and Elimination. Unlike NK which shows only active assumptions, SK provides a notation to keep track of the open assumptions at any inferential step. That is why SK can be seen as a formal representation of the derivability relation in NK. Based on this latter feature, SK has been adopted to study the derivabality relations characterizing logical systems such as Classical, Intuitionistic and Relevant logics (among others). This is the program of Substructural logics, which defines each of these logical systems based on the set of *structural rules* they adopt. The idea was first introduced by (Došen, 1989, p. 362), according to which:

> "The approach starts with the assumption that logic is the science of formal deductions, and that basic formal deductions are structural deductions, i.e., deductions independent of any constant of the language to which the premises and conclusions belong."

Below I will introduce the structural rules for Classical logic, which allows for multi-conclusions sequents − as presented by Restall (2019). Moreover, structural rules are distinguished from *operational rules*, which comprehends the SK Left and Right rules aforementioned.

It is now possible to appreciate how the proof theoretic semantics thesis presented in §X is formulated within the framework of SK. Indeed, (Došen, 1989, p. 362) goes on stating that: "Logical constants, on which the remaining formal deductions are dependent, may be said to serve as 'punctuation marks' for some structural features of deductions." Consider for example the Intuitionistic implication, which can be obtained from classical implication by abolishing Contraction on the Right (the KR rule mentioned below), which entails only single-conclusion sequent. This structural description of the Intuitionistic implication can be interpreted as the requirement of constructive proofs in Intuitionistic Logic. In this sense, the structural characterization of logical constants highlights the structural rules adopted by the logical system to which they belong. I will refer to this as the structural characterization of logical constants.

According to the structural characterization, the SK Right and Left rules are derived from more primitive rules, called *double-line rules*. Even if Došen (1989) understands his double-line rules as a criterion for demarcating logical constants, Sambin et al. (2000) makes a further step asserting that double-line rules implicitly define the meaning of logical constants. The core idea of Sambin et al. (2000) proposal can be summarized as follows. Double-lines rules differ from standard sequent rules regarding two points: i) double-lines rules allow for both a top-down and bottom-up readings; ii) double-lines characterize the logical constant on either the left or right side of the turnstile. The other side of the sequent rules can be retrieved from the double-lines rule through the application of structural and Cut rules - as explained below. Therefore, double-line rules implicitly define the meaning of logical constants it in the sense that any other operation satisfying the same rule is provably equivalent to it. Indeed, Schroeder-Heister (2018) comments:

> "Sambin et al. (2000), in their Basic Logic, explicitly understand what Dosen calls double-line rules as fundamental meaning giving rules. [...] So Sambin et al. (2000) use the same starting point as Dosen, but interpret it not as a structural description of the behaviour of constants, but semantically as their implicit definition."

8.1 Double-lines and Structural Rules

In this section, I will first introduce the double-line rules for the quantifiers ∀ and ∃ and then state the structural rules characterizing Classical logic − based on the system of Restall (2019). The double-line rules for the classical quantifiers correspond to ∀R and ∃L, as:

$$\frac{\Gamma \vdash \varphi(y), \Delta}{\Gamma \vdash \forall(x)\varphi(x), \Delta}\ (\forall Def) \qquad \frac{\Gamma, \varphi(y) \vdash \Delta}{\Gamma, \exists(x)\varphi(x) \vdash \Delta}\ (\exists Def)$$

Where the variable y does not occur in the bottom sequent of both rules.

The restriction on the quantifiers double-line rules is called the *eigenvariables condition* and in the next paragraph I will explain why is a key notion to understand the structural characterization of ∀ and ∃. According to Restall (2019), the structural rules for Classical logic are respectively *Identity*, *Weakening, Contraction* and *Cut*[21]:

$$\Delta \vdash \Delta \ \text{(ID)}$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, \Phi \vdash \Delta}\ \text{(KL)} \quad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Phi, \Delta}\ \text{(KR)}$$

$$\frac{\Gamma, \Phi, \Phi \vdash \Delta}{\Gamma, \Phi \vdash \Delta}\ \text{(WL)} \quad \frac{\Gamma \vdash \Phi, \Phi, \Delta}{\Gamma \vdash \Phi, \Delta}\ \text{(WR)}$$

$$\frac{\Gamma \vdash \Phi, \Delta \qquad \Gamma, \Phi \vdash \Delta}{\Gamma \vdash \Delta}\ \text{(Cut)}$$

Based on these structural rules, it is possible to infer from ∀*Def* and ∃*Def*, respectively, the ∀L and ∃R rules as follows: first the quantifiers ∀ or ∃ is iterated by the application of (ID), then after applying the other structural rules, the inferential equation is resolved by applying (Cut) − for the proof see Restall (2019). However, there is a further structural rule which is strictly connected to

[21] Following the literature, I refer to Weakening as K and Contraction as W.

the ones of $\forall$ and $\exists$, namely the rule of Substitution. The rule of Substitution is structural because i) no other logical constants occur in it and, ii) is a global rule which modify the entire sequent, rather than local rules that introduce formulas. In the literature, the structural rules of substitution is implemented in the double-lines rules of $\forall$ and $\exists$ with the addition of substitution boxes. However, given that in the next paragraph I will stress how the structural rule of Substitution affects the binding operation of $\forall$ and $\exists$, I opt for stating the rule explicitly, as:

$$\frac{\Gamma \vdash \Delta}{\Gamma(t/x) \vdash \Delta(t/x)} \text{(Sub.L)} \quad \frac{\Gamma(t/x) \vdash \Delta(t/x)}{\Gamma \vdash \Delta} \text{(Sub.R)}$$

8.2 Arbitrariness and the ε-operator

In this section, I will claim that the structural characterization of $\forall$ and $\exists$ relies on the notion of arbitrariness as displayed by the ε-operator. In order to do so, I will adopt the structural characterization of logical constants in order to investigate the meaning of the classical quantifiers $\forall$ and $\exists$.

More precisely, I will first explain how the quantifiers $\forall$ and $\exists$ can be analysed based on the structural rule of substitution. However, I will stress that the Substitution rule must be restricted in the case of $\forall$R and $\exists$L, as made by the eigenvariables condition. The eigenvariables condition expresses the notion of arbitrariness as involved in $\forall Def$ and $\exists Def$. Then, I will claim that due to the eigenvariables condition, $\forall$Def and $\exists$Def plus the structural rule of Substitution are not sufficient to implicitly define the meaning of the classical quantifiers. Finally, I will prove that if the ε-operator is added to $\forall Def$, $\exists Def$ and the structural rule of substitution, then the symmetry between right and left sequent rules is restored. Therefore, I will conclude claiming that reasoning about arbitrary instances − as realized through ε-terms − is part of the proof theoretical characterization of the classical quantifiers $\forall$ and $\exists$.

The quantifiers $\forall$ and $\exists$ are variable-binding operators, which form sentences from open formulas. I will first explain how the binding process is carried out through the structural rule of substitution. Following the setting of §X, I assume that formulas might contain occurrences of free variables. Therefore, as remarked above, variables are considered as terms, together with constants and functional terms. [22]. If Φ does not contain any free variable x, then $\Phi(t/x) = \Phi$. Adopting such a wide class of terms can be appreciated by considering:

$$\frac{\Gamma, \varphi(t) \vdash \Delta}{\Gamma, \forall x \varphi(x) \vdash \Delta} (\forall \text{L})$$

Indeed, from a universal sentence, both singular and functional terms satisfying the predicate can be inferred. However, when a term t is substituted for a variable x in a formula Φ, it is important that no variables of t 'get caught' by the quantifiers in Φ. In order to avoid so, the principle for renaming bound variables (α-conversion) is defined, based on the structural rule of Substitution (Negri et al., 2008):

Def.7 **(α-conversion)** We say that a term t is free for x in Φ if no variable of t becomes bound as an effect of the substitution of t for x in Φ. This condition can always be met by appropriate renaming of bound variables in the formula Φ: If Φ is,for example, $\forall x \varphi(x)$ and y is a variable not occurring in Φ, α-conversion guarantees that we can identify Φ with $\forall y \varphi(y/x)$.

The α-conversion definition guarantees an important fact about the syntax of quantified formulas, namely that no variable can occur both bound and free in the same formula. Therefore,

[22]It should be remarked that this assumption is not necessary for the following argument. Restricting formulas to contain only bound variables would require to distinguish a further substitution rule which governs the substitution of constants and functions terms − as for Restall (2019).

the structural rule of Substitution prescribe the (syntactical) behaviour of quantifiers as binding devices[23]. However, not all the SK rules for quantifiers allow for unrestricted substitutions. Indeed, this is not the case for ∀R and ∃L where a proof of either $\forall x\varphi(x)$ or $\exists x\varphi(x)$ consists of a proof of $\varphi(y/x)$ for an arbitrary y. In SK, for a variable to be arbitrary means that nothing else apart its range of values is assumed. This point was remarked by Došen (1989) − one of the first advocates of the structural characterization of logical constants:

> "Since the rule of substitution for variables permits us to read the x which may occur free in Φ as 'any', according to [$\forall Def$] and [$\exists Def$], the two quantifiers express something about the place of 'any' in deductions. If 'any' is in a conclusion, and nowhere else, it becomes 'every', and if it is in a premise, and nowhere else, it becomes 'some'. So the logical form of $\forall x\Phi$ and $\exists x\Phi$ mirrors a structural feature of deductions, viz. the presence of a variable in a conclusion Φ, or in a premise Φ, a variable which doesn't occur free anywhere else in the deduction."

The notion of arbitrariness is represented by the eigenvariables condition, which guarantees that the variable introduced is fresh, namely not already adopted in previous inferential steps. This should not be surprising: clearly while ∀R corresponds to UG/∀I, ∃L corresponds to EI/∃E. So, both in SK and NK, these rules come with restrictions on their application. However, it is worth mentioning a key difference between the two cases: while NK allows a 'semantic ascent' over the inferential rules in the way that allows for discharged assumptions, SK is fully syntactical in the way that the derivability relation is locally represented by sequents. Yet, the eigenvariables restriction on $\forall Def$ and $\exists Def$ poses a challenge to the structural characterization of logical constant. Indeed, in the case of ∀ and ∃, the double-lines rules and the rule of Substitution are not sufficient for the implicit definition of the classical quantifiers. This could be proved by *reductio ad absurdum*: if $\forall Def$ and $\exists Def$ (plus the rule of substitution) were sufficient for the implicit definition of ∀ and ∃, then ∀L and ∃R would work as well as double-lines rules. However, it is not possible to infer the eigernvariables condition from these latter rules, which would entails an invalid derivation from ∀R and ∃L. So, the eigenvariables condition breaks the symmetry between the Right and Left sequent rules of ∀ and ∃.

Instead, I propose to adopt the resources of the ε-operator in order to formalize the eigenvaribales condition. This proposal is intuitively supported by the interpretation of $\varepsilon_x\varphi(x)$ as 'an arbitrary x such that $\varphi(x)$, if any'. Indeed, I will show that if the $\forall Def$ and $\exists Def$ rules are formulated with the ε-operator, then the eigenvariables condition is not required as a further restriction.

$$\frac{\Gamma \vdash \Delta, \varphi(\varepsilon_x \neg\varphi(x))}{\Gamma \vdash \Delta, \forall x\varphi(x)} \ (\forall Def_\varepsilon) \qquad \frac{\Gamma, \varphi(\varepsilon_x \varphi(x)) \vdash \Delta}{\Gamma \exists x\varphi(x) \vdash \Delta} \ (\exists Def_\varepsilon)$$

Those rules have no eigenvariable condition. Therefore, while in $\forall Def$ and $\exists Def$ the eigenvariable condition is imposed through the 'semantic ascent' in order to capture the idea that the variables in the upper sequent are arbitrary, in $\forall Def_\varepsilon$ and $\exists Def_\varepsilon$ the eigenvariable condition is incorporated at the syntactic level of formulas. Moreover, the class of terms t in the structural rule of Substitution is expanded with ε-terms (called Sub_ε). Then, in SK_ε, it is possible to prove the symmetry between Right and Left rules based on $\forall Def_\varepsilon$ and $\exists Def_\varepsilon$. Indeed, consider how ∃R is inferred from the latter[24]:

[23] It is worth mentioning that Alechina and Van Lambalgen (1996) shows how several generalized quantifiers can be obtained in the setting of SK by restricting the rule of Substitution.

