J. Aust. Math. Soc. **84** (2008), 39–61
doi:10.1017/S1446788708000086

CANCELLATIVE AND MALCEV PRESENTATIONS FOR FINITE REES INDEX SUBSEMIGROUPS AND EXTENSIONS

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(Received 29 September 2006; revised 17 May 2007)

Communicated by D. Easdown

Abstract

It is known that, for semigroups, the property of admitting a finite presentation is preserved on passing to subsemigroups and extensions of finite Rees index. The present paper shows that the same holds true for Malcev, cancellative, left-cancellative and right-cancellative presentations. (A Malcev (respectively, cancellative, left-cancellative, right-cancellative) presentation is a presentation of a special type that can be used to define any group-embeddable (respectively, cancellative, left-cancellative, right-cancellative) semigroup.)

2000 *Mathematics subject classification*: 20M05.

Keywords and phrases: Maclev presentation, cancellative, subsemigroup, finite index, rewriting.

1. Introduction

Many properties of groups are preserved under passing to finite extensions or to subgroups of finite index. Examples include the concepts of finite generation and finite presentability, as asserted by the celebrated Reidemeister–Schreier theorem (see, for example, [18, Section II.4]). There have been various efforts to find an analogous concept of index for semigroups whose finiteness also preserves such properties.

The earliest such semigroup index to be defined was the Rees index, introduced by Jura [17]. The *Rees index* of a subsemigroup T in a semigroup S is the cardinality of $S - T$, and therefore is clearly not a generalization of the group index. If T is a subsemigroup of S of finite Rees index, then S is called a *small extension* of T and T a *large subsemigroup* of S. Passing to large subsemigroups and small extensions does, however, preserve the properties of finite generation and finite presentability.

THEOREM 1.1 [10]. *Let S be a semigroup and let T be a subsemigroup of S of finite Rees index. Then S is finitely generated if and only if T is finitely generated.*

THEOREM 1.2 [21]. *Let S be a semigroup and let T be a subsemigroup of S of finite Rees index. Then S is finitely presented if and only if T is finitely presented.*

The purpose of this paper is to prove that passing to large subsemigroups or small extensions preserves the property of admitting a finite Malcev, cancellative, left-cancellative, or right-cancellative presentation. (A Malcev presentation is a special type of semigroup presentation that can be used to define any group-embeddable semigroup. Similarly, a cancellative (respectively, left-cancellative, right-cancellative) presentation can be used to defined any cancellative (respectively, left-cancellative, right-cancellative) semigroup.) More formally, the following theorems are the principal results of the paper.

THEOREM 1. *Let S be a semigroup that embeds into a group. Let T be a subsemigroup of S of finite Rees index. Then S has a finite Malcev presentation if and only if T has a finite Malcev presentation.*

THEOREM 2. *Let S be a cancellative semigroup and let T be a subsemigroup of S of finite Rees index. Then S admits a finite cancellative presentation if and only if T admits a finite cancellative presentation.*

THEOREM 3. *Let S be a left-cancellative (respectively, right-cancellative) semigroup and let T be a subsemigroup of S of finite Rees index. Then S admits a finite left-cancellative (respectively, right-cancellative) presentation if and only if T admits a finite left-cancellative (respectively, right-cancellative) presentation.*

2. Species of presentations

2.1. Informal discussion An 'ordinary' semigroup presentation defines the semigroup by means of generators and defining relations. Informally, a Malcev presentation defines a semigroup by means of generators, defining relations, and a rule of group-embeddability. Similarly, a cancellative (respectively, left-cancellative, right-cancellative) presentation defines a semigroup by means of generators, defining relations, and a rule of cancellativity (respectively, left-cancellativity, right-cancellativity). (These loose definitions will be formalized later in this section, but will suffice for the present discussion.)

Spehner [22] introduced Malcev presentations, although they are based on Malcev's necessary and sufficient condition for the embeddability of a semigroup in a group [19]. The concept of a cancellative presentation first appears in a paper by Croisot [13]. Left-cancellative and right-cancellative presentations were introduced by Adjan [1].

Adjan [1] was the first to compare cancellative, left-cancellative, right-cancellative, and 'ordinary' semigroup presentations. Spehner [22] made similar comparisons of these types of presentations and also of Malcev presentations. Spehner and Adjan showed that a rule of group-embeddability, cancellativity, left-cancellativity, or right-cancellativity is worth an infinite number of defining relations. More formally, a finitely generated semigroup may admit:

- a finite Malcev presentation, but no finite cancellative presentation (see [22, Theorem 3.4]);
- a finite cancellative presentation, but no finite left- or right-cancellative presentation (see [22, Theorem 3.1(ii)] and [1, Theorem I.4]);
- a finite left-cancellative presentation, but no finite 'ordinary' or right-cancellative presentation (see [22, Theorem 3.1(i)] and [1, Theorem I.2]);
- a finite right-cancellative presentation, but no finite 'ordinary' or left-cancellative presentation (see [22, Theorem 3.1(i)] and [1, Theorem I.2]).

After Spehner's work comparing the various species of presentations, very little work was carried out on Malcev, cancellative, left-cancellative, and right-cancellative presentations, whilst the theory of 'ordinary' semigroup presentations flourished. Except for one additional paper by Spehner [23], proving that every finitely generated submonoid of a free monoid admits a finite Malcev presentation, the study of the other types of presentations remained in abeyance until the recent work of Cain *et al.* [8, 9] and Cain [4–6] on Malcev presentations. This recent work has shown that the theory of Malcev presentations is an interesting and worthwhile research area that has perhaps been unfairly neglected. For a survey of the theory of Malcev presentations, see [7].

The present paper continues the authors' earlier work on Malcev presentations and expands it to encompass cancellative, left-cancellative, and right-cancellative presentations.

2.2. Formal definitions Following [14], the notation used in this paper distinguishes a word from the element of the semigroup it represents. Let A be an alphabet representing a set of generators for a semigroup S. For any word $w \in A^+$, denote by $\overline{w}$ the element of S represented by w. (The symbol A^+ denotes the free semigroup over A; similarly A^* denotes the free monoid over A. The empty word is denoted by ε.) For any set of words W, let $\overline{W}$ be the set of all elements of S represented by words in W.

This section contains the definitions and results about Malcev, cancellative, left-cancellative, and right-cancellative presentations required for the rest of the paper. However, the reader is assumed to be familiar with the basic theory of (ordinary) semigroup presentations. (Refer to [15] or [20] for background information on the theory of semigroup presentations. For a fuller exposition of the foundations of the theory of Malcev presentations, see [4, Ch. 1].)

DEFINITION 2.1. Let S be any semigroup. A congruence σ on S is:

- a *Malcev congruence* if S/σ is embeddable in a group;
- a *cancellative congruence* if S/σ is a cancellative semigroup;
- a *left-cancellative congruence* if S/σ is a left-cancellative semigroup;
- a *right-cancellative congruence* if S/σ is a right-cancellative semigroup.

If $\{\sigma_i : i \in I\}$ is a set of Malcev congruences on S, then $\sigma = \bigcap_{i\in I} \sigma_i$ is also a Malcev congruence on S. This is true because S/σ_i embeds in a group G_i for each $i \in I$, so S/σ embeds in $\prod_{i\in I} S/\sigma_i$, which in turn embeds in $\prod_{i\in I} G_i$.

Similarly, if $\{\sigma_i : i \in I\}$ is a set of cancellative (respectively, left-cancellative, right-cancellative) congruences on S, then $\sigma = \bigcap_{i \in I} \sigma_i$ is also a cancellative (respectively, left-cancellative, right-cancellative) congruence on S [12, Lemma 9.49].

DEFINITION 2.2. Let A^+ be a free semigroup; let $\rho \subseteq A^+ \times A^+$ be any binary relation on A^+. Make the following definitions.

(1) ρ^{M} denotes the smallest Malcev congruence containing ρ—namely,

$$\rho^{\mathrm{M}} = \bigcap \{\sigma : \sigma \supseteq \rho,\ \sigma \text{ is a Malcev congruence on } A^+\}.$$

Then $\mathrm{SgM}\langle A \mid \rho\rangle$ is a *Malcev presentation* for (any semigroup isomorphic to) A^+/ρ^{M};

(2) ρ^{C} denotes the smallest cancellative congruence containing ρ—namely,

$$\rho^{\mathrm{C}} = \bigcap \{\sigma : \sigma \supseteq \rho,\ \sigma \text{ is a cancellative congruence on } A^+\}.$$

Then $\mathrm{SgC}\langle A \mid \rho\rangle$ is a *cancellative presentation* for (any semigroup isomorphic to) A^+/ρ^{C};

(3) ρ^{LC} denotes the smallest left-cancellative congruence containing ρ—namely,

$$\rho^{\mathrm{LC}} = \bigcap \{\sigma : \sigma \supseteq \rho,\ \sigma \text{ is a left-cancellative congruence on } A^+\}.$$

Then $\mathrm{SgLC}\langle A \mid \rho\rangle$ is a *left-cancellative presentation* for (any semigroup isomorphic to) A^+/ρ^{LC};

(4) ρ^{RC} denotes the smallest right-cancellative congruence containing ρ—namely,

$$\rho^{\mathrm{RC}} = \bigcap \{\sigma : \sigma \supseteq \rho,\ \sigma \text{ is a right-cancellative congruence on } A^+\}.$$

Then $\mathrm{SgRC}\langle A \mid \rho\rangle$ is a *right-cancellative presentation* for (any semigroup isomorphic to) A^+/ρ^{RC}.

