ASSOUAD TYPE DIMENSIONS AND DIMENSION SPECTRA

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Assouad type dimensions and dimension spectra

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For Shuang Li

whose love lights up my life

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Abstract

In the first part of this thesis we introduce a new dimension spectrum motivated by the Assouad dimension; a familiar notion of dimension which, for a given metric space, returns the minimal exponent $\alpha \geqslant 0$ such that for any pair of scales $0 < r < R$, any ball of radius R may be covered by a constant times $(R/r)^\alpha$ balls of radius r. To each $\theta \in (0,1)$, we associate the appropriate analogue of the Assouad dimension with the restriction that the two scales r and R used in the definition satisfy $\log R/\log r = \theta$. The resulting 'dimension spectrum' (as a function of θ) thus gives finer geometric information regarding the scaling structure of the space and, in some precise sense, interpolates between the upper box dimension and the Assouad dimension. This latter point is particularly useful because the spectrum is generally better behaved than the Assouad dimension. We also consider the corresponding 'lower spectrum', motivated by the lower dimension, which acts as a dual to the Assouad spectrum. We conduct a detailed study of these dimension spectra; including analytic and geometric properties. We also compute the spectra explicitly for some common examples of fractals including decreasing sequences with decreasing gaps and spirals with sub-exponential and monotonic winding. We also give several applications of our results, including: dimension distortion estimates under bi-Hölder maps for Assouad dimension. We compute the spectrum explicitly for a range of well-studied fractal sets, including: the self-affine carpets of Bedford and McMullen, self-similar and self-conformal sets with overlaps, Mandelbrot percolation, and Moran constructions. We find that the spectrum behaves differently for each of these models and can take on a rich variety of forms. We also consider some applications, including the provision of new bi-Lipschitz invariants and bounds on a family of 'tail densities' defined for subsets of the integers.

The second part of this thesis, we study the Assouad dimension of sets of integers and deduce a weak solution to the Erdős-Turán conjecture. Let $F \subset \mathbb{N}$. If $\sum_{n\in F} n^{-1} = \infty$ then F "asymptotically" contains arbitrarily long arithmetic progressions.

Contents

1 Introduction

We begin by introducing basic definitions and results of this thesis. This thesis is based on the author's co-authored papers [FY16a], [FY16b] and [FY18].

The Assouad dimension is a fundamental notion of dimension used to study fractal objects in a wide variety of contexts. It was popularised by Assouad in the 1970s [A77, A79] and subsequently took on significant importance in embedding theory. Recall the famous *Assouad Embedding Theorem* which states that if (X, d) is a metric space with the doubling property (equivalently, with finite Assouad dimension), then (X, d^{ε}) admits a bi-Lipschitz embedding into some finite dimensional Euclidean space for any $\varepsilon \in (0, 1)$. The notion we now call Assouad dimension does go back further, however, to Larman's work in the 1960s [L67a, L67b] and even to Bouligand's 1928 paper [B28]. It is also worth noting that, due to its deep connections with tangents (see [MT10]), it is intimately related to pioneering work of Furstenberg on micro-sets which goes back to the 1960s, see [F08]. Roughly speaking, the Assouad dimension assigns a number to a given metric space which quantifies the most difficult location and scale at which to cover the space. More precisely, it considers two scales $0 < r < R$ and finds the maximal exponential growth rate of $N(B(x, R), r)$ as R and r decrease, where $N(E, r)$ is the minimal number of r-balls required to cover a set E.

The Assouad dimension has found important applications in a wide variety of contexts, including a sustained importance in embedding theory, see [O02, O10, R11]. It is also central to quasi-conformal geometry, see [H01, T01, MT10], and has recently been gaining significant attention in the literature on fractal geometry and geometric measure theory, see for example [FO17, KLV13, KR16, LDR15, F14, L98, M11]. There are also connections between the Assouad dimension and problems in arithmetic combinatorics, for example the existence of arithmetic progressions or asymptotic arithmetic progressions which will be discussed in Chapter 12.

Since it is an *extremal* quantity, the Assouad dimension gives rather coarse information about the space and is often very large; larger than the other familiar notions of dimension such as Hausdorff and box-counting dimension. Also, despite the fact that two scales are used (r and R), the Assouad dimension returns no information about *which* scales 'see' the maximal exponential growth rate described above. In this thesis we propose a programme to tackle these problems: we fix the relationship between the scales R and r and then compute the corresponding *restricted* Assouad dimension by only considering pairs of scales with this fixed relationship. More precisely, for a fixed $\theta \in (0, 1)$, we look for the maximal exponential growth rate of $N(B(x, R), r)$ as R decreases and r is defined by $\log R/ \log r = \theta$. One can then vary θ and obtain a spectrum of dimensions for the given metric space which can be viewed as providing finer geometric information about the (lack of) homogeneity present and a more complete picture of how the space scales. One may also be able to pick out θs which 'see' the Assouad dimension, i.e., values where the spectrum reaches the true Assouad dimension. If the Assouad dimension is 'seen' by the spectrum, then we are able to glean more information about the Assouad dimension because the spectrum is generally better behaved than the Assouad dimension, see for example Theorem 4.12.

Another key motivation for this work is that the finer the information we are able to glean concerning the scaling structure of the space, the better the applications should be. In particular, we believe that the notions we introduce and study here should bear fruit in other areas where the Assouad dimension already plays a role; such as embedding theory, quasi-conformal geometry, and geometric measure theory.

We begin by considering how these spectra behave as functions of θ for arbitrary sets. Some of the notable results we obtain in this direction include:

1. There are non-trivial (and sharp) bounds on the spectra in terms of familiar dimensions, see Propositions 3.1 and 3.9, and also Corollaries 3.2 and 3.3.

2. The spectra are continuous in θ, see Corollaries 3.5 and 3.10.

3. The Assouad spectrum interpolates between upper box dimension and Assouad dimension. In particular, as $\theta \to 0$ the spectrum always approaches the upper box dimension, and as $\theta \to 1$ the spectrum always approaches its maximal value, which is often the Assouad dimension. See Corollary 3.2, Corollary 3.6, and Proposition 3.7.

4. The spectra are often, but not necessarily, monotonic, see Proposition 3.7 and Chapter 7.

5. The spectra have good distortion properties under bi-Hölder functions, which is in contrast to the Assouad dimension, see Proposition 4.8.

6. We analyse how the spectra behave under standard geometric operations such as unions, closures and products, see Propositions 4.1 and 4.5.

Although the main purpose here is to introduce, and conduct a thorough investigation of, our new dimension spectra, we also obtain several results as corollaries or bi-products of our work, which are not *a priori* related to the spectra. We summarise some of these results here:

1. We provide new bi-Hölder distortion results for Assouad dimension, see Theorem 4.12. In particular, if the Assouad spectrum reaches the Assouad dimension, then we can give a bound on how the Assouad dimension distorts under a bi-Hölder map. No such bounds exist for general sets.

2. We prove that sub-exponential spirals cannot be 'unwound' to line segments via certain bi-Hölder functions, see Corollary 6.3. This provides a natural extension to work of Fish and Paunescu concerning bi-Lipschitz unwinding [FP16], as well as classical unwinding theorems of Katznelson, Subhashis and Sullivan [KSS90].

3. We prove that a spiral with 'monotonic winding' either has Assouad dimension 1 or 2, see Theorem 6.1.

The key motivation behind these Assouad type spectra is that they provide more detailed and precise information about the scaling structure of the space. In particular, one obtains a spectrum of exponents, rather than a single one. We will explicitly compute the spectra in a variety of important contexts. What we find is that the spectra display a wide variety of different features and forms, reflecting the differences between the models we consider. Specifically, our main results are contained in four virtually stand alone sections including:

1. Self-affine carpets in Chapter 8.

2. Self-similar and self-conformal sets with overlaps in Chapter 9 .

3. Mandelbrot percolation in Chapter 10.

4. Moran constructions in Chapter 11.

In Chapter 12 we study the Assouad dimension of integer sets and provide a weak answer to the Erdős-Turán conjecture. Finally, in Chapter 13 we collect several open questions and discuss possible directions for future work.

2 Notation and preliminaries

2.1 Assouad type spectra

We begin by recalling the precise definition of the Assouad dimension, which serves to motivate our new definition and will be used in Chapter 12. Let $F \subseteq X$ where X is a fixed metric space. The *Assouad dimension* of F is defined by

$$\dim_{\mathrm{A}} F \; = \; \inf \Bigg\{ \alpha \; : \; (\exists C > 0) \, (\forall 0 < r < R) \, (\forall x \in F) \\ N\big(B(x,R) \cap F, r\big) \; \leqslant \; C \left(\frac{R}{r}\right)^{\alpha} \Bigg\}.$$

As described above, we will modify this definition by taking the infimum over the less restrictive condition that the scaling property only holds for scales $0 < r < R \leqslant 1$ satisfying a particular relationship. For $\theta \in (0,1)$, we define

$$\dim_{\mathrm{A}}^{\theta} F \; = \; \inf \Bigg\{ \alpha \; : \; (\exists C > 0) \, (\forall 0 < R \leqslant 1) \, (\forall x \in F) \\ N\big(B(x,R) \cap F, R^{1/\theta}\big) \; \leqslant \; C \left(\frac{R}{R^{1/\theta}}\right)^{\alpha} \Bigg\}.$$

We are particularly interested in the function $\theta \mapsto \dim_{\mathrm{A}}^{\theta} F$ which we refer to as the *Assouad spectrum* (of F). For convenience we extend $\dim_{\mathrm{A}}^{\theta} F$ to $\theta \in (0,\infty)$ by setting $\dim_{\mathrm{A}}^{\theta} F = 0$ for $\theta \geqslant 1$.

Of course there are many other ways to fix the relationship between the scales r and R. However, it turns out that if one wants to develop a rich theory, the most natural way to do this is what we propose here. See the discussion in Chapter 13 at the end of the thesis for more details on this point.

The lower dimension, introduced by Larman [L67a, L67b], is the natural dual of the Assouad dimension. We refer the reader to [F14] for an in-depth discussion of the relationships and differences between these dimensions. Let $F \subseteq X$ be as above. The *lower dimension* of F is defined by

$$\dim_{\mathrm{L}} F = \sup\Bigg\{\alpha \ : \ (\exists C > 0)\,(\forall 0 < r < R \leqslant 1)\,(\forall x \in F) \quad N\big(B(x,R) \cap F, r\big) \geqslant C\left(\frac{R}{r}\right)^{\alpha}\Bigg\}.$$

Notice that the above definition is not exactly a dual version of Assouad dimension in which we did not have the requirement $R < 1$. Due to the local nature of this definition, it has many strange properties which may not be seen as desirable for a 'dimension' to satisfy. For example, it is not monotone as the presence of a single isolated point renders the lower dimension 0, and it may take the value 0 for an open subset of Euclidean space, see [F14, Example 2.5]. One can modify the definition to get rid of these (perhaps) strange properties by defining the *modified lower dimension* by

$$\dim_{\mathrm{ML}} F = \sup\{\dim_{\mathrm{L}} E \ : \ \emptyset \neq E \subseteq F\}.$$

For $\theta \in (0,1)$, we define

$$\dim_{\mathrm{L}}^{\theta} F = \sup\Bigg\{\alpha \ : \ (\exists C > 0)\,(\forall 0 < R \leqslant 1)\,(\forall x \in F) \quad N\big(B(x,R) \cap F, R^{1/\theta}\big) \geqslant C\left(\frac{R}{R^{1/\theta}}\right)^{\alpha}\Bigg\}.$$

Again, we are particularly interested in the function $\theta \mapsto \dim_{\mathrm{L}}^{\theta} F$ which we refer to as the *lower spectrum* (of F). As before, we extend $\dim_{\mathrm{L}}^{\theta} F$ to $\theta \in (0,\infty)$ by setting $\dim_{\mathrm{L}}^{\theta} F = 0$ for $\theta \geqslant 1$. We can also modify this definition to force it to be monotone and to take on the ambient spatial dimension for open sets. The *modified lower spectrum* (of F) is defined by

$$\dim_{\mathrm{ML}}^{\theta} F = \sup\big\{\dim_{\mathrm{L}}^{\theta} E \ : \ \emptyset \neq E \subseteq F\big\}.$$

The key motivation behind these new definitions is that the geometric information provided by the Assouad and lower dimensions is too coarse. We gain more information by understanding how the inhomogeneity depends on the scales one is considering. Alternative approaches to getting more out of these dimensions are possible. For example, Fraser and Todd [FT18] recently considered a quantitative analysis of the Assouad dimension where they looked to understand how inhomogeneity varies in space, i.e. as one changes the point $x \in F$ around which one is trying to cover the set. They found that for some natural examples this inhomogeneity could be described by a *Large Deviations Principle*. In a certain sense our approach is dual to that of [FT18] in that we put restrictions on *scale* but still maximise over *space*, whereas in [FT18] restrictions were put on *space*, but the quantities were still maximised over all *scales*.

2.2 Other notions of dimensions

The Assouad and lower dimensions are closely related to the upper and lower box dimension. The *upper* and *lower box dimensions* of a totally bounded set F are defined by

$$\overline{\dim}_{\mathrm{B}} F \;=\; \limsup_{R \to 0} \frac{\log N\big(F,R\big)}{-\log R} \qquad \text{and} \qquad \underline{\dim}_{\mathrm{B}} F \;=\; \liminf_{R \to 0} \frac{\log N\big(F,R\big)}{-\log R}$$

and if the upper and lower box dimensions coincide we call the common value the *box dimension* of F and denote it by $\dim_{\mathrm{B}} F$. We refer the reader to [F03, Chapter 3] for more details on the upper and lower box dimensions and their basic properties. In particular we note the following general relationships which hold for any totally bounded set F:

$$\dim_{\mathrm{L}} F \;\leqslant\; \dim_{\mathrm{ML}} F \;\leqslant\; \underline{\dim}_{\mathrm{B}} F \;\leqslant\; \overline{\dim}_{\mathrm{B}} F \;\leqslant\; \dim_{\mathrm{A}} F,$$

see [F14, L67a, L67b]. The upper and lower box dimensions will play an important role in our analysis.

Unlike the Assouad and lower dimensions, we can give simple explicit formulae for the dimension spectra. Indeed, it follows immediately from the definitions that

$$\dim_{\mathrm{A}}^{\theta} F \;=\; \limsup_{R \to 0}\; \sup_{x \in F} \frac{\log N\big(B(x,R) \cap F, R^{1/\theta}\big)}{(1-1/\theta)\log R}$$

and

$$\dim_{\mathrm{L}}^{\theta} F \;=\; \liminf_{R \to 0}\; \inf_{x \in F} \frac{\log N\big(B(x,R) \cap F, R^{1/\theta}\big)}{(1-1/\theta)\log R}.$$

As with the definitions of Assouad and lower dimension, as well as the upper and lower box dimensions, the definition of $N(\cdot, r)$ may be replaced with a number of related concepts without altering any of the definitions. For example, if working in $\mathbb{R}^d$ we could use the number of r-cubes in an r-mesh which intersect the given set. Another possibility is to let $N(E, r)$ be the maximal cardinality of an r-packing of E, where an r-packing is a collection of closed pairwise disjoint balls of radius r with centres in E. Also, using the explicit formulae given above, we see that letting $R \to 0$ through an exponential sequence of scales, such as 2^{-k} ($k \in \mathbb{N}$), yields the same limits. We leave it to the reader to show that these variations lead to the same dimensions and spectra and refer to [F03, Chapter 3]) for more details. It is often useful to adopt these different definitions of $N(\cdot, r)$.

