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Moments in Pearson's Four-Step Uniform Random Walk Problem and Other Applications of Very Well-Poised Generalized Hypergeometric Series

J. Roderick McCrorie
University of St Andrews, Scotland, UK

Abstract

This paper considers the representation of odd moments of the distribution of a four-step uniform random walk in even dimensions, which are based on both linear combinations of two constants representable as contiguous very well-poised generalized hypergeometric series and as even moments of the square of the complete elliptic integral of the first kind. Neither constants are currently available in closed form. New symmetries are found in the critical values of the *L*-series of two underlying cusp forms, providing a sense in which one of the constants has a formal counterpart. The significant roles this constant and its counterpart play in multidisciplinary contexts is described. The results unblock the problem of representing them in terms of lower-order generalized hypergeometric series, offering progress towards identifying their closed forms. The same approach facilitates a canonical characterization of the hypergeometry of the parbelos, adding to the characterizations outlined by Campbell, D'Aurozio and Sondow (2020, *The American Mathematical Monthly* **127(1)**, 23-32). The paper also connects the econometric problem of characterizing the bias in the canonical autoregressive model under the unit root hypothesis to very well-poised generalized hypergeometric series. The confluence of ideas presented reflects a multidisciplinarity that accords with the approach and philosophy of Prasanta Chandra Mahalanobis.

Keywords. Four-step uniform random walk in the plane, Dickey-Fuller distribution, very well-poised generalized hypergeometric series, elliptic integral, universal parabolic constant, moments
AMS (2000) subject classification. Primary 33C20, 60G50, 62M10, Secondary 11Y60

1 Introduction

In his assessment of the impact of Karl Pearson's work in the development of Statistics in India, Nayak (2009) outlined the nascent role it played in attracting Prasanta Chandra Mahalanobis to the discipline, and in inspiring

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him towards his eventual leading role in its development across the Indian subcontinent. See also Ghosh (1994). C.R. Rao (1973) saw Mahalanobis existentially as ". . . a physicist by training, a statistician by instinct and a planner by conviction." He argued that

> "He did not consider statistics as a narrow subject confined to the mathematical theory of probability, or routine analysis of data in applied research, or collection of data as an aid to administrative decisions . . . But he took a wider view of statistics as 'a new technology for increasing the efficiency of human efforts in the widest sense'. This has naturally aroused his interest in various fields and enabled him to enrich the science of statistics with a practical base of great depth and spread. (*ibid.*, p. 463)

This paper is also influenced by an early Pearson contribution: a statement of the problem on the random walk. While it does not deal directly with an issue inspired by a statistical contribution by Mahalanobis and has more scientific than practical application, it attempts to offer a multidisciplinary approach, including an application to Econometrics, that is consistent with his wider statistical philosophy.

Pearson (1905) initiated work on a now-celebrated problem: characterizing the distribution of the distance travelled in an N-step random walk in the plane. The walk starts at the origin and entails N steps of unit length, each taken in a uniformly random direction. The problem is classical and attracted some immediate contributions, e.g. Rayleigh (1905) who, referring to his earlier work, gave the distribution for large N; and Kluyver (1905) who expressed the density function for general N in terms of an integral involving Bessel functions. Finding closed-form expressions for the densities for small N, however, has proved to be considerably more challenging, given their lack of smoothness and the radical differences in their shapes compared with the densities for larger N. Indeed, it could be argued the small-N problem has only received a satisfactory treatment given recent developments in mathematical and computational technology. General approaches to the small-N problem are Borwein, Nuyens, Straub and Wan (2011), Borwein, Straub, Wan and Zudilin (2012), Borwein, Straub and Vignat (2016) and Joyce (2017). Related papers, including some of a more technical nature, are Borwein, Straub and Wan (2013), Borwein and Straub (2013), Borwein and Sinnamon (2016) and Zhou (2019a). Borwein (2016) offers an introduction and perspective on recent work. The problem is also discussed in Chapter 6 of the recent monograph by Brunault and Zudilin (2020).

We first consider the N-step uniform random walk problem in the plane, although our results pertain to higher dimensions. Let X_N be the distance to the origin after N steps. Borwein, Straub, Wan and Zudilin (2012) showed the s-th moments $\mathcal{W}_N(s)$ of X_N can be computed by

$$\mathcal{W}_N(s) = \int \dots \int_{[0,1]^N} \left| e^{2\pi i \theta_1} + \dots + e^{2\pi i \theta_N} \right|^s d\theta_1 \dots d\theta_N . \tag{1.1}$$

Let $p_N(x)$ be the probability density function of X_N. Then $p_N(x)$ and $\mathcal{W}_N(s)$ are related via

$$\mathcal{W}_N(s) = \int_0^\infty x^s p_N(x)\, dx = \int_0^N x^s p_N(x)\, dx, \tag{1.2}$$

where, accordingly, $\mathcal{W}_N(s-1)$ is the Mellin transform of $p_N(x)$. Kluyver (1905) showed that

$$p_N(x) = \int_0^\infty xt\, J_0(xt)\, J_0(t)^N\, dt, \tag{1.3}$$

where $J_0(t) := \frac{2}{\pi}\int_0^{\pi/2} \cos(t \cos\theta)\; d\theta$ is the Bessel function of the first kind and zero-th order.

As a probability density function, $p_N(x)$ is supported on $[0, N]$. While closed forms are available for $N = 1$ and 2, the small-N behaviour of (1.3) makes finding closed-forms or even just tractable expressions for the densities and moments in the short-walk problems analytically and numerically challenging. This is especially so for $N = 3$, 4, 5 and 6. Borwein, Nuyens, Straub and Wan (2011) illustrate the dramatically different shapes of the small-N density functions, including the so-called "shark-fin curve" of the $N = 4$ case of interest here. The densities become smooth only at $N = 6$. Remarkably, the small-N case has thrown up features that have numerous analogies in multidisciplinary contexts, including Analytic Number Theory and Physics.

Borwein, Nuyens, Straub and Wan (2011) showed by two different methods that for general N in the plane,

$$\mathcal{W}_N(s) = N^s \sum_{m=0}^{\infty} (-1)^m \binom{s/2}{m} \sum_{k=0}^{m} \frac{(-1)^k}{N^{2k}} \binom{m}{k} \sum_{a_1+\dots+a_N=k} \binom{k}{a_1, \dots, a_N}^2, \; \mathrm{Re}(s) > 0, \tag{1.4}$$

where $\binom{k}{a_1,\dots,a_N}$ is the multinomial coefficient $\frac{k!}{a_1!\dots a_N!}$ and the function Re(.) denotes the real part of a complex number. The even moments take on a simpler form. Borwein, Straub, Wan and Zudilin (2012) showed that

$$\mathcal{W}_N(2s) = \sum_{a_1+\dots+a_N=s} \binom{s}{a_1, \dots, a_N}^2. \tag{1.5}$$

Borwein, Nuyens, Straub and Wan (2011) found this same expression in Richmond and Shallit (2009), indicating that the even moments in the plane count Abelian squares. They also satisfy a recursion that has an analytic continuation to the complex plane. In the $N = 4$ case,

$$(s+4)^3 \mathcal{W}_4(s+4) - 4(s+3)(5s^2+30s+48)\mathcal{W}_4(s+2) + 64(s+2)^3 \mathcal{W}_4(s) = 0. \tag{1.6}$$

The analytic continuation, based on a three-term recurrence relation, implies that the odd moments can be written in terms of two initial values:

$$\mathcal{W}_4(-1) = \int_0^\infty J_0(t)^4 \, dt = \frac{4}{\pi^3}\int_0^\infty I_0(t) K_0(t)^3 \, dt , \tag{1.7}$$

and

$$\mathcal{W}_4(1) = 4\int_0^\infty J_1(t) J_0(t)^3 \frac{dt}{t} = \frac{256}{\pi^3}\int_0^\infty t^2 I_0(t) K_0(t)^3 \, dt , \tag{1.8}$$

where $J_0(t)$ are the Bessel functions of the first kind in (1.3), $J_1(t) \coloneqq -\, dJ_0(t)/dt$, and $I_0(t) \coloneqq \frac{1}{\pi}\int_0^\pi \exp(t\cos\phi)\, d\phi$ and $K_0(t) \coloneqq \int_0^\infty \exp(-t\cosh u)\, du$ are the modified Bessel functions of first and second kinds, of zero-th order respectively.[1] Borwein, Straub and Vignat (2016) showed through a generalization of (1.5) and (1.6) that the constants given by (1.7) and (1.8) provide a basis for all the odd moments in arbitrary even dimensions in the sense that all such moments are linear combinations of them with weights that are rational numbers.

The primary purpose of the paper is to develop the connection between moments in the four-step random walk problem and very well-poised (VWP) generalized hypergeometric series. The search for closed forms for (1.7) and (1.8) has been an important element of this problem because of the connection it provides to numerous other disciplines. Here, we contribute directly to the problem in two ways. Firstly, we provide a sense in which the constant represented by (1.7) has a formal counterpart in generalized hypergeometric series and describe the multidisciplinary role that both play; and secondly, using methods based on elliptic integrals, we derive new series representations for the constant and its counterpart

[1] The expressions involving I_0 and K_0 are included to elicit a connection to work by Bailey, Borwein, Broadhurst and Glasser (2008) on Feynman diagrams in two-dimensional Quantum Field Theory; Laporta (2008, 2018) on four-loop integrals in Quantum Electrodynamics; and Broadhurst (2013, 2016), Broadhurst and Mellit (2016), Broadhurst and Roberts (2018) and Zhou (2019b) on critical L-values attached to modular forms (as defined in Section 2).

that provide another step towards finding their closed forms. The efficacy of the method proposed in the paper is demonstrated through its successful application to the problem of characterizing the hypergeometry of the parbelos recently discussed by Campbell, D'Aurizio and Sondow (2020). Finally, a connection is made between the econometric problem of characterizing the bias in the canonical autoregressive model under the unit root hypothesis and VWP generalized hypergeometric series. This elucidates a shared characteristic with the four-step random walk problem that could be exploited in future work. While the methods presented here are appropriate to the times, they represent a confluence of ideas having a flavour of multidisciplinarity in the spirit of the work and philosophy of Prasanta Chandra Mahalanobis.

The paper is organized as follows. Section 2 provides the notation and definitions of generalized hypergeometric series and the other machinery, including modular forms, used in the later sections. Section 3 contains the main results, providing new expressions for the moment (1.7) and its counterpart defined there. Section 4 contains the application to the hypergeometry of the parbelos. Section 5 offers the econometric application. Section 6 concludes and offers suggestions for further work. Proofs are given in the Appendix.

2 Notation

As usual, we let $\mathbb{N} := \{1, 2, \ldots\}$ be the set of natural numbers, $\mathbb{Z}$ the set of integers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{R}$ the set of real numbers and $\mathbb{C}$ the set of complex numbers.

2.1 Generalized Hypergeometric Series and Special Functions For $p, q \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and interpreting empty products as 1,

$$pFq(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_p; z) \equiv pFq\left[\begin{matrix}\alpha_1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_q\end{matrix}; z\right] := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} \tag{2.1}$$

$(p \leq q+1; p \leq q \textit{ and } |z| < \infty; p = q+1 \textit{ and } |z| < 1; p = q+1; |z| = 1 \textit{ and } \mathrm{Re}(s) > 0)$,

where Pochhammer's symbol $(\lambda)_n$ denotes the shifted factorial function

$$(\lambda)_n = \begin{cases} 1 & (n = 0, \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (n \in \mathbb{N}, \lambda \in \mathbb{C}), \end{cases} \tag{2.2}$$

which, in terms of the gamma function, $\Gamma(\mathrm{z}) := \int_0^\infty e^{-t} t^{z-1}\, dt$ ($\mathrm{Re}(z) > 0$), is given by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \qquad (\lambda \in \mathbb{C}\setminus\mathbb{Z}_0^-,\ \mathbb{Z}_0^- := \{0, -1, -2, \ldots\}), \tag{2.3}$$

This allows us to define the binomial coefficient as

$$\binom{\lambda}{n} := \frac{\Gamma(\lambda+1)}{n!\Gamma(\lambda-n-1)} = \frac{(-1)^n(-\lambda)_n}{n!}. \tag{2.4}$$

The *parameter excess*, ω, is

$$\omega := \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} a_j \left(\beta_j \notin Z_0^-;\ j = 1, \ldots, q\right) \tag{2.5}$$

Specifically, when $p = q + 1$, (2.1) is

(i) absolutely convergent for $|z| = 1$ if $\mathrm{Re}(\omega) > 0$;
(ii) conditionally convergent for $|z| = 1 (z \neq 1)$ if $-1 < \mathrm{Re}(\omega) \leq 0$; and
(iii) divergent for $|z| = 1$ if $\mathrm{Re}(\omega) \leq -1$.

Both standard notations in (2.1) will be used: the second emphasizes the distinction between numerator and denominator parameters but the first is more parsimonious.

