Self-stabilizing processes based on random signs

K.J. Falconer

Mathematical Institute, University of St Andrews, North Haugh, St Andrews, Fife, KY16 9SS, Scotland, UK

and

J. Lévy Véhel

Case Law Analytics & Inria,

Université Nantes, Laboratoire de Mathématiques Jean Leray

2 Rue de la Houssinière - BP 92208 - F-44322 Nantes Cedex, France

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Abstract

A self-stabilizing processes $\{Z(t), t \in [t_0, t_1)\}$ is a random process which when localized, that is scaled to a fine limit near a given $t \in [t_0, t_1)$, has the distribution of an $\alpha(Z(t))$-stable process, where $\alpha : \mathbb{R} \to (0, 2)$ is a given continuous function. Thus the stability index near t depends on the value of the process at t. In another paper [5] we constructed self-stabilizing processes using sums over plane Poisson point processes in the case of $\alpha : \mathbb{R} \to (0, 1)$ which depended on the almost sure absolute convergence of the sums. Here we construct pure jump self-stabilizing processes when α may take values greater than 1 when convergence may no longer be absolute. We do this in two stages, firstly by setting up a process based on a fixed point set but taking random signs of the summands, and then randomizing the point set to get a process with the desired local properties.

1 Introduction and background

For a fixed $0 < \alpha \leq 2$ *symmetric α-stable Lévy motion* $\{L_\alpha(t), t \geq 0\}$ is a stochastic process characterized by having stationary independent increments with $L(0) = 0$ almost surely, and $L_\alpha(t) - L_\alpha(s)$ $(s > t)$ having the distribution of $S_\alpha((t-s)^{1/\alpha}, 0, 0)$, where $S_\alpha(c, \beta, \mu)$ denotes a stable random variable with stability-index α, with scale parameter c, skewness parameter β and shift μ. A detailed account of such processes may be found in [12] but we summarize here the features we need. Stable motion L_α is $1/\alpha$-self-similar in the sense that $L_\alpha(ct)$ and $c^{1/\alpha} L_\alpha(t)$ are equal in distribution so in particular have the same finite-dimensional distributions. There is a version of L_α such that its sample paths are càdlàg, that is right continuous with left limits.

One way of representing symmetric α-stable Lévy motion L_α is as a sum over a plane point process. Throughout the paper we write

$$r^{\langle s \rangle} = \text{sign}(r)|r|^s \text{ for } r \in \mathbb{R}, s \in \mathbb{R}.$$

Then

$$L_\alpha(t) = C_\alpha \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} 1_{(0,t]}(\mathsf{X})\mathsf{Y}^{\langle -1/\alpha\rangle}, \tag{1.1}$$

where C_α is a normalising constant given by

$$C_\alpha = \Big(\int_0^\infty u^{-\alpha}\sin u\, du\Big)^{-1/\alpha}$$

and where Π is a Poisson point process on $\mathbb{R}^+ \times \mathbb{R}$ with plane Lebesgue measure $\mathcal{L}^2$ as mean measure, so that for a Borel set $A \subset \mathbb{R}^+ \times \mathbb{R}$ the number of points of Π in A has a Poisson distribution with parameter $\mathcal{L}^2(A)$, independently for disjoint A. The sum (1.1) is almost surely absolutely uniformly convergent if $0 < \alpha < 1$, but if $\alpha \geq 1$ then (1.1) must be taken as the limit as $n \to \infty$ of symmetric partial sums

$$L_{\alpha,n}(t) = C_\alpha \sum_{(\mathsf{X},\mathsf{Y})\in\Pi:|\mathsf{Y}|\leq n} 1_{(0,t]}(\mathsf{X})\mathsf{Y}^{\langle -1/\alpha\rangle},$$

in the sense that $\|L_{\alpha,n} - L_\alpha\|_\infty \to 0$ almost surely.

Several variants of α-stable motion have been considered. For example, for *multistable Lévy motion* $\{M_\alpha(t), t \geq 0\}$ the stability index α in (1.1) can depend on X so that the local behaviour changes with t, see [3, 4, 6, 7, 9, 10, 11]. Thus given a continuous $\alpha : \mathbb{R}^+ \to (0, 2)$,

$$M_\alpha(t) = \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} 1_{(0,t]}(\mathsf{X})C_{\alpha(\mathsf{X})}\mathsf{Y}^{\langle -1/\alpha(\mathsf{X})\rangle}.$$

Then M_α is a Markov process. Under certain conditions it is *localisable* with *local form* $L_{\alpha(t)}$, in the sense that near t the process 'looks like' an $\alpha(t)$-stable process, that is for each $t > 0$ and $u \in \mathbb{R}$,

$$\frac{M_\alpha(t + ru) - M_\alpha(t)}{r^{1/\alpha(t)}} \overset{\text{dist}}{\to} L_{\alpha(t)}(u)$$

as $r \searrow 0$, where convergence is in distribution with respect to the Skorohod metric and consequently is convergent in finite dimensional distributions, see [3, 4].

The local stability parameter of multistable Lévy motion depends on the time t but in some contexts, for example in financial modelling, it may be appropriate for the local stability parameter to depend instead (or even as well) on the *value* of the process at time t. Such a process might be called 'self-stabilizing'. Thus, for suitable $\alpha : \mathbb{R} \to (0, 2)$, we seek a process $\{Z(t), t \geq 0\}$ that is localisable with local form $L^0_{\alpha(Z(t))}$, in the sense that for each t and $u > 0$,

$$\frac{Z(t + ru) - Z(t)}{r^{1/\alpha(Z(t))}}\bigg|\,\mathcal{F}_t \overset{\text{dist}}{\to} L^0_{\alpha(Z(t))}(u) \tag{1.2}$$

as $r \searrow 0$, where convergence is in distribution and finite dimensional distributions and where $\mathcal{F}_t$ indicates conditioning on the process up to time t. (For notational simplicity it is easier to construct Z_α with the non-normalised α-stable processes $L^0_\alpha = C_\alpha^{-1}L_\alpha$ as its local form.)

Throughout the paper we write $D[t_0, t_1)$ for the càdlàg functions on the interval $[t_0, t_1)$, that is functions that are right continuous with left limits; this is the natural space for functions defined by sums over point sets.

In an earlier paper [5] we constructed self-stabilizing processes for $\alpha : \mathbb{R}^+ \to (0,1)$ by first showing that there exists a deterministic function $f \in D[t_0, t_1)$ satisfying the relation

$$f(t) = a_0 + \sum_{(x,y)\in\Pi} 1_{(t_0,t]}(x) y^{\langle -1/\alpha(f(x_-))\rangle}$$

for a *fixed* point set Π, and then randomising to get a random function Z such that

$$Z(t) = a_0 + \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} 1_{(t_0,t]}(\mathsf{X})\, \mathsf{Y}^{\langle -1/\alpha(Z(\mathsf{X}_-))\rangle} \qquad (t_0 \le t < t_1).$$

Then, for all $t \in [t_0, t_1)$ this random function satisfies (1.2) almost surely. However, this approach depends on the infinite sums being absolutely convergent, which need not be the case if $\alpha(t) \ge 1$ for some t.