2.3 Miscellaneous notation

Here we summarise some notation which we will use throughout the thesis. For positive real-valued functions f, g, we write $f(x) \lesssim g(x)$ to mean that there exists a universal constant $M > 0$, independent of x, such that $f(x) \leqslant Mg(x)$. Some readers may be more familiar with the notation $f(x) = O(g(x))$, which is sometimes more convenient and means the same thing. Similarly, $f(x) \gtrsim g(x)$ means that $f(x) \geqslant Mg(x)$ with a universal constant $M > 0$, independent of x. If both $f(x) \lesssim g(x)$ and $f(x) \gtrsim g(x)$, then we write $f(x) \asymp g(x)$. Generally one should think of x as being the tuple consisting of all variables in the expression $f(x)$. Usually x will be a length scale but could sometimes also incorporate points in the metric space in question or other independent length scales.

For a real number a, we write a^+ to denote a real number that is strictly larger than a but can be chosen as close to a as we wish. Similarly, we write a^- to denote a real number that is strictly less than a but can be chosen as close to a as we wish.

For real numbers a, b, we write $a \wedge b$ for the minimum of the two numbers and $a \vee b$ for the maximum. Also, for a non-negative real number $x \geqslant 0$, we write $[x]$ for the integer part of x.

3 Analytic properties and general bounds

Our first proposition gives general (and sharp) bounds on the Assouad spectrum in terms of the Assouad and box dimensions.

Proposition 3.1. *Let F be a totally bounded set. Then for all $\theta \in (0,1)$ we have*

$$\overline{\dim}_{\mathrm{B}} F \;\leqslant\; \dim_{\mathrm{A}}^{\theta} F \;\leqslant\; \frac{\overline{\dim}_{\mathrm{B}} F}{1-\theta} \wedge \dim_{\mathrm{A}} F.$$

Proof. We will write B for the upper box dimension of F. First of all there is a clear upper bound holding for any $x \in F$ and small enough R:

$$N(B(x,R)\cap F, R^{1/\theta}) \leqslant N(F, R^{1/\theta}) \lesssim R^{-B^{+}/\theta}.$$

This implies that

$$\sup_{x\in F} N(B(x,R), R^{1/\theta}) \leqslant N(F, R^{1/\theta}) \lesssim R^{-B^{+}/\theta}. \tag{3.1}$$

Whenever we have a covering of F by R-balls, if we further cover each R-ball with $R^{1/\theta}$-balls then we get a cover of F by $R^{1/\theta}$-balls and an upper bound for $N(F, R^{1/\theta})$. We can cover F with $N(F,R)$ R-balls, and all those R-balls can be covered by at most $\sup_{x\in F} N(B(x,R)\cap F, R^{1/\theta})$ many $R^{\frac{1}{\theta}}$-balls, therefore

$$\sup_{x\in F} N(B(x,R)\cap F, R^{1/\theta}) N(F,R) \geqslant N(F, R^{1/\theta})$$

and so

$$\frac{N(F,R^{1/\theta})}{N(F,R)} \leqslant \sup_{x\in F} N(B(x,R)\cap F, R^{1/\theta}) \leqslant N(F, R^{1/\theta}).$$

Since $N(F,R) \lesssim R^{-B^+}$ for all small enough R and $N(F,R) \gtrsim R^{-B^-}$ for infinitely many $R \to 0$ we have that

$$\sup_{x \in F} N(B(x,R) \cap F, R^{1/\theta}) \gtrsim N(F, R^{1/\theta}) R^{B^+} \gtrsim R^{-B^-/\theta + B^+} \tag{3.2}$$

holds for a sequence of $R \to 0$. It now follows from (3.1) and (3.2) that

$$\frac{B^-/\theta - B^+}{1/\theta - 1} \leqslant \dim_{\mathrm{A}}^{\theta} F \leqslant \frac{B^+/\theta}{1/\theta - 1} = \frac{B^+}{1-\theta}.$$

Finally, since $\dim_{\mathrm{A}} F$ is a trivial upper bound for $\dim_{\mathrm{A}}^{\theta} F$ for all $\theta \in (0,1)$, the desired conclusion follows. □

The above estimates show that if the upper box and Assouad dimensions of a set coincide, then the Assouad spectrum is constantly equal to the common value for all $\theta \in (0,1)$. Such sets are highly homogeneous and therefore it is not surprising that the Assouad spectrum yields no new information. Fortunately, sets with distinct upper box and Assouad dimensions abound and we will focus on such examples. In this case, the above estimates show that, in some sense, the Assouad spectrum *must* yield finer information than the upper box and Assouad dimensions alone. Indeed, we will see by Corollary 3.2 that the only way the spectrum can be constant is if it is constantly equal to the upper box dimension, but such behaviour would be quite striking since the definition is more similar to the Assouad dimension than the upper box dimension. In such cases, the Assouad dimension is not 'seen' by any θ and this shows that to obtain the Assouad dimension, one must use a complicated collection of pairs (R,r) without any clear exponential relationship.

We also note that these general bounds are sharp. In particular, we show that the upper bound is always attained for a natural family of decreasing sequences, see Chapter 5. We also give examples where the *lower* bound is always attained (and the Assouad dimension is strictly larger than the upper box dimension), see Example 5.3. We also note that for many natural examples the spectrum lies strictly between these upper and lower bounds. For example, in Chapter 8 we show that the spectra necessarily lie strictly between the general upper and lower bounds for the self-affine carpets studied by Bedford and McMullen (provided the construction has non-uniform fibres).

Letting $\theta \to 0$ in the previous result we obtain the following corollary:

Corollary 3.2. *For any totally bounded set F, we have*

$$\dim_{\mathrm{A}}^{\theta} F \to \overline{\dim}_{\mathrm{B}} F$$

as $\theta \to 0$.

The limit of $\dim_{\mathrm{A}}^{\theta} F$ as $\theta \to 1$ was studied in [FHHTY18]. It is known that this limit coincides with the *quasi-Assouad dimension* of F. In this thesis, we do not need the notion of quasi-Assouad dimension. See [FHHTY18] for precise definitions.

Proposition 3.1 has the following immediate corollary, which at first sight looks surprising since the definition of the Assouad spectrum does not appear to depend so sensitively on the upper box dimension. This corollary was also obtained in [GH17, Proposition 15].

Corollary 3.3. *For any totally bounded set F with $\overline{\dim}_{\mathrm{B}} F = 0$, we have*

$$\dim_{\mathrm{A}}^{\theta} F = 0$$

for all $\theta \in (0,1)$.

We now move towards analytic properties of the spectra. Our first result is a technical regularity observation, which has some useful consequences.

Proposition 3.4. *For any set F and $0 < \theta_1 < \theta_2 < 1$ we have*

$$\dim_{\mathrm{A}}^{\theta_2} F \left(\frac{\frac{1}{\theta_2} - 1}{\frac{1}{\theta_1} - 1} \right) \leqslant \dim_{\mathrm{A}}^{\theta_1} F \leqslant \dim_{\mathrm{A}} F \left(\frac{\frac{1}{\theta_1} - \frac{1}{\theta_2}}{\frac{1}{\theta_1} - 1} \right) + \dim_{\mathrm{A}}^{\theta_2} F \left(\frac{\frac{1}{\theta_2} - 1}{\frac{1}{\theta_1} - 1} \right).$$

Proof. Following similar ideas as in the proofs above, for $0 < \theta_1 < \theta_2 < 1$, we have for any $R > 0$

$$\sup_{x \in F} N(B(x,R) \cap F, R^{1/\theta_1}) \geqslant \sup_{x \in F} N(B(x,R) \cap F, R^{1/\theta_2}).$$

Within F we can cover any $R^{\frac{1}{\theta_2}}$-ball with

$$\lesssim \left(\frac{R^{\frac{1}{\theta_2}}}{R^{\frac{1}{\theta_1}}} \right)^{(\dim_{\mathrm{A}} F)^+}$$

many $R^{\frac{1}{\theta_1}}$-balls. Then for small enough $R > 0$ we have

$$\left(\frac{R^{\frac{1}{\theta_2}}}{R^{\frac{1}{\theta_1}}} \right)^{(\dim_{\mathrm{A}} F)^+} \sup_{x \in F} N(B(x,R) \cap F, R^{1/\theta_2}) \geqslant \sup_{x \in F} N(B(x,R) \cap F, R^{1/\theta_1}).$$

Therefore,

$$\begin{aligned} R^{(\dim_{\mathrm{A}} F)^+ (\frac{1}{\theta_2} - \frac{1}{\theta_1})} \sup_{x \in F} N(B(x,R) \cap F, R^{1/\theta_2}) &\geqslant \sup_{x \in F} N(B(x,R) \cap F, R^{1/\theta_1}) \\ &\geqslant \sup_{x \in F} N(B(x,R) \cap F, R^{1/\theta_2}) \end{aligned}$$

Also notice that for a sequence of $R \to 0$ we have

$$\sup_{x \in F} N(B(x,R) \cap F, R^{1/\theta_2}) \gtrsim R^{(1-1/\theta_2)(\dim_{\mathrm{A}}^{\theta_2} F)^-}$$

as well as for any sufficiently small $R > 0$ we have

$$\sup_{x \in F} N(B(x,R) \cap F, R^{1/\theta_2}) \lesssim R^{(1-1/\theta_2)(\dim_{\mathrm{A}}^{\theta_2} F)^+}.$$

The desired bounds then follow immediately from the definitions. □

The bounds in Proposition 3.4 have some very useful consequences, such as continuity of the spectrum.

Corollary 3.5. *For any* $0 < \theta_1 \leqslant \theta_2 < 1$ *we have*

$$|\dim_{\mathrm{A}}^{\theta_1} F - \dim_{\mathrm{A}}^{\theta_2} F| \ \leqslant \ \frac{\dim_{\mathrm{A}} F}{\theta_2(1-\theta_1)}|\theta_1 - \theta_2|.$$

In particular, the function $\theta \mapsto \dim_{\mathrm{A}}^{\theta} F$ *is continuous in* $\theta \in (0,1)$ *and for any* $\varepsilon > 0$*, the function* $\theta \mapsto \dim_{\mathrm{A}}^{\theta} F$ *is Lipschitz on the interval* $[\varepsilon, 1-\varepsilon]$*.*

Proof. This follows immediately from the bounds presented in Proposition 3.4. □

Continuity of the dimension spectra is a useful property, especially when dealing with random fractals such as Mandelbrot percolation: a *continuous* function is determined by its values on a *countable* dense set. We point out that the spectra are not any more regular than continuous, as for most of our examples the spectra exhibit phase transitions where they fail to be differentiable.

Another useful consequence of Proposition 3.4 is that 'if the spectrum reaches the Assouad dimension of F, then it stays there'.

Corollary 3.6. *If for some* $\theta \in (0,1)$*, we have* $\dim_{\mathrm{A}}^{\theta} F = \dim_{\mathrm{A}} F$*, then*

$$\dim_{\mathrm{A}}^{\theta'} F = \dim_{\mathrm{A}} F$$

for all $\theta' \in [\theta, 1)$*.*

Proof. Starting with the right hand inequality from Proposition 3.4, assume that $\dim_{\mathrm{A}}^{\theta_1} F = \dim_{\mathrm{A}} F$. This immediately gives that $\dim_{\mathrm{A}}^{\theta_2} F \geqslant \dim_{\mathrm{A}} F$ which, together with Proposition 3.1, proves the result. □

Another natural question concerns monotonicity. Indeed, all of the 'natural' examples we consider have monotone spectra, i.e. the spectrum is non-decreasing in θ. Surprisingly, this is not always the case. We exhibit this by constructing an example in Chapter 7. The following result shows that one does have some sort of 'quasi-monotonicity', however.

Proposition 3.7. *For any* F *and* $0 < \theta_1 < \theta_2 < 1$ *we have*

$$\dim_{\mathrm{A}}^{\theta_1} F \ \leqslant \ \left(\frac{1-\theta_2}{1-\theta_1}\right) \dim_{\mathrm{A}}^{\theta_2} F \ + \ \left(\frac{\theta_2-\theta_1}{1-\theta_1}\right) \dim_{\mathrm{A}}^{\theta_1/\theta_2} F.$$

In particular, by setting $\theta_2 = \sqrt{\theta_1}$*, we have*

$$\dim_{\mathrm{A}}^{\theta_1} F \ \leqslant \ \dim_{\mathrm{A}}^{\sqrt{\theta_1}} F$$

for any $\theta_1 \in (0,1)$*. Furthermore, this implies that for any* $\theta \in (0,1)$*, we can find* θ' *arbitrarily close to* 1 *such that* $\dim_{\mathrm{A}}^{\theta'} F \geqslant \dim_{\mathrm{A}}^{\theta} F$*.*

Proof. Fix $0 < \theta_1 < \theta_2 < 1$ and notice that $0 < \frac{\theta_1}{\theta_2} < 1$. For sufficiently small $R > 0$, we have:

$$\sup_{x \in F} N(B(x,R) \cap F, R^{1/\theta_i}) \lesssim \left(R^{1-\frac{1}{\theta_i}}\right)^{(\dim_{\mathrm{A}}^{\theta_i} F)^+}$$

for $i = 1, 2$. We can also find infinitely many $R \to 0$ such that:

$$\sup_{x \in F} N(B(x,R) \cap F, R^{1/\theta_i}) \gtrsim \left(R^{1-\frac{1}{\theta_i}}\right)^{(\dim_{\mathrm{A}}^{\theta_i} F)^-} .$$

Let $R' = R^{\theta_2/\theta_1}$, and observe that $(R')^{1/\theta_2} = R^{1/\theta_1}$. We can cover any R ball with at most $\lesssim (R/R')^{(\dim_{\mathrm{A}}^{\theta_1/\theta_2} F)^+}$ balls with radius R' for small enough R.

Then we need no more than $\sup_{x \in F} N(B(x,R') \cap F, R^{1/\theta_1})$ balls with radius R^{1/θ_1} to cover any R'-ball.

