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On Exponential Densities and Limit Ratios of Subsets of $\mathbb{N}$

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Abstract. Given $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha \leq \beta$, we prove that there exists a subset of $\mathbb{N}$ such that its lower and upper exponential densities and its lower and upper limit ratios are equal to α, β, γ and 1, respectively. This result provides an affirmative answer to an open problem posed by Grekos et al. (Unif Distrib Theory 6:117–130, 2011).

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1. Introduction and Statement of Results

The purpose of this paper is to provide an affirmative solution to a problem posed by Grekos et al. [5] about the interrelationship between the exponential densities and the limit ratios of subsets of integers. We start by recalling the definitions of exponential densities and limit ratios of subsets of integers.

Definition. (*Exponential densities*) For an infinite subset A of $\mathbb{N}$, write $N_n(A) = \big|\{1, \ldots, n\} \cap A\big|$; here and below, we write $|A|$ for the cardinality of a set A. We define the lower and upper exponential densities of an infinite subset A of $\mathbb{N}$ by

$$\underline{\varepsilon}(A) = \liminf_n \frac{\log N_n(A)}{\log n}$$

and

$$\overline{\varepsilon}(A) = \limsup_n \frac{\log N_n(A)}{\log n},$$

respectively.

Definition. (*Limit ratios*) For an infinite subset $A = \{a_1, a_2, \ldots\}$ of $\mathbb{N}$ with $a_1 < a_2 < \cdots$, we define the lower and upper limit ratios of A by

$$\underline{\varrho}(A) = \liminf_n \frac{a_n}{a_{n+1}}$$

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and

$$\overline{\varrho}(A) = \limsup_n \frac{a_n}{a_{n+1}}\,,$$

respectively.

The exponential densities and the limit ratios are fundamental in many diverse areas of pure and applied mathematics, including analytic number theory [1,11], metric number theory [4,6,7], and more recently in fractal geometry of discrete sets [2,9] and theoretical computer science [3]; the reader is referred to the remarks below for more details. Because of their important and ubiquitous role, it is natural to investigate their interrelationship. For example, Grekos et al. [5] proved that if $\overline{\varrho}(A) < 1$, then $\overline{\varepsilon}(A) = 0$, and they also present examples showing that if $\overline{\varrho}(A) = 1$, then nothing can be said about the value of $\overline{\varepsilon}(A)$. This observation led Grekos et al. [5] to ask the following question.

Question. [5, Problem 2.7] *Given $\alpha, \beta, \gamma \in [0,1]$ with $\alpha \le \beta$, does there exist an infinite subset A of $\mathbb{N}$ such that*

$$\underline{\varepsilon}(A) = \alpha\,, \quad \overline{\varepsilon}(A) = \beta\,, \quad \underline{\varrho}(A) = \gamma\,, \quad \overline{\varrho}(A) = 1\,?$$

The main purpose of this paper is to provide an affirmative answer to this question. We formally state our result as follows.

Theorem 1.1. *Given $\alpha, \beta, \gamma \in [0,1]$ with $\alpha \le \beta$, then there exists an infinite subset A of $\mathbb{N}$ such that*

$$\underline{\varepsilon}(A) = \alpha, \overline{\varepsilon}(A) = \beta, \underline{\varrho}(A) = \gamma, \overline{\varrho}(A) = 1\,.$$

The proof of Theorem 1.1 is given in Sects. 2–4; Sect. 2 contains some auxiliary results; in Sect. 3 we prove Theorem 1.1 for $\beta = 0$; and in Sect. 4 we prove Theorem 1.1 for $\beta > 0$.

The main difficulty in the proof of Theorem 1.1 is to ensure that $\underline{\varrho}(A) = \gamma$. Indeed, it is not difficult to see that there is a subset A of $\mathbb{N}$ with $\underline{\varepsilon}(A) = \alpha$ and $\overline{\varepsilon}(A) = \beta$. To see this we note that it is not difficult to show that we can choose positive integers p_n and q_n, for $n \in \mathbb{N}$, with $p_1 < q_1 < p_2 < q_2 < \dots$ such that if we let

$$A = \Big((p_1, q_1] \cap \mathbb{N} \Big) \cup \Big((p_2, q_2] \cap \mathbb{N} \Big) \cup \Big((p_3, q_3] \cap \mathbb{N} \Big) \cup \cdots\,, \tag{1.1}$$

then $\frac{\log N_{p_n}(A)}{\log p_n} \to \alpha$ and $\frac{\log N_{q_n}(A)}{\log q_n} \to \beta$. It is now clear that the set A satisfies $\underline{\varepsilon}(A) = \alpha$ and $\overline{\varepsilon}(A) = \beta$. Of course, the set A in (1.1) does not necessarily satisfy $\underline{\varrho}(A) = \gamma$. To achieve this, we "insert" finitely many numbers of the form

$$\left[\frac{r_n}{\gamma_n^0}\right], \left[\frac{r_n}{\gamma_n^1}\right], \left[\frac{r_n}{\gamma_n^2}\right], \dots \tag{1.2}$$

into the gap $(q_{n-1}, p_n] \cap \mathbb{N}$, where r_n is a sufficiently large integer and $0 < \gamma_n < 1$ with $\gamma_n \to \gamma$ (in (1.2) we write $[x]$ for the integer part of the real number x); since $[\frac{r_n}{\gamma_n^l}]/[\frac{r_n}{\gamma_n^{l+1}}]$ is "close" to γ for all integers l and all large values of n, this will guarantee that $\underline{\varrho}(A) = \gamma$. However, considerable care needs to be taken when "inserting" the points from (1.2) into the gap $(q_{n-1}, p_n] \cap \mathbb{N}$: if too many points from (1.2) are "inserted", then the lower density $\underline{\varepsilon}(A)$ will increase and can become strictly greater than α, and if too few points from (1.2) are "inserted", then the lower limit ratio $\underline{\varrho}(A)$ will decrease and can become strictly smaller than γ.

We close this section with some remarks about exponential densities and limit ratios.

Zeta functions and exponential densities. There is an alternative and important analytic characterisation of the upper exponential density. Namely, for a infinite subset A of $\mathbb{N}$, let $\tau(A)$ denote the abscissa of convergence of the "zeta function" $\zeta_A(s) = \sum_{x \in A} \frac{1}{x^s}$ for $s \in \mathbb{C}$, i.e., we put

$$\tau(A) = \sup \left\{ t \geq 0 \,\middle|\, \sum_{x \in A} \frac{1}{x^t} = \infty \right\}.$$

It is well known that

$$\overline{\varepsilon}(A) = \tau(A), \tag{1.3}$$

and a proof of this can be found in many texts in analytic number theory; see, for example, [1, p. 165, Theorem 8.2], [8, p. 8, Theorem 7] or [11, p. 115, Theorem 11]. Because of (1.3), the upper exponential density plays a fundamental role in analytic number theory (see [1,11]) and metric number theory (see [7]).

Fractal geometry and exponential densities. Exponential densities have been rediscovered several times recently in areas of mathematics outside number theory. For example, the lower and upper exponential densities, and their higher-dimensional analogues, were rediscovered by Bedford and Fisher [2] in the early 1990s as a motivation for their work on higher-order densities of fractals, and now play a fundamental role in fractal geometry [9] and in the study of (fractal) geometry of discrete subsets of $\mathbb{R}^d$. The exponential densities were also rediscovered in 2005 by Doty et al. [3] in their work on effective fractal dimensions of discrete sets in theoretical computer science and now play an important part in several areas of computer science.

