Supplementary Material for the paper:

Sparse designs for estimating variance components of nested factors with random effects

By authors:

R. A. Bailey, Célia Fernandes and Paulo Ramos

Table SM.1: Summary for the balanced nested design $m/2/2$

$a_1 = m$ $a_2 = 2$ $a_3 = 2$	$c_1 = m$ $c_2 = 2m$ $c_3 = 4m$	$b_1 = 4$ $b_2 = 2$ $b_3 = 1$
$d_1 = m - 1$ $d_2 = m$ $d_3 = 2m$	$\gamma_1 = 4\sigma_1^2 + 2\sigma_2^2 + \sigma_3^2$ $\gamma_2 = 2\sigma_2^2 + \sigma_3^2$ $\gamma_3 = \sigma_3^2$	$\hat{\sigma}_1^2 = \frac{1}{4}\left(\hat{\gamma}_1 - \hat{\gamma}_2\right)$ $\hat{\sigma}_2^2 = \frac{1}{2}\left(\hat{\gamma}_2 - \hat{\gamma}_3\right)$ $\hat{\sigma}_3^2 = \hat{\gamma}_3$
$\mathrm{Var}\left(\hat{\sigma}_1^2\right) = \frac{1}{2m(m-1)}\left[4m\sigma_1^4 + (2m-1)\,\sigma_2^4 + \left(\frac{1}{2}m - \frac{1}{4}\right)\sigma_3^4 + 4m\sigma_1^2\sigma_2^2 + 2m\sigma_1^2\sigma_3^2 + (2m-1)\,\sigma_2^2\sigma_3^2\right]$ $\mathrm{Var}\left(\hat{\sigma}_2^2\right) = \frac{1}{2m(m-1)}\left[(4m-4)\,\sigma_2^4 + \left(\frac{3}{2}m - \frac{3}{2}\right)\sigma_3^4 + (4m-4)\,\sigma_2^2\sigma_3^2\right]$ $\mathrm{Var}\left(\hat{\sigma}_3^2\right) = \frac{1}{2m(m-1)}\left[(2m-2)\,\sigma_3^4\right]$		

Table SM.2: Summary for the staggered design n/T_3

$d_1 = n-1$	$\gamma_1 = 3\sigma_1^2 + \frac{5}{3}\sigma_2^2 + \sigma_3^2$	$\hat{\sigma}_1^2 = \frac{1}{12}\left(4\hat{\gamma}_1 - 5\hat{\gamma}_2 + \hat{\gamma}_3\right)$
$d_2 = n$	$\gamma_2 = \frac{4}{3}\sigma_2^2 + \sigma_3^2$	$\hat{\sigma}_2^2 = \frac{3}{4}\left(\hat{\gamma}_2 - \hat{\gamma}_3\right)$
$d_3 = n$	$\gamma_3 = \sigma_3^2$	$\hat{\sigma}_3^2 = \hat{\gamma}_3$
$\text{Var}\left(\hat{\sigma}_1^2\right) = \frac{1}{2n(n-1)}\left[4n\sigma_1^4 + \frac{180n-80}{81}\sigma_2^4 + \frac{21n-13}{18}\sigma_3^4 + \frac{40n}{9}\sigma_1^2\sigma_2^2 + \frac{8n}{3}\sigma_1^2\sigma_3^2 + \frac{90n-50}{27}\sigma_2^2\sigma_3^2\right]$		
$\text{Var}\left(\hat{\sigma}_2^2\right) = \frac{1}{2n(n-1)}\left[4(n-1)\sigma_2^4 + \frac{9(n-1)}{2}\sigma_3^4 + 6(n-1)\sigma_2^2\sigma_3^2\right]$		
$\text{Var}\left(\hat{\sigma}_3^2\right) = \frac{1}{2n(n-1)}\left[4(n-1)\sigma_3^4\right]$		