[24] By [K] I mean a single application of both KL and KR.

$$\dfrac{\dfrac{\dfrac{\dfrac{\exists x\varphi(x) \vdash \exists x\varphi(x)}{\varphi(\varepsilon_x\varphi(x)) \vdash \exists x\varphi(x)}\,(\exists Def_\varepsilon)}{\varphi(t) \vdash \exists x\varphi(x)}\,(\text{Sub.L}_\varepsilon)}{\Gamma, \varphi(t) \vdash \exists x\varphi(x), \Delta}\,[\text{K}] \qquad \dfrac{\Gamma \vdash \Delta, \varphi(t)}{\Gamma \vdash \varphi(t), \exists x\varphi(x), \Delta}\,(\text{KR})}{\Gamma \vdash \exists x\varphi(x), \Delta}\,(\text{Cut})$$

Which generates:

$$\dfrac{\Gamma \vdash \Delta, \varphi(t)}{\Gamma \vdash \Delta, \exists x\varphi(x)}\,(\exists\text{R})$$

However, if the symmetry between Right and Left rules is a condition for the claim that double-line rules define the meaning of logical constants and if the symmetry of the Right and Left rules for $\forall$ and $\exists$ can be proved only by implementing the ε-terms in the structural rule of Substitution, then the ε-operator is a meaning constituent of the classical quantifiers $\forall$ and $\exists$. Otherwise, if we take double-line rules as descriptions of the structural behaviour of logical constants – as Došen (1989) does – the conclusion can be reformulated as follows: deriving (from) arbitrary variables of which nothing is assumed except their range of values is a constitutive part of the structural characterization of the classical quantifiers $\forall$ and $\exists$.

9 Summary

In this chapter, I have adopted the methodological framework of proof theoretic semantics in order to investigate the meaning of the classical quantifiers $\forall$ and $\exists$. The main thesis of this chapter is that the understanding of the inferential rules of $\forall$ and $\exists$ relies on the understanding of the inferential rules of the ε-operator. Moreover, if the SK rules are taken as implicit definitions of the meaning of $\forall$ and $\exists$, then the ε is a meaning constituent of the classical quantifiers. More precisely, I have formulated the proof theoretic semantics thesis within the frameworks of both NK and SK, so as to give different perspective over the notions involved by the inferential rules of $\forall$ and $\exists$. While NK by the discharge of assumptions captures the dynamic process of proofs, SK characterize the structural level of logical constants. Based on the NK and SK settings, I have highlighted two notions involved by the inferential rules of the classical quantifiers, respectively, term-dependence and arbitrariness. The connection between these two notions is well expressed by Tiu (2005):

> "In proof search for a universal quantified formula, e.g., $\forall x\varphi(x)$, the quantified variable x is replaced by a new constant c, and proof search is continued on $\varphi(c)$. Such constants are called eigenvariables, and in traditional intuitionistic or classical logic, they play the role of scoped constants as they are created dynamically as proof search progresses and are not instantiated during the proof search. In the meta theory of the logic, eigenvariables play the role of place holder for values, since from a proof for $\varphi(c)$ where c is an eigenvariable, one can obtain a proof of $\varphi(t)$ for any term t by substituting t into c. In the proof theory of definitions, these dual roles of eigenvariables are internalized in the proof rules of the logic."

Following these leading idea, the main arguments of this chapter can be summarized as follows. Based on the NK setting:

1. The distinction between direct (EI, UG) and indirect ($\exists$E, $\forall$I) rules relies on the representation of the order relation of term-dependence as, respectively, index on the instantial variables or subordinate proofs. Moreover, while in NK_ε term-dependence is instantieted at the syntactic level of nested ε-terms, in NK term-dependence is represented at the semantic level of variables and subordinate proofs values.

2. If the understanding of the NK rules is sufficient to grasp also the meaning of a logical constants, and if the relation of terms-dependence is instatieted by both NK_ε and NK (in)direct rules, then the notion of choice process (as represented by the term-dependence relation) is part of the meaning of both the classical quantifiers $\forall$, $\exists$ and the ε-operator.

Based on the SK setting:

1. If double-lines rules can be analysed according to the structural rules, then the structural rule of Substitution has to be restricted by the eigenvariable condition for the rules of $\forall$R and $\exists$L. Moreover, the eigenvariables condition captures the idea by which the variable in the upper sequent of $\forall$R and $\exists$L is arbitrary, namely nothing else apart its range of values is assumed.

2. If double-lines rules are implicit definition for the meaning of the logical constant, then the ε-operator has to be considered as part of the syntactical category of terms, so as to recover the symmetry between the Left and Right sequents of $\forall$ and $\exists$. In this sense, the ε-operator is a meaning constituent of the classical quantifiers.

Part III

Epsilon Calculus, Set Theory and (Im)predicative Definitions

So far, I have compared the interpretation of quantifiers as higher-order predicates (represented by $\forall$ and $\exists$) with that of choice functions (represented by the ε-operator), based on both the model and proof theoretic semantics. I have argued that the latter interpretation should replace the former in order to formalize different relations of dependence, respectively, functional and term dependences as arising within formulas and inferences. These arguments should have provided the reader with a different framework by which thinking about quantification theory. However, given the ubiquity of quantified logic for mathematical theorizing, it is reasonable to wonder about the consequences of adopting EC instead of FOL as framework for mathematical logic.

That is why, in this last chapter, I will adopt EC and the ε-operator in order to shed light on debates in the philosophy of mathematics. More precisely, I will focus on Set theory and logical definitions. These two theories are clearly connected: axiomatic Set theory can accommodate the foundational role prompted by many authors in the literature if and only if it provides a framework for logical definitions which does not entail contradictions. More precisely, I will focus on the issue of whether adopting impredicative definitions in the theory of logical definitions and the opposition between the combinatorial and logical view of collections in Set theory. Again, these two problems are related: if impredicative definitions generate paradoxes, then axiomatic Set theory has to distinguish between collections that determine sets and those that do not.

However, for the sake of clarity, this last chapter is divided into two self-standing parts. I will initially consider the theory of logical definitions, claiming that, in the classical framework of FOL, paradoxes arise due to the violation of patterns of quantifiers independence in the defying formulas − as suggested by Hintikka (2012). Instead, I will argue that these patterns can be represented in EC, thus turning the issue of whether accepting impredicative definitions from ontological to semantic grounds. Then, in the second part, I will introduce the axiomatic Set theory formalized in EC, as proposed by Bourbaki (1968). Then, through a comparison with both ZFC and NBG, I will argue that Bourbaki (1968) system captures the logical conceptions of classes providing a novel abstractionist account for cardinal numbers close to that of Cantor (1915).

However, it should be remarked that, given the ubiquity of quantified logic for mathematical theorizing mentioned above, this last chapter is a case-study of the consequences of EC for the philosophy of mathematics, which go far beyond the scope of the present work. Nevertheless, while the ε-operator has been usually considered in the context of Hilbert's finitary program (Zach, 2003), the literature as focused only recently on the advantages of the ε-operator for several programs in the philosophy of mathematics, such as: neologicism, like Woods (2014), Boccuni and Woods (2018) and structuralism, such as Schiemer and Gratzl (2016), Leitgeb (2020) − which will be mentioned below.

10 Theory of Definitions

Searching for fruitful definitions − either stipulative definitions of new expressions or redefinitions of established ones − is a central part of mathematical practice. That is why I will first introduce the topic of logical definitions (Gupta, 2019). Those kind of definitions are called logical because: i) they are part of an axiomatized theory $\mathcal{T}$ and ii) they are formalized in the language $\mathcal{L}$ of either FOL or SOL with identity. In this respect, the discussion in §2.1 on *definable* Skolem functions concerns the theory of logical definitions. Indeed, in §2, I have considered examples of (Skolem) function symbols introduced to name the *functional dependencies* detected in the axioms of a theory − such as uniform and uniform continuous functions in analysis or the inverse operator in group theory. This should be already enough to convince the reader of the relevance of quantifiers interaction for the theory of logical definitions.

Indeed, the aim of this section is to argue for the advantages of EC over FOL as framework for logical definitions. More precisely, the comparison will focus on how the interpretation of the quantifiers in the defining formulas − as either higher-order predicates (∀ and ∃) or choice functions (the ε-operator) − affects logical and philosophical aspects of definitions. However, it is important to remark that, while §Chapter I focused on the definition of functional symbols (as introduced by FI), I will here consider only the definitions of individuals − which are 0-ary function symbols. This focus will lead the discussion to evaluate a prominent mathematical and philosophical issue, namely impredicative definitions, by which individuals are defined by reference to the totality to which they belong. The literature on the topic mainly focuses on two problems[25]: i) why impredicative definitions are the source of paradoxes like Russell's and Richard's ones and, ii) whether impredicative definitions can be adopted or not. Concerning (i), Hintikka (2012) argues that (different criteria of) predicativity relies on (different patterns of) quantifiers independence. Then the violation of these patterns implies a contradiction, as in the form of either Russell or Richard paradoxes. In §10.1 I will support Hintikka's claim by pointing out that the two main accounts of impredicativity − namely, Russell (1908) and Poincaré (2012) ones − can be interpreted as arising from different patterns of quantifiers independence in the defining formulas. Then, in §10.2, I will extend Hintikka (1996) proposal to EC, explaining how the different patterns of quantifiers (in)dependence can be represented as (un)nested ε-terms. Finally, in §10.3, I will evaluate the consequences of the EC formulation of (im)predicative definition for (ii). *Contra* the received view of Gödel (1944), I will argue that the issue of adopting (im)predicative definitions relies on semantic rather than ontological grounds, which in the case of the EC formulation correspond to whether adopting empty terms or not.

Logical definitions, as adopted in mathematics, are said to be nominal because they concern the notational representation of mathematical objects rather than their 'essence' (as for real definition). Indeed logical definitions are statements that establish the meaning of the novel expressions introduced in the theory. Logical definitions are composed by three elements: i) the defined term t; ii) a formula containing the defined term t, called the *definiens*; iii) a formula equated with (ii) called the *definiendum* . Such as:

$$t =_{df} \forall x \exists y ((x = y) \leftrightarrow \Phi[x]) \tag{12}$$

Some remarks are in order. First of all, while $=_{df}$ is a metalinguistic expression asserting that the defining formula (12) can always be substituted for the term t, $\leftrightarrow$ is an expression of the object language, asserting the identity of the truth values between the definiens and definiendum. Secondly, the square brackets in the *definiendum* $\Phi[x]$ remark that the formula Φ may contain (one or more) quantifiers ∀ and ∃. Finally, definitions like (12) are interpreted according a theory, namely a set of sentences in the signature of FOL with identity containing non-logical axioms.

As remarked above, logical definitions are always formulated as part of the axioms of a theory. That is why, in the following section, I will assume as a case study Set theory. Indeed, in (naïve) Set theory, formula (12) correspond to the Unrestricted Comprehension Axiom, which asserts the existence of an object (a set or class) satisfying the definiens condition. However, during this section, I will not consider a specific axiomatic system of Set theory. This choice is justified by the fact that I am here mainly interested on how the logical interpretation of quantifiers might affect the

[25] I acknowledge that this distinction leaves out the works done by Feferman (2007) on predicative analysis and number theory. It is worth mentioning that there is a strict connection between Feferman predicative program and Hilbert finitist program − where the ε-operator was first put at work. However, a discussion of this connection would bring us too far away from the topic at issue, namely the advantages of the ε-operator over the classical quantifiers for the theory of logical definitions. That is why I refer the reader to Zach (2007).

mathematical and philosophical issues concerning (im)predicative definitions. Instead, in §11 I will carry out a full comparison between ZFC formulated in FOL and the Set Theory formulated in EC. Moreover, in order to facilitate the comparison between Classical Logic and EC over logical definitions, I will consider only definitions formulated in FOL.

A necessary condition for a definition like (12) to be legitimate is that of non-circularity, namely: the defined term t must not occur in the *definiendum*[26]. However, since the works on the foundations of mathematics happened at the beginning of the twenty century, authors like Russell and Whitehead (1910) have remarked that the Unrestricted Comprehension Axiom leads to paradoxes. That is why Russell (1908) and Poincaré (2012) have proposed further restrictions on (12) in order to avoid paradoxes like Richard's and Russell's ones − which will be discussed in the next paragraph.

10.1 Poincaré and Russell Criteria of Predicativity

First of all, some terminology. Predicativity is a syntactic and semantic criterion which disallows the definition of a mathematical entity by quantification over a totality it belongs to. Predicativity is a mathematical criterion which should be distinguished from the philosophical stance of predicativism, which comprehends the arguments against impredicative definitions. Those arguments are usually summed up as the Vicious Circle Principle (VCP), which will be presented below. Predicativity as a mathematical criterion is primarily supposed to stop the raising of paradoxes. More precisely, following Ramsey (1926), we distinguish two kinds of paradoxes: the *logical* ones, which arise only within a mathematical or logical theory and involves terms such as classes or numbers, and the *semantic* ones, which involve notions of reference and definability that are relative to an interpreted language. In this paragraph, I will consider examples of both kinds of paradoxes, respectively, Russell's and Richard's ones.