2. The proof of Theorem 1.1. Auxiliary results

The section contains two auxiliary results that are used in the proof of Theorem 1.1. We begin with a definition.

Definition. (*Eventually strictly increasing sequence*) A sequence $(a_n)_n$ of real numbers is called eventually strictly increasing if there is a positive integer N such that $a_n < a_{n+1}$ for all $n \geq N$.

An eventually strictly increasing sequence $(a_n)_n$ of positive integers determine in a natural way an unbounded subset A of $\mathbb{N}$ by setting $A = \{a_1, a_2, \ldots\}$. Indeed, the set A in Theorem 1.1 will be constructed in this way. For this reason, it is useful to have expressions for the exponential densities and the limit ratios of subsets of $\mathbb{N}$ constructed in this way, and the first result in this section lists some explicit and well-known expressions for the exponential densities and the limit ratios of subsets of $\mathbb{N}$ obtained in this way.

Lemma 2.1. *Let $(a_n)_n$ be an eventually strictly increasing sequence of positive integers and put $A = \{a_1, a_2, \ldots\}$.*

(1) *We have*

$$\underline{\varepsilon}(A) = \liminf_n \frac{\log n}{\log a_n}\,,$$
$$\overline{\varepsilon}(A) = \limsup_n \frac{\log n}{\log a_n}\,.$$

(2) *We have*

$$\underline{\varrho}(A) = \liminf_n \frac{a_n}{a_{n+1}}\,,$$
$$\overline{\varrho}(A) = \limsup_n \frac{a_n}{a_{n+1}}\,.$$

Proof. (1) This result is well known; see, for example, the text [10, Part I, Exercise 113].
(2) This result follows immediately from the definitions of $\underline{\varrho}(A)$ and $\overline{\varrho}(A)$. □

The statements in Lemma 2.1 will be used frequently, and often without mentioning, in Sect. 3. The second result in this section gives a useful estimate for the difference $|\frac{\log N_{n+1}(A)}{\log(n+N)} - \frac{\log N_n(A)}{\log n}|$ for $A \subseteq \mathbb{N}$ and integers n and N.

Lemma 2.2. *For $A \subseteq \mathbb{N}$ and all positive integers n and N, we have*

$$\left|\frac{\log N_{n+1}(A)}{\log(n+N)} - \frac{\log N_n(A)}{\log n}\right| \le \frac{1}{\log(n+1)}\left(\log 2 + \log\left(1 + \frac{N}{n}\right)\right).$$

In particular, we have

$$\left|\frac{\log N_{n+1}(A)}{\log(n+1)} - \frac{\log N_n(A)}{\log n}\right| \le 2\frac{\log 2}{\log(n+1)}\,.$$

Proof. For brevity, write $\delta_{n,N} = \frac{\log N_{n+1}(A)}{\log(n+N)} - \frac{\log N_n(A)}{\log n}$, and note that a straightforward calculation shows that

$$\begin{aligned}\delta_{n,N} &= \frac{\log(\frac{N_{n+1}(A)}{N_n(A)})\,\log n - \log(1 + \frac{N}{n})\,\log N_n(A)}{\log(n+N)\,\log n}\\ &= \frac{\log(\frac{N_{n+1}(A)}{N_n(A)})}{\log(n+N)} - \frac{\log(1 + \frac{N}{n})\,\log N_n(A)}{\log(n+N)\,\log n}\\ &= \frac{1}{\log(n+N)}\left(\log\left(\frac{N_{n+1}(A)}{N_n(A)}\right) - \log\left(1 + \frac{N}{n}\right)\frac{\log N_n(A)}{\log n}\right). \qquad (2.1)\end{aligned}$$

Next, since clearly $N_{n+1}(A) \le N_n(A) + 1 \le N_n(A) + N_n(A) = 2N_n(A)$ and $N_n(A) \le n$, we have $\left|\log(\frac{N_{n+1}(A)}{N_n(A)})\right| \le \left|\log(\frac{2N_n(A)}{N_n(A)})\right| = \log 2$ and $\left|\frac{\log N_n(A)}{\log n}\right| \le 1$, and it therefore follows from (2.1) that

$$|\delta_{n,N}| \le \frac{1}{\log(n+N)}\left(\left|\log\left(\frac{N_{n+1}(A)}{N_n(A)}\right)\right| + \log\left(1+\frac{N}{n}\right)\left|\frac{\log N_n(A)}{\log n}\right|\right)$$
$$\le \frac{1}{\log(n+1)}\left(\log 2 + \log\left(1+\frac{N}{n}\right)\right).$$

This completes the proof. □

3. The proof of Theorem 1.1. The case: $\beta = 0$.

The purpose of this section is to prove Theorem 1.1 for $\beta = 0$. We first prove a small auxiliary lemma, namely, Lemma 3.1 below; we note that the statement and proof of Lemma 3.1 for $\gamma = 1$ also appears in [5, Example 2.6]. Recall that if x is a real number, then we write $[x]$ for the integer part of x.

Lemma 3.1. *Let* $\gamma \in [0,1]$. *Then there is an eventually strictly increasing sequence* $(b_n)_{n\in\mathbb{N}}$ *with* $b_{n+1} - b_n \to \infty$ *such that if we put* $B = \{b_n \,|\, n \in \mathbb{N}\}$, *then* $\underline{\varepsilon}(B) = \overline{\varepsilon}(B) = 0$ *and* $\underline{\varrho}(B) = \overline{\varrho}(B) = \gamma$.

Proof. Define the sequence $(\gamma_n)_n$ by

$$\gamma_n = \begin{cases} \frac{1}{n} & \text{for } \gamma = 0; \\ \gamma & \text{for } 0 < \gamma < 1; \\ \frac{1}{1+\sqrt{\frac{1}{n}}} & \text{for } \gamma = 1, \end{cases}$$

and let $b_n = [\frac{1}{\gamma_n^n}]$. We will now show that the sequence $(b_n)_n$ has the desired properties. For brevity, write $x_n = \frac{\gamma_{n+1}^{n+1}}{\gamma_n^n}$ and $y_n = \frac{\log n}{\log \gamma_n^n}$, and note that $x_n \to \gamma$, $y_n \to 0$, $\gamma_n^n \to 0$ and $\frac{1}{\gamma_{n+1}^{n+1}} - \frac{1}{\gamma_n^n} \to \infty$ (indeed, this is immediate if $0 \le \gamma < 1$, and if $\gamma = 1$, then the statements follow from routine calculus arguments; see also [5, Example 2.6] for more details for the case $\gamma = 1$). Next, observe that $b_{n+1} - b_n \to \infty$ (because $\frac{1}{\gamma_{n+1}^{n+1}} - \frac{1}{\gamma_n^n} \to \infty$) and the sequence $(b_n)_n$ is therefore, in particular, eventually strictly increasing. Finally, to prove that $\underline{\varepsilon}(B) = \overline{\varepsilon}(B) = 0$ and $\underline{\varrho}(B) = \overline{\varrho}(B) = \gamma$, we note that $b_n = \frac{1}{\gamma_n^n} - u_n$ where $u_n \in [0,1]$.

First, we prove that $\underline{\varepsilon}(B) = \overline{\varepsilon}(B) = 0$. Indeed, since it is clear that $\frac{\log n}{\log b_n} = \frac{\log n}{\log(\frac{1}{\gamma_n^n} - u_n)} = \frac{y_n}{-1+\frac{1}{\log \gamma_n^n}\log(1-\gamma_n^n u_n)} \to 0$ (because $y_n \to 0$ and $\gamma_n^n \to 0$), we conclude from Lemma 2.1 that $\underline{\varepsilon}(B) = \overline{\varepsilon}(B) = 0$.