Table SM.3: Summary for the stair nested design $n/1/1 + 1/n/1 + 1/1/n$

$d_1 = n-1$	$\gamma_1 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2$	$\hat{\sigma}_1^2 = \hat{\gamma}_1 - \hat{\gamma}_2$
$d_2 = n-1$	$\gamma_2 = \sigma_2^2 + \sigma_3^2$	$\hat{\sigma}_2^2 = \hat{\gamma}_2 - \hat{\gamma}_3$
$d_3 = n-1$	$\gamma_3 = \sigma_3^2$	$\hat{\sigma}_3^2 = \hat{\gamma}_3$
$\text{Var}\left(\hat{\sigma}_1^2\right) = \frac{1}{2n(n-1)}\left[4n\sigma_1^4 + 8n\sigma_2^4 + 8n\sigma_3^4 + 8n\sigma_1^2\sigma_2^2 + 8n\sigma_1^2\sigma_3^2 + 16n\sigma_2^2\sigma_3^2\right]$		
$\text{Var}\left(\hat{\sigma}_2^2\right) = \frac{1}{2n(n-1)}\left[4n\sigma_2^4 + 8n\sigma_3^4 + 8n\sigma_2^2\sigma_3^2\right]$		
$\text{Var}\left(\hat{\sigma}_3^2\right) = \frac{1}{2n(n-1)}\left[4n\sigma_3^4\right]$		

Table SM.4: Summary for the sparse component design $n/1/2 + 1/n/1$

$d_1 = n-1$	$\gamma_1 = 2\sigma_1^2 + 2\sigma_2^2 + \sigma_3^2$	$\hat{\sigma}_1^2 = \frac{1}{2}\hat{\gamma}_1 - \hat{\gamma}_2 + \frac{1}{2}\hat{\gamma}_3$
$d_2 = n-1$	$\gamma_2 = \sigma_2^2 + \sigma_3^2$	$\hat{\sigma}_2^2 = \hat{\gamma}_2 - \hat{\gamma}_3$
$d_3 = n$	$\gamma_3 = \sigma_3^2$	$\hat{\sigma}_3^2 = \hat{\gamma}_3$
$\text{Var}\left(\hat{\sigma}_1^2\right) = \frac{1}{2n(n-1)}\left[4n\sigma_1^4 + 8n\sigma_2^4 + (6n-1)\,\sigma_3^4 + 8n\sigma_1^2\sigma_2^2 + 4n\sigma_1^2\sigma_3^2 + 12n\sigma_2^2\sigma_3^2\right]$		
$\text{Var}\left(\hat{\sigma}_2^2\right) = \frac{1}{2n(n-1)}\left[4n\sigma_2^4 + (8n-4)\,\sigma_3^4 + 8n\sigma_2^2\sigma_3^2\right]$		
$\text{Var}\left(\hat{\sigma}_3^2\right) = \frac{1}{2n(n-1)}\left[(4n-4)\,\sigma_3^4\right]$		

Table SM.5: Summary for the sparse component design $n/2/1 + 1/1/n$

$d_1 = n - 1$	$\gamma_1 = 2\sigma_1^2 + \sigma_2^2 + \sigma_3^2$	$\hat{\sigma}_1^2 = \frac{1}{2}\hat{\gamma}_1 - \frac{1}{2}\hat{\gamma}_2$
$d_2 = n$	$\gamma_2 = \sigma_2^2 + \sigma_3^2$	$\hat{\sigma}_2^2 = \hat{\gamma}_2 - \hat{\gamma}_3$
$d_3 = n - 1$	$\gamma_3 = \sigma_3^2$	$\hat{\sigma}_3^2 = \hat{\gamma}_3$
$\mathrm{Var}\left(\hat{\sigma}_1^2\right) = \frac{1}{2n(n-1)}\left[4n\sigma_1^4 + (2n-1)\,\sigma_2^4 + (2n-1)\,\sigma_3^4 + 4n\sigma_1^2\sigma_2^2 + 4n\sigma_1^2\sigma_3^2 + (4n-2)\,\sigma_2^2\sigma_3^2\right]$		
$\mathrm{Var}\left(\hat{\sigma}_2^2\right) = \frac{1}{2n(n-1)}\left[(4n-4)\,\sigma_2^4 + (8n-4)\,\sigma_3^4 + (8n-8)\,\sigma_2^2\sigma_3^2\right]$		
$\mathrm{Var}\left(\hat{\sigma}_3^2\right) = \frac{1}{2n(n-1)}\left[4n\sigma_3^4\right]$		

