Eductive Stability in Real Business Cycle Models*

George W. Evans
University of Oregon,
University of St. Andrews

Roger Guesnerie
Paris School of Economics
Collège de France

Bruce McGough
University of Oregon

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Abstract

Within the standard RBC model we examine issues of expectational coordination on the unique rational expectations equilibrium. We show the sensitivity of agents' plans and decisions to their short-run and long-run expectations is too great to trigger eductive coordination in a world of rational agents who are endowed with knowledge of the economic structure and contemplate the possibility of small deviations from equilibrium: eductive stability never obtains. We conclude adaptive learning must play a role in real-time dynamics. Our eductive instability theorem has a counterpart under adaptive learning: even with asymptotic stability the transition dynamics can involve large departures from rational expectations.

JEL Classifications: D84, D83, E32

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1 Introduction

The question of expectational coordination has returned to the forefront of intellectual debate in the context of the economic crisis that began in 2007. Some people have argued that the overly optimistic view of the world conveyed by our models arises in particular from the universal adoption of rational expectations (RE). In fact, the axiomatic use of RE has been under critical examination for some time: what is at stake in this debate is the robustness of expectational coordination on RE. This issue, the subject of the present paper, has been discussed using a variety of approaches.

Problems of multiplicity of rational expectations equilibria (REE) have received considerable attention, as in the sunspot literature, e.g. Benhabib and Farmer (1994) and Chiappori and Guesnerie (1991), and in the global games literature, e.g. Morris and Shin (1998). The existence of multiple REE arises also in the New Keynesian model, most dramatically in connection with the zero lower bound, e.g. Benhabib, Schmitt-Grohe and Uribe (2001). The possibility of multiple REE most obviously and forcefully raises the issue of coordination of expectations.

However, coordination of expectations on RE is crucial also in models with a unique REE. Rational expectations must be viewed as an equilibrium concept. In game theoretic terms the REE is a Nash equilibrium, not a dominant strategy equilibrium. Put differently, it is rational to have RE only if the expectations of other agents are in accordance with RE. This unavoidably leads to the question of how agents would come to coordinate on the REE.

One approach to this question examines the stability of the REE under adaptive learning. See, for example, Marcet and Sargent (1989), Woodford (1990), Brock and Hommes (1997) and Evans and Honkapohja (2001). Most prominently, under the least-squares learning approach, agents form expectations like statisticians or econometricians, updating their forecasting model over time as new data become available. This is a "bounded rationality" approach because the agents do not explicitly take account of the self-referential aspect of the model; and, consequently, agents can learn to have RE only asymptotically as they update their forecast models to observed data over time.

A second approach, called the "eductive" viewpoint, is the main focus of the current paper. See Evans and Guesnerie (1993, 2005) and Guesnerie (2002) for an introductory conceptual assessment in dynamic models. Under this approach, agents are assumed to be rational and to fully understand the structure of the economy. These authors argue that in some cases a mental reasoning process allows agents to deduce that the REE will obtain, in which case the REE is said to be eductively stable. This would provide strong support for RE, because RE is then attainable through mental reasoning and does not require the gradual accumulation of data over time.

Consider, for example, an unexpected permanent change in fiscal policy. One of the attractions of the RE approach is that, in the event of such a structural change, agents fully incorporate into their decision-making the resulting future changes in the equilibrium paths of wages and interest-rates. Eductive stability of the equilibrium would provide support for the assertion that coordination on this revised RE path could be quickly achieved by our fully rational agents: the usual textbook RE solutions would be fully vindicated.

In this paper we focus on eductive stability and expectational coordination in models in which agents are long-lived, or more precisely, far-sighted. The logical framework and the central results for eductive learning, as well as the connections with adaptive (or "evolutive") learning and the issue of sunspot multiplicity, are well understood in many contexts where agents are short-lived, such as simple overlapping generations models: see Gauthier and Guesnerie (2004). With long-lived agents, and this is a key message of our findings in this paper, the picture changes significantly.

The assumption that agents are long-lived is not economically innocuous. For example, long-lived agents take into account their lifetime income rather than income over a short horizon, a fact that of course has a key impact on the understanding and design of macroeconomic policies. This suggests, unless the RE-hypothesis is accepted axiomatically, that the effect of the presence of long-lived agents on eductive stability should receive careful attention. Does this make expectational coordination more or less robust?

To explore this complex question, we consider in this paper the simplest version of a Real Business Cycle (RBC) model and its standard focal point, the REE, which is presented in Section 2.1. In Section 2.2, we investigate a background question that is central for any assessment of the plausibility of expectational coordination on the REE: how do actions and plans of agents react to out-of-equilibrium beliefs? The answer, for small changes of beliefs, is provided by the planning theorem, Theorem 1, which appears to be new to the literature. This theorem identifies the changes in consumption and savings decisions associated with a small change in the whole infinite trajectory of beliefs, providing rich and comprehensive information on this question. We obtain a key expectational feedback parameter ξ, depending on the underlying model specification, which governs the response of plans to beliefs.

In Section 3, we develop the eductive approach. We define (local) eductive stability in an abstract setting, illustrate it using the cobweb model, and then show how to apply the framework to the RBC model. The question is whether rationality and knowledge of the economic structure can trigger a mental reasoning process leading to coordination on the REE. The reasoning process we envision is as follows. Suppose agents believe that the time-path of the economy will be in a neighborhood of the REE, and they consider the set of possible outcomes based on plans consistent with those beliefs. If this set of outcomes lies in a strictly smaller neighborhood of the REE then it would be natural for rational agents to revise their beliefs and reconsider the implied outcomes. Eductive coordination obtains if iteration then selects the REE.

We find that coordination on the REE in the RBC model is difficult. Indeed, Theorem 2 of Section 4 establishes the impossibility of local eductive stability. Thus, complete coordination on the REE time path cannot be expected. There is no collective view of the future – no natural set of paths close but not identical to the self-fulfilling, equilibrium path – that is able to trigger coordination on the equilibrium, along the eductive lines just described. Taking a real-time perspective, every such collective view is subject, at some stage, to be invalidated by facts. Put more dramatically, in this world, a "crisis," here an expectational crisis, is unavoidable, although the time required for the crisis to become manifest, i.e. the extent of the weakness of expectational coordination, depends upon expectational feedback parameter ξ.

What should we conclude from the instability results of Section 4? The REE remains the natural benchmark since it is uniquely the one in which agents would be unable to improve on their forecasts. However, the clear implication of eductive instability is that coordination on the REE can only be attained by an adaptive learning process, a process in which expectations respond to data over time. Returning to our hypothetical example of an unanticipated novel policy change that disturbs an RE steady state, renewed expectational coordination after the structural break cannot be immediate and can only take place over time.

Section 5 therefore takes up adaptive learning. Although it is known that the REE in the RBC model can be asymptotically stable under adaptive learning rules, our objective is to determine the links between our eductive instability result and adaptive learning dynamics. The planning theorem of Section 2 again plays a central role. In addition, drawing inspiration from the framework of Section 3, we develop a new stability notion for adaptive learning that emphasizes the extent of deviation from RE.

Specifically, in Section 5 we examine what we call "B-stability": consistency of the realized path with initial beliefs that the path will remain in a specified neighborhood of the equilibrium. As is well-known, adaptive learning dynamics depend on the "gain" parameter $0 < \alpha \leq 1$ that governs the extent to which expectations respond to observed data. The equilibrium paths under learning depend both on α and ξ.

Section 5 obtains both negative results, reflecting the earlier eductive instability results, and some more positive results that reflect the potential for stability under adaptive learning. The negative result, Theorem 3, is that the REE is never robustly B-stable under adaptive learning: the consistency of conjectured beliefs paths is not robust in the sense that this collective view of the future will be falsified, at some point in time, for a large set of adaptive learning rules indexed by the gain α. In particular, despite the REE being asymptotically stable for sufficiently small α, the deviation of the learning dynamics from the REE can become arbitrarily large. Theorem 4 provides more complete results on asymptotic and partial B-stability.

The results of Section 4, complemented by those of Section 5, indicate that full coordination on the REE is implausible: starting from an RE steady state, if there is an unanticipated structural or policy change, then, even if the nature of this structural change is fully understood, it will take time for agents to coordinate on the REE, and deviations during the learning transition can be large. The concluding Section 6 discusses directions for future research.

2 Model and Planning Theorem

We consider a standard RBC model, except that for simplicity we assume a fixed labor supply and omit exogenous productivity shocks.[1] These simplifications, which amount to a focus on a nonstochastic discrete-time Ramsey model, are not critical to our results and are made in order to clarify the central features of our analysis. Elimination of both random shocks and labor-supply response to disequilibrium expectations can be expected to facilitate coordination on the rational expectations equilibrium (REE).[2] Despite eliminating these influences we establish that eductive stability fails.

2.1 The model and equilibrium

There is a unit mass of identical infinitely-lived households, indexed by $\omega \in I$. At time t each household ω holds capital $k_t(\omega)$, resulting from previous decisions, and supplies inelastically one unit of labor. At time $t = 0$, facing interest rate r_0 and prospects of future interest rates r_t and wages q_t, household

[1] Seminal RBC papers include Kydland and Prescott (1982) and Long and Plosser (1983).

[2] For example, the expectational feedback parameter, which we denote by ξ below, can be shown to be larger in magnitude when labor supply is elastic.

ω determines his actions today and plans for the future by solving

$$\max \quad E_0(\omega)\sum_{t=0}^{\infty}\beta^t U(c_t(\omega)), \text{ where } 0<\beta<1, \tag{1}$$

$$\text{subject to} \quad k_{t+1}(\omega) = (1+r_t)k_t(\omega) + q_t - c_t(\omega), \tag{2}$$

with initial wealth $k_0(\omega)$ given. Here $E_0(\omega)$ captures the expectations of agent ω formed using his subjective distribution.

We will focus on the case in which $k_0(\omega)$ is the same for all agents, but we do not impose this initially. The utility function $U(c)$ is increasing, strictly concave and smooth. We further impose a No Ponzi Game (NPG) condition that the present value of their limiting lifetime wealth be nonnegative. The household's Euler equation is

$$U'(c_t(\omega)) = \beta E_t(\omega)\left((1+r_{t+1})U'(c_{t+1}(\omega))\right). \tag{3}$$

When the future is viewed as deterministic, expectations of future interest rates r_t and wages q_t are point expectations. Iterating forward the household flow budget constraint, and imposing the NPG and transversality conditions, gives the lifetime budget constraint of the household:

$$\sum_{t=0}^{\infty} R_t c_t(\omega) = \sum_{t=0}^{\infty} R_t q_t + (1+r_0)k_0(\omega), \text{ where } R_t = \prod_{i=1}^{t}(1+r_i)^{-1},\ R_0 = 1. \tag{4}$$

Goods are produced by firms from capital and labor using a constant returns to scale production function $f(K,L)$, satisfying the usual assumptions, under conditions of perfect competition. In particular, firms hire labor and rent capital in competitive factor markets. Because labor is inelastically supplied, r_t and q_t are given by $r_t = f_K(K_t,1) - \delta$ and $q_t = f_L(K_t,1)$, where $K_t = \int_I k_t(\omega)d\omega$, $f_K = \partial f/\partial K$, $f_L = \partial f/\partial L$, and $0 \leq \delta \leq 1$ is the depreciation rate. It follows that prices in time t are entirely determined by the level of aggregate capital K_t, which itself is predetermined, i.e. determined by aggregate savings in period $t-1$.

For convenience, below, we also write $f(K)$ in place of $f(K,1)$ and use the notation $f' = f_K$ and $f'' = f_{KK}$. In addition we have the aggregate capital accumulation equation

$$K_{t+1} = (1-\delta)K_t + f(K_t) - C_t, \text{ where } C_t = \int_I c_t(\omega)d\omega. \tag{5}$$

In this model, the definition of an intertemporal perfect foresight equilibrium requires that agents satisfy their Euler equations and flow budget constraints at each t, their intertemporal budget constraint, and that their expectations are consistent with the actual path of the economy. The model has a unique a (unique) perfect foresight steady state, on which we focus attention.

Definition 1 *The perfect foresight steady state* $K_t = k_t(\omega) = \bar{K}$, $C_t = c_t(\omega) = \bar{C}$, $r_t = \bar{r}$ *and* $q_t = \bar{q}$ *is given by* $1 = \beta(1+\bar{r})$, $\bar{r} = f_K(\bar{K}, 1) - \delta$, $\bar{q} = f_L(\bar{K}, 1)$, $\bar{C} = f(\bar{K}, 1) - \delta\bar{K} = \bar{r}\bar{K} + \bar{q}$.

If $K_0 = \bar{K}$ then under perfect foresight the economy stays in the steady state for all t, and if $K_0 \neq \bar{K}$, then there is a unique perfect foresight path that converges to the steady state as $t \to \infty$. We now assume that the economy is initially in the steady state, with $k_0(\omega) = \bar{K}$ for all ω, and we examine the robustness of expectational coordination on this equilibrium.

2.2 Beliefs, actions, plans and realizations

Consider an individual agent facing the consumption/savings problem (1)-(2). The behavior of the agent is in part determined by his beliefs about the future values of wages and interest rates. In general, an agent's beliefs could be stochastic, summarized by a sequence of joint density functions $\{F_t(q^t, r^t)\}$, where q^t and r^t are the time t wage and interest rate histories, respectively. We focus on deterministic beliefs, i.e. on point expectations, for reasons given later. This is natural given that our reference solution is nonstochastic and our analysis is local; also, this facilitates tractability.

The beliefs of agent ω may be summarized by real sequences of expected wages and interest rates. We assume the agent understands the relationship between aggregate capital and input prices, that is, the agent knows $r_t = f'(K_t) - \delta$, and $q_t = f(K_t) - f'(K_t)K_t$. Hence his beliefs are completely and consistently captured by a sequence identifying his point expectations of future capital stock. We denote these beliefs by $K^e(\omega) = \{K^e_t(\omega)\}_{t \geq 1}$. A beliefs profile is the collection of all agents' beliefs: $K^e = \{K^e(\omega) : \omega \in I\}$.