Here we use an alternative approach to construct self-stabilizing processes where $\alpha : \mathbb{R}^+ \to (0,2)$ and in general we cannot assume absolute convergence of the sums. We show in Section 2 that for a fixed point set $\Pi^+ \subset (t_0, t_1) \times \mathbb{R}^+$ and independent random 'signs' $S(x,y) = \pm 1$ there exists almost surely a random function $Z \in D[t_0, t_1)$ satisfying

$$Z(t) = a_0 + \sum_{(x,y)\in\Pi^+} 1_{(t_0,t]}(x) S(x,y) y^{-1/\alpha(Z(x_-))} \qquad (t_0 \le t < t_1),$$

see Theorem 2.2. To achieve this we work with partial sums

$$Z_n(t) = a_0 + \sum_{(x,y)\in\Pi^+ : |y|\le n} 1_{(t_0,t]}(x) S(x,y) y^{-1/\alpha(Z(x_-))} \qquad (t_0 \le t < t_1)$$

and show that the limit as $n \to \infty$ exists in a norm given by $\mathbb{E}\big(\|\cdot\|_\infty^2\big)^{1/2}$, where $\mathbb{E}$ denotes expectation. This is more awkward than it might seem at first sight since, as n increases, if a new point $(x,y) \in \Pi^+$ enters the sum, then $Z_n(t)$ will change for $t \ge x$ so for all (x', y') with $x' > x$ and $y' < y$ the summands $y'^{\langle -1/\alpha(Z_n(x'_-))\rangle}$ will change, with a knock on effect so that the change in $Z_n(t)$ may be considerably amplified as t increases past further x with $(x,y) \in \Pi^+$ and $y < n$.

In Section 3 we randomise the construction further by taking Π^+ to be a Poisson point process on $(t_0, t_1) \times \mathbb{R}^+$ with mean measure $2\mathcal{L}^2$ which, combined with the random signs, gives a point process with the same distribution as Π on $(t_0, t_1) \times \mathbb{R}$. We show that the resulting process Z satisfies a Hölder continuity property and is self-stabilizing in the sense of (1.2), see Theorem 3.6.

1.1 Basic facts used throughout the paper

For the rest of the paper we fix $a_0 \in \mathbb{R}$ and $0 < a < b < 2$ together with a function $\alpha : \mathbb{R} \to [a,b]$ that is continuously differentiable with bounded derivative. By the mean value theorem,

$$y^{-1/\alpha(v)} - y^{-1/\alpha(u)} \;=\; (v-u) y^{-1/\alpha(\xi)} \log y \frac{\alpha'(\xi)}{\alpha(\xi)^2} \qquad (y > 0,\ u, v \in \mathbb{R}),$$

where $\xi \in (u,v)$. In particular this gives the estimate we will use frequently:

$$\left| y^{-1/\alpha(v)} - y^{-1/\alpha(u)} \right| \;\le\; M\, |v-u|\, y^{-1/(a,b)} \qquad (y > 0,\ u, v \in \mathbb{R}), \tag{1.3}$$

where

$$M \;=\; \sup_{\xi\in\mathbb{R}} \frac{|\alpha'(\xi)|}{\alpha(\xi)^2},$$

and for convenience we write

$$y^{-1/(a,b)} \;=\; \max\left\{y^{-1/a}\big(1+|\log y|\big), y^{-1/b}\big(1+|\log y|\big)\right\} \qquad (y>0)$$

and

$$y^{-2/(a,b)} \;=\; (y^{-1/(a,b)})^2.$$

For $t_0 < t_1$ and a suitable probability space Ω (to be specified later), we will work with functions $F : \Omega \times [t_0,t_1) \to \mathbb{R}\cup\{\infty\}$ which we assume to be measurable (taking Lebesgue measure on $[t_0,t_1)$). Writing $F_\omega(t)$ for the value of F at $\omega\in\Omega$ and $t\in[t_0,t_1)$, we think of F_ω as a random function on $[t_0,t_1)$ in the natural way (most of the time we will write F instead of F_ω when the underlying randomness is clear). In particular we will work in the space

$$\mathcal{D} \;=\; \left\{F : F_\omega \in D[t_0,t_1) \text{ for almost all } \omega\in\Omega \text{ with } \mathbb{E}\big(\|F\|_\infty^2\big) < \infty\right\},$$

where $\mathbb{E}$ is expectation, $D[t_0,t_1)$ denotes the càdlàg functions, and $\|\cdot\|_\infty$ is the usual supremum norm. By identifying F and F' if $F_\omega = F'_\omega$ for almost all $\omega\in\Omega$, this becomes a normed space under the norm

$$\Big(\mathbb{E}\big(\|F\|_\infty^2\big)\Big)^{1/2}. \tag{1.4}$$

A routine check shows that (1.4) defines a complete norm on $\mathcal{D}$.

2 Point sums with random signs

In this section we fix a discrete point set $\Pi^+ \subset (t_0,t_1)\times\mathbb{R}^+$ and form sums over values at the points of Π^+ with an independent random assignment of sign $+$ or $-$ at each point of Π^+.

We will assume that the point set Π^+ satisfies

$$\sum_{(x,y)\in\Pi^+} y^{-2/(a,b)} < \infty;$$

this will certainly be the case if $\sum_{(x,y)\in\Pi^+} y^{-2/b'} < \infty$ for some b' with $b<b'<2$.

Our first aim is to show that if $\{S(x,y)\in\{-1,1\} : (x,y)\in\Pi^+\}$ are random 'signs', that is independent random variables taking the values 1 and -1 with equal probability $\frac{1}{2}$, then, almost surely, there exists a random function $Z\in D[t_0,t_1)$ satisfying

$$Z(t) = a_0 + \sum_{(x,y)\in\Pi^+} 1_{(t_0,t]}(x) S(x,y) y^{-1/\alpha(Z(x_-))} \qquad (t_0\le t<t_1). \tag{2.1}$$

in an appropriate sense. This Z will be the limit in norm of the random functions obtained by restricting the sums to $y\le n$. Thus we define for $n\in\mathbb{N}$

$$Z_n(t) = a_0 + \sum_{(x,y)\in\Pi^+ :\, y\le n} 1_{(t_0,t]}(x) S(x,y) y^{-1/\alpha(Z_n(x_-))} \qquad (t_0\le t<t_1). \tag{2.2}$$

Here, and throughout this section, our probability space has Ω as the set $\{-1,1\}^{\Pi^+} \equiv \{S(x,y) \in \{-1,1\} : (x,y) \in \Pi^+\}$ of all assignments of signs ± 1 to the points of Π^+, and σ-field generated by the subsets of the form $S \times \{-1,1\}^{\Pi^+ \setminus X}$ for each finite $X \subset \Pi^+$ and each $S \subset \{-1,1\}^X$. In particular, the probability that a given set of k points of Π^+ having any particular assignment of signs is 2^{-k}.

Note that the sum in (2.2) is over the finite set $\{(x,y) \in \Pi^+ : y \le n\}$ so, given the $S(x,y)$, the piecewise constant $Z_n(t) \in D[t_0,t_1)$ can be evaluated inductively over increasing x with $(x,y) \in \{\Pi^+ : y \le n\}$ in a finite number of steps. Nevertheless, as has been remarked, we need to be careful about taking limits of $\{Z_n\}_n$ as $n \to \infty$ since on increasing n the contributions from the summands with $0 < y < n$ change as the values of $Z_n(x_-)$ change. Note that if $0 < a < b < 1$ the sum (2.1) is absolutely convergent, and this case is considered in [5], but if $0 < a < b < 2$ then $\alpha(z)$ may take values greater than 1 and care is needed in defining Z. We will show that there exists $Z \in \mathcal{D}$ such that $\mathbb{E}\big(\|Z_n - Z\|_\infty^2\big) \to 0$ and Z satisfies (2.1) in an appropriate sense. In particular there is a sequence of integers $n_j \nearrow \infty$ such that almost surely $\|Z_{n_j} - Z\|_\infty \to 0$, i.e. Z_{n_j} converges uniformly to Z.

2.1 Existence of functions defined by random signs

To obtain such a Z we show that $\{Z_n\}_n$ is a Cauchy sequence in the complete norm (1.4). $\mathbb{E}\big(\|\cdot\|_\infty^2\big)^{1/2}$ on $\mathcal{D}$.