Russell (1908) and Poincaré (2012) are widely considered as the first authors to endorse predicativism. Yet, many authors have highlighted that Poincaré and Russell adopt different criteria to distinguish predicative and impredicative definitions. For instance, (Chihara, 1973, 142) asserts: "Notice that the above sense of nonpredicativity [the one of Poincarè] is distinct from Russell's. We thus have two senses of impredicativity to deal with". That is why I will first consider Russell (1908) and Poincaré (2012) diagnosis of, respectively, Russell's and Richard's paradox, claiming that they pin down to two different criteria of predicativity. More precisely, I will stress that Russell (1908) and Poincaré (2012) criteria of predicativity rest on two different patterns of quantifiers (in)dependence in the defining formulas. In the next paragraph, I will come back to the question of whether this is the interpretation that Russell and Poincaré had in mind.

Jung (1999) points out that Russell (1908) offers at least six different formulations of VCP. However, since Gödel (1944) the literature have focused mainly on two of them: the definitional and presuppositional readings of VCP. However, I will here focus only on the former, given that the presuppotitional reading of VCP is tied up with the background theory, as either Set or Type theory, and so not suited for the present general discussion on logical definitions. (Russell, 1908, p.63) states the principle as follows:

> **Definability VCP:** If, provided a certain collection had a total, it would have members only definable in terms of that total, then the said collection has no total.

In the discussion below, I will adopt **Definability VCP** in order to point out that Russell (1908) and Poincaré (2012) detect two different ways by which an object can be defined in terms of a total to which it belongs, respectively by being a member of the universe stated by the *definiendum* or by being a member of the conditions stated by the *definiens*. In the next paragraph, following

[26]Yet, this condition is not sufficient. Other restrictions for valid definitions are conservativeness and eliminability − see Gupta (2019).

Hintikka (2012) insights, I will claim that Russell (1908) and Poincaré (2012) criteria of predicativity correspond to two different patterns of quantifiers independence in the defining formula (12).

I will consider first Russell's paradox, which is one of the most well-known logical or set-theoretical paradoxes − in the sense of Ramsey (1926) explained above. Russell's paradox is carried out in the *naïve* set theory with the Unrestricted Comprehension axiom, according to which any conditions can be substituted for $\Phi[x]$ in (12). But from the assumption of this axiom, Russell's contradiction follows. For example, if we let $\Phi[x]$ stand for $\neg(x \in x)$ and let $R = \{x : \neg(x \in x)\}$, then R is the set whose members are exactly those objects that are not members of themselves. Is R a member of itself? If it is, then it must satisfy the condition of not being a member of itself and so it is not. If it is not, then it must not satisfy the condition of not being a member of itself, and so it must be a member of itself. On the assumption of the Law of Excluded Middle, the contradiction follows. I will now explain − based on **Definability VCP** − what is the condition of predicativity which Russell (1908) takes as the source of the paradox.

In order to understand Russell (1908) diagnosis of the paradox, I will first set some terminology. In Russell and Whitehead (1910), the Comprehension Axiom (12) is composed by two distinct syntactical categories: while the *definiens* correspond to a propositional function, the *definiendum* is a class. Roughly, while propositional functions are open sentences, classes are the extensions of predicates. Based on this, I will claim that, according to Russell (1908), Russell's paradox arise because the class $R : \{x|x \notin x\}$ is defined according to the totality to which it belongs. Indeed, the predicative criterion proposed by Russell forbids the defined object (namely, a class) to be the value of the universal quantifier in the *definiendum*, as stated by (Russell and Whitehead, 1910, p. 102) himself: "Whatever contains an apparent variable must not be a possible value of that variable." That is why Russell criteria of predicativity prevents the definition from being plainly circular, as defining the (unknown) class R by specifying what it itself must be like. Therefore, **Definability VCP** can be reformulated according to Russell criterion of predicativity as: no totality T may contain an object which is definable only by universally quantifying over T. As stated by (Russell, 1908, p.62) himself:

> "That there is no such class [namely, R] results from the fact that, if we suppose that there is, the supposition gives rise (as in the above contradiction) to new classes lying outside the supposed total of all classes."

Someone could object that Russell (1908) predicative criterion does not concern the defined class being a (potential) value of the universal quantifier in the *definiendum*, but rather the (type theoretic) order of the *definiens* not being lower of than the (type theoretic) order of the defined class R − as stated by Russell and Whitehead (1910) Theory of Types. In a nutshell, Russell and Whitehead (1910) Theory of Type − as either simple or ramified − classifies the entities of a theory in a hierarchy, with individuals at the lowest level, sets of individuals at the next level, sets of sets further up again. Moreover, sets containing members of different types are disallowed, so the Russell class R cannot be formulated. However, this objection confuses the diagnosis of Russell's paradox, with its solution − as formalized by the Theory of Types. Indeed, Russell analysis of his own paradox comes in two parts: the *pars destruens* of VCP and the *pars construens* of Type Theory. More precisely, the Theory of Types must provide a natural and conservative explanation of why mathematical statements about supposed totalities are disallowed − as stated by **Definability VCP**. This point is well expressed by (Chihara, 1973, p. 10):

> "[...] for Russell, the vicious-circle principle was 'purely negative in its scope'; he felt that an adequate solution to the paradoxes must provide a positive theory which would 'exclude' totalities in accordance with the vicious-circle principle."

I will now move to consider Poincaré (2012) evaluation of Richard's paradox. Richard's paradox asserts that the set of real numbers definable in a finite number of words is denumerable (called E). But then we can describe in finitely many words another real number formed from an enumeration of the members of this set by means of diagonalization (called N). So there is a real number that is defined in the enumeration if and only if it is not. Richard's paradox is a semantic paradox − in the sense of Ramsey (1926) mentioned above. However, it should be remarked that, while Russell's paradox is carried out within ZFC, Richard's paradox concern the metatheory of ZFC because the notion of 'definability' is not part of the object language (Fraenkel et al., 1973).

I will now claim that Poincaré (2012) diagnosis of the circularity of Richard's paradox differs from the one detected by Russell (1908) of his own paradox. It might be useful to distinguish the two definitional steps of Richard's paradox. We first define the set E and then produce N by diagonalization on E. However, Poincaré (2012) claims that neither of the two steps can be completed:

> "This reasoning rests on a classification of integers into two categories: those which can be defined by a sentence with fewer than one hundred [English] words and those which cannot be. In asking the question, we proclaim implicitly that this classification is immutable and that we begin our reasoning only after having established it definitively. But that is not possible [...] the classification of numbers can be fixed only after the selection of the sentences is completed, and this selection can be completed only after the classification is determined, so that neither the classification nor the selection can ever be terminated."

Indeed, as remarked by Poincaré himself, it is clear that the definition of N depends on that of E. More precisely, the definition of N would appear somewhere in the sequence of English sentences. However, it would not be meaningful at that place in the sequence, because it mentions the set E which has not yet been defined. Precisely the definition of E is for Poincaré (2012) the source of Richard's paradox:

> "E is the aggregate of all the numbers that can be defined by a finite number of words, without introducing the notion of the aggregate E itself, otherwise the definition of E would contain a vicious circle, for we cannot define E by the aggregate E itself."

That is why the problem concerns how the defined set E is specified by the *definiens*. Indeed, also the set E is defined by an instance of the Comprehension Axiom (12), where the definiens $\Phi[x]$ corresponds to the condition of being definable by a finite number of English words. Therefore, for Poincarè, the defined set E should not be a possible member of the conditions specified by the definiens. Otherwise, we would not be able to specify the object (namely, a set or class) without first checking whether the condition is satisfied. But if so, then checking whether the condition is satisfied would require to determine first whether there is such a set or class − thus producing a vicious circle. That is why, **Definability VCP** can be reformulated according to Poincaré criterion of predicativity as: if the definition of a supposed totality would be modified by the introduction of a new element, then that collection has no total.

Finally, the difference between the criteria of predicativity of Russell and Poincaré is highlighted by their respective diagnosis of Richard's paradox. Indeed, while Poincaré (2012) blames the definition of the set E of definable numbers, as explained above, (Russell and Whitehead, 1910, p.67) considers N as the source of the paradox:

> "[...] in Richard's paradox, when we confine ourselves, as we must, to decimals that have a definition of a given type, the number N, which causes the paradox, is found to have a definition which belongs to a higher type, and thus not to come within the scope of our previous definitions."

I take this as further evidence supporting the claim that Russell and Poincaré considered, respectively, Russell and Richard paradoxes as arising from different criteria of predicativity. In the next paragraph, following the insights of Hintikka (2012), I will claim that these criteria can be interpreted as different patterns of quantifiers independence in the defining formulas.

10.2 (Im)predicative Definitions as Quantifiers (In)dependence

I will now argue that Poincaré (2012) and Russell (1908) different criteria of predicativity support Hintikka (2012) claim, by which those conditions correspond to different patterns of quantifiers independence. By patterns of quantifiers independence, I mean the syntactic order of the quantifiers in the defying formulas, by which quantified variables fall in the scope of another one (more on this below). However, I have to explain what it means for the value of a quantified variable to be independent from that of another one. Indeed, in §Chapter I, I have focused only on the notion of *functional dependence*, which is well represented by the process of Skolemization for FOL formulas. However, the process of Skolemization can be extended so as to shed light on the notion of variables independence. Indeed, Skolemization represents functional dependences by replacing the inner existential quantifier with a Skolem function having as arguments the external universal variables. However, if we restrict the arguments of the Skolem function, then the value of the existential variable is said to be independent from that of a specific universal variable. Consider for example the formula $\forall x \forall y \exists z \varphi(x, y, z)$, where its Skolem Normal Form is $\forall x \forall y \varphi(x, y, f(x, y))$. However, by restricting the argument of the Skolem function introduced, we get for example $\forall x \forall y \varphi(x, y, f(y))$ where the value of $f(y)$ is independent from that of x[27]. In FOL, given the linear order of formulas, no restriction on the arguments of Skolem functions is possible. That is why, we have to extend FOL with either branching quantifiers or the slash operator − both considered by Hintikka (1996). For the sake of clarity, I will adopt the latter method. Then, the aforementioned formula would be represented as $\forall x \forall y \exists z \varphi(x, y, z_{/x})$. However, both formal systems have the same expressive power of Σ-SOL (Väänänen and Hodges, 2010), and so the two formulations are equivalent.

Based on these notions, I will argue that Russell (1908) and Poincaré (2012) criteria of predicativity correspond to two different patterns of quantifiers independence. As explained above, for Russell (1908) definitions are impredicative if the specified object is a (potential) value of the universal quantifier in the *definiendum*. For Poincaré (2012), definitions are impredicative if the specified object is a (potential) value of the quantifiers in the *definines*. That is why Hintikka (2012) distinguishes two criteria of predicativity, formalized as:

- **Russell's VCP:** $\forall x \exists y_{/x}((x = y) \leftrightarrow \Phi[x])$
- **Poincaré's VCP:** $\forall x \exists y((x = y) \leftrightarrow \Phi[x_{/y}])$

I will argue that in EC, unlike FOL, the patterns of quantifiers (in)dependence can be formulated as (un)nested ε-terms. The comparison with EC concerning the patterns of quantifiers independence is suggested by **Th.7**, by which every EC formula is equivalent to some Σ-SOL formulas (Blass and Gurevich, 2000). However, as explained above, both FOL with either branching quantifiers or the slash operator is expressively equivalent to Σ-SOL, which represent the independence among variables values as restrictions on the arguments of the Skolem functions. That is why, it is reasonable to ask how the patterns of quantifiers independence of (im)predicative definitions detected by Hintikka (2012) are formalized within EC. Secondly, in order to appreciate EC formulation of (im)predicative definitions, I have to recall from §7.2 what nested ε-terms are. The ε-operator is a variable-binding operator and so ε-terms that fall in the scope of another are said to be nested. Moreover, the order of the nested ε-terms in a formula determines also how the value of an ε-term depends on that of another one. Therefore, given that the ε-operator is a term-forming operator, the values of unnested ε-terms in a formula are independent of each other. Based on these notions, it is possible to formalize Russell (1908) and Poincaré (2012) criteria of predicativity as follows:

[27] This formula correspond to the Skolem Normal Form of the branched quantified formula:

$$\begin{pmatrix} \forall x \\ \forall y \exists z \end{pmatrix} \varphi(x, y, z)$$

- **Russell's VCP$_{EC}$:** $\varepsilon_y \forall x((x = y) \leftrightarrow \Phi[x])$
- **Poincaré's VCP$_{EC}$:** $\varepsilon_y \forall x((x = y) \leftrightarrow \Phi[\varepsilon_x \Phi(x)])$

Even if the (philosophical) consequences for **Definability VCP** will be discussed below, it should be appreciated already why the EC formulation of predicative criteria is an improvement over the FOL one. In FOL a set-valued quantifiers Q_S can depend on a set S only by either $S \in Q_S$ or $S \subseteq Q_S$. Therefore, when a set-valued quantifiers Q_S is adopted to define a set S − as for the Comprehension Axiom (12) − the latter should not be neither a member nor a subset of the former. Instead, in EC a quantifier can informationally depend on a set, as defined by the (extensional) semantics of the ε-operator. If a set A is empty, then the formula $\varepsilon_x \varphi(x)$ − where the unary predicate $\varphi(x)$ denotes the set A − refers to an arbitrary individual drown from the domain. That is why in EC (im)predicative definitions become indefinite description, interpreted as "an arbitrary set y such that $\Phi[x]$ is true". However, if the condition $\Phi[x]$ denotes an empty set − as $(x \notin x)$ in Russell's paradox − then the impredicative description is an empty term. I will come back in §11.1 to explain why, in the context of Set Theory formulated in EC, the above impredicative descriptions does not produce paradoxes.