If A and ρ are both finite, then the presentations $\mathrm{SgM}\langle A \mid \rho\rangle$, $\mathrm{SgC}\langle A \mid \rho\rangle$, $\mathrm{SgLC}\langle A \mid \rho\rangle$, and $\mathrm{SgRC}\langle A \mid \rho\rangle$ are said to be finite.

REMARK 2.3. Definition 2.2 could be rephrased in terms of universal algebra as follows.

Let $\mathcal{Q}$ be a prevariety (a nonempty class of semigroups closed under isomorphism, direct product, and subsemigroups). Then, for any binary relation ρ on A^+, there is a smallest congruence $\rho^{\mathcal{Q}}$ containing ρ with $A^+/\rho^{\mathcal{Q}}$. (This follows from the fact that at least one such congruence exists, namely the universal relation $\rho = A^+ \times A^+$, and the intersection ρ' of a family $\{\rho_i : i \in I\}$ of such congruences is also such a congruence, since A^+/ρ' embeds into $\prod_{i \in I} A^+/\rho'$.) A $\mathcal{Q}$-presentation $\mathrm{Sg}\mathcal{Q}\langle A \mid \rho\rangle$ presents the semigroup A^+/ρ.

The four semigroups defined in Definition 2.2 are the special cases when $\mathcal{Q}$ is taken to be in turn the class of group-embeddable, cancellative, left-cancellative, and right-cancellative semigroups.

The notation in Definition 2.2 distinguishes Malcev, cancellative, left-cancellative, and right-cancellative presentations with generators A and defining relations ρ from the ordinary semigroup presentation $\mathrm{Sg}\langle A \mid \rho\rangle$, which defines $A^+/\rho^\#$. (Recall that $\rho^\#$ denotes the smallest congruence containing ρ.) Similarly, $\mathrm{Gp}\langle A \mid \rho\rangle$ denotes the group presentation with the same set of generators and defining relations.

Let X be a subset of a particular semigroup. Denote by $\mathrm{Sg}\langle X\rangle$ the subsemigroup generated by X.

2.3. Universal groups and Malcev presentations This section contains the few necessary facts about universal groups of semigroups and their connection to Malcev presentations. For further background on universal groups, refer to [12, Ch. 12]; for their interaction with Malcev presentations, see [4, Section 1.3].

DEFINITION 2.4. Let S be a group-embeddable semigroup. The *universal group* U of S is the largest group into which S embeds and which S generates, in the sense that all other such groups are homomorphic images of U.

The concept of a universal group can be defined for all semigroups, not just those that are group-embeddable. However, the definition above will suffice for the purposes of this paper. The universal group of a (not necessarily group-embeddable) semigroup is unique up to isomorphism.

PROPOSITION 2.5 [12, Construction 12.6]. *Let S be a semigroup. Suppose that S is presented by $\mathrm{Sg}\langle A \mid \rho\rangle$ for some alphabet A and set of defining relations ρ. Then $\mathrm{Gp}\langle A \mid \rho\rangle$ is (isomorphic to) the universal group of S.*

The following two results show the connection between universal groups and Malcev presentations. The proof of the first result is long and technical; the second is a corollary of the first.

PROPOSITION 2.6 [4, Proposition 1.3.1]. *Let S be a semigroup that embeds into a group. If $\mathrm{SgM}\langle A \mid \rho\rangle$ is a Malcev presentation for S, then the universal group of S is presented by $\mathrm{Gp}\langle A \mid \rho\rangle$. Conversely, if $\mathrm{Gp}\langle A \mid \rho\rangle$ is a presentation for the universal group of S, where A represents a generating set for S and $\rho \subseteq A^+ \times A^+$, then $\mathrm{SgM}\langle A \mid \rho\rangle$ is a Malcev presentation for S.*

PROPOSITION 2.7 [4, Corollary 1.3.2]. *If a group-embeddable semigroup S has a finite Malcev presentation, then its universal group G is finitely presented. Conversely, if the universal group of S is finitely presented and S itself is finitely generated, then S admits a finite Malcev presentation.*

PROOF. Any finite Malcev presentation for S is a finite presentation for G by Proposition 2.6.

To prove the second statement, let $\mathrm{Sg}\langle A \mid \rho\rangle$ be any presentation for S with A being finite. Then the universal group G of S is presented by $\mathrm{Gp}\langle A \mid \rho\rangle$, by Proposition 2.6. Since G is finitely presented, there is a finite subset σ of ρ such that $\mathrm{Gp}\langle A \mid \sigma\rangle$ is a presentation for G. Using Proposition 2.6 again, $\mathrm{SgM}\langle A \mid \sigma\rangle$ is a finite Malcev presentation for S. ☐

Example 0.9.3 of [4] exhibits a group-embeddable semigroup that is not itself finitely generated but whose universal group is finitely presented. The finite generation condition in the second part of Proposition 2.7 is therefore not superfluous.

2.4. Syntactic rules for cancellative presentations The purpose of this subsection and the next is to describe syntactic rules that govern when two words over an alphabet A are related by a cancellative or left- or right-cancellative congruence on A^+ generated by a set of pairs of words over A. Similar syntactic rules exist for Malcev congruences, but these are not needed. (For a description of the omitted rules, see any of [4, 7–9].)

Let S be the semigroup defined by the presentation $\mathrm{Sg}\langle A \mid \mathcal{P}\rangle$. Two words $u, v \in A^+$ represent the same element of S—that is, are $\mathcal{P}^\#$-related— if and only if there is a sequence

$$u = u_0 \to u_1 \to \cdots \to u_l = v,$$

where, for each $i = 0, \ldots, n-1$, there exists $p_i, q_i, q_i', r_i \in A^*$ with $u_i = p_i q_i r_i$, $u_{i+1} = p_i q_i' r_i$ and either $(q_i, q_i') \in \mathcal{P}$ or $(q_i', q_i) \in \mathcal{P}$. Thus two words represent the same element of S if and only if it is possible to transform one word to the other by a finite number of replacements of a subword that forms one side of a pair in $\mathcal{P}$ by the word forming the other side of that pair.

Now let S be defined by the cancellative presentation $\mathrm{SgC}\langle A \mid \mathcal{P}\rangle$. The syntactic rules that govern when two words $u, v \in A^+$ represent the same element of S, that is, are $\mathcal{P}^{\mathsf{C}}$-related, are necessarily more complex than those for ordinary presentations.

Let A^{L}, A^{R} be two sets in bijection with A under the mappings $a \mapsto a^{\mathsf{L}}$, $a \mapsto a^{\mathsf{R}}$, respectively, with A, A^{L}, A^{R} being pairwise disjoint.

Extend the mappings $a \mapsto a^{\mathsf{L}}$ and $a \mapsto a^{\mathsf{R}}$ to anti-isomorphisms from A^* to $(A^{\mathsf{L}})^*$ and $(A^{\mathsf{R}})^*$ in the obvious way: for $w = a_1 a_2 \ldots a_n \in A^*$, with $a_i \in A$, define

$$w^{\mathsf{L}} = (a_1 \ldots a_n)^{\mathsf{L}} = a_n^{\mathsf{L}} a_{n-1}^{\mathsf{L}} \ldots a_1^{\mathsf{L}} \quad \text{and} \quad w^{\mathsf{R}} = (a_1 \ldots a_n)^{\mathsf{R}} = a_n^{\mathsf{R}} a_{n-1}^{\mathsf{R}} \ldots a_1^{\mathsf{R}}.$$

Two words $u, v \in A^+$ are $\mathcal{P}^{\mathsf{C}}$-related if and only if there is a *cancellative $\mathcal{P}$-chain* from u to v. A cancellative $\mathcal{P}$-chain from u to v is a sequence

$$u = u_0 \to u_1 \to \cdots \to u_l = v,$$

with each $u_i \in (A \cup A^{\mathsf{L}} \cup A^{\mathsf{R}})^+$ satisfying the following properties.

(1) For each i, $u_i = p_i q_i r_i$ and $u_{i+1} = p_i q_i' r_i$, where $p_i \in (A \cup A^{\mathsf{L}})^*$, $r_i \in (A \cup A^{\mathsf{R}})^*$, and one of the following statements holds:

(a) either $(q_i, q_i') \in \mathcal{P}$ or $(q_i', q_i) \in \mathcal{P}$;

(b) $q_i = \varepsilon$ and $q_i' = aa^{\mathsf{R}}$ for some $a \in A^+$;

(c) $q_i = aa^{\mathsf{R}}$ and $q_i' = \varepsilon$ for some $a \in A^+$;

(d) $q_i = \varepsilon$ and $q_i' = a^{\mathsf{L}}a$ for some $a \in A^+$; or
(e) $q_i = a^{\mathsf{L}}a$ and $q_i' = \varepsilon$ for some $a \in A^+$.