The key ingredient of our analysis is the understanding of the effect of changes in individual expectations on changes in individual actions or plans (particularly when these changes occur near the equilibrium). Taking as reference point the perfect foresight steady-state path $K_0 = \bar{K}, K_t = \bar{K}$, for all t, we examine the solution to the program, which gives the agent's present actions and future plans. This is simply the program (1) subject to (4), where q_t and r_t are replaced by $q^e_t(\omega)$ and $r^e_t(\omega)$, derived as just explained from $K^e_t(\omega)$.

We focus on small changes, around the steady-state values, in the individual agent's initial capital $k_0(\omega)$ and point expectations $K^e_t(\omega)$, and on the effect of these small changes on the agent's initial plans. We measure these small changes in capital as deviations from steady state: we set $dk_0(\omega) = k_0(\omega) - \bar{K}$ and $dK_t^e(\omega) = K_t^e(\omega) - \bar{K}$. We also write $dK^e(\omega) = \{dK_t^e(\omega)\}_{t\geq 1}$ for the beliefs path of agent ω, and $dK^e = \{dK^e(\omega) : \omega \in I\}$. Similarly, for the agent's corresponding optimal plans $k_t(\omega)$ and $c_t(\omega)$, which as we will see are fully determined by their beliefs and their initial capital holdings, we write $dk(\omega) = \{dk_t(\omega)\}_{t\geq 1}$ where $dk_t(\omega) = k_t(\omega) - \bar{K}$ and $dc(\omega) = \{dc_t(\omega)\}_{t\geq 0}$ where $dc_t(\omega) = c_t(\omega) - \bar{C}$. At this stage we maintain the assumption that each agent's plans are well captured by first-order approximations. We justify this assumption later. Given this assumption, we can identify a particular variable's time path with its first-order approximation, and thus use our deviation notation to capture this identification.

2.2.1 Expectations and plans of agents

Our objective is to determine the agent's plans as functions of the agent's expectations and initial savings. We begin with two fundamental lemmas.

Lemma 1 *Given a time path of beliefs $\{dK_t^e(\omega)\}_{t\geq 1}$, the optimal consumption and savings paths for agent ω satisfy*

$$\begin{aligned} dc_{t+1}(\omega) &= dc_t(\omega) + \sigma^{-1}\beta\bar{C}f'' dK_{t+1}^e(\omega) & (6)\\ dk_{t+1}(\omega) &= \beta^{-1}dk_t(\omega) - dc_t(\omega), & (7) \end{aligned}$$

where $\sigma = -\bar{C}U''(\bar{C})/U'(\bar{C})$ is the consumption elasticity of marginal utility and where f'' denotes $f''(\bar{K})$.

Lemma 2 *(Welfare Lemma). Given a time path of beliefs $\{dK_t^e(\omega)\}_{t\geq 1}$, initial saving $dk_0(\omega)$ and aggregate capital stock dK_0, the optimal consumption path for agent ω satisfies*

$$\beta^{-1}dk_t(\omega) = \sum_{s\geq 0}\beta^s dc_{t+s}(\omega), \text{ for all } t \geq 0. \qquad (8)$$

The proofs of these and all results are in the Appendix. Lemma 1 specifies the agent's Euler equation and flow budget constraint to first order. We remark that the simple form of linearized budget constraint (7) arises from the absence of an income effect, i.e. in a competitive equilibrium $dq_t + \bar{K}dr_t = 0$. Lemma 2, which is derived using this observation and the linearized life-time budget constraint (LTBC), can be made stronger: it is not necessary that the consumption path be optimal, it need only satisfy the LTBC. However, an important implication of Lemma 2 is that, to first order, changes in beliefs do not impact welfare: $dU(\omega) = \sum_{t\geq 0} \beta^t U'(\bar{C}) dc_t(\omega) = 0$.

It is useful to define the "expectations feedback" parameter

$$\xi = -\sigma^{-1}(1-\beta)^{-1}\beta^2 \bar{C} f'' > 0,$$

which will be of considerable interest, as it plays a central role in determining the responsiveness of consumption and savings plans to changes in expectations of the aggregate capital path.

An implication of the preceding results is that an agent's optimal time t consumption $dc_t(\omega)$ and saving $dk_{t+1}(\omega)$ are functions of his future beliefs $\left\{dK^e_{t+n}(\omega)\right\}_{n\geq 1}$ and his realized time t capital holdings $dk_t(\omega)$:

Lemma 3 *Given beliefs path $dK^e(\omega)$, the optimal plans for agent ω satisfy*

$$dc_t(\omega) = \beta^{-1}(1-\beta)dk_t(\omega) + \beta^{-1}(1-\beta)\xi \sum_{n\geq 1} \beta^n dK^e_{t+n}(\omega) \tag{9}$$

$$dk_{t+1}(\omega) = dk_t(\omega) - \beta^{-1}(1-\beta)\xi \sum_{n\geq 1} \beta^n dK^e_{t+n}(\omega). \tag{10}$$

Finally we give the planning theorem, which specifies the optimal plans $dk(\omega) = \{dk_t(\omega)\}_{t\geq 1}$ and $dc(\omega) = \{dc_t(\omega)\}_{t\geq 0}$ as functions just of the beliefs path $dK^e(\omega)$. The planning theorem is most conveniently expressed in terms of the following semi-infinite matrices:

$$\Theta^k = \begin{pmatrix} 1-\beta & \beta-\beta^2 & \beta^2-\beta^3 & \cdots \\ 1-\beta & 1-\beta^2 & \beta-\beta^3 & \cdots \\ 1-\beta & 1-\beta^2 & 1-\beta^3 & \cdots \\ 1-\beta & 1-\beta^2 & 1-\beta^3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ and } \Theta^c = \begin{pmatrix} -\beta & -\beta^2 & -\beta^3 & \cdots \\ 1-\beta & -\beta^2 & -\beta^3 & \cdots \\ 1-\beta & 1-\beta^2 & -\beta^3 & \cdots \\ 1-\beta & 1-\beta^2 & 1-\beta^3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

that is, $\Theta^k = \left(\Theta^k_{ij}\right)$ and $\Theta^c = \left(\Theta^c_{ij}\right)$, where

$$\Theta^k_{ij} = \begin{cases} \beta^{j-i} - \beta^j & \text{if } 1 \leq i < j \\ 1-\beta^j & \text{if } i \geq j \end{cases} \text{ and } \Theta^c_{ij} = \begin{cases} -\beta^j & \text{if } 1 \leq i \leq j \\ 1-\beta^j & \text{if } i > j \end{cases}.$$

Recalling that $dk(\omega) = \{dk_t(\omega)\}_{t\geq 1}$ and $dc(\omega) = \{dc_t(\omega)\}_{t\geq 0}$, agent ω's optimal plan in terms of his belief path $dK^e(\omega)$, is given by the following theorem:

Theorem 1 *(Planning Theorem). Assume $dk_0(\omega) = 0$. Given beliefs path $dK^e(\omega)$, the optimal plans for agent ω are given by*

$$\begin{aligned} dk(\omega) &= \Gamma^k\left(dK^e(\omega)\right) \equiv -\xi\Theta^k dK^e(\omega), \\ dc(\omega) &= \Gamma^c\left(dK^e(\omega)\right) \equiv -\xi\beta^{-1}(1-\beta)\Theta^c dK^e(\omega). \end{aligned}$$

Figure 1 illustrates the results for $dK_N^e(\omega) > 0$ and $dK_t^e(\omega) = 0$ for $t \neq N$. Note that the impact on $dk_t(\omega)$ is negative for all $t \geq 1$ while the impact on $dc_t(\omega)$ changes sign over time. We emphasize this point in the next section.

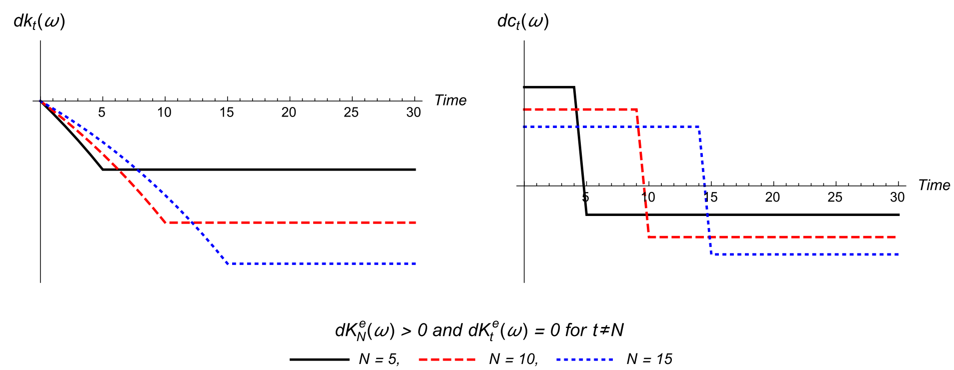

Figure 1: Impact on planned $dk_t(\omega), dc_t(\omega)$ of a future one-period increase of $dK_N^e\ (\omega)$ for alternative values of N.

Intuitively, $dK_N^e(\omega) > 0$ and $dK_t^e(\omega) = 0$ for $t \neq N$ implies that expected dr_N is negative while $dr_t = 0$ for $t \neq N$. From the Euler equation (3) consumption will be constant both before and after $t = N$, and from period $N - 1$ to period N consumption must fall. The lifetime budget constraint (4) implies that $dc_t(\omega) > 0 > dc_s$ for $t < N \leq s$: the low interest rate in period N induces households to move consumption from later to earlier periods. To understand the corresponding path of savings $dk_t(\omega)$, observe that to support the relatively high level of consumption for $t < N$, households must reduce their savings over time; after period N, facing interest rates that remain at the steady-state level, households consume an amount equal to their total labor and capital earnings and hence their savings remain constant.

2.2.2 Implications of the planning theorem

The planning theorem, through the expectations feedback parameter ξ and the matrices Θ^c and Θ^k, fully describes the connection between expectations and the planned actions of the individual agents. We will later exploit this knowledge to obtain consequences of these beliefs for describing the possible corresponding paths of aggregate capital. At this stage we provide intuition for later results, limiting attention to discussion of the effects of beliefs on the plans of individual agents.

First we note that the *general sensitivity* of decisions to expectations is governed by the value of ξ. A high ξ means that the individual decisions and plans react strongly to expectations. We discuss later the interpretation of ξ, but it is intuitively clear that a high f'' or a low σ increases expectational sensitivity. In addition to this general expectational sensitivity, our results also stress *the profile effect* of beliefs, which are fully captured by the above matrices. Taken together, we get rich information – indeed exhaustive to first-order approximation – giving the effects of beliefs on the agent's plans.

This information can be used to provide a more comprehensive intuition about agents' plans. Here we pick out three insights,[3] Corollaries 1 – 3, which are particularly relevant for what follows.

First, as foreshadowed above, a striking and immediate implication from inspecting Θ^k is the following corollary on strategic substitutability.[4]

Corollary 1 *The map Γ^k exhibits local strategic substitutability, i.e.*

$$\frac{\partial \left(dk_t(\omega)\right)}{\partial \left(dK_T^e(\omega)\right)} < 0 \text{ for all } t, T \geq 1.$$

Second, consider beliefs of the form $-e < dK_t^e(\omega) < e$, for all $t \in \mathbb{N}$, where $e > 0$. Implications for savings in period one, $dk_1(\omega)$, which then follow from the first row of Θ^k, are given by Corollary 2.

Corollary 2 *If $-e < dK_t^e(\omega) < e$ for all $t \geq 1$ then $-\xi e < dk_1(\omega) < \xi e$.*

Third, the connection between the infinite path of beliefs and the individual savings decision is further clarified by the following example. Suppose that expected capital is above the steady state by a fixed amount $e > 0$ for $N > 0$ periods before reverting to the steady state. That is, suppose the agent ω's beliefs profile, $dK^e(\omega)$, satisfies

$$dK_T^e(\omega) = \begin{cases} e & \text{for} \quad T = 1, \ldots, N \\ 0 & \text{for} \quad T > N \end{cases} . \tag{11}$$

Because of its later importance, we call these "N-period deviation" beliefs. By acting on $dK^e(\omega)$, the operators Γ^k and Γ^c yield the savings and consumption plans of agent ω, respectively: $dk(\omega) = \Gamma^k\left(dK^e(\omega)\right)$ and $dc(\omega) = \Gamma^c\left(dK^e(\omega)\right)$.

[3] There are many others. For example, the reader is invited to compute the sum of the rows of each matrix to evaluate the effect of extreme beliefs.

[4] This property suggests that we could identify the set of rationalizable trajectories in the RBC model using the findings of Guesnerie-Jara-Moroni (2010), who characterize rationalizable equilibria in finite-dimensional models with strategic substitutabilities. Because our model is infinite dimensional, the results in our setting are not immediate, however, and left for future research. The results of next section suggest that the set under discussion is (very) large.

Agent ω believes the real interest rate will be at its steady-state value after period N; thus he will hold consumption and savings constant from period N onward. These constant values can be determined using the operators Γ^k and Γ^c. Appendix A shows that we obtain:

Corollary 3 *Suppose an agent has N-period deviation beliefs (11). Then*

$$dk_t(\omega) = \begin{cases} -\xi\left(t - \bar{r}^{-1}(1-\beta^t)\beta^{N-t}\right)e & \text{for } 1 \le t < N \\ -\xi\left(N - \bar{r}^{-1}(1-\beta^N)\right)e & \text{for } t \ge N \end{cases} \tag{12}$$

$$dc_t(\omega) = \begin{cases} \xi\left(1-\beta^N - \bar{r}t\right)e & \text{for } 0 \le t < N \\ \bar{r}dk_t(\omega) & \text{for } t \ge N \end{cases}. \tag{13}$$

Note that for $e > 0$ it follows that $dc_0(\omega) > 0$, that $dc_t(\omega)$ declines over time for $t < N$, and that $dc_t(\omega) < 0$ for $t \ge N$. For $t \ge N$ agent ω consumes the net return to savings, leaving the savings stock unchanged.

