

Normalisers of primitive permutation groups in quasipolynomial time

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Abstract

We show that given generators for subgroups G and H of S_n , if G is primitive then generators for $N_H(G)$ may be computed in quasipolynomial time, namely $2^{O(\log^3 n)}$. The previous best known bound was simply exponential.

1 Introduction and outline

The *normaliser problem* has as input generating sets X and Y for subgroups G and H of S_n , and asks one to return generators for $N_H(G)$. The asymptotically best known algorithm is due to Wiebking [16], and runs in simply exponential time $2^{O(n)}$. In practice, there are no efficient general algorithms to solve the normaliser problem.

Any generating set X for a subgroup G of S_n can be reduced to one of size at most n in time $O(|X|n^2 + n^5)$ (see Lemma 2.5(a)), so we shall generally assume that our generating sets have size at most n . A problem about subgroups of S_n , input via their generating sets, can be solved in *polynomial time* if there exists an algorithm to solve it in time bounded polynomially in n . It can be solved in *quasipolynomial time* if there exists an algorithm to solve it in time bounded by $2^{O(\log^c n)}$, for some absolute constant c .

Polynomial time solutions are known for many permutation group problems. However, in addition to the normaliser problem, polynomial time algorithms for computing set stabilisers, centralisers and intersections of permutation groups have so far proven elusive. Amongst these problems, the normaliser problem occupies a special place. Luks showed in [12] that the other problems listed above are polynomial time equivalent: we call their polynomial time equivalence class the *Luks class*. Luks also proved in [12] that the set-stabiliser problem is polynomial time reducible to the normaliser problem, but no reduction in the other direction is known.

Although the normaliser problem appears to be harder than the problems in the Luks class, they are not without similarities. Luks showed that the set-stabiliser problem is solvable in polynomial time if one restricts the composition factors of the group in question. This yielded a polynomial time algorithm for testing isomorphism of graphs with bounded degree [11]. Almost 30 years later, Luks and Miyazaki were able to show that the same holds for the normaliser

problem: if one restricts the composition factors of H then one can compute $N_H(G)$ in polynomial time [13].

The majority of our paper consists of the proof of the following theorem, concerning the solution of the normaliser problem in S_n .

Theorem 1.1. *Let $G = \langle X \rangle \leq S_n$ be a primitive permutation group. Then $N_{S_n}(G)$ can be computed in time $|X| \cdot 2^{O(\log^3 n)}$.*

Proof. This will follow from Theorem 3.4 if G is not large (see Definition 2.2), and from Corollary 4.8 if G is large. \square

The *string isomorphism* problem (as stated in [1, Definition 3.2]) asks one to find the elements σ in a group $G \leq S_n$ that map a function $f : \{1, \dots, n\} \rightarrow \Sigma$ to a function $g : \{1, \dots, n\} \rightarrow \Sigma$, where the action is $(i)f^\sigma = (i^{\sigma^{-1}})f$. In a dramatic breakthrough, Babai proved (see [1, 2]) that the string isomorphism problem can be solved in quasipolynomial time. Babai also gave a polynomial time reduction from the graph isomorphism problem to the string isomorphism problem, and hence showed that the graph isomorphism problem can also be solved in quasipolynomial time. The set stabiliser problem is a special case of the string isomorphism problem, where $f = g$ and $|\Sigma| = 2$, and so it also follows that all problems in the Luks class can be solved in quasipolynomial time.