We now consider special cases where the numerator and denominator parameters are related in certain ways. The generalized hypergeometric series (2.1), with $p = q + 1$,

$${}_{q+1}F_q\left[\begin{matrix}\alpha_1, \ldots, \alpha_{q+1} \\ \beta_1, \ldots, \beta_q\end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)\ldots(\alpha_{q+1})}{(\beta_1)\ldots(\beta_q)} \frac{z^n}{n!}, \tag{2.6}$$

is said to be

(i) *balanced* if $\alpha_1 + \ldots + \alpha_{q+1} + 1 = \beta_1 + \ldots + \beta_q$;
(ii) *nearly-poised (of the first kind)* if $\alpha_1 + \beta_1 = \ldots = \alpha_q + \beta_q$;
(iii) *well-poised* if $\alpha_{q+1} + 1 = \alpha_1 + \beta_1 = \ldots = \alpha_q + \beta_q$; and
(iv) *very well-poised* (VWP) if it is well-poised and $\alpha_1 = \frac{1}{2}\alpha_{q+1} + 1$.

Our focus here is on VWP non-terminating ${}_7F_6(1)$ series,[2] which take the general form

[2] The literature on generalized hypergeometric series divides between non-terminating (or never-ending) series and terminating series where the series terminates after a finite number of terms. The assumption in the terminating case that one of the numerator entries c, d, e, f or g is a negative integer is therefore not made here.

$$W(a;c,d,e,f,g) := {}_7F_6\left[\begin{matrix} a, 1+\frac{1}{2}a, c, d, e, f, g \\ \frac{1}{2}a, 1+a-c, 1+a-d, 1+a-e, 1+a-f, 1+a-g \end{matrix};1\right], \tag{2.7}$$

with $s := c+d+e+f+g-2a-1$ being subject to the convergence condition $s<1$.

Bailey (1935) showed in his classic tract that, under conditions, a series such as (2.7) can be decomposed into two balanced ${}_4F_3(1)$ series. See the Appendix for details.

Other special functions of interest include the complete elliptic integral of the first and second kinds:

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \frac{1}{2}\pi\ {}_2F_1\left[\begin{matrix}\frac{1}{2},\frac{1}{2}\\1\end{matrix};k^2\right], \qquad |k|<1, \tag{2.8}$$

and

$$E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}\ d\theta = \frac{1}{2}\pi\ {}_2F_1\left[\begin{matrix}-\frac{1}{2},\frac{1}{2}\\1\end{matrix};k^2\right], \quad |k|<1, \tag{2.9}$$

where the real number k is called the *modulus* of the elliptic integral. $K'(k) := K(k')$ and $E'(k) := E(k')$ are *complementary functions*, with $k' := \sqrt{1-k^2}$ called the *complementary modulus.*

2.2 Modular forms and L-series Let $SL_2(\mathbb{Z})$ denote the special *linear group*

$$SL_2(\mathbb{Z}) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{Z} \text{ and } ad-bc=1\right\}. \tag{2.10}$$

For $k \in \mathbb{Z}$, a *modular form of weight k* is an analytic function f defined on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ that transforms according to the rule

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \quad \text{for all} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), \tag{2.11}$$

and whose Fourier series

$$f(z) = \sum_{n=-\infty}^{\infty} \gamma(n)\exp(2\pi i z), \qquad i=\sqrt{-1}, \tag{2.12}$$

satisfies $\gamma(n)=0$ for all $n<0$. If, in addition, $\gamma(0)=0$, then f is said to be *a cusp form of weight k*. For $\mathrm{Im}(\tau)>0$, the *Dedekind eta function* is defined by

$$\eta(\tau) := q^{1/24} \prod_{m=1}^{\infty} (1-q^m) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(6n+1)^2/24}, \text{where } q = \exp(2\pi i \tau). \quad (2.13)$$

This function offers a way of generating modular forms. See Martin (1996) for a classification.

Given $N \in \mathbb{N}$, we can define the following congruence subgroup

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \bmod N \right\}. \quad (2.14)$$

Then a *weight k modular form* (respectively, *cusp form*) *of level N*, $f_{k,\,N}$, is defined as before, but with $SL_2(\mathbb{Z})$ replaced by $\Gamma_0(N)$. Its *L-function* can be defined through a Mellin transform:

$$L(f_{k,N}, s) := \frac{(2\pi)^s}{\Gamma(s)} \int_0^{\infty} f_{k,N}(iy) y^{s-1} dy. \quad (2.15)$$

A special L-value $L(f_{k,\,N}, s_0)$ is said to be *critical* if $s_0 \in \mathbb{N} \cap [1, k-1]$. See Koblitz (1993) for more background to this section.

2.3 Basis constants and integer-relation algorithms Our generic problem is whether the generalized hypergeometric series involved in the representation of the moments of the four-step random walk problem via (1.7) and (1.8) can be evaluated in terms of known constants or reduced to a more fundamental, tractable or accessible form. Moments are, of course, just numbers and so notions of a "closed form" of a number, as discussed by Borwein and Crandall (2013), are important. Here, we shall work within what they called the *ring of hyperclosure*, which essentially means numbers or constants that are representable as generalized hypergeometric series. An important question then is whether such numbers are *further reducible* as products of gamma functions of rational argument, or more generally in terms of a linear combination of a set of notionally fundamental constants, including such gamma values, with weights in $\mathbb{Q}$. The general question of identifying such fundamental constants, or *basis constants*, has become an important topic in Quantum Field Theory. See, e.g., Ablinger and Blümlein (2013), Ablinger, Blümlein and Schneider (2011, 2013) and Laporta (2018). The premise that basis constants exist is entirely consistent with the philosophy of integer-relation algorithms such as the PSLQ algorithm (Bailey and Broadhurst, 2000). Given a set of real numbers known to a given precision, such algorithms involve a search that seeks integer relations among them, or seeks to determine that no such relation exists within a certain bound.[3] McCrorie (2020b) takes the idea a step farther and explicitly proposes a notion of *duality* between series and integrals on one hand and basis constants on the other. Instead of just classically summing series and evaluating integrals in terms of rational linear combinations of basis constants, the dual approach sees linear combinations of basis constants *decomposed* into series and integrals in such a way that different series and integrals can be systematically and taxonomically related. The potential relevance to the current problem is demonstrated in Section 4 where it is successfully applied to construct the hypergeometry of the parbelos, albeit in a context where closed form summation is already available.

The following constants, which are *periods* in the sense of Kontsevich and Zagier (2001), are treated here as basis constants:

$$\sqrt{2}{:=}\int_{x^2\le\frac{1}{2}}dx\cong 1.41421\ 35623\ 73095\ 04880\ 16887\ 24209\ 69807\ 85696\ldots \tag{2.16}$$

$$\boldsymbol{\pi}{:=}\iint_{x^2+y^2\le 1}dx\ dy=\Gamma\left(\frac{1}{2}\right)^2\cong 3.14159\ 26535\ 89793\ 23846\ 26433\ 83279\ 502\ldots \tag{2.17}$$

$$L{:=}\iint_{x^4+y^4\le 1}dx\ dy=\Gamma\left(\frac{1}{4}\right)^2/2\sqrt{2\boldsymbol{\pi}}\cong 2.62205\ 75542\ 92119\ 81046\ 48395\ 89\ldots \tag{2.18}$$

$$\boldsymbol{\varpi}_8{:=}\iint_{x^8+y^8\le 1}dx\ dy=\Gamma\left(\frac{1}{8}\right)^2/2^{7/4}\Gamma\left(\frac{1}{4}\right)\cong 4.65437\ 02848\ 73077\ 50121\ 007257\ldots \tag{2.19}$$

$$\log 2{:=}\int_{0<x<\frac{1}{2}}\frac{dx}{1-x}\cong 0.69314\ 71805\ 59945\ 30941\ 72321\ 21458\ 17656\ 80755\ldots \tag{2.20}$$

$$\log\left(1+\sqrt{2}\right){:=}\int_{0<x<1}\frac{dx}{\sqrt{1+x^2}}\cong 0.88137\ 35870\ 19543\ 02523\ 26093\ 24979\ 79230\ldots \tag{2.21}$$

The constants $\boldsymbol{\pi}$, L and $\boldsymbol{\varpi}_8$ have the geometric interpretation of being arclengths of the circle, lemniscate and quatrefoil respectively. The algebraic independence of $\boldsymbol{\pi}$ and L follows from Chudunovsky's theorem which established the algebraic independence of $\Gamma\left(\frac{1}{2}\right)$ and $\Gamma\left(\frac{1}{4}\right)$ (see Waldschmidt, 2008). In accordance with the approach taken in integer-relation algorithms, we shall avoid using mixed expressions involving, say, $\boldsymbol{\pi}$ and $\Gamma\left(\frac{1}{4}\right)$, or using $\Gamma\left(\frac{1}{4}\right)$ alongside $\Gamma\left(\frac{3}{4}\right)$, and instead use $\boldsymbol{\pi}$, L and $\boldsymbol{\varpi}_8$, and their powers and reciprocal powers, as basis constants.[4] McCrorie

[3] While any positive result represents, at best, the basis of a conjecture, through the identification process such an algorithm may *facilitate* the construction of a formal proposition and a proof or refutation.
[4] For example, instead of $\Gamma\left(\frac{3}{4}\right)$, we use $\boldsymbol{\pi}\sqrt{2}/\Gamma\left(\frac{1}{4}\right)$, through which a term in the *reciprocal* of L emerges.

(2020b) discusses a possible alternative approach that utilizes the singular values of the elliptic integral K. Both approaches are essentially based on values of the central beta function and, while the singular value approach would in certain circumstances usefully facilitate a connection with theta functions, they are not required here.

3 Main results

In this section, we first state an expression for the density p_4 and the series governing its odd moments in even dimensions. We then show how the series for $\mathscr{W}_4(-1)$ has a counterpart in analysis that arises through the consideration of an intertwinement between certain cusp forms. New symmetries between the pair of constants are then established, which at the same time reveal new expressions for each in terms of fundamental VWP${}_7F_6(1)$ series that are reducible to sums of lower-order ${}_4F_3(1)$ series with entries in $\frac{1}{4}\mathbb{N}$ and $\frac{1}{8}\mathbb{N}$.

3.1 Generalized hypergeometric series representation of even moments in the four-step random walk problem With the notation in Section 2, we begin with the expression for p_4 derived by Borwein, Straub, Wan and Zudilin (2012), which is valid for all $x \in (0, 4]$:

$$p_4(x) = \left(\frac{2}{\pi^2}\right)\frac{\sqrt{(4-x)(4+x)}}{x}\mathrm{Re}\,{}_3F_2\left[\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ \frac{5}{6},\ \frac{7}{6}\end{matrix};\frac{(4-x)^3(4+x)^3}{108x^4}\right]. \quad (3.1)$$

The above expression embodies an analytic continuation as the ${}_3F_2$ function on the interval (0, 2) is complex, and masks the fact that the expansion of p_4 at 0 involves logarithmic terms. Joyce (2017) discusses this aspect in some detail and provides other expressions for p_4. Unfortunately, none of the hitherto derived expressions for p_4 readily lends itself towards deriving tractable expressions for the moments of p_4 via a direct application of (1.2). The even moments in odd dimensions can be expressed via (1.7) and (1.8) in terms of linear combinations VWP non-terminating ${}_7F_6(1)$ series, viz.

$$\mathscr{W}_4(-1) = \frac{\pi}{4} W\left(\frac{1}{2}; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \tag{3.2}$$

$$= \frac{\pi}{4}\, {}_7F_6\left[\begin{matrix} \frac{1}{2}, & \frac{5}{4}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2} \\ & \frac{1}{4}, & 1, & 1, & 1, & 1, & 1 \end{matrix}; 1\right],$$

and

$$\mathscr{W}_4(1) = 4\mathscr{W}_4(-1) - \frac{9\pi}{16} W\left(\frac{3}{2}; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}\right) \tag{3.3}$$

See Borwein, Straub and Vignat (2016) for details. To focus on the topics of interest in the following section, we shall only concentrate on the series in (3.2). Further results, including a discussion of (3.3), are provided in the companion paper by McCrorie (2020a) where series allied to the series in (3.2) and (3.3) are constructed taxonomically.

3.2 $\mathscr{W}_4(-1)$ and a counterpart as critical values of the L-series of cusp forms Consider the pair of weight 4 cusp forms

$$f_{4,8}(\tau) = \eta(2\tau)^4\eta(4\tau)^4, \quad f_{4,16}(\tau) = \frac{\eta(4\tau)^{16}}{\eta(2\tau)^4\eta(8\tau)^4}, \tag{3.4}$$

with levels 8 and 16 (LMFDB labels 8.4.1a and 16.4.1.a) respectively.[5] Rogers, Wan and Zucker (2015, p.128) noted the following intertwinement among their critical L-values:[6]

$$L(f_{4,16},3) = \frac{\pi}{2} L(f_{4,8},2) = \frac{\pi^2}{8} L(f_{4,16},1) = \frac{1}{4}\int_0^1 K(k)K'(k)\,dk = \frac{1}{8}\int_0^1 \frac{K(k)^2}{k'}\,dk \tag{3.5}$$

$$= \frac{\pi^3}{32}\, {}_4F_3\left[\begin{matrix} \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2} \\ & 1, & 1, & 1 \end{matrix}; 1\right]$$

and

[5] The LMFDB collaboration provides the L-functions and Modular Forms Database, where modular forms and their properties are classified by label. See: http://www.lmfdb.org.