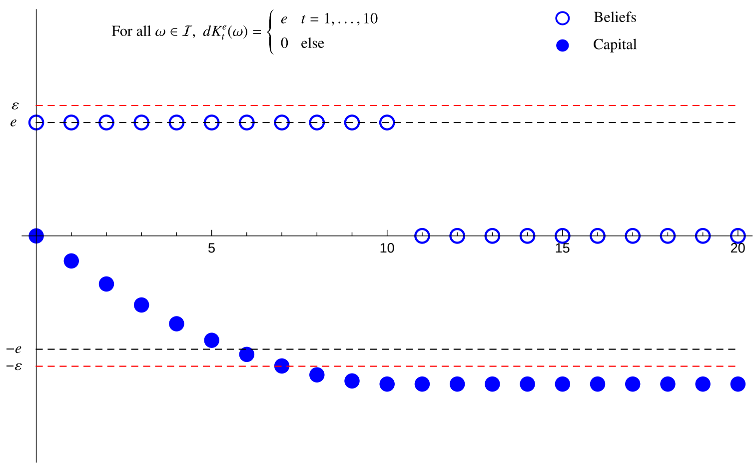

Figure 2: Savings plan for N-period beliefs when $N = 10$.

The expectation of a period of low interest rates associated with $dK^e_T(\omega) = e > 0$ for $T \le N$ leads agents to shift consumption from the future to the present. However, for our purposes equation (12) is more significant. Most strikingly, for $t \ge N$ the right-hand side implies that $dk_t(\omega)$ becomes large (in magnitude) as N gets large. Thus even if agent ω thinks that capital deviates from steady state by only some small e, and only for some finite period of time, his savings path will move away from, and remain arbitrarily far from, the steady state, provided that period is long enough. The implications of Corollary 3 are shown in Figure 2 in which we set $\beta = 0.95$, $\xi = 0.5$ and $N = 10$. The horizontal dashed lines at $\pm\varepsilon$ will be explained in Section 4.

With the operators Γ^k and Γ^c in hand we can now turn to our central concern, which is the possibility of rational agents coordinating on the rational expectations steady-state path.

3 Eductive Stability

We first present a standard definition of eductive stability based on simple game-theoretical considerations, and by doing so we provide a "hyper-rationality" view on expectational coordination. We then pursue a less sophisticated view of expectational coordination, which reflects first-level rationality considerations. These two viewpoints provide complementary insights on the problem of expectational coordination. Our analysis initially will initially abstract from the time dimension of the problem; then, after illustrating the concepts in a simple cobweb framework, we reintroduce time and adapt the general ideas to our infinite-horizon setting.

3.1 Local strong eductive stability

We begin by considering an abstract economy populated with a continuum of rational economic agents. The agents know the logic of the collective economic interactions (the underlying model). Both the rationality of the agents and the model are Common Knowledge (CK). The state of the system is denoted E and belongs to some subset $\mathcal{E}$ of a topological vector space.

Emphasizing the expectational aspects of the problem, we view an equilibrium of the system as a state E^* such that if everybody believes that it prevails, it does prevail.

Under eductive learning, as described below, each agent contemplates the possible states of the economy implied by the beliefs and associated actions of the economy's agents. Coordination on a particular equilibrium outcome obtains when this contemplation, together with the knowledge that all agents are engaged in the same contemplation, rules out all potential economic outcomes except the equilibrium. If coordination on an equilibrium is implied by the eductive learning process, then we say that the equilibrium is strongly eductively stable.[5] The argument can be either global or local. We now introduce the local version of eductive stability. As we will see, even local stability is at issue in our infinite-horizon model.

We say that E^* is locally strongly eductively stable if and only if given any open neighborhood $U \subset \mathcal{E}$ of E^*, there exists an open neighborhood

[5] The term "strongly rational" is also used. For details on the game theoretic background of our investigation see Guesnerie and Jara-Moroni (2011). Matsui and Oyama (2006) consider rationalizability of perfect foresight paths in continuous-time implementations of repeated symmetric two-player normal-form games with finite actions. Other related investigations include Jara-Moroni (2012) and Yu (2014). We view eductive stability as a zero-one criterion. Less stringent indices of stability are developed in Desgranges and Ghosal (2010).

$V(E^*) \subset U$ such that Assertion A implies Assertion B:

Assertion A: It is CK that $E \in V(E^*)$.

Assertion B: It is CK that $E = E^*$.

We now describe a mental reasoning process induced by CK that can lead from Assertion A to Assertion B:

1. Because all agents know that $E \in V(E^*)$ it follows that all agents know that each agent chooses a best response to their subjective probability distribution over $V(E^*)$. It follows that all agents know that the state of the system will be in a set $\mathcal{E}(1)$.

2. If $\mathcal{E}(1)$ is a proper subset of $V(E^*)$, the mental reasoning process goes on as in step 1, but based now on $\mathcal{E}(1)$ instead of $V(E^*)$. In this case it follows that all agents know that the state of the system will be in a set $\mathcal{E}(2) \subset V(E^*)$.

3. The mental reasoning process continues provided that at each stage, $\mathcal{E}(n)$ is a proper subset of $\mathcal{E}(n-1)$.

If the sequence $\{\mathcal{E}(n)\}_{n\geq 1}$ is strictly decreasing and converges to $\{E^*\}$, i.e. $\mathcal{E}(n) \subsetneq \mathcal{E}(n-1)$ and $\cap_{n\geq 1}\mathcal{E}(n) = \{E^*\}$, the equilibrium is locally strongly rational or locally strongly eductively stable.[6]

Inspired by game-theoretic considerations,[7] the ideas of eductive stability have provided new tools for assessing the stability of expectational coordination in many different contexts: for early references see Guesnerie (2005) and for some further references see Desgranges (2014). The connections of the economically oriented applications to the game theoretical viewpoint, for example the relationship between eductive stability and uniqueness of rationalizable strategies, are systematically assessed in Guesnerie and Jara-Moroni (2011).[8]

3.2 Example: cobweb model

To illustrate the eductive approach, consider the cobweb model, studied by Bray and Savin (1986) under adaptive learning and by Guesnerie (1992) under

[6]If the initial neighbourhood were equal to the whole space $\mathcal{E}$, then the word global would replace the word local. Note that a globally strongly eductively stable equilibrium is also locally strongly eductively stable.

[7]The early discussion of rationalizability, putting emphasis on rationalizable beliefs, is usually associated with the work of Bernheim (1984) and Pearce (1984).

[8]Most of their analysis is conducted in a finite-dimensional state space $\mathcal{E}$. Thus their results would require an extension to an infinte-dimensional setting to ensure applicability to our RBC framework.

eductive learning. The model is interpreted as a producers' game in which the strategy of each firm is its output and the optimal choice of output depends on expected price.

Thus consider the following static, nonstochastic model. There is a unit mass of firms I with identical technologies given by the cost function $c(q) = (2C)^{-1}q^2$, with $C > 0$. Each firm $\omega \in I$ must make its production decision before the price is known, and because it maximizes expected profits, it chooses its quantity $q(\omega)$ based on its expected market price $p^e(\omega)$, so that $q(\omega) = Cp^e(\omega)$. Assume that market demand $d(p)$ is twice continuously differentiable and decreasing in price p. Market clearing requires

$$d(p) = \int_I q(\omega)d\omega = C\int_I p^e(\omega)d\omega.$$

By inverting demand, we obtain

$$p = d^{-1}\left(C\int_I p^e(\omega)d\omega\right). \tag{14}$$

Because demand is downward-sloping there is a unique perfect foresight equilibrium p^* satisfying $d(p^*) = Cp^*$, with $p^e(\omega) = p^*$ for all $\omega \in I$. Equation (14) to first order may be written

$$p - p^* = -\phi\int \left(p^e(\omega) - p^*\right)d\omega,$$

where $\phi = -C/d'(p^*) > 0$ measures the expectational feedback.

We now ask under what conditions p^* is locally strongly eductively stable.[9] We consider here the linearized model. The argument is as follows. Let U be a small open neighborhood containing p^*, and let $V(p^*)$ be an open interval contained in U and centered at p^*. Suppose each agent believes that $p \in V(p^*)$, i.e. has a subjective probability distribution over $V(p^*)$. In the cobweb model only the mean of the distribution, which can be viewed as the point expectation, is needed for decision-making and thus for stability analysis.[10] It follows that $p \in \mathcal{E}(1) \equiv \phi V(p^*)$, and if $\phi < 1$ then $\mathcal{E}(1)$ is a proper subset of $V(p^*)$. Now suppose that it is common knowledge that $p \in V(p^*)$. It then follows that it is common knowledge that $p^e(\omega) \in V(p^*)$ for all ω, and thus it is common knowledge that $p \in \mathcal{E}(1)$. The argument may be repeated replacing $V(p^*)$ with $\mathcal{E}(1)$. Iterating this argument it follows that $p^e(\omega) \in$

[9] Guesnerie (1992) developed the global argument in a more general version of the model.

[10] In fact, it can be shown that point expectations are sufficient for stability analysis in much more general settings: see Guesnerie and Jara-Moroni (2011) for details.

$\mathcal{E}(N) \equiv \phi^N V(p^*)$ for all $N = 0, 1, 2, \ldots$, so that local strong eductive stability is implied. In contrast, local eductive stability fails if $\phi > 1$ since, even under the local common knowledge assumption $p \in V(p^*)$, hyper-rational agents are unable to coordinate on $p = p^*$ through mental reasoning. Guesnerie (1992) generalizes this argument to a global nonlinear set-up. See Appendix B for the nonlinear argument using the language and notation developed here.

3.3 Local weak eductive stability

The above definition, based on the successive deletion of non-best responses and starting under the assumption that the state of the system is close to the equilibrium state, reflects the local version of a high-tech, i.e. hyper-rationality, viewpoint. Another plausible intuitive definition of local expectational stability, based only on individual-agent rationality, is the following: there exists a non-trivial neighborhood of the equilibrium such that, if everybody believes that the state of the system is in this neighborhood, it is necessarily the case, whatever the specific form taken by each agent's belief, that the state is in the given neighborhood.[11]

Using the above terminology, we say that E^* is locally weakly eductively stable if and only if given any open neighborhood $U \subset \mathcal{E}$ of E^*, there exists an open neighborhood $V(E^*) \subset U$ such that Assertion A implies Assertion B:

Assertion A: All agents believe that $E \in V(E^*)$.

Assertion B: The state of the system E generated by any collection of such beliefs cannot contradict these beliefs, i.e. $E \in V(E^*)$.

Again, $V(E^*)$ is an initial belief assumption, a universally shared conjecture on the set of possible states, and for local weak eductive stability we require that this belief cannot be falsified by actual outcomes resulting from individual actions that are best responses to some probability distributions over $V(E^*)$.[12] The argument is here low-tech in the sense that it refers to the rationality of agents, but not to CK of rationality or of the model:[13] the criterion focuses on agents' actions which depend only on their beliefs about the state of the system, and not on their beliefs about other agents' beliefs.

To put it another way, the criterion appeals only to the results of the first step of the mental reasoning process specified in Section 3.1. It is then obvious

[11] The conjectural equilibrium bounds discussed by Benhabib and Bull (1988), in the context of the overlapping generations model of money, has a similar motivation.

[12] Equivalently, in the absence of such a neighborhood $V(E^*)$, facts may falsify any collective conjecture, whatever the proximity of the conjectured set to the equilibrium (unless the conjecture is reduced to the equilibrium E^* itself).

[13] It does not even require full knowledge of the model.

that local strong eductive stability implies local weak eductive stability. The close connections between the strong and weak criteria, as suggested in the literature[14] are fairly intuitive, and indeed obvious in the linear cobweb model, where the reader can check that local strong eductive stability obtains when $\phi < 1$, whereas local weak eductive stability requires only $\phi \leq 1$.

We regard local weak eductive stability as a natural stability condition in its own right: if E^* is is locally weakly eductively stable then beliefs near E^* lead to outcomes near E^*. Indeed our results in the RBC model in Section 4 are presented in terms of weak eductive stability.

Finally, concerning the connections between the eductive and the "evolutive" or adaptive learning viewpoint, we note that failure to find a set $V(E^*)$ for which the equilibrium is locally eductively stable signals a tendency for any near-equilibrium states of beliefs to be driven away, thus threatening the convergence of reasonable learning rules.[15]

3.4 Local eductive stability in the RBC model

Consider now the application of eductive stability to the RBC framework. The equilibrium E^* under consideration is given by $K_t = \bar{K}$ for all $t \geq 0$, and the first issue concerns the notion of a neighborhood $V(E^*)$. We begin with the assumption that agents believe that the path of capital will lie within a "tube" or "cylinder" of radius $\varepsilon > 0$ centered at the steady state, i.e. $K_t \in \left(\bar{K} - \varepsilon, \bar{K} + \varepsilon\right)$ for all $t \geq 1$. These belief restrictions are captured by considering as neighborhoods ε-balls in $l^{\infty}(\mathbb{N})$, centered at the steady-state path. More generally we will consider belief spaces associated with topologies induced by weighted l^{∞} norms. Specifically, let $\varphi \in \mathbb{R}^{\mathbb{N}}$ be a sequence of positive weights bounded away from zero, that is, $\varphi_t > 0$ for all $t \in \mathbb{N}$, and $\varphi_{\min} = \inf_{t \in \mathbb{N}} \varphi_t > 0$. For $x \in \mathbb{R}^{\mathbb{N}}$, define

$$\|x\|_{\infty}^{\varphi} = \sup_{t \in \mathbb{N}} \varphi_t |x_t|,$$

and let $(l^{\infty}(\mathbb{N}, \varphi), \|\cdot\|_{\infty}^{\varphi}) \equiv l_{\varphi}^{\infty}$ be the Banach space of all real sequences x with finite weighted norm $\|x\|_{\infty}^{\varphi}$. Observe that $\varphi_t = 1$ for all t corresponds to the usual l^{∞} norm. Allowing φ to be any sequence of positive weights bounded away from zero facilitates the analysis of eductive stability for beliefs that may converge to the steady state along possibly complex paths. With this norm

[14] A formal statement of quasi-equivalence requires additional technical assumptions such as the (weak) assumptions stressed of Guesnerie and Jara-Moroni (2011).

[15] More on this subject can be found in Guesnerie (2002), Guesnerie and Woodford (1991) and Gauthier and Guesnerie (2004).

the belief restrictions are captured by ε-balls, $B_\varphi(\varepsilon)$, of the form

$$B_\varphi(\varepsilon) = \left\{ \{K_t\}_{t\geq 1} \in l_\varphi^\infty : \sup_{t\in\mathbb{N}} \varphi_t |K_t - \bar{K}| < \varepsilon \right\}.$$

As we note below, if $\varphi_t = \lambda^t$ for $\lambda > 1$ a ball can be visualized as a tube with shrinking radius.