Helfgott went on to prove a more precise bound in [9], namely that the string isomorphism problem can be solved in time $2^{O(\log^3 n)}$. Hence, given subgroups G and H of S_n , the intersection $G \cap H$ can be computed in time $2^{O(\log^3 n)}$. Our main result now follows immediately:

Theorem 1.2. *Let $G = \langle X \rangle \leq S_n$ and $H = \langle Y \rangle \leq S_n$ be permutation groups, with G primitive. Then $N_H(G)$ can be computed in time $(|X| + |Y|)2^{O(\log^3 n)}$.*

A *base* of a permutation group $G \leq S_n$ is a sequence B of elements of $\{1, \dots, n\}$, such that the pointwise stabiliser in G of B is trivial. We write $b(G)$ for the size of the smallest base for G . A technical tool in our proof, which may be of independent interest, is the following easy corollary of beautiful recent work [5, 8] of Duyan, Halasi, Liebeck and Maróti. This bound is noted in [8, Proof of Corollary 1.3], but we state and prove it in slightly more generality.

Corollary 1.3. *Let G be primitive and not large. Then*

$$b(G) \leq 2\lceil \log n \rceil + 26.$$

(Throughout the paper, logarithms are to base 2, unless specified otherwise.)

In future work, we plan to consider the normaliser problem for imprimitive groups, but the required techniques will be quite different. The proof of Theorem 1.1 relies on the fact that primitive groups have generating sets of size $\max\{2, \log n\}$ (see [10]), and either have a small base, as in Corollary 1.3, or have a very precisely specified structure: in general, imprimitive groups have no logarithmic bound on generating sets or base size.

The paper is structured as follows. In Section 2 we divide the primitive groups into small, large and almost simple groups, state the results from which

Corollary 1.3 will follow, and collect various well known facts about the complexity of permutation group algorithms. In Section 3 we present an algorithm to solve the normaliser problem in S_n when G is not large. In Section 4 we first show that the normaliser in S_n of the socle of any large primitive group of type PA can be computed in polynomial time. We then give a “moderately” quasipolynomial time algorithm (time $2^{O(\log n \log \log n)}$) to solve the normaliser problem in S_n when G is large and of type PA, and a polynomial time method when G is almost simple.

2 Small and large primitive groups

In this section we first divide the primitive groups into three (non-disjoint) families: small groups, large groups and almost simple groups. We then present some standard complexity results.

2.1 Small and large primitive groups

We write $\text{Sym}(\Omega)$ to denote the symmetric group on an arbitrary (finite) set Ω , and reserve S_n for the symmetric group on $\{1, \dots, n\}$.

Definition 2.1. Let $H \leq \text{Sym}(\Omega)$ and $K \leq \text{Sym}(\Gamma)$. A *permutation isomorphism* from H to K is a pair (f, ϕ) , where $f : \Omega \rightarrow \Gamma$ is a bijection, $\phi : H \rightarrow K$ is an isomorphism, and $(\alpha^h)f = (\alpha f)^{h\phi}$ for all $\alpha \in \Omega$ and $h \in H$.

Definition 2.2. Let $G \leq S_n$ be primitive. Then G is *large* if there exist natural numbers m, k and ℓ such that $n = \binom{m}{k}^\ell$, and G is permutation isomorphic to a group \widehat{G} with $A_m^\ell \leq \widehat{G} \leq S_m \wr S_\ell$, where the action of A_m and S_m is on k -element subsets of $\{1, \dots, m\}$ and \widehat{G} is in product action. (For a detailed description of product action, see [4, Section 4.5].)

Theorem 2.3 ([14, Theorem 1.1]). *Let $G \leq S_n$ be primitive. Then one of the following holds:*

- a) G is large;
- b) $G = M_{11}, M_{12}, M_{23}$ or M_{24} with their 4-transitive actions; or
- c) $|G| < n^{1+\lceil \log n \rceil}$.

We shall say that G is *small* if $|G| < n^{1+\lceil \log n \rceil}$. Thus each primitive group G is small, large, or almost simple.

We shall combine Theorem 2.3 with the following recent result, which builds on Duyan, Halasi and Maróti’s proof [5] of Pyber’s base size conjecture.

