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The intersection graph of a finite simple group has diameter at most 5

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Abstract. Let G be a non-abelian finite simple group. In addition, let Δ_G be the intersection graph of G, whose vertices are the proper non-trivial subgroups of G, with distinct subgroups joined by an edge if and only if they intersect non-trivially. We prove that the diameter of Δ_G has a tight upper bound of 5, thereby resolving a question posed by Shen (Czechoslov Math J 60(4):945–950, 2010). Furthermore, a diameter of 5 is achieved only by the baby monster group and certain unitary groups of odd prime dimension.

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1. Introduction. For a finite group G, let Δ_G be the *intersection graph* of G. This is the graph whose vertices are the proper non-trivial subgroups of G, with two distinct vertices S_1 and S_2 joined by an edge if and only if $S_1 \cap S_2 \neq 1$. We write $d(S_1, S_2)$ to denote the distance in Δ_G between vertices S_1 and S_2, and if these vertices are joined by an edge, then we write $S_1 \sim S_2$. Additionally, $\mathrm{diam}(\Delta_G)$ denotes the diameter of Δ_G.

Csákány and Pollák [5] introduced the graph Δ_G in 1969 as an analogue of the intersection graph of a semigroup defined by Bosák [1] in 1964. For finite non-simple groups G, Csákány and Pollák determined the cases where Δ_G is connected, and proved that, in these cases, $\mathrm{diam}(\Delta_G) \leqslant 4$ (see also [14, Lemma 5]). It is not known if there exists a finite non-simple group G with $\mathrm{diam}(\Delta_G) = 4$.

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Suppose now that G is a non-abelian finite simple group. In 2010, Shen [14] proved that Δ_G is connected, and asked two questions: does $\mathrm{diam}(\Delta_G)$ have an upper bound? If yes, does the upper bound of 4 from the non-simple case also apply here? In the same year, Herzog, Longobardi, and Maj [7] independently showed that the subgraph of Δ_G induced by the *maximal* subgroups of G is connected with diameter at most 62. As each proper non-trivial subgroup of G is adjacent in Δ_G to some maximal subgroup, this implies an upper bound of 64 for $\mathrm{diam}(\Delta_G)$, resolving Shen's first question. Ma [12] reduced this upper bound to 28 in 2016. In the other direction, Shahsavari and Khosravi [13, Theorem 3.7] proved in 2017 that $\mathrm{diam}(\Delta_G) \geqslant 3$.

In this paper, we significantly reduce the previously known upper bound of 28 for $\mathrm{diam}(\Delta_G)$, and show that the new bound is best possible. In particular, we prove the following theorem, which resolves Shen's second question with a negative answer.

Theorem 1.1. *Let G be a non-abelian finite simple group.*

(i) Δ_G is connected with diameter at most 5.

(ii) If G is the baby monster group $\mathbb{B}$, then $\mathrm{diam}(\Delta_G) = 5$.

(iii) If $\mathrm{diam}(\Delta_G) = 5$ *and $G \not\cong \mathbb{B}$, then G is a unitary group* $\mathrm{U}_n(q)$, *with n an odd prime and q a prime power.*

Remark 1.2. Using information from the ATLAS [4], we can show that if S_1 and S_2 are vertices of $\Delta_{\mathbb{B}}$ with $d(S_1, S_2) = 5$, then $|S_1| = |S_2| = 47$.

Remark 1.3. If $G \in \{\mathrm{U}_3(3), \mathrm{U}_3(5), \mathrm{U}_5(2)\}$, then G has no maximal subgroup of odd order [11, Theorem 2]. As we will explain in the proof of Theorem 1.1, this implies that $\mathrm{diam}(\Delta_G) \leqslant 4$. Indeed, we can use information from the ATLAS [4] to show that $\mathrm{diam}(\Delta_{\mathrm{U}_3(3)}) = 3$. Furthermore, even though $\mathrm{U}_3(7)$ has a maximal subgroup of odd order, we deduce from calculations in Magma [2] that $\mathrm{diam}(\Delta_{\mathrm{U}_3(7)}) = 4$. On the other hand, we can adapt the proof of Theorem 1.1(ii), with the aid of several Magma calculations, to show that $\mathrm{diam}(\Delta_{\mathrm{U}_7(2)}) = 5$.