[6] The penultimate expression of (3.5) has been replaced by an equivalent expression involving K and K' that arguably makes the symmetry clearer.

$$L(f_{4,8},3)=\frac{\pi}{2}L(f_{4,16},2)=\frac{\pi^2}{4}L(f_{4,8},1)=\frac{1}{8}\int_0^1\frac{K(k)K'(k)}{k'}\,dk=\frac{1}{4}\int_0^1 K(k)^2\,dk$$
$$=\frac{\pi^4}{128}\,{}_7F_6\left[\begin{matrix}\frac{1}{2},\frac{5}{4},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ \frac{1}{4},1,1,1,1,1\end{matrix};1\right], \tag{3.6}$$

where $K(k)$ is the complete integral of the first kind and $K'(k)$ is its complementary function. The constants represented by the generalized hypergeometric series in (3.5) and (3.6) appear in a number of different fields. Laporta (2008) treated both constants as simplest cases in his analysis of spin integrals related to Feynman diagrams. His constant A is a multiple $\pi^3/8$ of the ${}_4F_3(1)$ series in (3.5) and his constant B is a multiple $\pi^4/16$ of the ${}_7F_6(1)$ series in (3.6). As noted, Rogers, Wan and Zucker (2015) found both constants in the L-series evaluations discussed above, as did Wan and Zucker (2016) in certain eight-dimensional lattice sum evaluations. The first constant is a multiple of the constant $c_{4,0}:=\int_0^\infty K_0(t)^4\,dt$ that arose in evaluations by Bailey, Borwein, Broadhurst and Glasser (2008) of integrals that arise in Quantum Field Theory. Seen as a four-loop sunrise integral, it relates directly to Laporta's (2008) paper and indeed their constant $c_{4,\,0}$ is a multiple of 2π of Laporta's constant A. The second constant here, represented by the VWP ${}_7F_6(1)$ series in (3.6), is a multiple of their constant $s_{4,\,0}$ which is expressed in terms of the integral in (1.7) involving modified Bessel functions. Their $s_{4,\,0}$ is the same as Laporta's constant B. Guttmann (2010) demonstrated a connection of the first constant to the lattice Green's function of the four-dimensional hyper-body-centred cubic lattice. It has also appeared recently, and more fundamentally, in the areas of supercongruences and Calabi-Yau threefolds (Zagier, 2018; Zudilin, 2018; and Osburn and Straub, 2019). The second constant, of direct interest here given its role in the four-step random walk problem, is central in the theory of elliptic integrals. See especially Wan (2012).

The possibility of whether the two generalized hypergeometric series can be reduced to $\mathbb{Q}$.-linear spans of sets of more basic mathematical constants remains an open problem and, indeed, the hypergeometric form in (3.2) has not hitherto been found to be especially amenable to decomposition or analysis.

Borwein and Straub (2013) note that an immediate expression in terms of balanced ${}_6F_5(1)$ series is available via standard contiguous series relations:

$$W\left(\frac{1}{2};\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) = {}_6F_5\left[\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2};1\\1,1,1,1,1\end{matrix}\right] + \frac{1}{16}\,{}_6F_5\left[\begin{matrix}\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2};1\\2,2,2,2,2\end{matrix}\right]. \quad (3.7)$$

Following an intensive search based on the PSLQ algorithm, Wan (2013) and Borwein, Straub and Wan (2013) found a conjectural three-term relation whose validity they then established using known integrals associated with W. Zudilin and Y.V. Nesterenko in work on the irrationality of special odd values of the Riemann zeta function and related constants:[7]

$$W\left(\frac{1}{2};\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) = \frac{3}{16}W\left(\frac{3}{2};\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2}\right) + \frac{3}{4}W\left(\frac{3}{2};\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{3}{2}\right). \quad (3.8)$$

The first right-hand-side series in involved in (3.3) but, again, neither series on the right-hand-side readily decomposes into two balanced ${}_4F_3(1)$ series using standard methods.

Our principal result shows that lower-order generalized hypergeometric series are available for both constants and that there is a formal sense in which they are *exact counterparts of each other*. This idea is implicit in certain decompositions of integrals of elliptic integrals in the Ph.D. thesis by Wan (2013) although the idea was not developed there. See also Wan (2012). Expression (3.9) below in its essential form was obtained by Wan (2013, p. 115) as an expression involving ${}_4F_3(1)$ series, although here we show it is more fundamentally an expression involving a three-term relation between VWP ${}_7F_6(1)$ series. (3.10) is new. Decomposing each series into two balanced ${}_4F_3(1)$ series reveals a remarkable symmetry: a chosen multiple of the ${}_7F_6(1)$ series describing each constant is seen to decompose into the *same two* ${}_4F_3(1)$ series, the magnitudes of whose weights are *exactly the same*.

THEOREM 1.

(a) $$\frac{\pi^3}{4}W\left(\frac{1}{2};\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) = L^2 W\left(\frac{1}{4};\frac{1}{8},\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{5}{8}\right) + \frac{\pi^2}{L^2}W\left(\frac{3}{4};\frac{3}{8},\frac{1}{2},\frac{1}{2},\frac{3}{4},\frac{7}{8}\right) \quad (3.9)$$

$$\frac{\pi^2}{2}W\left(\frac{1}{2};\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{4}\right) = L^2 W\left(\frac{1}{4};\frac{1}{8},\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{5}{8}\right) - \frac{\pi^2}{L^2}W\left(\frac{3}{4};\frac{3}{8},\frac{1}{2},\frac{1}{2},\frac{3}{4},\frac{7}{8}\right) \quad (3.10)$$

[7] See Borwein, Straub and Wan (2013).

(b) $$\pi^2\mathscr{W}_4(-1)=\frac{8}{\pi}\int_0^1 K(k)^2\,dk=\frac{32}{\pi}L(f_{4,8},3)=8L(f_{4,16},2)=8\pi L(f_{4,8},1)$$

$$=\frac{\pi^3}{4}\,{}_7F_6\left[\begin{matrix}\frac{1}{2}, & \frac{5}{4}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2} \\ & \frac{1}{4}, & 1, & 1, & 1, & 1, & 1\end{matrix};1\right]$$

$$=L^2\,{}_4F_3\left[\begin{matrix}\frac{1}{4}, & \frac{1}{4}, & \frac{1}{2}, & \frac{1}{2} \\ & \frac{3}{4}, & \frac{3}{4}, & 1\end{matrix};1\right]+\frac{\pi^2}{L^2}\,{}_4F_3\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2}, & \frac{3}{4}, & \frac{3}{4} \\ & 1, & \frac{5}{4}, & \frac{5}{4}\end{matrix};1\right] \tag{3.11}$$

$$\frac{16}{\pi}L\big(f_{4,16},3\big)=8L\big(f_{4,8},2\big)=2\pi L\big(f_{4,16},1\big)=\frac{\pi^2}{2}\,{}_4F_3\left[\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2} \\ 1,1,1\end{matrix};1\right]$$

$$=L^2\,{}_4F_3\left[\begin{matrix}\frac{1}{4},\frac{1}{4},\frac{1}{2},\frac{1}{2} \\ \frac{3}{4},\frac{3}{4},1\end{matrix};1\right]-\frac{\pi^2}{L^2}\,{}_4F_3\left[\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{3}{4},\frac{3}{4} \\ 1,\frac{5}{4},\frac{5}{4}\end{matrix};1\right] \tag{3.12}$$

Theorem 1 expresses the same idea in two ways. In part (a), a chosen multiple of each VWP series is seen to satisfy a three-term relation involving the same VWP ${}_7F_6(1)$ series, whose weights, L^2 and π^2/L^2, are the same up to a sign. The series $W\big(\frac{1}{4};\frac{1}{8},\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{5}{8}\big)$ and $W\big(\frac{3}{4};\frac{3}{8},\frac{1}{2},\frac{1}{2},\frac{3}{4},\frac{7}{8}\big)$ relate to each other in the same way that π relates to $1/\pi$ and $\Gamma\big(\frac{1}{4}\big)$ relates to $\Gamma\big(\frac{3}{4}\big)$. Note that towards the ultimate aim of obtaining a closed form for both left-hand-side series, the decomposition on the right-hand-side has entries in $\frac{1}{8}\mathbb{N}$. Part (b) expresses the same idea in ${}_4F_3(1)$ series. While the above results unblock the problem of finding decompositions of $W\big(\frac{1}{2};\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\big)$ in terms of entries in $\frac{1}{4}\mathbb{N}$ and $\frac{1}{8}\mathbb{N}$, none of the series on the right-hand-side of the above expressions has a known closed form.

A further decomposition of the series in (3.5) and (3.6) can be obtained using the decomposition of the moments of a different elliptic integral.

THEOREM 2.

(a) $$\frac{\pi^4}{2}W\left(\frac{1}{2};\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)=L^4W\left(\frac{1}{4};\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)-3\left(\frac{\pi}{L}\right)^4W\left(\frac{3}{4};\frac{3}{4},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) \tag{3.13}$$

$$2\pi^3 W\left(\frac{1}{2};\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{4}\right) = L^4 W\left(\frac{1}{4};\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) + 3\left(\frac{\pi}{L}\right)^4 W\left(\frac{3}{4};\frac{3}{4},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) \quad (3.14)$$

(b) $$\pi^3\mathcal{W}_4(-1) = 8\int_0^1 K(k)^2\,dk = 32L(f_{4,8},3) = 8\pi L(f_{4,16},2) = 8\pi^2 L(f_{4,8,}1)$$

$$= \frac{\pi^4}{4}\,{}_7F_6\left[\begin{matrix}\frac{1}{2}, & \frac{5}{4}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2} \\ & \frac{1}{4}, & 1, & 1, & 1, & 1, & 1\end{matrix};1\right] \quad (3.15)$$

$$= 2L^3\,{}_4F_3\left[\begin{matrix}\frac{1}{4}, & \frac{1}{4}, & \frac{1}{4}, & \frac{1}{4} \\ & \frac{1}{2}, & \frac{3}{4}, & \frac{3}{4}\end{matrix};1\right] + \left(\frac{\pi}{L}\right)^3\,{}_4F_3\left[\begin{matrix}\frac{3}{4}, & \frac{3}{4}, & \frac{3}{4}, & \frac{3}{4} \\ & \frac{5}{4}, & \frac{5}{4}, & \frac{3}{2}\end{matrix};1\right]$$

$$- \pi^2\,{}_4F_3\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2} \\ & \frac{3}{4}, & 1, & \frac{5}{4}\end{matrix};1\right].$$

$$32L(f_{4,16},3) = 16\pi L(f_{4,8},2) = 4\pi^2 L(f_{4,16},1) = \pi^3\,{}_4F_3\left[\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2} \\ 1,1,1\end{matrix};1\right]$$

$$= 2L^3\,{}_4F_3\left[\begin{matrix}\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4} \\ \frac{1}{2},\frac{3}{4},\frac{3}{4}\end{matrix};1\right] - \left(\frac{\pi}{L}\right)^3\,{}_4F_3\left[\begin{matrix}\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4} \\ \frac{5}{4},\frac{5}{4},\frac{3}{2}\end{matrix};1\right]. \quad (3.16)$$

Again, the decomposition is seen fundamentally to involve three-term relations involving VWP ${}_7F_6(1)$ series and involves remarkable symmetries. While there is a generalized hypergeometric series in (3.15) that appears not to be common to (3.16), this is only because of cancellation, as can be seen when directly applying the decomposition of the VWP ${}_7F_6(1)$ series in (3.14) into balanced ${}_4F_3(1)$ series:

$$
\begin{aligned}
\pi^3\, {}_4F_3\left[\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ 1,1,1\end{matrix};1\right]
&= 2L^3\, {}_4F_3\left[\begin{matrix}\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\\ \frac{1}{2},\frac{3}{4},\frac{3}{4}\end{matrix};1\right] - \frac{1}{2}\pi^2\, {}_4F_3\left[\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ \frac{3}{4},1,\frac{5}{4}\end{matrix};1\right] \\
&\quad + \frac{1}{2}\pi^2\, {}_4F_3\left[\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ \frac{3}{4},1,\frac{5}{4}\end{matrix};1\right] - \left(\frac{\pi}{L}\right)^3 {}_4F_3\left[\begin{matrix}\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4}\\ \frac{5}{4},\frac{5}{4},\frac{3}{2}\end{matrix};1\right]. \qquad (3.17)
\end{aligned}
$$

Consideration of which VWP ${}_7F_6(1)$ series can potentially support the latter ${}_4F_3(1)$ series in (3.16) in their decomposition into ${}_4F_3(1)$ series leads to the following two-term relation, which supports our final result on the representation of $\mathscr{W}_4(-1)$ and related series. This provides a remarkable connection between the series involved in (3.5) and (3.6), and a balanced ${}_5F_4(1)$ series. More expressions involving entries in $\frac{1}{4}\mathbb{N}$ and $\frac{1}{8}\mathbb{N}$ are provided in the companion paper by McCrorie (2020a).

THEOREM 3.