Next, we prove that $\underline{\varrho}(B) = \overline{\varrho}(B) = 0$. However, since $\frac{b_n}{b_{n+1}} = \frac{\frac{1}{\gamma_n^n} - u_n}{\frac{1}{\gamma_{n+1}^{n+1}} - u_{n+1}} = \frac{x_n - \gamma_{n+1}^{n+1} u_n}{1 - \gamma_{n+1}^{n+1} u_{n+1}} \to \gamma$ (because $x_n \to \gamma$ and $\gamma_n^n \to 0$), we deduce that $\underline{\varrho}(B) = \overline{\varrho}(B) = \gamma$. □

Proof of Theorem 1.1. The case: $\beta = 0$

It follows from Lemma 3.1 that we can choose an eventually strictly increasing sequence $(b_n)_{n\in\mathbb{N}}$ with $b_{n+1} - b_n \to \infty$ such that if we put $B = \{b_n \,|\, n \in \mathbb{N}\}$, then $\underline{\varepsilon}(B) = \overline{\varepsilon}(B) = 0$ and $\underline{\varrho}(B) = \overline{\varrho}(B) = \gamma$. Now, let

$$\begin{aligned} A &= \Big\{b_n \,|\, n \in \mathbb{N}\Big\} \cup \Big\{b_{2k-1} + 1 \,|\, k \in \mathbb{N}\Big\} \\ &= \Big\{b_1, b_1 + 1, b_2, \\ &\qquad\qquad b_3, b_3 + 1, b_4, \\ &\qquad\qquad\qquad b_5, b_5 + 1, b_6, \dots\Big\}\,; \end{aligned}$$

i.e., the set A is obtained in the following way: for each k, we "insert" the point $b_{2k-1} + 1$ into the sequence $(b_n)_{n\in\mathbb{N}}$ between the points b_{2k-1} and b_{2k}. Next, we write the set A in the form $A = \{a_1, a_2, \dots\}$ with $a_1 < a_2 < \cdots$. Observe that since the sequence $(b_n)_{n\in\mathbb{N}}$ is eventually strictly increasing sequence with $b_{n+1} - b_n \to \infty$, we conclude that $b_{2k-1} < b_{2k-1} + 1 < b_{2k}$ for all sufficiently large k. In particular, this implies that there is a positive integer K such that for all k, we have

$$a_{3k-2+K} = b_{2k-1}\,, \quad a_{3k-1+K} = b_{2k-1} + 1\,, \quad a_{3k+K} = b_{2k}\,. \tag{3.1}$$

We will now prove that $\underline{\varepsilon}(A) = \alpha$, $\overline{\varepsilon}(A) = \beta$, $\underline{\varrho}(A) = \gamma$ and $\overline{\varrho}(A) = 1$.

We first show that $\underline{\varepsilon}(A) = \alpha$ and $\overline{\varepsilon}(A) = \beta$. To prove this, we note that (3.1) implies that

$$\begin{aligned} \limsup_k \frac{\log(3k-2+K)}{\log a_{3k-2+K}} &= \limsup_k \frac{\log(3k-2+K)}{\log b_{2k-1}} \\ &= \limsup_k \frac{\log(2k-1)}{\log b_{2k-1}} \\ &\le \limsup_n \frac{\log n}{\log b_n} \\ &= \overline{\varepsilon}(B) \\ &= 0\,. \end{aligned} \tag{3.2}$$

A very similar argument shows that

$$\limsup_k \frac{\log(3k-1+K)}{\log a_{3k-1+K}} = 0\,, \tag{3.3}$$

$$\limsup_k \frac{\log(3k+K)}{\log a_{3k+K}} = 0\,. \tag{3.4}$$

It follows from (3.2)–(3.4) that $\frac{\log n}{\log a_n} \to 0$, and so $\underline{\varepsilon}(A) \le \overline{\varepsilon}(A) = \limsup_n \frac{\log n}{\log a_n} = 0$, whence $\underline{\varepsilon}(A) = 0 = \alpha$ and $\overline{\varepsilon}(A) = 0 = \beta$.

Next, we show that $\underline{\varrho}(A) = \gamma$ and $\overline{\varrho}(A) = 1$. Indeed, using (3.1) we see that

$$\frac{a_{3k-2+K}}{a_{3k-1+K}} = \frac{b_{2k-1}}{b_{2k-1} + 1} \to 1\,. \tag{3.5}$$

Also, using (3.1) and the fact that $\underline{\varrho}(B) = \overline{\varrho}(B) = \gamma$ (and so $\frac{b_n}{b_{n+1}} \to \gamma$), we see that

$$\begin{aligned}\liminf_k \frac{a_{3k-1+K}}{a_{3k+K}} &= \liminf_k \frac{b_{2k-1}+1}{b_{2k}} \\ &= \liminf_k \frac{b_{2k-1}}{b_{2k}} \\ &= \gamma\,. && [\text{since } \tfrac{b_n}{b_{n+1}} \to \gamma] \qquad (3.6)\end{aligned}$$

Finally, using (3.1) and the fact that $\underline{\varrho}(B) = \overline{\varrho}(B) = \gamma$ (and so $\frac{b_n}{b_{n+1}} \to \gamma$), we see that

$$\begin{aligned}\liminf_k \frac{a_{3k+K}}{a_{3k+1+K}} &= \liminf_k \frac{a_{3k+K}}{a_{3(k+1)-2+K}} \\ &= \liminf_k \frac{b_{2k}}{b_{2(k+1)-1}} \\ &= \liminf_k \frac{b_{2k}}{b_{2k+1}} \\ &= \gamma\,. && [\text{since } \tfrac{b_n}{b_{n+1}} \to \gamma] \qquad (3.7)\end{aligned}$$

We conclude from (3.5) to (3.7) that $\liminf_n \frac{a_n}{a_{n+1}} = \gamma$ and $\limsup_n \frac{a_n}{a_{n+1}} = 1$, whence $\underline{\rho}(A) = \gamma$ and $\overline{\rho}(A) = 1$. □

4. The proof of Theorem 1.1. The case: $\beta > 0$.

The purpose of this section is to prove Theorem 1.1 for $\beta > 0$.
Proof of Theorem 1.1. The case $\beta > 0$
We may clearly choose two sequences $(\alpha_n)_n$ and $(\beta_n)_n$ such that:

$$0 < \alpha_n < 1 \quad \text{and} \quad 0 < \beta_n < 1 \quad \text{for all } n; \qquad (4.1)$$
$$\alpha_n < \beta_n \quad \text{and} \quad \alpha_{n+1} < \beta_n \quad \text{for all } n; \qquad (4.2)$$
$$\alpha_n \to \alpha \quad \text{and} \quad \beta_n \to \beta\,, \qquad (4.3)$$
$$|\alpha_{n+1} - \beta_n| \geq 4\tfrac{\log 2}{\log n} \quad \text{for all sufficiently large } n; \qquad (4.4)$$
$$|\alpha_n - \beta_n| \geq 4\tfrac{\log 2}{\log n} \quad \text{for all sufficiently large } n. \qquad (4.5)$$

