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Optical eigenmode description of single-photon light-matter interactions

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ABSTRACT

When light scatters from an object, it can impart some physical quantity such as momentum or angular momentum. This can act as a measurement on the photon, which collapses on to an eigenstate of the measurement operator. However the corresponding operator is not the same as that describing the total linear or angular momentum in free space. Optical eigenmodes provide a powerful method to describe this interaction by expanding the field as a linear combination of some basis modes and examining the eigenvalues and eigenvectors of the quadratic measure in question. We extend this to the quantum case by writing the quantum operator corresponding to a given measurement such as energy, momentum or angular momentum as a superposition of creation and annihilation operators for each eigenmode. Upon measurement we find that the possible states of a single photon are simply the classical eigenmodes of the measurement. As an application, we examine the force and torque on a general, possibly anisotropic, material. By looking at eigenvalues of the measurement operator we show that the amount of a given quantity transferred in an interaction with matter is not in general the expected amount which a photon carries in free space, even at the single photon level. In particular the difference in linear or angular momentum from before and after is in general not equal to $\hbar k$ or $\hbar$ which are the eigenvalues of these quantities in free space.

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1. INTRODUCTION

The decomposition of fields into eigenmodes is a common technique to solve various proplems. Such a decomposition can be used to describe the propagation of light in a variety of setups such as waveguides, photonic crystals, and optical cavities. The method of Optical Eigenmodes (OEi) has been introduced as a generalization of this mode decomposition going beyond describing the propagation properties.[1] This method applies to any quadratic measure of the electromagnetic field such as energy density, momentum, angular momentum, etc.[2] In addition, we can describe the change in these quantities when interacting with matter, giving the forces or torques experienced by particles. This method consists of finding the eigenmodes of the quadratic measure in question. These input light fields contribute to the measure independently, without interference, allowing for example the addressing of individual plasmonic nano-antennas of an array without cross-talk.[3] It is worth noting that as this method results in a set of eigenmodes which contribute an amount equal to the corresponding real eigenvalue, optimisation of a quantity such as spot size or energy transmission is as trivial as choosing the mode with the highest or lowest eigenvalue.

While a single photon is known to carry both spin and orbital angular momentum quantised in units of $\hbar$,[4,5] and momentum quantised in units of $\hbar k$, where k is the wave-vector, the photon need not be in an eigenstate of any of these quantites before or after an interaction with matter. Hence, it is not necessarily the case that an integer number of these units is transferred when a photon scatters from a nanoparticle. As mechanical resonators have been brought well into the quantum regime,[6,7] the interaction between single quanta of optical and mechanical degrees of freedom becomes increasingly important. In this paper we apply the method of optical

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eigenmodes to quantum measurements, which correspond to Hermitian operators acting on quantum states. We show that it is these eigenmodes which deliver a quantised amount of linear or angular momentum which is differrent to the quanta of these observables in free space. For a given input state, each eigenmode will be selected with a probability proportional to its overlap with the input. After the measurement, the quantum state will collapse exactly into one of the classical optical eigenmodes of the measurement. Hence, optical eigenmodes provide a natural framework to consider the interaction of quantum light states with macroscopic matter.

2. OPTICAL EIGENMODES FOR QUANTUM OPERATORS

Expanding the electric field as a superposition of some basis fields $\mathbf{E} = \sum_i \alpha_i \mathbf{E}_i$, we can write a general quadratic function of the fields as a Hermitian form of these coefficients $m(\mathbf{E}) = \alpha^* \mathcal{M} \alpha$ where $\mathcal{M}$ is a Hermitian matrix. For the expansion to be exact we require a complete set of basis fields; in practice we can keep only a finite number of basis functions which span a subspace, for example a set of low-order Hermite-Gauss beams. Since $\mathcal{M}$ is Hermitian, it has a spectrum of real eigenvalues which can be ordered to find the maximum or minimum possible value of the quantity of interest and the corresponding optical eigenmode which produces it.