(a)
$$
W\left(1;\frac{1}{2},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4}\right) = \frac{3\pi}{8}\, W\left(\frac{3}{4};\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{4}\right) \qquad (3.18)
$$

(b)
$$
\frac{8\pi^3}{L^4}\, {}_5F_4\left[\begin{matrix}\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},1\\ \frac{5}{4},\frac{5}{4},\frac{5}{4},\frac{5}{4}\end{matrix};1\right] = \pi^2\, {}_4F_3\left[\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ \frac{3}{4},1,\frac{5}{4}\end{matrix};1\right] - \left(2\frac{\pi}{L}\right)^3 {}_4F_3\left[\begin{matrix}\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4}\\ \frac{5}{4},\frac{5}{4},\frac{3}{2}\end{matrix};1\right] \qquad (3.19)
$$

(c)
$$
\begin{aligned}
\mathscr{W}_4(-1) &= \frac{8}{\pi^3}\int_0^1 K(k)^2\, dk \\
&= \frac{\pi}{4}\, {}_7F_6\left[\begin{matrix}\frac{1}{2}, & \frac{5}{4}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}\\ & \frac{1}{4}, & 1, & 1, & 1, & 1, & 1\end{matrix};1\right] \qquad (3.20) \\
&= {}_4F_3\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}\\ 1, & 1, & 1 & \end{matrix};1\right] - \frac{8}{L^4}\, {}_5F_4\left[\begin{matrix}\frac{3}{4}, & \frac{3}{4}, & \frac{3}{4}, & \frac{3}{4}, & 1\\ \frac{5}{4}, & \frac{5}{4}, & \frac{5}{4}, & \frac{5}{4} & \end{matrix};1\right].
\end{aligned}
$$

Again, the result is seen fundamentally to be a relation between VWP ${}_7F_6(1)$ series but the key to (3.19) is that the series $W\left(1;\frac{1}{2},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4}\right)$ is a ${}_5F_4(1)$ series in disguise. Closed-form results in the context of ${}_4F_3(1)$ and ${}_5F_4(1)$ series are sparse in the literature and, while recent progress has been made by Campbell, D'Aurizio and Sondow (2019), none of the series in Theorems 1 to 3 is currently known in closed form. Nevertheless, we now demonstrate that, at least in principle, the approach followed in this paper *does* potentially offer progress. In the context of constructing the hypergeometry of the parbelos, we show that finding transformed series with entries in $\frac{1}{4}\mathbb{N}$ and $\frac{1}{8}\mathbb{N}$ provides a route towards finding the closed form of a relevant generalized hypergeometric series. Our approach adds to the four approaches to the characterization problem offered by Campbell, D'Aurizio and Sondow (2020).

4 Hypergeometry of the Parbelos

Sondow (2013) formally introduced the *parbelos* as a parabolic analogue of the arbelos, a classical shape bounded by three pairwise tangent semicircles with collinear diameters. The parbelos corresponding to a given arbelos is obtained by replacing the semicircles of the arbelos with the latus rectum arcs of parabolas opening in the same direction, the foci of which are the centres of the semicircles of the arbelos. Sondow showed that the ratio of the length of the boundary of a parbelos to the length of the corresponding arbelos is always P/π, where P is the *universal parabolic constant*. For any parabola, this constant is the ratio of the arc length of the parabolic segment formed by the latus rectum to the focal parameter. It is given by

$$P = \sqrt{2} + \log\left(1+\sqrt{2}\right) \cong 2.29558\ 71493\ 92638\ 07403\ 42980\ 49189\ 4903... \tag{4.1}$$

The problem of characterizing the hypergeometry of the parbelos is to find generalized hypergeometric series that sum to P/π. We could say, given the discussion in Section 2.3, that the problem more generally seeks to find allied series up to multiplication by a rational number, or even multiplication by a product or ratio of gamma factors. Campbell, D'Aurizio and Sondow (2020) recently outlined methods in which generalized hypergeometric series relating to the universal parabolic constant could be derived, their first establishing that

$${}_3F_2\left[\begin{matrix}-\frac{1}{2},\frac{1}{4},\frac{3}{4} \\ \frac{1}{2},\quad 1\end{matrix};1\right] = \frac{P}{\pi}. \tag{4.2}$$

Here, we will use the duality concept discussed earlier to derive this series based on a *decomposition* of the basis constant $\log\left(1+\sqrt{2}\right)$. The method can also be applied, in principle, to characterize the *equilateral hyperbolic constant*, which is the constant that, for any equilateral (or rectangular) hyperbola, i.e. one whose semi-axes are similar, is the ratio of the area of the latus rectum segment to the square of its semi-axis. It is given by

$$H=\sqrt{2}-\log\left(1+\sqrt{2}\right)\cong 0.53283\ 99753\ 53552\ 02356\ 90793\ 99229\ 9057\ldots \tag{4.3}$$

We will also show how deriving series with entries in $\frac{1}{8}\mathbb{N}$ drives this characterization.

A classical result by Watson (1918) in a mildly reparametrized form states that

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\,{}_3F_2\left[\begin{matrix} a,\ b,\ \frac{a+b-1}{2} \\ a+b,\ \frac{a+b+1}{2}\end{matrix};1\right]=\frac{1}{2}\left(\psi\left(\frac{a+1}{2}\right)+\psi\left(\frac{b+1}{2}\right)-\psi\left(\frac{a}{2}\right)-\psi\left(\frac{b}{2}\right)\right), \tag{4.4}$$

where $\psi(x) := d\log\,\Gamma(z)/dz = \Gamma'(x)/\Gamma(x)$ is the digamma function. Kölbig (1996) shows that

$$\psi\left(\frac{p}{q}\right)=-\gamma-\log\,(q)+\sum_{j=1}^{q-1}\exp(-2\pi ijp/q)\log(1-\exp(2\pi ij/q)), \tag{4.5}$$

where $\varepsilon_j=\exp(2\pi ij/q)$ $(j=0,1,\ldots,q-1)$ are the q-th roots of unity. This expression motivates taking special values of logarithms of the *cyclotomic polynomials* (polynomials fundamentally factorized using the roots of unity) as basis constants for the class of ${}_3F_2(1)$ generalized hypergeometric series. See also Ablinger, Blümlein and Schneider (2011, 2013).

McCrorie (2020c) argued that closed-form summation in ${}_3F_2(1)$ series involving P or H could be naturally seen from the *initial standpoint* of Watson's result, taking $a=b=\frac{1}{4}$ and, at first, using *known values* of $\{\psi(k/8),\ 1\leq k<8,\ k\in\mathbb{N}\}$, which involve the basis constant $\log\left(1+\sqrt{2}\right)$. For example,

$$\psi(1/8)=-\gamma-\frac{\pi}{2}\left(\sqrt{2}+1\right)-4\log(2)-\sqrt{2}\log\left(1+\sqrt{2}\right), \tag{4.6}$$

$$\psi(5/8)=-\gamma+\frac{\pi}{2}\left(\sqrt{2}-1\right)-4\log(2)+\sqrt{2}\log\left(1+\sqrt{2}\right), \tag{4.7}$$

where $\gamma := \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log(n)\right)$ is the Euler-Mascheroni constant. Application of (4.4) gives upon simplification of the gamma factors the implicit three-term relation

$$ {}_3F_2\left[\begin{matrix}\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}\\ \frac{1}{2}, \quad \frac{3}{4}\end{matrix}; 1\right] = \frac{1}{L}\left(\frac{\pi}{2} + \log\left(1+\sqrt{2}\right)\right). \tag{4.8}$$

From this base, allied generalized hypergeometric series can be constructed *taxonomically* using transformations applied to the left-hand-side of (4.8) through various methods. McCrorie (2020c) shows that

$$ {}_3F_2\left[\begin{matrix}\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}\\ \frac{1}{2}, \quad \frac{3}{4}\end{matrix}; 1\right] = \frac{\sqrt{2}}{L}\, {}_3F_2\left[\begin{matrix}\frac{1}{4}, \frac{1}{2}, 1\\ \frac{3}{4}, \frac{5}{4}\end{matrix}; 1\right] = \frac{1}{L}\left(\frac{\pi}{2} + \frac{\sqrt{2}}{3}\, {}_3F_2\left[\begin{matrix}\frac{1}{2}, 1, 1\\ \frac{5}{4}, \frac{7}{4}\end{matrix}; 1\right]\right)$$

and

$$ {}_3F_2\left[\begin{matrix}\frac{1}{2}, 1, 1\\ \frac{5}{4}, \frac{7}{4}\end{matrix}; 1\right] = \frac{\pi}{2}\, {}_3F_2\left[\begin{matrix}\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\\ \frac{3}{2}, 1\end{matrix}; 1\right].$$

Campbell, D'Aurizio and Sondow (2020) showed that

$$ {}_3F_2\left[\begin{matrix}\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\\ \frac{3}{2}, 1\end{matrix}; 1\right] = \frac{4}{\pi}\ln\left(1+\sqrt{2}\right), \tag{4.9}$$

and that (4.9) is contiguous to (4.2). The hypergeometry of the parbelos is therefore established from the natural base position given by (4.8).

McCrorie (2020c) motivates the consideration of $\{\psi(k/8), 1 \le k < 8, k \in \mathbb{N}\}$ through a reduction of series with entries in $\frac{1}{4}\mathbb{N}$ to series with entries in $\frac{1}{8}\mathbb{N}$, in the spirit of Section 3. He derived the following relationship which he argued should be viewed as fundamental:

$$\sqrt{2}\ln\left(1+\sqrt{2}\right) = {}_3F_2\left[\begin{matrix}\frac{1}{4}, \frac{3}{4}, 1\\ \frac{5}{4}, \frac{3}{2}\end{matrix}; 1\right] = \frac{8}{1\times 5}\, {}_3F_2\left[\begin{matrix}\frac{1}{8}, \frac{5}{8}, 1\\ \frac{9}{8}, \frac{13}{8}\end{matrix}; 1\right] - \frac{8}{3\times 7}\, {}_3F_2\left[\begin{matrix}\frac{3}{8}, \frac{7}{8}, 1\\ \frac{11}{8}, \frac{15}{8}\end{matrix}; 1\right]. \tag{4.10}$$

The two ${}_3F_2(1)$ series on the right-hand-side are free of Pochhammer symbols, viz.

$$\frac{1}{5}\,{}_3F_2\left[\begin{matrix}\frac{1}{8},\frac{5}{8},1\\ \frac{9}{8},\quad\frac{13}{8}\end{matrix};1\right]=\sum_{k=0}^{\infty}\frac{1}{(8k+1)(8k+5)},\tag{4.11}$$

and

$$\frac{1}{21}\,{}_3F_2\left[\begin{matrix}\frac{3}{8},\frac{7}{8},1\\ \frac{11}{8},\quad\frac{15}{8}\end{matrix};1\right]=\sum_{k=0}^{\infty}\frac{1}{(8k+3)(8k+7)}.\tag{4.12}$$

With entries in $\frac{1}{8}\mathbb{N}$, the series are therefore of a form that is summable, up to a constant that is the product of a rational number and ϖ_8 or its reciprocal, *in terms of* the digamma functions $\{\psi(k/8), 1\leq k<8, k\in\mathbb{N}\}$ (see Al-Saqabi, Kalla and Srivastava, 1991). The method supports an approach to series summation whose evaluation involves the logarithm of $1+\sqrt{2}$, and extrapolates via (4.4) and (4.5) to a wider class of basis constants based on the cyclotomic polynomials. Ablinger, Blümlein and Schneider (2011, 2013) set this idea in the context of sums relevant to Mathematical Physics. As noted, the issue of finding (or defining) basis constants for the series in Section 3 is more challenging because the underlying series are of ${}_4F_3(1)$ type.

5 An Analogous Problem in Econometrics

We now establish a connection between a classical, still-open problem in Econometrics and VWP ${}_7F_6(1)$ series. Consider the first-order autoregressive, AR(1), process[8]

$$x_t=\rho x_{t-1}+\varepsilon_t(t=1,\ldots,T)\tag{5.1}$$

where $\varepsilon_t\sim NID(0,\ 1)$, $x_0=0$ and $|\rho|=1$. Although (5.1) is highly idealized from today's standpoint, it is canonical as a building block for more useful and realistic econometric time series models in a sense explicitly outlined by Abadir (1992). The ordinary least squares (OLS) estimator of ρ based on a sample of size T coincides with the Gaussian maximum likelihood estimator and is given by

$$\widehat{\rho}_T:=\sum_{t=1}^{T}x_tx_{t-1}/\sum_{t=1}^{T}x_{t-1}^2.\tag{5.2}$$

[8] The material in this section is based on the framework adopted by McCrorie (2020b).