Table SM.6: Summary for the sparse component design $n/1/1 + 1/n/2$

$d_1 = n - 1$	$\gamma_1 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2$	$\hat{\sigma}_1^2 = \hat{\gamma}_1 - \frac{1}{2}\hat{\gamma}_2 - \frac{1}{2}\hat{\gamma}_3$
$d_2 = n - 1$	$\gamma_2 = 2\sigma_2^2 + \sigma_3^2$	$\hat{\sigma}_2^2 = \frac{1}{2}\hat{\gamma}_2 - \frac{1}{2}\hat{\gamma}_3$
$d_3 = n$	$\gamma_3 = \sigma_3^2$	$\hat{\sigma}_3^2 = \hat{\gamma}_3$
$\mathrm{Var}\left(\hat{\sigma}_1^2\right) = \frac{1}{2n(n-1)}\left[4n\sigma_1^4 + 8n\sigma_2^4 + (6n-1)\,\sigma_3^4 + 8n\sigma_1^2\sigma_2^2 + 8n\sigma_1^2\sigma_3^2 + 12n\sigma_2^2\sigma_3^2\right]$		
$\mathrm{Var}\left(\hat{\sigma}_2^2\right) = \frac{1}{2n(n-1)}\left[4n\sigma_2^4 + (2n-1)\,\sigma_3^4 + 4n\sigma_2^2\sigma_3^2\right]$		
$\mathrm{Var}\left(\hat{\sigma}_3^2\right) = \frac{1}{2n(n-1)}\left[(4n-4)\,\sigma_3^4\right]$		

Table SM.7: Summary for the balanced nested design $m/2/2/2$

$a_1 = m$	$c_1 = m$	$b_1 = 8$
$a_2 = 2$	$c_2 = 2m$	$b_2 = 4$
$a_3 = 2$	$c_3 = 4m$	$b_3 = 2$
$a_4 = 2$	$c_4 = 8m$	$b_3 = 1$
$d_1 = m - 1$	$\gamma_1 = 8\sigma_1^2 + 4\sigma_2^2 + 2\sigma_3^2 + \sigma_4^2$	$\hat{\sigma}_1^2 = \frac{1}{8}\left(\hat{\gamma}_1 - \hat{\gamma}_2\right)$
$d_2 = m$	$\gamma_2 = 4\sigma_2^2 + 2\sigma_3^2 + \sigma_4^2$	$\hat{\sigma}_2^2 = \frac{1}{4}\left(\hat{\gamma}_2 - \hat{\gamma}_3\right)$
$d_3 = 2m$	$\gamma_3 = 2\sigma_3^2 + \sigma_4^2$	$\hat{\sigma}_3^2 = \frac{1}{2}\left(\hat{\gamma}_3 - \hat{\gamma}_4\right)$
$d_4 = 4m$	$\gamma_4 = \sigma_4^2$	$\hat{\sigma}_4^2 = \hat{\gamma}_4$
$\mathrm{Var}\left(\hat{\sigma}_1^2\right) = \frac{1}{2m(m-1)}\left[4m\sigma_1^4 + (2m-1)\,\sigma_2^4 + \left(\frac{1}{2}m - \frac{1}{4}\right)\sigma_3^4 + \left(\frac{1}{8}m - \frac{1}{16}\right)\sigma_4^4 + 4m\sigma_1^2\sigma_2^2 + 2m\sigma_1^2\sigma_3^2 + m\sigma_1^2\sigma_4^2 + (2m-1)\,\sigma_2^2\sigma_3^2 + \left(m - \frac{1}{2}\right)\sigma_2^2\sigma_4^2 + \left(\frac{1}{2}m - \frac{1}{4}\right)\sigma_3^2\sigma_4^2\right]$		
$\mathrm{Var}\left(\hat{\sigma}_2^2\right) = \frac{1}{2m(m-1)}\left[(4m-4)\,\sigma_2^4 + \left(\frac{3}{2}m - \frac{3}{2}\right)\sigma_3^4 + \left(\frac{3}{8}m - \frac{3}{8}\right)\sigma_4^4 + (4m-4)\,\sigma_2^2\sigma_3^2 + (2m-2)\,\sigma_2^2\sigma_4^2 + \left(\frac{3}{2}m - \frac{3}{2}\right)\sigma_3^2\sigma_4^2\right]$		
$\mathrm{Var}\left(\hat{\sigma}_3^2\right) = \frac{1}{2m(m-1)}\left[(2m-2)\,\sigma_3^4 + \left(\frac{3}{4}m - \frac{3}{4}\right)\sigma_4^4 + (2m-2)\,\sigma_3^2\sigma_4^2\right]$		
$\mathrm{Var}\left(\hat{\sigma}_4^2\right) = \frac{1}{2m(m-1)}\left[(m-1)\,\sigma_4^4\right]$		