The quantization of the free electro-magnetic field is similarly done by expanding over some basis, for example plane waves,

$$\mathbf{E}(\mathbf{r}, t) = i\omega \sum_{\mathbf{k},\sigma} \mathbf{e}_{\mathbf{k},\sigma} a_{\mathbf{k},\sigma}(t) e^{i\mathbf{k}\cdot\mathbf{r}} - \text{h.c.}, \tag{1}$$

and then promoting the coefficients to operators, $\hat{a}_{\mathbf{k},\sigma}$ with canonical commutation relations $\left[\hat{a}_{\mathbf{k},\sigma} \hat{a}^\dagger_{\mathbf{k}',\sigma'}\right] = \delta_{\mathbf{k}\mathbf{k}'} \delta_{\sigma\sigma'}$.[8]

In analogy with the classical case, an observable in quantum mechanics is describes by some operator $\hat{m}$. If the classical observable is quadratic, then we also have the quantum operator as a Hermitian form of the creation and annihilation operators $\hat{m} = \hat{\mathbf{a}}^\dagger \mathcal{M} \hat{\mathbf{a}}$. Again, $\mathcal{M}$ is Hermitian and has real eigenvalues which may be ordered. These eigenvalues have an additional significance; they are the only possible results of a measurement of m, and after the measurement the system will collapse into the corresponding eigenvector.

To expand the concept of optical eigenmodes to the quantum case, we find a unitary transformation U such that $D = U^\dagger \mathcal{M} U$ is diagonal. The creation and annihilation operators transform in a corresponding way, $\hat{A}_i = U_{ij} \hat{a}_j$. Since U is unitary, this preserves the canonical commutation relations $\left[\hat{A}_i, \hat{A}^\dagger_j\right] = \left[\hat{a}_i, \hat{a}^\dagger_j\right] = \delta_{ij}$. These operators $\hat{A}^\dagger_i$, $\hat{A}_i$ correspond to creating and annihilating a single photon in the eigenmode i, which will contribute an exact amount to the measurement m. Crucially, performing the measurement will project the quantum state on to one of these eigenmodes, which are just those described by the classical matrix $\mathcal{M}$. Again, the real eigenvalues can be ordered to find the state which maximises or minimises any quantum observable.

3. APPLICATIONS

As an example of the above we look at the quantum of force or torque when light is incident on a slab of homogenous, but anisotropic material. The quantum operator of interest here is the difference in linear momentum or angular momentum between the incoming and outgoing beams. The creation and annihilation operators corresponding to the outgoing beams are related to those in the incoming beams via the transfer matrix method.[9] Consider a birefringent, uniaxial material whose optic axis is perpendicular to its surface such that s and p polarized rays have independent transmission amplitudes $t_{s,p}$ and reflection amplitudes $r_{s,p}$. The scattering is illustrated in Fig. 1. We consider a plane wave incoming from the left, labeled by the operator $\hat{a}$, being transmitted and reflected in to plane waves labeled by $\hat{c}$ and $\hat{d}$ respectively. We must also take in to account the corresponding input $\hat{b}$ which is incident on the opposite side with the same angle of incidence. For the majority of this section we will further simplify to a wave-plate which has $|t_p| = |t_s| = t$, with $t_s = \exp{(i\phi)} t_p$ and similarly $r_s = -\exp{(i\phi)} r_p$.

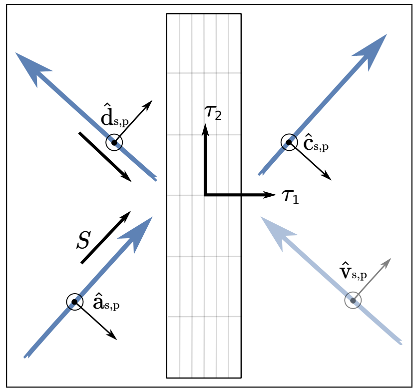

Figure 1. Illustration of light scattering from a material labeling the incoming and outgoing beams.

We have the linear and spin angular momentum operators for the incoming beam;

$$\hat{\mathbf{P}}_a = \sum_{\mathbf{k}} \hbar\mathbf{k}\left(\hat{a}^\dagger_{\mathbf{k},+}\hat{a}_{\mathbf{k},+} + \hat{a}^\dagger_{\mathbf{k},-}\hat{a}_{\mathbf{k},-}\right) \tag{2}$$

$$\hat{\mathbf{S}}_a = \sum_{\mathbf{k}} \hbar\frac{\mathbf{k}}{|k|}\left(\hat{a}^\dagger_{\mathbf{k},+}\hat{a}_{\mathbf{k},+} - \hat{a}^\dagger_{\mathbf{k},-}\hat{a}_{\mathbf{k},-}\right) \tag{3}$$