It is convenient to make a further assumption on Π^+, that if $(x,y),(x',y') \in \Pi^+$ then $x \neq x'$. Without this assumption the results remain valid with an essentially identical proof, but the notation becomes more cumbersome as the single terms added in (2.7) have to be replaced by sums over several terms corresponding to each point (x_i,y_i) with a common value of x_i. In any case, when in Section 3 we let Π^+ be a realisation of a Poisson point process, this assumption will hold almost surely.

Proposition 2.1. *Let a_0, α and Π^+ be as above and let Z_n be given by* (2.2)*. Then for $m \ge n \ge 1$,*

$$\mathbb{E}\big(\|Z_m - Z_n\|_\infty^2\big) \;\le\; 4 \prod_{(x,y)\in\Pi^+ : 0<y\le n} (1 + M^2 y^{-2/(a,b)}) \sum_{(x,y)\in\Pi^+ : n<y\le m} y^{-2/b}. \tag{2.3}$$

In particular $\{Z_n\}_n$ is a Cauchy sequence in $\mathcal{D}$ under the norm $\mathbb{E}(\|\cdot\|_\infty^2)^{1/2}$.

Proof. Let $m > n$. We list the points

$$\{(x,y) \in \Pi^+ : y \le m\} = \{(x_1,y_1), \ldots, (x_N,y_N)\},$$

with $t_0 < x_1 < \cdots < x_N < t_1$, where as mentioned we assume that the x_i are distinct. For notational convenience we set $x_0 := t_0$ and $x_{N+1} := t_1$. We write $i_1 < i_2 < \cdots < i_K$ for the indices such that $y_{i_k} \le n$, and let $i_0 := 0$ and $i_{K+1} := N+1$ With this notation, (2.2) restricts to the jump points x_i as

$$Z_m(x_i) = a_0 + \sum_{0<j\le i} S(x_j,y_j) y_j^{-1/\alpha(Z_m(x_{j-1}))}, \quad Z_n(x_i) = a_0 + \sum_{k:0<i_k\le i} S(x_{i_k},y_{i_k}) y_{i_k}^{-1/\alpha(Z_n(x_{i_k-1}))}. \tag{2.4}$$

Write

$$c_k = M^2\, y_{i_k}^{-2/(a,b)} \qquad (1 \le k \le K) \tag{2.5}$$

and

$$\epsilon_k = \sum_{i_{k-1}+1 \le i \le i_k - 1} y_i^{-2/b} \qquad (1 \le k \le K+1). \tag{2.6}$$

Let $\mathcal{F}_i$ be the minimal σ-field of subsets of Ω such that the sign assignments $\{S(x_j, y_j) : 1 \le j \le i\}$ are Borel measurable; thus conditioning on $\mathcal{F}_i$ is equivalent to taking the finite set of values $\{S(x_j, y_j) : 1 \le j \le i\}$ as known. We consider $\big\{Z_n(x_i) - Z_m(x_i), \mathcal{F}_i\big\}_{i=0}^{N}$ which from (2.4) and that $\mathbb{E}(S(x_j, y_j)) = 0$ is a bounded martingale. Indeed, from (2.4), for $1 \le i \le N$,

$$\begin{aligned} Z_m(x_i) - Z_n(x_i) \;&=\; Z_m(x_{i-1}) - Z_n(x_{i-1}) \\ &+ \begin{cases} S(x_i, y_i)\, y_i^{-1/\alpha(Z_m(x_{i-1}))} & \text{if } i \ne i_k \text{ for all } k \\ S(x_i, y_i)\big(y_i^{-1/\alpha(Z_m(x_{i-1}))} - y_i^{-1/\alpha(Z_n(x_{i-1}))}\big) & \text{if } i = i_k \text{ for some } k \end{cases}. \end{aligned} \tag{2.7}$$

We will show by induction on i that for all $1 \le k \le K+1$ and all $i_{k-1} \le i < i_k$,

$$\begin{aligned} \mathbb{E}\big((Z_m(x_i) - Z_n(x_i))^2\big) \;&\le\; (1 + c_1)\cdots(1 + c_{k-1})(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_{k-1}) \\ &\quad + y_{i_{k-1}+1}^{-2/b} + y_{i_{k-1}+2}^{-2/b} + \cdots + y_i^{-2/b} \qquad (2.8) \\ &\le\; (1 + c_1)\cdots(1 + c_{k-1})(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_{k-1}) \,+\, \epsilon_k. \qquad (2.9) \end{aligned}$$

Note that (2.9) follows immediately from (2.8) using (2.6). Inequality (2.8) is trivially true when $i = 0$. Let $0 < i \le N$ and assume inductively that (2.8) holds with i replaced by $i - 1$. There are two cases.

(a) If $i \ne i_k$ for all k then from (2.7)

$$\begin{aligned} &\mathbb{E}\big((Z_m(x_i) - Z_n(x_i))^2 \big| \mathcal{F}_{i-1}\big) \\ &\qquad = \mathbb{E}\Big(\Big(Z_m(x_{i-1}) - Z_n(x_{i-1}) + S(x_i, y_i)\, y_i^{-1/\alpha(Z_m(x_{i-1}))}\Big)^2 \Big| \mathcal{F}_{i-1}\Big) \\ &\qquad = \big(Z_m(x_{i-1}) - Z_n(x_{i-1})\big)^2 + y_i^{-2/\alpha(Z_m(x_{i-1}))} \\ &\qquad \le \big(Z_m(x_{i-1}) - Z_n(x_{i-1})\big)^2 + y_i^{-2/b} \end{aligned}$$

as $y_i \ne y_{i_k}$ for all k implies $y_i > n \ge 1$. Thus taking the unconditional expectation and using (2.8) for $i - 1$ gives (2.8) for i.

(b) If $i = i_k$ for some $1 \le k \le K$, then from (2.7),

$$\begin{aligned} &\mathbb{E}\big((Z_m(x_i) - Z_n(x_i))^2 \big| \mathcal{F}_{i-1}\big) \\ &\quad = \mathbb{E}\Big(\Big(Z_m(x_{i-1}) - Z_n(x_{i-1}) + S(x_i, y_i)\big(y_i^{-1/\alpha(Z_m(x_{i-1}))} - y_i^{-1/\alpha(Z_n(x_{i-1}))}\big)\Big)^2 \Big| \mathcal{F}_{i-1}\Big) \\ &\quad = \big(Z_m(x_{i-1}) - Z_n(x_{i-1})\big)^2 + \big(y_i^{-1/\alpha(Z_m(x_{i-1}))} - y_i^{-1/\alpha(Z_n(x_{i-1}))}\big)^2 \\ &\quad \le \big(Z_m(x_{i-1}) - Z_n(x_{i-1})\big)^2 \big(1 + M^2 y_{i_k}^{-2/(a,b)}\big) \\ &\quad \le \big(Z_m(x_{i-1}) - Z_n(x_{i-1})\big)^2 \big(1 + c_k\big), \end{aligned}$$

using (1.3) and (2.5). Again, taking the unconditional expectation and using (2.9) for $i - 1$ gives (2.8) for i with a vacuous sum of terms $y_j^{-2/b}$, completing the induction.

It follows from (2.9) that for all $0 \le i \le N$,

$$\begin{aligned}\mathbb{E}\big((Z_m(x_i) - Z_n(x_i))^2\big) &\le \prod_{k=1}^{K}(1+c_k)\sum_{k=1}^{K+1}\epsilon_k \\ &\le \prod_{(x,y)\in\Pi^+ : y\le n}(1+M^2y^{-2/(a,b)})\sum_{(x,y)\in\Pi^+ : n<y\le m} y^{-2/b} \qquad (2.10)\end{aligned}$$

using (2.5) and (2.6).