With these notions and notations in place, local strong eductive stability is given as follows: for any open neighborhood $U \subset l_\varphi^\infty$ of $\{\bar{K}\}_{t\geq 1}$ there exists $\varepsilon > 0$ such that Assertion A implies Assertion B:

Assertion A: It is CK that $\{K_t\}_{t\geq 1} \in B_\varphi(\varepsilon) \subset U$.

Assertion B: It is CK that $\{K_t\}_{t\geq 1} = \{\bar{K}\}_{t\geq 1}$.

Similarly, local weak eductive stability is given as follows: for any open neighborhood $U \subset l_\varphi^\infty$ of $\{\bar{K}\}_{t\geq 1}$ there exists $\varepsilon > 0$ such that Assertion A implies Assertion B:

Assertion A: Each agent believes $\{K_t\}_{t\geq 1} \in B_\varphi(\varepsilon) \subset U$.

Assertion B: Agents' plans imply $\{K_t\}_{t\geq 1} \in B_\varphi(\varepsilon)$.

In the next Section we will show the failure of local weak eductive stability in the RBC model, and hence the failure of strong stability as well.

The notion of weak eductive stability just defined concerns the trajectory of aggregate capital generated by plans chosen in time zero and adhered to in perpetuity. Alternatively we could consider the trajectory of aggregate capital generated by the choices made in real time, i.e. period by period, holding beliefs $\{K_t^e(\omega)\}_{t\geq 1}$ about future aggregate capital fixed, but with agents conditioning their time t choices on their realized time t capital holdings.[16] The corresponding paths, to first order, of agent ω's realized consumption and saving are given by Lemma 3. The steady state is said to be weakly eductively stable in the alternative sense provided that the corresponding realized path of aggregate capital does not contradict their beliefs that it remains in the initial conjectured neighborhood. In fact, Lemma 3 immediately implies that this weak stability alternative is equivalent to our original notion of local weak eductive stability. Indeed, to first order, it follows from equation (10) that agent ω's time t saving decision, $dk_{t+1}(\omega)$, depends only on his time t capital holdings and his time t beliefs about the future path of aggregate capital: the time t realization of aggregate capital does not impact his time t decision.

[16] That is, agent ω's time t savings decision, $k_{t+1}(\omega)$ depends on his time t capital holdings $k_t(\omega)$, and his beliefs path $\{K_{t+s}^e(\omega)\}_{s\geq 1}$.

4 Eductive Instability of the RBC Model

We are now ready to obtain our main result.

Theorem 2 *Let $\varphi \in \mathbb{R}^{\mathbb{N}}$ be a sequence of positive weights bounded away from zero. The perfect foresight steady state is not locally weakly eductively stable with respect to the topology induced by the weighted l^{∞}-norm $\|\cdot\|_{\infty}^{\varphi}$.*

The intuition for the proof is straightforward for the case of uniform weights $\varphi_t = 1$ in the linearized model. While an argument for stability would require considering heterogeneous and possibly stochastic beliefs, because the intention is to prove instability it is sufficient to assume homogeneity of beliefs and to consider point expectations. The goal is to provide a collection of beliefs paths that results in a corresponding path for aggregate capital that escapes the ε-ball. We consider homogeneous N-period deviation beliefs, as illustrated in Figure 2 of Section 2. First note that, with homogeneous beliefs, individual savings plan coincide with the implied aggregate path of capital. For $\varepsilon > 0$, the ε-ball $B(\varepsilon)$ around the origin is indicated by the outer (red) dashed lines, and the agents' N-period deviation beliefs dK^e, with $N = 10$ and $0 < dK_t = e < \varepsilon$ for $t \leq 10$, are shown by the open (blue) circles. The solid (blue) disks now represent the implied path of aggregate capital dK. The essential observation is that dK^e is in $B(\varepsilon)$ whereas dK is not.

The economic intuition relies on Corollary 3. For expectations $dK_s^e(\omega) = e > 0$ for $s \leq N$, the optimal plan for agents is to increase consumption today at the expense of future consumption. If $\xi > 1$ then for sufficiently large N the positive impact on dC_0 will be large enough to be immediately destabilizing in the short run in the sense that $dK_1 < -e$. However, even if ξ is small, homogeneous expectations $dK_s^e(\omega) = e > 0$ for $s \leq N$ and any $e \in (0, \varepsilon)$, we eventually have $dK_t < -\varepsilon$ if N is sufficiently large. The planned future reduction in consumption is not enough to avoid, and is consistent with, dK_t leaving $B(\varepsilon)$. Note that ε and e are chosen so that agents' beliefs are consistent with $B(\varepsilon)$ while the aggregate planned trajectory of capital escapes it. While the intuition is provided using the linearized model the proof given in the Appendix allows for the nonlinear framework.

As just noted, given N, whether and when the implied path of aggregate capital exits the ε-neighborhood depends on the magnitude of the expectational feedback parameter ξ. This is illustrated in Figure 3, where the implied paths of dK are shown for four different values of ξ and the same belief path as used in Figure 2. Observe that for small feedback $\xi = 0.2$ the implied path remains in the ball $B(\varepsilon)$, though of course for sufficiently large N the path would exit $B(\varepsilon)$ for $\xi = 0.2$. In this example $N = 17$ is sufficient. We remark also that when ξ is sufficiently small the first N such that $|dk_N(\omega)| > \varepsilon$ can be relatively large. More specifically, under these beliefs it can be shown that for $t < \xi^{-1}$ aggregate capital dK_t remains inside $B(\varepsilon)$.

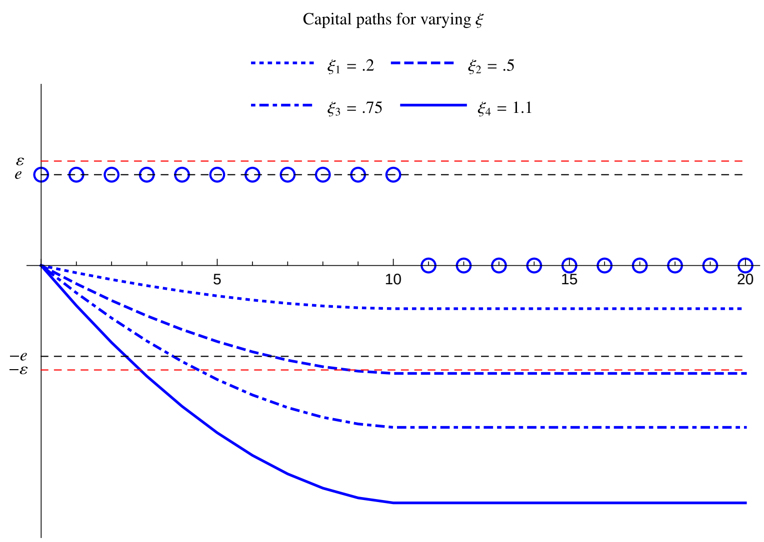

Figure 3: Implied paths of aggregate capital for N-period deviation beliefs with $N = 10$ and with varying values of expectational feedback parameter ξ.

Allowing for time-varying weights facilitates analysis of more general notions of nearness, thus making our arguments robust to alternative topologies. Note that the unit ball B in l^∞ can be envisioned as a tube of radius one about the horizontal axis. Given positive weights φ, the unit ball B in l^∞_φ can be similarly envisioned as a tube about the horizontal axis, but here the radius at time t is φ_t^{-1}. Thus increasing weights result in a tube of shrinking radius. Figure 4 provides an illustration based on the weights $\varphi_t = \lambda^t$ where $\lambda > 1$. The intuition for the instability result in this case is as follows: the ε-ball can be envisioned as a tube with radius decreasing to zero; the image of this ball under the action of Γ^k represents the collection of all possible implied aggregate paths. As depicted in Figure 4, this image takes the form of a tube of increasing radius ρ_t, which asymptotes to $\Xi = ((\lambda - \beta)(\lambda - 1))^{-1} \xi \lambda (1 - \beta) \varepsilon$. Since $\Xi > 0$ it follows that $\Gamma^k (B_\varphi(\varepsilon)) \nsubseteq B_\varphi(\varepsilon)$.

The negative result of Theorem 2 means that it is unlikely that rational agents will be able to coordinate on the REE using eductive mental reasoning: rational agents cannot convince themselves that the rational expectations equilibrium will necessarily prevail.[17] In light of this result we now turn to a more flexible perspective in which boundedly rational agents revise expectations over time in response to observed data.

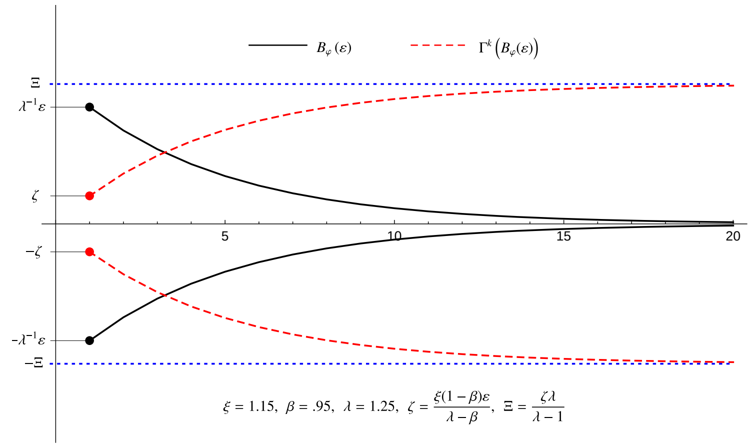

Figure 4: Image of the tube $B_\varphi(\varepsilon)$ under the action of Γ^k for increasing weights $\varphi_t = \lambda^t$, $\lambda > 1$, i.e. the radius of the tube is shrinking at rate λ^{-1}.

5 Eductive Reasoning and Adaptive Learning

The steady state $\bar{K}$ is not eductively stable: knowledge by rational agents that the path must be near $\bar{K}$ cannot trigger coordination through mental reasoning. Bounded rationality therefore seem unavoidable and this suggests introducing real-time adaptive learning. At the same time, it is known that, for a suitable and natural class of adaptive learning rules, the steady state is locally asymptotically stable. This suggests a disconnect between the adaptive and eductive approaches that is much more significant than previously noted in the literature. However, a better way to proceed is to combine key aspects of the two approaches, mixing adaptive learning with the considerations on proximity of beliefs to the steady state associated with the eductive viewpoint.

[17] Instability results also appear in the adaptive learning literature. For example, Howitt (1992) and Evans and Honkapohja (2003) show instability for a class of interest-rate rules in monetary models. However, these models can also suffer from indeterminacy. The generic instability result of the current paper is particularly striking since the RBC model is in general well-behaved.

5.1 The framework

We again endow agents with expectations about the future path of the aggregate capital stock. These expectations are restricted to belong to a set, which for convenience we will take to be $B(\varepsilon)$, the ε-ball for $\{dK_t\}_{t=1}^{\infty}$ associated with uniform weights $\varphi_t = 1$ for all $t \in \mathbb{N}$. In this Section we restrict attention to the linearized system. The set $B(\varepsilon)$ can here be viewed as describing a collective belief that provides bounds on individual beliefs. As in Section 2.2 and the eductive approach of Sections 3.4 and 4, agents' decisions are based on an assessment of the whole future,[18] but now we look at the system in real time. Our focus remains on whether the path for aggregate capital lies within $B(\varepsilon)$. Now, however, in accordance with the evolutive or adaptive learning viewpoint, the expected trajectory at time t is assumed not only to reflect initial beliefs but also to respond over time to observed actual capital.

More precisely, we specify a set of adaptive learning rules that determine the way initial expectations change along the real-time trajectory of aggregate capital. Then, as in the eductive approach, we ask if the implied path $dK = \{dK_t\}_{t=1}^{\infty}$ will necessarily lie in $B(\varepsilon)$, i.e. if the collective belief, which serves as a frame for the individual beliefs, is subject to real-time falsification. If for some nonempty subset of adaptive learning rules, falsification is impossible, then we say that the steady state is B-stable under evolutive (adaptive) learning for those learning rules, and if this occurs for all adaptive learning rules within the set of rules under consideration, we say the steady state is robustly B-stable under evolutive learning.

5.2 The real-time system

In the real-time system we assume that at each time t each agent ω solves anew their dynamic optimization problem. That is, at each time t agent ω chooses $dk_{t+1}(\omega)$ optimally given their savings, their expectations (which are now revised each period) and the aggregate capital stock.

Equation (10) can be interpreted as the time t saving function of agent ω, since it relates the agent's saving at t to his wealth $dk_t(\omega)$, and to expected future prices, captured by $dK^e_{t+n}(\omega)$. This equation motivates our interpretation of ξ as an expectations feedback parameter: ξ measures the impact

[18] In the adaptive learning literature, within infinite-horizon models, this approach has been followed, for example, in Sargent (1993, pp. 122-125), Preston (2006), Eusepi and Preston (2011) and Evans, Honkapoha and Mitra (2009). An alternative approach in the adaptive learning literature is based on one-step-ahead "Euler equation" learning. See, e.g. Evans and Honkapohja (2001), Ch. 10.

on savings of a permanent unit increase in expected future aggregate capital. Allowing expectations to evolve over time, and using Lemma 3, the real-time description of savings behavior is given by

$$dk_{t+1}(\omega) = dk_t(\omega) - \beta^{-1}(1-\beta)\xi \sum\nolimits_{j\geq 1} \beta^j dK^e_{t,t+j}(\omega), \qquad (15)$$

where now $dK^e_{t,t+j}(\omega)$ is the point expectation of aggregate capital in period $t+j$ held by agent ω in period t.