Table SM.8: Summary for the staggered design n/T_{4a}

$d_1 = n - 1$	$\gamma_1 = 4\sigma_1^2 + \frac{5}{2}\sigma_2^2 + \frac{3}{2}\sigma_3^2 + \sigma_4^2$	$\hat{\sigma}_1^2 = \frac{1}{12}\left(3\hat{\gamma}_1 - 5\hat{\gamma}_2 + \hat{\gamma}_3 + \hat{\gamma}_4\right)$
$d_2 = n$	$\gamma_2 = \frac{3}{2}\sigma_2^2 + \frac{7}{6}\sigma_3^2 + \sigma_4^2$	$\hat{\sigma}_2^2 = \frac{1}{12}\left(8\hat{\gamma}_2 - 7\hat{\gamma}_3 - \hat{\gamma}_4\right)$
$d_3 = n$	$\gamma_3 = \frac{4}{3}\sigma_3^2 + \sigma_4^2$	$\hat{\sigma}_3^2 = \frac{3}{4}\left(\hat{\gamma}_3 - \hat{\gamma}_4\right)$
$d_4 = n$	$\gamma_4 = \sigma_4^2$	$\hat{\sigma}_4^2 = \hat{\gamma}_4$
$\mathrm{Var}\left(\hat{\sigma}_1^2\right) = \frac{1}{2n(n-1)}\left[4n\sigma_1^4 + \frac{40n-15}{16}\sigma_2^4 + \frac{24n-15}{16}\sigma_3^4 + \frac{4n-3}{4}\sigma_4^4 + 5n\sigma_1^2\sigma_2^2 + 3n\sigma_1^2\sigma_3^2 + 2n\sigma_1^2\sigma_4^2 + \frac{280n-145}{72}\sigma_2^2\sigma_3^2 + \frac{40n-25}{12}\sigma_2^2\sigma_4^2 + \frac{88n-61}{36}\sigma_3^2\sigma_4^2\right]$		
$\mathrm{Var}\left(\hat{\sigma}_2^2\right) = \frac{1}{2n(n-1)}\left[4(n-1)\sigma_2^4 + \frac{14(n-1)}{3}\sigma_3^4 + \frac{19(n-1)}{6}\sigma_4^4 + \frac{56(n-1)}{9}\sigma_2^2\sigma_3^2 + \frac{16(n-1)}{3}\sigma_2^2\sigma_4^2 + \frac{70(n-1)}{9}\sigma_3^2\sigma_4^2\right]$		
$\mathrm{Var}\left(\hat{\sigma}_3^2\right) = \frac{1}{2n(n-1)}\left[4(n-1)\sigma_3^4 + \frac{9(n-1)}{2}\sigma_4^4 + 6(n-1)\sigma_3^2\sigma_4^2\right]$		
$\mathrm{Var}\left(\hat{\sigma}_4^2\right) = \frac{1}{2n(n-1)}\left[4(n-1)\sigma_4^4\right]$		