Given an arbitrary R-ball, first cover it with R'-balls, and then cover those R'-balls by R^{1/θ_1}-balls using optimal covers as indicated above. This yields

$$\sup_{x \in F} N(B(x,R) \cap F, R^{1/\theta_1}) \ \leqslant \ \left(\frac{R}{R'}\right)^{(\dim_{\mathrm{A}}^{\theta_1/\theta_2} F)^+} \sup_{x \in F} N(B(x,R') \cap F, R^{1/\theta_1})$$

but since $(R')^{1/\theta_2} = R^{1/\theta_1}$ we have

$$\sup_{x \in F} N(B(x,R') \cap F, R^{1/\theta_1}) \lesssim \left((R')^{1-\frac{1}{\theta_2}}\right)^{(\dim_{\mathrm{A}}^{\theta_2} F)^+}$$

for all R' small enough and also

$$\sup_{x \in F} N(B(x,R) \cap F, R^{1/\theta_1}) \gtrsim \left(R^{1-\frac{1}{\theta_1}}\right)^{(\dim_{\mathrm{A}}^{\theta_1} F)^-}$$

for some arbitrarily small R.

Therefore for some arbitrarily small R we get:

$$\left(\frac{R'}{R}\right)^{(\dim_{\mathrm{A}}^{\theta_1/\theta_2} F)^+} \left(R^{1-\frac{1}{\theta_1}}\right)^{(\dim_{\mathrm{A}}^{\theta_1} F)^-} \ \lesssim \ \left((R')^{1-\frac{1}{\theta_2}}\right)^{(\dim_{\mathrm{A}}^{\theta_2} F)^+}$$

and so replacing R' by R^{θ_2/θ_1}, taking logs, and dividing through by $\log R$ yields

$$\left(\frac{\theta_2}{\theta_1} - 1\right) (\dim_{\mathrm{A}}^{\theta_1/\theta_2} F)^+ \ + \ \left(1 - \frac{1}{\theta_1}\right) (\dim_{\mathrm{A}}^{\theta_1} F)^- \ \geqslant \ \frac{\theta_2}{\theta_1} \left(1 - \frac{1}{\theta_2}\right) (\dim_{\mathrm{A}}^{\theta_2} F)^+.$$

This, in turn, yields

$$\dim_{\mathrm{A}}^{\theta_1} F \ \leqslant \ \left(\frac{1-\theta_2}{1-\theta_1}\right) \dim_{\mathrm{A}}^{\theta_2} F \ + \ \left(\frac{\theta_2 - \theta_1}{1-\theta_1}\right) \dim_{\mathrm{A}}^{\theta_1/\theta_2} F$$

as required. □

Remark 3.8. *The strategy of the above proof is to cover a large ball with middle sized balls and further cover the middle sized balls with smaller balls. This can be generalized to arbitrarily many levels of covering to obtain more general results.*

We can cover an R-ball with $R^{\frac{1}{\theta_n}}$-balls, then each of these balls with $R^{\frac{1}{\theta_{n-1}}}$-balls and so on and then applying the same proof strategy as above we end up with the following inequality: for

$$0 < \theta_1 < \theta_2 < \cdots < \theta_n < 1$$

we have

$$\dim_{\mathrm{A}}^{\theta_1} F \;\leqslant\; \left(\frac{1-\theta_n}{1-\theta_1}\right) \dim_{\mathrm{A}}^{\theta_n} F \;+\; \sum_{i=2}^{n} \left(\frac{\theta_i-\theta_{i-1}}{1-\theta_1}\right) \dim_{\mathrm{A}}^{\theta_{i-1}/\theta_i} F.$$

By setting $\theta_i = \theta_1^{\frac{n-i+1}{n}}$ we end up with:

$$\dim_{\mathrm{A}}^{\theta_1} F \;\leqslant\; \dim_{\mathrm{A}}^{\sqrt[n]{\theta_1}} F$$

for any $\theta_1 \in (0,1)$ and any natural number n, which is a slightly stronger result.

We will now discuss the analogous properties for the lower (and modified lower) spectrum.

Proposition 3.9. *Let F be a totally bounded set. Then for all $\theta \in (0,1)$ we have*

$$\dim_{\mathrm{L}} F \;\leqslant\; \dim_{\mathrm{L}}^{\theta} F \;\leqslant\; \underline{\dim}_{\mathrm{B}} F$$

and

$$\dim_{\mathrm{ML}} F \;\leqslant\; \dim_{\mathrm{ML}}^{\theta} F \;\leqslant\; \underline{\dim}_{\mathrm{B}} F.$$

Proof. First note that it follows immediately from the definitions that $\dim_{\mathrm{L}} F \leqslant \dim_{\mathrm{L}}^{\theta} F$ and therefore also $\dim_{\mathrm{ML}} F \leqslant \dim_{\mathrm{ML}}^{\theta} F$.

We will now prove the upper bounds and during the proof we will write b for the lower box dimension of F. Fix $\theta \in (0,1)$ and $R \in (0,1)$. Let $M(E,r)$ denote the largest possible cardinality of an r-packing of a set E by closed balls of radius r. That is, a maximal collection $\mathcal{M}$ of disjoint r-balls with centres in E. Here the maximality means that for each $x \in E$, the ball $B(x,R)$ must intersect with at least one ball in $\mathcal{M}$. Take an optimal $2R$-packing of F by closed balls and then inside each of these balls construct an optimal $R^{1/\theta}$-packing of the smaller ball centered at the same point but with radius R. The resulting $R^{1/\theta}$-balls are centered in F and are pairwise disjoint and, therefore, one obtains an $R^{1/\theta}$-packing of F by more than

$$M(F,2R) \inf_{x\in F} M\left(B(x,R),\, R^{1/\theta}\right)$$

balls. This yields

$$\inf_{x\in F} M\left(B(x,R),\, R^{1/\theta}\right) \;\leqslant\; \frac{M(F,R^{1/\theta})}{M(F,2R)} \;\lesssim\; \frac{R^{-b^+/\theta}}{R^{-b^-}} \;=\; R^{-(b^+/\theta-b^-)}$$

for arbitrarily small R. It then follows from the definitions

$$\dim_{\mathrm{L}}^{\theta} F \leqslant \frac{b^+/\theta - b^-}{1/\theta - 1}$$

from which the desired upper bound follows. This also passes to the modified lower spectrum, completing the proof. □

Since the lower dimension is bounded above by the modified lower dimension (and the Hausdorff dimension if the set is compact, see [L67a]) it is natural to ask if this is also (uniformly) true for the lower spectrum, i.e., if the upper bounds in Proposition 3.9 can be improved? Perhaps surprisingly, this is not the case. In particular, for self-affine carpets the lower spectrum approaches the box dimension as $\theta \to 0$, see Chapter 8.

Theorem 3.10. *The functions $\theta \mapsto \dim_{\mathrm{L}}^{\theta} F$ and $\theta \mapsto \dim_{\mathrm{ML}}^{\theta} F$ are continuous in $\theta \in (0,1)$. Moreover, they are Lipschitz on any closed subinterval of $(0,1)$. More precisely for $0 < \theta_1 \leqslant \theta_2 < 1$ we have*

$$|\dim_{\mathrm{L}}^{\theta_1} F - \dim_{\mathrm{L}}^{\theta_2} F| \leqslant \frac{\dim_{\mathrm{A}} F}{\theta_2(1-\theta_1)}|\theta_1 - \theta_2|$$

and

$$|\dim_{\mathrm{ML}}^{\theta_1} F - \dim_{\mathrm{ML}}^{\theta_2} F| \leqslant \frac{\dim_{\mathrm{A}} F}{\theta_2(1-\theta_1)}|\theta_1 - \theta_2|.$$

Proof. For any $0 < R < 1$ and $0 < \theta_1 < \theta_2 < 1$ we have $R^{1/\theta_1} < R^{1/\theta_2}$, therefore it is clear that for any $0 < R < 1$:

$$\inf_{x \in F} N(B(x,R) \cap F, R^{1/\theta_2}) \leqslant \inf_{x \in F} N(B(x,R) \cap F, R^{1/\theta_1}).$$

Now notice that,

$$\inf_{x \in F} N(B(x,R) \cap F, R^{1/\theta_1}) \lesssim \inf_{x \in F} N(B(x,R) \cap F, R^{1/\theta_2}) \left(\frac{R^{1/\theta_2}}{R^{1/\theta_1}}\right)^{(\dim_{\mathrm{A}} F)^+}.$$

This is because we may cover at least one R-ball with

$$\lesssim \inf_{x \in F} N(B(x,R) \cap F, R^{1/\theta_2}) \left(\frac{R^{1/\theta_2}}{R^{1/\theta_1}}\right)^{(\dim_{\mathrm{A}} F)^+}$$

balls of radius R^{1/θ_1}, and therefore this number is no smaller than $\inf_{x \in F} N(B(x,R) \cap F, R^{1/\theta_1})$. The above two inequalities imply that for infinitely many $R \to 0$ we have

$$R^{(1-1/\theta_2)(\dim_{\mathrm{L}}^{\theta_2} F)^-} \lesssim R^{(1-1/\theta_1)()\dim_{\mathrm{L}}^{\theta_1} F)^+}$$

and also for infinitely many $R \to 0$ we have

$$R^{(1-1/\theta_1)(\dim_{\mathrm{L}}^{\theta_1} F)^-} \lesssim R^{(1-1/\theta_2)(\dim_{\mathrm{L}}^{\theta_2} F)^+} R^{(\dim_{\mathrm{A}} F)^+(1/\theta_2 - 1/\theta_1)}.$$

It follows that

$$\begin{aligned}\left(1-\frac{1}{\theta_2}\right)\dim_{\mathrm{L}}^{\theta_2} F+\dim_{\mathrm{A}} F\left(\frac{1}{\theta_2}-\frac{1}{\theta_1}\right) &\leqslant \left(1-\frac{1}{\theta_1}\right)\dim_{\mathrm{L}}^{\theta_1} F\\ &\leqslant \left(1-\frac{1}{\theta_2}\right)\dim_{\mathrm{L}}^{\theta_2} F\end{aligned}$$

Dividing through by $(1-1/\theta_1)$ and then letting $\theta_1 \nearrow \theta_2$ establishes lower semicontinuity of $\theta \mapsto \dim_{\mathrm{L}} F$ at θ_2 and letting $\theta_2 \searrow \theta_1$ establishes upper semicontinuity of $\theta \mapsto \dim_{\mathrm{L}} F$ at θ_1. Since θ_1 and θ_2 are arbitrary the desired continuity follows.

The above discussion holds for any metric space F, and in particular for any subspace $E \subseteq F$ we have

$$\begin{aligned}\left(1-\frac{1}{\theta_2}\right)\dim_{\mathrm{L}}^{\theta_2} E+\dim_{\mathrm{A}} E\left(\frac{1}{\theta_2}-\frac{1}{\theta_1}\right) &\leqslant \left(1-\frac{1}{\theta_1}\right)\dim_{\mathrm{L}}^{\theta_1} E\\ &\leqslant \left(1-\frac{1}{\theta_2}\right)\dim_{\mathrm{L}}^{\theta_2} E\end{aligned}$$

Taking the supremum over all $E \subseteq F$ throughout, we get

$$\begin{aligned}\left(1-\frac{1}{\theta_2}\right)\dim_{\mathrm{ML}}^{\theta_2} F-\dim_{\mathrm{A}} F\left(\frac{1}{\theta_1}-\frac{1}{\theta_2}\right) &\leqslant \left(1-\frac{1}{\theta_1}\right)\dim_{\mathrm{ML}}^{\theta_1} F\\ &\leqslant \left(1-\frac{1}{\theta_2}\right)\dim_{\mathrm{ML}}^{\theta_2} F\end{aligned}$$

and therefore the modified lower spectrum is also continuous. Finally, the fact that the lower spectrum and modified lower spectrum are Lipschitz on any closed subinterval of $(0,1)$ also follows immediately by applying the above bounds. $\square$

4 Geometric properties

In this chapter we investigate how the various dimension spectra are affected by standard geometric operations such as products, unions, and images under Hölder continuous maps.

It is clear that the spectra satisfy the following properties and we leave the proofs to the reader.

Proposition 4.1 (Closure, monotonicity, and finite stability)**.**

1. *For any set F in a metric space and any $\theta \in (0,1)$, we have:*

$$\begin{aligned} \dim_{\mathrm{A}}^{\theta} F &= \dim_{\mathrm{A}}^{\theta} \overline{F} \\ \dim_{\mathrm{L}}^{\theta} F &= \dim_{\mathrm{L}}^{\theta} \overline{F}. \end{aligned}$$

2. *For any $F' \subseteq F$ and any $\theta \in (0,1)$, we have:*

$$\begin{aligned} \dim_{\mathrm{A}}^{\theta} F' &\leqslant \dim_{\mathrm{A}}^{\theta} F \\ \dim_{\mathrm{ML}}^{\theta} F' &\leqslant \dim_{\mathrm{ML}}^{\theta} F. \end{aligned}$$

3. *For any finite collection of sets $\{F_i\}_{i=1}^n$ we have, for all $\theta \in (0,1)$,*

$$\dim_{\mathrm{A}}^{\theta}\left(\bigcup_{i=1}^{n} F_i\right) = \max_{i=1,2,\ldots,n} \dim_{\mathrm{A}}^{\theta} F_i.$$

Interestingly, the modified lower dimension and modified lower spectrum are not stable under taking closure as the following example illustrates.

Example 4.2. *Let*

$$X \;=\; \left\{(p/q, 1/q) \;:\; p,q \in \mathbb{N}^+,\, p \leqslant q,\, \gcd(p,q) = 1\right\} \;\subseteq\; [0,1]^2$$

and observe that every point $x \in X$ is isolated and therefore any subset of X has an isolated point. This implies that for any $\theta \in (0,1)$

$$\dim_{\mathrm{ML}}^{\theta} X \;=\; \dim_{\mathrm{ML}} X \;=\; 0.$$

However, $[0,1] \times \{0\} \subseteq \overline{X}$ and so

$$\dim_{\mathrm{ML}}^{\theta} \overline{X} \;=\; \dim_{\mathrm{ML}} \overline{X} \;=\; 1.$$

Clearly the Assouad spectrum is not stable under countable unions. For example $\mathbb{Q} \cap [0,1]$ is a countable union of point sets, all of which have Assouad spectrum constantly equal to 0, but $\mathbb{Q} \cap [0,1]$ has Assouad spectrum constantly equal to 1 by the closure property. The lower spectrum is not even stable under finite unions: consider the union of $[0,1] \cup \{2\}$ and $\{0\} \cup [1,2]$. One can say more if the sets in the union are properly separated.

Proposition 4.3 (Unions of properly separated sets)**.** *Let E, F be 'properly separated' subsets of a metric space (X,d), i.e. sets such that*

$$\inf_{x \in E, y \in F} d(x,y) > 0.$$

Then,

$$\dim_{\mathrm{ML}} E \cup F \;=\; \dim_{\mathrm{ML}} E \vee \dim_{\mathrm{ML}} F$$

and, for all $\theta \in (0,1)$,

$$\dim_{\mathrm{ML}}^{\theta} E \cup F \;=\; \dim_{\mathrm{ML}}^{\theta} E \vee \dim_{\mathrm{ML}}^{\theta} F$$

and

$$\dim_{\mathrm{L}}^{\theta} E \cup F \;=\; \dim_{\mathrm{L}}^{\theta} E \wedge \dim_{\mathrm{L}}^{\theta} F.$$

Moreover, these results extend to arbitrary finite unions of pairwise 'properly separated' sets where the maximum/minimum is taken over all sets in the union.