A step in the sequence of type (a) above is a *$\mathcal{P}$-step*. Steps of types (b) and (d) are respectively *right* and *left insertions*; those of types (c) and (e) are respectively *right* and *left deletions*. Notice that, since relations in $\mathcal{P}$ only include letters from A, each insertion $u_j \to u_{j+1}$ must have an associated deletion $u_{j\delta} \to u_{j\delta+1}$ (with $j\delta > j$) where the letter a^{L} or a^{R} just introduced is removed.

(The restriction on the letters that can appear in p_i and r_i simply means that no change is made to the left of a letter a^{L} or to the right of a letter a^{R}.)

(2) If $u_i \to u_{i+1}$ and $u_j \to u_{j+1}$ are both insertions, with $i < j$, then $j\delta < i\delta$. That is, the letter a^{L} or a^{R} most recently inserted must be deleted before any of the earlier such letters introduced can be removed. (Thus one can imagine the insertions being 'pushed' onto a stack and the associated deletions being 'popped' off.)

For the purposes of this paper, a slightly more flexible form of the cancellative $\mathcal{P}$-chain is needed. An *extended cancellative $\mathcal{P}$-chain* allows the insertion and deletion of words ww^{R} and $w^{\mathsf{L}}w$ for any word $w \in A^+$, with the insistence that each insertion $u_i \to u_{i+1}$ of ww^{R} (respectively $w^{\mathsf{L}}w$) has a dual deletion $u_{i\delta} \to u_{i\delta+1}$. That is, if a word w^{R} is introduced in a single step of an extended cancellative $\mathcal{P}$-chain, it must be removed in a single step. (This definition is simply a terminological tool to simplify the statements of certain upcoming technical results.)

2.5. Syntactic rules for left-/right-cancellative presentations Let $S = \mathrm{SgLC}\langle A \mid \mathcal{P}\rangle$ be a semigroup. Two words $u, v \in A^+$ represent the same element of S, that is, are $\mathcal{P}^{\mathrm{LC}}$-related, if and only if there is a *left-cancellative $\mathcal{P}$-chain* from u to v. A left-cancellative $\mathcal{P}$-chain is a cancellative $\mathcal{P}$-chain that does not involve any letters from A^{R}.

Similarly, if $S = \mathrm{SgRC}\langle A \mid \mathcal{P}\rangle$ is a semigroup, then two words $u, v \in A^+$ represent the same element of S, that is, are $\mathcal{P}^{\mathrm{RC}}$-related, if and only if there is a *right-cancellative $\mathcal{P}$-chain* from u to v. A right-cancellative $\mathcal{P}$-chain is a cancellative $\mathcal{P}$-chain that does not involve any letters from A^{L}.

3. Malcev presentations—extensions and subsemigroups

The first of the main results to be proven is the preservation under passing to large subsemigroups or small extensions of the property of admitting a finite Malcev presentation. The proof technique for this result is fundamentally different from that used for cancellative, left-cancellative, and right-cancellative presentations. The latter results are proven using rewriting techniques. The present section avoids such syntactical arguments.

THEOREM 3.1. *Let S be a semigroup that embeds in a group. Let T be a subsemigroup of S. Suppose that $|T| > |S - T|$. (This includes the possibility that T is infinite and $S - T$ finite.) Then the universal groups of S and T are isomorphic.*

Proof. Let $\mathrm{Sg}\langle T \mid \tau\rangle$ and $\mathrm{Sg}\langle S \mid \sigma\rangle$ be the Cayley table presentations for T and S, respectively. Let G be the universal group of S, and view S and T as subsemigroups of G. By Proposition 2.5, G is presented by $\mathrm{Gp}\langle S \mid \sigma\rangle$. Use the sets S and T as both symbols in presentations and as elements of G, and suspend (for the duration of this proof) the notational distinction between a symbol and the element it represents. The key to the proof is the following lemma.

Lemma 3.2. *For each $s \in S - T$, there exist elements u_s, v_s, w_s, and x_s of T such that $s = u_s v_s^{-1}$ and $s = w_s^{-1} x_s$ in G.*

Proof. Let $k = |S - T|$. Let $k < l \leq |T|$. Pick distinct elements $t_1, t_2, \ldots, t_l$ of T. Suppose that the elements $st_1, st_2, \ldots, st_l$ are not all distinct. Then for some i, j with $i \neq j$, $st_i = st_j$, which means that $t_i = t_j$, contradicting the choice of $t_1, t_2, \ldots, t_l$. Therefore, the elements $st_1, st_2, \ldots, st_l$ are all distinct, and so at least one of them lies in T since $l > |S - T|$. Let h be such that $st_h \in T$. Let $u_s = st_h$ and $v_s = t_h$. Then $u_s, v_s \in T$ and $s = u_s v_s^{-1}$ in G. Similar reasoning yields w_s and x_s. □

Lemma 3.2 shows that T generates G as a group, since the subgroup of G generated by T contains S, and S is certainly a group generating set for G. The strategy of the remainder of the proof is to show that G has a presentation $\mathrm{Gp}\langle T \mid \rho\rangle$ such that all of the defining relations in ρ are between positive words and are valid in T. All the relations in ρ must then be consequences of those in τ. Therefore, $\mathrm{Gp}\langle T \mid \tau\rangle$ will present G and so G will be isomorphic to the universal group of T.

Recall that

$$\tau = \{(pq, r) : p, q, r \in T, \overline{pq} = \overline{r}\}.$$

Let

$$\omega = \sigma - \tau = \{(pq, r) : p \in S - T \text{ or } q \in S - T, \overline{pq} = \overline{r}\} \subseteq SS \times S,$$

so that σ is partitioned as $\tau \cup \omega$ and let

$$\mathcal{P} = \{(sv_s, u_s), (w_s s, x_s) : s \in S - T\},$$

where u_s, v_s, w_s, and x_s are as in Lemma 3.2. Notice that $\mathcal{P} \subseteq \omega$.

Create a new set of defining relations ω' as follows. For each relation $(pq, r) \in \omega$, add relations to ω' in accordance with the appropriate case below.

(1) $p \in S - T$, $q \in T$, $r \in T$. Use $\mathcal{P}$ to see that the relation $(w_p^{-1} x_p q, r)$ is valid in G. Therefore, add the relation $(x_p q, w_p r)$, which is valid in T, to ω'. Observe that (pq, r) is a consequence (in the group G) of this new relation and $(w_p p, x_p) \in \mathcal{P}$.

(2) $p \in T$, $q \in S - T$, $r \in T$. The relation $(pu_q v_q^{-1}, r)$ is valid in G. Add (pu_q, rv_q) to ω' and observe that the original relation is once again a consequence of the new one and $(qv_q, u_q) \in \mathcal{P}$.

(3) $p \in S - T$, $q \in S - T$, $r \in T$. The relation $(w_p^{-1} x_p u_q v_q^{-1}, r)$ is valid; add $(x_p u_q, w_p r v_q)$ to ω'. Once more the original relation is a consequence of the new one and $(w_p p, x_p)$, $(q v_q, u_q) \in \mathcal{P}$.

(4) $p \in S - T$, $q \in T$, $r \in S - T$. The relation $(w_p^{-1} x_p q, u_r v_r^{-1})$ is valid; add $(x_p q v_r, w_p u_r)$ to ω'. The original relation is a consequence of the new one and $(w_p p, x_p)$, $(r v_r, u_r) \in \mathcal{P}$.

(5) $p \in T$, $q \in S - T$, $r \in S - T$. The relation $(p u_q v_q^{-1}, w_r^{-1} x_r)$ is valid; add $(w_r p u_q, x_r v_q)$ to ω'. The original relation is a consequence of the new one and $(w_r r, x_r)$, $(q v_q, u_q) \in \mathcal{P}$.

(6) $p \in S - T$, $q \in S - T$, $r \in S - T$. Now, $pq = r$ in S, so $pq = u_r v_r^{-1}$ in G. Therefore, $pq v_r = u_r$. Now consider two sub-cases.
   (a) $q v_r = s \in S - T$. Then $ps = u_r$, and so (pq, r) is a consequence in G of $(r v_r, u_r)$, $(q v_r, s)$ and (ps, u_r). The set $\mathcal{P}$ contains the first of these three relations. The second and third are in ω and are of types 4 and 3 above.
   (b) $q v_r = t \in T$. Then $pt = u_r$, and so (pq, r) is a consequence of $(r v_r, u_r)$, $(q v_r, t)$ and (pt, u_r). Again, the set $\mathcal{P}$ contains the first of these three relations. The second and third are both in ω and are of type 1 above.

   In either sub-case, (pq, r) is a consequence of relations that are either in $\mathcal{P}$ or are in ω, but are of the form (1)–(5) above. Therefore, do not add any relations to ω'.

Now, $G = \mathrm{Gp}\langle S \mid \sigma \rangle = \mathrm{Gp}\langle S \mid \sigma \cup \mathcal{P} \rangle = \mathrm{Gp}\langle S \mid \tau \cup \omega \cup \mathcal{P} \rangle$. Each relation in ω' is a consequence of those in $\mathcal{P}$ and those in ω. On the other hand, each relation in ω is a consequence of those in $\mathcal{P}$ and those in ω'. So the group G is also presented by $\mathrm{Gp}\langle S \mid \tau \cup \omega' \cup \mathcal{P} \rangle$.