Table SM.9: Summary for the generalized staggered design n/T_{4b}

$d_1 = n-1$	$\gamma_1 = 4\sigma_1^2 + 2\sigma_2^2 + \frac{3}{2}\sigma_3^2 + \sigma_4^2$	$\hat{\sigma}_1^2 = \frac{1}{4}\left(\hat{\gamma}_1 - \hat{\gamma}_2\right)$
$d_2 = n$	$\gamma_2 = 2\sigma_2^2 + \frac{3}{2}\sigma_3^2 + \sigma_4^2$	$\hat{\sigma}_2^2 = \frac{1}{4}\left(2\hat{\gamma}_2 - 3\hat{\gamma}_3 + \hat{\gamma}_4\right)$
$d_3 = n$	$\gamma_3 = \sigma_3^2 + \sigma_4^2$	$\hat{\sigma}_3^2 = \hat{\gamma}_3 - \hat{\gamma}_4$
$d_4 = n$	$\gamma_4 = \sigma_4^2$	$\hat{\sigma}_4^2 = \hat{\gamma}_4$
$\text{Var}\left(\hat{\sigma}_1^2\right) = \frac{1}{2n(n-1)}\left[4n\sigma_1^4 + (2n-1)\,\sigma_2^4 + \frac{16n-7}{16}\sigma_3^4 + \frac{2n-1}{4}\sigma_4^4 + 4n\sigma_1^2\sigma_2^2 + 3n\sigma_1^2\sigma_3^2 + 2n\sigma_1^2\sigma_4^2 + \frac{6n-3}{2}\sigma_2^2\sigma_3^2 + (2n-1)\,\sigma_2^2\sigma_4^2 + \frac{6n-3}{4}\sigma_3^2\sigma_4^2\right]$		
$\text{Var}\left(\hat{\sigma}_2^2\right) = \frac{1}{2n(n-1)}\left[4(n-1)\sigma_2^4 + \frac{9(n-1)}{2}\sigma_3^4 + \frac{7(n-1)}{2}\sigma_4^4 + 6(n-1)\sigma_2^2\sigma_3^2 + 4(n-1)\sigma_2^2\sigma_4^2 + \frac{15(n-1)}{2}\sigma_3^2\sigma_4^2\right]$		
$\text{Var}\left(\hat{\sigma}_3^2\right) = \frac{1}{2n(n-1)}\left[4(n-1)\sigma_3^4 + 8(n-1)\sigma_4^4 + 8(n-1)\sigma_3^2\sigma_4^2\right]$		
$\text{Var}\left(\hat{\sigma}_4^2\right) = \frac{1}{2n(n-1)}\left[4(n-1)\sigma_4^4\right]$		

Table SM.10: Summary for the stair nested design $n/1/1/1{+}1/n/1/1/{+}1/1/n/1{+}1/1/1/n$

$d_1 = n-1$	$\gamma_1 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2$	$\hat{\sigma}_1^2 = \hat{\gamma}_1 - \hat{\gamma}_2$
$d_2 = n-1$	$\gamma_2 = \sigma_2^2 + \sigma_3^2 + \sigma_4^2$	$\hat{\sigma}_2^2 = \hat{\gamma}_2 - \hat{\gamma}_3$
$d_3 = n-1$	$\gamma_3 = \sigma_3^2 + \sigma_4^2$	$\hat{\sigma}_3^2 = \hat{\gamma}_3 - \hat{\gamma}_4$
$d_4 = n-1$	$\gamma_4 = \sigma_4^2$	$\hat{\sigma}_4^2 = \hat{\gamma}_4$
$\text{Var}\left(\hat{\sigma}_1^2\right) = \frac{1}{2n(n-1)}\left[4n\sigma_1^4 + 8n\sigma_2^4 + 8n\sigma_3^4 + 8n\sigma_4^4 + 8n\sigma_1^2\sigma_2^2 + 8n\sigma_1^2\sigma_3^2 + 8n\sigma_1^2\sigma_4^2 + 16n\sigma_2^2\sigma_3^2 + 16n\sigma_2^2\sigma_4^2 + 16n\sigma_3^2\sigma_4^2\right]$		
$\text{Var}\left(\hat{\sigma}_2^2\right) = \frac{1}{2n(n-1)}\left[4n\sigma_2^4 + 8n\sigma_3^4 + 8n\sigma_4^4 + 8n\sigma_2^2\sigma_3^2 + 8n\sigma_2^2\sigma_4^2 + 16n\sigma_3^2\sigma_4^2\right]$		
$\text{Var}\left(\hat{\sigma}_3^2\right) = \frac{1}{2n(n-1)}\left[4n\sigma_3^4 + 8n\sigma_4^4 + 8n\sigma_3^2\sigma_4^2\right]$		
$\text{Var}\left(\hat{\sigma}_4^2\right) = \frac{1}{2n(n-1)}\left[4n\sigma_4^4\right]$		