Remark 4.4. *In [F14, Theorem 2.2], it was shown that under the same conditions we have*

$$\dim_{\mathrm{L}} E \cup F \;=\; \dim_{\mathrm{L}} E \wedge \dim_{\mathrm{L}} F.$$

Proof. The argument for the lower spectrum is similar to [F14, Theorem 2.2] and is omitted. For the modified lower dimension and modified lower spectrum, the proof is straightforward and we only briefly give the modified lower dimension argument. The lower bound ($\geqslant$) follows from monotonicity. For the upper bound, we have

$$\begin{aligned}
\dim_{\mathrm{ML}} E \cup F \;&=\; \sup_{\emptyset \neq Z \subseteq E \cup F} \dim_{\mathrm{L}} (Z \cap E) \cup (Z \cap F) \\
&=\; \sup_{\emptyset \neq Z \subseteq E \cup F} \Big(\dim_{\mathrm{L}} (Z \cap E) \wedge \dim_{\mathrm{L}} (Z \cap F) \Big)
\end{aligned}$$

$$\begin{aligned} &\leqslant \left(\sup_{\emptyset \neq Z \subseteq E} \dim_{\mathrm{L}} Z \right) \vee \left(\sup_{\emptyset \neq Z \subseteq F} \dim_{\mathrm{L}} Z \right) \\ &= \dim_{\mathrm{ML}} E \vee \dim_{\mathrm{ML}} F \end{aligned}$$

as required. Note that we used the fact that the lower dimension of the union of two properly separated sets is given by the minimum of the individual dimensions, which is provided in [F14, Theorem 2.2]. We also adopt the convention that $\dim_{\mathrm{L}} \emptyset = +\infty$. □

There are many results in dimension theory related to how the dimension of a product space depends on the dimensions of the marginals. A common phenomenon is that dimensions are best considered in pairs and the following standard formula has been verified for many 'dimension pairs' dim and Dim:

$$\begin{aligned} \dim X + \dim Y \leqslant \dim(X \times Y) \leqslant \dim X + \operatorname{Dim} Y &\leqslant \operatorname{Dim}(X \times Y) \\ &\leqslant \operatorname{Dim} X + \operatorname{Dim} Y. \end{aligned}$$

Such examples include Hausdorff and packing dimension, see Howroyd [H96]; lower and upper box dimension; and lower and Assouad dimension. For recent works on such product formulae see [ORS16, OR15, F14]. We show below that the Assouad and lower spectra give rise to a continuum of 'dimension pairs'.

There are many natural 'product metrics' to impose on the product $X \times Y$ of metric spaces (X, d_X) and (Y, d_Y), with a natural choice being the sup metric $d_{X\times Y}$ on $X \times Y$ defined by

$$d_{X\times Y}\big((x_1, y_1), (x_2, y_2)\big) = d_X(x_1, x_2) \vee d_Y(y_1, y_2).$$

In particular, this metric is compatible with the product topology and bi-Lipschitz equivalent with many other commonly used product metrics, such as those induced by p norms.

Proposition 4.5 (Products)**.** *Let E, F be metric spaces and equip the product $E \times F$ with any suitable product metric. For any $\theta \in (0, 1)$ we have*

$$\dim_{\mathrm{ML}}^{\theta} E + \dim_{\mathrm{A}}^{\theta} F \leqslant \dim_{\mathrm{A}}^{\theta}(E \times F) \leqslant \dim_{\mathrm{A}}^{\theta} E + \dim_{\mathrm{A}}^{\theta} F$$

$$\dim_{\mathrm{L}}^{\theta} E + \dim_{\mathrm{L}}^{\theta} F \leqslant \dim_{\mathrm{L}}^{\theta}(E \times F) \leqslant \dim_{\mathrm{L}}^{\theta} E + \dim_{\mathrm{A}}^{\theta} F$$

and

$$\dim_{\mathrm{ML}}^{\theta} E + \dim_{\mathrm{ML}}^{\theta} F \leqslant \dim_{\mathrm{ML}}^{\theta}(E \times F) \leqslant \dim_{\mathrm{ML}}^{\theta} E + \dim_{\mathrm{A}}^{\theta} F.$$

Proof. For the purposes of this proof we use the sup metric on the product space. In particular this means that the product of two covering sets of diameter r is a set of diameter r and so covers of parts of E and F can be easily combined to provide covers of the corresponding parts of $E \times F$. Let P_E and P_F denote the projection on to E and F respectively. Then clearly for any $R > 0$ and $x \in E \times F$ we have

$$\begin{aligned} N(B(x,R) \cap E \times F, R^{1/\theta}) &\leqslant \sup_{y \in E} N(B(y,R) \cap E, R^{1/\theta}) \sup_{z \in F} N(B(z,R) \cap F, R^{1/\theta}) \\ &\lesssim R^{(1-1/\theta)((\dim_{\mathrm{A}}^{\theta} E)^{+} + (\dim_{\mathrm{A}}^{\theta} F)^{+})} \end{aligned}$$

which proves that $\dim_{\mathrm{A}}^{\theta}(E \times F) \leqslant \dim_{\mathrm{A}}^{\theta} E + \dim_{\mathrm{A}}^{\theta} F$. On the other hand for any $E' \subset E$:

$$\begin{aligned} N(B(x,R) \cap E \times F, R^{1/\theta}) &\geqslant N(B(x,R) \cap E' \times F, R^{1/\theta}) \\ &\geqslant \inf_{y \in E'} N(B(y,R) \cap E', R^{1/\theta}) N(B(P_F x, R) \cap F, R^{1/\theta}). \end{aligned}$$

Since $x \in E' \times F$ can be chosen such that

$$N(B(P_F x, R) \cap F, R^{1/\theta}) \geqslant \sup_{z \in F} N(B(z,R) \cap F, R^{1/\theta})^-$$

we have

$$\begin{aligned} \sup_{x \in E \times F} N(B(x,R), R^{1/\theta}) &\geqslant \sup_{x \in E' \times F} N(B(x,R) \cap E' \times F, R^{1/\theta}) \\ &\geqslant \inf_{y \in E'} N(B(y,R) \cap E', R^{1/\theta}) \sup_{z \in F} N(B(z,R) \cap F, R^{1/\theta})^-. \end{aligned}$$

This implies that:

$$\sup_{x \in E \times F} N(B(x,R), R^{1/\theta}) \geqslant \inf_{y \in E'} N(B(y,R) \cap E', R^{1/\theta}) \sup_{z \in F} N(B(z,R) \cap F, R^{1/\theta})$$

which, similar to above, yields $\dim_{\mathrm{A}}^{\theta}(E \times F) \geqslant \dim_{\mathrm{ML}}^{\theta} E + \dim_{\mathrm{A}}^{\theta} F$ as required. The second chain of inequalities (which concern the lower spectrum) follow by a similar argument, which we omit. The third chain of inequalities (which concern the modified lower spectrum) follow easily from the second. In particular, for the lower bound choose nonempty subsets $E' \subset E$ and $F' \subset F$ such that $\dim_{\mathrm{L}}^{\theta} E' \geqslant (\dim_{\mathrm{ML}}^{\theta} E)^-$ and $\dim_{\mathrm{L}}^{\theta} F' \geqslant (\dim_{\mathrm{ML}}^{\theta} F)^-$ and then apply monotonicity and the result for the lower spectrum to obtain

$$\begin{aligned} \dim_{\mathrm{ML}}^{\theta}(E \times F) \geqslant \dim_{\mathrm{ML}}^{\theta}(E' \times F') \geqslant \dim_{\mathrm{L}}^{\theta}(E' \times F') &\geqslant \dim_{\mathrm{L}}^{\theta} E' + \dim_{\mathrm{L}}^{\theta} F' \\ &\geqslant (\dim_{\mathrm{ML}}^{\theta} E)^- + (\dim_{\mathrm{ML}}^{\theta} F)^- \end{aligned}$$

which proves the desired lower bound. For the upper bound, the upper bound concerning the lower spectrum implies that

$$\sup_{E' \subseteq E} \dim_{\mathrm{L}}^{\theta}(E' \times F) \leqslant \sup_{E' \subseteq E} \dim_{\mathrm{L}}^{\theta} E' + \dim_{\mathrm{A}}^{\theta} F = \dim_{\mathrm{ML}}^{\theta} E + \dim_{\mathrm{A}}^{\theta} F$$

which is almost what we want, apart from that it is not *a priori* obvious that the quantity on the left is equal to $\dim_{\mathrm{ML}}^{\theta}(E \times F)$. However, this follows since for any $K \subseteq E \times F$ we have for any $x \in K$ and $R > 0$ that

$$B(x,R) \cap K \subseteq B(x,R) \cap P_E K \times F$$

which yields that $\dim_{\mathrm{L}}^{\theta} K \leqslant \dim_{\mathrm{L}}^{\theta}(P_E K \times F)$ completing the proof. □

We also obtain a sharp result for ‘self-products’.

Proposition 4.6 (Self-products). *Let F be a metric space, $n \in \mathbb{N}$, and equip the n-fold product $F^n = F \times \cdots \times F$ with any suitable product metric. For any $\theta \in (0,1)$ we have*

$$\dim_{\mathrm{A}}^{\theta}(F^n) = n \dim_{\mathrm{A}}^{\theta} F,$$

$$\dim_{\mathrm{L}}^{\theta}(F^n) = n \dim_{\mathrm{L}}^{\theta} F$$

and

$$\dim_{\mathrm{ML}}^{\theta}(F^n) = n \dim_{\mathrm{ML}}^{\theta} F.$$

Proof. This proof is similar to the general case and we omit the details. The key point is that, for a self-product, one may choose $x \in F$ which witnesses the extremal behaviour at some scale and then consider the point $(x, \ldots, x) \in F^n$. The projection of this point onto every coordinate then witnesses extremal behaviour and this passes to F^n. □

We note that using a similar approach one may also obtain the following minor, but useful, improvement on [F14, Theorem 2.1]. Specifically, we upgrade lower dimension to modified lower dimension which is useful in situations where the lower dimension is small for reasons which do not affect other dimensions, for example when the set E contains an isolated point.

Proposition 4.7. *For metric spaces E, F we have*

$$\dim_{\mathrm{ML}} E + \dim_{\mathrm{A}} F \leqslant \dim_{\mathrm{A}}(E \times F) \leqslant \dim_{\mathrm{A}} E + \dim_{\mathrm{A}} F.$$

Another important aspect of a dimension is how it behaves under distortion by maps which are 'not too wild'. Indeed, all of the standard notions of dimension, such as the Hausdorff, box, packing, Assouad and lower dimension, are stable under bi-Lipschitz distortion, for example. Relaxing bi-Lipschitz to simply Lipschitz or even Hölder, there are elementary bounds which show that under distortion by an α-Hölder map the Hausdorff or box dimensions cannot increase by more than a factor of $1/\alpha$, see [F03, Chapter 2-3]. Assouad and lower dimension do not enjoy such stability and can wildly increase under distortion by even a Lipschitz map, see [F14]. The reason for this is that because one is trying to control two scales (one in each direction), one needs bounds on the distortion of the map in *both* directions. Here we conduct a detailed analysis of how the dimension spectra distorts under bi-Hölder maps, i.e., Hölder maps with Hölder inverses. It is noteworthy that one cannot relate the value of the Assouad spectrum of the set and its image at a particular value θ, but rather at two different values of θ which are related according to the Hölder parameters. This makes the theory of dimension distortion for our spectra rather more subtle than for a dimension which returns a single exponent. Recall that a doubling metric space is one for which there is a uniform constant C such that any ball may be covered by fewer than C balls of half the radius. This is easily seen to be equivalent to having finite Assouad dimension, see [R11, Lemma 9.4].

Proposition 4.8 (Hölder maps)**.** *Let $S : X \to Y$ be a map between doubling metric spaces (X, d_X) and (Y, d_Y) such that for all $x, y \in X$ with $d_X(x, y)$ sufficiently small,*

$$d_X(x,y)^\beta \;\lesssim\; d_Y\big(S(x), S(y)\big) \;\lesssim\; d_X(x,y)^\alpha$$

for some fixed constants $\beta \geqslant 1 \geqslant \alpha > 0$. Then, for any $F \subseteq X$ and $\theta \in (0,1)$, we have

$$\frac{1-\frac{\beta}{\alpha}\theta}{\beta(1-\theta)} \dim_{\mathrm{A}}^{\frac{\beta}{\alpha}\theta} F \;\leqslant\; \dim_{\mathrm{A}}^{\theta} S(F) \;\leqslant\; \frac{1-\frac{\alpha}{\beta}\theta}{\alpha(1-\theta)} \dim_{\mathrm{A}}^{\frac{\alpha}{\beta}\theta} F$$

$$\frac{1-\frac{\beta}{\alpha}\theta}{\beta(1-\theta)} \dim_{\mathrm{L}}^{\frac{\beta}{\alpha}\theta} F \;\leqslant\; \dim_{\mathrm{L}}^{\theta} S(F) \;\leqslant\; \frac{1-\frac{\alpha}{\beta}\theta}{\alpha(1-\theta)} \dim_{\mathrm{L}}^{\frac{\alpha}{\beta}\theta} F$$

and

$$\frac{1-\frac{\beta}{\alpha}\theta}{\beta(1-\theta)}\,\dim_{\mathrm{ML}}^{\frac{\beta}{\alpha}\theta} F \;\leqslant\; \dim_{\mathrm{ML}}^{\theta} S(F) \;\leqslant\; \frac{1-\frac{\alpha}{\beta}\theta}{\alpha(1-\theta)}\,\dim_{\mathrm{ML}}^{\frac{\alpha}{\beta}\theta} F.$$

Proof. First notice that S is invertible and for all $x, y \in S(X)$ with $d_Y(x,y)$ sufficiently small we have

$$d_Y(x,y)^{1/\alpha} \;\lesssim\; d_X\big(S^{-1}(x), S^{-1}(y)\big) \;\lesssim\; d_Y(x,y)^{1/\beta}.$$

By the assumptions on S there are uniform constants $C, c > 0$ such that for any sufficiently small $r > 0$ and $x \in X$ and $y \in S(X)$ we have

$$B(S(x), cr^{\beta}) \subseteq S(B(x,r)) \subseteq B(S(x), Cr^{\alpha})$$

and

$$B(S^{-1}(y), cr^{1/\alpha}) \subseteq S^{-1}(B(y,r)) \subseteq B(S^{-1}(y), Cr^{1/\beta}).$$

Therefore, for $0 < r < R$ with R small enough (recall that our metric space has the doubling property) we have

$$N(B(x, R^{1/\alpha}), r^{1/\beta}) \lesssim N(B(S(x),R), r) \lesssim N(B(x, R^{1/\beta}), r^{1/\alpha})$$

for any $x \in X$. The left inequality holds because any r-cover of $B(S(x), R)$ can be mapped under S^{-1} to yield an (up to multiplicative constants) $r^{1/\beta}$-cover of $B(x, R^{1/\alpha})$ by the same number of sets (up to another multiplicative constant depending on the doubling property of the space). Similarly, the right inequality holds because any $r^{1/\alpha}$-cover of $B(x, R^{1/\beta})$ can be mapped under S to yield an (up to multiplicative constants) r-cover of $B(S(x), R)$ by the same number of sets (up to another multiplicative constant depending on the doubling property of the space).