It can be seen from equation (15) that the agent's decision $dk_{t+1}(\omega)$ depends on a single sufficient statistic for $\left\{dK^e_{t,t+j}(\omega)\right\}_{j=1}^{\infty}$, given by

$$d\hat{K}^e_t(\omega) = \beta^{-1}(1-\beta) \sum\nolimits_{j\geq 1} \beta^j dK^e_{t,t+j}(\omega).$$

The normalization factor $\beta^{-1}(1-\beta)$ ensures that the sum of the weights on $dK^e_{t,t+j}(\omega)$ is one, so that $\left\{dK^e_{t,t+j}(\omega)\right\}_{j=1}^{\infty} \in B(\varepsilon)$ implies $\left|d\hat{K}^e_t(\omega)\right| < \varepsilon$. We thus rewrite (15) as $dk_{t+1}(\omega) = dk_t(\omega) - \xi d\hat{K}^e_t(\omega)$, and, following the boundedly-rational adaptive learning approach, interpret this equation as providing a behavioral decision rule in which we re-envision the agent's saving choice as depending solely on current wealth and the sufficient statistic $d\hat{K}^e_t(\omega)$. For convenience we refer to $d\hat{K}^e_t(\omega)$ as "expected future capital."

Finally we specify a simple adaptive scheme for the revisions of expectations over time: $d\hat{K}^e_t(\omega) = (1-\alpha)d\hat{K}^e_{t-1}(\omega) + \alpha dK_t$, where $0 < \alpha \leq 1$, called the gain parameter in the literature, describes how expectations reflect current information about the actual capital stock.[19] We note that this adaptive scheme is equivalent to regressing aggregate capital on an intercept, allowing for discounting of past data. This can be viewed as a special case of discounted least-squares in nonstochastic models and is standard in the literature.

We are now in a position to describe the real-time evolution of the system. For the sake of simplicity, we start from an initial situation in which the time-zero belief is the same for everybody:

$$d\hat{K}^e_0(\omega) \equiv \beta^{-1}(1-\beta) \sum\nolimits_{j\geq 1} \beta^j dK^e_{0,j}(\omega) = d\hat{K}^e_0,$$

[19] Adaptive learning in nonstochastic models with infinite horizons often assumes that forecasts are the same at all horizons. See, for example, Evans, Honkapohja and Mitra (2009). In the current context this means $dK^e_{t,t+j} = e$ at t for all j. Our formulation in terms of $d\hat{K}^e_t$ allows for greater generality, while retaining a single sufficient statistic that is updated over time. In stochastic models, the time pattern of variables can be estimated and updated using recursive least squares. For technical reasons this procedure cannot be used in nonstochastic systems. Intuitively, in a nonstochastic equilibrium the asymptotic lack of temporal variation makes impossible consistent estimation of time-series parameters. See Evans and Honkapohja (2001, pp. 152-154).

and thus the sufficient statistics are initially, and remain, homogeneous across agents. Finally, in line with our earlier analysis, we assume that $dK_0 = 0$.[20]

The homogeneity assumption allows us to calculate the resulting time path, and is also illuminating in the sense that we would hope the system to be stable under learning if we start with small expected future capital $\left|d\hat{K}_0^e\right|$. Under homogeneity the system's dynamics, which depend only on initial $d\hat{K}_0^e$ and on the parameters α and ξ, can be written as

$$dK_{t+1} = dK_t - \xi d\hat{K}_t^e \text{ and } d\hat{K}_{t+1}^e = (1-\alpha)d\hat{K}_t^e + \alpha dK_{t+1}. \tag{16}$$

We can now return to the previously suggested concept of B-stability under adaptive learning and give formal definitions.

Definition 2 *The steady state is B-stable under adaptive learning for a given* $0 < \alpha \leq 1$ *if, for all* $\varepsilon > 0$ *sufficiently small,* $|d\hat{K}_0^e| < \varepsilon$ *implies that the trajectory* $\{dK_t\}_{t=1}^{\infty}$, *generated by (16) remains in* $B(\varepsilon)$.

Definition 3 *The steady state is robustly B-stable under adaptive learning if it is B-stable under adaptive learning for all* $0 < \alpha \leq 1$.

If the steady state is B-stable for some nonempty subset of $0 < \alpha \leq 1$, but is not robustly B-stable, then we will say that it is partially B-stable under adaptive learning.

5.3 The results

The first result is again an impossibility theorem.

Theorem 3 *Under adaptive learning the steady state is not robustly B-stable.*

In fact the result might be expected in view of Theorem 2. The trajectory under adaptive learning is continuous in α (for small $\alpha > 0$ the α-trajectory is close to the $\alpha = 0$ trajectory for a long period of time). The path of dK_t under adaptive learning and when $\alpha = 0$ corresponds to the path that would be realized by the actions of agents with the constantly held beliefs $dK_t^e(\omega) = d\hat{K}_0^e$. Failure of robust B-stability would then follow by continuity as a result of the impossibility of eductive stability.[21]

[20] Our results are qualitatively robust to small perturbations of intial aggregate capital.

[21] As shown by Lemma 3 the plans of agent ω are to first order independent of aggregate capital. The same must then be true for actual decisions when there are unchanged beliefs. Hence if the trajectory of planned capital exits $B(\varepsilon)$ then the same must be true for the actual trajectory.

A striking feature of Theorem 3 is that instability arises for small α, which in the adaptive and least-squares learning literature is usually viewed as a stabilizing case (the "small gain" limit).[22] In our approach, the problem is that in this case the initial collective belief will be falsified, which we view as a fragility of expectational coordination.

The instability result is stronger than stated in the following sense, which can be verified from the proof of Theorem 3. Given $\varepsilon > 0$, consider *any* initial beliefs $0 < |d\hat{K}_0^e| < \varepsilon$. Then there exists $\alpha > 0$ such that the corresponding trajectory under adaptive learning will have $|dK_t| > \varepsilon$ for some $t \geq 1$. That is, starting from the steady state, all initial expectations, for appropriate α, lead to paths dK_t that leave $B(\varepsilon)$.

We next turn to partial B-stability under adaptive learning. We will see that a natural necessary condition is given by asymptotic stability of the system (16). Asymptotic stability is the classical stability criterion used for adaptive learning. We therefore start with the following result:

Lemma 4 *System (16) is asymptotically stable if and only if* $\xi < 4\alpha^{-1} - 2$.

Asymptotic stability is a necessary condition for partial B-stability under adaptive learning.[23] Another necessary condition is given by $\xi \leq 1$: indeed, since $dK_0 = 0$ we have $dK_1 = -\xi d\hat{K}_0^e$. A more complete picture is provided by the following Theorem.

Theorem 4 *We have the following results on partial B-stability and asymptotic stability for the adaptive learning system (16) with* $\xi > 0$ *and* $0 < \alpha \leq 1$*:*

1. *Given* α*, the steady state is B-stable for sufficiently small* ξ*.*

2. *Given* ξ *satisfying* $\xi \leq 1$*:*

    (a) *the steady state is not B-stable for sufficiently small* α*;*

[22] The connection between eductive stability and adaptive learning has been discussed in Evans and Guesnerie (1993), Guesnerie (2002) and Hommes and Wagener (2010). In short-horizon set-ups, eductive instability is usually reflected in adaptive instability for large gains (here $\alpha < 1$ large). This is seen for the overlapping generations model with money in Guesnerie and Woodford (1991) and Evans and Honkapohja (1995). We also note that in experiments the complex expectational dynamics found in Hommes (2011) for the cobweb model arise in the eductively unstable case. For related work see Bao and Duffy (2016).

[23] To see this, assume the system is not asymptotically stable. The eigenvalues in this case are real. (The proof of Lemma 4 implies that in the complex case the modulus of the eigenvalues is $1 - \alpha$.) We must show that $dK_0 = 0$, $d\hat{K}^e \neq 0$ leads to a divergent path. Noting that $(0, 1)$ is not an eigenvector it follows that all initial conditions $(0, d\hat{K}^e)$ lead to divergent paths provided at least one of the eigenvalues has magnitude larger than one.

(b) *the steady state is B-stable for sufficiently large* α*;*

(c) *the steady state is asymptotically stable for all* α*.*

3. *Given* ξ *satisfying* $\xi > 1$*:*

(a) *the steady state is not B-stable for any* α*;*

(b) *for* $1 < \xi < 2$*, the steady state is asymptotically stable for all* α*;*

(c) *for* $\xi \geq 2$*, it is asymptotically stable for* $\alpha < 4(2 + \xi)^{-1}$*.*

This theorem emphasizes the relevance of the expectations feedback parameter ξ for understanding real-time learning. Indeed, the coefficient ξ, stressed in Sections 2 and 3, plays a key role in both partial B-stability and asymptotic stability under adaptive learning: small ξ is stabilizing and large ξ is destabilizing.[24] The left panel of Figure 5 captures the role of ξ for fixed α. Asymptotic stability holds for all examined values of ξ, but B-stability only for $\xi = 0.1$.

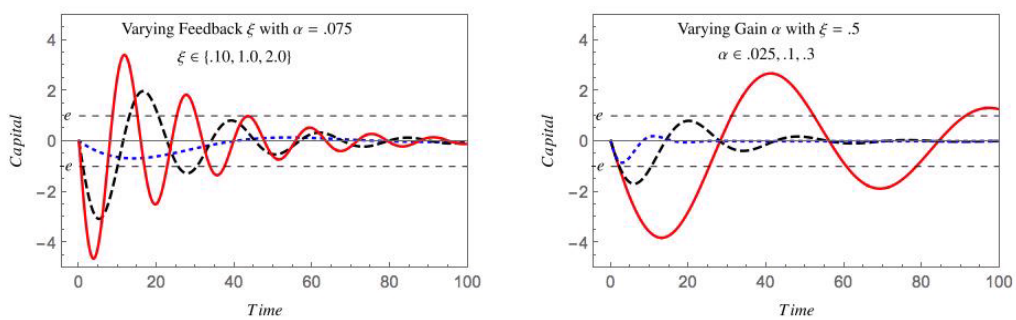

Figure 5: Aggregate capital paths under adaptive learning. Left panel: maximum amplitude of paths increasing in feedback parameter ξ. Right panel: maximum amplitude of paths decreasing in gain parameter α.

The right panel of Figure 5 looks at the role of α for fixed ξ. Small values of α generate the failure of robust B-stability under adaptive learning, even though the asymptotic stability condition $\xi < 4\alpha^{-1} - 2$ is easier to satisfy

[24]Partial B-stability results for $\xi < 1$ exhibit a trade-off between ξ and α: as $\xi \to 1$ from below, the partial B-stability region tends to $\alpha > 1/2$. Numerical results indicate partial B-stability for $\alpha \in (\gamma(\xi), 1]$ where $\gamma(\xi)$ is continuous and monotonically increasing in ξ with $\gamma(\xi) \to 0$ as $\xi \to 0$. When $\xi = 1$ we have partial B-stability for $\alpha \in [1/2, 1]$. We note a discontinuity in B-stability for given α, which is caused by first-period behavior: $dK_1 = -\xi dK_0^e$ implies B-instability for any $\xi > 1$.

when $\alpha > 0$ is small. The reason for this is that small α under adaptive learning leads to a cumulative movement of aggregate capital away from the steady-state value, which over finite time periods, as $\alpha \to 0$, track the possible dK_t paths deduced by agents in our eductive setting.[25]

5.4 Discussion

We have assessed expectational coordination from two competing yet complementary viewpoints – eductive and adaptive – and in both cases we have shown that coordination is difficult to achieve. This difficulty relates to the elasticities of economic outcomes to agents' expectations, and the results of Section 2 demonstrate that in the presence of far-sighted agents these elasticities, which depend in part on the expectational feedback parameter ξ, are very high. The planning theorem implies a generic failure of local eductive stability in the RBC model: rational agents would be unable to deduce coordination on the REE.[26] Hence full coordination on the REE, as of time zero, cannot be expected based on full rationality and mental reasoning. This negative result holds independently of ξ and the system characteristics, in sharp contrast to previous eductive learning studies that emphasize significant classes of "good cases" in which eductive considerations provide support for the possibility of expectational coordination on the REE – see Guesnerie (2005).

Small ξ still favours coordination: as noted in Section 4, the length of time for which plans will necessarily be compatible with initial beliefs that K_t is near the steady state is inversely related to ξ. However, under eductive reasoning, even with small ξ the incompatibility of aggregate plans with those beliefs will eventually emerge; and, under adaptive learning with small gain α the path of K_t will falsify those beliefs. In either case a crisis of beliefs could result, and under adaptive learning, large cyclical swings can arise.

Our instability result is in striking contrast to results for dynamic models with short-lived agents. In Appendix D, available in the Supplementary Materials, we study eductive stability in the standard overlapping generations (OLG) model with capital in which agents live for two periods. We find that

[25] An interesting feature of the lack of robust B-stability under adaptive learning, which can be seen in the proof of Theorem 3 given in the Appendix, is that the instability is associated with long cyclical movements in dK_t.

[26] We do not claim that this difficulty occurs in every model with infinitely-lived agents. For example, it would not occur in the world of Lucas (1978) for the reasons, which have an eductive flavor, stressed in Section 6 of his paper. The difference in conclusions reflects that in the Lucas tree model the dividend of a tree is exogenous, while in the RBC setting beliefs of agents affect rental rates though the endogeneity of aggregate capital.

the short planning horizon makes expectational coordination easier. In particular we obtain an eductive stability condition that is satisfied for frequently employed parametric specifications of preferences and technology.

Returning to our RBC framework, a logical consequence of the failure of eductive stability is that the evolution of the economy requires bounded rationality considerations that incorporate adaptive learning rules. Theorem 3 provides results closely linked to eductive instability: while real-time adaptive learning under small gain is asymptotically stable, the resulting time paths will include periods of large deviations from the REE, even if initial expectations are close to the steady state.

These results suggest that non-negligible deviations from the REE are likely. Suppose, starting from a steady state, that there is a small shock to expectations created by some news event. Not only will coordination on the REE be impossible using eductive learning, but also some plausible asymptotically stable adaptive learning rules will necessarily first lead the economy further *away* from the steady state.

Our eductive and adaptive results thus go hand-in-hand. The lack of eductive stability in the RBC model is reflected in a lack of robust adaptive B-stability. Thus the eductive viewpoint opens another door to the territories being investigated in adaptive learning studies with long-horizon agents and provides a powerful tool for their exploration.[27]

6 Conclusions

In this paper we have examined the issue of coordination of expectations on the benchmark rational expectations solution in the benchmark RBC model. The difficulties of expectational coordination can be ascertained from two sides, the eductive one and the adaptive one. In both cases, far-sighted agents are sensitive to the whole path of expectations.