It is an open problem to classify the finite simple unitary groups G with $\mathrm{diam}(\Delta_G) = 5$.

2. Proof of Theorem 1.1. In order to prove Theorem 1.1 in the unitary case, we will require the following proposition. For a prime power q, let f be the unitary form on the vector space $V := \mathbb{F}_{q^2}^3$ whose Gram matrix is the 3×3 identity matrix, and let $\mathrm{SU}_3(q)$ be the associated special unitary group. Then the standard basis for (V, f) is orthonormal, and a matrix $A \in \mathrm{SL}_3(q^2)$ lies in $\mathrm{SU}_3(q)$ if and only if $A^{-1} = A^{\sigma\mathsf{T}}$, where σ is the field automorphism $\alpha \mapsto \alpha^q$ of $\mathbb{F}_{q^2}$. For a subspace U of V, we will write $\mathrm{SU}_3(q)_U$ to denote the stabiliser of U in $\mathrm{SU}_3(q)$.

Proposition 2.1. *Let q be a prime power greater than* 2, *and let X and Y be one-dimensional subspaces of the unitary space (V, f), with X non-degenerate. Then* $\mathrm{SU}_3(q)_X \cap \mathrm{SU}_3(q)_Y$ *contains a non-scalar matrix.*

Proof. We may assume without loss of generality that X contains the vector $(1,0,0)$. Let (a,b,c) be a non-zero vector of Y. In addition, let ω be a primitive element of $\mathbb{F}_{q^2}$, and let $\lambda := \omega^{q-1}$. Then $|\lambda| = q+1 > 3$. If at least one of a, b, and c is equal to 0, then $\mathrm{SU}_3(q)_X \cap \mathrm{SU}_3(q)_Y$ contains a non-scalar diagonal matrix with two diagonal entries equal to λ and one equal to λ^{-2} (not necessarily in that order).

Suppose now that a, b, and c are all non-zero, and let $\mu := b^{-1}c$. We may assume that $a = 1$. The trace map $\alpha \mapsto \alpha + \alpha^q$ from $\mathbb{F}_{q^2}$ to $\mathbb{F}_q$ is $\mathbb{F}_q$-linear, and hence has a non-trivial kernel. In particular, there exists $\beta \in \mathbb{F}_{q^2}$ such that $\beta \neq 1$ and $\beta + \beta^q = 2$. It follows from simple calculations that if $\mu^{q+1} = -1$, then $\mathrm{SU}_3(q)_X \cap \mathrm{SU}_3(q)_Y$ contains

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & \mu(1-\beta^q) \\ 0 & \mu^{-1}(1-\beta) & \beta^q \end{pmatrix}.$$

If instead $\mu^{q+1} \neq -1$, then we can define $\gamma := \lambda^{-2}(\lambda^3 + \mu^{q+1})(1+\mu^{q+1})^{-1}$. In this case, $\mathrm{SU}_3(q)_X \cap \mathrm{SU}_3(q)_Y$ contains

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \gamma & \mu(\lambda - (\gamma\lambda)^q) \\ 0 & \mu^{-1}(\lambda-\gamma) & (\lambda\gamma)^q \end{pmatrix}.$$

Note that $\lambda \neq \gamma$ since $|\lambda| > 3$. □

Proof of Theorem 1.1. Let S_1 and S_2 be proper non-trivial subgroups of G, and let M_1 and M_2 be maximal subgroups of G that contain S_1 and S_2, respectively. Since $d(M_1, M_2) \leqslant d(S_1, M_2) \leqslant d(S_1, S_2)$, we may assume that S_1 and S_2 are not maximal in G. We may also assume that $M_1 \neq M_2$, as otherwise $S_1 \sim M_1 \sim S_2$ and $d(S_1, S_2) \leqslant 2$.