Setting $r = R^{1/\theta}$ from here we notice that for any sufficiently small $R > 0$ we have by definition

$$N(B(x, R^{1/\beta}), r^{1/\alpha}) \;\lesssim\; \left(\frac{R^{1/\beta}}{(R^{1/\beta})^{\frac{\beta}{\alpha\theta}}}\right)^{(\dim_{\mathrm{A}}^{\frac{\alpha\theta}{\beta}} F)^+} \;=\; (R^{1-1/\theta})^{\frac{1-\frac{\alpha}{\beta}\theta}{\alpha(1-\theta)}(\dim_{\mathrm{A}}^{\frac{\alpha\theta}{\beta}} F)^+}$$

and, similarly, for infinitely many $R \to 0$ we have

$$N(B(x, R^{1/\alpha}), r^{1/\beta}) \;\gtrsim\; \left(\frac{R^{1/\alpha}}{(R^{1/\alpha})^{\frac{\alpha}{\beta\theta}}}\right)^{(\dim_{\mathrm{A}}^{\frac{\beta\theta}{\alpha}} F)^-} \;=\; (R^{1-1/\theta})^{\frac{1-\frac{\beta}{\alpha}\theta}{\beta(1-\theta)}(\dim_{\mathrm{A}}^{\frac{\beta\theta}{\alpha}} F)^-}.$$

Recall that if $\beta\theta/\alpha \geqslant 1$, then $\dim_{\mathrm{A}}^{\frac{\beta\theta}{\alpha}} F = 0$. Also by definition, for any sufficiently small $R > 0$ we have

$$N(B(S(x),R), r) \;\lesssim\; (R^{1-1/\theta})^{(\dim_{\mathrm{A}}^{\theta} S(F))^+}$$

and for infinitely many $R \to 0$ we have

$$N(B(S(x),R), r) \;\gtrsim\; (R^{1-1/\theta})^{(\dim_{\mathrm{A}}^{\theta} S(F))^-}.$$

Together these estimates yield that for infinitely many $R \to 0$ we have

$$(R^{1-1/\theta})^{\dim_{\mathrm{A}}^{\theta} S(F)^-} \lesssim (R^{1-1/\theta})^{\frac{1-\frac{\alpha}{\beta}\theta}{\alpha(1-\theta)}(\dim_{\mathrm{A}}^{\frac{\alpha\theta}{\beta}} F)^+}$$

and

$$(R^{1-1/\theta})^{\frac{1-\frac{\beta}{\alpha}\theta}{\beta(1-\theta)} \dim_{\mathrm{A}}^{\frac{\beta\theta}{\alpha}} F^-} \lesssim (R^{1-1/\theta})^{(\dim_{\mathrm{A}}^{\theta} S(F))^+}$$

which gives

$$\frac{1-\frac{\beta}{\alpha}\theta}{\beta(1-\theta)} \dim_{\mathrm{A}}^{\frac{\beta\theta}{\alpha}} F \leqslant \dim_{\mathrm{A}}^{\theta} S(F) \leqslant \frac{1-\frac{\alpha}{\beta}\theta}{\alpha(1-\theta)} \dim_{\mathrm{A}}^{\frac{\alpha\theta}{\beta}} F$$

as required. The argument for the lower spectrum is similar and omitted. Since subsets of F are in one to one correspondence with subsets of $S(F)$ through the map S, we may take supremum over nonempty subsets of F throughout, which yields the analogous estimates for the modified lower spectrum, completing the proof. □

The lower bounds for the spectra of $S(F)$ all become equal to 0 (and thus trivial) when $\theta \geqslant \alpha/\beta$ and the upper bounds for the spectra of $S(F)$ blow up as $\theta \to 1$. These are unfortunate properties but are indicative of the complex relations between the spectra at different values of θ once the set has been distorted by S. One can rectify this situation somewhat by combining our estimates with Lemmas 3.1 and 3.9 and the classical results that upper and lower box dimension cannot increase by more than a factor of $1/\alpha$ under distortion by an α-Hölder map.

Corollary 4.9. *Let $S : X \to Y$ be as in Proposition 4.8. Then for any $F \subseteq X$ and $\theta \in (0,1)$*

$$\frac{1-\frac{\beta}{\alpha}\theta}{\beta(1-\theta)} \dim_{\mathrm{A}}^{\frac{\beta\theta}{\alpha}} F \vee \frac{\overline{\dim}_{\mathrm{B}} F}{\beta} \leqslant \dim_{\mathrm{A}}^{\theta} S(F) \leqslant \frac{1-\frac{\alpha}{\beta}\theta}{\alpha(1-\theta)} \dim_{\mathrm{A}}^{\frac{\alpha}{\beta}\theta} F$$

$$\frac{1-\frac{\beta}{\alpha}\theta}{\beta(1-\theta)} \dim_{\mathrm{L}}^{\frac{\beta\theta}{\alpha}} F \leqslant \dim_{\mathrm{L}}^{\theta} S(F) \leqslant \frac{1-\frac{\alpha}{\beta}\theta}{\alpha(1-\theta)} \dim_{\mathrm{L}}^{\frac{\alpha}{\beta}\theta} F \wedge \frac{\underline{\dim}_{\mathrm{B}} F}{\alpha}$$

and

$$\frac{1-\frac{\beta}{\alpha}\theta}{\beta(1-\theta)} \dim_{\mathrm{ML}}^{\frac{\beta\theta}{\alpha}} F \leqslant \dim_{\mathrm{ML}}^{\theta} S(F) \leqslant \frac{1-\frac{\alpha}{\beta}\theta}{\alpha(1-\theta)} \dim_{\mathrm{ML}}^{\frac{\alpha}{\beta}\theta} F \wedge \frac{\underline{\dim}_{\mathrm{B}} F}{\alpha}.$$

Note that in the above we could also bound $\dim_{\mathrm{A}}^{\theta} S(F)$ from above by $\dim_{\mathrm{A}} S(F)$, but the point is to bound dimensions of $S(F)$ by expressions involving only dimensions of F and the bi-Hölder restrictions on S are not enough to yield bounds for the *Assouad dimension* of $S(F)$ in terms of F.

Also, we could have used Lemma 3.1 to apparently improve the upper bound for the Assouad spectrum to include the bound

$$\dim_{\mathrm{A}}^{\theta} S(F) \leqslant \frac{\overline{\dim}_{\mathrm{B}} S(F)}{(1-\theta)} \leqslant \frac{\overline{\dim}_{\mathrm{B}} F}{\alpha(1-\theta)}$$

but by virtue of Lemma 3.1 it follows that

$$\overline{\dim}_{\mathrm{B}} F \geqslant \left(1 - \frac{\alpha}{\beta}\theta\right) \dim_{\mathrm{A}}^{\frac{\alpha}{\beta}\theta} F$$

for all $\theta \in (0, 1)$ and so this estimate cannot improve the one we already have.

It is important to comment on the sharpness of the estimates from Corollary 4.9. A first observation is that such estimates cannot possibly be sharp in any precise sense, although letting $\alpha, \beta \to 1$ shows that they are at least asymptotically sharp. The reason for this is that they are based on knowledge of the extremal distortion of F over the whole space and the spectra are only sensitive to the extremal properties of the set in question. Indeed, the thickest part of the set F (and $S(F)$), which determines the spectra, may occur at a location in the domain of S where the distortion is less than the global extreme. We will consider our estimates in detail for a natural family of sets and bi-Hölder maps in Section 5.1.

Setting $\alpha = \beta = 1$ in Proposition 4.8, we obtain bi-Lipschitz stability as another immediate corollary.

Corollary 4.10. *The Assouad, lower, and modified lower, spectra are bi-Lipschitz invariant.*

Being bi-Lipschitz invariant is a useful property and one possible application is in classifying metric spaces up to bi-Lipschitz equivalence. There has been considerable interest in this problem since the seminal paper of Falconer and Marsh [FM92] which sought to determine for which pairs of self-similar subsets of the line one can find a bi-Lipschitz function taking one to the other. Having the same Hausdorff dimension is a necessary condition, since Hausdorff dimension is a bi-Lipschitz invariant, but it is not sufficient: there are self-similar subsets of the line which have the same Hausdorff dimension but which are not bi-Lipschitz equivalent. As such, it is useful to find other bi-Lipschitz invariants, such as the other notions of dimension mentioned above. Corollary 4.10 provides a new continuum of bi-Lipschitz invariants and so has potential applications in proving that certain metric spaces are not bi-Lipschitz equivalent, even if their Hausdorff, box, packing, Assouad and lower dimensions are equal.

Suppose S is a map on F such that

$$\frac{\log |x - y|}{\log |S(x) - S(y)|} \to 1$$

uniformly as $|x - y| \to 0$. Such maps are sometimes called *quasi-Lipschitz*, see [LX14, LX16]. Rather than setting $\alpha = \beta = 1$ in Proposition 4.8, if we just let $\alpha, \beta \to 1$ we see that the spectra are also all invariant under *quasi-Lipschitz* maps.

Corollary 4.11. *The Assouad, lower, and modified lower, spectra are quasi-Lipschitz invariant.*

4.1 Bi-Hölder distortion for Assouad dimension

In Proposition 4.8 we gave some estimates for the Assouad spectrum of a set after distortion by a bi-Hölder map. Similar, but simpler, estimates hold for the other standard notions of dimension, such as the Hausdorff dimension and upper and lower box dimension. In light of the other known results, one might expect that:

$$\frac{1}{\beta}\dim_{\mathrm{A}} F \quad \leqslant \quad \dim_{\mathrm{A}} S(F) \quad \leqslant \quad \frac{1}{\alpha}\dim_{\mathrm{A}} F$$

for any bi-Hölder map with parameters $0 < \alpha \leqslant 1 \leqslant \beta < \infty$. In particular, these bounds hold if Assouad dimension is replaced by Hausdorff, packing, upper box or lower box dimension. The situation turns out to be more subtle for Assouad dimension. In particular, the Assouad dimension may be distorted significantly for bi-Hölder maps with parameters arbitrarily close to 1, i.e., maps which are arbitrarily close to being bi-Lipschitz. More precisely, Lü and Xi [LX16, Proposition 1.2] proved that for any $s, t \in (0,1]$ one may find subsets of $[0,1]$ with Assouad dimension s and t respectively, such that one is a quasi-Lipschitz image of the other and vice versa. This shows that there do not exist general bounds on the Assouad dimension of $S(F)$ in terms of the Assouad dimension of F and the Hölder parameters α and β. However, we prove that if one assumes that the Assouad spectrum reaches the Assouad dimension, then one *can* give non-trivial dimension bounds for distortion under bi-Hölder maps. In particular, to obtain some bounds one needs additional assumptions about the set F.

Theorem 4.12. *Let $S : X \to Y$ be as in Proposition 4.8 and let*

$$\theta_0 = \inf\left\{\theta \in [0,1] : \dim_{\mathrm{A}}^{\theta} F = \dim_{\mathrm{A}} F\right\}$$

assuming the set of suitable θs is non-empty. Then for any $F \subseteq X$ we have

$$\dim_{\mathrm{A}} S(F) \geqslant \frac{\dim_{\mathrm{A}} F}{\beta - \theta_0\alpha}(1-\theta_0).$$

In particular, if S is quasi-Lipschitz and $\dim_{\mathrm{A}}^{\theta} F = \dim_{\mathrm{A}} F$ for some θ, then $\dim_{\mathrm{A}} S(F) \geqslant \dim_{\mathrm{A}} F$.

Before we prove the result, note that if θ_0 exists and $\dim_{\mathrm{A}} F > \overline{\dim}_{\mathrm{B}} F$, then Proposition 3.1 implies that

$$\theta_0 \geqslant 1 - \frac{\overline{\dim}_{\mathrm{B}} F}{\dim_{\mathrm{A}} F} > 0.$$

Proof. We have for any $\theta \in [0,1]$ that

$$\dim_{\mathrm{A}} S(F) \geqslant \dim_{\mathrm{A}}^{\theta} S(F)$$

and Proposition 4.8 further implies that

$$\dim_{\mathrm{A}}^{\theta} S(F) \geqslant \frac{1-\frac{\beta}{\alpha}\theta}{\beta(1-\theta)}\dim_{\mathrm{A}}^{\frac{\beta}{\alpha}\theta} F.$$

Therefore, applying these inequalities with a sequence $\theta_i \to \frac{\alpha}{\beta}\theta_0 \in (0,1)$ from above proves the desired lower bound. $\square$

Observe that the smaller θ_0 is, the better our lower bound for the Assouad dimension of $S(F)$. It is natural to consider an analogous upper bound, but this would require *a priori* knowledge of the set $S(F)$, i.e., we have to know $\theta_0' = \inf\{\theta \in [0,1] : \dim_{\mathrm{A}}^{\theta} S(F) = \dim_{\mathrm{A}} S(F)\}$. One may obtain a precise analogue by applying the above theorem with S replaced by S^{-1}, but we do not pursue the details here.

We can also derive similar results for the lower and modified lower dimension using Proposition 4.8, but we leave the precise formulations to the reader.

5 Decreasing sequences with decreasing gaps

In this chapter we will consider a simple family of countable compact fractal subsets of the line which allow explicit calculation of the Assouad spectra. Observe that any countable compact set has lower spectra and modified lower spectra equal to zero and so we omit discussion of these spectra for the duration of this chapter. Despite being relatively simple, these fractal sequences have several useful properties. First they provide us with a continuum of examples where the upper bound from Proposition 3.1 is attained. Secondly, they provide examples where the *lower* bound from Proposition 3.1 is attained, see Example 5.3. Thus we demonstrate the sharpness of Proposition 3.1. They also allow us to analyse the bounds for dimension distortion under bi-Hölder maps given in Corollary 4.9 in an explicit and representative way, see Section 5.1.

More precisely, we study *decreasing sequences with decreasing gaps*, which we formulate as follows. Let f be a function from $\mathbb{R}^+$ to $[0,1]$ such that $f(x)$ and $g(x) := f(x) - f(x+1)$ are both strictly decreasing functions and they converge to 0 as $x \to \infty$. We also assume for convenience that both f and g are smooth. Our set of interest is then

$$F \;=\; \{f(n)\}_{n\geqslant 1} \;\cup\; \{0\}.$$

The following result (stated using our notation) was proved by García, Hare and Mendivil [GHM15, Proposition 4]. We also obtained this result, using a different proof, in answer to a question posed to us by Chris Miller (Ohio State University), but we omit our argument and refer the reader to [GHM15]. We also refer the reader to our proof of Theorem 6.1, which proves a similar dichotomy in a different setting.