where $+$ and $-$ refer to left and right circular polarization respectively, $\hat{a}_\pm = (\hat{a}_p \pm i\hat{a}_s)$, and similarly for the other incoming and outgoing beams. While these operators have the expected eigenvalues $\hbar\mathbf{k}$ and $\hbar$ respectively, if we measure the deflection or rotation of a particle it is the difference in these quantities before and after the interaction which is transferred. Defining $\hat{\mathbf{F}} \equiv \hat{\mathbf{P}}_c + \hat{\mathbf{P}}_d - \hat{\mathbf{P}}_a - \hat{\mathbf{P}}_b$ and $\hat{\mathbf{T}} \equiv \hat{\mathbf{S}}_c + \hat{\mathbf{S}}_d - \hat{\mathbf{S}}_a - \hat{\mathbf{S}}_b$, we can write these as Hermitian forms of the input operators, i.e. $\hat{\mathbf{F}} = \psi^\dagger F\psi$, $\hat{\mathbf{T}} = \psi^\dagger T\psi$, with $\psi = (\hat{a}_+, \hat{a}_-, \hat{b}_+, \hat{b}_-)^T$. We find

$$\mathcal{F}_x = \hbar k_x \begin{pmatrix} |t|^2 - |r|^2 - 1 & 0 & 2i|r||t| & 0 \\ 0 & |t|^2 - |r|^2 - 1 & 0 & -2i|r||t| \\ -2i|r||t| & 0 & 1 + |r|^2 - |t|^2 & 0 \\ 0 & 2i|r||t| & 0 & 1 + |r|^2 - |t|^2 \end{pmatrix}. \tag{4}$$

This has eigenvalues $\pm 2\hbar k_x\sqrt{1-|t|^2}$. That is, a photon reflecting from a partially reflective surface does not transfer a momentum $2\hbar k_x$ with some probability, but rather transfers a fractional amount of this with 100% probability. This may seem suprising, but it is a result of measuring only the difference in momentum before and after the interaction; where we to measure the momentum of the photon itself before and after the scattering, both results would have magnitude $\hbar k$.

A similar form exists for the torque on the plate, defined as the difference in angular momentum between the incoming and outgoing beams, which is more complex due to the mixing of polarizations in the birefringent material. The eigenvalues of $\mathcal{T}_x$ are $\pm\hbar k_x\sqrt{2(1-\cos\phi)}$, again corresponding to a fractional amount of spin angular momentum of the incoming photon. The eigenvalues of $\mathcal{T}_y$ are $\pm\sqrt{2}\sqrt{1+\cos\phi(|r|^2-|t|^2) \pm 2|r||t|\sin\phi}$.

These values can be understood by considering some special cases. For a plate with no birefringence ($\phi = 0$) there is no torque around the x-axis as there is no change in angular momentum. However there is a torque in the y-direction of $\pm 2\hbar|r|$ due to the reversal of circular polarization upon reflection. For a quarter-wave plate ($\phi = \pi/2$) the eigenvalues of $\mathcal{T}_x$ are $\pm\sqrt{2}\hbar$ while those of $\mathcal{T}_y$ are $\pm\sqrt{2(1 \pm |r||t|)}$. For a half-wave plate ($\phi = \pi$) these are $\pm 2\hbar$ and $\pm 2|t|\hbar$, respectively, corresponding to the conversion of right circular polarization to left and vice-versa.

It is important to note that the various components of force and torque need not commute in general. If two components do then there is a basis of simultaneous eigenvectors and they can be simultaneously diagonalized. If they do not then there will be some minimum uncertainty relation between those values. For the case we consider here we have $[\mathcal{T}_x, \mathcal{T}_y]$, $[\mathcal{F}_x, \mathcal{T}_y] \neq 0$. However we find $[\mathcal{T}_x, \mathcal{F}_x] = 0$. This is because the force depends on interference between the different input ports, but not the different polarizations which each have the same momentum, while the torque in the x-direction depends on the coupling between different polarizations, but not the different input ports. This is a consequence of the choice $|t_p| = |t_s|$, and will no longer be true if these two differ.

Another simple limiting case is that where the plate acts as a polarizing beam-splitter, i.e. $t_p = r_s = 0$, $t_s = r_p = 1$. In this case $\mathcal{T}_x$ and $\mathcal{T}_y$ have the same eigenvalues of $0, \pm 2\hbar$, while $\mathcal{F}_x$ has the similar set of $0, \pm 2\hbar k_x$. However these three operators all have different eigenvectors and so do not commute. The eigenvectors of $\mathcal{F}_x$ correspond to a single input with a polarization which is either reflected or transmitted. The eigenvalues of $\mathcal{F}_x$ and $\mathcal{F}_y$ correspond to an input in both ports with either opposite or identical circular polarizations respectively. The interference between the reflected p polarization and the transmitted s polarization from each circular state leads to another circular polarization in each output.

