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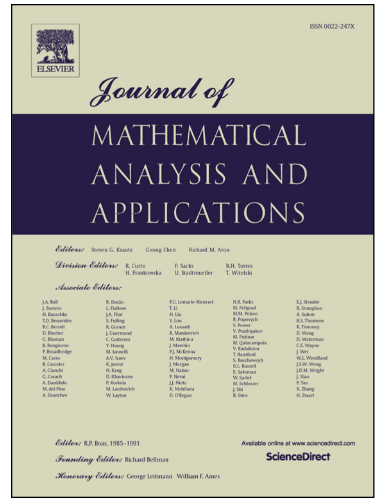

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ON THE AVERAGE L^q-DIMENSIONS OF TYPICAL MEASURES BELONGING TO THE GROMOV-HAUSDORFF-PROHOROFF SPACE

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Abstract. We study the average L^q-dimensions of typical Borel probability measures belonging to the Gromov-Hausdorff-Prohoroff space (of all Borel probability measures with compact supports) equipped with the Gromov-Hausdorff-Prohoroff metric.

1. Introduction.

Recall that a subset E of a (complete) metric space M is called co-meagre if its complement is meagre, and we say that a typical element $x \in M$ has property P if the set $E = \{x \in M \,|\, x \text{ has property } \mathsf{P}\}$ is co-meagre, see Oxtoby [Ox] for more details. In this paper we study the L^q-dimensions of a typical Borel probability measures belonging the Gromov-Hausdorff-Prohoroff space (of all Borel probability measures with compact support) equipped with the Gromov-Hausdorff-Prohoroff metric. In Section 1.1 we recall the definition of the Gromov-Hausdorff-Prohoroff space and the Gromov-Hausdorff-Prohoroff metric, and in Section 1.2 we recall the definitions of the L^q-dimensions. The main results are presented in Section 1.3 and Section 1.4.

1.1. The Gromov-Hausdorff-Prohoroff space P_{GHP} and the Gromov-Hausdorff-Prohoroff metric d_{GHP}. For a compact metric space X, we denote the family of all Borel probability measures on X by $\mathcal{P}(X)$, i.e. we write

$$\mathcal{P}(X) = \left\{ \mu \,\middle|\, \mu \text{ is a Borel probability measure on } X \right\}. \tag{1.1}$$

The pre-Gromov-Hausdorff-Prohoroff space $\mathcal{P}_{\mathsf{GHP}}$ is now defined by

$$\begin{aligned}\mathcal{P}_{\mathsf{GHP}} &= \bigcup_{\substack{X \text{ is a compact}\\ \text{metric space}}} \mathcal{P}(X)\\ &= \left\{ \mu \,\middle|\, \mu \text{ is a Borel probability measure on a compact metric space} \right\}.\end{aligned}$$

Next, we define the equivalence relation $\sim$ in $\mathcal{P}_{\mathsf{GHP}}$ as follows. Namely, for $\mu, \nu \in \mathcal{P}_{\mathsf{GHP}}$, we write

$$\mu \sim \nu \quad \Leftrightarrow \quad \begin{array}{c}\text{there is a bijective isometry } f : \operatorname{supp}\mu \to \operatorname{supp}\nu\\ \text{such that } \nu = \mu \circ f^{-1}\end{array}.$$

It is clear that $\sim$ is an equivalence relation in $\mathcal{P}_{\mathsf{GHP}}$, and the Gromov-Hausdorff-Prohoroff space P_{GHP} is defined by

$$P_{\mathsf{GHP}} = {}^{\mathcal{P}_{\mathsf{GHP}}}\!/\!\sim,$$

see, for example, [AbDeHo, p. 4] or [Mi, Section 6.2]. While elements of P_{GHP} are equivalence classes of measures, we will use the standard convention and identify an equivalence class with its representative, i.e. we will regard the elements of P_{GHP} as measures and not as equivalence classes of measures.

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Next, we define the Gromov-Hausdorff-Prohoroff metric d_{GHP} on P_{GHP}. For a compact metric space, let $\mathcal{K}(X)$ denote the family of non-empty compact subsets of X. For $A, B \in \mathcal{K}(X)$, the Hausdorff distance $d_{\mathsf{H}}(A, B)$ between A and B is defined by

$$d_{\mathsf{H}}(A,B) = \max\left(\sup_{x\in A}\operatorname{dist}(x,B)\,,\,\sup_{y\in B}\operatorname{dist}(y,A)\right), \tag{1.2}$$

where $\operatorname{dist}(x,E) = \inf_{z\in E} d(x,z)$ for $x \in X$ and $E \subseteq X$. Also, for $\mu, \nu \in \mathcal{P}(X)$, the Prohoroff distance $d_{\mathsf{P}}(\mu,\nu)$ between μ and ν is defined as follows. Let $\operatorname{Lip}(X)$ denote the family of Lipschitz functions $f : X \to \mathbb{R}$ with $|f| \le 1$ and $\operatorname{Lip}(f) \le 1$ where $\operatorname{Lip}(f)$ denotes the Lipschitz constant of f, i.e. $\operatorname{Lip}(X) = \{f : X \to \mathbb{R} \,|\, |f| \le 1\,,\, \operatorname{Lip}(f) \le 1\}$. The Prohoroff distance $d_{\mathsf{P}}(\mu,\nu)$ between μ and ν is defined by

$$d_{\mathsf{P}}(\mu,\nu) = \sup_{f\in\operatorname{Lip}(X)}\left|\int f\,d\mu - \int f\,d\nu\right|. \tag{1.3}$$

The Gromov-Hausdorff-Prohoroff metric d_{GHP} on P_{GHP} is now defined by

$$\begin{aligned} d_{\mathsf{GHP}}(\mu,\nu) = \inf\Big\{ d_{\mathsf{H}}(f(\operatorname{supp}\mu), g(\operatorname{supp}\nu)) \,+\, d_{\mathsf{P}}(\mu\circ f^{-1}, \nu\circ g^{-1}) \\ \Big|\; X \text{ is a compact metric space} \\ \text{and } f : \operatorname{supp}\mu \to X \\ \text{and } g : \operatorname{supp}\nu \to X \text{ are isometries} \Big\}. \end{aligned} \tag{1.4}$$

for $\mu, \nu \in P_{\mathsf{GHP}}$, see, for example, [AbDeHo p. 4] or [Mi, Section 6.2] (see also [Vi]). The Gromov-Hausdorff-Prohoroff metric extends the Hausdorff metric, the Gromov-Hausdorff metric on the space of all compact metric spaces, and the Prohoroff metric. It can be shown that Gromov-Hausdorff-Prohoroff metric space $(P_{\mathsf{GHP}}, d_{\mathsf{GHP}})$ is complete; the reader is referred to [AbDeHo, p. 4] or [Mi, Section 6.2] for a proof of this and for a discussion of the Gromov-Hausdorff-Prohoroff metric space.

1.2. L^q-dimensions. For a probability measure μ on a compact metric space, the L^q-dimensions of μ are defined as follows, see, for example, [Fa,Pe]. For $r > 0$ and a real number q, write

$$I_r^q(\mu) = \int \mu(B(x,r))^{q-1}\,d\mu(x)\,, \tag{1.5}$$

where $B(x,r)$ denotes the open ball with centre at x and radius equal to r. The lower and upper L^q-dimensions of order q are now defined by

$$\begin{aligned} \underline{D}^q(\mu) &= \liminf_{r\searrow 0}\frac{\log I_r^q(\mu)}{-\log r} \\ \overline{D}^q(\mu) &= \limsup_{r\searrow 0}\frac{\log I_r^q(\mu)}{-\log r}\,. \end{aligned} \tag{1.6}$$

There is an alternative expression for the L^q-dimensions using closed balls in stead of open balls. Since this expression will be used in the statements of the main results in the paper, we will now provide the required definitions and notation. For a metric space (X,d) and $x \in X$ and $r > 0$, we write $C(x,r) = \{y \in X \,|\, d(x,y) \le r\}$ for the closed ball with centre at x and radius equal to r. Next, for $r > 0$ and a real number q, write

$$J_r^q(\mu) = \int \mu(C(x,r))^{q-1}\,d\mu(x)\,. \tag{1.7}$$

It is not difficult to see that the L^q-dimensions can be computed using J_μ^q (instead of I_μ^q). Indeed, for a straightforward argument shows that

$$\begin{aligned} \underline{D}^q(\mu) &= \liminf_{r\searrow 0}\frac{\log J_r^q(\mu)}{-\log r}\,, \\ \overline{D}^q(\mu) &= \limsup_{r\searrow 0}\frac{\log J_r^q(\mu)}{-\log r}\,. \end{aligned}$$

The main significance of the L^q-dimensions is their relationship with the multifractal spectrum of μ. In the 1980's it was conjectured in the physics literature that for "good" measures μ the multifractal spectrum of μ equals the Legendre transform of the L^q-dimensions. This result is known as the Multifractal Formalism. During the 1990's there has been an enormous interest in verifying the Multifractal Formalism and computing the multifractal spectra and L^q-dimensions of measures in the mathematical literature, see [Fa,Pe] and the references therein.