Theorem 5.1 (García, Hare and Mendivil)**.** *Let $F = \{0\} \cup \{f(n)\}_{n\in\mathbb{N}}$ be a decreasing sequence with decreasing gaps as described above. Then the Assouad dimension of F is either 0 or 1. Moreover, the Assouad dimension is 0 if and only if $f(n)$ decays to 0 at least exponentially fast, i.e. $\liminf_{n\to\infty} \log((f(n))^{-1})/n > 0$.*

Our main result on dimensions of decreasing sequences with decreasing gaps is as follows and shows that the Assouad spectrum only depends on the upper box dimension of the set. A pleasant ancillary benefit of the explicit formula we obtain is that for all such sets the upper bound in Proposition 3.1 is sharp. Also, we note that the upper box dimensions of such sets are well studied and can be computed effectively. For example, the upper (and lower) box dimensions can be estimated in terms of the exponential decay rate of the gap lengths $g(n)$. See the 'cut-out sets' discussed in [F97, Chapter 3] for more information.

Theorem 5.2. *Let $F = \{0\} \cup \{f(n)\}_{n\in\mathbb{N}}$ be a decreasing sequence with decreasing gaps as described above. Then for all $\theta \in (0,1)$ we have*

$$\dim_{\mathrm{A}}^{\theta} F \;=\; \frac{\overline{\dim}_{\mathrm{B}} F}{1-\theta} \wedge 1.$$

Proof. If either $\overline{\dim}_{\mathrm{B}} F = 0$ or $\overline{\dim}_{\mathrm{B}} F = 1$ then the result follows immediately from Proposition 3.1 and therefore we may assume from now on that

$$\overline{\dim}_{\mathrm{B}} F = B \in (0,1).$$

We start by giving some general bounds. Let $0 < r < R < 1$ and consider the number

$$\sup_{x\in F} N(B(x,R)\cap F, r).$$

For any $r > 0$, there is a smallest number n_r such that $g(n_r) < r$. Notice that by definition $n_r = [g^{-1}(r)] = g^{-1}(r) + O(1)$. If $R \leqslant f(n_r)$ then we will need approximately (R/r) many r-balls to cover $[0,R)\cap F$, and this is already of the largest order possible. Therefore we will focus on the case where $R > f(n_r)$. The following bound follows from the fact that the sequence has decreasing gaps:

$$\begin{aligned} N(B(0,R/2)\cap F, r) \;&\leqslant\; \sup_{x\in F} N(B(x,R/2)\cap F, r) \;\leqslant\; N(B(0,R)\cap F, r) \\ &\leqslant\; 2N(B(0,R/2)\cap F, r). \end{aligned}$$

If $R \geqslant f(n_r)$, then we have the key formula:

$$N(B(0,R)\cap F, r) \;\asymp\; \frac{f\circ g^{-1}(r)}{r} + g^{-1}(r) - f^{-1}(R). \tag{5.1}$$

This is because for points smaller than $f\circ g^{-1}(r)$ the gaps are smaller than r, and therefore we need approximately $\frac{f\circ g^{-1}(r)}{r}$ many r-balls to cover them. Moreover, for points between $f\circ g^{-1}(r)$ and R, of which there are approximately $g^{-1}(r) - f^{-1}(R)$ many, we need one r-ball for each of them.

Let $\theta \in (0, 1-B)$ and observe that if $R > 0$ is sufficiently small, then $g^{-1}(R^{1/\theta}) - f^{-1}(R) \geqslant 0$. It therefore follows from the key formula (5.1) that

$$\sup_{x\in F} N(B(x,R)\cap F, R^{1/\theta}) \;\asymp\; \frac{f\circ g^{-1}(R^{1/\theta})}{R^{1/\theta}} + g^{-1}(R^{1/\theta}) - f^{-1}(R).$$

If we can find infinitely many $R \to 0$ such that

$$g^{-1}(R^{1/\theta}) \leqslant f^{-1}\left(R^{(1-B^-)/\theta}\right)$$

then

$$\frac{f \circ g^{-1}(R^{1/\theta})}{R^{1/\theta}} \geqslant R^{-B^-/\theta} \tag{5.2}$$

and we get $\dim_{\mathrm{A}}^{\theta} F \geqslant \frac{B^-}{1-\theta}$ as required. Therefore assume that for all sufficiently small $R > 0$ we have

$$g^{-1}(R^{1/\theta}) > f^{-1}\left(R^{(1-B^-)/\theta}\right).$$

This implies that for R small enough we have

$$f \circ g^{-1}(R) < R^{1-B^-}$$

which in turn implies that for n large enough we have

$$f(n) < g(n)^{1-B^-} = (f(n) - f(n+1))^{1-B^-}$$

and

$$g(n) = f(n) - f(n+1) > f(n)^{1/(1-B^-)}.$$

This holds for all large enough n and therefore we can assume it holds for all n without loss of generality. For simplicity, we write $\alpha = \frac{1}{1-B^-} > 1$. We have

$$\begin{aligned}
f(n+1)^{1-\alpha} - f(n)^{1-\alpha} &= (f(n) + f(n+1) - f(n))^{1-\alpha} - f(n)^{1-\alpha} \\
&= f(n)^{1-\alpha}\left(\left(1 + \frac{f(n+1) - f(n)}{f(n)}\right)^{1-\alpha} - 1\right) \\
&\geqslant f(n)^{1-\alpha}\left((1-\alpha)\frac{f(n+1) - f(n)}{f(n)}\right) \\
&= (\alpha - 1)\frac{g(n)}{f(n)^{\alpha}} \\
&> (\alpha - 1).
\end{aligned}$$

Iterating the above inequality yields

$$f(n)^{1-\alpha} - f(1)^{1-\alpha} > (\alpha - 1)(n-1)$$

and therefore

$$f(n) < \left(\frac{1}{(\alpha-1)(n-1) + f(1)^{1-\alpha}}\right)^{\frac{1}{\alpha-1}}.$$

This implies that for all large enough n we have

$$f(n) \lesssim \left(\frac{1}{n}\right)^{\frac{1}{\alpha-1}} = \left(\frac{1}{n}\right)^{\frac{1-B^-}{B^-}}$$

and for small enough x we have

$$f^{-1}(x) \lesssim \left(\frac{1}{x}\right)^{\frac{B^-}{1-B^-}}. \tag{5.3}$$

Recalling that $B \in (0,1)$ is the upper box dimension, we can find a sequence $r_i \to 0$ such that:

$$\frac{f(n_{r_i})}{r_i} + n_{r_i} \gtrsim N(F, r_i) \gtrsim {r_i}^{-B^-}.$$

If for infinitely many i we have

$$\frac{f(n_{r_i})}{r_i} \gtrsim r_i^{-B^-}$$

then the situation is the same as in the beginning of the proof (see (5.2)) and we get our conclusion. Otherwise we have infinitely many i such that

$$n_{r_i} \gtrsim {r_i}^{-B^-}.$$

It follows from the key formula (5.1), that for infinitely many i we have

$$\begin{aligned}
\sup_{x\in F} N(B(x, R_i) \cap F, r_i) &\gtrsim g^{-1}(r_i) - f^{-1}(R_i) \\
&= g^{-1}(r_i) - f^{-1}({r_i}^{\theta}) \\
&\gtrsim {r_i}^{-B^-} - \left(\frac{1}{r_i^{\theta}}\right)^{B^-/(1-B^-)} \qquad \text{by (5.3)} \\
&= r_i^{-B^-} - r_i^{-B^-\theta/(1-B^-)}.
\end{aligned}$$

Since $\theta < 1 - B^-$ we have $B^- > \frac{B^-\theta}{1-B^-}$ and so

$$\sup_{x\in F} N(B(x, R_i), r_i) \gtrsim r_i^{-B^-}$$

for infinitely many i. It follows that for $\theta \in (0, 1-B)$ we have $\dim_{\mathrm{A}}^{\theta} F \geqslant B^-/(1-\theta)$ which, combined with Proposition 3.1, yields the desired result for this range of θ. Finally, continuity of the spectrum (Corollary 3.5) gives that for $\theta = 1 - B$ we have $\dim_{\mathrm{A}}^{\theta} F = 1 = \dim_{\mathrm{A}} F$ and Corollary 3.6 yields that this also holds for all $\theta \in [1-B, 1)$ completing the proof. □

Example 5.3. *The set* $E = \{e^{-\sqrt{n}} : n \in \mathbb{N}\} \cup \{0\}$ *is a simple example where the spectrum does not peak at the Assouad dimension. Straightforward computations, which we omit, yield that* $\dim_{\mathrm{B}} E = \dim_{\mathrm{A}}^{\theta} E = 0 < \dim_{\mathrm{A}} E = 1$*. Moreover, this example can be modified to provide constructions demonstrating the sharpness of the lower bound from Proposition 3.1, even if the box dimension is positive. For example, consider* $F := [0,1] \times E$*. It follows from the discussion here and Proposition 4.5 that*

$$\dim_{\mathrm{B}} F = \dim_{\mathrm{A}}^{\theta} F = 1 < \dim_{\mathrm{A}} F = 2.$$

5.1 Sequences with polynomial decay

In this section we provide our first concrete example where we can compute the Assouad spectrum explicitly. We specialise to a particular continuously parameterised family of decreasing sequences with prescribed polynomial decay. In particular, for a fixed $\lambda > 0$ we study the set

$$F_\lambda \;=\; \{0\} \cup \left\{\frac{1}{n^\lambda}\right\}_{n\in\mathbb{N}}$$

and give an explicit formula for $\dim_{\mathrm{A}}^\theta F_\lambda$. These sets F_λ are some of the first examples one considers when studying the box and Assouad dimensions (in particular F_1) and elementary calculations reveal that for any $\lambda > 0$

$$\dim_{\mathrm{B}} F_\lambda \;=\; \frac{1}{\lambda+1} \;<\; 1 = \dim_{\mathrm{A}} F_\lambda.$$

We therefore have the following immediate Corollary of Theorem 5.2.

Corollary 5.4. *For all $\lambda > 0$ and $\theta \in (0,1)$ we have*

$$\dim_{\mathrm{A}}^\theta F_\lambda \;=\; \frac{1}{(\lambda+1)(1-\theta)} \wedge 1.$$

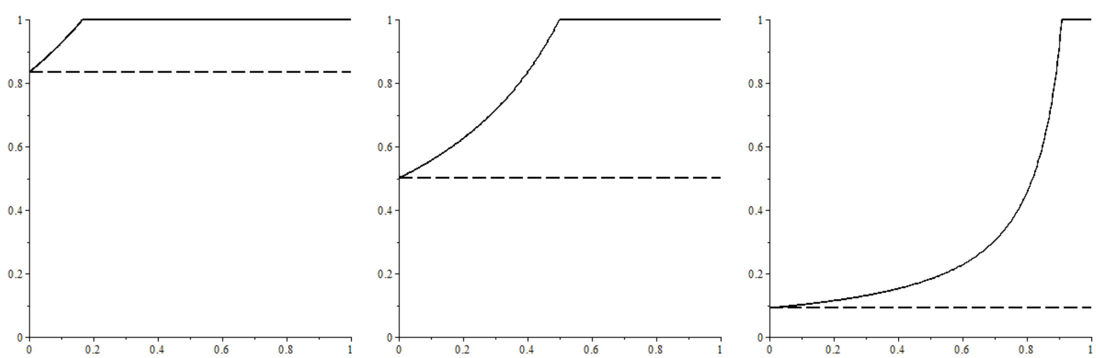

Figure 1: Three plots of the Assouad spectrum of F_λ for $\lambda = 1/5, 1, 10$ going from left to right. The plotted curves are $\dim_{\mathrm{A}}^\theta F$ as a functions of θ. The bounds from Proposition 3.1 are shown as dashed lines, although the upper bound is obtained in each case.

The family of sets $\{F_\lambda\}_{\lambda>0}$ studied in this section provide us with a simple continuum of sets with the property that any one can be mapped onto any other by a bi-Hölder map. Since we have a very simple explicit formula for the Assouad spectrum of each F_λ, this provides an excellent opportunity to test the bounds obtained in Corollary 4.9. For $\alpha > 0$, let $S_\alpha : [0,1] \to [0,1]$ be defined by $S_\alpha(x) = x^\alpha$ and observe that, for any $\lambda > 0$, we have $S_\alpha(F_\lambda) \;=\; F_{\alpha\lambda}$. Also note that if $\alpha \in (0,1)$ then S is α-Hölder with a Lipschitz inverse, and if $\alpha > 1$ then S is Lipschitz with a $1/\alpha$-Hölder inverse. In the

$\alpha \in (0,1)$ region, the bounds on the Assouad spectrum from Corollary 4.9 yield that for $\theta \in (0,1)$

$$\frac{1-\theta/\alpha}{(1-\theta)} \dim_{\mathrm{A}}^{\theta/\alpha} F_\lambda \vee \overline{\dim}_{\mathrm{B}} F_\lambda \leqslant \dim_{\mathrm{A}}^{\theta} S_\alpha(F_\lambda) \leqslant \frac{1-\alpha\theta}{\alpha(1-\theta)} \dim_{\mathrm{A}}^{\alpha\theta} F_\lambda \wedge 1$$

and applying the explicit formulae for the Assouad spectra derived above gives:

$$\begin{aligned}\frac{1-\theta/\alpha}{(1-\theta)}\left(\frac{1}{(\lambda+1)(1-\theta/\alpha)} \wedge 1\right) \vee \frac{1}{(\lambda+1)} &\leqslant \frac{1}{(\alpha\lambda+1)(1-\theta)} \wedge 1 \\ &\leqslant \frac{1-\alpha\theta}{\alpha(1-\theta)} \wedge \frac{1}{\alpha(\lambda+1)(1-\theta)} \wedge 1\end{aligned}$$

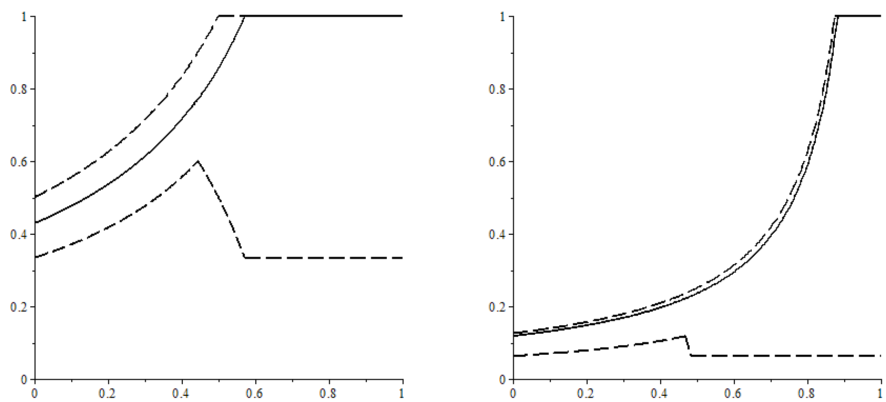

Figure 2: Two plots of the bounds on the Assouad spectrum of the polynomial sequence F_λ under bi-Hölder distortion by S_α. On the left $\lambda = 2$ and $\alpha = 2/3$ and on the right $\lambda = 15$ and $\alpha = 1/2$. The actual spectrum of $S_\alpha(F_\lambda)$ is shown as a solid line and the bounds are dashed.