Noting that $Z_m(t) - Z_n(t)$ is constant except at the jump points x_i, and applying Doob's maximal inequality [14] to the martingale $\big\{Z_m(x_i) - Z_n(x_i), \mathcal{F}_i\big\}_{i=0}^{N}$, we obtain

$$\begin{aligned}\mathbb{E}\big(\sup_{t_0\le t<t_1}(Z_m(t) - Z_n(t))^2\big) &= \mathbb{E}\big(\max_{0\le i\le N}(Z_n(x_i) - Z_m(x_i))^2\big) \\ &\le 4\,\mathbb{E}\big((Z_m(x_N) - Z_n(x_N))^2\big)\end{aligned}$$

for all $m \ge n$. Combining with (2.10) gives (2.3). Since

$$\prod_{(x,y)\in\Pi^+}(1+M^2y^{-2/(a,b)}) \le \exp\Big(M^2\sum_{(x,y)\in\Pi^+} y^{-2/(a,b)}\Big) < \infty$$

and $\sum_{(x,y)\in\Pi^+} y^{-2/(a,b)}$ and $\sum_{(x,y)\in\Pi^+} y^{-2/b}$ are convergent by assumption, $\{Z_n\}_n$ is a Cauchy sequence. □

We now deduce the existence of Z as the norm limit of the Z_n.

Theorem 2.2. *Let a_0, α and Π^+ be as above. Then there exists $Z \in \mathcal{D}$ satisfying* (2.1)*, with $\lim_{n\to\infty}\mathbb{E}\big(\|Z_n - Z\|_\infty^2\big) = 0$ where Z_n as in* (2.2)*; more specifically*

$$\mathbb{E}\big(\|Z_n - Z\|_\infty^2\big) \le 4\prod_{(x,y)\in\Pi^+}(1+M^2y^{-2/(a,b)})\sum_{(x,y)\in\Pi^+ : y>n} y^{-2/b} \to 0. \qquad (2.11)$$

Moreover, there exists a sequence $n_j \nearrow \infty$ such that almost surely $\|Z_{n_j} - Z\|_\infty \to 0$ i.e. $Z_{n_j} \to Z$ uniformly. If $0 < b < 1$ then almost surely $\|Z_n - Z\|_\infty \to 0$.

Proof. Since $\{Z_n\}_n$ is Cauchy in the complete norm $\mathbb{E}\big(\|\cdot\|_\infty^2\big)^{1/2}$ the desired limit $Z \in \mathcal{D}$ exists. Letting $m \to \infty$ in (2.3) gives (2.11).

Moreover, choosing any increasing sequence $\{n_j\}_j$ such that

$$\sum_{(x,y)\in\Pi^+ : y>n_j} y^{-2/b} < 2^{-j} \qquad (2.12)$$

for all sufficiently large j, then by (2.3) $\mathbb{E}\big(\|Z_{n_{j+1}} - Z_{n_j}\|_\infty^2\big) < c\ 2^{-j}$ so almost surely $Z = Z_{n_1} + \sum_{j=1}^{\infty}(Z_{n_{j+1}} - Z_{n_j})$ is convergent in $\|\cdot\|_\infty$. If $0 < b < 1$ then

$$\begin{aligned}\mathbb{E}\big(\|Z_{n+1} - Z_n\|_\infty\big) &\le \big(\mathbb{E}\big(\|Z_{n+1} - Z_n\|_\infty^2\big)\big)^{1/2} \\ &\le c\Big(\sum_{(x,y)\in\Pi^+ : n<y\le n+1} y^{-2/b}\Big)^{1/2} \le c\sum_{(x,y)\in\Pi^+ : n<y\le n+1} y^{-1/b},\end{aligned}$$

so $\mathbb{E}\big(\sum_{n=1}^{\infty}\|Z_{n+1} - Z_n\|_\infty\big) < \infty$, giving that $Z = Z_1 + \sum_{j=1}^{\infty}(Z_{n+1} - Z_n)$ is almost surely uniformly convergent. □

2.2 Local properties of functions defined by random signs

We next examine local properties of the random function Z constructed in Section 2.1. We will show that for a given $t \in [t_0, t_1)$, almost surely Z satisfies a Hölder condition to the right of t and also is locally approximable by a random function L defined in a similar way to Z, but with a fixed exponent $\alpha(Z(t))$. Throughout this section we fix $t \in [t_0, t_1)$ and throughout this subsection we restrict Ω to the subspace of full probability (by Theorem 2.2) such that there exists a sequence $n_j \nearrow \infty$ such that $Z_{n_j}(t) \to Z(t)$. Let $\mathcal{F}_t$ be the σ-field underlying the signs $\{S(x,y) \in \{-1,1\} : (x,y) \in \Pi^+, x \leq t\}$.

Let Z be given by the norm limit of the partial sums Z_n in (2.2), as in Section 2.1. We also construct $L_n, L \in \mathcal{D}$ restricted to $[t, t_1)$ in a similar way using the point set $\{(x,y) \in \Pi^+ : (t < x < t_1)\}$ and a (conditional) fixed index $\alpha(Z(t)) \in [a,b]$; thus L_n is the piecewise constant random function defined by

$$L_n(u) \;\equiv\; \sum_{(x,y)\in\Pi^+ \,:\, y \leq n} 1_{(t,u]}(x) S(x,y) y^{-1/\alpha(Z_n(t))} \qquad (t \leq u < t_1)$$

where the $S(x,y)$ are independent random signs, and with $L \in \mathcal{D}$ defined by

$$\mathbb{E}\big(\|L_n - L\|_\infty^2\big) \to 0, \tag{2.13}$$

as a particular case of Theorem 2.2, where here $\|\cdot\|_\infty$ is the supremum norm on $[t, t_1)$.

Proposition 2.3. *Let $\alpha : \mathbb{R} \to [a,b]$, Π^+ be as before and let $Z_n, Z, L_n, L \in \mathcal{D}$ restricted to $[t, t_1)$ be as above, taking the same realisations of $S(x,y)$ for Z_n and L_n. Then, there are constants c_1, c_2 depending only on Π^+ and α such that, conditional on $\mathcal{F}_t$, for all $0 \leq h < t_1 - t$,*

$$\mathbb{E}\Big(\sup_{0\leq h'\leq h} \big(Z(t+h') - Z(t)\big)^2\Big) \;\leq\; c_1 \sum_{(x,y)\in\Pi^+ \,:\, t<x\leq t+h} y^{-2/\alpha(Z(t))} \tag{2.14}$$

and

$$\mathbb{E}\Big(\sup_{0\leq h'\leq h} \big((Z(t+h') - Z(t)) - L(t+h')\big)^2\Big) \\ \leq\; c_2\Big(\sum_{(x,y)\in\Pi^+ \,:\, t<x\leq t+h} y^{-2/(a,b)}\Big)\Big(\sum_{(x,y)\in\Pi^+ \,:\, t<x\leq t+h} y^{-2/\alpha(Z(t))}\Big). \tag{2.15}$$

Proof. For brevity write

$$W_n(u) = Z_n(u) - Z_n(t) \quad \text{and} \quad D_n(u) = W_n(u) - L_n(u) \qquad (t \leq u < t_1).$$