Partition $\mathcal{P}$ as $\mathcal{P}' \cup \mathcal{P}''$, where

$$\mathcal{P}' = \{(s v_s, u_s) : s \in S\},$$
$$\mathcal{P}'' = \{(w_s s, x_s) : s \in S\}.$$

Every relation in

$$\mathcal{Q} = \{(x_s v_s, w_s u_s) : s \in S\}$$

is a consequence of those in $\mathcal{P}$; every relation in $\mathcal{P}''$ is a consequence of those in $\mathcal{P}' \cup \mathcal{Q}$. Therefore, the group G is presented by $\mathrm{Gp}\langle S \mid \tau \cup \omega' \cup \mathcal{P}' \cup \mathcal{Q} \rangle$.

Finally, use the relations in $\mathcal{P}'$ to eliminate the generators contained in $S - T$. This gives a presentation $\mathrm{Gp}\langle T \mid \tau \cup \omega' \cup \mathcal{Q} \rangle$ for G. Observing that $\tau \cup \omega' \cup \mathcal{Q}$ consists of positive relations between elements of T completes the proof. □

THEOREM 1. *Let S be a semigroup that embeds into a group. Let T be a subsemigroup of S of finite Rees index. Then S has a finite Malcev presentation if and only if T has a finite Malcev presentation.*

PROOF. If T is finite, then S is also finite and therefore S and T trivially both admit finite Malcev presentations. Therefore, assume that T is infinite, in which case

$|S - T| < |T|$. Theorem 3.1 applies to show that the universal groups of S and T are isomorphic.

By Theorem 1.1, S is finitely generated if and only if T is finitely generated. Let S admit a finite Malcev presentation. Then S is finitely generated and its universal group is finitely presented by Proposition 2.7. Therefore, T is finitely generated and its universal group—isomorphic to that of S—is finitely presented. Proposition 2.7 applies again to show that T admits a finite Malcev presentation. Similar reasoning shows that S has a finite Malcev presentation if T does. □

4. Left-/right-/two-sided cancellative presentations—extensions

THEOREM 4.1. *Let S be a $\mathfrak{P}$ semigroup, where $\mathfrak{P}$ means one of 'ordinary', 'left-cancellative', 'right-cancellative', or 'cancellative'. Let T be a subsemigroup of S of finite Rees index. (So T also has the property $\mathfrak{P}$.) If T has a finite $\mathfrak{P}$ presentation, then S has a finite $\mathfrak{P}$ presentation.*

PROOF. Let $\langle B \mid \mathcal{T} \rangle$ be a finite $\mathfrak{P}$ presentation for T. (An abuse of notation allows $\langle B \mid \mathcal{T} \rangle$ to denote an unspecified type of presentation.) Let $A = B \cup C$, where C bijectively represents the elements of $S - T$. The semigroup S is clearly generated by $\overline{A}$.

Let $c \in C$ and $a \in A$. Let $u_{ac}, u_{ca} \in B^+ \cup C$ be such that $\overline{ac} = \overline{u_{ac}}$ and $\overline{ca} = \overline{u_{ca}}$. The aim is now to show that S has the finite $\mathfrak{P}$ presentation

$$\langle A \mid \mathcal{T} \cup \{(ac, u_{ac}), (ca, u_{ca}) : c \in C, a \in A\}\rangle.$$

Let σ be the $\mathfrak{P}$ congruence generated by the defining relations in this presentation.

All the additional defining relations certainly hold in S. Let $u, v \in A^+$ with $\overline{u} = \overline{v}$. The intention is to show that (u, v) is in σ. First of all, notice that there exist words $u', v' \in B^+ \cup C$ such that (u, u') and (v, v') lie in σ. Now, either $u', v' \in C$ and so $u' = v'$, or $u', v' \in B^+$, in which case (u', v') lies in the $\mathfrak{P}$ congruence generated by $\mathcal{T}$. In either case, it follows that $(u, v) \in \sigma$.

Thus the given $\mathfrak{P}$ presentation defines S. □

One could also follow the proof of Theorem 4.1 using Malcev presentations and group-embeddable semigroups to establish anew that the property of admitting a finite Malcev presentation is preserved under passing to small extensions.

REMARK 4.2. Theorem 4.1 also holds (with the same proof) in the more general setting of the $\mathcal{Q}$-presentations discussed in Remark 2.3.

5. Rewriting techniques

Before embarking on the proofs of the large subsemigroup results for cancellative and left- and right-cancellative presentations, it is necessary to review some known results on rewriting semigroup presentations.

Let S be a $\mathfrak{P}$ semigroup, where $\mathfrak{P}$ means one of 'ordinary', 'left-cancellative', 'right-cancellative', or 'cancellative'. Let T be a subsemigroup of S of finite Rees index. Observe that T necessarily inherits the property $\mathfrak{P}$.

Suppose that S has a $\mathfrak{P}$ presentation $\langle A \mid \mathcal{P}\rangle$. Without loss of generality, make the following assumptions:

(1) $A = B \cup C$, where $\mathrm{Sg}\langle B\rangle = T$ and $\overline{C} = S - T$ (this assumption does not affect whether A is finite);

(2) $\mathcal{P}$ contains all valid relations lying in $AA \times C$ (this assumption does not affect whether the presentation under discussion is finite).

Define

$$L(A, T) = \{w \in A^+ : \overline{w} \in T\}.$$

For any word $w \in A^* - L(A, T)$, let $\underline{w} \in C \cup \{\varepsilon\}$ be the unique element of C representing $\overline{w}$, or ε if $w = \varepsilon$.

Suppose that the alphabet D is the set

$$\{d_{\rho,a,\sigma} : \rho, \sigma \in C \cup \{\varepsilon\}, a \in A, \rho a, \rho a\sigma \in L(A, T)\},$$

and that, for all ρ, a, and σ,

$$\overline{d_{\rho,a,\sigma}} = \overline{\rho a \sigma}.$$

Notice that if A is finite, D too must be finite.

THEOREM 5.1 [10]. *The subsemigroup T is generated by $\overline{D}$.*

Define a mapping $\phi : L(A, T) \to D^+$ as follows. Let $w \in L(A, T)$ with $w'a$ being the shortest prefix of w lying in $L(A, T)$ and w'' being the remainder of w. Then

$$w\phi = \begin{cases} d_{\underline{w'},a,\underline{w''}} & \text{if } w'' \notin L(A, T),\\ d_{\underline{w'},a,\varepsilon}(w''\phi) & \text{if } w'' \in L(A, T).\end{cases}$$

This mapping ϕ rewrites words in $L(A, T)$ to words over D representing the same element of T. Define another mapping $\psi : D^+ \to L(A, T)$ by extending the mapping

$$d_{\rho,a,\sigma} \mapsto \rho a \sigma,$$

to D^+ in the natural way. Notice that $\overline{w} = \overline{w\psi}$.

In [10], it is proven that if $\mathfrak{P}$ is 'ordinary', then the subsemigroup T has 'ordinary' presentation $\mathrm{Sg}\langle D \mid \mathcal{Q}\rangle$, where $\mathcal{Q}$ contains the following infinite collection of defining relations:

$$(\rho a \sigma)\phi = d_{\rho,a,\sigma}, \tag{1}$$

$$(w_1 w_2)\phi = (w_1\phi)\,(w_2\phi), \tag{2}$$

$$(w_3 u w_4)\phi = (w_3 v w_4)\phi \tag{3}$$

(where $\rho, \sigma \in C \cup \{\varepsilon\}$, $a \in A$, ρa, $\rho a \sigma \in L(A, T)$, $w_1, w_2 \in L(A, T)$, $w_3, w_4 \in A^*$, $(u, v) \in \mathcal{P}$, $w_3 u w_4 \in L(A, T)$).

THEOREM 5.2 [21]. *Suppose that the* $\mathfrak{P}$ *presentation* $\langle A \mid \mathcal{P}\rangle$ *for* S *is finite. Then there is a finite set of relations* $\mathcal{S} \subseteq D^+ \times D^+$*—with* $\overline{u} = \overline{v}$ *for all* $(u, v) \in \mathcal{S}$*—such that the relations (1), (2), and (3) all lie in* $\mathcal{S}^\#$.

Theorem 5.2 is a restatement of [21, Theorem 6.1] in purely syntactic terms. (In the original result, the subsemigroup T is presented by $\mathrm{Sg}\langle D \mid \mathcal{S}\rangle$.) The use of this result in syntactic rather than semantic terms is similar to the technique employed in [3].

6. Large subsemigroups

The remainder of the paper is dedicated to proving that large subsemigroups of semigroups with finite $\mathfrak{P}$ presentations also admit finite $\mathfrak{P}$ presentations when $\mathfrak{P}$ is 'cancellative', 'left-cancellative', or 'right-cancellative'.

The proof of this result when $\mathfrak{P}$ is 'ordinary' is long and technical, involving consideration of a large number of cases. However, when $\mathfrak{P}$ is cancellative, one can reduce the proof to the case where the subsemigroup T is an ideal (see Theorem 7.1 and the discussion following its proof). Similarly, when $\mathfrak{P}$ is 'left-cancellative' or 'right-cancellative', one can reduce respectively to the cases where T is a left or right ideal (see Theorem 8.1 *et seq.*).

7. Cancellative presentations

The first step towards a proof of Theorem 2 is the following result, which reduces the proof of the general result to the case where the subsemigroup is in fact an ideal.