Since $\beta > 0$, we can also choose a real number δ with $0 < \delta \leq \frac{\beta}{2}$. We now define the sequence $(\gamma_n)_n$ by

$$\gamma_n = \begin{cases} \frac{1}{n^{\frac{\delta}{4}}} & \text{for } \gamma = 0; \\ \gamma & \text{for } 0 < \gamma < 1; \\ 1 - \frac{1}{n^{\frac{\delta}{4}}} & \text{for } \gamma = 1; \end{cases}$$

note that $\gamma_n \to \gamma$ and that $0 < \gamma_n < 1$ for all n. For each positive integer n, let N_n be the unique positive integer such that $\frac{1}{1-\gamma_n} \leq N_n < \frac{1}{1-\gamma_n} + 1$. Next, we define inductively three sequences $(p_n)_{n\in\mathbb{N}}$, $(q_n)_{n\in\mathbb{N}}$ and $(k_n)_{n\in\mathbb{N}}$ of positive integers such that if we write $r_n = N_n + q_n$ and

$$A_n = \big((p_1, q_1] \cap \mathbb{N} \big) \quad \cup \quad \left\{ \left[\frac{r_1}{\gamma_1^0} \right], \left[\frac{r_1}{\gamma_1^1} \right], \ldots, \left[\frac{r_1}{\gamma_1^{k_1}} \right] \right\}$$

$$\vdots$$

$$\cup \quad \big((p_{n-1}, q_{n-1}] \cap \mathbb{N} \big) \quad \cup \quad \left\{ \left[\frac{r_{n-1}}{\gamma_{n-1}^0} \right], \left[\frac{r_{n-1}}{\gamma_{n-1}^1} \right], \ldots, \left[\frac{r_{n-1}}{\gamma_{n-1}^{k_{n-1}}} \right] \right\}$$

$$\cup \quad \big((p_n, q_n] \cap \mathbb{N} \big),$$

then the following four conditions are satisfied for all positive integers n:
Condition 4.1: $p_1 < q_1 < p_2 < q_2 < \cdots < p_n < q_n$;
Condition 4.2: For all $0 \le m < k_{n-1}$, *we have*

$$\frac{\log \left(N_{\left[\frac{r_{n-1}}{\gamma_{n-1}^{k_{n-1}}} \right]} \left(A_{n-1} \cup \left\{ \left[\frac{r_{n-1}}{\gamma_{n-1}^0} \right], \left[\frac{r_{n-1}}{\gamma_{n-1}^1} \right], \ldots, \left[\frac{r_{n-1}}{\gamma_{n-1}^{k_{n-1}}} \right] \right\} \right) \right)}{\log \left[\frac{r_{n-1}}{\gamma_{n-1}^{k_{n-1}}} \right]}$$

$$\le \alpha_n < \frac{\log \left(N_{\left[\frac{r_{n-1}}{\gamma_{n-1}^{m}} \right]} \left(A_{n-1} \cup \left\{ \left[\frac{r_{n-1}}{\gamma_{n-1}^0} \right], \left[\frac{r_{n-1}}{\gamma_{n-1}^1} \right], \ldots, \left[\frac{r_{n-1}}{\gamma_{n-1}^{m}} \right] \right\} \right) \right)}{\log \left[\frac{r_{n-1}}{\gamma_{n-1}^{m}} \right]};$$

Condition 4.3: $p_n = \left[\frac{r_{n-1}}{\gamma_{n-1}^{k_{n-1}}} \right]$;
Condition 4.4: For all $p_n < m < q_n - 1$, *we have*

$$\frac{\log N_m \left(A_{n-1} \cup \left\{ \left[\frac{r_{n-1}}{\gamma_{n-1}^0} \right], \left[\frac{r_{n-1}}{\gamma_{n-1}^1} \right], \ldots, \left[\frac{r_{n-1}}{\gamma_{n-1}^{k_{n-1}}} \right] \right\} \cup ((p_n, m] \cap \mathbb{N}) \right)}{\log m}$$

$$< \beta_n \le \frac{\log N_{q_n} \left(A_{n-1} \cup \left\{ \left[\frac{r_{n-1}}{\gamma_{n-1}^0} \right], \left[\frac{r_{n-1}}{\gamma_{n-1}^1} \right], \ldots, \left[\frac{r_{n-1}}{\gamma_{n-1}^{k_{n-1}}} \right] \right\} \cup ((p_n, q_n] \cap \mathbb{N}) \right)}{\log q_n}.$$

The start of the induction. We put $p_1 = 1$ and note that since $\beta_1 < 1$ and $\frac{\log N_m((1,\infty)\cap\mathbb{N}))}{\log m} \nearrow 1$ as $m \to \infty$, we can choose an integer q_1 with $q_1 > p_1$ such that for all $p_1 < m < q_1 - 1$, we have

$$\frac{\log N_m((1, m] \cap \mathbb{N})}{\log m} < \beta_1 \le \frac{\log N_{q_1}((1, q_1] \cap \mathbb{N})}{\log q_1}.$$

It is now clear that Conditions 4.1 and 4.4 are satisfied for $n = 1$, and Conditions 4.2 and 4.3 are vacuously true for $n = 1$. This completes the start of the induction.
The inductive step. Let l be a positive integer and assume that the integers $p_1, \ldots, p_l$, $q_1, \ldots, q_l$ and $k_1, \ldots, k_{l-1}$ have been chosen such that Conditions 4.1–4.4 are satisfied for $n = l$. We must now choose integers p_{l+1}, q_{l+1} and k_l such that the integers $p_1, \ldots, p_l, p_{l+1}$, $q_1, \ldots, q_l, q_{l+1}$ and $k_1, \ldots, k_{l-1}, k_l$ satisfy Conditions 4.1–4.4 for $n = l + 1$.

We first choose k_l and p_{l+1}. Observing that $N_{\left[\frac{r_l}{\gamma_l^m}\right]}\big(A_l \cup \big\{\big[\frac{r_l}{\gamma_l^0}\big], \big[\frac{r_l}{\gamma_l^1}\big], \ldots\big\}\big) = N_{q_l}(A_l) + m + 1$, we conclude that

$$\frac{\log\left(N_{\left[\frac{r_l}{\gamma_l^m}\right]}\left(A_l \cup \left\{\left[\frac{r_l}{\gamma_l^0}\right], \left[\frac{r_l}{\gamma_l^1}\right], \ldots\right\}\right)\right)}{\log\left[\frac{r_l}{\gamma_l^m}\right]} = \frac{\log(N_{q_l}(A_l) + m + 1)}{\log\left[\frac{r_l}{\gamma_l^m}\right]} \to 0\,. \quad (4.6)$$

Also, note that

$$\frac{\log N_{q_l}(A_l)}{\log q_l} \geq \beta_l > \alpha_{l+1}\,. \quad (4.7)$$

Since $\alpha_{l+1} > 0$, it follows from (4.6) and (4.7) that we can choose a positive integer k_l such that for all $0 \leq m < k_l$, we have

$$\frac{\log\left(N_{\left[\frac{r_l}{\gamma_l^{k_l}}\right]}\left(A_l \cup \left\{\left[\frac{r_l}{\gamma_l^0}\right], \left[\frac{r_l}{\gamma_l^1}\right], \ldots, \left[\frac{r_l}{\gamma_l^{k_l}}\right]\right\}\right)\right)}{\log\left[\frac{r_l}{\gamma_l^{k_l}}\right]} \leq \alpha_{l+1} < \frac{\log\left(N_{\left[\frac{r_l}{\gamma_l^m}\right]}\left(A_l \cup \left\{\left[\frac{r_l}{\gamma_l^0}\right], \left[\frac{r_l}{\gamma_l^1}\right], \ldots, \left[\frac{r_l}{\gamma_l^m}\right]\right\}\right)\right)}{\log\left[\frac{r_l}{\gamma_l^m}\right]}. \quad (4.8)$$