It is not surprising that long-run concerns influence present decisions and future plans. However, the sensitivity to expectations of long-run plans envisaged today is extreme, and this is at the heart of the impossibility of eductive stability: fully rational agents with knowledge of the structure and rationality of other agents are unable to coordinate on RE. While rational expectations is the benchmark solution, the failure of eductive stability in RBC models indicates that real-time adaptive learning dynamics must play a role.

Under adaptive learning dynamics the sources of instability obtained in the

[27] For RBC models, important adaptive learning dynamics have been noted in Eusepi and Preston (2011), Branch and McGough (2011) and Mitra, Evans and Honkapohja (2013).

eductive framework remain pivotal. If the adaptation parameter is large then unstable overshooting can arise in the short run, while if the adaptation rate is small then low-frequency swings over the medium run will necessarily generate instability during the learning transition. Because occasional structural shifts, policy changes or news events will in practice disrupt the equilibrium, transitional learning dynamics will continue to be important.

Our findings indicate several directions for future research. First, our generic eductive instability in the RBC model contrasts strikingly with earlier research in short-horizon dynamic models, and with eductive stability in a parametric class of overlapping generations growth models. The standard assumption of infinite-lived agents in the RBC model clearly plays a major role in the analysis of the RBC model, but of course the infinite-lifetime assumption is made for theoretical convenience rather than empirical realism. Future research should examine eductive stability in models with long but finite lifetimes.

The other line of research suggested by our analysis is the incorporation into adaptive learning of considerations that are central to eductive analysis. Adaptive learning is a bounded rationality approach in which agents make forecasts based on statistical or econometric models that are updated over time. An advantage of this approach is that the agents require minimal knowledge of the economic structure and do not need to think explicitly about strategic interaction, factors that are central to the eductive viewpoint. This is an advantage but it may also be a disadvantage. If agents do understand the economic structure and how the expectations of others help to determine key economic aggregates, then it may be possible for them to use this information within an adaptive econometric forecasting framework. Particularly for cases in which eductive stability fails to hold, integration of eductive considerations into adaptive approaches is a natural avenue for future research.

Appendix A. Proofs.

Proof of Lemma 1. From the agent's flow budget constraint (2) we have, to first order, that

$$dk_{t+1}(\omega) = (1+\bar{r})dk_t(\omega) + \bar{K}dr_t^e(\omega) + dq_t^e(\omega) - dc_t(\omega). \qquad (17)$$

By constant returns to scale, $q = f(K) - Kf'(K)$, it follows that $dq + \bar{K}dr = 0$. Applying this observation to (17) yields (7). Under point expectations equation (3) is

$$U'(c_t(\omega)) = \beta(1 + r_{t+1}^e(\omega))U'(c_{t+1}(\omega)).$$

Thus to first order

$$U''(\bar{C})dc_t(\omega) = \beta(1+\bar{r})U''(\bar{C})dc_{t+1}(\omega) + \beta U'(\bar{C})dr_{t+1}^e(\omega).$$

Using $dr_{t+1}^e(\omega) = f''(\bar{K})dK_{t+1}^e(\omega)$, $\beta(1+\bar{r}) = 1$ and the definition of σ gives (6). ∎

Proof of Lemma 2. The agent's lifetime budget constraint, with his transversality condition incorporated, is given by

$$\sum_{t=0}^{\infty} R_t^e(\omega)c_t(\omega) = \sum_{t=0}^{\infty} R_t^e(\omega)q_t^e(\omega) + (1+r_0)k_0(\omega), \text{ where } R_t^e(\omega) = \prod_{i=1}^{t}(1+r_i^e(\omega))^{-1}$$

and $R_0^e(\omega) = 1$. We compute total derivatives at the steady state. Noting from $dR_t^e(\omega) = -\beta^{t+1}\sum_{i=1}^t dr_i^e(\omega)$ that $\sum_{t\geq 0} dR_t^e(\omega) = -\frac{1}{\bar{r}}\sum_{i\geq 1}\beta^i dr_i^e(\omega)$, we obtain

$$\sum_{t\geq 0}\beta^t dc_t(\omega) = \beta^{-1}dk_0(\omega) + dq_0 + \bar{K}dr_0 + \sum_{t\geq 1}\beta^t dq_t^e(\omega) + \bar{r}^{-1}\left(\bar{C} - \bar{q}\right)\sum_{t\geq 1}\beta^t dr_t^e(\omega).$$

At the steady state, we have $\bar{C} = \bar{q} + \bar{r}\bar{K}$. Also, recalling $dq + \bar{K}dr = 0$ it follows that

$$\begin{aligned}\sum_{t\geq 0}\beta^t dc_t(\omega) &= \beta^{-1}dk_0(\omega) + \sum_{t\geq 1}\beta^t\left(dq_t^e(\omega) + \bar{K}dr_t^e(\omega)\right), \text{ thus} \\ \sum_{t\geq 0}\beta^t dc_t(\omega) &= \beta^{-1}dk_0(\omega). \qquad (18)\end{aligned}$$

Recursively applying equation (7) from Lemma 1 to equation (18) completes the argument. ∎

Proof of Lemma 3. By recursive substitution the linearized Euler equation (6) taken at time t gives $dc_{t+n}(\omega) = dc_t(\omega) + \frac{\beta \bar{C} f''}{\sigma} \sum_{s=1}^{n} dK^e_{t+s}(\omega)$ for $n \geq 1$. Applying Lemma 2 gives

$$\beta^{-1} dk_t(\omega) = (1-\beta)^{-1} dc_t(\omega) + \frac{\beta \bar{C} f''}{\sigma} \sum_{n \geq 1} \beta^n \sum_{s=1}^{n} dK^e_{t+s}(\omega).$$

Using $\sum_{n\geq 1} \beta^n \sum_{s=1}^{n} dK^e_{t+s}(\omega) = (1-\beta)^{-1} \sum_{n \geq 1} \beta^n dK^e_{t+n}(\omega)$ yields the optimal plan for consumption (9). Combining this with (7) yields $dk_{t+1}(\omega) = dk_t(\omega) + \frac{\beta \bar{C} f''}{\sigma} \sum_{n\geq 1} \beta^n dK^e_{t+n}(\omega)$, which is equivalent to (10). ■

Proof of Theorem 1. We start with the savings plan**.** It suffices to show that

$$dk_t(\omega) = -\xi \sum_{n \geq 1} \Theta^k_{tn} \cdot dK^e_n(\omega) \tag{19}$$

satisfies (10) from Lemma 3, where we recall that

$$\xi = -\frac{\beta^2 \bar{C}}{\sigma(1-\beta)} f'' \text{ and } \Theta^k_{tn} = \begin{cases} \beta^{n-t}(1-\beta^t) & \text{if} \quad 1 \leq t < n \\ 1-\beta^n & \text{if} \quad t \geq n \end{cases}.$$

We compute that $dk_t(\omega) + \frac{\beta \bar{C} f''}{\sigma} \sum_{n \geq 1} \beta^n dK^e_{t+n}(\omega)$

$$\begin{aligned}
&= -\xi \sum_{n\geq 1} \Theta^k_{tn} \cdot dK^e_n(\omega) - \xi \left(\frac{1-\beta}{\beta}\right) \sum_{m \geq 1} \beta^m dK^e_{t+m}(\omega) \\
&= -\xi \sum_{n=1}^{t} \Theta^k_{tn} \cdot dK^e_n(\omega) - \xi \sum_{m\geq 1} \left(\beta^{m-1}(1-\beta) + \beta^m(1-\beta^t)\right) dK^e_{t+m}(\omega) \\
&= -\xi \sum_{n=1}^{t} \Theta^k_{tn} \cdot dK^e_n(\omega) - \xi \sum_{m\geq 1} \left(\beta^{m-1} - \beta^{t+m}\right) dK^e_{t+m}(\omega) \\
&= -\xi \sum_{n=1}^{t} \Theta^k_{t+1\,n} \cdot dK^e_n(\omega) - \xi \sum_{m\geq 1} \beta^{m-1} \left(1 - \beta^{t+1}\right) dK^e_{t+m}(\omega) \\
&= -\xi \sum_{n=1}^{t} \Theta^k_{t+1\,n} \cdot dK^e_n(\omega) - \xi \sum_{m\geq 1} \Theta_{t+1\,t+m} \cdot dK^e_{t+m}(\omega) \\
&= -\xi \sum_{n\geq 1} \Theta^k_{t+1\,n} \cdot dK^e_n(\omega) = dk_{t+1}(\omega).
\end{aligned}$$

This establishes the first part of Theorem 1.

To prove the second part concerning the consumption plan, we combine (9) from Lemma 3 with (19) to obtain

$$\begin{aligned} dc_0(\omega) &= \left(\frac{1-\beta}{\beta}\right) \xi \sum_{n\geq 1} \beta^n dK_n^e(\omega) \\ dc_t(\omega) &= -\left(\frac{1-\beta}{\beta}\right) \xi \left(\sum_{n\geq 1} \Theta_{tn}^k dK_n^e(\omega) - \sum_{n\geq 1} \beta^n dK_{t+n}^e(\omega)\right), \text{ for } t \geq 1. \end{aligned}$$

It is helpful to define $\Theta_{0n}^k = 0$, for $n \geq 1$, and let $\chi(\star)$ be the indicator function for the truth value of statement $\star$. Then agent ω's consumption plan may be summarized as

$$dc_t(\omega) = \left(\frac{\beta - 1}{\beta}\right) \xi \sum_{n\geq 1} \left(\Theta_{tn}^k - \chi(n \geq t+1)\, \beta^{n-t}\right) dK_n^e(\omega) \tag{20}$$

for $t \geq 0$. Recall that the first row of Θ^c corresponds to time $t = 0$ consumption. It follows from equation (20) that

$$\begin{aligned} t &= 0 \Rightarrow \Theta_{t+1,n}^c = -\beta^n \\ t &> 0 \Rightarrow \Theta_{t+1,n}^c = \begin{cases} \Theta_{t+1,n}^k - \beta^{n-(t+1)} & \text{if} \quad t+1 \leq n \\ 1 - \beta^n & \text{if} \quad t+1 > n \end{cases}. \end{aligned}$$

The result follows. ■

Proof of Corollary 3. We apply Theorem 1**.** For $1 \leq t < N$ we have $dk_t(\omega)$ equals $-\xi e$ times

$$\begin{aligned} \sum_{T=1}^{t} \Theta_{tT}^k + \sum_{T=t+1}^{N} \Theta_{tT}^k &= \sum_{T=1}^{t} (1 - \beta^T) + \sum_{T=t+1}^{N} (\beta^{T-t} - \beta^T) \\ &= \sum_{T=1}^{t} (1 - \beta^T) + (1 - \beta^t) \sum_{T=1}^{N-t} \beta^T \\ &= t - \beta \frac{1 - \beta^t}{1 - \beta} + (1 - \beta^t) \frac{\beta}{1 - \beta} (1 - \beta^{N-t}) \\ &= t - \bar{r}^{-1} (1 - \beta^t) \beta^{N-t}. \end{aligned}$$

For notational convenience, write $\theta_{tT}^c = \Theta_{t+1\,T}^c$. Then for $0 \leq t < N$ we have

$dc_t(\omega)$ equals $-\xi e$ times

$$\begin{aligned}
\bar{r}\sum_{T=1}^{t}\theta_{tT}^{c}+\bar{r}\sum_{T=t+1}^{N}\theta_{tT}^{c} &= \bar{r}\sum_{T=1}^{t}(1-\beta^{T})-\bar{r}\sum_{T=t+1}^{N}\beta^{T} \\
&= \bar{r}\left(t-\beta\frac{1-\beta^{t}}{1-\beta}\right)-\bar{r}\beta^{t+1}\frac{1-\beta^{N-t}}{1-\beta} \\
&= \bar{r}t-1+\beta^{N}.
\end{aligned}$$

For $t \geq N$ we have $dk_t(\omega)$ equals $-\xi e$ times

$$\sum_{T=1}^{N}\theta^{k}(t,T)=\sum_{T=1}^{N}(1-\beta^{T})=N-\frac{\beta}{1-\beta}(1-\beta^{N})=N-\bar{r}^{-1}(1-\beta^{N})$$

and $dc_t(\omega)$ equals $-\xi e$ times

$$\begin{aligned}
\bar{r}\sum_{T=1}^{N}\theta_{tT}^{c} &= \bar{r}\sum_{T=1}^{N}(1-\beta^{T}) \\
&= \bar{r}\left(N-\frac{\beta}{1-\beta}(1-\beta^{N})\right) \\
&= \bar{r}\left(N-\bar{r}^{-1}(1-\beta^{N})\right),
\end{aligned}$$

which implies $dc_t(\omega)=\bar{r}dk_t(\omega)$. ■

Proof of Theorem 2. Recall that we assume all agents hold initial savings at the steady-state level, i.e. $k_0(\omega)=\bar{K}$. It is useful to work in deviation from steady state form. Let

$$\begin{aligned}
\Delta K &= \{K_t-\bar{K}\}_{t=1}^{\infty}\equiv\{\Delta K_t\}_{t=1}^{\infty} \\
\Delta K^{e}(\omega) &= \{K_t^{e}(\omega)-\bar{K}\}_{t=1}^{\infty}\equiv\{\Delta K_t^{e}(\omega)\}_{t=1}^{\infty} \\
\Delta k(\Delta K^{e}(\omega)) &= \left\{k_t(\Delta K^{e}(\omega))-\bar{K}\right\}_{t=0}^{\infty}\equiv\{\Delta k_t(\omega)\}_{t=1}^{\infty} \\
dk(\Delta K^{e}(\omega)) &= \{dk_t(\Delta K^{e}(\omega))\}_{t=1}^{\infty}\equiv\{dk_t(\omega)\}_{t=1}^{\infty}.
\end{aligned}$$

Here, ΔK is the path of aggregate capital, $\Delta K^{e}(\omega)$ is the belief path for agent ω of aggregate capital, and $\Delta k(\Delta K^{e}(\omega))$ is the optimal savings plan for agent ω given beliefs $\Delta K^{e}(\omega)$. Finally $dk(\Delta K^{e}(\omega))=\Gamma^{k}(\Delta K^{e}(\omega))$ captures the first-order approximation to $\Delta k(\Delta K^{e}(\omega))$. We remark that in Section 2.2, where we used first-order approximations, we identified Δk with dk, whereas here we distinguish them.