Theorem 2.4 ([8]). *Let $G \leq S_n$ be primitive. Then $b(G) \leq \frac{2 \log |G|}{\log n} + 24$.*

PROOF OF COROLLARY 1.3: Since G is primitive and not large, G falls in Case (b) or (c) of Theorem 2.3. If G is in Case (b) then $b(G) \leq 7$, and the result is immediate. If G is in Case (c) then we use Theorem 2.4 to see that

$$b(G) \leq \frac{2 \log |G|}{\log n} + 24 \leq \frac{2(1 + \lceil \log n \rceil) \log n}{\log n} + 24 \leq 2 \lceil \log n \rceil + 26$$

as required. □

2.2 Complexity preliminaries

The following results are classical: see, for example [6, 15]. Recall from Section 1 that if we say that a permutation group algorithm runs in *polynomial time*, we mean polynomial in the degree n .

Lemma 2.5. *Let $G = \langle X \rangle \leq S_n$.*

- a) *In time $O(|X|n^2 + n^5)$ we can replace X by a generating set for G of size at most n .*
- b) *Given a generating set for G of size at most n , in polynomial time we can: compute the stabiliser in G of any given point; compute a base and strong generating set for G ; show that G is primitive; compute $|G|$; find (at most n) generators for the socle $\text{soc } G$; and compute the composition factors of G .*
- c) *Given a base $B = (\beta_1, \dots, \beta_b)$ and a strong generating set for G , in time $O(n^2)$ we can test whether for a tuple $D \in \{1, \dots, n\}^b$ there exists $y \in G$ such that $B^y = D$, and if there exists such a $y \in G$ then determine y .*

Let $H = \langle a_1, \dots, a_k \rangle \leq S_m$ and $K = \langle b_1, \dots, b_\ell \rangle \leq S_n$. We encode a homomorphism $\varphi: H \rightarrow K$ by a list $[a_1, \dots, a_k, b_1, \dots, b_\ell, (a_1)\varphi, \dots, (a_k)\varphi]$. We say that φ is *given by generator images*.

The following lemma is standard, and follows from Lemma 2.5.

Lemma 2.6. *Let $G_i = \langle X_i \rangle \leq S_{n_i}$, for $i = 1, 2, 3$. Furthermore let $\varphi: G_1 \rightarrow G_2$ and $\psi: G_2 \rightarrow G_3$ be homomorphisms given by generator images. Then in time polynomial in $|X_1|n_1 + |X_2|n_2 + |X_3|n_3$ we can: evaluate φ ; compute $\varphi\psi$; compute φ^{-1} , if φ is an isomorphism; compute the restriction of φ to H_1 , for any subgroup H_1 of G_1 with given generators.*

We shall also need to be able to compute a permutation isomorphism when given a suitable group isomorphism. The following lemma is well known: see for example [4, Lemma 1.6B] for a statement equivalent to Part (a), and [13, Lemma 3.5] for a somewhat more general statement than Part (b).

Lemma 2.7. *Let $H = \langle A \rangle \leq \text{Sym}(\Omega)$ and $K = \langle B \rangle \leq \text{Sym}(\Gamma)$ both be transitive, and let $\varphi: H \rightarrow K$ be an isomorphism.*

- a) *There exists a bijection f such that (f, φ) is a permutation isomorphism from H to K if and only if φ maps each point stabiliser of H to a point stabiliser of K .*
- b) *If such an f exists, and φ is given by generator images, then f can be computed in time polynomial in $(|A| + |B|)|\Omega|$.*

3 The normaliser problem for small groups

In this section, we present an algorithm to compute the normaliser in S_n of a transitive group of degree n , in time exponential in $\log n$ and the sizes of a given

base and (not necessarily strong) generating set. For primitive groups which are not large we show that this yields a quasipolynomial time algorithm.

First we show that we can decide in polynomial time whether a primitive group G is small.