In contrast to the uniform random walk problem, where in the multi-dimensional case each step corresponds to a random vector distributed on the unit sphere, here it is the model, parametrized by ρ, which nests a random walk (along with other types of behaviour). Our objective is to estimate this parameter on the basis of data and, in this context, it is the *estimator*, $\widehat{\rho}_T$, that connects to the hypersphere through its being a ratio of quadratic forms.[9] Clearly, the properties of the estimator $\widehat{\rho}_T$ vary across different regions in the parameter space. The characterization of, for example, the bias of the estimator,

$$b_T(\rho) := E\left(\widehat{\rho}_T\right) - \rho, \tag{5.3}$$

has been a classical problem in Econometrics, see e.g. Hurwicz (1950), White (1961) and Shenton and Johnson (1965). As $T \to \infty$,

$$\begin{aligned}
&\text{If } |\rho| < 1,\ T b_T(\rho) \text{ converges in probability to} - 2\rho;\\
&\text{If } \rho = \pm 1,\ T b_T(\rho) \text{ converges to} \pm m\,\rho, \ \text{ where } m \text{ is a negative constant;}\\
&\text{If } |\rho| > 1,\ T^{-1/2}|\rho|^T b_T(\rho) \text{ converges to } -2^{-1/2}\pi^{1/2}\rho^{-1}\left(\rho^2-1\right)^{3/2}.
\end{aligned} \tag{5.4}$$

See also Le Breton and Pham (1989) and Abadir (1993a). The problem of characterizing the constant m appearing in (5.4), which is *negative* in the positive parameter case (and positive in the negative case), remains open. An equivalent way of expressing the problem is that m is the mean of the asymptotic distribution of the test statistic under the unit root null hypothesis $\rho = 1$. This distribution, called the Dickey-Fuller distribution after Dickey and Fuller (1979, 1981), has become pervasive in the area of econometric time series analysis,[10] a discipline whose underlying (economic) variables seemingly exhibit trending behaviour. It is surprising that, forty years on, little is known about its mean when viewed as a mathematical constant. Renewed interest in the constant has followed from work on predictive regression by Phillips (2012, 2015). The model is typified by the specification

$$y_t = \beta' x_{t-1} + u_{0t}, \tag{5.5}$$

$$x_t = \rho x_{t-1} + u_{xt}, \tag{5.6}$$

[9] See Hillier (2001), Forchini (2002) and references therein.
[10] See also Rao (1978), Phillips (1987) and Abadir (1993b). Tanaka (2017) provides a textbook treatment.

where the objective is to predict the scalar time series y_t given past information embodied in a set of regressors x_{t-1} under an assumption such as $(u_t, \mathcal{F}_t)$ is a martingale difference sequence with

$$E(u_t u_t' | \mathcal{F}_{t-1}) = \begin{bmatrix} \sigma_{00} & \sigma_{0x} \\ \sigma_{x0} & \Sigma_{xx} \end{bmatrix},$$

where $\mathcal{F}_{t-1} = \sigma(u_t, u_{t-1}, \ldots)$ is the natural filtration associated with innovations $u_t = \left(u_{0t}, u_{xt}'\right)'$. Applying OLS to (5.5) under the assumption that x_t is scalar, and setting $u_{0.xt} = u_{0t} - \sigma_{0x}\Sigma_{xx}^{-1}u_{xt}$, the estimation error decomposes as

$$\widehat{\beta} - \beta = \frac{\sum_{t=1}^{T} x_{t-1} u_{0.xt}}{\sum_{t=1}^{T} x_{t-1}^2} + \sigma_{0x}\Sigma_{xx}^{-1}\left(\widehat{\rho} - \rho\right), \tag{5.8}$$

where $\widehat{\rho} = \left(\sum_{t=1}^{T} x_{t-1}^2\right)^{-1} \sum_{t=1}^{T} x_{t-1}\ x_t$. Taking expectations,

$$E\left(\widehat{\beta} - \beta\right) = \sigma_{0x}\Sigma_{xx}^{-1} E\left(\widehat{\rho} - \rho\right) = \sigma_{0x}\Sigma_{xx}^{-1} B_T(\rho), \qquad (5.9) \quad (5.7)$$

where the autoregressive bias function

$$B_T(\rho) := E\left(\widehat{\rho} - \rho\right) \tag{5.10}$$

depends only on ρ and T. Phillips (2012) provides an exact formula for $B_T(\rho)$ under Gaussianity and the following complete set of large-T asymptotic expansions

$$B_T(\rho) = \begin{cases} -2\rho/T + O\left(T^{-2}\right) & |\rho| < 1 \\ m/T + O\left(T^{-2}\right) & \rho = 1 \\ -g(c)/T + O\left(T^{-2}\right) & \rho = 1 + c/T \\ O\left(1/|\rho|^T\right) & |\rho| > 1 \end{cases}, \tag{5.11}$$

where $g(c)$ is a continuous function of a constant c given explicitly by Phillips (2012) and $\lim_{c \to 0} g(c) = -m$.

McCrorie (2020b) classified a set of generalized hypergeometric series that are allied to the constant m, defining the canonical member of the set to be

$$M := 1 - m = {}_3F_2\left[\begin{matrix} \frac{1}{2}, 1, 1 \\ \frac{3}{2}, \frac{5}{4} \end{matrix}; 1\right] \cong 2.78143\ 01712\ 77837\ 42085\ 88648\ 61173\ 56\ldots \tag{5.12}$$

Treated as basis constants, M and another constant constructed in parallel[11] were seen along with π and L to be involved in an expression for a critical value

[11] This constant relates to M in precisely the way $\Gamma\left(\frac{1}{4}\right)$ relates to $\Gamma\left(\frac{3}{4}\right)$, and the series $W\left(\frac{1}{4}; \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{5}{8}\right)$ and $W\left(\frac{3}{4}; \frac{3}{8}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}\right)$ in Theorem 1 relate to each other.

of the L-series of an elliptic curve of conductor 32, one of whose isogenies is the "congruent number elliptic curve" (Tian, 2015) and another which has the modular form parametrization $f_{2,\,32}(\tau)=\eta(4\tau)^2\eta(8\tau)^2$. They are LMFDB elliptic curves 32.a3 and 32.a4 respectively.[12] See Rodriguez Villegas (1999), Zudilin (2013) and Ito (2018) for a discussion of this L-series evaluation problem from the point of view of Analytic Number Theory.

We now show that, like the moments in the four-step uniform random walk problem, the constant M is representable in terms of VWP ${}_7F_6(1)$ series. This is facilitated by the following lemma which involves a transformation to a ${}_3F_2(1)$ series that is allied to the series in (5.12).

LEMMA 4.

$$ {}_3F_2\left[\begin{matrix}\frac{3}{4},1,1\\ \frac{3}{2},\ \frac{7}{4}\end{matrix};1\right]=3\left(M-\frac{\pi}{4}L\right). \tag{5.13}$$

THEOREM 5.

$$M-\frac{\pi}{4}L=\frac{1}{3}W\left(\frac{3}{2};\frac{3}{4},\frac{3}{4},1,1,\frac{5}{4}\right) \tag{5.14}$$

$$=\frac{1}{2}\left(\frac{\pi}{L}\right)^2W\left(\frac{1}{2};\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{2},\frac{1}{2}\right) \tag{5.15}$$

$$=\frac{3}{16}\frac{\pi^2}{L}W\left(\frac{3}{4};\frac{1}{4},\frac{1}{4},\frac{1}{2},\frac{3}{4},\frac{3}{4}\right) \tag{5.16}$$

$$=\frac{2}{3}W\left(1;\frac{1}{4},\frac{1}{2},\frac{3}{4},\frac{3}{4},1\right) \tag{5.17}$$

[12] $f(\tau)\coloneqq f_{2,\,32}$ is an example of a type of multiplicative η-product discussed by Voskresenskaya (2012) such that $f^2(\tau/2)$ is also a multiplicative η-product. Here, $f^2(\tau/2)=\eta(2\tau)^4\eta(4\tau)^4=f_{4,\,8}$.

$$= \frac{5}{9}\, W\left(\frac{5}{4}; \frac{1}{2}, \frac{1}{2}, 1, 1, 1\right) \tag{5.18}$$

$$= \frac{5L}{24}\, W\left(\frac{5}{4}; \frac{1}{2}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{5}{4}\right) \tag{5.19}$$

The expressions can be read in two ways. On one hand, M and so m are expressed in terms of a number of VWP ${}_7F_6(1)$ series. On the other, if M is taken to be a basis constant, the series themselves, which are not symbolically evaluated in term of known constants by software such as *Mathematica 11.2*, can be expressed in closed form within the $\mathbb{Q}$-linear span of $\{\boldsymbol{\pi}, L, M\}$. McCrorie (2020b) shows that each ${}_7F_6(1)$ series can be split-up into balanced ${}_4F_3(1)$ series with entries in $\frac{1}{4}\mathbb{N}$ and $\frac{1}{8}\mathbb{N}$, and closed form expressions for such series involving a wider class of constants including M can be obtained. These evaluations complement those that were obtained for allied ${}_3F_2(1)$ series.

6 Conclusion

This paper has provided new results and perspective on a mathematical constant that appears in the representation of the odd moments in even dimensions of Pearson's four-step uniform random walk problem. The symmetries in (3.11), (3.12), (3.15), (3.16) and (3.20) formally reveal the sense in which the constant defined by the series (3.2) has a counterpart in the generalized hypergeometric series in (3.5). As described above, both constants are manifest in a variety of fields. While the question of whether either constant can be reduced to a $\mathbb{Q}$-linear span of a more fundamental set of constants (known or to be defined) remains open, relating both constants to VWP ${}_7F_6(1)$ series with entries in $\frac{1}{4}\mathbb{N}$ and $\frac{1}{8}\mathbb{N}$ allowed a number of series of lower order to be related to them. While the results are of interest in their own right, they also open up the possibility of establishing the constants in terms of more elemental closed forms following the approach taken by McCrorie (2020c) to the universal parabolic constant summarized in Section 4.

The paper also motivates questions for possible future work. By providing a formal sense in which the two constants in Section 3 were counterparts of each other, the paper goes farther than Rogers, Wan and Zucker (2016) in describing how the critical values of the L-series of two related, even-weight cusp forms are intertwined. It would be interesting to investigate whether the type of the symmetries exhibited here apply more widely in the interplay between modular forms and generalized hypergeometric series and, if so, whether they can be built up in a similar way. Recent results by Straub and Zudilin (2020) and Brunault and Zudilin (2020) also suggest a connection with Mahler measure.

Given for any weight 4 modular form f, the functional equation relates $L(f, s)$ and $L(f, 4-s)$, the value at $s=4$ may have special importance. Papanikolas, Rogers and Samart (2014) and Wan and Zucker (2016) have derived hypergeometric series expressions for the L-series of $f_{4,\ 8}$ evaluated at 4, one being

$$L\left(f_{4,8}, 4\right) = \frac{\pi^4}{192}\left\{ {}_5F_4\left[\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, \quad 1, \quad 1, \quad \frac{3}{2} \end{matrix} ; 1\right] + \frac{7\zeta(3)}{\pi^2}\right\}, \tag{6.1}$$

where $\zeta(s) \coloneqq \sum_{n=1}^{\infty} \frac{1}{n^s}$ $(\mathrm{Re}(s) > 1)$ is the Riemann zeta function. It would be interesting to explore whether (6.1) encodes any information beyond that encoded in the critical values discussed here. A generalized hypergeometric series representation for $L(f_{4,\ 16}, 4)$ could also be derived and its relation to (6.1) explored.

Another direction for future work might relate the results in the paper more strongly to the property that the densities of the short-walk problems are recursively related. Indeed, following Hughes (1995), if $\phi_N(x) = p_N(x)/2\pi x$, then for integers $N \geq 2$,

$$\phi_N(x) = \frac{1}{2\pi}\int_0^{2\pi} \phi_{N-1}\left(\sqrt{x^2 - 2x\cos\alpha + 1}\right) d\,\alpha. \tag{6.2}$$

While this aspect has already been exploited, the underlying complexity of the short-walk problem for small N has meant that tractable expressions have still been difficult to obtain. On the other hand, some sharp results have been obtained on certain special values of the density functions. For example, Borwein, Straub, Wan and Zudilin (2012) and Joyce (2017) show using different methods that

$$p_4(1) = p_5'(0) = \frac{1}{8\pi^4\sqrt{5}}\Gamma\left(\frac{1}{15}\right)\Gamma\left(\frac{2}{15}\right)\Gamma\left(\frac{4}{15}\right)\Gamma\left(\frac{8}{15}\right) = \frac{3\sqrt{5}}{\pi^3}\frac{\left(\sqrt{5}-1\right)}{2}K_{15}^2, \tag{6.3}$$

where K_{15} is the complete elliptic integral evaluated at the 15-th singular value. The five-step random walk problem centres around a weight 3 level 15 cusp form

$$f_{3,15}(\tau) = [\eta(\tau)\eta(15\tau)]^3 + [\eta(3\tau)\eta(5\tau)]^3. \tag{6.4}$$

See especially Bloch, Kerr and Vanhove (2015), Samart (2016) and Zhou (2019a). This context offers another instance of the observation of Rogers, Wan and Zucker (2015) of an L-series of an odd-weighted modular form having critical values that are products of gamma functions. Because the functional equation here relates $L(f_{3,15}, s)$ and $L(f_{3,15}, 3-s)$, *all* the critical values relate to each other in a way that avoids intertwinement with another modular form as was exemplified here by (3.5) and (3.6). Specifically,

$$\begin{aligned}\frac{\pi^2}{15}L(f_{3,15},1) &= \frac{1}{2}\sqrt{\frac{\pi^2}{15}}L(f_{3,15},2) = \frac{\pi^4}{30}\int_0^\infty J_0(t)^4\,dt = \int_0^\infty tI_0(t)K_0(t)^4\,dt \\ &= \frac{1}{240\sqrt{5}}\Gamma\left(\frac{1}{15}\right)\Gamma\left(\frac{2}{15}\right)\Gamma\left(\frac{4}{15}\right)\Gamma\left(\frac{8}{15}\right).\end{aligned} \tag{6.5}$$

The ratio of the above value to π^2 is the so-called *Bologna constant* introduced by Laporta (2008) *op. cit.* and independently by Broadhurst (2007) in work that was developed in the jointly authored paper by Bailey, Borwein, Broadhurst and Glasser (2008). Characterizing the commonalities and differences in the types of expressions for moments that relate to even versus odd weight modular forms remains an important open problem.