1.3. ***L^q*-dimensions of typical measures.** The purpose of this paper is to investigate the L^q-dimensions of a typical element of P_{GHP}, i.e. of a typical measure. We note that for a fixed compact metric space X, the L^q-dimensions of a typical measure $\mu \in \mathcal{P}(X)$ on X have been studied earlier [Ba1,Ba2,MyRu,Ol1,Ol2]. For example, Bayart [Ba1], Myjak & Rudnicki [MyRu] (for $q = 2$) and Olsen [Ol1] show that if X is a compact Ahlfors regular subset of $\mathbb{R}^d$, then a typical measure $\mu \in \mathcal{P}(X)$ satisfies

$$\begin{aligned} \overline{D}^q(\mu) &= 0 && \text{for all } q > 1, \\ \underline{D}^q(\mu) &= \dim_{\mathsf{H}}(X)\,(1-q) && \text{for all } q > 1. \end{aligned} \tag{1.8}$$

where $\dim_{\mathsf{H}}(X)$ denotes the Hausdorff dimension of X. In particular, (1.8) shows that a typical measure on X is as "irregular" as possible: lower dimension $\underline{D}^q(\mu)$ is as small as possible and the upper dimension $\overline{D}^q(\mu)$ is as big as possible. Surprisingly, our results show that the behaviour of a typical measure is even more "irregular" than suggested by (1.8). Namely, by shifting the viewpoint from the study of typical measures belonging to $\mathcal{P}(X)$ for a fixed compact metric space X to the study of typical measures belonging to the "enlarged" space P_{GHP} of *all* measures, then the "irregular" behaviour of a typical measure is amplified very dramatically. In particular, we prove the following result.

Theorem 1.1.

(1) *All measures $\mu \in P_{\mathsf{GHP}}$ satisfy*

$$-\infty \le \underline{D}^q(\mu) \le \overline{D}^q(\mu) \le 0 \qquad \textit{for all } q > 1.$$

(2) *A typical measure $\mu \in P_{\mathsf{GHP}}$ satisfies*

$$\begin{aligned} \overline{D}^q(\mu) &= 0 && \textit{for all } q > 1, \\ \underline{D}^q(\mu) &= -\infty && \textit{for all } q \ge 2. \end{aligned}$$

Theorem 1.1 follows immediately from the more general result in Theorem 1.3. We note that Theorem 1.1 does not provide any information about the lower L^q-dimension $\underline{D}^q(\mu)$ of a typical measure μ for $q \in (1, 2)$. However, under an additional semi-continuity assumption, we are able to show that the lower L^q-dimension $\underline{D}^q(\mu)$ of a typical measure μ is also equal to $-\infty$ for all $q \in (1, 2)$; this is the contents of the next theorem.

Theorem 1.2. *Let $q \in (1, 2)$ and assume that the map $J_r^q : P_{\mathsf{GHP}} \to \mathbb{R}$ is upper semi-continuous for all $r > 0$. Then a typical measure $\mu \in P_{\mathsf{GHP}}$ satisfies*

$$\underline{D}^q(\mu) = -\infty.$$

Unfortunately, we have not been able to show that $J_r^q : P_{\mathsf{GHP}} \to \mathbb{R}$ is upper semi-continuous for $q \in (1, 2)$. However, in Lemmas 3.1–3.2 we prove that if $q \in \mathbb{N}$, i.e. if q is a positive integer, then:

the map $I_r^q : P_{\mathsf{GHP}} \to \mathbb{R}$ is lower semi-continuous for all $r > 0$

and

the map $J_r^q : P_{\mathsf{GHP}} \to \mathbb{R}$ is upper semi-continuous for all $r > 0$,

and this result is sufficient to prove the statement in Theorem 1.1. While we have only been able to prove semi-continuity of the maps I_r^q and J_r^q for $q \in \mathbb{N}$, we believe that these maps are always semi-continuous and make the following conjecture.

Conjecture 1.3. *The maps $I_r^q : P_{\mathsf{GHP}} \to \mathbb{R}$ and $J_r^q : P_{\mathsf{GHP}} \to \mathbb{R}$ are lower semi-continuous and upper semi-continuous, respectively, for all q and all $r > 0$.*

Theorem 1.1 shows that the lower L^q-dimension of a typical measure is as small as possible and that the upper L^q-dimension of a typical measure is as big as possible. Other results, including, for example, (1.8) from [Ba1,Ol1] as well as the results in [Ba2,Ge,Has,MyRu,Ol2], investigating the typical L^q-dimensions and other dimensions of measures show a similar dichotomy. The purpose of this paper is to analyse the intriguing dichotomy in Theorem 1.1, in more detail. In order to do so, we introduce the following notation. Namely, for a Borel probability measure μ with compact support and a real number q, we define the q'th moment scaling function $f_\mu^q : (0,\infty) \to [0,\infty]$ of μ by

$$f_\mu^q(t) = \frac{\log I_{e^{-t}}^q(\mu)}{-\log e^{-t}} = \frac{\log I_{e^{-t}}^q(\mu)}{t}\,. \tag{1.9}$$

Using this notation, the L^q-dimensions of μ are now given by

$$\underline{D}^q(\mu) = \liminf_{t\to\infty} f_\mu^q(t)\,,$$
$$\overline{D}^q(\mu) = \limsup_{t\to\infty} f_\mu^q(t)\,,$$

and Theorem 1.1 therefore shows that the moment scaling function $f_\mu^q(t)$ of a typical measure $\mu \in P_{\mathsf{GHP}}$ diverges in the worst possible way as $t \to \infty$. In this paper we will prove that the behaviour of the moment scaling function $f_\mu^q(t) = \frac{\log I_{e^{-t}}^q(\mu)}{t}$ of a typical measure $\mu \in P_{\mathsf{GHP}}$ is spectacularly more irregular than suggested by Theorem 1.2. Namely, there are standard techniques, known as averaging systems, that (at least in some cases) can assign limiting values to divergent functions (the precise definitions will be given below), and the purpose of this paper is to show the following surprising result: not only is the moment scaling function $f_\mu^q(t) = \frac{\log I_{e^{-t}}^q(\mu)}{t}$ of a typical measure μ divergent as $t \to \infty$, but it is so irregular that it remains spectacularly divergent as $t \to \infty$ even after being "averaged" or "smoothened out" using powerful averaging systems including, for example, *all* higher order Hölder and Cesaro averages, see Section 2.

1.4. Average L^q-dimensions of typical measures. We start by recalling the definition of an averaging (or summability) system; the reader is referred to Hardy's classical text [Har] for a systematic treatment of averaging systems.

Definition. Averaging system. *An averaging system is a family $\Pi = (\Pi_t)_{t\geq t_0}$ with $t_0 > 0$ such that:*

(i) *Π_t is a finite Borel measure on $[t_0,\infty)$;*
(ii) *Π_t has compact support;*
(iii) *The Consistency Condition: If $f : [t_0,\infty) \to [0,\infty)$ is a positive measurable function and there is a real number a such that*

$$f(t) \to a \quad \text{as } t \to \infty,$$

then

$$\int f\, d\Pi_t \to a \quad \text{as } t \to \infty.$$

If $f : [t_0,\infty) \to [0,\infty)$ is a positive measurable function, then we define lower and upper Π-average of f by

$$\underline{A}_\Pi f = \liminf_{t\to\infty} \int f\, d\Pi_t$$

and

$$\overline{A}_\Pi f = \limsup_{t\to\infty} \int f\, d\Pi_t\,,$$

respectively.

Applying averaging systems to the moment scaling function $f_\mu^q(t)$ in (1.9) leads to our key definition, namely, the definition of average L^q-dimensions.