Finally, it should be remarked that Hintikka (2012) thesis does not attempt to reflect the arguments for (im)predicative definitions as originally advocated by Russell (1908) and Poincaré (2012). Indeed the notions of quantifiers independence − on which I have argued Russell (1908) and Poincaré (2012) criteria of predicaticity rely − was only introduced by Henkin (1961). Instead, by supporting Hintikka (2012) claim, I intend to suggest a common interpretation of the early accounts of predicaticity as a logical thesis. This claim goes against Feferman (2007) valuation:

> "What is predicaticity? While this term suggests that there is a single idea involved, what history will show is that there are a number of ideas of predicativity which may lead to different analyses."

Trough the analysis of Russell (1908) and Poincaré (2012) diagnosis of, respectively, Russell's and Richard's paradox, I wanted to highlight to what extent the early accounts of predicaticity rely on their logical formalization, specifically on the interpretation of quantifiers $\forall$ and $\exists$ as higher-order predicates. However, both mine and Hintikka (2012) claims so far concern only the criteria of predicativity as a mathematical position, namely the one formulated to avoid Russell's and Richard's paradoxes. The philosophical position of predicativism, as summed up by VCP, will be considered in the next paragraph in the light of the novel formulation of impredicative descriptions in EC.

10.3 The Vicious Circle Principle Revised

In this paragraph I will discuss where the proposal of §10.2 leaves the philosophical discussion on predicativism, as summed up by the VCP principle. I will argue that, *contra* Gödel (1944), the issue of adopting impredicative definitions is primarily a matter of semantics rather than ontology. In this respect, I will adopt Quine (1985) distinction between impredicative *specification* and *individuation* in order to argue that in EC the issue of whether to accept (im)predicative definitions rest on the commitment to empty terms.

Indeed, Gödel (1944) argues that the cogency of **Definability VCP** − and thus whether or not adopting predicative criteria − rest on ontological issues, namely whether someone adopts a platonic (by which all mathematical objects exist mind-independently) or constructivist (by which mathematical objects are constructed by the theorem-proving activity of mathematicians) framework. Indeed, (Gödel, 1944, p.456) states that:

> "[...] the vicious circle principle [...] applies only if the entities involved are constructed by us. In this case, there clearly must exist a definition (namely the description of a construction) which does not refer to a totality to which the object

defined belongs, because the construction of a thing can certainly not be based on a totality of things to which the thing to be constructed itself belongs. If, however, it is a question of objects that exist independently of our constructions, there is nothing in the least absurd in the existence of totalities containing members, which can be described (i.e., uniquely characterized) only by reference to this totality."

Gödel's view of impredicative definitions is nowadays the received point of view. However, I will object the order of the explanation: given that different criteria of predicativity correspond to different patterns of quantifiers independence, then the reliability of **Definability VCP** primarily concerns how the quantifiers ∀ and ∃ in the defining formulas are interpreted. Once the interpretation of quantifiers is settled for the higher-order predicates one, then the issue of accepting impredicative definitions becomes one of ontological matter, namely whether once take a realist or constructivist point of view. Yet, the issue of impredicative definitions is primarily a semantic one. This is well remarked by the comparison between the FOL and EC formulation of (im)predicative definitions driven in §10.2: indeed, as I will further explain below, once the EC formulation of (im)predicative definitions is considered, then the issue of whether adopting (im)predicative definitions relies on whether accepting empty terms.

Further evidence of this claim can be found in the section 'All and Any' of the first edition of Russell and Whitehead (1910), where Russell foreshadowed how (im)predicative definitions are linked to the interpretation of quantifiers. Even if Russell does not talk about different interpretations of quantifiers − a technical notion which would be only later introduced by Tarski − scholars like Hazen (1983) argues that Russell and Whitehead (1910) remarks on the nature of quantification represent a first step towards a metalinguistic point of view. However, it is here informative to note how Russell and Whitehead (1910) links the comparison between the pronouns *all* and *any* with the debate over (im)predicative definitions and VCP. For Russell and Whitehead (1910) the distinction between all and any corresponds to that of bound ($\forall x \varphi(x)$) and free variables ($\varphi(x)$). The importance of this distinction for Russell has to do with the injunction against illegitimate totalities. (Russell, 1908, pp. 229-230) points out that the vicious circle involved by the Law of Excluded Middle can be avoided by formulating the principle with free variables instead of bound ones.

> "Thus we may say: 'p is true or false, where p is any proposition,' though we can not say 'all propositions are true or false.' The reason is that, in the former, we merely affirm an undetermined one of the propositions of the form 'p is true or false', whereas in the latter we affirm (if anything) a new proposition, different from all the propositions of the form 'p is true or false'. Thus we may admit 'any value' of a variable in cases where 'all values' would lead to reflexive fallacies; for the admission of 'any value' does not in the same way create new values. Hence the fundamental laws of logic can be stated concerning any proposition, though we can not significantly say that they hold of all propositions."

I take this as further evidence supporting the claim that the issue of accepting impredicative definitions primarily relies on the interpretation of the quantifiers in the defining formulas. Moreover, it is worth mentioning that many authors like Fine (1985), Martino (2001) and Horsten (2019) have understood Russell and Whitehead (1910) remarks as a first insight into the notion of *arbitrary reference/objects*, which is adopted to formalize indefinite descriptions − for a brief overview see §7.1.

Finally, I will consider how the EC formulation of (im)predicative definitions sheds new light over **Definability VCP**. More precisely, I will break down the notions of defineability into two parts, following the insights of Quine (1985). Indeed, a definition in the sense of (3) accomplishes two tasks: on the one hand, it *specifies* a new notation as a short hand for an older one. On the other hand, a definition *individuates* the (only) object satisfying it. Based on this distinction, I will follow Quine (1985) by claiming that in EC, even if it is possible to *specify* impredicatively, it is not possible to *individuate* impredicatively. However, it should be remarked that I will argue for this distinction on different grounds than Quine, who is motivated by concerns regarding the identity conditions for defined objects. Instead, I will support my claim according to the extensional semantics of the ε-operator.

More precisely, I have already explained in §10.2 that an object (class or set) can be specified impredicatively as long as the patterns of quantifiers independence detected in §10.1 are correctly formalized within EC as unnested ε-terms. Moreover, in EC impredicative definitions are bounded by an initial ε-operator, which is a referring term. This supports the claim that − *contra* Gödel (1944) − adopting (im)predicative definitions is a semantic rather than an ontological issue.

Secondly, I will claim that objects cannot be individuated impredicatively: in EC even if the patterns of quantifiers independence are respected, there are impredicative definitions that fails to individuate an object or, more precisely, they do not determine a set. Indeed, in §11.1, I will explain why the Russell's paradox is harmless in the Set theory formulated in EC. Yet, in EC those impredicative definitions are still referring descriptions. Indeed, it should be kept in mind that the ε-operator is a total choice operator which refers to an arbitrary individual from the domain in case the set is empty. Indeed, as argued by Donnellan (1966), descriptions can be used in (at least) two different ways: according to the attributive use, descriptions ascribe properties to objects. Instead, according to the referential use, descriptions make claims even if no object satisfies the descriptive content. Then the extensional semantics of the ε-operator allows to formalize this distinction, with impredicative definitions not determining sets falling in the referential use. However, developing a theory of descriptions based on the ε-operator is beyond the scope of the present discussion[28].

Instead, the claim that in EC impredicative descriptions fail to determine sets while still being referential, is well supported by the suppositional function of the class $R : \{x|x \notin x\}$ in the deduction of Russell's paradox. Indeed, as remarked by Boccuni (2015), within inferences adopting *reductio ad absurdum*, like Russell's paradox, we temporarily assign an individual as the value of the class R $\{x|x \notin x\}$. But then we find out that the individual we picked does not, and in fact cannot, satisfy the condition 'set of all the non self-membered sets'. Since the individual chosen is an arbitrary individual, that the Russell condition is not satisfied holds of all the individuals of the domain. But in the process, 'r' has indeed been referential: it referred to an arbitrary individual that does not satisfy the Russell condition. This description fit very well with the EC formulation of impredicative definitions proposed above, which specify a new notation for a class while failing to individuate the corresponding set − as it will further explained in the next paragraph.

11 Set and Class Theory: the case of BK

So far the discussion has focused only on the logical issue of (im)predicative definitions, which I have argued relies on different patterns of quantifiers (in)dependence. However, in the previous section, I have adopted several notions of Set and Class theory in order to consider examples of impredicative definitions, such as Russell and Richard paradoxes. Instead, in this section, I will carry out a full comparison between the Set Theory formulated in First-order Logic − namely, Zermelo Frankael with the Axiom of Choice (ZFC) − and the one formulated in the Epsilon Calculus. I will adopt Bourbaki (1968) Theory of Sets (which I will refer to as BK) as paradigm of the latter[29].

[28] It is worth mentioning that, in order to make room for Donnellan (1966) distinction between attributive and referential uses of descriptions, the semantic condition for empty ε-terms should be changed. Another possible way to go is suggested by Leisenring (1969), by which $\varepsilon_x A$ assigns the same individual to the empty set as to the universe of discourse: $f(\varnothing) = f(\mathcal{D})$. However, this solution would imply to add a further axiom defining the interpretation of the ε-operator, namely $\varepsilon_x(x = x) = \varepsilon_x\neg(x = x)$.

[29] For historical accuracy, I should mention that Bourbaki (1968) adopts the τ-operator rather than the ε one. However, this is a matter of notation, and so I will keep the ε-operator discussed so far.

Bourbaki Theory of Sets has been considered as an outdated theory with a cumbersome notations (Mathias, 1992). However, the rising attention on the pioneering work of Bourbaki in Category Theory (Heinzmann and Petitot, 2020) and the implementation of BK for automated proof assistant (Grimm, 2010) demands further attention on their foundational axiomatic system. Yet, a comprehensive evaluation of Bourbaki's work is beyond the scope of the present discussion − see Anacona et al. (2014). Instead, I will focus below on how the implementation of the ε-operator in the formal language of an axiomatic theory of sets affects both the status of logical classes and cardinal numbers. Therefore, this last section is a further case study of the logical properties of the ε-operator in philosophy of mathematics, where Set theory has played a prevalent role. Finally, it should be remarked that while the discussion so far has focused on the theory of quantification − arguing for the advantages of the ε-operator over the classical quantifiers $\forall$ and $\exists$ − I will discuss below the improvements gained by adopting the ε-operator in axiomatic Set theory.

This last sections is organized as follows: I will first introduce the formal notation and the axiomatic system of BK. Then, I will focus on two formal definitions of BK − namely, those of collectivizing relations and cardinal numbers − in order to carry out a comparison between BK and ZFC. The main claims can be summarized as follow: in §11.1, I will argue that BK formalized a logical view of classes, by which sets are defined by the resources of the ε-operator. Moreover, I will argue that the system of BK allows to distinguish the logical grammar of classes from the set theoretical operations − unlike NBG. In §11.2, I will argue that the BK definition of cardinal numbers avoids the objections moved to the classical proposals of Frege/Russell and Zermelo/Von Neumann: unlike the former, the BK definition of cardinal numbers is *representational* − in the sense of Hallett (1984) explained below − by which the cardinal-set is equinumerous with the given set. Unlike the latter account, the BK definition of cardinal number is not *ambivalent*: the ε-term in the definiendum refers to an arbitrary set for each equivalence class, which can be either considered as Zermelo or Von Neumann ordinal. Moreover, based on the representational feature of the BK definition of cardinal numbers, I will characterize an abstractionist account similar to that of Cantor (1915), through an evaluation of the objections moved by Frege (1879).

Before continuing, I will introduce the axiomatic system of BK, which is a first-order set theory replacing the classical quantifiers $\forall$ and $\exists$ with the ε-operator. The ε-operator is introduced by Bourbaki according to the axioms of Critical Formula and Extensionality presented in §1. Therefore, the interpretation of $\varepsilon_x\varphi(x)$ is the usual one 'an x such that $\varphi(x)$, if any'. More precisely:

Def.8 The language $\mathcal{L}_{BK}$ is formulated within the Epsilon Calculus and contains: i) first-order variables (e.g. x, y, etc.); ii) terms (e.g. a, b, etc.); iii) n-ary relations (e.g. P(x), P(x,y), etc.). Every well-formed formula is either a *term* − which represents an object of the theory − or a *relation* − which represents a statement about these objects, i.e. a proposition.