Last, but by no means least, this paper has provided a connection between moment properties of the distribution of a short-walk problem and the Dickey-Fuller distribution that is now pervasive in Econometrics. The distribution theory underlying statistics like (5.2) that are ratios of quadratic forms is still incompletely characterized in the asymptotic case, and is less resolved in the finite sample case (see e.g. Hillier, 2001; Forchini, 2002; and references therein). Very few analytical expressions are available for the relevant densities and distributions beyond what K.M. Abadir derived in a number of papers in the 1990s (see, e.g. Abadir, 1993b; Abadir and Lucas, 2004). In spite of work by Van Garderen (1999, 2000), the underlying geometry of non-stationary autoregressive models is still not fully characterized. The shared characteristics between the four-step random walk problem and least-squares estimation in autoregressive models could motivate new approaches towards the latter that embrace machinery currently commonplace in Analytic Number Theory and

Mathematical Physics, and more systematically encompass numerical and experimental methods. The fundamental problem in autoregressive models, in the absence of a well-developed finite-sample theory, is the polarization in their properties that manifests itself in large samples. Some recent methods seek to confront this problem directly. The IVX endogenous instrumentation procedure introduced by Phillips and Magdalinos (2009) and developed by Phillips and Lee (2013, 2016), Kostakis, Magdalinos and Stamatogiannis (2015) and Phillips (2015) is central to this effort.

The intention of writing the current paper was to use the setting of the four-step random walk problem to offer a confluence of ideas around the constant represented by the VWP ${}_7F_6(1)$ series in (3.2) involved in the representation of the odd moments in even dimensions. The approach underpinned a multidisciplinary perspective of a type encouraged by the journal's founding editor, P.C. Mahalanobis. The *MacTutor History of Mathematics Archive* – a resource that provides, *inter alia*, summaries of the historical contribution of important mathematicians in the last few centuries – is held in the School of Mathematical Sciences in the author's current institution, the University of St Andrews. While not a substitute for Rao's (1973) important biographical article or the biography by Rudra (1996), its summary entry for Prasanta Chandra Mahalanobis can be accessed here: http://www-groups.dcs.st-and.ac.uk/history/Biographies/Mahalanobis.html.

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References

Abadir, K.M. (1992) A distribution generating equation for unit root statistics. *Oxford Bulletin of Economic Statistics* **54**, 305-323.

Abadir, K.M. (1993a) OLS bias in a nonstationary autoregression. *Econometric Theory* **9**, 81-93.

Abadir, K.M. (1993b) The limiting distribution of the autocorrelation coefficient under a unit root. *Annals of Statistics* **21(2)**, 1058-1070.

Abadir, K.M. and Lucas, A. (2004) A comparison of minimum MSE and maximum power for the nearly integrated non-Gaussian model. *Journal of Econometrics* **119**, 45-71.

Ablinger, J. and Blümlein, J. (2013) Harmonic sums, polylogarithms, special numbers, and their generalizations. In C. Schneider and J. Blümlein (eds), *Computer Algebra in Quantum Field Theory*, Texts & Monographs in Symbolic Computation, pp. 1-32. Springer.

Ablinger, J., Blümlein, J. and Schneider, C. (2011) Harmonic sums and polylogarithms generated by cyclotomic polynomials. *Journal of Mathematical Physics* **52**, 102301 (52pp.)

Ablinger, J., Blümlein, J. and Schneider, C. (2013) Analytic and algorithmic aspects of generalized harmonic sums and polylogarithms. *Journal of Mathematical Physics* **54**, 082031 (74pp.)

Al-Saqabi, B.N., Kalla, S.L. and Srivastava, H.M. (1991) A certain family of infinite series associated with digamma functions. *Journal of Mathematical Analysis and Applications* **159**, 361-372.

Bailey, D.H., Borwein, J.M., Broadhurst, D. and Glasser, M.L. (2008) Elliptic integral evaluations of Bessel moments and applications. *Journal of Physics A: Mathematical and Theoretical* **41(20)**, 5203-5231.

Bailey, D.H. and Broadhurst, D.J. (2000) Parallel integer relation detection: techniques and applications. *Mathematics of Computation* **70**, 1719-1736.

Bailey, W.N. (1935) *Generalized Hypergeometric Series*. Cambridge Tracts in Mathematics and Mathematical Physics, No. 32. Cambridge University Press.

Bloch, S., Kerr, M. and Vanhove, P. (2015) A Feynman integral via higher normal functions. *Compositio Mathematica* **151(12)**, 2329-2375.

Borwein, J.M. (2016) A short walk can be beautiful. *Journal of Humanistic Mathematics* **6(1)**, 86-109.

Borwein, J.M. and Crandall, R.E. (2013) Closed forms: what they are and why we care? *Notices of the American Mathematical Society* **60**, 50-65.

Borwein, J.M., Nuyens, D., Straub A. and Wan, J. (2011) Some arithmetic properties of random walk integrals. *The Ramanujan Journal* **26(1)**, 109-132.

Borwein, J.M. and Sinnamon, C. W. (2016) A closed form for the density functions of random walks in odd dimensions. *Bulletin of the Australian Mathematical Society* **93(2)**, 330-339.

Borwein, J.M. and Straub, A. (2013) Mahler measures, short walks and log-sine integrals. *Theoretical Computer Science* **479**, 4-21.

Borwein, J.M., Straub, A. and Vignat, C. (2016) Densities of short uniform random walks in higher dimensions. *Journal of Mathematical Analysis and its Applications* **437(1)**, 668-707.

Borwein, J.M., Straub, A. and Wan, J. (2013) Three-step and four-step random walk integrals. *Experimental Mathematics* **22(1)**, 1-14.

Borwein, J.M., Straub, A., Wan, J. and Zudilin, W. (2012) Densities of short uniform random walks. *Canadian Journal of Mathematics* **64(5)**, 961-990. (With an appendix by D. Zagier).

Broadhurst, D. (2007) Reciprocal PSLQ and the tiny nome of Bologna. International Workshop in *Frontiers in Perturbative Quantum Field Theory*, Zentrum für Interdisziplinäre Forschung in Bielefeld, June 14 2007. Available at: http://www.physik.uni-bielefeld.de/igs/schools/ZiF2007/Broadhurst.pdf.

Broadhurst, D. (2013) Multiple zeta values and modular forms in quantum field theory. In C. Schneider and J. Blümlein (eds), *Computer Algebra in Quantum Field Theory*, Texts & Monographs in Symbolic Computation, pp. 33–73. Springer.

Broadhurst, D. (2016) Feynman integrals, L-series and Kloosterman moments. *Communications in Number Theory and Physics* **10(3)**, 527-569.

Broadhurst, D. and A. Mellit (2016) Perturbative quantum field theory informs algebraic geometry. Loops and Legs in Quantum Field Theory 2016. Proceedings of Science **260**, 079 (8pp.). https://doi.org/10.22323/1.260.0079.

Broadhurst, D. and D.P. Roberts (2018) Quadratic relations between Feynman integrals. Loops and Legs in Quantum Field Theory 2018, *Proceedings of Science* **303**, 053 (9pp). https://doi.org/10.22323/1.303.0053.

Brunault, F. and W. Zudilin (2020) Many Variations of Mahler Measures: A Lasting Symphony Australian Mathematical Society Lecture Series 28, Cambridge University Press.

Campbell, J.M., D Aurizio, J. and Sondow, J. (2019) On the interplay among hypergeometric functions, complete elliptic integrals and Fourier-Legendre expansions. *Journal of Mathematical Analysis and Applications* **479(1)**, 90-121.

Campbell, J.M., D Aurizio, J. and Sondow, J. (2020) Hypergeometry of the parbelos. *The American Mathematical Monthly* **127(1)**, 23-32.

Dickey, D.A. and Fuller, W.A. (1979) Distribution of the estimators for autoregressive time series with a unit root. *Journal of the American Statistical Association* 74, 427-431.

Dickey, D.A. and Fuller, W.A. (1981) Likelihood ratio statistics for autoregressive time series with a unit root. *Econometrica* **49**, 1057-1072.

Forchini, G. (2002) The exact cumulative distribution function of a ratio of quadratic forms in normal variables, with application to the AR(1) model. *Econometric Theory* **18**, 823-852.

Ghosh, J.K. (1994) Mahalanobis and the art and science of statistics: the early days. *Indian Journal of History of Science* **29(1)**, 89-98.

Guttmann, A.J. (2010) Lattice Green's functions in all dimensions. *Journal of Physics A: Mathematical and Theoretical* **43**, 305205 (26pp).

Hillier, G. (2001) The density of a quadratic form in a vector uniformly distributed on the n-sphere. *Econometric Theory* **17**, 1-28.

Hughes, L. (1995) *Random Walks and Random Environments: Vol. 1. Random Walks*. Oxford University Press.

Hurwicz, L. (1950) Least squares estimation in time series models. In T. Koopmans (ed.), *Statistical Inference in Dynamic Economic Models*, pp. 365-383. Wiley.

Ito, R. (2018) The Beilinson conjectures for CM elliptic curves via hypergeometric functions. *The Ramanujan Journal* **45(2)**, 433-449.

Joyce, G.S. (2017) Exact results for the diamond lattice Green function with applications to uniform random walks in the plane. *Journal of Physics A: Mathematical and Theoretical* **50**, 423001 (29pp).

Kluyver, J.C. (1905) A local probability problem. *Nederlandsche Akademie van Wetenschappen Proceedings* **8(1)**, 341-350.

Koblitz, N. (1993) *Introduction to Elliptic Curves and Modular Forms*. Second edition. Springer.

Kölbig, K.S. (1996) The polygamma function and the derivatives of the cotangent function for rational arguments. CERN-CN-96-005, CERN Computing and Networks Division. Available at: http://cds.cern.ch/record/298844.

Kontsevich, M. and Zagier, D. (2001) Periods. In B. Engquist and W. Schmid (eds) *Mathematics Unlimited – 2001 and Beyond*, pp. 771-808. Springer.

Kostakis, A., Magdalinos, T. and Stamatogiannis, M.P. (2015) Robust econometric inference for stock return predictability. *The Review of Financial Studies* **28(5)**, 1506-1553.

Laporta, S. (2008) Analytical expressions of 3- and 4-loop sunrise Feynman integrals and 4-dimensional lattice integrals. *International Journal of Modern Physics A* **23(31)**, 5007-502.

Laporta, S. (2018) Building bases for analytical fits of four-loop QED master integrals. Loops and Legs in Quantum Field Theory 2018. *Proceedings of Science* **303**, 073 (8pp.) https://doi.org/10.22323/1.303.0073.

Le Breton, A. and Pham, D.T. (1989) On the bias of the least squares estimator for the first order autoregressive process. *Annals of the Institute of Statistical Mathematics* **41(2)**, 555-563.

Martin, Y. (1996) Multiplicative η-products. *Transactions of the American Mathematical Society* **348**, 4825-4856.

McCrorie, J.R. (2020a) Generalized hypergeometric series allied to moments in Pearson's four-step uniform random walk problem. In preparation.

McCrorie, J.R. (2020b) The mean of the Dickey-Fuller distribution viewed as a mathematical constant. In preparation.

McCrorie, J.R. (2020c) Duality in the hypergeometry of the parbelos. In preparation.

Nayak, T.K. (2009) Impact of Karl Pearson's work in the development of statistics in India. *International Statistical Review* **77**, 72-80.

Osburn, R. and Straub, A. (2019) Interpolated sequences and critical L-values of modular forms. In Blümlein, J., Schneider, C. and Paule, P. (eds) *Elliptic Integrals, Elliptic Functions and Modular Forms in Quantum Field Theory*, pp. 327-349. Springer.

Papanikolas, M.A., Rogers, M.D. and Samart, D. (2014) The Mahler measure of a Calabi-Yau threefold and special L-values. *Mathematische Zeitschrift* **276**, 1151-1163.

Pearson, K. (1905) The problem of the random walk. *Nature* **72**, 294, 342.

Phillips, P.C.B. (1987) Time series regression with a unit root. *Econometrica* **55**, 277-301.

Phillips, P.C.B. (2012) Folklore theorems, implicit maps and indirect inference. *Econometrica* **80(1)**, 425-454.

Phillips, P.C.B. (2015) Pitfalls and possibilities in predictive regression. Halbert White Memorial Lecture. *Journal of Financial Econometrics* **13(3)**, 521-555.

Phillips, P.C.B. and Lee, J. H. (2013) Predictive regression under varying degrees of persistence and robust long-horizon regression. *Journal of Econometrics* **177(2)**, 250-24.