THEOREM 7.1. *Let* S *be an infinite cancellative semigroup and let* T *be a subsemigroup of* S *of finite Rees index. Then*

$$I = \{t \in T : S^1 t S^1 \subseteq T\}$$

is an ideal of S *contained in* T *and has finite Rees index in* T*. Indeed,* I *is the largest ideal of* S *contained in* T.

(The symbol S^1 denotes the semigroup S with a two-sided identity adjoined.)

PROOF. Since S is infinite and $S - T$ is finite, T too must be infinite.

Suppose, with the aim of obtaining a contradiction, that $T - I$ is infinite. Then there are infinitely many elements $t \in T - I$ such that $sts' \in S - T$ for some $s, s' \in S^1$. Since $S - T$ is finite, there exists $w \in S - T$ such that the set

$$K_w = \{t \in T - I : (\exists s, s' \in S^1)\ (sts' = w)\}$$

is infinite. Fix such a w. For each $t \in K_w$, fix $s_t, s_t' \in S^1$ such that $s_t t s_t' = w$. Suppose that $s_t t \in S - T$. Then $s_t \in S - T$. So there can be only finitely many distinct $t \in K_w$

with $s_t t \in S - T$. (If there were infinitely many, one could find distinct $t, u \in K_w$ with $s_t t = s_u u$ and $s_t = s_u$, which would contradict cancellativity.) A similar comment applies to ts'_t. So the set

$$K'_w = \{t \in K_w : s_t t, ts'_t \in T\}$$

is infinite.

Now, if $t \in K'_w$, then s_t and s'_t both lie in $S - T$, which is a finite set. (If $s_t \in T$, then $s_t ts'_t = s_t(ts'_t) \in T$, which is a contradiction since $w \in S - T$. Similar reasoning applies to s'_t.) So there are distinct $t, u \in S - T$ with $s_t = s_u$ and $s'_t = s'_u$. But $s_t ts'_t = w = s_u us'_u$ and cancellativity then gives $t = u$, which is a contradiction. Therefore, $T - I$ is finite. In particular, this implies that I is nonempty.

Suppose that $t \in I$ and let $s, s' \in S^1$. Then $sts' \in T$ by the definition of I. Furthermore, $sts' \in I$ since otherwise there would exist $r, r' \in S^1$ such that $rsts'r' \notin T$, which would contradict $t \in I$. Thus I is an ideal of S. That it is contained in T is obvious; that I has finite Rees index in T has already been proven.

Suppose that J is another ideal of S contained in T with $I \subseteq J$. Suppose that $I \subset J$. Fix $t \in J - I$. Then there exist $s, s' \in S^1$ such that $sts' \notin T$. Then $sts' \notin J$, contradicting the fact that J is an ideal of S. So I is the largest ideal of S contained in T. □

Observe that cancellativity is a necessary hypothesis in Theorem 7.1: if one lets S be an infinite right zero semigroup and lets T be a large subsemigroup of S, then I is empty.

In light of Theorem 7.1, to prove Theorem 2 it suffices to prove that if S has a finite cancellative presentation and T is an *ideal*, then T too has a finite cancellative presentation. For, when S is infinite, one then recovers the general result for a subsemigroup T by first of all showing that the largest ideal I of S contained in T has a finite cancellative presentation, then applying Theorem 4.1 to show that T itself has a finite cancellative presentation. When S is finite, so is T, and thus T trivially has a finite cancellative presentation.

So, for the remainder of this section, adopt the notation from Section 5 with $\mathfrak{P}$ being ‘cancellative’. Suppose that $\mathrm{SgC}\langle A \mid \mathcal{P}\rangle$ is a finite cancellative presentation for S.

Consider an extended cancellative $\mathcal{P}$-chain from u to v, where $u, v \in A^+$. Every word in this chain has a factorization

$$\delta_1\gamma_1^{\mathsf{L}}\delta_2\gamma_2^{\mathsf{L}} \ldots \delta_n\gamma_n^{\mathsf{L}} w\alpha_m^{\mathsf{R}}\beta_m \ldots \alpha_2^{\mathsf{R}}\beta_2\alpha_1^{\mathsf{R}}\beta_1, \tag{4}$$

where each word $w, \alpha_i, \beta_i, \gamma_i, \delta_i$ is in A^* and each α_i^{R} and γ_i^{L} arises from the insertion of $\alpha_i\alpha_i^{\mathsf{R}}$ or $\gamma_i^{\mathsf{L}}\gamma_i$ earlier in the chain (and hence α_i and γ_i are nonempty). Call such a chain T-*consistent* if it satisfies the following two conditions:

(1) every word in the chain has a factorization (4) in which each w, α_i, β_i, γ_i, δ_i lies in $L(A, T) \cup \{\varepsilon\}$;

(2) each right insertion

$$\delta_1\gamma_1^{\mathsf{L}}\delta_2\gamma_2^{\mathsf{L}} \dots \delta_n\gamma_n^{\mathsf{L}} w\alpha_m^{\mathsf{R}}\beta_m \dots \alpha_2^{\mathsf{R}}\beta_2\alpha_1^{\mathsf{R}}\beta_1 \\ \to \delta_1\gamma_1^{\mathsf{L}}\delta_2\gamma_2^{\mathsf{L}} \dots \delta_n\gamma_n^{\mathsf{L}} w'\alpha_{m+1}\alpha_{m+1}^{\mathsf{R}} w''\alpha_m^{\mathsf{R}}\beta_m \dots \alpha_2^{\mathsf{R}}\beta_2\alpha_1^{\mathsf{R}}\beta_1$$

has the property that w' and w'' lie in $L(A, T)$, and similarly for left insertions.

(Notice that the two conditions are largely—but not wholly—interdependent.)

LEMMA 7.2. *If $u, v \in L(A, T)$ and $\overline{u} = \overline{v}$, then there is a T-consistent extended cancellative $\mathcal{P}$-chain from u to v.*

PROOF. Since $\overline{u} = \overline{v}$, there is an extended $\mathcal{P}$-cancellative chain

$$u = u_0 \to u_1 \to \cdots \to u_l = v.$$

The aim is to transform this chain into one that is T-consistent.

Observe first of all that the trivial subchain u_0 is T-consistent. Now suppose that the subchain

$$u_0 \to u_1 \to \cdots \to u_k$$

is T-consistent, where $0 \leq k < l$. Let the decomposition (4) of u_k be

$$u_k = \delta_1\gamma_1^{\mathsf{L}}\delta_2\gamma_2^{\mathsf{L}} \dots \delta_n\gamma_n^{\mathsf{L}} w\alpha_m^{\mathsf{R}}\beta_m \dots \alpha_2^{\mathsf{R}}\beta_2\alpha_1^{\mathsf{R}}\beta_1.$$

If $u_k \to u_{k+1}$ is an application of a relation from $\mathcal{P}$, then the decomposition of u_{k+1} is

$$u_{k+1} = \delta_1\gamma_1^{\mathsf{L}}\delta_2\gamma_2^{\mathsf{L}} \dots \delta_n\gamma_n^{\mathsf{L}} w'\alpha_m^{\mathsf{R}}\beta_m \dots \alpha_2^{\mathsf{R}}\beta_2\alpha_1^{\mathsf{R}}\beta_1,$$

where w' differs from w by a single application of $\mathcal{P}$. Now, since $w \in L(A, T)$ and $\overline{w} = \overline{w'}$, w' also lies in $L(A, T)$. So the chain is T-consistent up to u_{k+1}.

Now suppose that $u_k \to u_{k+1}$ is the insertion of $\alpha_{m+1}\alpha_{m+1}^{\mathsf{R}}$. (The reasoning for left insertions is symmetric.) Then the decomposition of u_{k+1} is

$$u_{k+1} = \delta_1\gamma_1^{\mathsf{L}}\delta_2\gamma_2^{\mathsf{L}} \dots \delta_n\gamma_n^{\mathsf{L}} w'\alpha_{m+1}\alpha_{m+1}^{\mathsf{R}} w''\alpha_m^{\mathsf{R}}\beta_m \dots \alpha_2^{\mathsf{R}}\beta_2\alpha_1^{\mathsf{R}}\beta_1,$$

where $w = w'w''$. Suppose that the corresponding deletion is $u_l \to u_{l+1}$ (where $l > k$), so that

$$u_l = \delta_1\gamma_1^{\mathsf{L}}\delta_2\gamma_2^{\mathsf{L}} \dots \delta_n\gamma_n^{\mathsf{L}} t\alpha_{m+1}\alpha_{m+1}^{\mathsf{R}} w''\alpha_m^{\mathsf{R}}\beta_m \dots \alpha_2^{\mathsf{R}}\beta_2\alpha_1^{\mathsf{R}}\beta_1$$

and

$$u_{l+1} = \delta_1\gamma_1^{\mathsf{L}}\delta_2\gamma_2^{\mathsf{L}} \dots \delta_n\gamma_n^{\mathsf{L}} tw''\alpha_m^{\mathsf{R}}\beta_m \dots \alpha_2^{\mathsf{R}}\beta_2\alpha_1^{\mathsf{R}}\beta_1,$$

and there is a cancellative $\mathcal{P}$-chain from $w'\alpha_{m+1}$ to $t\alpha_{m+1}$.