Let $0 \leq h \leq t_1 - t$. For each n, order the points $(x,y) \in \{\Pi^+ : t < x \leq t+h, y \leq n\}$ as (x_i, y_i) with $x_1 < x_2 < \cdots < x_K$ and let $x_0 = t$ (as before we lose little other than awkward notation by assuming that the x_i are distinct). Let $\mathcal{F}_i$ be the σ-field underlying the signs $\{S(x_j, y_j) : 1 \leq j \leq i\}$. Then for $1 \leq i \leq K$,

$$W_n(x_i) \;=\; W_n(x_{i-1}) + S(x_i, y_i) y_i^{-1/\alpha(Z_n(x_{i-1}))}$$

and

$$D_n(x_i) \;=\; D_n(x_{i-1}) + S(x_i, y_i)\big(y_i^{-1/\alpha(Z_n(x_{i-1}))} - y_i^{-1/\alpha(Z_n(t))}\big)$$

so that $\{W_n(x_i), \mathcal{F}_i\}$ and $\{D_n(x_i), \mathcal{F}_i\}$ are bounded martingales. For each $1 \leq i \leq K$, conditioning on $\mathcal{F}_{i-1}$ gives

$$\begin{aligned}
\mathbb{E}\big(W_n(x_i)^2\big|\mathcal{F}_{i-1}\big) &= W_n(x_{i-1})^2 \;+\; y_i^{-2/\alpha(Z_n(x_{i-1}))} \\
&\leq W_n(x_{i-1})^2 \;+\; \big(y_i^{-1/\alpha(Z_n(x_{i-1}))} - y_i^{-1/\alpha(Z_n(t))} + y_i^{-1/\alpha(Z_n(t))}\big)^2 \\
&\leq W_n(x_{i-1})^2 \;+\; 2\big(y_i^{-1/\alpha(Z_n(x_{i-1}))} - y_i^{-1/\alpha(Z_n(t))}\big)^2 \;+\; 2y_i^{-2/\alpha(Z_n(t))}, \\
&\leq W_n(x_{i-1})^2 \;+\; 2M^2 y_i^{-2/(a,b)}\big(Z_n(x_{i-1}) - Z_n(t)\big)^2 \;+\; 2y_i^{-2/\alpha(Z_n(t))} \\
&\leq W_n(x_{i-1})^2\big(1 + \; 2M^2 y_i^{-2/(a,b)}\big) \;+\; 2y_i^{-2/\alpha(Z_n(t))} \qquad (2.16)
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\big(D_n(x_i)^2\big|\mathcal{F}_{i-1}\big) &= D_n(x_{i-1})^2 \;+\big(y_i^{-1/\alpha(Z_n(x_{i-1}))} - y_i^{-1/\alpha(Z_n(t))}\big)^2 \\
&\leq D_n(x_{i-1})^2 + M^2 y_i^{-2/(a,b)}(Z_n(x_{i-1}) - Z_n(t))^2 \\
&= D_n(x_{i-1})^2 + M^2 y_i^{-2/(a,b)} W_n(x_{i-1})^2 \qquad (2.17)
\end{aligned}$$

where we have used (1.3). Then induction in decreasing j using (2.16) gives

$$\mathbb{E}\big(W_n(x_K)^2\big|\mathcal{F}_j\big) \;\leq\; \prod_{k=j+1}^{K} \big(1 + 2M^2 y_k^{-2/(a,b)}\big)\Big(W_n(x_j)^2 + 2\sum_{k=j+1}^{K} y_k^{-2/\alpha(Z_n(t))}\Big)$$

for all $0 \leq j \leq K-1$, and induction in decreasing j using (2.16) and (2.17) gives

$$\begin{aligned}
\mathbb{E}\big(D_n(x_K)^2\big|\mathcal{F}_j\big) \;\leq\; & \Big(M^2 \sum_{k=j+1}^{K} y_k^{-2/(a,b)}\Big) \prod_{k=j+1}^{K-1} \big(1 + 2M^2 y_k^{-2/(a,b)}\big) \\
& + \Big(W_n(x_j)^2 + 2\sum_{k=j+1}^{K-1} y_k^{-2/\alpha(Z_n(t))}\Big) + D_n(x_j)^2.
\end{aligned}$$

for all $0 \leq j \leq K-1$. Setting $j = 0$ in these two estimates and noting that $W_n(x_0) = W_n(t) = 0$ and $D_n(x_0) = D_n(t) = 0$, we get expectations conditioned only on $\mathcal{F}_t$:

$$\mathbb{E}\big(W_n(x_K)^2\big) \;\leq\; 2\prod_{k=1}^{K} \big(1 + 2M^2 y_k^{-2/(a,b)}\big)\Big(\sum_{k=1}^{K} y_k^{-2/\alpha(Z_n(t))}\Big)$$

and

$$\mathbb{E}\big(D_n(x_K)^2\big) \;\leq\; 2M^2\Big(\sum_{k=1}^{K} y_k^{-2/(a,b)}\Big) \prod_{k=1}^{K-1} \big(1 + 2M^2 y_k^{-2/(a,b)}\big)\Big(\sum_{k=1}^{K} y_k^{-2/\alpha(Z_n(t))}\Big).$$

Noting that W_n and D_n are constant between the x_i and applying Doob's inequality to the martingales $W_n(x_i)$ and $D_n(x_i)$,

$$\begin{aligned}
\mathbb{E}\big(\sup_{0\leq h'\leq h} (Z_n(t+h) - Z_n(t))^2\big) &= \mathbb{E}\big(\sup_{0\leq h'\leq h} W_n(t+h')^2\big) \;\leq\; \mathbb{E}\big(\max_{0\leq k\leq K} W_n(x_k)^2\big) \\
&\leq\; 4\mathbb{E}\big(W(x_K)^2\big) \;\leq\; 8c_3 \sum_{k=1}^{K} y_k^{-2/\alpha(Z_n(t))} \;\leq\; 8c_3 \sum_{(x,y)\in\Pi^+ : t<x\leq t+h} y^{-2/\alpha(Z_n(t))},
\end{aligned}$$

where $c_3 = \prod_{k=1}^{K-1}\left(1+2M^2 y_k^{-2/(a,b)}\right)$, and

$$\begin{aligned}\mathbb{E}\Big(\sup_{0\le h'\le h}\left((Z_n(t+h')-Z_n(t))-L_n(t+h')\right)^2\Big) \;&=\; \mathbb{E}\big(\sup_{0\le h'\le h} D_n(t+h')^2\big)\\ &\le\; \mathbb{E}\big(\max_{0\le k\le K} D_n(x_k)^2\big) \;\le\; 4\mathbb{E}\big(D(x_K)^2\big)\\ &\le\; 8M^2c_3\big(\sum_{k=1}^{K} y_k^{-2/(a,b)}\big)\big(\sum_{k=1}^{K} y_k^{-2/\alpha(Z_n(t))}\big)\\ &\le\; 8M^2c_3\big(\sum_{(x,y)\in\Pi^+:t<x\le t+h} y_k^{-2/(a,b)}\big)\big(\sum_{(x,y)\in\Pi^+:t<x\le t+h} y_k^{-2/\alpha(Z_n(t))}\big).\end{aligned}$$

By Theorem 2.2, $\mathbb{E}\big(\|Z_n - Z\|_\infty^2\big)\to 0$ and $\mathbb{E}\big(\|L_n - L\|_\infty^2\big)\to 0$ and there is a sequence $n_j \nearrow \infty$ such that $Z_{n_j}(t)\to Z(t)$, so we can take the limit of these inequalities along this subsequence using dominated convergence to get (2.14) and (2.15) with $c_1 = 8c_3$ and $c_2 = 8M^2c_3$. □

We remark that versions of (2.14) and (2.15) with

$$4\sum_{(x,y)\in\Pi^+\,:\,t<x\le t+h} y^{-2/(a,b)} \qquad\text{and}\qquad 4M^2\Big(\sum_{(x,y)\in\Pi^+\,:\,t<x\le t+h} y^{-2/(a,b)}\Big)^2.$$

respectively as the right-hand side bounds can be obtained using a simpler induction, but the exponents are not so sharp.