Suppose first that $|M_1|$ and $|M_2|$ are even. Then, as observed in the proof of [7, Proposition 3.1], there exist involutions $x \in M_1$ and $y \in M_2$, with $\langle x, y\rangle$ equal to a (proper) dihedral subgroup D of G (with $|D| = 2$ allowed). Hence $S_1 \sim M_1 \sim D \sim M_2 \sim S_2$, and so $d(S_1, S_2) \leqslant 4$. In particular, if every maximal subgroup of G has even order, then $\mathrm{diam}(\Delta_G) \leqslant 4$, as noted in the proof of [12, Lemma 2.3].

It remains to consider the case where G contains a maximal subgroup of odd order. Liebeck and Saxl [11, Theorem 2] present a list containing all possibilities for G and its maximal subgroups of odd order. By the previous paragraph, we may assume that the maximal subgroup M_1 has odd order. However, $|M_2|$ may be even. In what follows, information about the sporadic simple groups is taken from the Atlas [4], except where specified otherwise.

(i) $G = A_p$, with p prime, $p \equiv 3 \pmod 4$, and $p \notin \{7, 11, 23\}$. By [5, Theorem 2] (see also [14, Assertion I]), the intersection graph of any simple alternating group has diameter at most 4.

(ii) $G = \mathrm{L}_2(q)$, with q a prime power and $q \equiv 3 \pmod 4$. The group G acts transitively on the set Ω of one-dimensional subspaces of the vector space $\mathbb{F}_q^2$. Additionally, $M_1 = G_U$ for some $U \in \Omega$, and $G_U \cap G_W \neq 1$ for each $W \in \Omega$.

If $|M_2|$ is odd, then $M_2 = G_W$ for some W, and it follows that $M_1 \sim M_2$ and $d(S_1, S_2) \leqslant 3$. We may therefore assume that M_2 contains an involution g. Then g fixes no subspace in Ω, and so $g \in G_{\{U,X\}}$ for some $X \in \Omega \setminus \{U\}$. Since the non-trivial subgroup $G_U \cap G_X$ lies in both $M_1 = G_U$ and $G_{\{U,X\}}$, we deduce that $S_1 \sim M_1 \sim G_{\{U,X\}} \sim M_2 \sim S_2$. Thus $d(S_1, S_2) \leqslant 4$.

(iii) $G = \mathrm{L}_n(q)$, with n an odd prime, q a prime power, and $G \not\cong \mathrm{L}_3(4)$. Similarly to the previous case, the group G and its overgroup $R := \mathrm{PGL}_n(q)$ act transitively on the set Ω of one-dimensional subspaces of the vector space $\mathbb{F}_q^n$. Here, $M_1 = G \cap N_R(K)$, where K is a Singer subgroup of R, i.e., a cyclic subgroup of order $(q^n - 1)/(q - 1)$ (see [8, §1–2]).

Now, M_1 contains a non-identity element m that fixes a subspace $X \in \Omega$ [8, p. 497]. Observe that $m^k \in M_1$ for each $k \in K$. The action of K on Ω is transitive, and hence each subspace in Ω is fixed by some non-identity element of M_1. Therefore, if a non-identity element of S_2 fixes a subspace $U \in \Omega$, then $S_1 \sim M_1 \sim G_U \sim S_2$ and $d(S_1, S_2) \leqslant 3$. Otherwise, since n is prime, there exists $g \in G$ such that $S_2 \cap M_1^g \neq 1$. Thus $S_1 \sim M_1 \sim G_X \sim M_1^g \sim S_2$ and $d(S_1, S_2) \leqslant 4$.