Definition. Average L^q-dimension. *Let $\Pi = (\Pi_t)_{t\geq t_0}$ be an averaging system. Let $q \in \mathbb{R}$ and let μ be a Borel probability measure with compact support. We define the lower and upper Π-average L^q-dimensions of μ by*

$$\underline{D}^q_\Pi(\mu) = \underline{A}_\Pi f^q_\mu = \liminf_{t\to\infty} \int \frac{\log I^q_{e^{-s}}(\mu)}{s}\, d\Pi_t(s)\,,$$

and

$$\overline{D}^q_\Pi(\mu) = \overline{A}_\Pi f^q_\mu = \limsup_{t\to\infty} \int \frac{\log I^q_{e^{-s}}(\mu)}{s}\, d\Pi_t(s)\,,$$

respectively.

Remark. We note that L^q-dimensions are, in fact, average L^q-dimensions. Indeed, if μ is a Borel probability measure with compact support and we let Π denote the average system defined by $\Pi = (\delta_t)_{t\geq 1}$ (where δ_t denotes the Dirac measure concentrated at t), then clearly

$$\underline{D}^q_\Pi(\mu) = \underline{D}^q(\mu)\,,\quad \overline{D}^q_\Pi(\mu) = \overline{D}^q(\mu)\,. \tag{1.10}$$

We can now state the main result in the paper, namely, Theorem 1.4 below. This result shows that the behaviour of the moment scaling function $f^q_\mu(t) = \frac{\log I^q_{e^{-t}}(\mu)}{t}$ of a typical measure $\mu \in P_{\mathsf{GHP}}$ is so irregular that it remains divergent as $t \to \infty$ even after being "averaged" using arbitrary averaging systems.

Theorem 1.4. *Let Π be an averaging system.*

(1) *All measures $\mu \in P_{\mathsf{GHP}}$ satisfy*

$$-\infty \leq \underline{D}^q_\Pi(\mu) \leq \overline{D}^q_\Pi(\mu) \leq 0 \qquad \text{for all } q > 1.$$

(2) *A typical measure $\mu \in P_{\mathsf{GHP}}$ satisfies*

$$\overline{D}^q_\Pi(\mu) = 0 \qquad \text{for all } q > 1, \tag{1.11}$$

$$\underline{D}^q_\Pi(\mu) = -\infty \qquad \text{for all } q \geq 2. \tag{1.12}$$

Note that the statement in Theorem 1.4.(1) is trivial and is only included for completeness. The proof of Theorem 1.2.(2) is given in Sections 3–5. Section 3 contains various technical auxiliary results. The proof of Theorem 1.4.(2) equation (1.11) is given in Section 4 and the proof of Theorem 1.4.(2) equation (1.12) is given in Section 5.

Similarly to Theorem 1.1, Theorem 1.4 does not provide any information about the lower average L^q-dimension $\underline{D}^q_\Pi(\mu)$ of a typical measure μ for $q \in (1,2)$. However, under the semi-continuity assumption from Theorem 1.2, we are able to show that the lower average L^q-dimension $\underline{D}^q_\Pi(\mu)$ of a typical measure μ is also equal to $-\infty$ for all $q \in (1,2)$; this is the content of the next theorem.

Theorem 1.5. *Let $q \in (1,2)$ and assume that the map $J^q_r : P_{\mathsf{GHP}} \to \mathbb{R}$ is upper semi-continuous for all $r > 0$. Let Π be a an averageing system. Then a typical measure $\mu \in P_{\mathsf{GHP}}$ satisfies*

$$\underline{D}^q_\Pi(\mu) = -\infty.$$

The proof of Theorem 1.5 is given in Section 5.

Remark. Note that if we apply Theorem 1.4 and Theorem 1.5 to the average system Π defined by $\Pi = (\delta_t)_{t\geq 1}$, then it follows from (1.10) that the statements in Theorem 1.4 and Theorem 1.5 reduce to the statements in Theorem 1.1 and Theorem 1.2, respectively.

As an application of Theorem 1.4 we will now consider higher order Hölder and Cesaro averages of the moment scaling function f^q_μ of a typical measure μ; this is done in next section.

2. Hölder and Cesaro averages of the L^q-dimension of a typical measure.

Two of the most commonly used averaging systems are Hölder averages and Cesaro averages. We will now define these average systems and apply them to the moment scaling function $f^q_\mu(t) = \frac{\log I_{e^{-t}}(\mu)}{t}$ of a Borel probability measure μ. For $a > 0$ and a positive measurable function $f : (a,\infty) \to [0,\infty)$, we define $Mf : (a,\infty) \to [0,\infty)$ by

$$(Mf)(t) = \frac{1}{t}\int_a^t f(s)\,ds\,.$$

For a positive integer n, we now define the lower and upper n'th order Hölder averages of f by

$$\begin{aligned}\underline{H}_n f &= \liminf_{t\to\infty}\,(M^n f)(t)\,,\\ \overline{H}_n f &= \limsup_{t\to\infty}(M^n f)(t)\,.\end{aligned}$$

The Cesaro averages are defined as follows. First, we define $If : (a,\infty) \to [0,\infty)$ by

$$(If)(t) = \int_a^t f(s)\,ds\,.$$

For a positive integer n, we now define the lower and upper n'th order Cesaro averages of f by

$$\begin{aligned}\underline{C}_n f &= \liminf_{t\to\infty}\,\frac{n!}{t^n}(I^n f)(t)\,,\\ \overline{C}_n f &= \limsup_{t\to\infty}\,\frac{n!}{t^n}(I^n f)(t)\,.\end{aligned}$$

It is well-known that that the Hölder and Cesaro averages satisfy the following inequalities, namely,

$$\begin{aligned}\liminf_{t\to\infty} f(t) = \underline{H}_0 f \le \underline{H}_1 f \le \underline{H}_2 f \le \ldots \le \overline{H}_2 f \le \overline{H}_1 f \le \overline{H}_0 f = \limsup_{t\to\infty} f(t)\,,\\ \liminf_{t\to\infty} f(t) = \underline{C}_0 f \le \underline{C}_1 f \le \underline{C}_2 f \le \ldots \le \overline{C}_2 f \le \overline{C}_1 f \le \overline{C}_0 f = \limsup_{t\to\infty} f(t)\,.\end{aligned} \tag{2.1}$$

It is also well-known that the Hölder and Cesaro averages can be expressed using averaging systems in the sense of the definition in Section 1.4. Indeed, if for a positive integer n, we define the averaging system $\Pi^{\mathsf{H}}_n = (\Pi^{\mathsf{H}}_{n,t})_{t\ge a}$ by

$$\Pi^{\mathsf{H}}_{n,t}(B) = \frac{1}{(n-1)!\,t}\int_{[a,t]\cap B}(\log t - \log s)^{n-1}\,ds$$

for Borel subsets B of $[a,\infty)$, then

$$\begin{aligned}\underline{H}_n f &= \liminf_t \int f\,d\Pi^{\mathsf{H}}_{n,t}\,,\\ \overline{H}_n f &= \limsup_t \int f\,d\Pi^{\mathsf{H}}_{n,t}\,,\end{aligned}$$

see, for example, [Ja, p. 675]. Similarly, if for a positive integer n, we define the averaging system $\Pi^{\mathsf{C}}_n = (\Pi^{\mathsf{C}}_{n,t})_{t\ge a}$ by

$$\Pi^{\mathsf{C}}_{n,t}(B) = \frac{n}{t^n}\int_{[a,t]\cap B}(t-s)^{n-1}\,ds$$

then

$$\begin{aligned}\underline{C}_n f &= \liminf_t \int f\,d\Pi^{\mathsf{C}}_{n,t}\,,\\ \overline{C}_n f &= \limsup_t \int f\,d\Pi^{\mathsf{C}}_{n,t}\,,\end{aligned}$$

see, for example, [Har, pp. 110-111].

Using Hölder and Cesaro averages we can now introduce average Hölder and Cesaro L^q-dimensions by applying the definitions of the Hölder and Cesaro averages to the function $f^q_\mu(t) = \frac{\log I_{e^{-t}}(\mu)}{t}$. This is the content of the next definition.