Lemma 3.1. *Let $G = \langle X \rangle \leq S_n$ be primitive, with $|X| \leq n$. We can decide in polynomial time whether G is small, and whether G is almost simple.*

Proof. By Lemma 2.5(b), we can verify that G is primitive and compute the order of G in polynomial time, and hence determine whether G is small. By the same lemma, we can also compute the composition factors of $\text{soc } G$ in polynomial time, and hence determine whether $\text{soc } G$ is non-abelian simple. \square

Next we show that for primitive groups that are not large we can find generating sets and bases of size $O(\log n)$ in quasipolynomial time.

Lemma 3.2. *Let $G = \langle X \rangle \leq S_n$ be primitive and not large, with $|X| \leq n$.*

a) *We can compute a generating set of size $t \leq \max\{\log n, 2\}$ for G in time $2^{O(\log^3 n)}$.*

b) *We can compute a base B of size $b \leq 2\lceil \log n \rceil + 26$ for G in time $2^{O(\log^2 n)}$.*

Proof. (a) By [10] the group G has such a generating set. The number of t -tuples of elements of G is at most $|G|^t$. If G is a Mathieu group then $|G|^t$ is bounded by a constant, and otherwise by Theorem 2.3

$$|G|^t < n^{t(1+\lceil \log n \rceil)} < 2^{2t \log^2 n} \leq 2^{2 \log^3 n}.$$

By Lemma 2.5(b), each such t -tuple can be tested for whether it generates G in polynomial time.

(b) By Corollary 1.3, such a B exists. There are fewer than $n^{2\lceil \log n \rceil + 26} \leq 2^{2 \log^2 n + 26 \log n}$ candidate b -tuples B of elements from $\{1, \dots, n\}$ to test. We can check, in polynomial time by Lemma 2.5(b), whether B is a base. \square

Our next result applies to all transitive groups, not just primitive ones.

Theorem 3.3. *Let $G \leq S_n$ be transitive. Assume that a generating set $X = \{x_1, \dots, x_t\}$ and a base $B = (\beta_1, \dots, \beta_b)$ for G are known. Then $N_{S_n}(G)$ can be computed in time $2^{O(tb \log n)}$.*

Proof. Let $\Omega = \{1, \dots, n\}$. We iterate over all possible choices of $A = (\alpha_1, \dots, \alpha_b) \in \Omega^b$, and $D = (\delta_{ij} : 1 \leq i \leq t, 1 \leq j \leq b) \in \Omega^{bt}$. For each such A and D we proceed as follows.

Let $\gamma_{ij} = \alpha_j^{x_i}$, and let $\bar{\sigma} : \alpha_j \mapsto \beta_j, \gamma_{ij} \mapsto \delta_{ij}$. We first test whether $\bar{\sigma}$ is a well-defined bijection from its domain $\Delta \subseteq \Omega$ to its image $\Gamma \subseteq \Omega$. To do so, we check whether the α_i are pairwise distinct, whether $\gamma_{i_1 j_1} = \gamma_{i_2 j_2}$ if and only if $\delta_{i_1 j_1} = \delta_{i_2 j_2}$, and whether $\gamma_{ij} = \alpha_k$ if and only if $\delta_{ij} = \beta_k$. If the answer is ever “no”, then we move on to the next choice of A and D . For a fixed A and D , this step requires time $O(n^2 t^2)$.

If $\sigma \in S_n$ satisfies $\sigma|_{\Delta} = \bar{\sigma}$, then $x_i^{\sigma} : \beta_j \mapsto \delta_{ij}$ for $1 \leq i \leq t$ and $1 \leq j \leq b$. Since B is a base, for $1 \leq i \leq t$ there is at most one $y_i \in G$ such that $B^{y_i} = (\delta_{i1}, \dots, \delta_{ib})$. The existence of such y_i , and their determination, can be calculated in time $O(n^2t)$ by Lemma 2.5(c). If for some i no corresponding y_i exists, then we move on to the next choice of A and D .