(iv) $G = \mathrm{U}_n(q)$, with n an odd prime, q a prime power, and $G \not\cong \mathrm{U}_3(3)$, $\mathrm{U}_3(5)$, or $\mathrm{U}_5(2)$. Here, G acts intransitively on the set of one-dimensional subspaces of the vector space $\mathbb{F}_{q^2}^n$. Let $(q + 1, n)$ denote the greatest common divisor of $q+1$ and n. The maximal subgroup M_1 is equal to $N_G(T)$, where T is a Singer subgroup of G, i.e., a cyclic subgroup of order $\frac{q^n+1}{(q+1)(q+1,n)}$ (see [8, §5]). In fact, each maximal subgroup of G of odd order is conjugate to M_1. Similarly to the linear case, M_1 contains a non-identity element that fixes a one-dimensional subspace X of $\mathbb{F}_{q^2}^n$ [8, p. 512].

Let $L := G_X$. Then $M_1 \sim L$, and we can calculate $|L|$ using [3, Table 2.3]. In particular, $|L|$ is even. Hence if $|M_2|$ is even, then G contains a dihedral subgroup D such that $S_1 \sim M_1 \sim L \sim D \sim M_2 \sim S_2$, and $d(S_1, S_2) \leqslant 5$. If $|M_2|$ is odd, then there exists an element $g \in G$ such that $M_2 = M_1^g$. Thus $L^g \sim M_2$. If $n = 3$ and X is non-degenerate, then it follows from Proposition 2.1 that $L \sim L^g$. Therefore, $S_1 \sim M_1 \sim L \sim L^g \sim M_2 \sim S_2$ and $d(S_1, S_2) \leqslant 5$. In the remaining cases, we will show that $|L|^2/|G| > 1$, and hence $|L|\,|L^g| > |G|$. It will follow that $L \cap L^g \neq 1$, again yielding $d(S_1, S_2) \leqslant 5$.

Observe that $|L|^2/|G| > 1$ if and only if $\log|G|/\log|G : L| > 2$. By [6, Proposition 3.2], if $n \geqslant 7$, then $\log|G|/\log|G : L| > 2$, as required. If instead $n = 3$, then we may assume that X is totally singular. Here, $q > 2$, and hence

$$|L|^2/|G| = \frac{q^3(q^2 - 1)}{(q^3 + 1)(q + 1, 3)} \geqslant \frac{q^3(q - 1)}{(q^3 + 1)} > 1.$$

Suppose finally that $n = 5$. If X is totally singular, then $|L|^2/|G|$ is equal to

$$\frac{q^{10}(q^2 - 1)^3(q^3 + 1)}{(q^4 - 1)(q^5 + 1)(q + 1, 5)} > \frac{q^{10}}{(q^4 - 1)(q^5 + 1)} = \frac{q^{10}}{q^9 - q^5 + q^4 - 1} > 1.$$

If instead X is non-degenerate, then

$$|L|^2/|G| = \frac{q^2(q+1)\prod_{i=1}^{4}(q^i-(-1)^i)}{(q^5+1)(q+1,5)} > \frac{q^2(q^4-1)}{q^5+1} = \frac{q^6-q^2}{q^5+1} > 1.$$

(v) $G = \mathrm{M}_{23}$. In this case, M_1 has shape 23:11. We argue as in the proof of [14, Assertion I]. There exists a maximal subgroup L of G isomorphic to M_{22}, and $|M_1|\,|L|$ and $|M_2|\,|L|$ are greater than $|G|$ (for any choice of M_2). It follows that $S_1 \sim M_1 \sim L \sim M_2 \sim S_2$, and so $d(S_1, S_2) \leqslant 4$.

(vi) $G = \mathrm{Th}$. Here, M_1 has shape 31:15. If the proper non-trivial subgroup S_1 of M_1 has order 31, then S_1 lies in a maximal subgroup of shape $2^5{\cdot}\mathrm{L}_5(2)$. Otherwise, $|C_G(S_1)|$ is even. Therefore, in each case, S_1 lies in a maximal subgroup of even order. The same is true for S_2, and thus $d(S_1, S_2) \leqslant 4$.