Now, put

$$p_{l+1} = \left[\frac{r_l}{\gamma_l^{k_l}}\right], \quad (4.9)$$

and note that (because $N_l \geq \frac{1}{1-\gamma_l} \geq 1$)

$$p_{l+1} = \left[\frac{r_l}{\gamma_l^{k_l}}\right] > \frac{r_l}{\gamma_l^{k_l}} - 1 \geq r_l - 1 = q_l + N_l - 1 \geq q_l\,. \quad (4.10)$$

Next, we choose q_{l+1} as follows. Observe that

$$\frac{\log N_m\left(A_l \cup \left\{\left[\frac{r_l}{\gamma_l^0}\right], \left[\frac{r_l}{\gamma_l^1}\right], \ldots, \left[\frac{r_l}{\gamma_l^{k_l}}\right]\right\} \cup ((p_{l+1}, \infty) \cap \mathbb{N})\right)}{\log m} \nearrow 1 \quad (4.11)$$

as $m \to \infty$ for $m \geq p_{l+1}$ and that

$$\frac{\log\left(N_{\left[\frac{r_l}{\gamma_l^{k_l}}\right]}\left(A_l \cup \left\{\left[\frac{r_l}{\gamma_l^0}\right], \left[\frac{r_l}{\gamma_l^1}\right], \ldots, \left[\frac{r_l}{\gamma_l^{k_l}}\right]\right\}\right)\right)}{\log\left[\frac{r_l}{\gamma_l^{k_l}}\right]} \leq \alpha_{l+1} < \beta_{l+1}\,. \quad (4.12)$$

Since $\beta_{l+1} < 1$, it follows from (4.11) and (4.12) that we can choose an integer q_{l+1} with $q_{l+1} > p_{l+1}$ such that for all $p_{l+1} < m < q_{l+1} - 1$, we have

$$\frac{\log N_m\left(A_l \cup \left\{\left[\frac{r_l}{\gamma_l^0}\right], \left[\frac{r_l}{\gamma_l^1}\right], \ldots, \left[\frac{r_l}{\gamma_l^{k_l}}\right]\right\} \cup ((p_{l+1}, m] \cap \mathbb{N})\right)}{\log m}$$

$$< \beta_{l+1} \le \frac{\log N_{q_{l+1}}\Big(A_l \cup \Big\{\Big[\frac{r_l}{\gamma_l^0}\Big],\Big[\frac{r_l}{\gamma_l^1}\Big],\dots,\Big[\frac{r_l}{\gamma_l^{k_l}}\Big]\Big\} \cup ((p_{l+1},q_{l+1}]\cap\mathbb{N})\Big)}{\log q_{l+1}}\,. \tag{4.13}$$

We conclude immediately from (4.8), (4.9), (4.10) and (4.13) that the integers $p_1,\dots,p_l,p_{l+1}$, $q_1,\dots,q_l,q_{l+1}$ and $k_1,\dots,k_{l-1},k_l$ satisfy Conditions 4.1–4.4 for $n=l+1$. This completes the inductive step.

We now define the set A by

$$A=\cup_n A_n\,.$$

Below, we will prove that $\underline{\varepsilon}(A)=\alpha$, $\overline{\varepsilon}(A)=\beta$, $\underline{\varrho}(A)=\gamma$ and $\overline{\varrho}(A)=1$. However, we first prove two technical claims.

Define the function $f_n:[0,\infty)\to\mathbb{R}$ by

$$f_n(x)=\frac{\log(N_{q_n}(A)+x+1)}{\log\frac{r_n}{\gamma_n^x}}\,.$$

Claim 1. The function f_n is decreasing for all sufficiently large n.
Proof of Claim 1. It clearly suffices to show the following statement:

$$\exists\Delta\in\mathbb{N}\;:\;\forall n\ge\Delta\;:\;\forall x\ge 0\;:\;f_n'(x)<0\,. \tag{4.14}$$

Computing f_n', it is easily seen that statement (4.14) is equivalent to the following statement:

$$\exists\Delta\in\mathbb{N}\;:\;\forall n\ge\Delta\;:\;\forall x\ge 0\;:\;\frac{\log r_n}{\log\frac{1}{\gamma_n}}-N_{q_n}(A_n)-1<\big(N_{q_n}(A_n)+x+1\big)$$
$$\log\frac{\big(N_{q_n}(A_n)+x+1\big)}{e}\,. \tag{4.15}$$

To prove statement (4.15), it clearly suffices to show that

$$\exists\Delta\in\mathbb{N}\;:\;\forall n\ge\Delta\;:\;\frac{\log r_n}{\log\frac{1}{\gamma_n}}-N_{q_n}(A_n)<0\,. \tag{4.16}$$

Since $\beta_n\to\beta$ and $\beta>0$, we conclude from Condition 4.4 that $\frac{\log N_{q_n}(A_n)}{\log q_n}\ge\beta_n>\frac{\beta}{2}\ge\delta$ for all sufficiently large n, and so $\log N_{q_n}(A_n)\ge q_n^\delta$ for all sufficiently large n. We also note that if $\gamma=0$, then $\frac{1}{\gamma_n}=n^{\frac{\delta}{4}}\le q_n^{\frac{\delta}{4}}$ (since $n\le q_n$); if $0<\gamma<1$, then $\frac{1}{\gamma_n}=\frac{1}{\gamma}\le\frac{1}{\gamma}q_n^{\frac{\delta}{4}}$; and that if $\gamma=1$, then $\frac{1}{\gamma_n}=\frac{1}{1-\frac{1}{n^{\frac{\delta}{4}}}}=\frac{n^{\frac{\delta}{4}}}{1-n^{\frac{\delta}{4}}}\le 2\le 2q_n^{\frac{\delta}{4}}$. Hence, there is a constant c_1 such that $\frac{1}{\gamma_n}\le c_1q_n^{\frac{\delta}{4}}$ for all n. A similar argument shows that there is a constant c_2 such that $\frac{1}{1-\gamma_n}\le c_2q_n^{\frac{\delta}{4}}$ for all n. Hence, writing $c=2c_1+c_2$, we conclude that

$$\frac{\log r_n}{\log\frac{1}{\gamma_n}}-N_{q_n}(A_n)\le\frac{\log r_n}{\log\frac{1}{\gamma_n}}-q_n^\delta\le\frac{\log(\frac{3}{1-\gamma_n}+\frac{q_n}{\gamma_n})}{\log\frac{1}{\gamma_n}}-q_n^\delta\le\frac{\log(cq_n^{1+\frac{\delta}{4}})}{\log\frac{1}{\gamma_n}}-q_n^\delta \tag{4.17}$$

for all sufficiently large n. It follows from (4.17) that in to prove (4.16), it suffices to show that

$$\exists \Delta \in \mathbb{N} \ : \ \forall n \geq \Delta \ : \ \frac{\log(c q_n^{1+\frac{\delta}{4}})}{\log \frac{1}{\gamma_n}} - q_n^{\delta} < 0 \,. \tag{3.41}$$

However, it is clear that $\frac{\log(c q_n^{1+\frac{\delta}{4}})}{\log \frac{1}{\gamma_n}} - q_n^{\delta} < 0$ if and only if $\frac{\log(c q_n^{1+\frac{\delta}{4}})}{q_n^{\delta} \log \frac{1}{\gamma_n}} < 1$. Consequently, to show (3.41), it suffices to prove that

$$\frac{\log(c q_n^{1+\frac{\delta}{4}})}{q_n^{\delta} \log \frac{1}{\gamma_n}} \to 0 \,. \tag{4.18}$$