Definition. Average Hölder and Cesaro L^q-dimensions. *Let $\mu \in P_{\mathsf{GHP}}$. For $q \in \mathbb{R}$, we define the lower and upper n'th order average Hölder L^q-dimension of μ, denoted by $\underline{D}^q_{\mathsf{H},n}(\mu)$ and $\overline{D}^q_{\mathsf{H},n}(\mu)$, as the lower and upper n'th order Hölder average of the function $f^q_\mu(t) = \frac{\log I_{e^{-t}}(\mu)}{t}$ for $t \geq 1$, i.e. we put*

$$\begin{aligned}\underline{D}^q_{\mathsf{H},n}(\mu) &= \underline{H}_n f^q_\mu\,,\\ \overline{D}^q_{\mathsf{H},n}(\mu) &= \overline{H}_n f^q_\mu\,.\end{aligned}$$

Similarly, we define the lower and upper n'th order average Cesaro L^q-dimension of μ, denoted by $\underline{D}^q_{\mathsf{C},n}(\mu)$ and $\overline{D}^q_{\mathsf{C},n}(\mu)$, by

$$\begin{aligned}\underline{D}^q_{\mathsf{C},n}(\mu) &= \underline{C}_n f^q_\mu\,,\\ \overline{D}^q_{\mathsf{C},n}(\mu) &= \overline{C}_n f^q_\mu\,.\end{aligned}$$

The higher order average Hölder and Cesaro L^q-dimensions form a double infinite hierarchy in (at least) countably infinite many levels, namely, we have (using (2.1))

$$\begin{aligned}\underline{D}^q(\mu) = \underline{D}^q_{\mathsf{H},0}(\mu) \leq \underline{D}^q_{\mathsf{H},1}(\mu) \leq \ldots \leq \overline{D}^q_{\mathsf{H},1}(\mu) \leq \overline{D}^q_{\mathsf{H},0}(\mu) = \overline{D}^q(\mu)\,,\\ \underline{D}^q(\mu) = \underline{D}^q_{\mathsf{C},0}(\mu) \leq \underline{D}^q_{\mathsf{C},1}(\mu) \leq \ldots \leq \overline{D}^q_{\mathsf{C},1}(\mu) \leq \overline{D}^q_{\mathsf{C},0}(\mu) = \overline{D}^q(\mu)\,.\end{aligned} \tag{2.2}$$

As an application of Theorem 1.4, we will now show that the behaviour of a typical measure $\mu \in P_{\mathsf{GHP}}$ is so irregular that not even the hierarchies in (2.2) formed by taking Hölder and Cesaro averages of *all* orders are sufficiently powerful to "smoothen out" the behaviour of the box counting function $f^q_\mu(t) = \frac{\log I^q_{e^{-t}}(\mu)}{t}$ as $t \to \infty$.

Theorem 2.1. *A typical measure $\mu \in P_{\mathsf{GHP}}$ satisfies*

$$\begin{aligned}\overline{D}^q_{\mathsf{H},n}(\mu) = \overline{D}^q_{\mathsf{C},n}(\mu) &= 0 \qquad \text{for all } q > 1,\\ \underline{D}^q_{\mathsf{H},n}(\mu) = \underline{D}^q_{\mathsf{C},n}(\mu) &= -\infty \qquad \text{for all } q \geq 2,\end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$.

Proof.
This statement follows immediately from Theorem 1.4. □

3. Proofs of Theorem 1.4 and Theorem 1.5: Preliminary results.

In this section we collect some basic notation and some technical auxiliary lemmas that will be used extensively in Sections 4–5. We first prove some useful auxiliary continuity results about L^q-dimensions. Let (X,d) be a compact metric space and $q\in\mathbb{R}$. Fix $\mu\in\mathcal{P}(X)$. Recall that for $r>0$, we write

$$I_r^q(\mu)=\int\mu(B(x,r))^{q-1}\,d\mu(x)\,.$$

Also recall that for $x\in X$, we let $C(x,r)$ denote the closed ball with centre at x and radius equal to r, i.e. $C(x,r)=\{y\in X\,|\,d(x,y)\le r\}$, and write

$$J_r^q(\mu)=\int\mu(C(x,r))^{q-1}\,d\mu(x)\,.$$

The next lemma collects some of the basic continuity properties of the functions I_r^q and J_r^q.

Lemma 3.1. *Let X be a compact metric space and $q\in\mathbb{N}$. Let $r>0$.*

(1) *The function $I_r^q:\mathcal{P}(X)\to\mathbb{R}$ is lower semi-continuous.*
(2) *The function $J_r^q:\mathcal{P}(X)\to\mathbb{R}$ is upper semi-continuous.*
(3) *$I_r^q(\mu)\le J_r^q(\mu)$ for all measures $\mu\in\mathcal{P}(X)$.*

Proof.
(1)–(2) Below we use the following notation, namely, if A is a subset of a set M, then we will write $1_A:M\to\mathbb{R}$ for the indicator function on A. First, define $F,G:X^q\to\mathbb{R}$ by

$$F(x_1,\dots,x_{q-1},x)=1_{B(x,r)\times\dots\times B(x,r)}(x_1,\dots,x_{q-1})\,,$$
$$G(x_1,\dots,x_{q-1},x)=1_{C(x,r)\times\dots\times C(x,r)}(x_1,\dots,x_{q-1})\,.$$

Next, note that for all $\mu\in\mathcal{P}(X)$, we have

$$\begin{aligned}I_r^q(\mu)&=\int\mu(B(x,r))^{q-1}\,d\mu(x)\\&=\int\left(\prod_{i=1}^{q-1}\int 1_{B(x,r)}(x_i)\,d\mu(x_i)\right)d\mu(x)\\&=\int\prod_{i=1}^{q-1}1_{B(x,r)}(x_i)\,d(\mu\times\dots\times\mu)(x_1,\dots,x_{q-1},x)\\&=\int 1_{B(x,r)\times\dots\times B(x,r)}(x_1,\dots,x_{q-1})\,d(\mu\times\dots\times\mu)(x_1,\dots,x_{q-1},x)\\&=\int F\,d(\mu\times\dots\times\mu)\,,\end{aligned}\tag{3.1}$$

and similarly

$$J_r^q(\mu)=\int G\,d(\mu\times\dots\times\mu)\,.\tag{3.2}$$

We can now prove the statements in (1) and (2). Let $(\mu_n)_n$ be a sequence in $\mathcal{P}(X)$ and $\mu\in\mathcal{P}(X)$ with $\mu_n\to\mu$. First, note that since X is separable (because X is compact) and $\mu_n\to\mu$, it follows from [Bi, p. 21, Theorem 3.2] that $\mu_n\times\dots\times\mu_n\to\mu\times\dots\times\mu$. Next, since F and G are easily seen to be lower semi-continuous and upper semi-continuous, respectively, we therefore conclude from (3.1) and (3.2) that

$$\begin{aligned}I_r^q(\mu)&=\int F\,d(\mu\times\dots\times\mu)\\&\le\liminf_n\int F\,d(\mu_n\times\dots\times\mu_n)\\&=\liminf_n I_r^q(\mu_n)\,,\end{aligned}$$

and

$$\begin{aligned}\limsup_n J_r^q(\mu_n) &= \limsup_n \int G\, d(\mu_n \times \cdots \times \mu_n)\\ &\leq \int G\, d(\mu \times \cdots \times \mu)\\ &= J_r^q(\mu)\end{aligned}$$

(3) This statement follows immediately from the definitions of I_r^q and J_r^q. □

Lemma 3.2. *Let $q \in \mathbb{N}$. Let $r > 0$.*

(1) *The function $I_r^q : P_{\mathsf{GHP}} \to \mathbb{R}$ is lower semi-continuous.*
(2) *The function $J_r^q : P_{\mathsf{GHP}} \to \mathbb{R}$ is upper semi-continuous.*
(3) *$I_r^q(\mu) \leq J_r^q(\mu)$ for all measures $\mu \in P_{\mathsf{GHP}}$.*

Proof.
This follows immediately from Lemma 3.1. □

The final auxiliary lemma in this section provides a useful approximation result.

Lemma 3.3. *Let X be a compact metric space and $\mu \in \mathcal{P}(X)$. Let $r > 0$. Then there is a measure $P \in \mathcal{P}(X)$ with finite support satisfying* $\operatorname{supp} P \subseteq \operatorname{supp} \mu$ *such that*

$$\begin{aligned}d_{\mathsf{H}}(\operatorname{supp}\mu, \operatorname{supp} P) &< r\,,\\ d_{\mathsf{P}}(\mu, P) &< r\,.\end{aligned}$$

(Recall that the Hausdorff metric d_{H} is defined in (1.2) and that the Prohoroff metric d_{P} is defined in (1.3).)