We now show how to determine the (unique) $\sigma \in S_n$ such that $x_i^{\sigma} = y_i$ for $1 \leq i \leq t$, or show that no such σ exists. The identity $\mu^{x_i\sigma} = \mu^{\sigma y_i}$ must hold for all $\mu \in \Delta$ and all $x_i \in X$. Since G is transitive, if $\Delta \subsetneq \Omega$ we can find $x_j \in X$ and $\mu \in \Delta$ with $\mu^{x_j} \notin \Delta$. Then $(\mu^{x_j})^{\sigma} = \mu^{\sigma y_j} = \mu^{\bar{\sigma} y_j} \notin \Gamma$. Hence we can define and check the image of one more point under σ by examining at most $|\Delta||X| = O(nt)$ images of points under permutations until we find a μ and x_j as specified. This must be carried out $O(n)$ times to fully specify σ , so the total time is $O(n^2t)$.

Hence, given A and D , in time $O(n^2t^2)$ we can either determine a corresponding element $\sigma \in N_{S_n}(G)$, or show that no such element exists. There are $n^{b+tb} = 2^{(t+1)b \log n}$ such sequences of elements A and D to test. Thus we can compute $N_{S_n}(G)$ in time $O(n^2t^2 \cdot 2^{(t+1)b \log n}) \subseteq 2^{O(tb \log n)}$. \square

Finally we present our main result for primitive groups that are not large.

Theorem 3.4. *Let $G = \langle X \rangle \leq S_n$ be primitive and not large. Then we can compute $N_{S_n}(G)$ in time $|X| \cdot 2^{O(\log^3 n)}$.*

Proof. By Lemma 2.5, in $O(|X|n^5)$ we can replace X by a generating set of size at most n , and in polynomial time we can compute a base and strong generating set for G . By Lemma 3.1 we can deduce that G is small or almost simple in polynomial time, and hence whether G is not large. The result now follows from Lemma 3.2 and Theorem 3.3. \square

4 The normaliser problem for large groups

Recall Definition 2.2 of a large primitive group. In this section we show that if G is large then we can construct $N_{S_n}(G)$ in polynomial time.

Definition 4.1. Let $G \leq S_n$ be primitive. The group G is of *type PA* if there exist an $\ell \geq 2$, an almost simple primitive group $A \leq \text{Sym}(\Delta)$ with socle T , and a group $\widehat{G} \leq \text{Sym}(\Delta^{\ell})$ permutation isomorphic to G with

$$\text{soc } G \cong T^{\ell} \leq \widehat{G} \leq A \wr S_{\ell},$$

in product action on Δ^{ℓ} .

From Definition 2.2, it is immediate that a large primitive group is either almost simple or of type PA.

4.1 Constructing the normaliser of the socle

In this subsection, we construct $N_{S_n}(\text{soc } G)$ when G is large and of type PA. We will require the following well known property: see [4, Lemma 4.5A], for example.

Lemma 4.2. *Let G be primitive of type PA. With the notation of Definition 4.1, $N_{\text{Sym}(\Delta^\ell)}(\text{soc } \widehat{G}) = N_{\text{Sym}(\Delta)}(T) \wr S_\ell$, in product action.*

Lemma 4.3. *Let $G = \langle X \rangle \leq S_n$ be a large primitive group of type PA, with $|X| \leq n$. Then we can compute a generating set of size four for $N_{S_n}(\text{soc } G)$ in polynomial time.*

Proof. Let $S = \text{soc } G$. Since G is large and of type PA, there exist an $\ell \geq 2$, m and k such that S is permutation isomorphic to A_m^ℓ acting component-wise on the set of ℓ -tuples of k -element subsets of $\{1, \dots, m\}$. Let Δ be the set of k -subsets of $\{1, \dots, m\}$, and write $A_{k,m}$ and $S_{k,m}$ for the actions of A_m and S_m on Δ . Let $\Gamma = \Delta^\ell$, and let $W = S_{k,m} \wr S_\ell \leq \text{Sym}(\Gamma)$, in product action. Let $A := \text{soc } W = A_{k,m}^\ell \leq \text{Sym}(\Gamma)$. Then S is permutation isomorphic to A . We shall proceed by first constructing the permutation isomorphism, and then using it to construct $N_{S_n}(S)$ as the pre-image of $N_{\text{Sym}(\Gamma)}(A)$. It is immediate from Lemma 4.2 that $N_{\text{Sym}(\Gamma)}(A) = W$.