In the $\alpha > 1$ region, the bounds on the Assouad spectrum from Corollary 4.9 yield that for $\theta \in (0,1)$

$$\frac{1-\alpha\theta}{\alpha(1-\theta)} \dim_{\mathrm{A}}^{\alpha\theta} F_\lambda \vee \frac{\overline{\dim}_{\mathrm{B}} F_\lambda}{\alpha} \leqslant \dim_{\mathrm{A}}^{\theta} S_\alpha(F_\lambda) \leqslant \frac{1-\theta/\alpha}{1-\theta} \dim_{\mathrm{A}}^{\theta/\alpha} F_\lambda \wedge 1$$

and applying the explicit formulae for the Assouad spectra derived above gives:

$$\begin{aligned}\frac{1-\alpha\theta}{\alpha(1-\theta)}\left(\frac{1}{(\lambda+1)(1-\alpha\theta)} \wedge 1\right) \vee \frac{1}{\alpha(\lambda+1)} &\leqslant \frac{1}{(\alpha\lambda+1)(1-\theta)} \wedge 1 \\ &\leqslant \frac{1-\theta/\alpha}{1-\theta} \wedge \frac{1}{(\lambda+1)(1-\theta)} \wedge 1.\end{aligned}$$

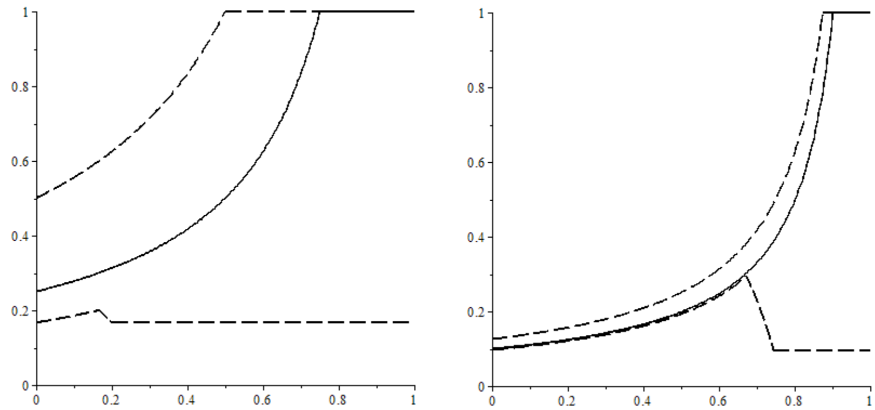

Figure 3: Two plots of the bounds on the Assouad spectrum of the polynomial sequence F_λ under bi-Hölder distortion by S_α. On the left $\lambda = 1$ and $\alpha = 3$ and on the right $\lambda = 7$ and $\alpha = 13/10$. The actual spectrum of $S_\alpha(F_\lambda)$ is shown as a solid line and the bounds are dashed.

6 Unwinding spirals

In this chapter we consider the problem of ‘unwinding spirals’ or, more precisely, the question of whether a given spiral can be mapped to a unit line segment via a homeomorphism with certain ‘metric’ restrictions; see [FP16] for an overview of recent results in this direction. A classical positive result is that the ‘logarithmic spiral’ can be ‘unwound’ to a unit line segment by a bi-Lipschitz map, see [KSS90].

We are interested in the more general question of whether a spiral can be unwound via a bi-Hölder homeomorphism and what restrictions are there on the bi-Hölder parameters? Our main result is in the negative direction: we show that if the bi-Hölder map is too close to being bi-Lipschitz, then it cannot unwind the spiral, where ‘too close’ is precisely characterised by the upper box dimension of the spiral. Our result is a simple application of our work on how the Assouad spectrum can change under bi-Hölder maps and, moreover, we show that one gets strictly better information than if one considers the box or Hausdorff dimensions directly.

In general a *spiral* S is defined to be the set:

$$S = \{\phi(\alpha)\exp(i\alpha) \,:\, \alpha \in [0,\infty)\} \cup \{0\}$$

where ϕ is any continuous decreasing real-valued function such that $\lim_{\alpha\to\infty}\phi(\alpha) = 0$ and for convenience we assume that $\phi(0) = 1$. We say a spiral is *convex differentiable* if ϕ is differentiable and its derivative is non-decreasing. We also say that the spiral has *monotonic winding* if the function

$$x \mapsto \phi(x) - \phi(x+2\pi)$$

is decreasing in $x \geqslant 0$. This is similar to the assumption that decreasing sequences have decreasing gaps. Indeed, monotonic winding guarantees that any ray starting at the origin intersects the spiral in a decreasing sequence with decreasing gaps. If

$\phi(x) = \exp(-cx)$ for some $c > 0$, then the resulting spiral is often referred to as the *logarithmic spiral* (mentioned above). If

$$\frac{\log \phi(x)}{x} \to 0$$

as $x \to \infty$, then the winding is said to be *sub-exponential*. This is the interesting case since then the spiral can have infinite length and thus cannot be unwound by a bi-Lipschitz homeomorphism.

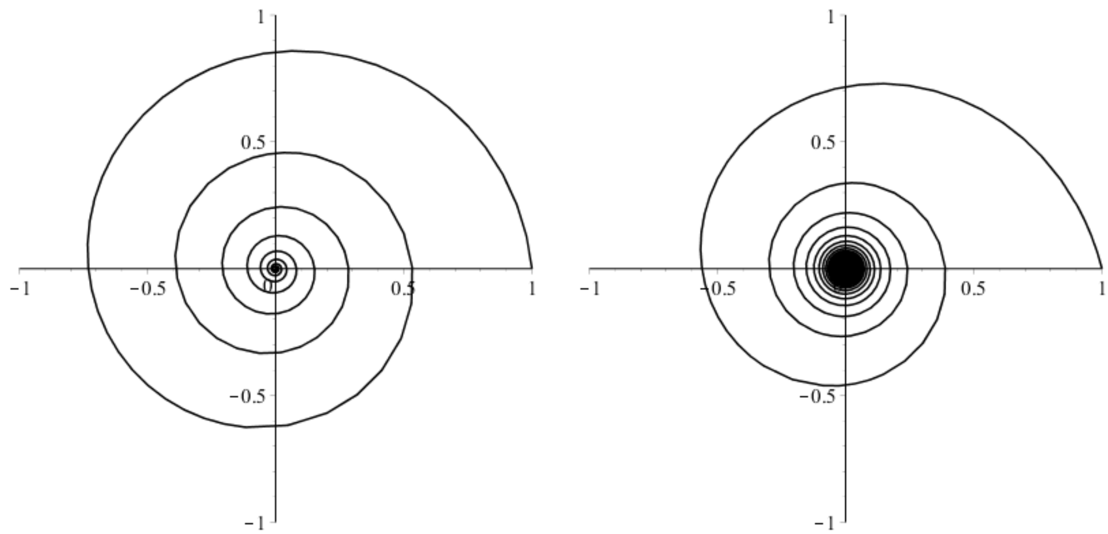

Figure 4: Two spirals: on the left, the *logarithmic spiral* with $c = 1/10$; and, on the right, a spiral with sub-exponential winding where $\phi(x) = 1/(x/4 + 1)$.

First we prove a dichotomy for spirals with monotonic winding.

Theorem 6.1. *Let S be a spiral with monotonic winding. If the winding is sub-exponential, then* $\dim_{\mathrm{A}} S = 2$ *and otherwise* $\dim_{\mathrm{A}} S = 1$.

To prove this result we prove that if the Assouad dimension is strictly less than 2, then the winding must be exponential. Note that this is yet another proof of the fact that spirals with sub-exponential winding cannot be unwound to a line segment by a bi-Lipschitz homeomorphism, since such maps preserve Assouad dimension.

Theorem 6.2. *Let S be a convex differentiable spiral with monotonic winding such that* $\overline{\dim}_{\mathrm{B}} S > 1$. *Then for all* $\theta \in (0, 1)$ *we have*

$$\dim_{\mathrm{A}}^{\theta} S = \frac{\overline{\dim}_{\mathrm{B}} S}{1-\theta} \wedge 2.$$

Note that such spirals must have sub-exponential winding.

Clearly any spiral S is homeomorphic to the unit line segment $[0, 1]$. Let f be a homeomorphism between these two sets and suppose that f is also bi-Hölder with parameters $\beta \geqslant 1 \geqslant \alpha > 0$, i.e. for all $x, y \in S$ we have

$$|x-y|^{\beta} \lesssim |f(x)-f(y)| \lesssim |x-y|^{\alpha}.$$

It follows immediately from the standard results for box dimension, that

$$\frac{1}{\beta}\overline{\dim}_{\mathrm{B}} S \leqslant \overline{\dim}_{\mathrm{B}} f(S) = \overline{\dim}_{\mathrm{B}}[0,1] = 1$$

and so, provided the spiral has upper box dimension larger than its topological dimension, f must be quantitatively far away from being bi-Lipschitz. In particular, we require

$$\beta \geqslant \overline{\dim}_{\mathrm{B}} S > 1.$$

Observe that if we consider Hausdorff dimension here, then we get no information on β because the Hausdorff dimension of *any* spiral is 1. Fortunately, we can get more information if we consider the Assouad dimension.

Corollary 6.3. *Let f be a bi-Hölder homeomorphism with parameters $\beta \geqslant 1 \geqslant \alpha > 0$ mapping a convex differentiable spiral S with sub-exponential and monotonic winding to a line segment. Also, assume that $\overline{\dim}_{\mathrm{B}} S > 1$. Then*

$$\beta \geqslant \alpha + \overline{\dim}_{\mathrm{B}} S\left(1-\frac{\alpha}{2}\right) \geqslant (1+\alpha/2) \vee \overline{\dim}_{\mathrm{B}} S > 1$$

and, if $\alpha = 1$, then

$$\beta \geqslant 1 + \frac{\overline{\dim}_{\mathrm{B}} S}{2}.$$

Proof. It follows from Theorem 4.12 that

$$\dim_{\mathrm{A}} f(S) \geqslant \frac{\dim_{\mathrm{A}} S}{\beta - \theta_0 \alpha}(1-\theta_0)$$

where

$$\theta_0 = \inf\left\{\theta \in [0,1] : \dim_{\mathrm{A}}^{\theta} S = \dim_{\mathrm{A}} S\right\} = 1 - \frac{\overline{\dim}_{\mathrm{B}} S}{2}$$

by Theorems 6.1 and 6.2. Therefore, by Theorem 4.12, we have

$$1 \geqslant \frac{2}{\beta - \alpha\left(1-\frac{\overline{\dim}_{\mathrm{B}} S}{2}\right)} \frac{\overline{\dim}_{\mathrm{B}} S}{2}$$

and solving for β yields

$$\beta \geqslant \alpha + \overline{\dim}_{\mathrm{B}} S\left(1-\frac{\alpha}{2}\right)$$

as required. $\square$

Note that Corollary 6.3 gives strictly better information than we get directly from the upper box dimension, provided the upper box dimension of S is strictly less than 2. Otherwise, both estimates reduce to $\beta \geqslant 2$.

For the purpose of the following proofs, define $f(x) = \phi(2\pi x)$ and $g(x) = f(x) - f(x+1)$. By definition f is decreasing and monotonic winding guarantees that g is also decreasing. Clearly $g(x) < f(x)$ for any $x \geqslant 0$. We will refer to the δ-neighbourhood of S (or part of S) as a δ-*sausage* for $\delta > 0$.

6.1 Proof of Theorem 6.1

Let $0 < r < R < g(0) < 1$ and observe that there is a unique $x_r > 0$ such that $g(x_r) = r$, and a unique $x_R > 0$ such that $f(x_R) = R$. If $x_r < x_R$, we see that the r-sausage of S will completely cover the ball $B(0, R)$, thus the number of r-balls needed to cover $B(0, R) \cap S$ is $\gtrsim (R/r)^2$. If $x_r > x_R$, the r-sausage of S will completely cover a ball smaller than $B(0, R)$. It is easy to see that the smaller ball can be taken to be $B(0, f(x_r))$, thus we need $\gtrsim (f(x_r)/r)^2$ many r-balls to cover $B(0, R) \cap S$.

Since g and f are continuous decreasing functions and $g < f$ we can deduce that the inverse functions g^{-1}, f^{-1} are also decreasing and $g^{-1} < f^{-1}$ (on the appropriate domain). Therefore $g^{-1}(R) < f^{-1}(R)$ and there exists a unique $r \in (0, R)$ such that $g^{-1}(r) = f^{-1}(R)$. We see that the number of r-balls needed to cover $B(0, R) \cap S$ is

$$\gtrsim \left(f(g^{-1}(r))/r\right)^2 = (R/r)^2$$

Therefore, if the Assouad dimension of S is strictly smaller than 2, then it must be true that for all small enough $R > 0$, the r defined above must be such that R/r is uniformly bounded from above. This reasoning is similar to the case of decreasing sequences. Suppose there exists $M > 1$ such that $R/r < M$ for all small enough R and the r chosen above. Then we conclude that

$$g^{-1}(R) < g^{-1}(r) = f^{-1}(R) < g^{-1}(R/M)$$

and so, applying g throughout, we get

$$R > g(f^{-1}(R)) > R/M.$$

Observe that $f^{-1}(R) = x_R$ and so

$$f(x_R) > g(x_R) > f(x_R)/M$$

and subtracting $f(x_R)$ throughout and taking negatives yields

$$f(x_R + 1) < f(x_R)(1 - 1/M).$$

In particular this holds for sufficiently small $R > 0$ and so by continuity of f we can deduce that

$$f(x + 1) < f(x)(1 - 1/M)$$

for any large enough x. From here it is clear that

$$\liminf_{x\to\infty} |\log f(x)/x| > 0.$$

As the winding is monotonic we in fact have

$$\lim_{x\to\infty} |\log f(x)/x| > 0.$$

because the limit holds for integral x, and since $f(x)$ is a decreasing function the limit holds in general. Therefore f, and hence ϕ, is at least exponential and since spirals with exponential winding are bi-Lipschitz equivalent to a line segment we deduce that $\dim_A S = 1$, see [FP16].