Proof.
Since $\operatorname{supp}\mu$ is compact, we can choose a finite subset E of $\operatorname{supp}\mu$ such that $d_{\mathsf{H}}(\operatorname{supp}\mu, E) < r$. Also, since $\operatorname{supp}\mu$ is compact there is a measure $Q \in \mathcal{P}(X)$ with finite support satisfying $\operatorname{supp} Q \subseteq \operatorname{supp}\mu$ such that $d_{\mathsf{P}}(\mu, Q) < \frac{r}{3}$, see, for example, [Bo, Example 8.1.6.(i)]. Now put $L = \sum_{x\in E} \frac{1}{|E|}\delta_x$ and $P = \frac{r}{3}L + (1-\frac{r}{3})Q$. It is clear that the support of P is finite and that $\operatorname{supp} P \subseteq \operatorname{supp}\mu$. We also have $\operatorname{supp} P = E \cup \operatorname{supp} Q$, whence $d_{\mathsf{H}}(\operatorname{supp}\mu, \operatorname{supp} P) = d_{\mathsf{H}}(\operatorname{supp}\mu, E\cup\operatorname{supp} Q) = d_{\mathsf{H}}(\operatorname{supp}\mu\cup\operatorname{supp} Q, E\cup \operatorname{supp} Q) \leq d_{\mathsf{H}}(\operatorname{supp}\mu, E) + d_{\mathsf{H}}(\operatorname{supp} Q, \operatorname{supp} Q) < r$. Finally, we show that $d_{\mathsf{P}}(\mu, P) < r$. Recall, that $\operatorname{Lip}(X) = \{f : X \to \mathbb{R} \,|\, |f| \leq 1\,,\, \operatorname{Lip}(f)| \leq 1\}$, and that the distance $d_{\mathsf{P}}(\nu, \lambda)$ between two measures $\nu, \lambda \in \mathcal{P}(X)$ is defined by $d_{\mathsf{P}}(\nu,\lambda) = \sup_{f\in\operatorname{Lip}(X)} |\int f\,d\nu - \int f\,d\lambda|$, see (1.3). We therefore conclude that

$$\begin{aligned}d_{\mathsf{P}}(\mu, P) &\leq d_{\mathsf{P}}(\mu, Q) + d_{\mathsf{P}}(Q, P)\\ &< \tfrac{r}{3} + \sup_{f\in\operatorname{Lip}(X)} \left|\int f\,dQ - \int f\,dP\right|\\ &= \tfrac{r}{3} + \sup_{f\in\operatorname{Lip}(X)} \left|\int f\,dQ - \int f\,d(\tfrac{r}{3}L + (1-\tfrac{r}{3})Q)\right|\\ &= \tfrac{r}{3} + \tfrac{r}{3} \sup_{f\in\operatorname{Lip}(X)} \left|\int f\,dQ - \int f\,dL\right|\\ &\leq \tfrac{r}{3} + \tfrac{r}{3} \sup_{f\in\operatorname{Lip}(X)} \left(\int |f|\,dQ + \int |f|\,dL\right)\\ &\leq \tfrac{r}{3} + 2\tfrac{r}{3}\\ &= r\,.\end{aligned}$$

This completes the proof. □

4. Proof of Theorem 1.4.(2) equation (1.11): $\overline{D}^q_\Pi(\mu) = 0$ for a typical $\mu \in P_{\mathsf{GHP}}$.

In this section we prove Theorem 1.4.(2) equation (1.11). The first lemma (i.e. Lemma 4.1) is standard; however, for the benefit of the reader we have decided to state it explicitly.

Lemma 4.1. The reverse Fatou's Lemma [St, Theorem 3.2.3]. *Let $(M, \mathcal{E}, \mu)$ be a measure space and let $(\varphi_n)_n$ be a sequence of positive measurable functions $\varphi_n : M \to [0, \infty]$. If $\int \sup_n \varphi_n \, d\mu < \infty$, then $\limsup_n \int \varphi_n \, d\mu \le \int \limsup_n \varphi_n \, d\mu$.*

Lemma 4.2. *Let $q \ge 1$ and assume that $I^q_r : P_{\mathsf{GHP}} \to \mathbb{R}$ is lower semi-continuous for all $r > 0$. Let $\Pi = (\Pi_t)_{t \ge t_0}$ be an averaging system. Let $c \in \mathbb{R}$ and $t \ge t_0$. Then the set*

$$\left\{ \mu \in P_{\mathsf{GHP}} \,\middle|\, \int \frac{\log I^q_{e^{-s}}(\mu)}{s} \, d\Pi_t(s) > c \right\}$$

is open in P_{GHP}.

Proof.
Write

$$\begin{aligned} F &= P_{\mathsf{GHP}} \setminus \left\{ \mu \in P_{\mathsf{GHP}} \,\middle|\, \int \frac{\log I^q_{e^{-s}}(\mu)}{s} \, d\Pi_t(s) > c \right\} \\ &= \left\{ \mu \in P_{\mathsf{GHP}} \,\middle|\, \int \frac{\log I^q_{e^{-s}}(\mu)}{s} \, d\Pi_t(s) \le c \right\}. \end{aligned}$$

We must now prove that F is closed in P_{GHP}. In order to show this, we fix a sequence $(\mu_n)_n$ in F and $\mu \in P_{\mathsf{GHP}}$ with $\mu_n \to \mu$. We must now prove that $\mu \in F$, i.e. we must prove that $\int \frac{\log I^q_{e^{-s}}(X)}{s} \, d\Pi_t(s) \le c$. For brevity define functions $\varphi, \varphi_n : [t_0, \infty) \to [0, \infty)$ by $\varphi(s) = \frac{\log I^q_{e^{-s}}(\mu)}{s}$ and $\varphi_n(s) = \frac{\log I^q_{e^{-s}}(\mu_n)}{s}$. We now prove the following three claims.

Claim 1. For all $s \ge t_0$, we have $\varphi(s) \le \liminf_n \varphi_n(s)$. In particular $\int \varphi \, d\Pi_t \le \int \liminf_n \varphi_n \, d\Pi_t$.
Proof of Claim 1. This follows from the fact that map $I^q_r : P_{\mathsf{GHP}} \to \mathbb{R}$ is lower semi-continuous for all $r > 0$. This completes the proof of Claim 1.

Claim 2. We have $\int \sup_n (-\varphi_n) \, d\Pi_t < \infty$.
Proof of Claim 2. The measure Π_t has compact support and we can therefore choose $T_0 \ge t_0$ such that $\operatorname{supp} \Pi_t \in [t_0, T_0]$. Next, since clearly $I^q_{e^{-T_0}}(\mu) > 0$ and $I^q_{e^{-T_0}} : P_{\mathsf{GHP}} \to \mathbb{R}$ is lower semi-continuous, we conclude that $0 < I^q_{e^{-T_0}}(\mu) \le \liminf_n I^q_{e^{-T_0}}(\mu_n)$, and we therefore deduce that $c_0 = \inf_n I^q_{e^{-T_0}}(\mu_n) > 0$. It follows from this that for all $s \in [t_0, T_0]$, we have $\sup_n(-\varphi_n(s)) = \sup_n -\frac{\log I^q_{e^{-s}}(\mu_n)}{s} = \sup_n \frac{\log I^q_{e^{-s}}(\mu_n)^{-1}}{s} \le \sup_n \frac{\log I^q_{e^{-T_0}}(\mu_n)^{-1}}{t_0} \le \frac{\log c_0^{-1}}{t_0}$. This clearly implies that $\int \sup_n(-\varphi_n) \, d\Pi_t = \int_{t_0}^{T_0} \sup_n(-\varphi_n) \, d\Pi_t \le \int_{t_0}^{T_0} \frac{\log c_0^{-1}}{t_0} \, d\Pi_t = \Pi_t([t_0, T_0]) \frac{\log c_0^{-1}}{t_0} < \infty$. This completes the proof of Claim 2.

Claim 3. We have $\int \liminf_n \varphi_n \, d\Pi_t \le c$.
Proof of Claim 3. Since $\mu_n \in F$, we conclude that $\int \varphi_n \, d\Pi_t = \int \frac{\log I^q_{e^{-s}}(\mu_n)}{s} \, d\Pi_t(s) \le c$ for all n, whence

$$\liminf_n \int \varphi_n \, d\Pi_t \le c \,. \tag{4.1}$$

We also note that since $-\varphi_n \ge 0$, it follows from Claim 2 and the reverse Fatou's lemma (Lemma 4.1) that

$$\limsup_n \int (-\varphi_n) \, d\Pi_t \le \int \limsup_n (-\varphi_n) \, d\Pi_t \,,$$

whence

$$\begin{aligned}\int \liminf_n \varphi_n \, d\Pi_t &= -\int \limsup_n (-\varphi_n) \, d\Pi_t \\ &\leq -\limsup_n \int -\varphi_n \, d\Pi_t \\ &= \liminf_n \int \varphi_n \, d\Pi_t \,. \end{aligned} \tag{4.2}$$

Combining the inequalities (4.1) and (4.2) we now conclude that $\int \liminf_n \varphi_n \, d\Pi_t \leq \liminf_n \int \varphi_n \, d\Pi_t \leq c$. This competes the proof of Claim 3.