There are three possible ways in which the T-consistency of the chain $u_0 \to \cdots \to u_k$ may not extend to u_{k+1}: w', α_{m+1}, or w'' may not lie in $L(A, T)$.

Suppose that $w' \notin L(A, T)$. Since T is an ideal, $w' \in C^*$. As $\overline{w'} = \overline{t}$, the word t also lies in C^*. So, by the fact that $\mathcal{P}$ contains all relations involving words of length 2 over C (see assumption 2 in Section 5), there is a chain from w' to t that involves no insertions and deletions. Thus one can use this chain to go from u_k to u_{l+1} without having to insert $\alpha_{m+1}\alpha_{m+1}^{\mathsf{R}}$. The new chain thus obtained is T-consistent up to and including u_{l+1}.

So assume that $w' \in L(A, T)$. Since T is an ideal, $w'\alpha_{m+1} \in L(A, T)$.

Suppose now that either $w'' \notin L(A, T)$ or $\alpha_{m+1} \notin L(A, T)$. Let $b \in B$. Observe that $w''b, \alpha_{m+1}b \in L(A, T)$ since T is an ideal. Modify the subchain $u_k \to \cdots \to u_{l+1}$ to the following form:

$$
\begin{aligned}
u_k &= \delta_1\gamma_1^{\mathsf{L}}\delta_2\gamma_2^{\mathsf{L}} \ldots \delta_n\gamma_n^{\mathsf{L}} w\alpha_m^{\mathsf{R}}\beta_m \ldots \alpha_2^{\mathsf{R}}\beta_2\alpha_1^{\mathsf{R}}\beta_1 \\
&\to \delta_1\gamma_1^{\mathsf{L}}\delta_2\gamma_2^{\mathsf{L}} \ldots \delta_n\gamma_n^{\mathsf{L}} wbb^{\mathsf{R}}\alpha_m^{\mathsf{R}}\beta_m \ldots \alpha_2^{\mathsf{R}}\beta_2\alpha_1^{\mathsf{R}}\beta_1 \\
&\to \delta_1\gamma_1^{\mathsf{L}}\delta_2\gamma_2^{\mathsf{L}} \ldots \delta_n\gamma_n^{\mathsf{L}} w'\alpha_{m+1}bb^{\mathsf{R}}\alpha_{m+1}^{\mathsf{R}}w''bb^{\mathsf{R}}\alpha_m^{\mathsf{R}}\beta_m \ldots \alpha_2^{\mathsf{R}}\beta_2\alpha_1^{\mathsf{R}}\beta_1 \\
&\ \ \vdots \\
&\to \delta_1\gamma_1^{\mathsf{L}}\delta_2\gamma_2^{\mathsf{L}} \ldots \delta_n\gamma_n^{\mathsf{L}} t\alpha_{m+1}bb^{\mathsf{R}}\alpha_{m+1}^{\mathsf{R}}w''bb^{\mathsf{R}}\alpha_m^{\mathsf{R}}\beta_m \ldots \alpha_2^{\mathsf{R}}\beta_2\alpha_1^{\mathsf{R}}\beta_1 \\
&\to \delta_1\gamma_1^{\mathsf{L}}\delta_2\gamma_2^{\mathsf{L}} \ldots \delta_n\gamma_n^{\mathsf{L}} tw''bb^{\mathsf{R}}\alpha_m^{\mathsf{R}}\beta_m \ldots \alpha_2^{\mathsf{R}}\beta_2\alpha_1^{\mathsf{R}}\beta_1 \\
&\to \delta_1\gamma_1^{\mathsf{L}}\delta_2\gamma_2^{\mathsf{L}} \ldots \delta_n\gamma_n^{\mathsf{L}} tw''\alpha_m^{\mathsf{R}}\beta_m \ldots \alpha_2^{\mathsf{R}}\beta_2\alpha_1^{\mathsf{R}}\beta_1 \\
&= u_{l+1}.
\end{aligned}
$$

This gives a chain that is T-consistent up to and including the point where α_{m+1} (or $\alpha_{m+1}b$) has been inserted, which is certainly after u_k. Notice that, although the chain may have been lengthened, the number of insertions in the part of the chain which may not be T-consistent has decreased. Notice that the subchain from the deletion of α_{m+1} (or $\alpha_{m+1}b$) to u_{l+1} is T-consistent. (In particular, $t \in L(A, T)$ since $w' \in L(A, T)$ and $\overline{t} = \overline{w'}$.)

This leaves the possibility that w', α_{m+1}, w'' all lie in $L(A, T)$. In this case, it is immediate that the chain is T-consistent up to and including u_{k+1}. (Observe that the corresponding deletion $u_l \to u_{l+1}$ forms a T-consistent subchain.)

Finally, suppose that $u_k \to u_{k+1}$ is a deletion. Then it arises from a corresponding earlier insertion, and so, by the observations in the last two paragraphs, the chain is T-consistent up to and including u_{k+1}.

This process of transforming a chain into a T-consistent one will terminate. This proves the result. □

Extend the rewriting mapping ϕ to words arising in cancellative $\mathcal{P}$-chains as follows. Define $u^{\mathsf{L}}\phi = (u\phi)^{\mathsf{L}}$ and $u^{\mathsf{R}}\phi = (u\phi)^{\mathsf{R}}$, and, for a word u having a decomposition (4)

$$\delta_1 \gamma_1^{\mathsf{L}} \delta_2 \gamma_2^{\mathsf{L}} \ldots \delta_n \gamma_n^{\mathsf{L}} w \alpha_m^{\mathsf{R}} \beta_m \ldots \alpha_2^{\mathsf{R}} \beta_2 \alpha_1^{\mathsf{R}} \beta_1,$$

let $w\phi$ be

$$(\delta_1\phi)\,(\gamma_1^{\mathsf{L}}\phi)\,(\delta_2\phi)\,(\gamma_2^{\mathsf{L}}\phi)\ldots(\delta_n\phi)\,(\gamma_n^{\mathsf{L}}\phi)(w\phi)\,(\alpha_n^{\mathsf{R}}\phi)\,(\beta_n\phi)$$
$$\ldots(\alpha_2^{\mathsf{R}}\phi)\,(\beta_2\phi)\,(\alpha_1^{\mathsf{R}}\phi)\,(\beta_1\phi).$$

LEMMA 7.3. *The semigroup T has cancellative presentation* $\mathrm{SgC}\langle D \mid \mathcal{S}\rangle$.

Recall that the finite set of relations $\mathcal{S}$ was introduced in the statement of Theorem 5.2 and is such that $\mathcal{S}^{\#}$ contains all relations (1), (2), and (3). The alphabet D was also introduced in Section 5.

PROOF. Notice first of all that every relation in $\mathcal{S}$ holds in T by the definition of $\mathcal{S}$ and the mapping ϕ. It remains to show that every relation that holds in D lies in the cancellative congruence $\mathcal{S}^{\mathsf{C}}$.

Let u and v be words over D representing the same element of T. Let $u' = u\psi$ and $v' = v\psi$. The words u' and v' lie in $L(A, T)$, so by Lemma 7.2 there is a T-consistent extended cancellative $\mathcal{P}$-chain

$$u' = u_0 \to u_1 \to \cdots \to u_l = v'.$$

Inductively construct a cancellative $\mathcal{S}$-chain from $u'\phi$ to $v'\phi$ as follows.

Trivially, such a chain exists from $u'\phi$ to $u_0\phi$. Suppose that such a chain $\mathfrak{C}_k$ has been constructed from $u'\phi$ to $u_k\phi$, and that the decomposition (4) of u_k is

$$u_k = \delta_1 \gamma_1^{\mathsf{L}} \delta_2 \gamma_2^{\mathsf{L}} \ldots \delta_n \gamma_n^{\mathsf{L}} w \alpha_m^{\mathsf{R}} \beta_m \ldots \alpha_2^{\mathsf{R}} \beta_2 \alpha_1^{\mathsf{R}} \beta_1.$$

Suppose that $u_k \to u_{k+1}$ is an application of a relation from $\mathcal{P}$. Then u_{k+1} has decomposition (4)

$$u_{k+1} = \delta_1 \gamma_1^{\mathsf{L}} \delta_2 \gamma_2^{\mathsf{L}} \ldots \delta_n \gamma_n^{\mathsf{L}} \hat{w} \alpha_m^{\mathsf{R}} \beta_m \ldots \alpha_2^{\mathsf{R}} \beta_2 \alpha_1^{\mathsf{R}} \beta_1,$$

where $\hat{w}$ differs from w by a single application of a relation from $\mathcal{P}$. So $\hat{w}\phi$ differs from $w\phi$ — and so $u_{k+1}\phi$ differs from $u_k\phi$—by a single application of a relation (3), which lies in $\mathcal{S}^{\#}$. So there is certainly a cancellative $\mathcal{S}$-chain from $u_k\phi$ to $u_{k+1}\phi$. Concatenating this with the existing $\mathcal{S}$-chain $\mathfrak{C}_k$ gives a chain from $u'\phi$ to $u_{k+1}\phi$.