Table SM.11: Summary for the sparse component design $n/2/1/1 + 1/1/n/2$

$d_1 = n-1$	$\gamma_1 = 2\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2$	$\hat{\sigma}_1^2 = \frac{1}{2}\hat{\gamma}_1 - \frac{1}{2}\hat{\gamma}_2$
$d_2 = n$	$\gamma_2 = \sigma_2^2 + \sigma_3^2 + \sigma_4^2$	$\hat{\sigma}_2^2 = \hat{\gamma}_2 - \frac{1}{2}\hat{\gamma}_3 - \frac{1}{2}\hat{\gamma}_4$
$d_3 = n-1$	$\gamma_3 = 2\sigma_3^2 + \sigma_4^2$	$\hat{\sigma}_3^2 = \frac{1}{2}\hat{\gamma}_3 - \frac{1}{2}\hat{\gamma}_4$
$d_4 = n$	$\gamma_4 = \sigma_4^2$	$\hat{\sigma}_4^2 = \hat{\gamma}_4$

$$\begin{aligned}
\mathrm{Var}\left(\hat{\sigma}_1^2\right) &= \tfrac{1}{2n(n-1)}\left[4n\sigma_1^4 + (2n-1)\,\sigma_2^4 + (2n-1)\,\sigma_3^4 + (2n-1)\,\sigma_4^4 + 4n\sigma_1^2\sigma_2^2 + 4n\sigma_1^2\sigma_3^2\right.\\
&\qquad \left. + 4n\sigma_1^2\sigma_4^2 + (4n-2)\,\sigma_2^2\sigma_3^2 + (4n-2)\,\sigma_2^2\sigma_4^2 + (4n-2)\,\sigma_3^2\sigma_4^2\right]\\
\mathrm{Var}\left(\hat{\sigma}_2^2\right) &= \tfrac{1}{2n(n-1)}\left[(4n-4)\,\sigma_2^4 + (8n-4)\,\sigma_3^4 + (6n-5)\,\sigma_4^4 + (8n-8)\sigma_2^2\sigma_3^2\right.\\
&\qquad \left. + (8n-8)\sigma_2^2\sigma_4^2 + (12n-8)\,\sigma_3^2\sigma_4^2\right]\\
\mathrm{Var}\left(\hat{\sigma}_3^2\right) &= \tfrac{1}{2n(n-1)}\left[4n\sigma_3^4 + (2n-1)\,\sigma_4^4 + 4n\sigma_3^2\sigma_4^2\right]\\
\mathrm{Var}\left(\hat{\sigma}_4^2\right) &= \tfrac{1}{2n(n-1)}\left[(4n-4)\,\sigma_4^4\right]
\end{aligned}$$

Table SM.12: Summary for the sparse component design $n/1/2/1 + 1/n/1/2$

$d_1 = n-1$	$\gamma_1 = 2\sigma_1^2 + 2\sigma_2^2 + \sigma_3^2 + \sigma_4^2$	$\hat{\sigma}_1^2 = \frac{1}{2}\hat{\gamma}_1 - \frac{1}{2}\hat{\gamma}_2 + \frac{1}{2}\hat{\gamma}_3 - \frac{1}{2}\hat{\gamma}_4$
$d_2 = n-1$	$\gamma_2 = 2\sigma_2^2 + 2\sigma_3^2 + \sigma_4^2$	$\hat{\sigma}_2^2 = \frac{1}{2}\hat{\gamma}_2 - \hat{\gamma}_3 + \frac{1}{2}\hat{\gamma}_4$
$d_3 = n$	$\gamma_3 = \sigma_3^2 + \sigma_4^2$	$\hat{\sigma}_3^2 = \hat{\gamma}_3 - \hat{\gamma}_4$
$d_4 = n$	$\gamma_4 = \sigma_4^2$	$\hat{\sigma}_4^2 = \hat{\gamma}_4$