It suffices to focus on homogeneous point expectations beliefs. Thus we consider the behavior of an individual agent and drop the identifier ω for notational convenience. We will find $\bar{\varepsilon} > 0$ so that if $0 < \varepsilon < \bar{\varepsilon}$ then $\Delta k(B(\varepsilon)) \not\subset B(\varepsilon)$. There are two cases.

Case 1: $\inf_{t\in\mathbb{N}} \varphi_t^{-1} = 0$. For any $\bar{\varepsilon} > 0$ let $0 < \varepsilon < \bar{\varepsilon}$ and define

$$\Delta K^e = \begin{cases} \varphi_1^{-1}\varepsilon & \text{if } t = 1 \\ 0 & \text{else} \end{cases},$$

so that $\Delta K^e \in B(\varepsilon)$. Since our representative agent believes that the real interest rate will be at its steady-state value from period 2 onward, it follows from the agent's first-order conditions and his lifetime budget constraint that for $t \geq 1, |\Delta k_t(\Delta K^e)| = |\Delta k_1(\Delta K^e)| > 0$. Since $\inf_{t\in\mathbb{N}} \varphi_t^{-1} = 0$, we may choose $\hat{t} > 0$ so that $\varphi_{\hat{t}}^{-1}\varepsilon < |\Delta k_1(\Delta K^e)|$. It follows that

$$\|\Delta k(\Delta K^e)\|_\varphi = \sup_{t\in\mathbb{N}} \varphi_t |\Delta k_t(\Delta K^e)| \geq \varphi_{\hat{t}} |\Delta k_{\hat{t}}(\Delta K^e)| = \varphi_{\hat{t}} |\Delta k_1(\Delta K^e)| > \varepsilon.$$

We conclude that $\Delta k(\Delta K^e) \notin B(\varepsilon)$.

Case 2: $\inf_{t\in\mathbb{N}} \varphi_t^{-1} = \bar{\varphi}^{-1} > 0$. Note that $\bar{\varphi} \geq \varphi_t$ for all $t \in \mathbb{N}$. For $\varepsilon > 0$, set

$$\Delta K^e(\varepsilon, N) = \begin{cases} \frac{\varepsilon}{2\bar{\varphi}} & \text{if } t = 1, \ldots, N \\ 0 & \text{else} \end{cases},$$

so that $\Delta K^e(\varepsilon, N) \in B(\varepsilon)$ for each $N \in \mathbb{N}$. Let $\psi > 1$, and recall that, by assumption, $\varphi_{\min} = \inf_{t\in\mathbb{N}} \varphi_t > 0$. Note also that, by Corollary 3,

$$dk_N(\Delta K^e(\varepsilon, N)) = -\xi\left(N - \bar{r}^{-1}(1-\beta^N)\right)\frac{\varepsilon}{2\bar{\varphi}}.$$

It follows that we may choose N so that for any $\varepsilon > 0$,

$$|dk_N(\Delta K^e(\varepsilon, N))| > \frac{\psi\varepsilon}{\varphi_{\min}}.$$

Finally, restricting attention to N-period deviation beliefs, we may interpret Δk_N as a map from $\mathbb{R}^N$ to $\mathbb{R}$, where $\mathbb{R}^N$ is endowed with the topology induced by the weighted norm $\|\cdot\|_\varphi$. It follows that we may write $\Delta k_N(x) = dk_N(x) + g(x)$ where g is $\mathcal{O}\left(\|x\|_\varphi^2\right)$, and use the second-order property of g to choose $\bar{\varepsilon} > 0$ so that

$$\|x\|_\varphi < \bar{\varepsilon} \implies |g(x)| < \left(\frac{\psi - 1}{2\varphi_{\min}}\right)\|x\|_\varphi.$$

It follows, for $0 < \varepsilon < \bar{\varepsilon}$, that

$$\begin{aligned}\left\|\Delta k\left(\Delta K^e(\varepsilon,N)\right)\right\|_\varphi &= \sup_{n\in\mathbb{N}} \varphi_n \left|\Delta k_n\left(\Delta K^e(\varepsilon,N)\right)\right| \\ &\geq \varphi_N \left|\Delta k_N\left(\Delta K^e(\varepsilon,N)\right)\right| \\ &\geq \varphi_N \left(\left|dk_N\left(\Delta K^e(\varepsilon,N)\right)\right| - \left|g\left(\Delta K^e(\varepsilon,N)\right)\right|\right) \\ &> \varphi_N \left(\frac{\psi\varepsilon}{\varphi_{\min}} - \left(\frac{\psi-1}{2\varphi_{\min}}\right)\left\|\Delta K^e(\varepsilon,N)\right\|_\varphi\right) \\ &> \frac{\varphi_N}{\varphi_{\min}}\left(\frac{1+\psi}{2}\right)\varepsilon > \varepsilon,\end{aligned}$$

which shows that $\Delta k\left(\Delta K^e(N)\right) \notin B(\bar{\varepsilon})$. ■

We remark that while the proof relies on the specific form of Γ^k, Appendix C of the on-line Supplementary Materials indicates that in a linearized setting local strategic substitutability is sufficient for weak eductive instability.

Proof of Theorem 3. To examine B-stability we consider the system (16) with the assumed initial conditions $dK_0 = 0$ and $dK_0^e(\omega) = e$, where $e = \pm\varepsilon$ for all ω. The dynamics of dK_t can equivalently be written as

$$dK_{t+2} = (2 - \alpha(1+\xi))dK_{t+1} - (1-\alpha)dK_t,$$

with $dK_0 = 0$ and $dK_1 = -\xi e$. The eigenvalues of the dynamic system are

$$\lambda_1, \lambda_2 = \frac{1}{2}\left\{2 - \alpha(1+\xi) \pm \sqrt{\alpha}\sqrt{\alpha(1+\xi)^2 - 4\xi}\right\}.$$

If $\alpha(1+\xi)^2 < 4\xi$ these eigenvalues are complex conjugates, and the solution is given by $dK_t = -\frac{\xi e}{r\sin\psi} r^t \sin(\psi t)$, where $r^2 = 1-\alpha$ and

$$\psi = \sin^{-1}\left(\frac{1}{2}\sqrt{(1-\alpha)^{-1}\left(4\xi\alpha - \alpha^2(1+\xi)^2\right)}\right).$$

If $\alpha(1+\xi)^2 > 4\xi$ the eigenvalues are real and distinct (we omit the non-generic case of repeated roots) and $dK_t = -(\lambda_1 - \lambda_2)^{-1}\xi e(\lambda_1^t - \lambda_2^t)$.

To show lack of B-stability for small $\alpha > 0$, note that for sufficiently small α the roots are complex and at $t = T = \frac{\pi}{2\psi}$

$$dK_T = -\frac{2\xi e}{\sqrt{\alpha}\sqrt{4\xi - \alpha(1+\xi)^2}}(1-\alpha)^{\pi/(4\psi)}.$$

It can be verified that $\lim_{\alpha\downarrow 0} dK_{T(\psi(\alpha))} = \pm\infty$, where the sign is opposite to the sign of e. It follows that for $\alpha > 0$ sufficiently small we have $|dK_t| > \varepsilon$ for values of t near T. ■

Proof of Lemma 4. The system can be written as

$$\begin{pmatrix} dK_{t+1} \\ d\hat{K}^e_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & -\xi \\ \alpha & 1-\alpha(1+\xi) \end{pmatrix} \begin{pmatrix} dK_t \\ d\hat{K}^e_t \end{pmatrix}.$$

Let A denote the 2×2 matrix that governs the dynamics. For asymptotic stability we need both eigenvalues within the unit circle. Equivalently (see LaSalle (1986), p. 28) we require $|\det(A)| < 1$ and $|\mathrm{tr}(A)| < 1 + \det(A)$. Since $\det(A) = 1 - \alpha$ the first condition is satisfied for all $0 < \alpha < 1$. Using $\mathrm{tr}(A) = 2 - \alpha(1+\xi)$ leads to the stated condition. ■

Proof of Theorem 4. 1) (Sketch) For ξ sufficiently small, the eigenvalues of the system (16) are real and positive. The proof of Theorem 3 shows that, in this case, the path dK_t is proportional to $(\lambda_1 - \lambda_2)(\lambda_1^t - \lambda_2^t)$, which, as a continuously differentiable function of t, has at most one critical point for $t > 0$. This, together with the fact that the maximum magnitude of dK_t is continuous in ξ, and approaches zero as $\xi \to 0$, yields the result. 2) Asymptotic stability is immediate from Lemma 4. Failure of B-stability for small $\alpha > 0$ follows from the proof of Theorem 3. To show B-stability for large $\alpha < 1$, note that for $\alpha = 1$ the solution is $dK_t = -(1-\xi)^t \xi e$. By continuity $\{dK_t\}_{t=0}^{\infty} \in D(\varepsilon)^{\infty}$ for all $\alpha < 1$ sufficiently large. 3) Lack of partial B-stability is immediate since $dK_1 = -\xi e$ for $dK_0 = 0$ and $dK_0^e(\omega) = e$, where $e = \pm\varepsilon$ for all ω. The asymptotic stability results follow from Lemma 4. ■

Appendix B. The cobweb model

Proposition B1 *Consider the model (14) with $\phi = -C/d'(p^*)$.*

1. *If $0 < \phi < 1$ then p^* is locally eductively stable.*
2. *If $\phi > 1$ then p^* is not locally eductively stable.*

The proof of Proposition B1 requires a Lemma concerning local contractions. If $f : \mathbb{R} \to \mathbb{R}$ and $n \in \mathbb{N}$ then let f^n denote the n^{th}-iterate of f. Let $B(x^*, \varepsilon)$ be the open ε-ball about x^*. We say that f is a local $\hat{\phi}$-contraction at x^* provided that there exists $\hat{\phi} \in (0,1)$ and $\varepsilon > 0$ so that

$$|x - x^*| < \varepsilon \implies |f(x) - x^*| \leq \hat{\phi}|x - x^*|.$$

Lemma B1 *(Contraction Lemma). Suppose $|f'(x^*)| < 1$ and $f'(x^*) \neq 0$. Then there exists $\hat{\phi} \in (0,1)$ so that f is a local $\hat{\phi}$-contraction. Further, there exists $\varepsilon > 0$ such that for all $n \in \mathbb{N}$, $f^n(B(x^*, \varepsilon)) \subset f^{n-1}(B(x^*, \varepsilon))$.*

Proof of Proposition B1. First let

$$\mathcal{F}\left(\{p^e(\omega)\}_{\omega\in I}\right) = d^{-1}\left(C\int_I p^e(\omega)d\omega\right)$$

and let $G(p^e) = d^{-1}(C\cdot p^e)$ be the supply given homogeneous expectations $p^e(\omega) = p^e$. Suppose $0 < \phi < 1$. Let A be an open neighborhood of p^*. We first claim that if A is connected then $\mathcal{F}^n\left(A^I\right) = G^n(A)$, where the superscript indicated recursive composition. Since G captures homogeneous expectations, it is immediate that $G(A)\subset\mathcal{F}^1\left(A^I\right)$; indeed, this holds regardless of whether A is connected. Now let $p\in\mathcal{F}^1\left(A^I\right)$. Then there exists a beliefs profile $\{p^e(\omega)\}\in A^I$ with

$$p = d^{-1}\left(C\cdot\int_I p^e(\omega)d\omega\right).$$

But

$$\{p^e(\omega)\}\in A^I \implies \inf\, A\le p^e\equiv\int_I p^e(\omega)d\omega\le\sup A,$$

with the respective inequalities holding strictly if $\inf\, A, \sup A\notin A$ respectively. Since A is connected, it follows that $p^e\in A$ whence $p = G(p^e)\in G(A)$. Having established $\mathcal{F}^1\left(A^I\right) = G(A)$, we may proceed by induction. Since the continuous image of connected sets are connected, we have by induction that $G^{n-1}(A) = \mathcal{F}^{n-1}\left(A^I\right)$ is connected. Thus

$$\mathcal{F}^n(A) = F\left(\left(\mathcal{F}^{n-1}(A)\right)^I\right) = \mathcal{F}^1\left(\mathcal{F}^{n-1}(A)\right) = \mathcal{F}^1\left(G^{n-1}(A)\right) = G^n(A),$$

and the result follows.

To complete the stability argument, let U be an open neighborhood of p^*. Now note that $G(p^*) = p^*$ and $|G'(p^*)| < 1$. It follows from Lemma B1 that there exists $\varepsilon > 0$ such that $B(p^*,\varepsilon)\subset U$ and $G^n\left(B(p^*,\varepsilon)\right)\subset G^{n-1}\left(B(p^*,\varepsilon)\right)$. Also, since G is a local $\hat{\phi}$-contraction for some $\hat{\phi}\in(0,1)$ we have that $G^n\left(B(p^*,\varepsilon)\right)\subset B\left(p^*,\hat{\phi}^n\varepsilon\right)$, so that

$$p^*\in\bigcap_n G^n\left(B(p^*,\varepsilon)\right)\subset\bigcap_n B\left(p^*,\hat{\phi}^n\varepsilon\right) = \{p^*\}.$$

Turning to part 2 of the proposition, suppose instead that $\phi > 1$. To show instability we may consider homogeneous expectations. Write

$$G(p^e) - p^* = -\phi(p^e - p^*) + g(p^e - p^*),$$

where g is $\mathcal{O}\left(|p^e - p^*|^2\right)$. Choose ε so that

$$p^e \in B(p^*, 2\varepsilon) \implies \frac{|g(p^e - p^*)|}{|p^e - p^*|} < \frac{\phi - 1}{2}.$$

Then $p^e = p^* + \varepsilon$ implies

$$\begin{aligned}
|G(p^e) - p^*| &> \phi|p^e - p^*| - |g(p^e - p^*)| \\
&= \left(\phi - \frac{|g(p^e - p^*)|}{|p^e - p^*|}\right) |p^e - p^*| \\
&> \left(\frac{1 + \phi}{2}\right) |p^e - p^*| > \varepsilon.
\end{aligned}$$

Since $p^* + \varepsilon$ is on the boundary of $B(p^*, \varepsilon)$ it follows by continuity that there exists $p^e \in B(p^*, \varepsilon)$ so that $|G(p^e) - p^*| > \varepsilon$. ■

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On-line Supplementary Material
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Eductive Stability in Real Business Cycle Models

George W. Evans
University of Oregon,
University of St. Andrews

Roger Guesnerie
Paris School of Economics
Collège de France

Bruce McGough
University of Oregon

March 5, 2018

Appendix B: Proof of Contraction Lemma

Lemma B1 *(Contraction Lemma). Suppose* $|f'(x^*)| < 1$ *and* $f'(x^*) \neq 0$. *Then there exists* $\hat{\phi} \in (0,1)$ *so that* f *is a local* $\hat{\phi}$*-contraction. Further, there exists* $\varepsilon > 0$ *such that for all* $n \in \mathbb{N}$, $f^n(B(x^*,\varepsilon)) \subset f^{n-1}(B(x^*,\varepsilon))$.