Next, noticing that $\limsup_n \frac{\log(c q_n^{1+\frac{\delta}{4}})}{q_n^{\delta} \log \frac{1}{\gamma_n}} \leq (\log c) \limsup_n \frac{1}{q_n^{\delta} \log \frac{1}{\gamma_n}} + (1 + \frac{\delta}{4}) \limsup_n \frac{\log q_n}{q_n^{\frac{\delta}{2}}} \frac{1}{q_n^{\frac{\delta}{2}} \log \frac{1}{\gamma_n}} = (\log c) \limsup_n \frac{1}{q_n^{\delta} \log \frac{1}{\gamma_n}} + 0 \limsup_n \frac{1}{q_n^{\frac{\delta}{2}} \log \frac{1}{\gamma_n}}$, we conclude that to prove (4.18), it suffices to show that

$$q_n^{d} \log \frac{1}{\gamma_n} \to \infty \tag{4.19}$$

for all $d \geq \frac{\delta}{2}$. However, if $\gamma = 0$, then we have $q_n^d \log \frac{1}{\gamma_n} = q_n^d \log n^{\frac{\delta}{4}} \to \infty$; if $0 < \gamma < 1$, then we have $q_n^d \log \frac{1}{\gamma_n} = q_n^d \log \frac{1}{\gamma} \to \infty$; and if $\gamma = 1$, then we have $q_n^d \log \frac{1}{\gamma_n} = q_n^d \log \frac{1}{1 - \frac{1}{n^{\frac{\delta}{4}}}} = q_n^d \log(1 + \frac{1}{n^{\frac{\delta}{4}} - 1}) \geq q_n^d \log(1 + \frac{1}{n^{\frac{\delta}{4}}}) \geq q_n^d \frac{1}{n^{\frac{\delta}{4}}} \log 2 \geq n^d \frac{1}{n^{\frac{\delta}{4}}} \log 2 = n^{d - \frac{\delta}{4}} \log 2 \to \infty$ (since $q_n \geq n$). This proves (4.19), and completes the proof of Claim 1.

Claim 2.

(1) *There is a sequence* $(\varepsilon_n)_n$ *with* $\varepsilon_n \to 0$ *such that for all sufficiently large* n *and all* m *with* $q_n < m \leq p_{n+1}$, *we have* $\frac{\log N_{q_n}(A)}{\log q_n} \geq \frac{\log N_m(A)}{\log m} \geq \frac{\log N_{p_{n+1}}(A)}{\log p_{n+1}} + \varepsilon_n$.

(2) *For all integers* n *and* m *with* $p_n < m \leq q_n$, *we have* $\frac{\log N_{p_n}(A)}{\log p_n} \leq \frac{\log N_m(A)}{\log m} \leq \frac{\log N_{q_n}(A)}{\log q_n}$.

(3) *We have* $\frac{\log N_{p_n}(A)}{\log p_n} \to \alpha$ *and* $\frac{\log N_{q_n}(A)}{\log q_n} \to \beta$.

(4) *For all positive integers* n *and* l, *we have* $[\frac{r_n}{\gamma_n^l}] < [\frac{r_n}{\gamma_n^{l+1}}]$.

(5) *For all sufficiently large* n, *we have* $|(q_n, p_{n+1}] \cap A| \geq 2$.

(6) *For all sufficiently large* n, *we have* $|(p_n, q_n] \cap A| \geq 2$.

Proof of Claim 2. (1) The left hand inequality is clear. We will now prove the right hand inequality. There is a unique integer l such that $[\frac{r_n}{\gamma_n^l}] \leq m < [\frac{r_n}{\gamma_n^{l+1}}]$. It now follows from the definition of A that $N_m(A) = N_{[\frac{r_n}{\gamma_n^{l+1}}]-1}(A)$, and

Lemma 2.2, and Claim 1 therefore implies that for all sufficiently large n we have

$$\begin{aligned}
\frac{\log N_m(A)}{\log m} &= \frac{\log N_{[\frac{r_n}{\gamma_n^{l+1}}]-1}(A)}{\log m}\\
&\geq \frac{\log N_{[\frac{r_n}{\gamma_n^{l+1}}]-1}(A)}{\log\left([\frac{r_n}{\gamma_n^{l+1}}]-1\right)}\\
&\geq \frac{\log N_{[\frac{r_n}{\gamma_n^{l+1}}]}(A)}{\log\left[\frac{r_n}{\gamma_n^{l+1}}\right]} - \frac{\log 4}{\log\left[\frac{r_n}{\gamma_n^{l+1}}\right]} && \text{[by Lemma 2.2]}\\
&\geq \frac{\log N_{[\frac{r_n}{\gamma_n^{l+1}}]}(A)}{\log \frac{r_n}{\gamma_n^{l+1}}} - \frac{\log 4}{\log\left[\frac{r_n}{\gamma_n^{l+1}}\right]}\\
&= f_n(l+1) - \frac{\log 4}{\log q_n} && \text{[since } q_n \leq [\tfrac{r_n}{\gamma_n^{l+1}}]\text{]}\\
&\geq f_n(k_n) - \frac{\log 4}{\log q_n} && \text{[by Claim 1 for all sufficiently large } n\text{]}\\
&= \frac{\log N_{[\frac{r_n}{\gamma_n^{k_n}}]}(A)}{\log \frac{r_n}{\gamma_n^{k_n}}} - \frac{\log 4}{\log\left[\frac{r_n}{\gamma_n^{l+1}}\right]}\\
&= \frac{\log[\frac{r_n}{\gamma_n^{k_n}}]}{\log \frac{r_n}{\gamma_n^{k_n}}}\,\frac{\log N_{[\frac{r_n}{\gamma_n^{k_n}}]}(A)}{\log[\frac{r_n}{\gamma_n^{k_n}}]} - \frac{\log 4}{\log\left[\frac{r_n}{\gamma_n^{l+1}}\right]}\\
&= \frac{\log[\frac{r_n}{\gamma_n^{k_n}}]}{\log \frac{r_n}{\gamma_n^{k_n}}}\,\frac{\log N_{p_{n+1}}(A)}{\log p_{n+1}} - \frac{\log 4}{\log q_n}\,.
\end{aligned} \tag{4.20}$$

Hence, putting $u_n = \frac{\log[\frac{r_n}{\gamma_n^{k_n}}]}{\log \frac{r_n}{\gamma_n^{k_n}}}$ and $\varepsilon_n = (u_u - 1)\,\frac{\log N_{p_{n+1}}(A)}{\log p_{n+1}} - \frac{\log 4}{\log q_n}$, we clearly have $u_n \to 1$ and $|\varepsilon_n| \leq |u_u - 1|\,|\frac{\log N_{p_{n+1}}(A)}{\log p_{n+1}}| + \frac{\log 4}{\log q_n} \leq |u_u - 1| + \frac{\log 4}{\log q_n} \to 0$, and (4.20) now implies that $\frac{\log N_m(A)}{\log m} \geq u_u\,\frac{\log N_{p_{n+1}}(A)}{\log p_{n+1}} - \frac{\log 4}{\log q_n} = \frac{\log N_{p_{n+1}}(A)}{\log p_{n+1}} + \varepsilon_n$.

(2) This statement follows immediately from the definition of the set A.