(vii) $G = \mathbb{B}$. In this case, M_1 has shape 47:23. Additionally, G has a maximal subgroup $K \cong \mathrm{Fi}_{23}$, which has even order, and $M_1 \sim K$. Hence if $|M_2|$ is even, then $S_1 \sim M_1 \sim K \sim D \sim M_2 \sim S_2$ for some dihedral subgroup D of G, yielding $d(S_1, S_2) \leqslant 5$. Otherwise, there exists an element $g \in G$ such that $M_2 = M_1^g$, and hence $K^g \sim M_2$. As $|K|^2/|G| > 1$, we conclude that $S_1 \sim M_1 \sim K \sim K^g \sim M_2 \sim S_2$ and $d(S_1, S_2) \leqslant 5$. Thus $\mathrm{diam}(\Delta_G) \leqslant 5$.

We now show that $\mathrm{diam}(\Delta_G)$ is equal to 5. Let H be a subgroup of M_1 of order 23. Then H is a Sylow subgroup of G. It follows from [15, p. 67] that each maximal subgroup of G that contains H is conjugate either to M_1, to K, or to a subgroup L of shape $2^{1+22}{\cdot}\mathrm{Co}_2$. We may assume that $H \leqslant M_1 \cap K \cap L$. Additionally, $N_G(H)$ has shape $(23{:}11) \times 2$ and $N_L(H) = N_G(H)$, while $|N_G(H) : N_{M_1}(H)| = 22$. Since the 22 non-identity elements of H fall into two K-conjugacy classes and $C_K(H) = H$, we conclude that $N_K(H)$ has shape 23:11, and so $|N_G(H) : N_K(H)| = 2$.

Consider the pairs (H', M'), where H' is a G-conjugate of H, M' is a G-conjugate of M_1, and $H' \leqslant M'$. As any two G-conjugates of H appear in an equal number of such pairs, we deduce that H lies in exactly $|N_G(H) : N_{M_1}(H)| = 22$ G-conjugates of M_1. Similarly, H lies in two G-conjugates of K and one G-conjugate of L.

As M_1 has shape 47:23, it contains a subgroup S of order 47. In fact, M_1 is the unique maximal subgroup of G that contains S. Hence if J is a maximal subgroup of G satisfying $J \neq M_1$ and $J \cap M_1 \neq 1$, then J contains a G-conjugate of H. Let $\mathcal{U}$ be the set of G-conjugates of H that lie in at least one such maximal subgroup J, or in M_1. There are 47 subgroups of order 23 in M_1, each of which lies in two G-conjugates of K, and there are $|K : N_K(H)|$ subgroups of order 23 in K. Therefore, there are fewer than $47 \cdot 2|K : N_K(H)|$ subgroups in $\mathcal{U}$ that lie in at least one G-conjugate of K. By considering the G-conjugates of M_1 and L similarly, we conclude that

$$|\mathcal{U}| < 47(2|K : N_K(H)| + 22 \cdot 47 + |L : N_L(H)|) < |G : M_1|/22.$$

Hence there exists $g \in G$ such that no subgroup of M_1^g lies in $\mathcal{U}$. This means that M_1 and M_1^g are not adjacent in Δ_G and have no common neighbours, and so $d(M_1, M_1^g) > 2$. As M_1 and M_1^g are the unique neighbours of S and S^g, respectively, it follows that $d(S, S^g) > 4$. Therefore, $\mathrm{diam}(\Delta_G) = 5$.

(viii) $G = \mathbb{M}$. Liebeck and Saxl list two possible maximal subgroups of odd order (up to conjugacy), of shape $59{:}29$ and $71{:}35$, respectively. However, these subgroups are not, in fact, maximal: the former lies in the maximal subgroup $\mathrm{L}_2(59)$ constructed in [9], and the latter lies in the maximal subgroup $\mathrm{L}_2(71)$ constructed in [10]. Hence G has no maximal subgroup of odd order, and so $\mathrm{diam}(\Delta_G) \leqslant 4$. □

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