4. SCATTERING FROM A FINITE OBJECT

While considering infinite homogeneous slabs allows some simple analytic results, we are primarily interested in the scattering of light from finite objects which may not be homogeneous or isotropic. To calculate the quantized forces and torques on such objects, we can consider a surface which fully encloses the object, and consider the momentum and angular momentum into and out of that surface.

As an example, consider an elliptical scatterer in two dimensions. To probe momentum transfer to this particle, we use a basis of Gaussian beams with a direction relative to the x axis which varies from 0 to 2π in steps of $5°$. For each input beam we calculate, using finite element method, the scattered field in the far-field. An example of the scattering of one such input beam is shown in Fig. 2. The momentum in each incoming and outgoing beam has the same form as Eq. 2. The angular spectrum decomposition allows a change of basis between plane waves and the input Gaussian beams. This results in a quadratic form for the total change of momentum, i.e. the force on the scatterer, in terms of the input basis,

$$F_x = \sum \hat{a}_i^\dagger f_{x,ij} \hat{a}_j \tag{5}$$

$$F_y = \sum \hat{a}_i^\dagger f_{y,ij} \hat{a}_j, \tag{6}$$

where $\hat{a}_i$ now label our basis of input beams. Diagonalising either of these operators via a unitary transformation leads to a corresponding change of basis of optical eigenmodes which are the quantum eigenmodes of the corresponding operator, and deliver the corresponding eigenvalue of force when that component is measured.

One such eigenmode is shown in Fig. 3 (a). This is an input optical eigenmode of the x component of force, i.e. if there is a single incident photon in this field distribution, and the x-component of scattering of the particle is measured, the result will be the corresponding eigenvalue with 100% probability. An arbitrary incident beam will in general not be in one of these eigenmodes. In that case the result of a single measurement will be one of the eigenvalues of the operator, with a probability given by the overlap of the incoming beam with each optical eigenmode. These probabilities are shown in Fig. 3 (b) for one of our original basis functions, a Gaussian beam with initial direction along the x axis.

The relative scattering modes and resulting forces are shown for three optical eigenmodes of F_x in Fig. 4 with relatively high overlap with a Gaussian directed along the x axis. The angular intensity distribution of the incoming optical eigenmode is shown along the top row in red, with that of the same optical eigenmode after the scattering in blue. In each case there is some forward-directed part of the incoming mode which leads to a reasonable overlap probability, though none of them approaches an overlap probability of 1. The average momentum of each optical eigenmode before and after the scattering is shown in the bottom row, again in red and blue respectively. The average change in momentum is proportional to the force on the scatterer shown in gray. If the force was measured many times in different directions, for example by interaction with the environment, this is the average force we would expect from this optical eigenmode. However, this is an eigenmode only of the x component of force, shown in black. The value of this component, in arbitrary units, is also shown in each column. For a single-photon measurement, we would record one of these values, with the respective probability, and immediately after the measurement the photon would be in the outgoing optical eigenmodes. These optical eigenmodes hence give a natural basis to describe scattering between quantum states of light and micro-particles.

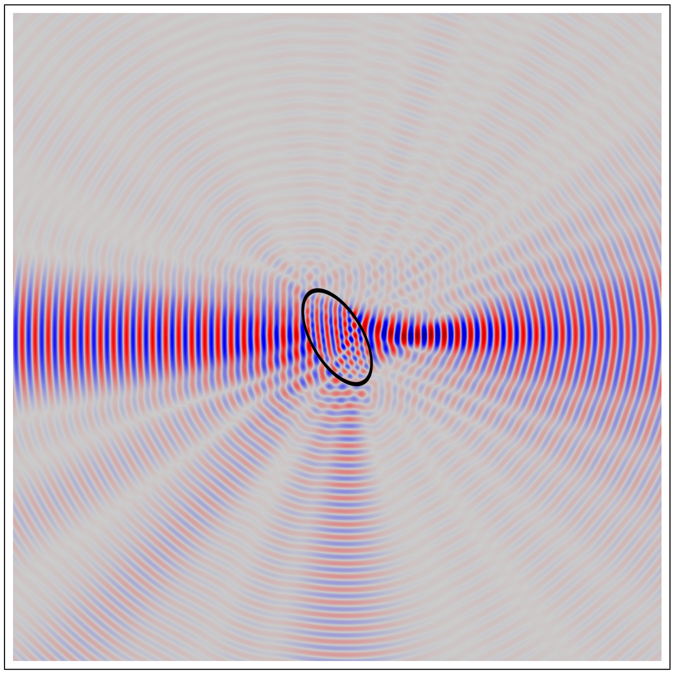

Figure 2. Numeric calculation of scattered field from an ellipse for an Gaussian beam incident along the $+x$ direction.