Bourbaki warns from the start, that the word 'set' must be considered strictly as a synonym for term − I will come back on this in §11.1. If a and b are terms, then the well-formed formula a $\in$ b is a relation, called the *membership relation*. Finally, BK is characterized by the following list of axioms and axiom schemata:

(Ax.1) $\forall x\forall y((x \subset y \wedge y \subset x) \rightarrow (x = y))$ (Extensionality)

(Ax.2) $\forall x\forall y\exists z(z = x \vee z = y)$ (Pairing)

(Ax.3) $\forall x\forall y\exists z(y \in z \leftrightarrow y \subset x)$ (Power set)

(Ax.4) There exists an infinite set N. (Infinity)

(Sc.1) $\forall w\exists y\forall z(R \rightarrow (z \in y)) \rightarrow \forall x\exists y\forall z(z \in y \leftrightarrow \exists w(w \in x \wedge R))$ (Selection and Union)

Two last remarks about BK are in order. First of all, as already explained in §2.2, the ε-operator is equivalent to the Axiom of Global Choice, which asserts the existence of a choice function defined on the class of every non-empty set. Moreover, in BK the Axiom of Selection and Union is not restricted, namely it might contains ε-term, like $z = \varepsilon_u(u \in w)$, which represents the choice of an arbitrary set u from w ranging over u. If this is the case, then the Axiom of Choice is derivable in BK (Leisenring, 1969). Indeed, Bourbaki (1968) proves AC in the version of *Zermelo's theorem*, by which every set can be well-ordered. In this sense in BK, the philosophical issue of whether to accept AC − which has been highly debated since Zermelo's formulation of the axiom (Moore, 2012) − is resolved at the syntactic level by the introduction of the ε-operator. This clearly suggest to consider AC as a logical principle − as argued by Hintikka (1999). However, this topic would require a self-standing discussion which will not pursued here.

Secondly, BK − unlike ZFC − lacks the Axiom of Foundation and thus it allows the existence of non well-founded set, as I will explain below. However, the Axiom of Foundation has been proved to be independent from ZFC (Bernays, 1991), therefore there are models of ZFC which already contain non well-founded sets. That is why, the axioms of BK could be justified according to either the iterative conception − as for ZFC − or by the graph conception of sets − as ZFC^- with some version of the Anti-Foundation Axiom. However, I will not discuss this issue here which would bring us too far away from the advantages of adopting the ε-operator in axiomatic Set theory. Instead, it is possible to prove the relative consistency of BK to that of ZFC^-. Indeed, while the explicit axioms Ax.1-4 clearly belong to ZFC^-, Anacona et al. (2014) prove that also the axiom schema Sc.1 is verified in ZFC^-. Therefore, BK and ZFC^- are equivalent.

11.1 Collectivizing Relations as Logical Classes

In this paragraph I will compare BK with both ZFC and NBG on the definitions of collections as either combinatorial sets or logical classes. I will argue that this comparison pins down to two main differences: on the one hand, while in ZFC adopts the combinatorial view of sets, BK formalize the logical notion of classes based on that of *collectivizing relation*. Then, sets are introduced in BK through the ε-operator which turns the (collectivizing) relation into a term. I will support my claim by showing that the non-existence of Russell's class R $\{x|x \notin x\}$ can be verified in BK − but not in ZFC. On the other hand, unlike Bernays (1991) two-sorted NBG, BK keeps the syntactic difference between sets (namely, terms) and classes (namely, relations) without introducing a specific membership relation for each. Indeed in BK, even the membership relation $\in$ is considered as a collectivizing relation, which can be applied to both sets and classes.In conclusion, I will argue that the syntactic distinction of BK resists Maddy (1983) objection to the classical accounts of sets and classes as formalized by NBG.

I will first introduce the distinction between the combinatorial and logical view of collections. Indeed, many authors in the literature have detected two different accounts of collections which have prompted the development of axiomatic set theory[30]. On the one hand, Frege, Russell and Quine (among others) advocate *logical* collections, which are the extension of predicates. I will refer to logical collections as *classes*. According to this view, a class is derived from a property, which tell us whether an object belongs to it or not. (Maddy, 1983, p. 121) well express this point saying:

> "The logical notion, beginning with Frege's extension of a concept, takes a number of different forms depending on exactly what sort of entity provides the principle of selection, but all these have in common the idea of dividing absolutely everything into two groups according to some sort of rule."

The logical account of classes is suggested by the fact that some class-theoretical operations mirror the logical connectives: intersection of extensions/conjunction of predicates, non-membership/negation of the predicate, union/disjunction, subset/indicative conditional.

[30]It should be remarked that substantial differences remain even among authors advocating the same account of collections. However, given the generality of the comparison here at issue, I will not discuss those differences any further.

On the other hand, Cantor, Zermelo and Fraenkel (among others) endorse the combinatorial view of collections, by which collections are obtained from some well-defined objects such as the natural numbers by enumerating their members in an arbitrary way. I will refer to combinatorial collections as *sets*. Such sets exist independently of our ability to provide a defining condition or rule that characterizes its members. Bernays (1983) famously remarked that:

> "The notion of set used in a 'quasi-combinatorial' sense, by which I mean: in the sense of an analogy of the finite to the infinite. Consider, for example, the different functions which assign to each member of the finite series 1, 2, ..., n a number of the same series. There are n^n functions of this sort, and each of them is obtained by n independent determinations. Passing to the infinite case, we imagine functions engendered by an infinity of independent determinations which assign to each integer an integer, and we reason about the totality of these functions. In the same way, one views a set of integers as the result of infinitely many independent acts deciding for each number whether it should be included or excluded."

The combinatorial view of sets is instead prompted by the set theoretical operations of separation, replacement and power-set (among others). I will explain below how these set-theoretical operations are formulated within the class talk of BK.

Therefore, the main question is: does BK adopts the logical or combinatorial view of collections? By checking the list of axiom presented above, it is clear that none of them contain terms denoting sets. The reason is that in BK sets are introduced through classes. Indeed, it should kept in mind from §11 that BK formulas are divided between terms and relations. Bourbaki warns from the start, that the word "set" must be considered strictly as a synonym for term, i.e., a well-formed formula which represents an object of the theory. Instead, relations between elements of one or more sets are denoted by inserting the appropriate terms into a scheme characteristic of the relation considered (and similarly for properties). More precisely, Bourbaki (1968) does not talk about 'classes' but of 'collectivizing relations':

Def.9 Let R be a relation and let x be a variable of the theory BK. If y denotes a letter different from x which does not appear in R, then the relation $\exists y \forall x(x \in y \leftrightarrow R)$ is denoted by Coll_xR. If Coll_xR is a theorem of BK, R is said to be *collectivizing* in x in BK.

Therefore, to say that R is collectivizing in x is to say that there exists a set a such that the objects x which possess the property R are precisely the elements of a. That is why in BK, the class talk is replaced with that of collectivizing relations, which are formulas of the formal system that represent collections of objects that verify a certain given property. I will explain below how set are derived from collectivizing relations (namely, classes). However, before continuing, it should be pointed out that in BK the definition of collectivizing relation is actually broader than the standard, set theoretic definition of relations as ordered pairs. Indeed, based on **Ax.1**, Bourbaki introduces the concept of functional relation as:

Th.9 Let R be a relation and x a letter. If R is collectivizing in x, the relation $\forall x(x \in y \leftrightarrow R)$, where y is a letter distinct from x which does not appear in R, is *functional* in y.

Proof. (Bourbaki, 1968, p. 68) □

If a collectivizing relation is functional, then there is a unique set with such and such properties. The notion of functional collectivizing relation allows to formulate some axioms of BK by the introduction of class formation. Indeed, Bourbaki (1968) states, for example, **Ax.2** as $\forall x \forall y Coll_z(z = x \vee z = y)$, which asserts that, for any objects x and y, there is a unique set whose only elements are x and y. In this way, the class talk which is usually restricted to logical operation like, for example, intersection/conjunction or union/disjunction (as explained above), is extended to genuinely set theoretic operations like pairing and power-set. I take this as further evidence that the logical view of classes is the one formalized by the BK notion of collectivizing relation. Yet, Coll_xR in **Def.9** is not a term, and so it does not actually refer to a set − based on **Def.8**. Indeed, a further step is required in order to move from the (collectivizing) relation to the set it represents, showing also how the ε-operator is adopted in BK:

Def.10 Let R be a relation and x,y two distinct letters which do not appear in R. If R is a functional relation collectivizing in x, then Coll_xR is represented by the term:

$$\varepsilon_y \forall x(x \in y \leftrightarrow R)$$

I will explain below why ε-terms can be introduced even when the relation R is not collectivizing. However, it should be already appreciated that **Def.8** provides the characterization of logical classes we were looking for. More precisely, as explained above, logical classes are derived from properties which collect the members of a class. Then, in BK, the notion of collectivizing predicate is taken as fundamental to define sets. As stated by (Bourbaki, 1968, p. 347):

> A *set* consists of *elements* which are capable of possessing certain *properties* and of having certain *relations* between themselves or with elements of other sets.[Italics in the original]

I will support my claim by showing that in BK no set correspond to the Russell's class R of all sets that do not belong to themself. In ZFC, the Russell class is "too large" to form a set, namely it does not exist at any stage of the cumulative hierarchy − thus according to the iterative conception of the axioms of ZFC which captures the combinatorial view of sets. Instead, in BK, a non collectivizing relation fails to determine a sets, as:

Th.10 The relation $x \notin x$ is not collectivizing in x. Therefore, $\neg(\text{Coll}_x(x \notin x))$ is a theorem in BK.

Proof. (Bourbaki, 1968, p. 68.) □

Based on the extensional semantics of the ε-operator, it is nonetheless possible to introduce the ε-term for the Russell class, as $\varepsilon_x(x \notin x)$. In this case the ε-term is still a well-formed term of BK, however it does not determine a set. This example highlight an important difference between ZFC and BK over the notions of set existence and definability, which is well expressed by (Wang, 1955, pp. 61-62):

> "Zermelo's set theory is, for example, a system in which every number set, if expressible in the system, can be proved to exist in the system. There is, however, a slight complication. While we assume that every number set, if expressible in [ZFC], can be proved to exist in [ZFC], it is not true that every number set, if expressible in [BK], can be proved to exist in [BK]. Thus, there are formulas $F(x)$ in [BK] which contain the Hilbert ε-symbol such that $\exists y \forall x(x \in y \leftrightarrow F(x))$ need not be a theorem of [BK]."

I will now pass to the comparison between BK and NBG − the latter considered as the main theory of (logical) classes. First of all, it should be remarked that the sets of axioms of these two theories are almost identical, with the only difference of the Axiom of Foundation (which has been proved to be independent also from NBG by Bernays (1991)). Instead, I will point out an advantage of BK over (what has been considered as) a notational matter in NBG. This notational matter will turn out to be philosophically insightful to distinguish combinatorial sets from logical classes.

As for BK, also in NBG classes are introduced through a class existence schema, which is however an axiom rather than a theorem. Then sets are introduced as a particular kind of classes, namely those that can be member of other classes. These sets are distinguished from proper classes, which are instead classes that are "too large" to form a set without implying a contradiction − as the cumulative hierarchy of sets V. Therefore, as for BK, also in NBG sets are derived from classes. However, regarding NBG, Maddy (1983) complains the definition of sets based on that of classes, which makes the distinction between the two vague:

> "The problem is that when proper classes are combinatorially determined just as sets are, it becomes very difficult to say why this layer of proper classes on top of V is not just another stage of sets we forgot to include."

Since Maddy (1983) criticisms, several accounts of classes as either mereological fusion or pluralities have been proposed in the literature, in order to stress the difference between classes and sets − which are nonetheless beyond the scope of the present discussion. Instead, I want to point out that one of the founder of NBG − namely, Bernays − foreshadowed (a version of) Maddy's objection. Indeed, Bernays (1991) formulates NBG in a two-sorted first-order language in order to keep track of the difference between sets and classes since the very syntactic level of formulas formation. In this sense, Bernays (1991) two-sorted NBG resists Maddy's objection. I will first briefly evaluate the pros and cons of Bernays (1991), concluding with the advantages of adopting BK. Indeed, concerning the relation between Set theory and extensional logic, (Bernays, 1991, p. 42) claims that:

> "But another way is to give up Frege's first assumption, that is to distinguish classes as extensions from sets as individuals. Then we have the advantage that the operation of forming a class $\{x|\Phi(x)\}$ from a predicate $\Phi(c)$ can be taken as an unrestricted logical operation, not depending on a specifying comprehension axiom. But from this then we have to separate the mathematical processes of set formation which in the way of Cantor are performed as generalizations of our intuitive operations on finite collections."

Even if Bernays proposal pins down to a syntactic matter, it is nonetheless justified by theoretical motivations: namely to distinguish the logical grammar − which gives rise to classes − from set theoretic formation. However, this syntactic distinction can be carried out only by assuming two different membership relations, one for sets ($\in$) and one for classes (η). Yet, this is an unnatural consequence which increases the complexity of formulas.