(3) It follows from Condition 4.2 and Lemma 2.2 that

$$\begin{aligned}
\left|\frac{\log N_{p_n}(A)}{\log p_n} - \alpha\right| &\leq \left|\frac{\log N_{p_n}(A)}{\log p_n} - \alpha_n\right| + |\alpha_n - \alpha|\\
&\leq \left|\frac{\log N_{p_n}(A)}{\log p_n} - \frac{\log N_{p_n-1}(A)}{\log(p_n-1)}\right| + |\alpha_n - \alpha| && \text{[by Condition 4.2]}\\
&\leq \tfrac{\log 4}{\log p_n} + |\alpha_n - \alpha| && \text{[by Lemma 2.2]}\\
&\to 0\,,
\end{aligned}$$

and it follows from Condition 4.4 and Lemma 2.2 that

$$\begin{aligned}\left|\frac{\log N_{q_n}(A)}{\log q_n}-\beta\right| &\leq \left|\frac{\log N_{q_n}(A)}{\log q_n}-\beta_n\right|+|\beta_n-\beta| \\ &\leq \left|\frac{\log N_{q_n}(A)}{\log q_n}-\frac{\log N_{q_n-1}(A)}{\log(q_n-1)}\right|+|\beta_n-\beta| \qquad \text{[by Condition 4.4]} \\ &\leq \tfrac{\log 4}{\log q_n}+|\beta_n-\beta| \qquad \text{[by Lemma 2.2]} \\ &\to 0\,.\end{aligned}$$

(4) Since $N_n > \frac{1}{1-\gamma_n}$, we conclude that $[\frac{r_n}{\gamma_n^{l+1}}]-[\frac{r_n}{\gamma_n^{l}}] \geq \frac{r_n}{\gamma_n^{l+1}}-\frac{r_n}{\gamma_n^{l}}-1 = (\frac{N_n+q_n}{\gamma_n^l})(\frac{1-\gamma_n}{\gamma_n})-1 \geq N_n(1-\gamma_n)-1>0$, whence $[\frac{r_n}{\gamma_n^{l+1}}]>[\frac{r_n}{\gamma_n^{l}}]$.
(5) Since $(q_n,p_{n+1}]\cap A = \{[\frac{r_n}{\gamma_n^0}],[\frac{r_n}{\gamma_n^1}],\ldots,[\frac{r_n}{\gamma_n^{k_n}}]\}$, we must show that $|\{[\frac{r_{n_i}}{\gamma_{n_i}^0}],[\frac{r_{n_i}}{\gamma_{n_i}^1}],\ldots,[\frac{r_{n_i}}{\gamma_{n_i}^{k_{n_i}}}]\}| \geq 2$ for all sufficiently large n, i.e., we must show that $k_n \geq 1$ for all sufficiently large n. We now assume, in order to reach a contradiction, that this is not the case, i.e., we assume that there is a strictly increasing sequence $(n_i)_i$ of integers such that $k_{n_i}=0$ for all i. This implies that $N_{[\frac{r_{n_i}}{\gamma_n^{k_{n_i}}}]}(A) = N_{[\frac{r_{n_i}}{\gamma_{n_i}^0}]}(A) = N_{q_{n_i}}(A)+1$ and $[\frac{r_{n_i}}{\gamma_{n_i}^{k_{n_i}}}]=[\frac{r_{n_i}}{\gamma_{n_i}^0}]=r_{n_i}=N_{n_i}+q_{n_i}$ for all i, whence (using Condition 4.2)

$$\frac{\log(N_{q_{n_i}}(A))}{\log(N_{n_i}+q_{n_i})} \leq \frac{\log(N_{q_{n_i}}(A)+1)}{\log(N_{n_i}+q_{n_i})} = \frac{\log N_{[\frac{r_{n_i}}{\gamma_n^{k_{n_i}}}]}(A)}{\log\,[\frac{r_{n_i}}{\gamma_n^{k_{n_i}}}]} = \frac{\log N_{p_{n_i+1}}(A)}{\log p_{n_i+1}} \leq \alpha_{n_i+1} \tag{4.21}$$

for all i. We also have (using Condition 4.4)

$$\frac{\log N_{q_{n_i}}(A)}{\log q_{n_i}} \geq \beta_{n_i}\,. \tag{4.22}$$

It follows immediately from (4.21) and (4.22) and Lemma 2.2 that for all sufficiently large i, we have

$$\begin{aligned}4\frac{\log 2}{\log n_i} \leq |\alpha_{n_i+1}-\beta_{n_i}| &\leq \left|\frac{\log(N_{q_{n_i}}(A))}{\log(N_{n_i}+q_{n_i})}-\frac{\log N_{q_{n_i}}(A)}{\log q_{n_i}}\right| \\ &\leq \frac{1}{\log q_{n_i}}\left(\log 2+\log\left(1+\frac{N_{n_i}}{q_{n_i}}\right)\right).\end{aligned} \tag{4.23}$$

However, if $\gamma=0$, then $\frac{N_n}{q_n} \leq \frac{1}{q_n}(\frac{1}{1-\gamma_n}+1) = \frac{1}{q_n}(\frac{1}{1-\frac{1}{n^{\frac{\delta}{4}}}}+1) \to 0$; if $0<\gamma<1$, then $\frac{N_n}{q_n} \leq \frac{1}{q_n}(\frac{1}{1-\gamma_n}+1) = \frac{1}{q_n}(\frac{1}{1-\frac{1}{\gamma}}+1) \to 0$; and if $\gamma=1$, then $\frac{N_n}{q_n} \leq \frac{1}{q_n}(\frac{1}{1-\gamma_n}+1) = \frac{1}{q_n}(n^{\frac{\delta}{4}}+1) \leq \frac{1}{n}(n^{\frac{\delta}{4}}+1) \to 0$. Hence $\frac{N_n}{q_n}\to 0$, and we therefore conclude that for all sufficiently large n, we have $\frac{N_n}{q_n} \leq 1$, and so for all sufficiently large n, we have

$$\frac{1}{\log q_n}\left(\log 2+\log\left(1+\frac{N_n}{q_n}\right)\right) \leq \tfrac{1}{\log q_n}(\log 2+\log 2) \leq 2\tfrac{\log 2}{\log n}\,. \tag{4.24}$$

The desired contradiction now follows from (4.23) and (4.24).
(6) We assume, in order to reach a contradiction, that this is not the case, i.e., we assume that there is a strictly increasing sequence $(n_i)_i$ of integers such that $|(p_{n_i}, q_{n_i}]| < 2$ for all i. This implies that $q_{n_i} = p_{n_i} + 1$ for all i, whence (using Condition 4.4)

$$\frac{\log N_{p_{n_i}+1}(A)}{\log(p_{n_i}+1)} = \frac{\log N_{q_{n_i}}(A)}{\log q_{n_i}} \geq \beta_{n_i} \tag{4.25}$$

for all i. We also have (using Condition 4.2)

$$\frac{\log N_{p_{n_i}}(A)}{\log p_{n_i}} \leq \alpha_{n_i} \tag{4.26}$$

for all i. It follows immediately from (4.25) and (4.26) and Lemma 2.2 that for all i, we have

$$4\frac{\log 2}{\log n_i} \leq |\alpha_{n_i} - \beta_{n_i}| \leq \left|\frac{\log N_{p_{n_i}+1}(A)}{\log(p_{n_i}+1)} - \frac{\log N_{p_{n_i}}(A)}{\log p_{n_i}}\right| \leq 2\frac{\log 2}{\log q_{n_i}} \leq 2\frac{\log 2}{\log n_i}\,. \tag{4.27}$$

However, (4.27) is a contradiction. This completes the proof of Claim 2.

We can now prove that $\underline{\varepsilon}(A) = \alpha$, $\overline{\varepsilon}(A) = \beta$, $\underline{\varrho}(A) = \gamma$ and $\overline{\varrho}(A) = 1$.