We first compute a base and strong generating set for G , and find generators for, and the composition factors of, S . Hence we determine ℓ and m , and calculate k from $n = \binom{m}{k}^\ell$. This can all be done in polynomial time by Lemma 2.5.

We shall now proceed in three steps. First we will compute an isomorphism ι from S to A , and generators for $W = N_{\text{Sym}(\Gamma)}(A)$. Secondly, we will use ι to construct a permutation isomorphism (f, φ) from S to A . Finally, we shall use f to construct generators for $N_{S_n}(S)$.

Let $W_1 \leq S_{m\ell}$ be the wreath product of S_m and S_ℓ , in imprimitive action. We generate W_1 with a set Z of size four: two permutations acting non-trivially only on the first block, to generate one copy of S_m , and two permutations generating the top group S_ℓ . Note that $m\ell \leq |\Delta|^\ell \leq n$, so Z can be written down in polynomial time. By [3, Theorem 4.1] we can compute in polynomial time an isomorphism $\iota_{1,1}: G \rightarrow K$ for some $K \leq W_1$. Notice that $\text{soc } K = \text{soc } W_1$. We can write down the corresponding two generators for the action of $S_{k,m}$ on Δ , and lift their action on Δ to an action on $\Gamma = \Delta^\ell$ by letting them act trivially on all but the first components of Γ . Then we write down two generators for the action of S_ℓ on the components of Γ , corresponding to those in Z . Let Y be this set of four permutations. Mapping Z to Y yields an isomorphism by generator images $\iota_{2,1}: W_1 \rightarrow W = S_{k,m} \wr S_\ell \leq \text{Sym}(\Gamma)$. Since W acts on $|\Delta|^\ell = n$ points we can compute and evaluate $\iota_{2,1}$ in polynomial time, by Lemma 2.6. We let ι_1 and ι_2 be the restrictions of $\iota_{1,1}$ and $\iota_{2,1}$ to the socles of the groups concerned: we can compute the socles and restrictions in polynomial time by Lemma 2.5 and Lemma 2.6, respectively. Finally, we let $\iota = \iota_1 \iota_2$ be the isomorphism from S to A given by the composition of these maps.

The groups A and S are permutation isomorphic: denote one such permutation isomorphism by

$$(p, \psi), \text{ with } \psi: A \rightarrow S, \quad p: \Delta^\ell \rightarrow \{1, \dots, n\}, \quad (1)$$

and notice that $\psi \iota \in \text{Aut}(A)$.

We shall now use ι to construct a permutation isomorphism (f, φ) from S to A . By Lemma 2.7(b), it suffices to find a suitable φ , as then f can be

constructed in polynomial time. By Lemma 2.7(a), such an f exists if and only if φ maps the point stabilisers of S to the point stabilisers of A .

To determine φ , we first consider the homomorphism:

$$\Psi: W \rightarrow \text{Aut}(A), \quad x \mapsto (g \mapsto x^{-1}gx).$$

Recall that $N_{\text{Sym}(\Gamma)}(A) = W$. Thus $\text{Im}(\Psi)$ consists of exactly those automorphisms of $A = A_{k,m}^\ell$ which are induced by permutations of Γ .

Assume first that $m = 5$ or $m \geq 7$, so that Ψ is surjective. In this case we shall define $\varphi := \iota$. To see that φ maps point stabilisers to point stabilisers, let ψ be as in (1), so that $\psi\iota = \psi\varphi \in \text{Aut}(A)$. Then there exists an element $w \in W \leq \text{Sym}(\Gamma)$ with $w\Psi = \psi\varphi$. Hence $\psi\varphi$ permutes the point stabilisers of A , and so by Lemma 2.7(a) the map φ induces a permutation isomorphism.