Instead, I want to claim that BK is a more natural framework to carry out Bernays (1991) proposal. Indeed in BK, even if $\in$ is assumed as a non-logical constant, the formula $x \in y$ is treated as a relation, which is indeed collectivizing in x (Bourbaki, 1968, p. 68.) In this sense, the membership relation $\in$ is derivative: we can say that an object a is a member of a collection b just in case b is the extension of some predicate, concept, or property that applies to a. Indeed, if there are some objects, however disparate, forming a collection in the combinatorial sense, then there is a property that characterizes just those things, namely the property of belonging to that collection. That is why in BK, the membership relation $\in$ can be adopted for both sets which belong to other sets ($x \in y$) and for sets belonging to collectivizing relations ($x \in Coll_x R$). The case of two relations $R$and S both collectivizing in x of which one is part of the other, is treated as: $\forall x(R \rightarrow S) \leftrightarrow Coll_x R \subset Coll_x S$ (Bourbaki, 1968, p. 69.). That is why I conclude that the BK (syntactic) distinction between terms produced by the set theoretical operations stated by the axioms and collectivizing relations clarify the distinction between, respectively, the combinatorial sets and the logical classes.

11.2 Cardinal Numbers as Arbitrary Sets

In this paragraph I will introduce and discuss the BK definition of cardinal numbers − which relies on the resources of the ε-operator − by comparing it with the other two main accounts of cardinal numbers in the literature, namely the Frege-Russell and Zermelo-von Neumann proposals.

More precisely, I will argue that the BK formulation of cardinal numbers has advantages over both Frege-Russell and Zermelo-von Neumann proposals: on the one hand, unlike Zermelo-von Neumann account, the BK definition of cardinal numbers does not require a prior definition of ordinal numbers. On the other hand, unlike the Frege-Russell account, the BK definition of cardinal number is *representational*, in the sense that cardinal numbers are sets equinumerous to the base set (Hallett, 1984). Based on this latter feature, I will develop an abstractionist account of cardinal numbers close to Cantor (1915), by which cardinal numbers are arbitrary sets determined by the only property of being equinumerous to the base set. Moreover, I will argue that the BK abstractionist account resists the objections originally made by Frege (1980) against Cantor (1915) proposal. Finally, I will point out that the BK definition of cardinal numbers is explicit, thus avoiding the 'Bad Company Objection' raised against Frege (1980) implicit definition of cardinal numbers (Wright, 1997).

I will first introduce the Frege-Russell and Zermelo-von Neumann definitions of cardinal and ordinal numbers. For the sake of clarity, this presentation is much indepted to Incurvati (2020). The Frege-Russell proposal is carried out in either SOL or Type theory (simple or ramified) and define the notion of ordinal number based on that of cardinal number. Intuitively, cardinal numbers are numbers which represent the size of collections. The idea is to associate with each set a an object – called the cardinality of a and denoted $|a|$ – which represents the size of a. Now say that sets a and b are equinumerous if there is a one-to-one correspondence between them, namely:

$$\forall x \forall y (x \approx y) \leftrightarrow (|x| = |y|) \tag{HP}$$

Based on the background logical theory of the Frege-Russell proposal, a cardinal is an equivalence class of equinumerous sets. That is to say, the cardinal of a set a is the set of all sets equinumerous with a. A similar definition is available for the ordinals. The idea here is that ordinals are numbers which represent the order structure of well-ordered collections.

Instead, the Zermelo-von Neumann account is carried out in ZFC and defines the notion of cardinal number based on that of ordinal number. Intuitively, ordinal numbers are numbers which represent places in a progression. According to the Zermelo-von Neumann definition, an ordinal is a transitive set well-ordered by membership, where a set a is transitive just in case every member of a member of a is also a member of a. Given the von Neumann definition of an ordinal, we can then define cardinals using the von Neumann cardinal assignment. Indeed, it is a theorem of ZFC that every set can be well-ordered. Then, von Neumann cardinal assignment defines $|a|$ to be the smallest ordinal equinumerous with a.

I will now turn to consider the framework of BK. Even if Bourbaki (1968) adopts the resources of the ε-operator to define cardinal numbers, the idea actually tracks back to (Ackermann, 1938, pp. 16-17) who states:

> "There is a certain vagueness in the assignment of a set and a cardinal number, for it is not explained how cardinal number and set are to be understood. In order to fix this vagueness, one can take different ways. [...] The other possibility is the one that one understands under the cardinal number as a set that is equivalent to the given set. [...] The advantage then is that you don't have any needs special axioms of abstraction, but the relevant formulas become provable. [...] From the axiomatic point of view, the mentioned indefiniteness is not disturbing, since all properties of the cardinal numbers can also be derived in this way." [Personal translation]

Then, both Ackermann (1938) and Bourbaki (1968) define the cardinal of a set based on the resources of the ε-operator, as:

Df.9 Let t be a set and x a variable not occurring free in t then the cardinal of t is defined as:

$$|t| =_{Df} \varepsilon_x (x \approx t)$$

The idea is that, for any equivalence relation $\approx$, **Df.9** can be used to specify a representative element from each equivalence class $\approx$. However, not much can be said about the set $|t|$ in **Df.9** except that it is equivalent to t and that it equals the cardinal number of any set which is equivalent to t. Instead, it is remarkable that the main property of cardinal numbers can be inferred from **Df.9**, as:

Th.11 Given that $\approx$ is an equivalence relation, HP follows from Df.9.

Proof. Let s and t be any two terms (i.e. sets). Since $t \approx t$, then $\exists z(z \approx t)$, and consequently, $\varepsilon_z(z \approx t) \approx t$, i.e. $|t| \approx t$ **(1)**. Similarly, we get $|s| \approx s$ **(2)**. From (1) and (2) and the fact that $\approx$ is an equivalence relation we obtain $|s| = |t| \rightarrow s \approx t$ **(3)**. On the other hand, the fact that $\approx$ is an equivalence relation implies $s \approx t \rightarrow \forall z(z \approx s \leftrightarrow z \approx t)$ **(4)**. By the EC axiom of Extensionality, we obtain $\forall z(z \approx s \leftrightarrow z \approx t) \rightarrow \varepsilon_z(z \approx s) = \varepsilon_z(z \approx t)$ **(5)**. Therefore, (4) and (5) yield $s \approx t \rightarrow |s| = |t|$ **(6)**. Consequently, from (3) and (6) we get $s \approx t \leftrightarrow |s| = |t|$ **(7)**. HP clearly follows from (7). This proof is adapted from (Leisenring, 1969, pp. 104-105)

□

First of all, I will compare with the BK formulation of cardinal numbers with the Zermelo-von Neumann account. It is straightforward that the former, unlike the latter, does not require a prior definition of ordinal numbers. This fact can be informally explained as follows: the ε-operator is equivalent to the Axiom of Global Choice (AGC) − see §2.2. Moreover, AGC is equivalent to the statement that there is a bijection between the class of all set V and the class of all ordinal numbers (Fraenkel et al., 1973). Therefore, the (arbitrary) set denoted by the ε-term in **Df.9** correspond to an ordinal number. Moreover, the (arbitrary) set − which corresponds to an ordinal number − is equinumerous to a given set. That is why, the notion of von Neumann cardinal assignment is implemented in **Df.9** by the ε-operator, which refers to an arbitrary set equinumerous to the given set. If this is the case[31], then the BK definition of cardinal numbers is able to resist to an objection made by Fine (1998) against the Zermelo-von Neumann account. The objection runs as follows: based on the von Neumann cardinal assignment presented above, cardinal numbers are identified by their representative ordinal numbers. But then, the Zermelo-von Neumann account is ambivalent: why the representatives of cardinal numbers should be defined as Zermelo's ordinal numbers ($\{\varnothing\}, \{\{\varnothing\}\}, \{\{\{\varnothing\}\}\}, etc.$) rather than von Neumann's ordinal numbers ($\{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, etc.$)[32]? Instead, the BK account of cardinal numbers bites the bullet of Fine (1998) objection, explaining why the identification is precisely arbitrary. Indeed, if the set denoted by the ε-term in **Df.9** corresponds to an ordinal number, then by being an arbitrary set there is no fact of the matter about whether it corresponds to either a Zermelo's or von Neumann's ordinal. Finally, given that **Df.9** does not require a prior definition of ordinal numbers, there is a further important difference between the Cantorian abstractionist account developed below and the original proposal of Cantor (1915). Indeed, Cantor (1915) claims that:

> "We will call by the name 'power' or 'cardinal number' of M the general concept which, by means of our active faculty of thought, arises from the aggregate M when we make abstraction of the nature of its various elements *m* and of the order in which they are given."

Therefore, for Cantor, the abstractionist process defining cardinal numbers is divided into two steps, namely abstraction from the order and from the nature of the elements of a set. Hallett (1984) argues at length that the Zermelo-von Neumann account of cardinal numbers presented above is a formalization of Cantor (1915) two-steps abstractionist process. Instead, in BK, the definition of cardinal number is operated directly on the unordered sets.

[31] I acknowledge that this is an hypothesis which requires a formal proof.

[32] I assume the reader to be familiar with the distinction between Zermelo and von Neumann definitions of ordinal numbers − otherwise, see Fraenkel et al. (1973).

Secondly, I will compare the BK definition of cardinal numbers with that of Frege-Russell. In order to do so, it is useful to consider Hallett (1984) distinction between *representational* and *non-representational* accounts of cardinal numbers. Informally, an account is representational if and only if the number set or class is itself the cardinal number − and non-representational otherwise. This difference can be better appreciated by pinning down the conditions for the definition of cardinal numbers. There are two minimal requirements:

i) The operation $|t|$ is defined for all sets t.

ii) $\forall x \forall y (x \approx y) \leftrightarrow (|x| = |y|)$.

Conditions (i-ii) are the ones adopted by the Frege-Russell account of cardinal number, where condition (ii) is clearly (HP). However, Hallett (1984) stresses that the Zermelo-von Neumann account endorses two further conditions, which were first introduced by Cantor (1915):

iii) For every set t, $|t|$ is a set.

iv) For every set t, $|t| \approx t$.

As explained above, according to the Frege-Russell account, the cardinal number of a set or concept is the set of all sets or concepts equinumerous to it. Therefore, it is clear why the Frege-Russell account is non-representational: every cardinal number (except 0) is way bigger than the set they number. Instead, the BK **Df.9** of cardinal numbers is clearly representational because it satisfies conditions (iii-iv). This highlights the similarities between the BK definition of cardinal numbers and Cantor (1915) proposal. However, a compelling question arise concerning *representational* accounts of cardinal numbers − as either formulated by BK or Cantor (1915). If $|t|$ is a set equinumerous to t − called the *cardinal-set* − then what are the elements of $|t|$? (Cantor, 1915, pp. 282-283) claims that:

> "We denote the cardinal number or power of [a set] M, the result of this twofold abstraction by $\bar{\bar{M}}$. Since each individual element m if we disregard its nature becomes a 'one', the cardinal number $\bar{\bar{M}}$ is itself a definite set composed of nothing but ones which exists in our mind as the intellectual image or projection of the given set M."

Cantor (1915) theory of units has been criticized since its formulation. However, a discussion of Cantor abstractionist account is beyond the scope of the present work − see Hallett (1984). Instead, given the similarities between the BK account and Cantor (1915) proposal, I will develop below an abstractionist account of cardinal numbers in the lines of Cantor. More precisely, I will present the BK abstractionist account through the criticisms originally made by Frege (1980) against Cantor (1915). For the sake of clarity, I will adopt Fine (1998) formulation of two of these objections[33]. The leading idea is that, given a set t, the abstractionist process is carried out by considering an arbitrary set $|t|$, which is identified by no property except that of being equinumerous to the given set t.