Suppose now that $u_k \to u_{k+1}$ is an insertion of $\alpha_{m+1}\alpha_{m+1}^{\mathsf{R}}$. Then u_{k+1} has decomposition

$$u_{k+1} = \delta_1 \gamma_1^{\mathsf{L}} \delta_2 \gamma_2^{\mathsf{L}} \ldots \delta_n \gamma_n^{\mathsf{L}} w' \alpha_{m+1} \alpha_{m+1}^{\mathsf{R}} w'' \alpha_m^{\mathsf{R}} \beta_m \ldots \alpha_2^{\mathsf{R}} \beta_2 \alpha_1^{\mathsf{R}} \beta_1,$$

where $w = w'w''$. To show the existence of a cancellative $\mathcal{S}$-chain from $u_k\phi$ to $u_{k+1}\phi$, it suffices to show that one exists from $w\phi$ to $(w'\alpha_{m+1})\phi(\alpha_{m+1}^{\mathsf{R}}\phi)\,(w''\phi)$. Using relations of type (2), one obtains the following cancellative $\mathcal{S}$-chain:

$$\begin{aligned} w\phi &= (w'w'')\phi \\ &\to (w'\phi)\,(w''\phi) \\ &\to (w'\phi)\,(\alpha_{m+1}\phi)\,(\alpha_{m+1}^{\mathsf{R}}\phi)\,(w''\phi) \\ &\to (w'\alpha_{m+1})\phi(\alpha_{m+1}^{\mathsf{R}}\phi)\,(w''\phi). \end{aligned}$$

Notice that w', α_{m+1}, w'' lie in $L(A, T)$ by the definition of T-consistency. This shows the existence of a cancellative chain from $u'\phi$ to $u_{k+1}\phi$.

If $u_k \to u_{k+1}$ is a deletion, then the reasoning for insertions applies in reverse to yield a cancellative $\mathcal{S}$-chain from $u'\phi$ to $u_{k+1}\phi$.

This process will eventually yield a cancellative $\mathcal{S}$-chain from $u'\phi$ to $v'\phi$. Relations of types (1) and (2) yield chains from u to $u'\phi$ and $v'\phi$ to v. Thus there is a cancellative $\mathcal{S}$-chain from u to v, that is, (u, v) lies in the cancellative congruence $\mathcal{S}^{\mathrm{C}}$.

Therefore, $\mathrm{SgC}\langle D \mid \mathcal{S}\rangle$ is a cancellative presentation for T. □

From the discussion following the proof of Theorem 7.1 and from Theorem 4.1, one obtains the second main result of the paper.

THEOREM 2. *Let S be a cancellative semigroup and let T be a subsemigroup of S of finite Rees index. Then S admits a finite cancellative presentation if and only if T admits a finite cancellative presentation.*

8. Right- and left-cancellative presentations

This section only contains proofs for right-cancellative presentations. The left-cancellative case is symmetric.

The first result is the analogue of Theorem 7.1.

THEOREM 8.1. *Let S be an infinite right-cancellative semigroup and let T be a subsemigroup of S of finite Rees index. Then*

$$I = \{t \in T : tS^1 \subseteq T\}$$

is a right ideal of S contained in T and has finite Rees index in T. Indeed, I is the largest right ideal of S contained in T.

PROOF. Since S is infinite and $S - T$ is finite, T too must be infinite. Suppose that $T - I$ is infinite. Then there are infinitely many elements $t \in T - I$ such that $ts \in S - T$ for some $s \in S$. Since $S - T$ is finite, there exists $w \in S - T$ such that the set

$$K_w = \{t \in T - I : (\exists s \in S)\ (ts = w)\}$$

is infinite. Fix such a w. For each $t \in K_w$, fix $s_t \in S$ with $ts_t = w$. Observe that each s_t must lie in $S - T$. So there exist distinct $t, u \in K_w$ with $s_t = s_u$, since $S - T$ is finite. So $ts_t = w = us_u$, whence $t = u$ by right-cancellativity, which is a contradiction. So $T - I$ is finite, which implies that I is nonempty.

Suppose that $t \in I$ and $s \in S^1$. Then $ts \in T$. Furthermore, $ts \in I$, since otherwise there would exist $u \in S^1$ with $t(su) \notin T$, contradicting the definition of I. So I is a right ideal of S. That T contains I is obvious; that $T - I$ is finite has already been proven.

Suppose that J is another right ideal of S contained in T with $I \subseteq J$. Suppose that $I \subset J$. Fix $t \in J - I$. Then there exists $s \in S^1$ with $ts \notin T$. So $ts \notin J$, contradicting the fact that J is a right ideal. So I is the largest right ideal of S contained in T. □

In light of Theorem 8.1, to prove Theorem 3 it suffices to prove that if S has a finite right-cancellative presentation and T is a *right ideal*, then T too has a finite right-cancellative presentation. For, when S is infinite, one then recovers the general result for a subsemigroup T by first of all showing that the largest ideal I of S contained in T has a finite right-cancellative presentation, then applying Theorem 4.1 to show that T itself has a finite right-cancellative presentation. When S is finite, so is T, and thus T trivially has a finite right-cancellative presentation.

LEMMA 8.2. *Let S be a right-cancellative semigroup; let T be a proper right ideal of S of finite Rees index. Then there exists a proper right ideal K of S that contains T and such that exactly one of the following two cases holds:*
(1) *$S - K$ is a subgroup and a right ideal of S;*
(2) *every element of $S - K$ has a right multiple that lies in K.*

The proof of this result uses some basic results regarding the Green's relations $\mathcal{H}$, $\mathcal{R}$, $\mathcal{L}$, $\mathcal{D}$, and $\mathcal{J}$ of a semigroup. For the definition of these relations and their basic properties, see [16, Ch. 2].

PROOF OF LEMMA 8.2. Choose an $\mathcal{R}$-maximal element x of S that lies outside T. Let R be its $\mathcal{R}$-class. Since x is $\mathcal{R}$-maximal, $K = S - R$ is a right ideal of S. The remainder of the proof is dedicated to showing that K has the desired properties.

Notice that, since T is a right ideal, the $\mathcal{R}$-class R must lie wholly within $S - T$. Thus K contains T.

Suppose that R is a right ideal. By the right-cancellativity of S, there must be a unique idempotent e in R. Let $x \in R$. Since $R \subseteq S - T$ is finite, some power of x is this idempotent e. Then, for any $x \in S$, $xe^2 = xe$, whence $xe = x$ by right-cancellativity. So e is a right identity for S. So, if $x \in R$, $x = xe$ lies in the same $\mathcal{H}$-class as e. Thus R consists of a single group $\mathcal{H}$-class.

On the other hand, suppose that some element x of R has a right multiple xs (where $s \in S$) that lies in K. Let $y \in R$. Then there exists $s' \in S^1$ with $ys' = x$. So $yss' = xs \in K$. So if some element of R has a right multiple lying in K (that is, if R is not a right ideal) then *every* element of R has this property. □

For the remainder of this section, let S be a right-cancellative semigroup, let T be a finite Rees index right ideal of S, and adopt the notation of Section 5 with $\mathfrak{P}$ being 'right-cancellative'. Suppose that the presentation $\mathrm{SgRC}\langle A \mid \mathcal{P}\rangle$ for S is finite. The aim is to show that T too must have a finite right-cancellative presentation. By Lemma 8.2, it suffices to prove this when $S - T$ is either a group or has the property that each of its elements has a right multiple lying in T. For one can then pass down through a descending sequence of finitely many (since $|S - T|$ is finite) right ideals to any right ideal of S.

Actually, the case when $S - T$ is a subgroup and a right ideal is very simple, as the following three results show.

PROPOSITION 8.3. *Suppose that $G = S - T$ is a subgroup and a right ideal of S. Then S is left-simple.*

PROOF. Let $L \subseteq S$ be a left ideal of S. The aim is to show that $L = S$.

Suppose that $L \subseteq T$. Pick $l \in L$ and $s \in G$. Then $sl \in G \cap L$ since G is a right ideal and L is a left ideal. This contradicts the assumption that T contains L. Thus $G \cap L$ is nonempty.

Now, $G \cap L$ is a left ideal of G, since L is a left ideal of S and G is a subsemigroup. But G is a group, and thus its only left ideal is G itself. Thus $G \cap L = G$, and so $G \subseteq L$. In particular, 1_G, the identity of G, lies in L.

Now, for $x \in S$, $x1_G = x1_G1_G$, whence $x = x1_G$. Since L is a left ideal, $x = x1_G$ lies in L. Therefore, $L = S$ as required. □

THEOREM 8.4 [11, Theorem 1.27]. *A semigroup is right-cancellative and left simple if and only if it is isomorphic to $G \times Z$, where G is a group and Z is a left zero semigroup.*

PROPOSITION 8.5. *Suppose that $S - T$ is a subgroup and a right ideal of S. Then $S \simeq (S - T) \times Z$, where Z is a left zero semigroup.*

PROOF. By Proposition 8.3, S is left simple. Since it is also right-cancellative, Theorem 8.4 applies to show that S is isomorphic to $G \times Z$, where G is a group and Z is a left zero semigroup. The $\mathcal{H}$-classes of S are $G \times \{z\}$ where $z \in Z$; its idempotents are $(1_G, z)$. So $S - T$, being an $\mathcal{H}$-class, is isomorphic to G. Thus $S \simeq (S - T) \times Z$ as required. □

COROLLARY 8.6. *Suppose that $S - T$ is a subgroup and a right ideal of S. Then S is finite.*

PROOF. By Proposition 8.5, $S \simeq (S - T) \times Z$, where Z is a left zero semigroup. The finite generating set $\overline{A}$ for S must project to a generating set for Z. Since Z is a left zero semigroup, its only generating set is Z itself. So Z is finite. Since $S - T$ is also finite, the semigroup S is finite. □

Therefore, by Corollary 8.6, if $S - T$ is a subgroup and a right ideal, then S, and so also T, is finite, and therefore trivially admits a finite right-cancellative presentation. This leaves the case where every element of $S - T$ has a right multiple that lies in T. So assume that $S - T$ has this property. The next lemma strengthens this property to show that any element of $S - T$ can be right-multiplied by an element of T to give an element of T.