Now let $m = 6$, so that the image of $S_{k,6}$ under $\Psi_0: S_{k,6} \rightarrow \text{Aut}(A_{k,6})$ has index 2 in $\text{Aut}(A_{k,6})$. Correspondingly the index $[\text{Aut}(A) : W\Psi] = 2^\ell$. In constant time we can compute generator images for an involutory automorphism $\tau \in \text{Aut}(A_{k,6}) \setminus (S_{k,6}\Psi_0)$. Hence we can compute generator images for $\tau_1, \dots, \tau_\ell \in \text{Aut}(A)$, where each τ_i induces τ on a distinct direct factor of A , and the identity on the other factors. Notice that $L := \langle \tau_1 \dots, \tau_\ell \rangle \cong C_2^\ell$, so that $|L| = 2^\ell \leq n$.

Let (p, ψ) be as in (1), so that $\psi\iota \in \text{Aut}(A)$. Then there exists a $\theta \in L$ such that $\psi\iota\theta \in W\Psi$. In particular, $\psi\iota\theta$ maps point stabilisers to point stabilisers, and hence by Lemma 2.7(a) $\iota\theta$ does so too. We can test in polynomial time whether the image of a point stabiliser is a point stabiliser, so we can check each of the $O(n)$ elements of L to find such a θ in polynomial time. We then set $\varphi = \iota\theta$.

Finally, the bijection f^{-1} induces an isomorphism from $\text{Sym}(\Gamma)$ to S_n that maps $W = S_{k,\ell} \wr S_\ell$ to the normaliser $N_{S_n}(S)$. Recall the set Y of generators for W . We can compute Yf^{-1} in polynomial time, and $\langle Yf^{-1} \rangle = N_{S_n}(S)$. \square

4.2 Computing the normaliser in S_n of a large group

Many of our results in this subsection apply directly to all primitive groups G that are either almost simple or of type PA. Our approach yields a “very moderately” quasipolynomial time algorithm: $O(2^{\log n \log \log n})$.

Throughout this subsection, let $G \leq S_n$, and let $M = N_{S_n}(\text{soc } G)$. We first relate the complexity of computing $N_{S_n}(G)$ to the index $[M : G]$.

Lemma 4.4. *Let $G = \langle X \rangle \leq S_n$ be primitive, with $|X| \leq n$. Assume that a base and strong generating set for G are known. Furthermore, let a generating set Y for M be given, with $|Y| \leq n$. Then we can compute the normaliser $N_{S_n}(G)$ in time $O(n^3[M : G]^2)$.*

Proof. First notice that $\text{soc } G$ is normal in $N_{S_n}(G)$, and so $N_{S_n}(G)$ is a subgroup of M . Since $G \leq M$, to compute $N_{S_n}(G)$ it suffices to test only the representatives of the right cosets $G \backslash M$ of G in M .

We shall compute the representatives of $G \backslash M$ by a standard orbit algorithm. To determine equality of cosets Ga and Gb we test whether $ab^{-1} \in G$, in

time $O(n^2)$ by Lemma 2.5(c). The cost of the computation of these coset representatives is then the product of the number of generators $|Y|$, the size of the orbit $[M : G]$, and the cost $[M : G] \cdot n^2$ for testing whether a representative of a newly computed coset $(Ga)y$ for $y \in Y$ is contained in one of the up to $[M : G]$ already computed cosets. In total this takes $O(|Y| \cdot n^2 \cdot [M : G] \cdot [M : G]) = O(n^3[M : G]^2)$ time.

Next, for h a coset representative, we can test whether $x^h \in G$ for all $x \in X$ in time $O(n^3)$. We can therefore test all such representatives h in time $O(n^3 \cdot [M : G])$. The result follows. \square

Luks and Miyazaki showed in [13, Corollary 3.24] that normalisers in S_n of non-abelian simple groups can be computed in polynomial time. We very slightly extend their argument, to almost simple groups: the following result does not require transitivity.