Finally, we deduce from Claim 1 and Claim 3 that

$$\begin{aligned}\int \frac{\log I^q_{e^{-s}}(\mu)}{s} \, d\Pi_t(s) &= \int \varphi \, d\Pi_t \\ &\leq \int \liminf_n \varphi_n \, d\Pi_t \\ &\leq c \,. \end{aligned}$$

This completes the proof. □

We now turn towards the proof of Theorem 1.4.(2) equation (1.11). We first prove the following result.

Theorem 4.3. *Let $q \geq 1$ and assume that $I^q_r : P_{\mathsf{GHP}} \to \mathbb{R}$ is lower semi-continuous for all $r > 0$. Let $\Pi = (\Pi_t)_{t \geq t_0}$ be an averaging system. Then a typical $\mu \in P_{\mathsf{GHP}}$ satisfies*

$$\overline{D}^q_\Pi(\mu) = 0 \,.$$

Proof.
We must prove that for a typical measure $\mu \in P_{\mathsf{GHP}}$, we have $\overline{D}^q_\Pi(\mu) = 0$. Since $\overline{D}^q_\Pi(\mu) \leq 0$, it suffices to prove that the set

$$U = \left\{ \mu \in P_{\mathsf{GHP}} \,\middle|\, \overline{D}^q_\Pi(\mu) < 0 \right\}$$

is meagre. For $u < 0$, write

$$U_u = \left\{ \mu \in P_{\mathsf{GHP}} \,\middle|\, \overline{D}^q_\Pi(\mu) < u \right\}.$$

Since

$$U = \bigcup_{\substack{u \in \mathbb{Q} \\ u < 0}} U_u \,,$$

it suffices to show that U_u is meagre for all $u \in \mathbb{Q}$ with $u < 0$.

We therefore fix $u \in \mathbb{Q}$ with $u < 0$. Next, in order to show that U_u is meagre, we note that it suffices to show that there is a countable family $(G_k)_k$ of open and dense subsets of P_{GHP} with $\cap_k G_k \subseteq P_{\mathsf{GHP}} \setminus U_u$. We now construct the sets G_k. For $t \geq t_0$, let

$$L_t = \left\{ \mu \in P_{\mathsf{GHP}} \,\middle|\, \int \frac{\log I^q_{e^{-s}}(\mu)}{s} \, d\Pi_t(s) > u \right\},$$

and for each positive integer k, put

$$G_k = \bigcup_{t \geq k} L_t \,.$$

Below we show that the family $(G_k)_k$ consists of open and dense subsets of P_{GHP} with $\cap_k G_k \subseteq P_{\mathsf{GHP}} \setminus U_u$; this is the content of the following three claims.

Claim 1. The set G_k is open in P_{GHP}.
Proof of Claim 1. Indeed, since it follows from Lemma 3.2 that L_t is open for all $t \geq t_0$, we immediately conclude that $G_k = \cup_{t \geq k} L_t$ is open. This completes the proof of Claim 1.

Claim 2. The set G_k is dense in P_{GHP}.
Proof of Claim 2. Let $\mu \in P_{\mathsf{GHP}}$ and let $r > 0$. We must now find $\nu \in P_{\mathsf{GHP}}$ such that $d_{\mathsf{GHP}}(\mu, \nu) < r$ and $\nu \in G_k$. Writing $X = \operatorname{supp}\mu$, it follows from Lemma 3.3 that we can find a measure $\nu \in \mathcal{P}(X)$ with finite support such that $d_{\mathsf{H}}(\operatorname{supp}\mu, \operatorname{supp}\nu) < \frac{r}{2}$ and $d_{\mathsf{P}}(\mu, \nu) < \frac{r}{2}$.

In particular, we conclude that $d_{\mathsf{GHP}}(\mu, \nu) \leq d_{\mathsf{H}}(\operatorname{supp}\mu, \operatorname{supp}\nu) + d_{\mathsf{P}}(\mu, \nu) < \frac{r}{2} + \frac{r}{2} = r$.

Next, we show that $\nu \in G_k$. Indeed, since the support of ν is finite, there is a finite subset E of X and a probability vector $(p_x)_{x \in E}$ such that $\nu = \sum_{x \in E} p_x \delta_x$, whence

$$I^q_{e^{-t}}(\nu) = \int \left(\sum_{y \in E} p_y \delta_y(B(x, e^{-t})) \right)^{q-1} d\nu(x) \tag{4.3}$$

for all $t > 0$. Next, write $r_E = \min_{x_1, x_2 \in E\,,\, x_1 \neq x_2} |x_1 - x_2|$, and note that $r_E > 0$ because E is finite. Choose t_E such that $e^{-t_E} = r_E$ and note that it follows from the definition of r_E that $(\sum_{y \in E} p_y \delta_y(B(x, e^{-t})))^{q-1} = \sum_{y \in E} p_y^{q-1} \delta_y(B(x, e^{-t}))^{q-1} = \sum_{y \in E} p_y^{q-1} \delta_y(B(x, e^{-t}))$ for all $x \in X$ and all $t \geq t_E$. We conclude from this and (4.3) that

$$\begin{aligned} I^q_{e^{-t}}(\nu) &= \sum_{y \in E} p_y^{q-1} \int \delta_y(B(x, e^{-t}))\, d\nu(x) \\ &= \sum_{y \in E} p_y^{q-1} \sum_{x \in E} p_x \delta_y(B(x, e^{-t})) \end{aligned} \tag{4.4}$$

for all $t \geq t_E$. However, since $\sum_{x \in E} p_x \delta_y(B(x, e^{-t})) = p_y$ for all $y \in E$ and all $t \geq t_E$, we deduce from (4.4) that

$$I^q_{e^{-t}}(\nu) = \sum_{y \in E} p_y^q \tag{4.5}$$

for all $t \geq t_E$. It follows from (4.5) that $\frac{\log I^q_{e^{-t}}(\nu)}{t} \to 0$, and the consistency condition therefore implies that $\int \frac{\log I^q_{e^{-s}}(\nu)}{s}\, d\Pi_t(s) \to 0$ as $t \to \infty$. We conclude immediately from this and the fact that $u < 0$ that there is a real number $t \geq k$ such that $\int \frac{\log I^q_{e^{-s}}(\nu)}{s}\, d\Pi_t(s) > u$, and so $\nu \in L_t \subseteq G_k$. This completes the proof of Claim 2.

Claim 3. We have $\cap_k G_k \subseteq P_{\mathsf{GHP}} \setminus U_u$.
Proof of Claim 3. Let $\mu \in \cap_k G_k$. Hence for each positive integer k, we can find $t_k \geq k$ such that $\mu \in L_{t_k}$, whence $\int \frac{\log I^q_{e^{-s}}(\mu)}{s}\, d\Pi_{t_k}(s) > u$ for all positive integers k. We conclude from this that $\overline{D}^q_{\Pi}(\mu) = \limsup_{t \to \infty} \int \frac{\log I^q_{e^{-s}}(\mu)}{s}\, d\Pi_t(s) \geq \limsup_k \int \frac{\log I^q_{e^{-s}}(\mu)}{s}\, d\Pi_{t_k}(s) \geq u$, whence $\mu \in P_{\mathsf{GHP}} \setminus U_u$. This completes the proof of Claim 3.

Combining Claim 1, Claim 2 and Claim 3, we now conclude that U_u is meagre. □

We can now prove Theorem 1.4.(2) equation (1.11).

Proof of Theorem 1.4.(2) equation (1.11).
Since Lemma 3.2 shows that $I^q_r : P_{\mathsf{GHP}} \to \mathbb{R}$ is lower semi-continuous for all $r > 0$ and all $q \in \mathbb{N}$, it follows from Theorem 4.3 that:

$$\text{A typical measure } \mu \in P_{\mathsf{GHP}} \text{ satisfies } \overline{D}^q_{\Pi}(\mu) = 0 \text{ for all } q \in \mathbb{N}. \tag{4.6}$$

Next, note that if $\mu \in P_{\mathsf{GHP}}$ and $1 < p \leq q$, then $\overline{D}^q_{\Pi}(\mu) \leq \overline{D}^p_{\Pi}(\mu) \leq 0$, and so:

$$\begin{aligned} &\text{If a typical measure } \mu \in P_{\mathsf{GHP}} \text{ satisfies } \overline{D}^q_{\Pi}(\mu) = 0, \\ &\text{then a typical measure } \mu \in P_{\mathsf{GHP}} \text{ satisfies } \overline{D}^p_{\Pi}(\mu) = 0 \text{ for all } 1 < p \leq q. \end{aligned} \tag{4.7}$$

Combining (4.6) and (4.7), we now conclude that a typical measure $\mu \in P_{\mathsf{GHP}}$ satisfies $\overline{D}^p_{\Pi}(\mu) = 0$ for all $1 < p$. □

5. Proof of Theorem 1.4.(2) equation (1.12) and Theorem 1.5: $\underline{D}_{\Pi}^{q}(\mu) = -\infty$ for a typical $\mu \in P_{\mathsf{GHP}}$.