$$\begin{aligned}
\mathrm{Var}\left(\hat{\sigma}_1^2\right) &= \tfrac{1}{2n(n-1)}\left[4n\sigma_1^4 + 8n\sigma_2^4 + (6n-1)\,\sigma_3^4 + (4n-2)\,\sigma_4^4 + 8n\sigma_1^2\sigma_2^2 + 4n\sigma_1^2\sigma_3^2\right.\\
&\qquad \left. + 4n\sigma_1^2\sigma_4^2 + 12n\sigma_2^2\sigma_3^2 + 8n\sigma_2^2\sigma_4^2 + (8n-2)\,\sigma_3^2\sigma_4^2\right]\\
\mathrm{Var}\left(\hat{\sigma}_2^2\right) &= \tfrac{1}{2n(n-1)}\left[4n\sigma_2^4 + (8n-4)\,\sigma_3^4 + (6n-5)\,\sigma_4^4 + 8n\sigma_2^2\sigma_3^2 + 4n\sigma_2^2\sigma_4^2\right.\\
&\qquad \left. + (12n-8)\,\sigma_3^2\sigma_4^2\right]\\
\mathrm{Var}\left(\hat{\sigma}_3^2\right) &= \tfrac{1}{2n(n-1)}\left[(4n-4)\,\sigma_3^4 + (8n-8)\,\sigma_4^4 + (8n-8)\,\sigma_3^2\sigma_4^2\right]\\
\mathrm{Var}\left(\hat{\sigma}_4^2\right) &= \tfrac{1}{2n(n-1)}\left[(4n-4)\,\sigma_4^4\right]
\end{aligned}$$

Table SM.13: Summary for the sparse component design $n/1/1/2+1/n/2/1$

$d_1 = n-1$	$\gamma_1 = 2\sigma_1^2 + 2\sigma_2^2 + 2\sigma_3^2 + \sigma_4^2$	$\hat{\sigma}_1^2 = \frac{1}{2}\hat{\gamma}_1 - \frac{1}{2}\hat{\gamma}_2 - \frac{1}{2}\hat{\gamma}_3 + \frac{1}{2}\hat{\gamma}_4$
$d_2 = n-1$	$\gamma_2 = 2\sigma_2^2 + \sigma_3^2 + \sigma_4^2$	$\hat{\sigma}_2^2 = \frac{1}{2}\hat{\gamma}_2 - \frac{1}{2}\hat{\gamma}_3$
$d_3 = n$	$\gamma_3 = \sigma_3^2 + \sigma_4^2$	$\hat{\sigma}_3^2 = \hat{\gamma}_3 - \hat{\gamma}_4$
$d_4 = n$	$\gamma_4 = \sigma_4^2$	$\hat{\sigma}_4^2 = \hat{\gamma}_4$

$$\begin{aligned}
\mathrm{Var}\left(\hat{\sigma}_1^2\right) &= \tfrac{1}{2n(n-1)}\left[4n\sigma_1^4 + 8n\sigma_2^4 + (6n-1)\,\sigma_3^4 + (4n-2)\,\sigma_4^4 + 8n\sigma_1^2\sigma_2^2 + 8n\sigma_1^2\sigma_3^2\right.\\
&\qquad\left. + 4n\sigma_1^2\sigma_4^2 + 12n\sigma_2^2\sigma_3^2 + 8n\sigma_2^2\sigma_4^2 + (8n-2)\,\sigma_3^2\sigma_4^2\right]\\
\mathrm{Var}\left(\hat{\sigma}_2^2\right) &= \tfrac{1}{2n(n-1)}\left[4n\sigma_2^4 + (2n-1)\,\sigma_3^4 + (2n-1)\,\sigma_4^4 + 4n\sigma_2^2\sigma_3^2 + 4n\sigma_2^2\sigma_4^2\right.\\
&\qquad\left. + (4n-2)\,\sigma_3^2\sigma_4^2\right]\\
\mathrm{Var}\left(\hat{\sigma}_3^2\right) &= \tfrac{1}{2n(n-1)}\left[(4n-4)\,\sigma_3^4 + (8n-8)\,\sigma_4^4 + (8n-8)\,\sigma_3^2\sigma_4^2\right]\\
\mathrm{Var}\left(\hat{\sigma}_4^2\right) &= \tfrac{1}{2n(n-1)}\left[(4n-4)\,\sigma_4^4\right]
\end{aligned}$$