Lemma 4.5. *Let $G = \langle X \rangle \leq S_n$ be almost simple, with $|X| \leq n$. Then we can compute $N_{S_n}(G)$ in polynomial time.*

Proof. By Lemma 2.5(b), we can compute a base and strong generating set for G , and generators for $T := \text{soc } G$, in polynomial time.

By [13, Lemma 3.3] we can compute $C := C_{S_n}(G)$, and then arguing as in the proof of [13, Corollary 3.24] we can determine in polynomial time which elements of $\text{Aut } T$ are induced by $N_{S_n}(T)$, and construct an almost simple group A with socle T which induces all of these automorphisms. Let $M = \langle C, A \rangle$. Then $M = N_{S_n}(T)$. By [7, Lemma 7.7], there exists an absolute constant κ such that $|\text{Out } T| \leq \kappa\sqrt{n}$. Hence $[M : G] \leq \kappa\sqrt{n}$. The result is now immediate from Lemma 4.4. \square

Next we consider the normalisers of the socles of groups of type PA.

Proposition 4.6. *There exists a constant c such that for all groups $G \leq S_n$ of type PA*

$$[M : \text{soc } G] \leq 2^{c \log n \log \log n}.$$

Proof. Let $T \leq \text{Sym } \Delta$ be a non-abelian simple group such that $S := \text{soc } G$ is permutation isomorphic to $T^\ell \leq \text{Sym } \Delta^\ell$. Then $M \cong N_{\text{Sym } \Delta}(T) \wr S_\ell$, by Lemma 4.2. Now $\ell = \log_{|\Delta|} n \leq \log n$ and by [7] there exists a constant κ such that $|\text{Out } T| \leq \kappa\sqrt{|\Delta|} = \kappa n^{1/(2\ell)}$. Let $\lambda = \log \kappa$. Then

$$\begin{aligned} [M : S] &\leq |\text{Out } T|^\ell |S_\ell| \leq \left(\kappa n^{1/(2\ell)}\right)^\ell \cdot \ell! \leq 2^{\lambda \log n} \cdot n^{1/2} \cdot (\log n)^{\log n} \\ &\leq 2^{\lambda \log n + (1/2) \log n + \log n \log \log n} = 2^{c \log n \log \log n}, \end{aligned}$$

for a suitable choice of c . \square

Proposition 4.7. *Let $G = \langle X \rangle \leq S_n$ be primitive of type PA, with $|X| \leq n$. If a generating set Y of size at most n is known for M , then we can compute the normaliser $N_{S_n}(G)$ in time $2^{O(\log n \log \log n)}$.*

Proof. By Lemma 2.5(b) we can compute a base and strong generating set for G in polynomial time. Given this data and Y , by Lemma 4.4 we can compute $N_{S_n}(G)$ in time $O(n^3[M : G]^2)$. By Proposition 4.6 the index $[M : G] \leq 2^{c \log n \log \log n}$ for some constant c . The result follows. \square

The following corollary completes the proof of Theorem 1.1.

Corollary 4.8. *Let $G = \langle X \rangle \leq S_n$ be a large primitive group. Then the normaliser $N_{S_n}(G)$ can be computed in time $|X| \cdot 2^{O(\log n \log \log n)}$.*

Proof. By Lemma 2.5 we can replace X by a set of at most n generators in $O(|X|n^2 + n^5)$, and we can determine whether G is almost simple in polynomial time. If G is almost simple, then we can compute $N_{S_n}(G)$ in polynomial time, by Lemma 4.5. Otherwise, G is of type PA. By Lemma 4.3 we can construct a generating set of size four for $M := N_{S_n}(\text{soc } G)$ in polynomial time. Hence by Proposition 4.7 we can compute $N_{S_n}(G)$ in time $2^{O(\log n \log \log n)}$. \square

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