In this section we prove Theorem 1.4.(2) equation (1.12) and Theorem 1.5.

Lemma 5.1. *Let $q \geq 1$ and assume that $J_r^q : P_{\mathsf{GHP}} \to \mathbb{R}$ is upper semi-continuous for all $r > 0$. Let $\Pi = (\Pi_t)_{t \geq t_0}$ be an averaging system. Let $c \in \mathbb{R}$ and $t \geq t_0$. Then the set*

$$\left\{ \mu \in P_{\mathsf{GHP}} \,\middle|\, \int \frac{\log J_{e^{-s}}^q(\mu)}{s} \, d\Pi_t(s) < c \right\}$$

is open in P_{GHP}.

Proof.
Write

$$\begin{aligned} F &= P_{\mathsf{GHP}} \setminus \left\{ \mu \in P_{\mathsf{GHP}} \,\middle|\, \int \frac{\log J_{e^{-s}}^q(\mu)}{s} \, d\Pi_t(s) < c \right\} \\ &= \left\{ \mu \in P_{\mathsf{GHP}} \,\middle|\, \int \frac{\log J_{e^{-s}}^q(\mu)}{s} \, d\Pi_t(s) \geq c \right\}. \end{aligned}$$

We must now prove that F is closed in P_{GHP}. In order to show this, we fix a sequence $(\mu_n)_n$ in F and $\mu \in P_{\mathsf{GHP}}$ with $\mu_n \to \mu$. We must now prove that $\mu \in F$, i.e. we must prove that $\int \frac{\log J_{e^{-s}}^q(\mu)}{s} \, d\Pi_t(s) \geq c$. For brevity define functions $\varphi, \varphi_n : [t_0, \infty) \to [0, \infty)$ by $\varphi(s) = \frac{\log J_{e^{-s}}^q(\mu)}{s}$ and $\varphi_n(s) = \frac{\log J_{e^{-s}}^q(\mu_n)}{s}$. We now prove the following two claims.

Claim 1. We have $c \leq \int \limsup_n \varphi_n \, d\Pi_t$.

Proof of Claim 1. Since $\mu_n \in F$, we conclude that $c \leq \int \frac{\log J_{e^{-s}}^q(\mu_n)}{s} \, d\Pi_t(s) = \int \varphi_n \, d\Pi_t$ for all n, whence $c \leq \limsup_n \int \varphi_n \, d\Pi_t$. Also, since $-\varphi_n \geq 0$, we note that it follows from Fatou's lemma that $\int \liminf_n (-\varphi_n) \, d\Pi_t \leq \liminf_n \int (-\varphi_n) \, d\Pi_t$, and so $\limsup_n \int \varphi_n \, d\Pi_t = -\liminf_n \int (-\varphi_n) \, d\Pi_t \leq -\int \liminf_n (-\varphi_n) \, d\Pi_t = \int \limsup_n \varphi_n \, d\Pi_t$. Combining the previous two inequalities we conclude that $c \leq \limsup_n \int \varphi_n \, d\Pi_t \leq \int \limsup_n \varphi_n \, d\Pi_t$. This completes the proof of Claim 2.

Claim 2. For all $s \geq t_0$, we have $\limsup_n \varphi_n(s) \leq \varphi(s)$. In particular $\int \limsup_n \varphi_n \, d\Pi_t \leq \int \varphi \, d\Pi_t$.
Proof of Claim 2. This follows from the fact that $J_r^q : P_{\mathsf{GHP}} \to \mathbb{R}$ is upper semi-continuous for all $r > 0$. This completes the proof of Claim 2.

Finally, we deduce from Claim 1 and Claim 2 that

$$\begin{aligned} c &\leq \int \limsup_n \varphi_n \, d\Pi_t \\ &\leq \int \varphi \, d\Pi_t \\ &= \int \frac{\log J_{e^{-s}}^q(\mu)}{s} \, d\Pi_t(s) \,. \end{aligned}$$

This completes the proof. □

We now turn towards the proofs of Theorem 1.4.(2) equation (1.12) and Theorem 1.5. We first prove the following result.

Theorem 5.2. *Let $q > 1$ and assume that $J^q_r : P_{\mathsf{GHP}} \to \mathbb{R}$ is upper semi-continuous for all $r > 0$. Let $\Pi = (\Pi_t)_{t \geq t_0}$ be an averaging system. Then a typical $\mu \in P_{\mathsf{GHP}}$ satisfies*

$$\underline{D}^q_{\Pi}(\mu) = -\infty\,.$$

Proof.
We must prove that for a typical measure $\mu \in P_{\mathsf{GHP}}$, we have $\underline{D}^q_{\Pi}(\mu) = -\infty$, i.e. we must prove that the set

$$U = \left\{\mu \in \mathcal{P}(X) \,\middle|\, \underline{D}^q_{\Pi}(\mu) > -\infty\right\}$$

is meagre. For $u \in \mathbb{R}$, write

$$U_u = \left\{\mu \in \mathcal{P}(X) \,\middle|\, \underline{D}^q_{\Pi}(\mu) > u\right\}.$$

Since

$$U = \bigcup_{u \in \mathbb{Q}} U_u\,,$$

it suffices to show that U_u is meagre for all $u \in \mathbb{Q}$ with $u > 0$.

We therefore fix $u \in \mathbb{Q}$ with $u > 0$, and note that it suffices to show that there is a countable family $(G_k)_k$ of open and dense subsets of P_{GHP} with $\cap_k G_k \subseteq P_{\mathsf{GHP}} \setminus U_u$. We now construct the sets G_k. For $t \geq t_0$, let

$$L_t = \left\{\mu \in P_{\mathsf{GHP}} \,\middle|\, \int \frac{\log J^q_{e^{-s}}(\mu)}{s}\, d\Pi_t(s) < u\right\},$$

and for each positive integer k, put

$$G_k = \bigcup_{t \geq k} L_t\,.$$

Below we show that the family $(G_k)_k$ consists of open and dense subsets of P_{GHP} with $\cap_k G_k \subseteq P_{\mathsf{GHP}} \setminus U_u$; this is the content of the following three claims.

Claim 1. The set G_k is open in P_{GHP}.
Proof of Claim 1. Indeed, since it follows from Lemma 3.2 that L_t is open for all $t \geq t_0$, we immediately conclude that $G_k = \cup_{t \geq k} L_t$ is open. This completes the proof of Claim 1.

Claim 2. The set G_k is dense in P_{GHP}.
Proof of Claim 2. Let $\mu \in P_{\mathsf{GHP}}$ and $r > 0$. We must now find $\nu \in P_{\mathsf{GHP}}$ with $d_{\mathsf{GHP}}(\mu, \nu) < r$ and $\nu \in G_k$. Write $M = \operatorname{supp}\mu$. It follows from Lemma 3.3 that there is a measure $P \in \mathcal{P}(M)$ with finite support such that $d_{\mathsf{H}}(\operatorname{supp}\mu, \operatorname{supp}P) < \frac{r}{4}$ and $d_{\mathsf{P}}(\mu, P) < \frac{r}{4}$. Also since $q > 1$, we can find a positive integer N with $N(1-q) < u$. We now write $C = \{x \in \mathbb{R}^N \mid |x| \leq \frac{r}{4}\}$ and let λ denote the normalized N-dimensional Lebesgue measure restricted to C. Finally, put

$$\nu = P \times \lambda\,.$$

Below we prove that $d_{\mathsf{GHP}}(\mu, \nu) < r$ and $\nu \in G_k$.