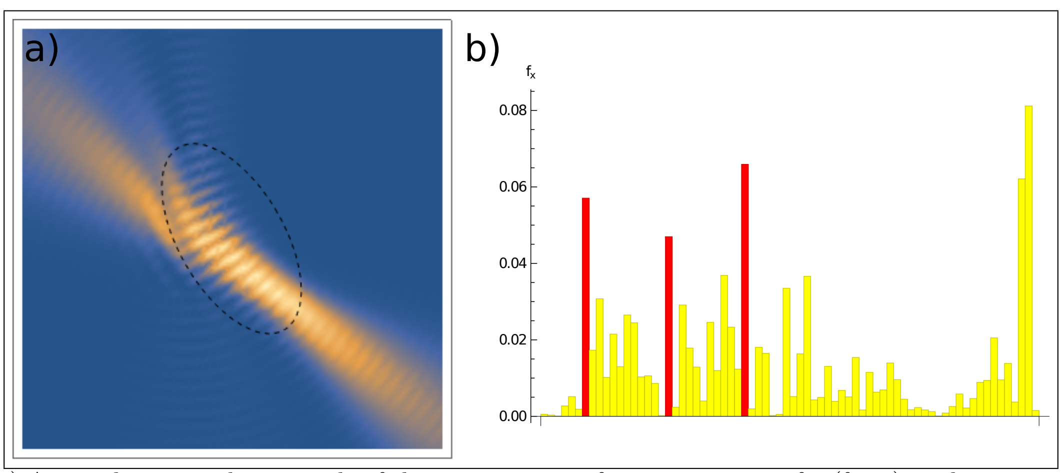

Figure 3. (a) An incident optical eigenmode of the x-component of momentum transfer (force) to the scatterer. (b) The probability of collapsing into each of the optical eigenmodes (ordered by decreasing eigenvalue) of the x-component of force. Those highlighted in red are explored further below.

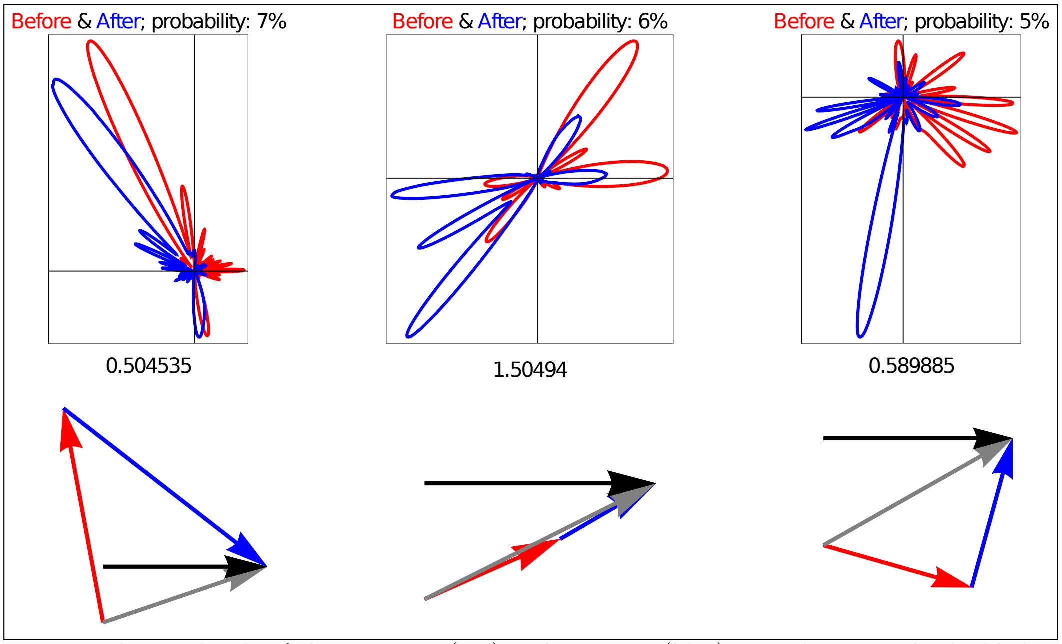

Figure 4. Top row: The amplitude of the incoming (red) and outgoing (blue) optical eigenmodes highlighted in Fig. 3. Bottom row: An illustration of the average momentum in the incoming (red) and outgoing (blue) optical eigenmodes. The difference between them (grey) gives the average force on the particle, but they are only eigenmodes of the x-component (black).

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