Proof of Lemma B1. Re-center so that $x^* = 0$ and let $\phi = f'(0)$. Now observe that the f is a local $\hat{\phi}$-contraction at the origin. To see this, write $f(x) = \phi x + g(x)$, where g is second order in $|x|$. Choose $\varepsilon > 0$ so that

$$|x| < \varepsilon \implies \left|\frac{g(x)}{x}\right| < \frac{1-|\phi|}{2},$$

and let

$$\hat{\phi} = |\phi| + \sup_{x \in B(\varepsilon)} \left|\frac{g(x)}{x}\right| < 1.$$

It follows that

$$|f(x)| \leq \left(|\phi| + \left| \frac{g(x)}{x} \right| \right) |x| < \hat{\phi}|x|,$$

as needed.

Turning to the main argument, first suppose $\phi > 0$. Choose ε sufficiently small so that f is strictly increasing on $B(\varepsilon)$. A simple induction shows that f^n is consequently strictly increasing on $B(\varepsilon)$, from which it follows that

$$\begin{aligned} f^n(B(\varepsilon)) &= (f^n(-\varepsilon), f^n(\varepsilon)) \subset \left(\hat{\phi} f^{n-1}(-\varepsilon), \hat{\phi} f^{n-1}(\varepsilon)\right) \\ &\subset \left(f^{n-1}(-\varepsilon), f^{n-1}(\varepsilon)\right) = f^{n-1}(B(\varepsilon)). \end{aligned}$$

Next assume that $\phi < 0$ and that $\varepsilon > 0$ is chosen to be small enough that f is strictly decreasing on $B(\varepsilon)$. A simple induction shows that f^n is consequently strictly decreasing on $B(\varepsilon)$ when n is odd and strictly increasing when n is even, from which it follows that $f^n(B(\varepsilon)) = (\delta_L^n, \delta_H^n)$ for appropriate $\delta_L^n < 0 < \delta_H^n$.

Observe that

$$\begin{aligned} (\delta_L^n, \delta_H^n) &= (f^n(-\varepsilon), f^n(\varepsilon)) \quad \text{if } n \text{ is even} \\ (\delta_L^n, \delta_H^n) &= (f^n(\varepsilon), f^n(-\varepsilon)) \quad \text{if } n \text{ is odd.} \end{aligned}$$

We work by induction. Since f is a local $\hat{\phi}$-contraction at the origin, it follows that $f(B(\varepsilon)) \subset B(\varepsilon)$. Assume the nesting holds for all $k \leq n$. By induction, $\delta_L^{n-1} < \delta_L^n$ and $\delta_H^{n-1} > \delta_H^n$. Since f is decreasing, it follows that $f(\delta_L^n) < f\left(\delta_L^{n-1}\right)$ and $f(\delta_H^n) > f\left(\delta_H^{n-1}\right)$.

Case 1: n is even.

$$\begin{aligned} \delta_H^{n+1} &= f^{n+1}(-\varepsilon) = f(f^n(-\varepsilon)) = f(\delta_L^n) < f\left(\delta_L^{n-1}\right) \\ &= f\left(f^{n-1}(\varepsilon)\right) = f^n(\varepsilon) = \delta_H^n \\ \delta_L^{n+1} &= f^{n+1}(\varepsilon) = f(f^n(\varepsilon)) = f(\delta_H^n) > f\left(\delta_H^{n-1}\right) \\ &= f\left(f^{n-1}(-\varepsilon)\right) = f^n(-\varepsilon) = \delta_L^n. \end{aligned}$$

Case 2: n is odd.

$$\begin{aligned} \delta_H^{n+1} &= f^{n+1}(\varepsilon) = f(f^n(\varepsilon)) = f(\delta_L^n) < f\left(\delta_L^{n-1}\right) \\ &= f\left(f^{n-1}(-\varepsilon)\right) = f^n(-\varepsilon) = \delta_H^n \\ \delta_L^{n+1} &= f^{n+1}(-\varepsilon) = f(f^n(-\varepsilon)) = f(\delta_H^n) > f\left(\delta_H^{n-1}\right) \\ &= f\left(f^{n-1}(\varepsilon)\right) = f^n(\varepsilon) = \delta_L^n. \end{aligned}$$

This establishes the result. ■

Appendix C: Strategic Substitutability

In this Appendix we establish the important connection between our instability results and a natural notion of strategic substitutability in our dynamic infinite-horizon setting. For simplicity, we restrict attention to the linearized environment; and we focus on the behavior of a representative agent, which allows us to drop reference to $\omega \in I$.

Recall that if $dK^e = \{dK_t^e\}$ is the beliefs path of a given agent then his corresponding savings plan is then given by $dk(dK^e) = \Gamma^k(dK^e)$, where $\Gamma^k = \{\Gamma^k_{t\,s}\}_{t,s\geq 1}$ is as defined in Section 2.2.1. Using this notation, we may write

$$dk_t = \sum_{s\geq 1} \Gamma^k_{t\,s} \cdot dK^e_s.$$

We say that the map Γ^k exhibits *local strategic substitutability* provided that $\Gamma^k_{t\,s} < 0$ for all $t, s \in \mathbb{N}$. The following result emphasizes that our main instability result is implied by the local strategic substitutability of Γ^k.

Theorem C1 *Let $\varphi \in R^{\mathbb{N}}$ be a sequence of positive weights satisfying $\varphi_{\min} = \inf_{n\in\mathbb{N}} \varphi_n > 0$. If Γ^k exhibits local strategic substitutability, and if*

$$\eta \equiv \inf_{n>0} |\Gamma^k_{nn}| > 0,$$

then there exists $\bar{\varepsilon} > 0$ so that $0 < \varepsilon < \bar{\varepsilon}$ implies $\Gamma^k(B(\varepsilon)) \not\subset B(\varepsilon)$.

Proof. Let $\bar{\varepsilon} > 0$ and $0 < \varepsilon < \bar{\varepsilon}$. Let $\varepsilon_t = \frac{\varepsilon}{2\varphi_t}$ and define

$$\Delta K_t^e(\varepsilon, N) = \begin{cases} \varepsilon_t & \text{if } t = 1, \ldots, N \\ 0 & \text{else} \end{cases},$$

so that $\Delta K^e(\varepsilon, N) \in B(\varepsilon)$.

Next, let $dk(\varepsilon, N) = dk\left(\Delta K^e(\varepsilon, N)\right)$, and notice that since $dK^e_{N+s}(\varepsilon, N) = 0$ for $s > 0$, it follows that $dk_{N+s}(\varepsilon, N) = dk_N(\varepsilon, N)$ for $s > 0$. This allows us to compute

$$\begin{aligned} dk_{N+1}(\varepsilon, N+1) &= dk_{N+1}(\varepsilon, N) + \Gamma^k_{N+1,N+1} \cdot \varepsilon_{N+1} \\ &= dk_N(\varepsilon, N) + \Gamma^k_{N+1,N+1} \cdot \varepsilon_{N+1} \\ &\leq dk_N(\varepsilon, N) - \eta \cdot \varepsilon_{N+1}. \end{aligned}$$

Substituting in recursively and using $dk_0(\varepsilon, N+1) = 0$, we have

$$dk_{N+1}(\varepsilon, N+1) \leq -\eta \sum_{m=1}^{N+1} \varepsilon_m.$$

Now set $s_N = \sum_{i=1}^{N} \varepsilon_i$. We claim that for any $\eta > 0$, there is some N so that $\eta s_N > 2\varepsilon_N$. Indeed, if s_N diverges the result follows from the fact that

$$2\varepsilon_N \leq \frac{\varepsilon}{\varphi_{\min}} < \infty;$$

otherwise, ηs_N is increasing and $2\varepsilon_N \to 0$. This shows

$$\begin{aligned} \|dk(\varepsilon, N+1)\|_{\varphi} &= \sup_{t \in \mathbb{N}} \varphi_t \left| dk_t(\varepsilon, N+1) \right| \geq \varphi_{N+1} |dk_{N+1}(\varepsilon, N+1)| \\ &\geq \varphi_{N+1} \cdot \left(\eta \sum_{m=1}^{N+1} \varepsilon_m \right) = \varphi_{N+1} \cdot (\eta s_{N+1}) \\ &> \varphi_{N+1} \cdot (2\varepsilon_{N+1}) = \varepsilon, \end{aligned}$$

so that $\Gamma^k \left(B(\varepsilon) \right) \not\subset B(\varepsilon)$. ■

We note that by Corollary 1 of the main text, Γ^k exhibits local strategic substitutability. Since $\inf_{n>0} |\Gamma^k_{nn}| = \xi(1-\beta) > 0$, Theorem C1 yields Theorem 2 of the main text as a corollary. Also, Corollary 1 and Theorem C1 focus on the implications of local strategic substitutability precisely because our map Γ^k exhibits this characteristic; however, analogous results with analogous proofs hold for maps that exhibit local strategic complementarity.

Appendix D: A Finite-horizon Model with Capital

Here we show that eductive stability can obtain in a finite horizon model with capital. To show this we examine a dynamic model with the shortest possible horizon: the two-period OLG model with capital.

Population is constant and normalized to one, and all markets are competitive. Let ω_t be an agent born at time t. He is endowed with one unit of labor, which he supplies inelastically for real wage q_t. He then allocates his income between savings $s(\omega_t) = k(\omega_t)$ and consumption $c_1(\omega_t)$. In period $t+1$, this agent is now old: he rents his savings for net real return r_{t+1}, consumes the gross return, plus his share of profits from ownership of firms, and dies. Thus agent ω_t solves the following problem

$$\begin{aligned} & \max E(\omega_t)\left\{u\left(c_1(\omega_t), c_2(\omega_t)\right)\right\} \\ \text{s.t.} \quad & c_1(\omega_t) + s(\omega_t) = q_t \qquad (1) \\ & c_2(\omega_t) = (1 + r^e_{t+1}(\omega_t))s(\omega_t) + \pi(\omega_t) \qquad (2) \end{aligned}$$

Notice that when agent ω_t makes his savings decision, he does not know the value of r_{t+1}. Below we assume constant returns to scale production so that $\pi = 0$.

The agent ω_t's first-order condition is given by

$$u_{c_1}\left(c_1(\omega_t), c_2(\omega_t)\right) = \beta(1 + r^e_{t+1}(\omega_t))u_{c_2}\left(c_1(\omega_t), c_2(\omega_t)\right). \qquad (3)$$

Equations (1)–(3) may be used to compute the savings decision of agent ω_t based on current and expected future factor prices:

$$s(\omega_t) = s(q_t, r^e_{t+1}(\omega_t)).$$

Firms hire workers and rent capital in competitive factor markets, and employ constant returns to scale technology to manufacture goods: $Y = f(K, L)$; thus profits are zero and factors prices are given by the respective marginal products: $q = f_L(K, 1) \equiv f_L(K)$ and $r = f_K(K, 1) \equiv f_K(K)$. Capital is inelastically supplied "in the morning" by the old and depreciation is zero: the capital accumulation equation is given accordingly by

$$K_{t+1} = \int s(\omega_t)d\omega_t = \int s(q_t, r^e_{t+1}(\omega_t))d\omega_t.$$

Assuming agents know the relationship between real interest rates and marginal products, and so form expectations of aggregate capital instead of real interest rates, we have

$$K_{t+1} = \int s\left(f_L(K_t), f_K(K^e_{t+1}(\omega_t))\right), \qquad (4)$$

where $K^e_{t+1}(\omega_t)$ is agent ω_t's forecast of aggregate capital tomorrow. Equation (4) captures the dynamics of the economy: given aggregate capital today and forecasts of aggregate capital tomorrow, the actual value of aggregate capital tomorrow can be determined. It also highlights a key difference between the OLG model and the RBC model: in the OLG model aggregate capital depends only on one-period-ahead forecasts; in the RBC model, aggregate capital depends on forecasts at all horizons.

Adopting for open sets the standard topology on $\mathbb{R}$, the following result characterizes stability:

Theorem D1 *The steady state $\bar{K}$ is locally strongly eductively stable if and only if*

$$\left|\partial \mathbf{s}/\partial \mathbf{r}\left(f_L(\bar{K},1), f_K(\bar{K},1)\right) \cdot f_{KK}\left(\bar{K},1\right)\right| < 1. \tag{5}$$

This theorem can be established using a formal argument analogous to the proof of Proposition B1, and for this reason, we omit the details here. Intuitively, if condition (5) holds then an iterative argument shows that period $t = 0$ young agents will conclude that $K_1 = \bar{K}$. For young agents at time $t = 1$ the situation is then identical to the situation at time $t = 0$. Thus at $t = 1$ agents will conclude that $K_2 = \bar{K}$ and this implies that it will be the case that $K_2 = \bar{K}$. An induction argument completes the proof.

We remark that this stability result is local and can provide a refinement criterion in the case of multiple steady states.

As an exercise for illustrating our results, we specify particular functional forms and conduct numerical analysis. Assume utility is time separable and takes the constant relative risk-aversion form

$$u(c_1, c_2) = \frac{1}{1-\sigma}\left(c_1^{1-\sigma} + c_2^{1-\sigma} - 2\right),$$

for $\sigma > 0$, and assume that production is Cobb-Douglas, $f(K, L) = K^{\theta}L^{1-\theta}$. In this case there is a unique positive steady-state level of capital, and parameter values for θ and σ completely characterize the model. For all parameters examined – $\theta \in (0, 1)$ and $\sigma \in (0, 100)$ – the steady state is strongly eductively stable.

This example provides a striking contrast to the coordination problems we have demonstrated for the RBC model with infinitely-lived agents.