Claim 3. $\underline{\varepsilon}(A) = \alpha$ *and* $\overline{\varepsilon}(A) = \beta$.

Proof of Claim 3. It follows immediately from Claim 2.(1) and Claim 2.(2) that

$$\begin{aligned} \underline{\varepsilon}(A) &= \liminf_n \frac{\log N_{p_n}(A)}{\log p_n}\,, \\ \overline{\varepsilon}(A) &= \limsup_n \frac{\log N_{q_n}(A)}{\log q_n}\,. \end{aligned} \tag{4.28}$$

We now conclude from (4.28) and Claim 2.(3) that $\underline{\varepsilon}(A) = \liminf_n \frac{\log N_{p_n}(A)}{\log p_n} = \alpha$ and $\overline{\varepsilon}(A) = \limsup_n \frac{\log N_{q_n}(A)}{\log q_n} = \beta$. This completes the proof of Claim 3.

Claim 4. $\underline{\varrho}(A) = \gamma$ *and* $\overline{\varrho}(A) = 1$.

Proof of Claim 4. We start by writing the set A in the form $A = \{a_1, a_2, \ldots\}$ with $a_1 < a_2 < \ldots$. Now let

$$\mathbb{H} = \left\{ n \in \mathbb{N} \,\middle|\, a_n \in \bigcup_i (p_i, q_i] \;\text{ and }\; a_{n+1} \in \bigcup_i (p_i, q_i] \right\}.$$

Also, let

$$\mathbb{I} = \left\{ n \in \mathbb{N} \,\middle|\, a_n \in \bigcup_i (q_i, p_{i+1}] \;\text{ and }\; a_{n+1} \in \bigcup_i (q_i, p_{i+1}] \right\},$$

$$\mathbb{J} = \left\{ n \in \mathbb{N} \,\middle|\, a_n \in \bigcup_i (q_i, p_{i+1}] \;\text{ and }\; a_{n+1} \notin \bigcup_i (q_i, p_{i+1}] \right\},$$

$$\mathbb{K} = \left\{ n \in \mathbb{N} \,\middle|\, a_n \notin \bigcup_i (q_i, p_{i+1}] \;\text{ and }\; a_{n+1} \in \bigcup_i (q_i, p_{i+1}] \right\}.$$

We note that if $n \in \mathbb{H}$, then $a_{n+1} = a_n + 1$, and since it follows from Claim 2.(6) that $\mathbb{H}$ is unbounded, we therefore conclude that

$$\lim_{\substack{n \\ n\in\mathbb{H}}} \frac{a_n}{a_{n+1}} = 1\,. \tag{4.29}$$

Next, we observe that if $n \in \mathbb{I}$, then there is a positive integer l_n such that $a_n = [\frac{r_n}{\gamma_n^{l_n}}]$ and $a_{n+1} = [\frac{r_n}{\gamma_n^{l_n+1}}]$. Also, there are numbers u_n and v_n with $u_n, v_n \in [0,1]$ such that $\frac{r_n}{\gamma_n^{l_n}} = [\frac{r_n}{\gamma_n^{l_n}}] + u_n$ and $\frac{r_n}{\gamma_n^{l_n+1}} = [\frac{r_n}{\gamma_n^{l_n+1}}] + v_n$. Since it follows from Claim 2.(5) that the set $\mathbb{I}$ is unbounded, we therefore conclude that (using the fact that the sequences $(\gamma_n^{l_n+1}u_n)_n$ and $(\gamma_n^{l_n+1}v_n)_n$ are bounded and that $r_n \to \infty$)

$$\begin{aligned}\liminf_{\substack{n \\ n\in\mathbb{I}}} \frac{a_n}{a_{n+1}} &= \liminf_{\substack{n \\ n\in\mathbb{I}}} \frac{[\frac{r_n}{\gamma_n^{l_n}}]}{[\frac{r_n}{\gamma_n^{l_n+1}}]} = \liminf_{\substack{n \\ n\in\mathbb{I}}} \frac{\frac{r_n}{\gamma_n^{l_n}} - u_n}{\frac{r_n}{\gamma_n^{l_n+1}} - v_n} \\ &= \liminf_{\substack{n \\ n\in\mathbb{I}}} \frac{\gamma_n - \frac{\gamma_n^{l_n+1}u_n}{r_n}}{1 - \frac{\gamma_n^{l_n+1}v_n}{r_n}} = \liminf_{\substack{n \\ n\in\mathbb{I}}} \gamma_n = \gamma\,. \end{aligned} \tag{4.30}$$

Also, it follows from the definition of A that if $n \in \mathbb{J}$, then there is a positive integer i_n such that $a_n = p_{i_n}$ and $a_{n+1} = p_{i_n} + 1$, whence

$$\liminf_{\substack{n \\ n\in\mathbb{J}}} \frac{a_n}{a_{n+1}} = \liminf_{\substack{n \\ n\in\mathbb{J}}} \frac{p_{i_n}}{p_{i_n}+1} = 1\,. \tag{4.31}$$

Furthermore, it follows from the definition of A that if $n \in \mathbb{K}$, then there is a positive integer j_n such that $a_n = q_{j_n}$ and $a_{n+1} = [\frac{r_{j_n}}{\gamma_{j_n}^0}]$, whence

$$\liminf_{\substack{n \\ n\in\mathbb{K}}} \frac{a_n}{a_{n+1}} = \liminf_{\substack{n \\ n\in\mathbb{K}}} \frac{q_{j_n}}{[\frac{r_{j_n}}{\gamma_{j_n}^0}]} = \liminf_{\substack{n \\ n\in\mathbb{K}}} \frac{q_{j_n}}{N_{j_n} + q_{j_n}} \geq \liminf_{\substack{n \\ n\in\mathbb{K}}} \frac{q_{j_n}}{\frac{2}{1-\gamma_{j_n}} + 1 + q_{j_n}}\,. \tag{4.32}$$

Next, observe that $\frac{q_n}{\frac{2}{1-\gamma_n}+1+q_n} \to 1$; indeed, if $\gamma = 0$, then $\gamma_n \to 0$, and so $\frac{q_n}{\frac{2}{1-\gamma_n}+1+q_n} \to 1$; if $0 < \gamma < 1$, then $\gamma_n = \gamma$ for all n, and so $\frac{q_n}{\frac{2}{1-\gamma_n}+1+q_n} \to 1$; and finally, if $\gamma = 1$, then $|\frac{q_n}{\frac{2}{1-\gamma_n}+1+q_n} - 1| = |\frac{q_n}{2n^{\frac{\delta}{4}}+1+q_n} - 1| = \frac{2n^{\frac{\delta}{4}}+1}{2n^{\frac{\delta}{4}}+1+q_n} \leq \frac{2n^{\frac{\delta}{4}}+1}{2n^{\frac{\delta}{4}}+1+n} \to 0$ because $q_n \geq n$. Since $\frac{q_n}{\frac{2}{1-\gamma_n}+1+q_n} \to 1$, we now conclude from (4.32) that

$$\liminf_{\substack{n \\ n\in\mathbb{K}}} \frac{a_n}{a_{n+1}} = 1\,. \tag{4.33}$$

Finally, since clearly $\mathbb{N} = \mathbb{H} \cup \mathbb{I} \cup \mathbb{J} \cup \mathbb{K}$, it follows from (4.29) to (4.31) and (4.33) that $\underline{\varrho}(A) = \gamma$ and $\overline{\varrho}(A) = 1$ This completes the proof of Claim 4.

Combining Claims 3–4 proves the statement in Theorem 1.1 for $\beta > 0$.

□

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