Phillips, P.C.B. and Lee, J. H. (2016) Robust econometric inference with mixed integrated and mildly explosive regressors. *Journal of Econometrics* **192(2)**, 433-450.

Phillips, P.C.B. and T. Magdalinos (2009) Econometric inference in the vicinity of unity. CoFie Working Paper No. 6, Singapore Management University.

Prudnikov, A.P., Brychkov, Y.A. and Marichev, O.I. (1986) *Integrals and Series: More Special Functions (Volume 3)*. Gordon and Beach.

Rao, C.R. (1973) Prasanta Chandra Mahalanobis. 1893-1972. *Biographical Memoirs of Fellows of the Royal Society* **19**, 454-492.

Rao, M.M. (1978) Asymptotic distribution of an estimator of the boundary parameter of an unstable process. *Annals of Statistics* **6(1)**, 185-190. (Correction: *Annals of Statistics* **8(6)**, 1403.)

Richmond, L.B. & J. Shallit (2009) Counting abelian squares. *The Electronic Journal of Combinatorics* **16(1)**, #R72.

Rayleigh L. (1905) The problem of the random walk. *Nature* **72**, 318.

Rodrigues Villegas, F. (1999) Modular Mahler measures I. In S.D. Ahlgren, G.E. Andrews & K. Ono (eds) *Topics in Number Theory*, pp. 17-48. Mathematics and Its Applications **467**, Springer.

Rogers, M., Wan, J.G. and Zucker, I.J. (2015) Moments of elliptic integrals and critical L-values. *The Ramanujan Journal* **37**, 113-130.

Rudra, A. (1996) *Prasanta Chandra Mahalanobis: A Biography*. Oxford University Press.

Samart, D. (2016) Feynman integrals and critical modular L-functions. *Communications in Number Theory and Physics* **10(1)**, 133-156.

Shenton, L.R. and N.L. Johnson (1965) Moments of a serial correlation coefficient. *Journal of the Royal Statistical Society Series B: Statistical and Methodological* **27**, 308-320.

Sondow, J. (2013) The parbelos, a parabolic analog of the arbelos. *American Mathematical Monthly* **120**, 929-935.

Straub, A. and W. Zudilin (2020) Short walk adventures. In D. H. Bailey, N.S. Borwein, R.P. Brent, R.S. Burachik, J.-A. H. Osborn, B. Sims & Q.J. Zhu (eds) *From Analysis to Visualization*. Jonathan Borwein Commemorative Conference 2017, Springer Proceedings in Mathematics and Statistics **313**, 423-439.

Tanaka, K. (2017) *Time Series Analysis: Nonstationary and Noninvertible Distribution Theory*, Second Edition. Wiley.

Tian, Y. (2015) Congruent number problem and arithmetic of elliptic curves. In S.Y. Cheng, L. Ji, Y.-S. Poon, J. Xiao, L. Yang & S.-T. Yau (eds.) *Introduction to Modern Mathematics: Advanced Lectures in Mathematics* **33**, pp. 409-423, Higher Education Press.

van Garderen, K.J. (1999) Exact geometry of first order autoregressive models. *Journal of Time Series Analysis* **20(1)**, 1-21.

van Garderen, K.J. (2000) An alternative comparison of classical tests: assessing the effects of curvature. In M. Salmon & P. Marriott (eds) *Applications of Differential Geometry to Econometrics*, pp. 230-280. Cambridge University Press.

Voskresenskaya, G.V. (2012) Arithmetic properties of Shimura sums related to several modular forms. *Journal of Mathematical Sciences* **182(4)**, 444-455.

Waldschmidt, M. (2008) Elliptic functions and transcendence. In K. Alladi (ed.), *Surveys in Number Theory*, pp. 148-188. Developments in Mathematics **17**, Springer.

Wan, J.G. (2012) Moments of products of elliptic integrals. *Advances in Applied Mathematics* **48**, 121-141.

Wan, J.G. (2013) Random walks, elliptic integrals and related constants. Ph.D. Thesis, University of Newcastle, New South Wales (Australia). Available at: http://docserver.carma.newcastle.edu.au/1487/1/jwthesis_rev.pdf.

Wan, J.G. and Zucker, I.J. (2016) Integrals of K and E from lattice sums. *The Ramanujan Journal* **40(2)**, 257-278.

Watson, G.N. (1918) The integral formala for generalized Legendre functions. *Proceedings of the London Mathematical Society* **17**, 241-246.

White, J.S. (1961) Asymptotic expansions for the mean and variance of the serial correlation coefficient. *Biometrika* **48**, 85-94.

Zagier, D.B. (2018) Arithmetic and topology of differential equations. In Mehrmann, V. and Skutella, M. (eds) *Proceedings of the Seventh European Congress of Mathematics* (Berlin, July 18-22, 2016), pp. 717-776. European Mathematical Society, Berlin.

Zhou, Y. (2019a) On Borwein's conjectures for planar uniform random walks. *Journal of the Australian Mathematical Society.* **107(3)**, 392-411.

Zhou, Y. (2019b) Some algebraic and arithmetic properties of Feynman diagrams. In J. Blümlein, C. Schneider & P. Paule (eds), *Elliptic Integrals, Elliptic Functions and Modular Functions in Quantum Field Theory*, pp. 485-509. Texts and Monographs in Symbolic Computation **18**, Springer.

Zudilin, W. (2013) Period(d)ness of L-values. In J.M. Borwein, I. Shparlinski & W. Zudilin (eds.), *Number Theory and Related Fields: In Memory of Alf van der Poorten*, Springer Proceedings in Mathematics & Statistics **43**, pp. 381-395.

Zudilin, W. (2018) A hypergeometric version of the modularity of rigid Calabi-Yau manifolds. Special Issue on Modular Forms and String Theory in Honor of Noriko Yui. *Symmetry, Integrability and Geometry: Methods and Applications (SIGMA)* **14**, 086 (14 pages) arXiv: 1805.00544v2[math.NT].

Appendix: Proofs

We shall use the following decomposition derived by Bailey (1935, p. 29) of a VWP non-terminating ${}_7F_6(1)$ series into two balanced ${}_4F_3(1)$ series under permutations of the parameters such that poles of the gamma function can be avoided and individual series converge.

$$
\begin{aligned}
W(a; c, d, e, f, g) &:= {}_7F_6\left[\begin{matrix} a, & 1+\frac{1}{2}a, & c, & d, & e, & f, & g \\ \frac{1}{2}a, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f, & 1+a-g \end{matrix}; 1\right] \\
&= \frac{\Gamma(1+a-e)\Gamma(1+a-f)\Gamma(1+a-g)\Gamma(1+a-e-f-g)}{\Gamma(1+a)\Gamma(1+a-f-g)\Gamma(1+a-g-e)\Gamma(1+a-e-f)} \\
&\times {}_4F_3\left[\begin{matrix} 1+a-c-d, e, f, g \\ 1+a-c, \quad 1+a-d, \quad e+f+g-a \end{matrix}; 1\right] \\
&+ \frac{\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)\Gamma(1+a-f)\Gamma(1+a-g)}{\Gamma(1+a)\Gamma(1+a-c-d)\Gamma(e)\Gamma(f)\Gamma(g)} \\
&\times \frac{\Gamma(e+f+g-a-1)\Gamma(2+2a-c-d-e-f-g)}{\Gamma(2+2a-c-e-f-g)\Gamma(2+2a-d-e-f-g)} \\
&\times {}_4F_3\left[\begin{matrix} 2+2a-c-d-e-f-g, & 1+a-f-g, & 1+a-g-e, & 1+a-e-f \\ 2+a-e-f-g, & 2+2a-c-e-f-g, & 2+2a-d-e-f-g \end{matrix}; 1\right] \\
&\mathrm{Re}(2+2a-c-d-e-f-g) > 0.
\end{aligned} \tag{A.1}
$$

We shall also use a two-term relation he derived between VWP ${}_7F_6(1)$ series (*ibid.*, p. 62):

$$W(a;c,d,e,f,g)=\frac{\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(2+2a-e-f-g)\Gamma(2+2a-c-d-e-f-g)}{\Gamma(1+a)\Gamma(1+a-c-d)\Gamma(2+2a-c-e-f-g)\Gamma(2+2a-d-e-f-g)} \times W(1+2a-e-f-g;c,d,1+a-f-g,1+a-e-g,1+a-e-f) \quad (A.2)$$

PROOF OF THEOREM 1.

Wan (2013, p.115) showed that

$$\int_0^1 K(k)^2\,dk=\int_0^1\frac{(1+k)K(k)K'(k)}{4\sqrt{k}}\,dk=\frac{1}{4}\int_0^1 k^{-1/2}K(k)K'(k)\,dx+\frac{1}{4}\int_0^1 k^{1/2}K(k)K'(k)\,dx. \quad (A.3)$$

Expression (6.65) in Wan (2013), which relates the left-hand-side to two balanced ${}_4F_3(1)$ series, follows from applying the Mellin transform given by Wan (2012, Proposition 2):

$$\int_0^1 k^n K(k)K'(k)\,dx=\frac{\pi^2}{8}\frac{\Gamma\left[\frac{1}{2}(n+1)\right]^2}{\Gamma\left[\frac{1}{2}(n+2)\right]^2}\,{}_4F_3\left[\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{n+1}{2},\frac{n+1}{2}\\ 1,\ \frac{n+2}{2},\ \frac{n+2}{2}\end{matrix};1\right]. \quad (A.4)$$

Evaluating the ratio of gamma functions in terms of powers of π and L, and reciprocals gives

$$\int_0^1 k^{-1/2}K(k)K'(k)\,dx=\frac{\pi L^2}{2}\,{}_4F_3\left[\begin{matrix}\frac{1}{4},\frac{1}{4},\frac{1}{2},\frac{1}{2}\\ \frac{3}{4},\ \frac{3}{4},\ 1\end{matrix};1\right], \quad (A.5)$$

$$\int_0^1 k^{1/2}K(k)K'(k)\,dx=\frac{\pi^3}{2L^2}\,{}_4F_3\left[\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{3}{4},\frac{3}{4}\\ 1,\ \frac{5}{4},\ \frac{5}{4}\end{matrix};1\right]. \quad (A.6)$$

Expressions (3.9) and (3.11) then follow from noting that

$$F_3\left[\begin{matrix}\frac{1}{4},\frac{1}{4},\frac{1}{2},\frac{1}{2}\\ \frac{3}{4},\frac{3}{4},1\end{matrix};1\right]={}_7F_6\left[\begin{matrix}\frac{1}{4},\frac{9}{8},\frac{1}{8},\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{5}{8}\\ \frac{1}{8},\frac{9}{8},1,\frac{3}{4},\frac{3}{4},\frac{5}{8}\end{matrix};1\right]=W\left(\frac{1}{4};\frac{1}{8},\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{5}{8}\right), \quad (A.7)$$

$$ {}_4F_3\left[\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{3}{4},\frac{3}{4}\\ 1,\frac{5}{4},\frac{5}{4}\end{matrix};1\right] = {}_7F_6\left[\begin{matrix}\frac{3}{4},\frac{11}{8},\frac{3}{8},\frac{1}{2},\frac{1}{2},\frac{3}{4},\frac{7}{8}\\ \frac{1}{8},\frac{11}{8},\frac{5}{4},\frac{5}{4},1,\frac{7}{8}\end{matrix};1\right] = W\left(\frac{3}{4};\frac{3}{8},\frac{1}{2},\frac{1}{2},\frac{3}{4},\frac{7}{8}\right), \quad (A.8) $$

and $\int_0^1 K(k)^2\,dk = \frac{\pi^4}{32}\,W\left(\frac{1}{2};\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$, upon multiplication by $2/\pi$. Thus (6.65) of Wan (2013), expressed in ${}_4F_3(1)$ series, is more fundamentally the three-term relation in VWP ${}_7F_6(1)$ series given by (3.9). Expressions (3.10) and (3.12) follow from consequences of the hitherto unnoticed fact that the ${}_4F_3(1)$ series in (3.5) is also a VWP ${}_7F_6(1)$ series:

$$ {}_4F_3\left[\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ 1,\quad 1,\quad 1\end{matrix};1\right] = {}_7F_6\left[\begin{matrix}\frac{1}{2}, & \frac{5}{4}, & \frac{1}{4}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{3}{4}\\ & \frac{1}{4}, & \frac{5}{4}, & 1, & 1, & 1, & \frac{3}{4}\end{matrix};1\right] $$

$$ = W\left(\frac{1}{2};\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{4}\right) = W\left(\frac{1}{2};\frac{3}{4},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{4}\right) $$

$$ = \frac{2L^2}{\pi^2}\,{}_4F_3\left[\begin{matrix}\frac{1}{4},\frac{1}{4},\frac{1}{2},\frac{1}{2}\\ \frac{3}{4},\quad\frac{3}{4},\quad 1\end{matrix};1\right] - \frac{2}{L^2}\,{}_4F_3\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2}, & \frac{3}{4}, & \frac{3}{4}\\ & 1, & \frac{5}{4}, & \frac{5}{4}\end{matrix};1\right] \text{by } (A.1). $$

(3.10) and (3.12) follow from applying (3.5), (A.7) and (A.8) and multiplying by $\pi^2/2$.