We can immediately deduce a local right Hölder bound for Z at t as well getting a comparison with L.

Proposition 2.4. *Let $Z\in\mathcal{D}$ be the random function given by Theorem 2.2 and let $t\in[t_0,t_1)$. Suppose that for some $\beta>0$,*

$$\sum_{(x,y)\in\Pi^+\,:\,t<x\le t+h} y^{-2/\alpha(Z(t))} \;=\; O(h^\beta) \qquad (0<h<t_1-t). \tag{2.18}$$

Then, conditional on $\mathcal{F}_t$, given $0<\epsilon<\beta$ there exist almost surely random numbers $C_1, C_2<\infty$ such that for all $0\le h<t_1-t$,

$$|Z(t+h)-Z(t)| \;\le\; C_1 h^{(\beta-\epsilon)/2}. \tag{2.19}$$

If, in addition to (2.18),

$$\sum_{(x,y)\in\Pi^+\,:\,t<x\le t+h} y^{-2/(a,b)} \;=\; O(h^\gamma) \qquad (0<h<t_1-t).$$

then

$$\big|\big(Z(t+h)-Z(t)\big)-L(t+h)\big| \;\le\; C_2 h^{(\beta+\gamma-\epsilon)/2} \qquad (0<h<t_1-t), \tag{2.20}$$

where L is as in (2.13) and defined using the same realisation of $\{S(x,y):(x,y)\in\Pi^+, t\le x<t_1\}$ as Z.

Proof. Setting $h=2^{-k}(t_1-t)$ in (2.14), multiplying by $2^{k(\beta-\epsilon)}$ and summing,

$$\mathbb{E}\Big(\sum_{k=0}^{\infty} 2^{k(\beta-\epsilon)} \sup_{0\le h'\le 2^{-k}(t_1-t)} \big(Z(t+h')-Z(t)\big)^2\Big) \;\le\; c\sum_{k=0}^{\infty} 2^{k(\beta-\epsilon)}2^{-k\beta} \;<\; \infty$$

for some constant c, giving (2.19). The bound (2.20) follows in a similar manner using (2.15). □

3 General Poisson point sums

We apply the conclusions of Section 2 to random functions where Π_2^+ is a realisation of a Poisson point processes in the half-plane and show that this gives a self-stabilizing processes. The key idea is that the distribution of the point sets $\{(\mathsf{X},\mathsf{Y}) \in \Pi\}$ where $\Pi \subset (t_0,t_1)\times\mathbb{R}$ is a Poisson point process with plane Lebesgue measure $\mathcal{L}^2$ as mean measure, is identical to that of $\{(\mathsf{X}, S(\mathsf{X},\mathsf{Y})\mathsf{Y}) : (\mathsf{X},\mathsf{Y}) \in \Pi_2^+, S(\mathsf{X},\mathsf{Y}) = \pm 1\}$, where Π_2^+ is a Poisson point process on $(t_0,t_1)\times\mathbb{R}^+$ with double Lebesgue measure $2\mathcal{L}^2$ as mean measure and with the $S(\mathsf{X},\mathsf{Y})$ independently taking the values ±1 with equal probability $\frac{1}{2}$ for each $(\mathsf{X},\mathsf{Y}) \in \Pi_2^+$; this follows from the superposition property of Poisson processes, see [8, Sections 2.2, 5.1]. Hence Π can be realised by first sampling (X,Y) from Π_2^+ and then assigning random signs to the Y coordinates.

3.1 Existence of random functions

Given a Poisson process $\Pi \subset (t_0,t_1)\times\mathbb{R}$ and $\alpha : \mathbb{R} \to [a,b]$ with $0 < a < b < 2$, we wish to show that there exist random functions $\mathcal{D}$ satisfying

$$Z(t) \;=\; a_0 + \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} 1_{(t_0,t]}(\mathsf{X})\,\mathsf{Y}^{\langle -1/\alpha(Z(\mathsf{X}_-))\rangle} \qquad (t_0 \le t < t_1) \tag{3.1}$$

in an appropriate sense. If $0 < a < b < 1$ then almost surely $\sum_{(\mathsf{X},\mathsf{Y})\in\Pi} |\mathsf{Y}|^{-1/\alpha(Z(\mathsf{X}_-))}$ converges, in which case the sum in (3.1) is almost surely absolutely convergent, but if $\alpha(z) \ge 1$ for some z there is no a priori guarantee of convergence. In a similar way to Section 2 we define $Z_n \in \mathcal{D}$ for $n \in \mathbb{N}$ by

$$Z_n(t) \;=\; a_0 + \sum_{(\mathsf{X},\mathsf{Y})\in\Pi:|\mathsf{Y}|\le n} 1_{(t_0,t]}(\mathsf{X})\,\mathsf{Y}^{\langle -1/\alpha(Z_n(\mathsf{X}_-))\rangle} \qquad (t_0 \le t < t_1). \tag{3.2}$$

Almost surely this sum is over a finite number of points and therefore, conditional on Π, Z_n is a well-defined piecewise-constant random function. We are interested in convergence of Z_n to a limiting function Z that satisfies (3.1) in some sense.

The following result, which is part of Campbell's theorem, will be useful in bounding Poisson sums.

Theorem 3.1 (Campbell's theorem). *Let Π be a Poisson process on $S \subset \mathbb{R}^n$ with mean measure μ and let $f : S \to \mathbb{R}$ be measurable. Then*

$$\mathbb{E}\Big(\sum_{\mathsf{P}\in\Pi} f(\mathsf{P})\Big) = \int_S f(u)d\mu(u)$$

and

$$\mathbb{E}\Big(\exp\sum_{\mathsf{P}\in\Pi} f(\mathsf{P})\Big) = \exp\int_S \big(\exp f(u) - 1\big)d\mu(u),$$

provided these integrals converge.

Proof. See [8, Section 3.2]. □

Lemma 3.2. *Let $\Pi \subset (t_0, t_1) \times \mathbb{R}$ be a Poisson process with mean measure $\mathcal{L}^2$ and let $\alpha : \mathbb{R} \to [a, b]$ where $0 < a < b < 2$. Then for all $0 < \eta < 2/b - 1$ there is almost surely a random $C < \infty$ such that*

$$\sum_{(\mathsf{X},\mathsf{Y})\in\Pi:|\mathsf{Y}|>n} |\mathsf{Y}|^{-2/b} \;<\; C\, n^{-\eta} \qquad (n \in \mathbb{N}). \tag{3.3}$$

Proof. By Theorem 3.1,

$$\mathbb{E}\Big(\sum_{(\mathsf{X},\mathsf{Y})\in\Pi:|\mathsf{Y}|>n} |\mathsf{Y}|^{-2/b} \Big) \;=\; \int_n^\infty y^{-2/b} \;\leq\; \frac{b}{2-b} n^{1-2/b}.$$

A Borel-Cantelli argument summing over $n = 2^{-k}$ completes the proof. □

We can now obtain develop Theorem 2.2 to sums over a Poisson point process.