Frege (1980) first objection amounts to a proof that according to Cantor (1915) abstractionist account there can be only two numbers, namely 0 and 1. If cardinal-sets are composed by indistinguishable units which correspond to the members of the given set, and if two identical cardinal-sets have identical units, then the units of any cardinal-set bigger than 2 would be equal to that of 1! Frege sums up the objection by saying that Cantor ascribes two contradictory properties to units, namely identity and distinguishability. Therefore, we must explain how a cardinal-set can result from abstracting on the elements of set even though no units of it can be uniquely associated with each element of the base set (Fine, 1998). As suggested by Tait (1996), there are two ways by which a Cantorian abstraction process can be carried out: according to the former, the units

[33]It is worth mentioning that also Fine (1998) advocates an abstractionist account in the lines of Cantor, which is formulated according to his theory of Arbitrary Objects (Fine, 1998). I leave the comparison between the BK abstractionist account and Fine (1985) one to a further work.

composing the cardinal-set $|t|$ are obtained from t by the abstraction of the properties characterizing each individual element of t. According to the latter, the units are obtained by abstraction of the property determining the set t itself. On this latter interpretation, the units of a cardinal-set $|t|$ are individuated by the only property of being members of $|t|$. While Frege (1980) first objection clearly applies to the former interpretation, the BK definition of cardinal numbers prompts for the latter. Indeed, in **Df.9** the base set t is equipollent to an arbitrary set picked out by the ε-operator. Bearing on the distinction between the combinatorial and logical view of collections presented in §11.1, it is clear that the arbitrary set picked out by the ε-operator in **Df.9** is a combinatorial set, individuated by the only property of being equinumerous to t. Therefore, the BK definition of cardinal numbers supports Tait (1996) view by which the abstraction process should be taken to apply to the elements of a set as a whole, conceived as a combinatorial collection rather than a logical extension of a predicate. This view seems to be shared by Cantor (1915) himself, when discussing Leibniz comments of numbers, asserts:

> "The addition of ones, however, can never serve for a definition of a number, since here the specification of the main thing, namely how often the ones must be added, cannot be achieved without using the number itself. This proves that the number is to be explained only as an organic unity of ones achieved by a *single act of abstraction*." [Italics added]

Frege (1980) second objection concerns the internal composition of cardinal-sets, which according to Cantor (1915) are made up by units. More precisely, we must explain how the units within a cardinal-set are indistinguishable from one another even though the units from different cardinal-sets are not (Fine, 1998). Based on **Df.9** and the Cantorian abstraction process presented above, it is possible to face this second objection as follows. I have already explained above why the members of a cardinal-set are indistinguishable from one another. This bear on the fact that in **Df.9** the base set t is equinumerous with an arbitrary set picked out by the ε-operator. So, its members are individuated by the only property of belonging to the set $-$ as for the combinatorial view of sets. Therefore, we are left to explain how the units of different cardinal-sets can be nonetheless distinguished from one another. This issue can be addressed based on the Extensionality axiom of EC. More precisely, units are individuated by the only property of being members of the arbitrary set picked out by the ε-operator. Moreover, co-extensional sets have the same members. Then, based on **Df.9**, an arbitrary set z equinumerous to t is co-extensional with an arbitrary set u equinumerous to s, only if t and s are equinumerous. But this is not the case for different cardinal sets, which explain why the units of their respective arbitrary sets are distinguished.

Finally, I will turn to the second argument supporting the BK definition of cardinal numbers over the Frege-Russell one. Indeed, Wright (1983) proves that the assumption of HP in SOL is sufficient to infer the Peano axioms for natural numbers, which realizes the logicist program of Frege (1980) by grounding arithmetical knowledge on logical basis. However, much of the argument rests of the status of HP. Indeed, HP is an implicit definition: the principle does not explicitly define $|x|$, but contextually defines it by defining contexts (in this case, identity statements) in which it occurs. This is the logical structure of abstraction principles, which comprehend also Frege's infamous Basic Law V[34]. However, given that all abstraction principles share the same logical form, how to distinguish the 'good' ones (like HP) from the 'bad' ones (like Basic Law V). Since Boolos (1987), this has been called the *Bad Company* objection to HP, which have given rise to a huge literature on the topic. However, most authors agree that even if the consistency is a necessary condition to demarcate 'good' abstraction principles from 'bad' ones, it is not sufficient. Indeed,

[34] According to which the extension of a concept/predicate is identical to the extension of another concept/predicate if and only if all and only the objects that fall under the former also fall under the latter. Basic Law V is an abstractionist principle because it implicitly defines the notion of 'extension of a concept/predicate'.

Wright (1997) argues that there are abstraction principle which are individually consistent, but not jointly consistent. Consider for example Wright (1997) *Nuisance Principle*:

$$\forall x \forall y(x \approx' y) \leftrightarrow (\S x = \S y) \tag{NP}$$

Where the relation $\approx'$ holds between the x and y if and only if the symmetric difference of the x and the y is finite. Then, NP is true *iff* the domain is finite, while HP is true only in infinite domain.

However, the Bad Company objection does not arise if someone endorse the BK definition of cardinal numbers. Indeed, **Df.9** is an explicit definition by which HP can be deduced. The cardinal-set $|t|$ is the definiendum specified by the definiens containing the ε-term. Moreover, in the context of BK, the further assumption of the Axiom of Infinity (see §11) rules out other abstraction principles like NP, which is true only in finite domain. The advantages of having an explicit definition for cardinal numbers has been pointed out also by Woods (2014), which however makes the claim for Second-order Logic rather than Set theory. However, a discussion of the ε-operator in SOL is beyond the scope of the present work.

12 Summary

In this chapter I have applied the ε-operator and EC to the theory of logical definitions and axiomatic Set theory, so as to provide a case study of the consequences of EC in the philosophy of mathematics. It is now possible to appreciate why these two topics are connected from the standpoint of EC. Indeed, in the Set theory formulated within EC (namely, BK), impredicative definitions are hamrless, namely they do not involve paradoxes. This provides a novel and consistent theory by which investigating several related philosophical issues such as the Vicious Circle Principle (VCP) and the opposition between logical and combinatorial collection. More precisely, concerning (im)predicative definitions, I have argued that:

1. Russell (1908) and Poincaré (2012) analysis of Russell's and Richard's paradoxes detects two different ways by which mathematical objects (sets and classes) can be defined according to the totality to which they belong, respectively: by being a (potential) value of the universal quantifiers contained in the *definiendum* or by being a (potential) value of the quantifiers contained in the *definiens*. These patterns of quantifiers (in)dependence can be represented in EC at the syntactic level of formulas as (un)nested ε-terms.
2. If different criteria of predicativity correspond to different patterns of quantifiers independence, then the issue of whether to adopt impredicative definitions (as stated by Definability VCP) primarily rests on semantics, rather than ontological, grounds. More precisely, if the quantifiers in the defying formula are interpreted according to EC, then the issue becomes of whether to adopt empty descriptions, which specify impredicatively even if they do not individuate impredicatively.

Instead, concerning Set theory, I have argue that:

3. The Set theory formalized in EC (namely, BK) captures the logical view of classes − as collectivizing relations − while determining combinatorial sets through the resources of the ε-operator. That is why the syntax of BK is able to keep track of the difference between logical classes (namely, collectivizing relations) and combinatorial sets (namely, arbitrary sets).
4. The BK definition of cardinal numbers provides several advantages over the Zermelo-von Neumann and Frege-Russell proposals. Unlike the former, the BK account does not require a

prior definitions of ordinal numbers. Unlike the latter, the cardinal-set is equinumerous to the given set. That is why, based on the BK definition of cardinal numbers, it is possible to develop an abstractionist account similar to that of Cantor which however resists Frege's objections. According to the BK abstractionist account, cardinal numbers are obtained by abstraction of arbitrary sets which are equinumerous to the given sets.

Conclusion

This work has carefully investigated the meaning of the classical quantifiers $\forall$ and $\exists$, as defined by both model and proof theoretic semantics. For a survey of the several claims supported so far, I refer the reader to the §Summary of each chapter. Instead, in conclusion, I will evaluate the main theoretical goal of the present work, discussing how they can be further investigated. Indeed, this work was aiming to meet three theoretical goals: i) to claim that the classical quantifiers $\forall$ and $\exists$ should be replaced by the ε-operator; ii) to highlight the notions of *functional-dependence* and *term-dependence* as arising, respectively, in model and proof theory; iii) to investigate the philosophical consequences of adopting the Epsilon Calculus as the framework for debates in the philosophy of mathematics. I think that only (i) has been extensively and satisfactorily discussed so far. Instead, (ii) and (iii) could be further investigated by focusing on two main issues, respectively:

1. The connections between the model theoretic notion of *functional-dependence* and the proof theoretic notion of *term-dependence*. Indeed, the main idea underlying both §Chapters I and II is that the expressive and inferential powers of quantification theory goes far beyond that of the classical quantifiers $\forall$ and $\exists$. Instead, by adopting the ε-operator of EC, I have explained how the dependence relations arising within formulas and inferences are formalized. Nevertheless, still much work needs to be done in order to provide a general framework for the logical representation of dependence relations, as arising in quantification theory. (Meyer Viol, 1995, p. 159) gives an insightful suggestion for further research:

   > "In constructing a semantics for dependence one is immediately confronted with the following fact. Dependence between terms in a proof does not reside in the denotations of these terms. Dependence arises in the way denotations of terms are chosen or constructed within a proof. This fits a dynamic or representational view of dependence. But as dependence is not inherent in term denotation, what kind of a semantics can we expect? It seems that the optimal semantics would be one which is closely linked to the structure of derivations. This suggests that the true semantics for dependence would be one in the vein of the Curry-Howard semantics for categorical proofs, rather than a Tarskian Semantics."

   The Curry-Howard correspondence establishes a direct relationship between natural deduction rules and terms in typed lambda calculus − see Richard (2019). However, I want to stress that the *dynamic* aspect pointed out by Meyer Viol arises from both the model and proof theoretic semantics of the classical quantifiers $\forall$ and $\exists$. On the one hand, according to *functional-dependence*, the value chosen for a quantified variable co-varies with that of another one. On the other hand, according to *term-dependence*, the choice of the value for an instantial variable depends on that made in a previous inferential step. That is why I conclude that that the dynamic process is a condition for the correct formalization of both functional-dependence and term-dependence.

2. The logicality of the Axiom of Global Choice (AGC). Indeed, the application of the ε-operator for debates in the philosophy of mathematics and the adoption of EC as logical framework has been triggered by the worries concerning the equivalence between the ε-axioms and AGC. More precisely, the concern runs as follows: given that no conclusive argument has yet been proposed to grounds the Axiom of Choice (AC) on either logical or set theoretical basis, then the stronger version of AGC (and so also the ε-operator) should be rejected for being too theoretically demanding. Given the poor results of supporting AC from a set theoretical perspective (Moore, 2012), I think that the only way to stop the concerns regarding the ε-operator is by arguing for the logicality of AGC. It should be remarked that this idea was

shared by some of the founders of modern axiomatic set theory, such as Cantor, Zermelo and (Hilbert, 1922, p. 152) himself, who concerning EC asserts that:

> "The essential idea on which the Axiom of Choice is based constitutes a general logical principle which, even for the first elements of mathematical inference, is necessary and indispensable. When we secure these first elements, we obtain at the same time the foundation for the Axiom of Choice. Both are done by means of my proof theory."

Indeed, the arguments for the logicality of AGC are strictly related to the constructive interpretation of quantifiers, by which quantifiers are primarily adopted as witnessing devices. Here there are two main issue that require further investigation: i) whether the constructive interpretation of quantifiers is the one formalized by the ε-operator and discussed in §Chapters I and II; ii) whether accepting the constructivist interpretation of quantifiers entails the commitment to a choice principle (like AGC) in the metalanguage.

References

Ackermann, W. (1938). Mengentheoretische begründung der logik. *Journal of Symbolic Logic*, 3(2):85–85.

Alechina, N. and Van Lambalgen, M. (1996). Generalized quantification as substructural logic. *The Journal of Symbolic Logic*, 61(3):1006–1044.

Anacona, M., Arboleda, L. C., and Pérez-Fernández, F. J. (2014). On bourbaki's axiomatic system for set theory. *Synthese*, 191(17):4069–4098.

Anellis, I. (1991). Forty years of "unnatural'natural deduction and quantification: A history of first-order systems of natural deduction from gentzen, to copi. *Modern Logic*, 2:113.

Bernays, P. (1983). On platonism in mathematics, 1935. *Philosophy of Mathematics: Selected Readings*, pages 274–286.

Bernays, P. (1991). *Axiomatic Set Theory*. Dover Books on Mathematics. Dover Publications.

Blass, A. and Gurevich, Y. (2000). The logic of choice. *Journal of Symbolic Logic*, 65(3):1264–1310.

Boccuni, F. (2015). Frege,Äôs grundgesetze and a reassessment of predicativity. In *From Logic to Practice*, pages 53–70. Springer.

Boccuni, F. and Woods, J. (2018). Structuralist neologicism. *Philosophia Mathematica*.

Bonnay, D. (2006). Logicality and invariance. *Bulletin of Symbolic Logic*, 14(1):29–68.

Boolos, G. (1987). The consistency of frege's foundations of arithmetic. In Thomson, J., editor, *On Being and Saying: Essays in Honor of Richard Cartwright*, pages 3–20. MIT Press.

Bourbaki, N. (1968). *Theory of sets*. Springer.

Brandom, R. (2009). *Articulating reasons: An introduction to inferentialism*. Harvard University Press.

Breckenridge, W. and Magidor, O. (2012). Arbitrary reference. *Philosophical Studies*, 158(3):377–400.

Caicedo, X. (1991). Hilbert ε-symbol in the presence of generalized quantifiers. *Bulletin of the Section of Logic*, 20(3/4):85–86.

Cantor, G. (1915). *Contributions to the Founding of the Theory of Transfinite Numbers*. Number 1. Open Court Publishing Company.