We first prove that $d_{\mathsf{GHP}}(\mu, \nu) < r$. Let $X = \operatorname{supp}\mu \times C$ and equip X with the maximum metric. Next, define $f : \operatorname{supp}\mu \to X$ and $g : \operatorname{supp}\nu = \operatorname{supp}P \times C \to X$ by $f(x) = (x, 0)$ and $g(x, y) = (x, y)$, and note that f and g are isometries. Next, we observe that

$$\begin{aligned}
d_{\mathsf{P}}(\mu \circ f^{-1}, \nu \circ g^{-1}) &= \sup_{F \in \mathrm{Lip}(X)} \left| \int F\, d(\mu \circ f^{-1}) - \int F\, d(\nu \circ g^{-1}) \right| \\
&= \sup_{F \in \mathrm{Lip}(X)} \left| \int F(x,0)\, d\mu(x) - \int F(x,y)\, d\nu(x,y) \right| \\
&= \sup_{F \in \mathrm{Lip}(X)} \left| \int_{\operatorname{supp}\mu} F(x,0)\, d\mu(x) - \int_C \int_{\operatorname{supp}\mu} F(x,y)\, dP(x)\, d\lambda(y) \right| \\
&= \sup_{F \in \mathrm{Lip}(X)} \left| \int_C \int_{\operatorname{supp}\mu} F(x,0)\, d\mu(x)\, d\lambda(y) - \int_C \int_{\operatorname{supp}\mu} F(x,y)\, dP(x)\, d\lambda(y) \right| \\
&\leq \sup_{F \in \mathrm{Lip}(X)} \int_C \left| \int_{\operatorname{supp}\mu} F(x,0)\, d\mu(x) - \int_{\operatorname{supp}\mu} F(x,y)\, dP(x) \right| d\lambda(y)\,. \qquad (5.1)
\end{aligned}$$

However, for all $F \in \mathrm{Lip}(X)$ and $y \in C$, we have

$$\begin{aligned}
&\left| \int_{\operatorname{supp}\mu} F(x,0)\,d\mu(x) - \int_{\operatorname{supp}\mu} F(x,y)\,dP(x) \right| \\
&\qquad \le \left| \int_{\operatorname{supp}\mu} F(x,0)\,d\mu(x) - \int_{\operatorname{supp}\mu} F(x,0)\,dP(x) \right| \\
&\qquad\qquad + \left| \int_{\operatorname{supp}\mu} F(x,0)\,dP(x) - \int_{\operatorname{supp}\mu} F(x,y)\,dP(x) \right| \\
&\qquad \le d_{\mathsf{P}}(\mu,P) + \int_{\operatorname{supp}\mu} |F(x,0)-F(x,y)|\,dP(x) \\
&\qquad \le d_{\mathsf{P}}(\mu,P) + \int_{\operatorname{supp}\mu} \mathrm{Lip}(X)\,|(x,0)-(x,y)|\,dP(x) \\
&\qquad \le d_{\mathsf{P}}(\mu,P) + \int_{\operatorname{supp}\mu} |y|\,dP(x) \\
&\qquad \le d_{\mathsf{P}}(\mu,P) + \tfrac{r}{4} \\
&\qquad < \tfrac{r}{4} + \tfrac{r}{4} \\
&\qquad = \tfrac{r}{2}\,. \qquad (5.2)
\end{aligned}$$

Combining (5.1) and (5.2), we conclude that

$$\begin{aligned}
d_{\mathsf{P}}(\mu\circ f^{-1}, \nu\circ g^{-1}) &< \sup_{F\in\mathrm{Lip}(X)} \int_C \tfrac{r}{2}\,d\lambda(y) \\
&= \tfrac{r}{2}\,. \qquad (5.3)
\end{aligned}$$

We also note that

$$\begin{aligned}
d_{\mathsf{H}}(f(\operatorname{supp}\mu), g(\operatorname{supp}\nu)) &= d_{\mathsf{H}}(\operatorname{supp}\mu\times\{0\}, \operatorname{supp}P\times C) \\
&\le d_{\mathsf{H}}(\operatorname{supp}\mu\times\{0\}, \operatorname{supp}P\times\{0\}) + d_{\mathsf{H}}(\operatorname{supp}P\times\{0\}, \operatorname{supp}P\times C) \\
&\le d_{\mathsf{H}}(\operatorname{supp}\mu, \operatorname{supp}P) + d_{\mathsf{H}}(\{0\}, C) \\
&< \tfrac{r}{4} + \tfrac{r}{4} \\
&= \tfrac{r}{2}\,. \qquad (5.4)
\end{aligned}$$

Finally, combining (5.3) and (5.4) immediately gives $d_{\mathsf{GHP}}(\mu,\nu) \le d_{\mathsf{H}}(f(\operatorname{supp}\mu), g(\operatorname{supp}\nu)) + d_{\mathsf{P}}(\mu\circ f^{-1}, \nu\circ g^{-1}) < r$.

Next, we prove that $\nu \in G_k$. Indeed, since λ is the normalized N-dimensional Lebesgue measure restricted to $C = \{x \in \mathbb{R}^N \,|\, |x| \le \frac{r}{4}\}$, a simple and straightforward calculation shows that $\frac{\log J_r^q(\mu)}{-\log r} \to N(1-q)$ as $r \searrow 0$. It follows from this and the fact that $\operatorname{supp}P$ is finite that $\frac{\log J_r^q(\nu)}{-\log r} = \frac{\log J_r^q(P\times\lambda)}{-\log r} \to N(1-q)$ as $r \searrow 0$, and so $\frac{\log J^q_{e^{-t}}(\nu)}{t} \to N(1-q)$ as $t\to\infty$. We deduce from this that $\int \frac{\log J^q_{e^{-s}}(\nu)}{s}\,d\Pi_t(s) \to N(1-q)$ as $t\to\infty$, and since $N(1-q) < u$, we can therefore find $t \ge k$ with $\int \frac{\log J^q_{e^{-s}}(\nu)}{s}\,d\Pi_t(s) < u$, whence $\nu \in L_t \subseteq G_k$. This completes the proof of Claim 2.

Claim 3. We have $\cap_k G_k \subseteq P_{\mathsf{GHP}} \setminus U_u$.

Proof of Claim 3. Let $\mu \in \cap_k G_k$. Hence for each positive integer k, we can find $t_k \ge k$ such that $\mu \in L_{t_k}$, whence $\int \frac{\log J^q_{e^{-s}}(\mu)}{s}\,d\Pi_{t_k}(s) < u$ for all positive integers k. We conclude from this and Lemma 3.3 that $\underline{D}^q_\Pi(\mu) = \liminf_{t\to\infty} \int \frac{\log I^q_{e^{-s}}(\mu)}{s}\,d\Pi_t(s) \le \liminf_{t\to\infty} \int \frac{\log J^q_{e^{-s}}(\mu)}{s}\,d\Pi_t(s) \le \liminf_k \int \frac{\log J^q_{e^{-s}}(\mu)}{s}\,d\Pi_{t_k}(s) \le u$, and so $\mu \in P_{\mathsf{GHP}} \setminus U_u$. This completes the proof of Claim 3.

Combining Claim 1, Claim 2 and Claim 3, we now conclude that U_u is meagre. □

We can now prove Theorem 1.4.(2) equation (1.12) and Theorem 1.5.

Proof of Theorem 1.4.(2) equation (1.12).
Since Lemma 3.2 shows that $J_r^q : P_{\mathsf{GHP}} \to \mathbb{R}$ is upper semi-continuous for all $r > 0$ and all $q \in \mathbb{N}$, it follows from Theorem 5.2 that:

$$\text{A typical measure } \mu \in P_{\mathsf{GHP}} \text{ satisfies } \underline{D}_{\Pi}^{q}(\mu) = -\infty \text{ for all } q \in \mathbb{N} \text{ with } q > 1. \tag{5.5}$$

Next, note that if $\mu \in P_{\mathsf{GHP}}$ and $1 < q \leq p$, then $-\infty \leq \underline{D}_{\Pi}^{p}(\mu) \leq \underline{D}_{\Pi}^{q}(\mu)$, and so:

$$\begin{aligned} &\text{If a typical measure } \mu \in P_{\mathsf{GHP}} \text{ satisfies } \overline{D}_{\Pi}^{q}(\mu) = -\infty, \\ &\text{then a typical measure } \mu \in P_{\mathsf{GHP}} \text{ satisfies } \overline{D}_{\Pi}^{p}(\mu) = -\infty \text{ for all } q \leq p. \end{aligned} \tag{5.6}$$

Combining (5.5) and (5.6), we now conclude that a typical measure $\mu \in P_{\mathsf{GHP}}$ satisfies $\underline{D}_{\Pi}^{p}(\mu) = -\infty$ for all $2 \leq p$. □

Proof of Theorem 1.5.
The statement in Theorem 1.5 follows immediately from Theorem 5.2. □

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