Theorem 3.3. *Let $\Pi \subset (t_0, t_1) \times \mathbb{R}$ be a Poisson point process with mean measure $\mathcal{L}^2$, let $\alpha : \mathbb{R} \to [a, b]$ where $0 < a < b < 2$ and let $a_0 \in \mathbb{R}$. Then there exists $Z \in \mathcal{D}$ satisfying (3.1) in the sense that $\lim_{n\to\infty} \mathbb{E}\big(\|Z_n - Z\|_\infty^2\big) = 0$ where Z_n is as in (3.2). Moreover, there exists a sequence $n_j \nearrow \infty$ such that almost surely $\|Z_{n_j} - Z\|_\infty \to 0$. If $0 < b < 1$ then almost surely $\|Z_n - Z\|_\infty \to 0$.*

Proof. Since $0 < b < 2$, the Poisson point set Π is almost surely a countable set of isolated points with

$$\sum_{(\mathsf{X},\mathsf{Y})\in\Pi} 1_{(t_0,t_1]}(\mathsf{X}) |\mathsf{Y}|^{-2/(a,b)} < \infty$$

and with the X distinct. As noted, Z_n in (3.2) has the same distribution as

$$Z_n(t) = a_0 + \sum_{(\mathsf{X},\mathsf{Y})\in\Pi_2^+:|\mathsf{Y}|\leq n} 1_{(t_0,t]}(\mathsf{X}) S(\mathsf{X},\mathsf{Y}) \mathsf{Y}^{-1/\alpha(Z(\mathsf{X}_-))},$$

where Π_2^+ is a Poisson point process on $(t_0, t_1) \times \mathbb{R}^+$ with $2\mathcal{L}^2$ as mean measure and $S(\mathsf{X},\mathsf{Y})$ are random signs on Π_2^+. Thus, by Theorem 2.2, for almost all realisations of Π_2^+ there almost surely exists a random function $Z \in \mathcal{D}$, such that $\mathbb{E}\big(\|Z_n - Z\|_\infty^2\big) \to 0$, and also such that there exists a sequence $n_j \nearrow \infty$ with $\|Z_{n_j} - Z\|_\infty \to 0$; note that by (3.3) and (2.12) we can take the same sequence n_j for all such realisations. Thus the conclusion holds for almost all sign combinations for almost all realisations of Π_2^+ and so for almost all $\{(\mathsf{X}, S(\mathsf{X},\mathsf{Y})\mathsf{Y}) : (\mathsf{X},\mathsf{Y}) \in \Pi_2^+, S(\mathsf{X},\mathsf{Y}) = \pm 1\}$, that is for almost all Π. □

For purposes of simulating these random functions we would like an estimate on how rapidly Z_n given by (3.2) converges to Z. However we cannot get useful estimates directly from Theorem 2.2 since allowing $(x, y) \in \Pi^+$ with y arbitrarily small leaves the sum in (2.11) unbounded. However, we can get some concrete estimates if we modify the setting slightly by assuming that there is $y_0 > 0$ such that $|\mathsf{Y}| \geq y_0$ if $(\mathsf{X},\mathsf{Y}) \in \Pi$, which ensures that the right hand side of (3.4) below converges. In practice this is a realistic assumption in that it excludes the possibility of Z having unboundedly large jumps.

Theorem 3.4. *Let $y_0 > 0$ and let Π be a Poisson point process on $(t_0, t_1) \times (-\infty, -y_0] \cup [y_0, \infty)$ with mean measure $\mathcal{L}^2$ restricted to this domain. Let $a_0 \in \mathbb{R}$, let $0 < a < b < 2$, let $\alpha : \mathbb{R} \to [a, b]$ and let $Z \in \mathcal{D}$ be the random function given by Theorem 3.3 using this Π with Z_n as in* (3.2)*. Then as $n \to \infty$*

$$\mathbb{E}\big(\|Z_n - Z\|_\infty^2\big) \;\le\; \frac{8b(t_1 - t_0)}{2 - b} \exp\left(2M^2(t_1 - t_0)\int_{y_0}^{\infty} y^{-2/(a,b)}\, dy\right) n^{-(2-b)/b} \;\to 0. \quad (3.4)$$

Proof. We proceed exactly as in the proofs of Proposition 2.1 and Theorem 2.2, except that we condition on realisations of a Poisson process Π_2^+ with mean measure $2\mathcal{L}$ on $(t_0, t_1) \times [y_0, \infty)$, to get (2.11) in this setting and then use Theorem 3.3 to get the process on this Π. Then for $n \ge y_0$, using (2.11), the independence of the Poisson point process Π_2^+ on $(t_0, t_1) \times [y_0, n]$ and on $(t_0, t_1) \times (n, \infty)$, and Theorem 3.1,

$$\begin{aligned}
\mathbb{E}\big(\|Z_n - Z\|_\infty^2\big) &= \mathbb{E}\Big(\mathbb{E}\big(\|Z_n - Z\|_\infty^2 \big| \Pi_2^+\big)\Big) \\
&\le 4\,\mathbb{E}\Bigg(\prod_{(\mathsf{X},\mathsf{Y})\in\Pi_2^+ : y_0 \le \mathsf{Y} \le n} (1 + M^2\mathsf{Y}^{-2/(a,b)}) \sum_{(\mathsf{X},\mathsf{Y})\in\Pi_2^+ : \mathsf{Y} > n} |\mathsf{Y}|^{-2/b}\Bigg) \\
&= 4\,\mathbb{E}\Bigg(\exp\Big(\sum_{(\mathsf{X},\mathsf{Y})\in\Pi_2^+ : y_0 \le \mathsf{Y} \le n} \log(1 + M^2\mathsf{Y}^{-2/(a,b)})\Big)\Bigg)\mathbb{E}\Bigg(\sum_{(\mathsf{X},\mathsf{Y})\in\Pi_2^+ : \mathsf{Y} > n} |\mathsf{Y}|^{-2/b}\Bigg) \\
&= 4\,\exp\left(\int_{y_0}^{n} 2(t_1 - t_0)\; M^2 y^{-2/(a,b)} dy\right)(t_1 - t_0)\int_{n}^{\infty} 2y^{-2/b} dy.
\end{aligned}$$

Letting $n \to \infty$ in the first integral and evaluating the second integral gives (3.4). $\square$

We remark that Theorem 3.4 allows us to quantify the rate of convergence in probability of $\|Z_n - Z\|_\infty \to 0$ in Theorem 3.3. By the Poisson distribution $\mathbb{P}\{\Pi \cap ((t_0, t_1) \times [0, y_0]) = \emptyset\} = \exp(-y_0(t_1 - t_0))$. Given $\epsilon > 0$ we can choose y_0 to make this probability at most $\epsilon/2$, then using (3.4) and Markov's inequality it follows that if n is sufficiently large then $\mathbb{P}\{\|Z_n - Z\|_\infty > \epsilon\} < \epsilon$. In practice, this leads to an enormous value of n.

3.2 Local properties and self-stabilising processes

We next obtain local properties of the random functions defined by a Poisson point process as in Theorem 3.3. Not only are the sample paths right-continuous, but they satisfy a local Hölder continuity estimate and are self-stabilizing, that is locally they look like α-stable processes.

We will use a bound provided by the *α-stable subordinator* which may be defined for each (constant) $0 < \alpha < 1$ by

$$S_\alpha(t) := \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} 1_{(t_0,t]}(\mathsf{X})\, |\mathsf{Y}|^{-1/\alpha} \qquad (t_0 \le t < t_1),$$

where the sum, which is almost surely convergent, is over a plane Poisson point process Π with mean measure $\mathcal{L}^2$. Then S_α on $[t_0, t_1)$ has stationary increments and for each $0 < \epsilon < 1/\alpha$ satisfies the Hölder property

$$S_\alpha(t) \;\le\; C(t - t_0)^{(1/\alpha) - \epsilon}, \quad (3.5)$$

where C is almost surely finite; this may be established using Campbell's Theorem 3.1 in a similar way to the proof of Lemma 3.2, or see [1, Section III.4] or [13].