Carnap, R. (1961). On the use of hilbert's ε-operator in scientific theories. In Bar-Hillel, Y., editor, *Essays on the Foundations of Mathematics*, pages 156–164. Magnes Press.

Chang, C. C. and Keisler, H. J. (1990). *Model Theory Vol. 73*. Elsevier.

Chihara, C. S. (1973). *Ontology and the Vicious Circle Principle*. Ithaca [N.Y.]Cornell University Press.

Copi, I. (1954). *Symbolic Logic*. New York: Macmillan.

Davidson, D. (1965). Theories of meaning and learnable languages. In Bar-Hillel, Y., editor, *Proceedings of the International Congress for Logic, Methodology, and Philosophy of Science*, pages 3–17. North-Holland.

DeVidi, D. and Korté, H. (2014). Naturalizing natural deduction. De Gruyter.

Donnellan, K. S. (1966). Reference and definite descriptions. *The philosophical review*, pages 281–304.

Došen, K. (1989). Logical constants as punctuation marks. *Notre Dame journal of formal logic*, 30(3):362–381.

Dummett, M. (1991). *The logical basis of metaphysics*. Harvard university press.

Egli, U., Von Heusinger, K., et al. (1995). The epsilon operator and e-type pronouns. *Amsterdam Studies in the Theory and History of Linguistic Science Series 4*, pages 121–121.

Feferman, S. (1965). Classifications of recursive functions by means of hierarchies. *Journal of Symbolic Logic*, 30(3):388–389.

Feferman, S. (1999). Logic, logics, and logicism. *Notre Dame Journal of Formal Logic*, 40(1):31–54.

Feferman, S. (2006). What kind of logic is ‚Äòindependence-friendly‚Äôlogic? volume 30, pages 453–469. Open Court ˆ eChicago Chicago.

Feferman, S. (2007). Predicativity. In *The Oxford Handbook of Philosophy of Mathematics and Logic*.

Fine, K. (1985). *Reasoning with Arbitrary Objects*. Blackwell.

Fine, K. (1998). Cantorian abstraction: A reconstruction and defense. *The journal of philosophy*, 95(12):599–634.

Fitch, F. B. (1952). *Symbolic Logic*. New York: Ronald Press Co.

Fraenkel, A. A., Bar-Hillel, Y., and Levy, A. (1973). *Foundations of set theory*. Elsevier.

Frege, G. (1879). Begriffsschrift, a formula language, modeled upon that of arithmetic, for pure thought. *From Frege to Gödel: A source book in mathematical logic*, 1931:1–82.

Frege, G. (1980). *The foundations of arithmetic: A logico-mathematical enquiry into the concept of number*. Northwestern University Press.

Gentzen, G. (1969). *The Collected Papers of Gerhard Gentzen*. Amsterdam: North-Holland Pub. Co.

Gödel, K. (1944). Russell's mathematical logic. In Feferman, S., Dawson, J., and Kleene, S., editors, *Journal of Symbolic Logic*, pages 119–141. Northwestern University Press.

Goldfarb, W. D. (1979). Logic in the twenties: The nature of the quantifier. *Journal of Symbolic Logic*, 44(3):351–368.

Grimm, J. (2010). Implementation of bourbaki's elements of mathematics in coq: Part one, theory of sets. *Journal of Formalized Reasoning*, 3(1):79–126.

Gupta, A. (2019). Definitions. In Zalta, E. N., editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, winter 2019 edition.

Haack, S. (1978). *Philosophy of Logics*. Cambridge University Press.

Hallett, M. (1984). *Cantorian Set Theory and Limitation of Size*. Clarendon Press.

Hazen, A. (1983). Predicative logics. In *Handbook of philosophical logic*, pages 331–407. Springer.

Hazen, A. (1987). Natural deduction and hilbert's ε-operator. *Journal of Philosophical Logic*, pages 411–421.

Heinzmann, G. and Petitot, J. (2020). The Functional Role of Structures in Bourbaki. In Reck, H., E., Schiemer, and G., editors, *The Prehistory of Mathematical Structuralism*, pages 187–214. Oxford University Press.

Henkin, L. (1949). The completeness of the first-order functional calculus. *J. Symbolic Logic*, 14(3):159–166.

Henkin, L. (1961). Some remarks on infinitely long formulas. In *Journal of Symbolic Logic*, pages 167–183. Pergamon Press.

Hilbert, D. (1922). The logical foundations of mathematics. *mathematical annals*, 88(1-2):151–165.

Hilbert, D. and Bernays, P. (1939). Grundlagen der mathematik, vol. 1 (1934), vol ii (1939).

Hintikka, J. (1996). *The Principles of Mathematics Revisited*. Cambridge University Press.

Hintikka, J. (1997). *The Place of C.S. Peirce in the History of Logical Theory*, pages 140–161. Springer Netherlands, Dordrecht.

Hintikka, J. (1998). Truth definitions, skolem functions and axiomatic set theory. *Bulletin of Symbolic Logic*, 4(3):303–337.

Hintikka, J. (1999). Is the axiom of choice a logical or set-theoretical principle? *Dialectica*, 53(3-4):283–290.

Hintikka, J. (2012). If logic, definitions and the vicious circle principle. *Journal of Philosophical Logic*, 41(2):505–517.

Horsten, L. (2019). *The Metaphysics and Mathematics of Arbitrary Objects*. Cambridge University Press.

Iacona, A. (2018). *Logical Form*. Springer.

Incurvati, L. (2020). *Conceptions of Set and the Foundations of Mathematics*. Cambridge University Press.

Jung, D. (1999). Russell, presupposition, and the vicious-circle principle. *Notre Dame Journal of Formal Logic*, 40(1):55–80.

Keenan, E. L. (1992). Beyond the frege boundary. *Linguistics and Philosophy*, 15(2):199–221.

Keisler, H. J. (1970). Logic with the quantifier ‚Äúthere exist uncountably many‚Äù. *Annals of Mathematical Logic*, 1(1):1–93.

King, J. C. (1991). Instantial terms, anaphora and arbitrary objects. *Philosophical Studies*, 61(3):239–265.

Krynicki, M. (1995). Henkin quantifiers,[w:] krynicki m., mostowski m., szczerba lw (red.). In Krynicki, M., Mostowski, M., and Szczerba, L., editors, *Quantifiers: Logics, Models and Computation*. Kluwer Academic Publishers.

Leisenring, A. C. (1969). *Mathematical Logic and Hilbert's & Symbol*. London: Macdonald Technical & Scientific.

Leitgeb, H. (2020). On non-eliminative structuralism. unlabeled graphs as a case study, part a. *Philosophia Mathematica*.

Lindström, P. (1966). First order predicate logic with generalized quantifiers. *Theoria*, 32(3):186–195.

Maddy, P. (1983). Proper classes. *Journal of Symbolic Logic*, 48(1):113–139.

Martino, E. (2001). Arbitrary reference in mathematical reasoning. *Topoi*, 20(1):65–77.

Mathias, A. R. (1992). The ignorance of bourbaki. *The mathematical intelligencer*, 14(3):4–13.

McGee, V. (1996). Logical operations. *Journal of Philosophical Logic*, 25(6):567–580.

Meyer Viol, W. (1995). Instantial logic. Master's thesis.

Mints, G. (2012). Intuitionistic existential instantiation and epsilon symbol.

Miraglia, P. (1996). *Can We Intend an Interpretation?* PhD thesis, The Ohio State University.

Moore, G. H. (2012). *Zermelo's axiom of choice: Its origins, development, and influence.* Courier Corporation.

Mostowski, A. (1957). On a generalization of quantifiers. *Fundamenta Mathematicae*, 44(2):12–36.

Neale, S. (1990). *Descriptions.* MIT Press.

Negri, S., Von Plato, J., and Ranta, A. (2008). *Structural proof theory.* Cambridge University Press.

Otto, M. (2000). Epsilon-logic is more expressive than first-order logic over finite structures. *Journal of Symbolic Logic*, 65(4):1749–1757.

Peacocke, C. (1988). Understanding logical constants: A realist's account. In *Proceedings of the British Academy, Volume 73: 1987*, page 153.

Pelletier, F. J. (1999). A brief history of natural deduction. *History and Philosophy of Logic*, 20(1):1–31.

Peters, S. and Westerståhl, D. (2006). *Quantifiers in Language and Logic.* Clarendon Press.

Poincaré, H. (2012). *The value of science: essential writings of Henri Poincaré.* Modern library.

Prawitz, D. (1965). *Natural Deduction: A Proof-Theoretical Study.* Dover Publications.

Quine, W. V. (1985). Events and reification. In Lepore, E. and McLaughlin, B., editors, *Actions and Events: Perspectives on the Philosophy of Davidson*, pages 162–71. Blackwell.

Quine, W. V. O. (1950). *Methods of Logic.* Harvard University Press.

Ramsey, F. P. (1926). The foundations of mathematics.

Restall, G. (2019). Generality and existence 1: Quantification and free logic. *The Review of Symbolic Logic*, 12(1):1–29.

Richard, Z. (2019). The significance of the curry-howard isomorphism. In Mras, G. M., Weingartner, P., and Ritter, B., editors, *Philosophy of Logic and Mathematics. Proceedings of the 41st International Ludwig Wittgenstein Symposium*, pages 313–326. Berlin: De Gruyter.

Russell, B. (1908). Mathematical logic as based on the theory of types. *American Journal of Mathematics*, 30(3):222–262.

Russell, B. and Whitehead, A. N. (1910). *Principia Mathematica Vol. I.* Cambridge University Press.

Sambin, G., Battilotti, G., and Faggian, C. (2000). Basic logic: Reflection, symmetry, visibility. *Journal of Symbolic Logic*, 65(3):979–1013.

Schiemer, G. and Gratzl, N. (2016). The epsilon-reconstruction of theories and scientific structuralism. *Erkenntnis*, 81(2):407–432.

Schroeder-Heister, P. (2018). Proof-theoretic semantics. *Stanford Encyclopedia of Philosophy.*

Sheffer, H. M. (1926). Principia mathematica. whitehead, alfred north, russell, bertrand.

Sher, G. (1991). *The Bounds of Logic: A Generalized Viewpoint.* MIT Press.

Sher, G. (2012). Logical quantifiers. In Fara, D. G. and Russell, G., editors, *Routledge Companion to Philosophy of Language*, pages 579–595. Routledge.

Shoenfield, J. R. (1967). *Mathematical Logic.* Reading, Mass., Addison-Wesley Pub. Co.

Tait, W. (1996). Frege versus cantor and dedekind: On the concept of number.

Tarski, A. and Corcoran, J. (1986). What are logical notions? *History and philosophy of logic*, 7(2):143–154.

Tennant, N. (1980). On epsilon and e. *Analysis*, 40(1):5.

Tiu, A. (2005). Model checking for π-calculus using proof search. In *International Conference on Concurrency Theory*, pages 36–50. Springer.

Väänänen, J. (2007). *Dependence Logic: A New Approach to Independence Friendly Logic.* Cambridge University Press.

Väänänen, J. and Hodges, W. (2010). Dependence of variables construed as an atomic formula. *Annals of Pure and Applied Logic*, 161(6):817–828.

van Dalen, D. (1980). *Logic and Structure.* Springer Verlag.

Van Heijenoort, J. (1967). *From Frege to Gödel: a source book in mathematical logic, 1879-1931*, volume 9. Harvard University Press.

von Plato, J. (2014). Generality and existence: Quantificational logic in historical perspective. *Bulletin of Symbolic Logic*, 20(4):417–448.

Wang, H. (1955). On denumerable bases of formal systems. *Mathematical interpretation of formal systems*, 16:57–84.

Wang, H. (1957). On denumerable bases of formal systems. *Journal of Symbolic Logic*, 22(3):292–293.

Wolf, R. S. (2005). *A tour through mathematical logic*, volume 30. American Mathematical Soc.

Woods, J. (2014). Logical indefinites. *Logique Et Analyse – Special Issue Edited by Julien Murzi and Massimiliano Carrara*, 227:277–307.

Wright, C. (1983). *Frege's Conception of Numbers as Objects.* Aberdeen University Press.

Wright, C. (1997). On the philosophical significance of frege's theorem. In Heck, R. G., editor, *Language, Thought, and Logic: Essays in Honour of Michael Dummett*, pages 201–44. Oxford University Press.

Zach, R. (2003). The practice of finitism: Epsilon calculus and consistency proofs in hilbert's program. *Synthese*, 137(1-2):211–259.

Zach, R. (2007). Hilbert‚Äôs program then and now. *Philosophy of Logic*, page 411‚Äì447.

Zach, R. (2017). Semantics and proof theory of the epsilon calculus. In Ghosh, S. and Prasad, S., editors, *Logic and Its Applications. ICLA 2017*, pages 27–47. Berlin, Heidelberg: Springer.

Zimmermann, T. E. (2019). *Model-theoretic semantics*, pages 181 – 231. De Gruyter Mouton, Berlin, Boston.