LEMMA 8.7. *For any element s of $S - T$, there exists an element p of T such that sp lies in T.*

PROOF. The element $s \in S - T$ has a right multiple sp' lying in T. If p' is also in T, set $p = p'$. If $p' \in S - T$, then it has a multiple $p'x$ lying in T. Since T is a right ideal, $s(p'x) = (sp')x \in T$. Setting $p = p'x$ gives the result. □

Every word in an extended cancellative chain has a decomposition (4). Similarly, every word in an extended right-cancellative chain has a factorization

$$w\alpha_m^{\mathsf{R}}\beta_m \dots \alpha_2^{\mathsf{R}}\beta_2\alpha_1^{\mathsf{R}}\beta_1, \tag{5}$$

where each word w, α_i, β_i is in A^* and each α_i^{R} arises from the insertion of $\alpha_i\alpha_i^{\mathsf{R}}$ earlier in the chain (and hence α_i is nonempty). The concept of T-consistency applies to right-cancellative chains.

LEMMA 8.8. *If $u, v \in L(A, T)$ and $\overline{u} = \overline{v}$, then there is a T-consistent extended right-cancellative $\mathcal{P}$-chain from u to v.*

PROOF. Since $\overline{u} = \overline{v}$, there is an right-cancellative $\mathcal{P}$ chain

$$u = u_0 \to u_1 \to \cdots \to u_l = v.$$

The aim is to transform this chain into a T-consistent one.

Observe firstly that the trivial subchain u_0 is T-consistent. Now suppose that the subchain

$$u_0 \to u_1 \to \cdots \to u_k,$$

is T-consistent, where $0 \le k < l$. Let the decomposition (5) of u_k be

$$u_k = w\alpha_m^{\mathsf{R}}\beta_m \dots \alpha_2^{\mathsf{R}}\beta_2\alpha_1^{\mathsf{R}}\beta_1.$$

If $u_k \to u_{k+1}$ is an application of a relation from $\mathcal{P}$, then the decomposition of u_{k+1} is

$$u_{k+1} = w'\alpha_m^{\mathsf{R}}\beta_m \dots \alpha_2^{\mathsf{R}}\beta_2\alpha_1^{\mathsf{R}}\beta_1,$$

where w' differs from w by a single application of $\mathcal{P}$. Now, since $w \in L(A, T)$ and $\overline{w} = \overline{w'}$, w' also lies in $L(A, T)$. So the chain is T-consistent up to and including u_{k+1}.

Now suppose that $u_k \to u_{k+1}$ is the insertion of $\alpha_{m+1}\alpha_{m+1}^{\mathsf{R}}$. Then the decomposition of u_{k+1} is

$$u_{k+1} = w'\alpha_{m+1}\alpha_{m+1}^{\mathsf{R}}w''\alpha_m^{\mathsf{R}}\beta_m \dots \alpha_2^{\mathsf{R}}\beta_2\alpha_1^{\mathsf{R}}\beta_1,$$

where $w = w'w''$. Suppose that the corresponding deletion is $u_l \to u_{l+1}$, so that

$$u_l = t\alpha_{m+1}\alpha_{m+1}^{\mathsf{R}}w''\alpha_m^{\mathsf{R}}\beta_m \dots \alpha_2^{\mathsf{R}}\beta_2\alpha_1^{\mathsf{R}}\beta_1$$

and

$$u_{l+1} = tw''\alpha_m^{\mathsf{R}}\beta_m \dots \alpha_2^{\mathsf{R}}\beta_2\alpha_1^{\mathsf{R}}\beta_1,$$

and there is a right-cancellative $\mathcal{P}$-chain from w' to t.

There are three possible ways in which the T-consistency of the chain $u_0 \to \cdots \to u_k$ may not extend to u_{k+1}: $w'\alpha_{m+1}$, α_{m+1}, or w'' may not lie in $L(A, T)$.

Suppose that $w' \notin L(A, T)$. Since T is a right ideal, $\overline{w'}$ must lie in CA^*. Similarly, as $\overline{w'} = \overline{t}$, the word t also lies in CA^*. Using relations of length two lying in the set $CA \times C$, one can work from left to right to transform w' into some $c \in C$. Similarly, one can transform t into that same c. Thus there is a chain from w' to t that involves no insertions and deletions. Thus one can use this chain to go from u_k to u_{l+1} without having to insert $\alpha_{m+1}\alpha_{m+1}^{\mathsf{R}}$. The new chain thus obtained is T-consistent up to and including u_{l+1}.

So assume that $w' \in L(A, T)$. Then $w'\alpha_{m+1} \in L(A, T)$ since T is a right ideal. Suppose now that either $w'' \notin L(A, T)$ or $\alpha_{m+1} \notin L(A, T)$. Applying Lemma 8.7, let η be a word in $L(A, T)$ such that $\overline{w''\eta} \in T$ and let $\zeta \in L(A, T)$ be such that $\overline{\alpha_{m+1}\zeta} \in T$. Modify the subchain $u_k \to \cdots \to u_{l+1}$ to the following form:

$$
\begin{aligned}
u_k &= w\alpha_m^{\mathsf{R}}\beta_m \ldots \alpha_2^{\mathsf{R}}\beta_2\alpha_1^{\mathsf{R}}\beta_1 \\
&\to w\eta\eta^{\mathsf{R}}\alpha_m^{\mathsf{R}}\beta_m \ldots \alpha_2^{\mathsf{R}}\beta_2\alpha_1^{\mathsf{R}}\beta_1 \\
&\to w'\alpha_{m+1}\zeta\zeta^{\mathsf{R}}\alpha_{m+1}^{\mathsf{R}}w''\eta\eta^{\mathsf{R}}\alpha_m^{\mathsf{R}}\beta_m \ldots \alpha_2^{\mathsf{R}}\beta_2\alpha_1^{\mathsf{R}}\beta_1 \\
&\ \ \vdots \\
&\to t\alpha_{m+1}\zeta\zeta^{\mathsf{R}}\alpha_{m+1}^{\mathsf{R}}w''\eta\eta^{\mathsf{R}}\alpha_m^{\mathsf{R}}\beta_m \ldots \alpha_2^{\mathsf{R}}\beta_2\alpha_1^{\mathsf{R}}\beta_1 \\
&\to tw''\eta\eta^{\mathsf{R}}\alpha_m^{\mathsf{R}}\beta_m \ldots \alpha_2^{\mathsf{R}}\beta_2\alpha_1^{\mathsf{R}}\beta_1 \\
&\to tw''\alpha_m^{\mathsf{R}}\beta_m \ldots \alpha_2^{\mathsf{R}}\beta_2\alpha_1^{\mathsf{R}}\beta_1 \\
&= u_{l+1}.
\end{aligned}
$$

This gives a chain that is T-consistent up to the point where α_{m+1} (or $\alpha_{m+1}\eta$) has been inserted, which is certainly after u_k. Notice that, although the chain may have been lengthened, the number of insertions in the part of the chain which may not be T-consistent has decreased. Notice that the subchain from the deletion of α_{m+1} (or $\alpha_{m+1}b$) to u_{l+1} is T-consistent. (In particular, $t \in L(A, T)$ since $w' \in L(A, T)$ and $\overline{t} = \overline{w'}$.)

This leaves the possibility that w', α_{m+1}, w'' all lie in $L(A, T)$. In this case, it is immediate that the chain is T-consistent up to and including u_{k+1}. (Observe that the corresponding deletion $u_l \to u_{l+1}$ forms a T-consistent subchain.)

Finally, suppose that $u_k \to u_{k+1}$ is a deletion. Then it arises from a corresponding earlier insertion, and so the chain is T-consistent up to and including u_{k+1}.

This process of transforming a chain into a T-consistent one will terminate. This proves the result. □

LEMMA 8.9. *The right ideal T has right-cancellative presentation* $\mathrm{SgC}\langle D \mid \mathcal{S}\rangle$.

PROOF. The reasoning for this result exactly parallels that of Lemma 7.3 except that one invokes Lemma 8.8 rather than Lemma 7.2. □

From the discussion following the proof of Theorem 8.1 and from Theorem 4.1, one obtains the third main result of the paper.

THEOREM 3. *Let S be a left-cancellative (respectively, right-cancellative) semigroup and let T be a subsemigroup of S of finite Rees index. Then S admits a finite left-cancellative (respectively, right-cancellative) presentation if and only if T admits a finite left-cancellative (respectively, right-cancellative) presentation.*

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