We write L^0_α for the non-normalized α-stable process, which has a representation

$$L^0_\alpha(t) = \sum_{(\mathsf{X},\mathsf{Y})\in\Pi} 1_{(0,t]}(\mathsf{X})\mathsf{Y}^{<-1/\alpha>} \tag{3.6}$$

where Π is a Poisson point process on $(t_1, t_2)\times\mathbb{R}$ with mean measure $\mathcal{L}^2$. This sum is almost surely absolutely convergent if $0<\alpha<1$ but for general $0<\alpha<2$ it is the limit as $n\to\infty$ of

$$L^0_{\alpha,n}(t) = \sum_{(\mathsf{X},\mathsf{Y})\in\Pi:|\mathsf{Y}|\le n} 1_{(0,t]}(\mathsf{X})\mathsf{Y}^{<-1/\alpha>}.$$

Whilst $\mathbb{E}\big(\|L^0_{\alpha,n}-L^0_\alpha\|^2_\infty\big)\to 0$ as a special case of Theorem 3.3, the constant value of α means that $L^0_{\alpha,n}$ and $L^0_{\alpha,m}-L^0_{\alpha,n}$ are independent for $m>n$ and also that $\{L^0_{\alpha,n}\}_n$ is a martingale, which ensures that $\|L^0_{\alpha,n}-L^0_\alpha\|_\infty\to 0$ almost surely.

In the same way to Z_n we can think of $L^0_{\alpha,n}(t)$ in terms of Π^+_2 and random signs, so that

$$L^0_{\alpha,n}(t) = \sum_{(\mathsf{X},\mathsf{Y})\in\Pi^+_2:\mathsf{Y}\le n} 1_{(0,t]}(\mathsf{X})S(\mathsf{X},\mathsf{Y})\mathsf{Y}^{<-1/\alpha>}.$$

The following proposition is the analogue of Proposition 2.4 in this context.

Proposition 3.5. *Let Z be the random function given by Theorem 3.3 and let $t\in[t_0,t_1)$. Then, conditional on $\mathcal{F}_t$, given $0<\epsilon<1/b$ there exist almost surely random numbers $C_1, C_2<\infty$ such that for all $0\le h<t_1-t$,*

$$|Z(t+h)-Z(t)| \le C_1 h^{1/\alpha(Z(t))\,-\,\epsilon}. \tag{3.7}$$

and

$$\big|\big(Z(t+h)-Z(t)\big)-\big(L^0_{\alpha(t)}(t+h)-L^0_{\alpha(t)}(t)\big)\big| \;\le\; C_2 h^{1/\alpha(Z(t))+1/b-\,\epsilon}, \tag{3.8}$$

where $L^0_{\alpha(t)}$ is the $\alpha(t)$-stable process (3.6) *defined using the same realisations of* Π *as* Z.

Proof. Let $\epsilon>0$. Let Π^+_2 be a Poisson point process on $\mathbb{R}$ with mean measure $2\mathcal{L}$. From (3.5)

$$\sum_{(X,Y)\in\Pi^+_2\,:\,t<x\le t+h} \mathsf{Y}^{-2/\alpha(Z(t))} \;=\; 2S_{\alpha(Z(t))/2}(h) \;\le\; Ch^{2/\alpha(Z(t))-\epsilon}$$

where $C<\infty$ for almost all realisations of Π^+_2. For such Π^+_2, Proposition 2.4 gives on randomising the signs,

$$\big|Z(t+h)-Z(t)|\Pi^+_2\big| \;\le\; C_1 h^{1/\alpha(Z(t))-\epsilon/2-\epsilon/2}$$

for some random C_1, and hence (3.7) holds almost surely.

In the same way,

$$\sum_{(X,Y)\in\Pi^+_2\,:\,t<x\le t+h} \mathsf{Y}^{-2/(a,b)} \;\le\; C'(h^{-1})^{-2/(a,b)-\epsilon/2} \;\le\; C'' h^{2/b-\epsilon}$$

where $C', C'' < \infty$ for almost all realisations of Π_2^+. Then in Proposition 2.4, $L(t+h) = L^0_{\alpha(t)}(t+h) - L^0_{\alpha(t)}(t)$, so for such Π_2^+, on randomising the signs,

$$\Big|\big(Z(t+h) - Z(t)\big) - \big(L^0_{\alpha(t)}(t+h) - L^0_{\alpha(t)}(t)\big)\big|\Pi_2^+\Big| \;\leq\; C_2 h^{1/\alpha(Z(t))+1/b-\epsilon}$$

for random $C_2 < \infty$, so (3.8) holds almost surely. □

We finally show that almost surely at each $t \in [t_0, t_1)$ the random function Z of Theorem 3.3 is right-localisable with local form an $\alpha(Z(t))$-stable process, so that Z may indeed be thought of as self-stablizing.

Theorem 3.6. *Let Z be the random function given by Theorem 3.3 and let $t \in [t_0, t_1)$. Then, conditional on $\mathcal{F}_t$, almost surely Z is strongly right-localisable at t, in the sense that*

$$\frac{Z(t+ru) - Z(t)}{r^{1/\alpha(Z(t))}}\bigg|\, \mathcal{F}_t \overset{\text{dist}}{\to} L^0_{\alpha(Z(t))}(u) \qquad (0 \leq u \leq 1)$$

as $r \searrow 0$, where convergence is in distribution with respect to $(D[0,1], \rho_S)$, with ρ_S is the Skorohod metric.

Proof. Let $0 < \epsilon < 1/b$. For $u \in [0,1]$ and $0 < r < t_1 - t$, almost surely

$$\begin{aligned}\big|\big(Z(t+ru) - Z(t)\big) - \big(L^0_{\alpha(t)}(t+ru) - L^0_{\alpha(t)}(t)\big)\big| \\ \leq\; C_2 (ru)^{1/\alpha(Z(t))+1/b-\epsilon} \;\leq\; C_2 r^{1/\alpha(Z(t))+1/b-\epsilon}\end{aligned}$$

for a random $C_2 < \infty$, by Proposition 3.5. Thus

$$\begin{aligned}\left\|\frac{Z(t+ru) - Z(t)}{r^{1/\alpha(Z(t))}} - \frac{L^0_{\alpha(Z(t))}(t+ru) - L^0_{\alpha(Z(t))}(t)}{r^{1/\alpha(Z(t))}}\right\|_\infty &\leq C_2 \frac{r^{1/\alpha(Z(t))+1/b-\epsilon}}{r^{1/\alpha(Z(t))}} \\ &= C_2 r^{1/b-\epsilon} \to 0\end{aligned}$$

almost surely as $r \searrow 0$. In particular, since $\|\cdot\|_\infty$ dominates ρ_S on $D[0,1]$,

$$\rho_S\left(\frac{Z(t+ru) - Z(t)}{r^{1/\alpha(Z(t))}}, \frac{L^0_{\alpha(Z(t))}(t+ru) - L^0_{\alpha(Z(t))}(t)}{r^{1/\alpha(Z(t))}}\right) \overset{\text{P}}{\to} 0$$

almost surely and in probability. Using that α-stable processes have stationary increments and scale with exponent $1/\alpha$,

$$\frac{L^0_{\alpha(Z(t))}(t+ru) - L^0_{\alpha(Z(t))}(t)}{r^{1/\alpha(Z(t))}} \overset{\text{dist}}{=} L^0_{\alpha(Z(t))}(u) - L^0_{\alpha(Z(t))}(0) \overset{\text{dist}}{=} L^0_{\alpha(Z(t))}(u),$$

so we conclude, using [2, Theorem 4.1] to combine convergence in probability and in distribution, that

$$\frac{Z(t+ru) - Z(t)}{r^{1/\alpha(Z(t))}}\bigg|\, \mathcal{F}_t \overset{\text{dist}}{\to} L^0_{\alpha(Z(t))}(u)$$

as $r \searrow 0$. □

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