PROOF OF THEOREM 2.

Wan (2013, p.115) also showed that

$$ \int_0^1 K(k)^2\,dk = \int_0^1 \frac{(1-k)K'(k)^2}{8\sqrt{k}}\,dk = \frac{1}{8}\int_0^1 k^{-1/2}K'(k)^2\,dk - \frac{1}{8}\int_0^1 k^{1/2}K'(k)^2\,dk. \quad (A.9) $$

Now we use the Mellin transform derived by Wan (2012, Proposition 1):

$$ \int_0^1 k^n K'(k)^2\,dx = \frac{2^{4n}(n+1)}{16}\frac{\Gamma\left[\frac{1}{2}(n+1)\right]^8}{\Gamma(n+1)^4}\,{}_7F_6\left[\begin{matrix}\frac{n+1}{2}, & \frac{n+5}{4}, & \frac{n+1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}\\ \frac{n+1}{4}, & 1, & \frac{n+2}{2}, & \frac{n+2}{2}, & \frac{n+2}{2}, & \frac{n+2}{2} & \end{matrix};1\right] \quad (A.10) $$

$$ = \frac{2^{4n}(n+1)}{16}\frac{\Gamma\left[\frac{1}{2}(n+1)\right]^8}{\Gamma(n+1)^4}\,W\left(\frac{n+1}{2};\frac{n+1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right). $$

Substituting for n and evaluating the ratio of gamma functions in terms of powers of π, L and their reciprocals gives

$$\int_0^1 k^{-1/2} K'(k)^2\,dx = \frac{L^4}{2}\, W\left(\frac{1}{4};\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right), \tag{A.11}$$

$$\int_0^1 k^{1/2} K'(k)^2\,dx = \frac{3}{2}\left(\frac{\pi}{L}\right)^4 W\left(\frac{3}{4};\frac{3}{4},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right). \tag{A.12}$$

Expression (3.13) follows from applying (A.9), (A.11) and (A.12) upon multiplication by 16, noting that $\int_0^1 K(k)^2\,dk = (\pi^4/32) \times W\left(\frac{1}{2};\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$. Applying the decomposition (A.1) to the W expressions in (A.11) and (A.12) gives

$$W\left(\frac{1}{4};\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) = -\frac{\pi^2}{L^4}\,{}_4F_3\left[\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ \frac{3}{4},1,\frac{5}{4}\end{matrix};1\right] + \frac{4}{L}\,{}_4F_3\left[\begin{matrix}\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\\ \frac{1}{2},\frac{3}{4},\frac{3}{4}\end{matrix};1\right], \tag{A.13}$$

$$W\left(\frac{3}{4};\frac{3}{4},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) = \frac{1}{3}\frac{L^4}{\pi^2}\,{}_4F_3\left[\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ \frac{3}{4},1,\frac{5}{4}\end{matrix};1\right] - \frac{2}{3\pi}L\,{}_4F_3\left[\begin{matrix}\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4}\\ \frac{5}{4},\frac{5}{4},\frac{3}{2}\end{matrix};1\right]. \tag{A.14}$$

(3.15) now follows from applying (A.9), (A.11) and (A.12) upon multiplication by 8.

Expressions (6.41) and (7.25) by Wan (2013) give

$$\frac{\pi^3}{4}\,{}_4F_3\left[\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ 1,\quad 1,\quad 1\end{matrix};1\right] = \int_0^1 \frac{(1+k)K(k)^2}{\sqrt{k}}\,dk = \int_0^1 \frac{(1+k)K'(k)^2}{4\sqrt{k}}\,dk. \tag{A.15}$$

Expression (3.14) then follows from $W\left(\frac{1}{2};\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{4}\right) = {}_4F_3\left[\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ 1,\ 1,\ 1\end{matrix};1\right]$ by applying (A.11) and (A.12) and multiplying by 8. (3.16) follows similarly from (A.15) applying (3.5), (A.13) and (A.14) and multiplying by 4.

PROOF OF THEOREM 3.

Part (a) follows by applying (A.2) to $W\left(1;\frac{1}{2},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4}\right)$. For (b) note that

$$W\left(1;\frac{1}{2},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4}\right)={}_7F_6\left[\begin{matrix}1,\frac{3}{2},\frac{1}{2},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4}\\ \frac{1}{2},\frac{3}{2},\frac{5}{4},\frac{5}{4},\frac{5}{4},\frac{5}{4}\end{matrix};1\right]={}_5F_4\left[\begin{matrix}\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},1\\ \frac{5}{4},\frac{5}{4},\frac{5}{4},\frac{5}{4}\end{matrix};1\right]. \quad (A.16)$$

Now apply (A.1) to $W\left(1;\frac{1}{2},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4}\right)$ to get

$${}_5F_4\left[\begin{matrix}\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},1\\ \frac{5}{4},\frac{5}{4},\frac{5}{4},\frac{5}{4}\end{matrix};1\right]=-\frac{L}{4}\,{}_4F_3\left[\begin{matrix}\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4}\\ \frac{5}{4},\frac{5}{4},\frac{3}{2}\end{matrix};1\right]+\frac{L^4}{8\pi}\,{}_4F_3\left[\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ \frac{3}{4},1,\frac{5}{4}\end{matrix};1\right],$$

on applying (A.16). (3.19) follows upon multiplication by $8\pi^3/L^4$. Finally, (3.20) follows from subtracting (3.19) from (3.15), using (3.16) and then dividing both sides by π^3.

PROOF OF LEMMA 4.

Expression (5.6) is (M.11a) in McCrorie (2020b) where it is one of a taxonomically constructed set of ${}_3F_2(1)$ series allied to M. It can be proved directly using (7.4.4.1) and (7.4.4.5) from Prudnikov, Brychkov and Marichev (1986). Specifically, the former is

$${}_3F_2\left[\begin{matrix}a,b,c\\ d,\quad e\end{matrix};1\right]=\frac{\Gamma(d)\Gamma(d+e-a-b-c)}{\Gamma(d+e-a-b)\Gamma(d-c)}\,{}_3F_2\left[\begin{matrix}e-a,\quad e-b,\quad c\\ d+e-a-b,\quad e\end{matrix};1\right]. \quad (A.17)$$

$$(\mathrm{Re}(d+e-a-b-c)>0 \text{ and } \mathrm{Re}(d-c)>0)$$

and the latter is

$${}_3F_2\left[\begin{matrix}a,b,c\\ d,e\end{matrix};1\right]=\frac{\Gamma(1+a-d)\Gamma(1+b-d)\Gamma(1+c-d)\Gamma(d)\Gamma(e)}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(1+e-d)\Gamma(2-d)} \quad (A.18)$$

$${}_3F_2\left[\begin{matrix}1+a-d,1+b-d,1+c-d\\ 1+e-d,2-d\end{matrix};1\right]+\frac{\Gamma(1+a-d)\Gamma(1+c-d)}{\Gamma(1-d)\Gamma(1+a+c-d)}$$

$${}_3F_2\left[\begin{matrix}a,c,e-b\\ 1+a+c-d,e\end{matrix};1\right].$$

$$(\mathrm{Re}(d+e-a-b-c)>0 \text{ and } \mathrm{Re}(1+b-d)>0)$$

In one application of (A.18) below, the ${}_3F_2(1)$ series reduces to a well-defined ${}_2F_1(1)$ series, which is summable using the Gauss Summation Theorem (e.g. Bailey (1935), equation 1.3.1):

$$ {}_2F_1\,(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,(\mathrm{Re}(c-a-b)>0). \tag{A.19}$$

By (A.17),

$$ M:={}_3F_2\left[\begin{matrix}\frac{1}{2},1,1\\ \frac{5}{4},\ \frac{3}{2}\end{matrix};1\right]=\frac{L^2}{\pi}\,{}_3F_2\left[\begin{matrix}\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\\ \frac{3}{4},\ \frac{3}{2}\end{matrix};1\right]. \tag{A.20}$$

By (A.18),

$$ {}_3F_2\left[\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{1}{2}\\ \frac{3}{4},\frac{3}{2}\end{matrix};1\right]=\frac{2\sqrt{2}}{3}\frac{\pi}{L^2}\,{}_3F_2\left[\begin{matrix}\frac{3}{4},\frac{3}{4},\frac{3}{4}\\ \frac{5}{4},\frac{7}{4}\end{matrix};1\right]=\frac{\pi}{L^2}\,{}_3F_2\left[\begin{matrix}\frac{1}{2},\frac{1}{2},1\\ \frac{5}{4},\frac{3}{2}\end{matrix};1\right]. \tag{A.21}$$

We now apply (A.17) to the first ${}_3F_2(1)$ series on the right-hand-side of (A.21):

$$ {}_3F_2\left[\begin{matrix}\frac{3}{4},\frac{3}{4},\frac{3}{4}\\ \frac{5}{4},\ \frac{7}{4}\end{matrix};1\right]=\frac{1}{\sqrt{2}}\,{}_3F_2\left[\begin{matrix}\frac{3}{4},\ 1,\ 1\\ \frac{3}{2},\ \frac{7}{4}\end{matrix};1\right] \tag{A.22}$$

We then apply (A.18) to the second:

$$ {}_3F_2\left[\begin{matrix}\frac{1}{2},\frac{1}{2},1\\ \frac{5}{4},\ \frac{3}{2}\end{matrix};1\right]=\sqrt{2}L\,{}_2F_1\left[\begin{matrix}\frac{1}{4},\ \frac{1}{4}\\ \frac{5}{4}\end{matrix};1\right]-{}_3F_2\left[\begin{matrix}\frac{1}{2},\ 1,\ 1\\ \frac{5}{4},\ \frac{3}{2}\end{matrix};1\right], \tag{A.23}$$

where ${}_2F_1\left[\begin{matrix}\frac{1}{4},\frac{1}{4}\\ \frac{5}{4}\end{matrix};1\right]=\frac{\pi}{2\sqrt{2}}$ (by A.19). Hence, by (A.20) – (A.23), we obtain (5.13):

$$ {}_3F_2\left[\begin{matrix}\frac{3}{4},1,1\\ \frac{3}{2},\ \frac{7}{4}\end{matrix};1\right]=3\Big(M-\frac{\pi}{4}L\Big).$$

PROOF OF THEOREM 5.

The advantage of using (5.13) is that it has a VWP ${}_7F_6(1)$ series representation that immediately reduces to the ${}_3F_2(1)$ series by the cancellation of numerator and denominator parameters:

$$ {}_3F_2\left[\begin{matrix}\frac{3}{4},1,1\\ \frac{3}{2},\frac{7}{4}\end{matrix};1\right] = {}_7F_6\left[\begin{matrix}\frac{3}{2},\frac{7}{4},\frac{3}{4},\frac{3}{4},1,1,\frac{5}{4}\\ \frac{3}{4},\frac{7}{4},\frac{7}{4},\frac{3}{2},\frac{3}{2},\frac{5}{4}\end{matrix};1\right] =: W\left(\frac{3}{2};\frac{3}{4},\frac{3}{4},1,1,\frac{5}{4}\right). \quad (A.24) $$

Additional VWP ${}_7F_6(1)$ series can be derived by permuting the entries in (A.24) and applying the two-term relation (A.2):

$$ \begin{aligned} W\left(\frac{3}{2};\frac{3}{4},\frac{3}{4},1,1,\frac{5}{4}\right) &= \frac{9}{16}\frac{\pi^2}{L}\,W\left(\frac{3}{4};\frac{3}{4},\frac{3}{4},\frac{1}{4},\frac{1}{4},\frac{1}{2}\right)\\ W\left(\frac{3}{2};\frac{3}{4},1,\frac{3}{4},1,\frac{5}{4}\right) &= 2\,W\left(1;\frac{3}{4},1,\frac{1}{4},\frac{1}{2},\frac{3}{4}\right)\\ W\left(\frac{3}{2};1,1,\frac{3}{4},\frac{3}{4},\frac{5}{4}\right) &= \frac{5}{3}\,W\left(\frac{5}{4};1,1,\frac{1}{2},\frac{1}{2},1\right)\\ W\left(\frac{3}{2};\frac{3}{4},\frac{5}{4},1,1,\frac{3}{4}\right) &= \frac{5L}{3}\,W\left(\frac{5}{4};\frac{5}{4},\frac{5}{4},\frac{3}{4},\frac{1}{2},\frac{3}{4}\right). \end{aligned} \quad (A.25) $$

Finally, applying (A.2) to (A.25) with its entries appropriately permuted gives

$$ W\left(1;\frac{1}{2},\frac{1}{4},1,\frac{3}{4},\frac{3}{4}\right) = \frac{3}{4}\left(\frac{\pi}{L}\right)^2 W\left(\frac{1}{2};\frac{1}{2},\frac{1}{4},\frac{1}{2},\frac{1}{4},\frac{1}{4}\right). $$

The six expressions in the statement of theorem are obtained by dividing the right-hand expressions by the factor 3 that appears in (5.13), reordering the entries of each ${}_7F_6(1)$ series in ascending order, and putting the series following the canonical series in ascending order in the parameter a.

J. Roderick McCrorie University of St Andrews, Scotland, UK

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