6.2 Proof of Theorem 6.2

Denote the upper box dimension of S by B and recall that by assumption and Theorem 6.1 we know $1 < B \leqslant 2 = \dim_{\mathrm{A}} S$. We will prove that

$$\dim_{\mathrm{A}}^{\theta} S \geqslant \frac{B}{1-\theta}$$

for $0 < \theta < 1 - B/2$ which, combined with Proposition 3.1 and Corollary 3.6, proves the result. Fix such a θ. If we can find a sequence of $r_i \to 0$ such that

$$g^{-1}(r_i) \leqslant f^{-1}(r_i^{1-B^-/2})$$

then as $f(g^{-1}(r_i)) \geqslant r_i^{1-B^-/2}$we would have:

$$r_i^{-B^-} \lesssim N(B(0, r^{1-B^-/2}) \cap S, r_i) \lesssim N(B(0, r_i^{\theta}) \cap S, r_i)$$

which proves the result. Therefore assume that for all $r > 0$ small enough we have the inequality

$$g^{-1}(r) > f^{-1}(r^{1-B^-/2})$$

which also implies that for all $r > 0$ small enough

$$g^{-1}(r) > f^{-1}(r^{\theta})$$

since $\theta < 1 - B/2$ and f^{-1} is decreasing. This also implies that $f(x) < g(x)^{1-B^-/2}$ for all large enough x. We assumed that $f(x)$ is differentiable and convex and therefore for all large enough $x > 0$ we have by the mean value theorem that

$$f(x)^{\frac{1}{1-B^-/2}} < g(x) = f(x) - f(x+1) = -f'(\zeta) < -f'(x)$$

where $\zeta \in [x, x+1]$. In fact it is true that for a suitable constant $C > 0$ and all $x > 0$ we have

$$Cf(x)^{\frac{1}{1-B^-/2}} < -f'(x).$$

For simplicity we write $\alpha = \frac{1}{1-B^-/2} > 1$. We have

$$-C > \frac{f'(x)}{f(x)^{\alpha}} = \frac{1}{1-\alpha}\left(f(x)^{1-\alpha}\right)'$$

which gives

$$\left(f(x)^{1-\alpha}\right)' > C(\alpha - 1).$$

Integrating both sides of this inequality yields

$$f(x)^{1-\alpha} - f(0)^{1-\alpha} > C(\alpha-1)x$$

and this implies for all large enough x that

$$f(x) < \left(\frac{1}{C(\alpha-1)x + f(0)^{1-\alpha}}\right)^{\frac{1}{\alpha-1}} \lesssim x^{-\frac{1}{\alpha-1}} = x^{-\frac{1-B^-/2}{B^-/2}} \tag{6.1}$$

and therefore
$$f^{-1}(x) \lesssim x^{-\frac{B^-/2}{1-B^-/2}}. \tag{6.2}$$
We now move towards bounding $N(S \cap B(x, r^\theta), r)$. First of all we need
$$\gtrsim \left(f(g^{-1}(r))/(r)\right)^2$$
many r-balls to cover
$$\{\phi(\alpha)\exp(i\alpha) \,:\, \alpha \in [2\pi g^{-1}(r), \infty)\}$$
because its r-neighbourhood will contain $B(0, g^{-1}(r))$. Now consider the subset of S given by
$$\{\phi(\alpha)\exp(i\alpha) \,:\, \alpha \in \left(2\pi f^{-1}(r^\theta), 2\pi g^{-1}(r)\right)\},$$
and decompose part of this set into the disjoint union of the sets
$$S_m = \{\phi(\alpha)\exp(i\alpha) \,:\, \alpha \in (2\pi m, 2\pi(m+1))\},$$
over integers $[f^{-1}(r^\theta)] + 1 \leqslant m \leqslant [g^{-1}(r)]$, where $[x]$ denotes the integer part of $x > 0$. The projection of each S_m onto the real axis contains an interval of length $f(m)$, so to cover S_m we need at least $\gtrsim f(m)/r$ many r-balls. Since distinct sets S_m are at least r separated, in order to cover all the sets in the above union we need at least
$$\gtrsim \frac{1}{r} \sum_{k=1}^{[g^{-1}(r)]-[f^{-1}(r^\theta)]} f([f^{-1}(r^\theta)] + k)$$
many r-balls. The sequence $f([f^{-1}(r^\theta)] + k)$ for $k = 1, \ldots, [g^{-1}(r)] - [f^{-1}(r^\theta)]$ is decreasing and the minimum of the sequence is $f([g^{-1}(r)]) \geqslant f(g^{-1}(r)) \geqslant r$. Since g is also decreasing we see that
$$f([f^{-1}(r^\theta)] + k) - f([f^{-1}(r^\theta)] + k + 1) \;\geqslant\; r$$
for $k = 1, \ldots, [g^{-1}(r)] - [f^{-1}(r^\theta)] - 1$. Therefore we have
$$f([f^{-1}(r^\theta)] + [g^{-1}(r)] - [f^{-1}(r^\theta)] - k) \;\geqslant\; (k+1)\, r$$
and applying this inequality to the above sum yields
$$\begin{aligned}
\sum_{k=1}^{[g^{-1}(r)]-[f^{-1}(r^\theta)]} f([f^{-1}(r^\theta)] + k) \;&\geqslant\; \sum_{k=1}^{[g^{-1}(r)]-[f^{-1}(r^\theta)]} kr \\
&=\; \frac{1}{2}([g^{-1}(r)] - [f^{-1}(r^\theta)])([g^{-1}(r)] - [f^{-1}(r^\theta)] + 1)\, r \\
&\gtrsim\; (g^{-1}(r) - f^{-1}(r^\theta))^2 r
\end{aligned}$$
where, in particular, the last $\gtrsim$ holds for all $r \to 0$. Therefore we have
$$N(B(0, r^\theta) \cap S, r) \gtrsim (g^{-1}(r) - f^{-1}(r^\theta))^2. \tag{6.3}$$

Now we will take into consideration the box dimension of the spiral. We have

$$N(S,r) \lesssim \left(f(g^{-1}(r))/(r)\right)^2 + L/r$$

where L is the length of the rectifiable part of the spiral corresponding to angles 0 to $2\pi g^{-1}(r)$. We may bound L from above by employing the classical length formula:

$$\begin{aligned} L &= \int_0^{2\pi g^{-1}(r)} \sqrt{\phi^2 + \dot{\phi}^2}\, d\alpha \\ &= \int_{K_1} \sqrt{\phi^2 + \dot{\phi}^2}\, d\alpha \ + \ \int_{K_2} \sqrt{\phi^2 + \dot{\phi}^2}\, d\alpha \end{aligned}$$

where $K_1 = \{\alpha \in (0, 2\pi g^{-1}(r)) \ : \ |\phi| > |\dot{\phi}|\}$, $K_2 = \{\alpha \in (0, 2\pi g^{-1}(r)) \ : \ |\phi| \leqslant |\dot{\phi}|\}$. By splitting the integral in this way we obtain

$$\begin{aligned} L &\leqslant \sqrt{2} \int_{K_1} \phi\, d\alpha \ - \ \sqrt{2} \int_{K_2} \dot{\phi}\, d\alpha \\ &\leqslant \sqrt{2} \int_0^{2\pi g^{-1}(r)} \phi\, d\alpha \ + \ \sqrt{2}\,\phi(0) \\ &\leqslant \sqrt{2} \sum_{k=0}^{[g^{-1}(r)+1]} f(k) \ + \ \sqrt{2} \end{aligned}$$

where the last inequality comes from the fact that f is decreasing. Therefore

$$N(S,r) \ \lesssim \ \left(f(g^{-1}(r))/(r)\right)^2 \ + \ \sum_{k=0}^{[g^{-1}(r)+1]} f(k)/r \ + \ 1/r.$$

Using (6.1) we can bound the middle term above by

$$\sum_{k=0}^{[g^{-1}(r)+1]} f(k)/r \ \lesssim \ \sum_{k=0}^{[g^{-1}(r)+1]} k^{-\frac{1-B^-/2}{B^-/2}}/r \ \lesssim \ r^{-1} g^{-1}(r)^{1-\frac{1-B^-/2}{B^-/2}}.$$

Therefore

$$N(S,r) \ \lesssim \ \left(f(g^{-1}(r))/(r)\right)^2 + r^{-1} g^{-1}(r)^{1-\frac{1-B^-/2}{B^-/2}} + 1/r.$$

Since the (upper) box dimension of S is B we can find a sequence of $r_i \to 0$ such that

$$\left(f(g^{-1}(r_i))/(r_i)\right)^2 + r_i^{-1} g^{-1}(r_i)^{1-\frac{1-B^-/2}{B^-/2}} + 1/r_i \ \gtrsim \ N(S,r_i) \ \gtrsim \ r_i^{-B^-}.$$

Since $B > 1$ we can assume $B^- > 1$, and then either the first term or the second term is $\gtrsim r_i^{-B^-}$ for infinitely many i. If this is true for the first term then by our initial observation we have $\dim_{\mathrm{A}}^{\theta} S \geqslant \frac{B^-}{1-\theta}$. If this is true for the second term, then we deduce that for infinitely many i we have

$$g^{-1}(r_i) \gtrsim r_i^{-B^-/2}.$$

Recalling the lower bound (6.3) and the assumption that $\theta < 1 - B^-/2$, we conclude that

$$
\begin{aligned}
N(B(0, r_i^\theta) \cap S, r_i) &\gtrsim (g^{-1}(r_i) - f^{-1}(r_i^\theta))^2 \\
&\gtrsim \left(r_i^{-B^-/2} - r_i^{-\theta \frac{B^-/2}{1-B^-/2}} \right)^2 \qquad \text{by (6.2)} \\
&\gtrsim r_i^{-B^-}
\end{aligned}
$$

which completes the proof.

7 An example with non-monotonic spectra

In this chapter we construct subsets of $\mathbb{R}$ whose spectra exhibit strange properties. In particular, we prove that the spectra are not necessarily monotone; monotonicity can be broken infinitely many times; and, both the Assouad and lower spectra can have infinitely many phase transitions, i.e. points where they fail to be differentiable.

Given an interval I of length $L < 1$ and numbers $\alpha > \beta > 1$, we construct a set $F_{\alpha,\beta} \subset I$ via the following inductive procedure:

1st step. We pack closed intervals of length L^α with gaps of length L^β inside L. The particular way of packing does not matter as long as it is by an optimal number. We let the union of the closed intervals of length L^α be denoted by I_1.

2nd step. For each interval of length L^α appearing at the first step, we optimally pack intervals of length L^{α^2} with gaps of length $L^{\alpha\beta}$. We let the union of the closed intervals of length L^{α^2} be denoted by I_2.

kth step. For each interval of length $L^{\alpha^{k-1}}$ appearing at the $(k-1)$th step, we optimally pack intervals of length $L^{\alpha^k}$with gaps of length $L^{\alpha^{k-1}\beta}$. We let the union of the closed intervals of length L^{α^k} be denoted by I_k.

We obtain a nested sequence of compact sets $I \supset I_1 \supset I_2 \supset I_3 \supset \dots$ and finally we let

$$F_{\alpha,\beta} = \bigcap_{k=1}^{\infty} I_k$$

which is a non-empty compact set. We can compute the dimensions of $F_{\alpha,\beta}$, the steps are standard and in fact similar to the arguments in the proof of Lemma 7.1. In particular,

$$\dim_{\mathrm{L}} F_{\alpha,\beta} = 0 < \underline{\dim}_{\mathrm{B}} F_{\alpha,\beta} = \frac{\beta-1}{\alpha-1} < \overline{\dim}_{\mathrm{B}} F_{\alpha,\beta} = \frac{\beta-1}{\alpha-1}\frac{\alpha}{\beta} < \dim_{\mathrm{A}} F_{\alpha,\beta} = 1.$$

The exact computation of the spectrum is complicated, but nevertheless we can get some information without much effort.

Lemma 7.1. *Whenever* $\log_\alpha \frac{1}{\theta}$ *is an integer, we have*

$$\dim_{\mathrm{L}}^\theta F_{\alpha,\beta} = \frac{\beta-1}{\alpha-1} = \underline{\dim}_{\mathrm{B}} F_{\alpha,\beta}$$

and

$$\dim_{\mathrm{A}}^\theta F_{\alpha,\beta} = \frac{\beta-1}{\alpha-1}\frac{\alpha}{\beta} = \overline{\dim}_{\mathrm{B}} F_{\alpha,\beta}.$$

In particular, the Assouad/lower spectrum is equal to the upper/lower box dimension infinitely often. This is the general lower/upper bound from Proposition 3.1/3.9.

Proof. Suppose $\log_\alpha \frac{1}{\theta} = m$ is an integer. For any $R \in (0, L)$, there is a unique integer $k \geqslant 1$ such that

$$L^{\alpha^k} \leqslant R < L^{\alpha^{k-1}}.$$

In particular, this means that

$$L^{\alpha^{k+m}} \leqslant R^{1/\theta} < L^{\alpha^{k-1+m}}.$$

There are two different cases to consider:

1. $L^{\alpha^k} \leqslant R < L^{\alpha^{k-1}\beta}$ and $L^{\alpha^{k+m}} \leqslant R^{1/\theta} < L^{\alpha^{k-1+m}\beta}$.

   For any $x \in F_{\alpha,\beta}$, the ball $B(x, R)$ contains one L^{α^k} interval, and any $R^{1/\theta}$-ball can cover at most one $L^{\alpha^{k+m}}$ interval because the distance between two disjoint intervals in the construction is $L^{\alpha^{k-1+m}\beta}$. By construction, in each L^{α^k} interval the number of intervals of length $L^{\alpha^{k+m}}$ is

   $$\asymp [L^{\alpha^k(1-\beta)}][L^{\alpha^{k+1}(1-\beta)}][L^{\alpha^{k+2}(1-\beta)}]\dots[L^{\alpha^{k+m}(1-\beta)}]$$

   where the $[.]$ denotes the integer part. Therefore, the above argument shows that

   $$N(B(x,R), R^{1/\theta}) \lesssim L^{(\alpha^k+\alpha^{k+1}+\alpha^{k+2}+\dots+\alpha^{k+m-1})(1-\beta)} = L^{\alpha^k \frac{1-\beta}{1-\alpha}(1-\alpha^m)}.$$

   We now derive a lower bound for $N(B(x,R), R^{1/\theta})$. Recall we only need the behaviour of

   $$N(B(x,R), R^{1/\theta})$$

   when R is sufficiently small. Choose R so small such that k will satisfy

   $$[L^{\alpha^k(1-\beta)}] > cL^{\alpha^k(1-\beta)}$$

   where $c > \frac{1}{2^{1/m}}$. This is possible because $[x] \geqslant x(1 - 1/x)$ for any positive x, and for x large enough we have $1 - 1/x > \frac{1}{2^{1/m}}$. Then for all small enough $R > 0$ we have

   $$N(B(x,R), R^{1/\theta}) \gtrsim c^m L^{\alpha^k \frac{1-\beta}{1-\alpha}(1-\alpha^m)} \geqslant \frac{1}{2} L^{\alpha^k \frac{1-\beta}{1-\alpha}(1-\alpha^m)}.$$

In summary, for all sufficiently small R satisfying the conditions of case (1) we have
$$N(B(x,R),R^{1/\theta}) \asymp L^{\alpha^k \frac{1-\beta}{1-\alpha}(1-\alpha^m)}.$$

2. $L^{\alpha^{k-1}\beta} \leqslant R < L^{\alpha^{k-1}}$ and $L^{\alpha^{k+m-1}\beta} \leqslant R^{1/\theta} < L^{\alpha^{k-1+m}}$.
   In this case any ball $B(x,R)$ contains
$$\left[\frac{R}{L^{\alpha^{k-1}\beta}}\right] \pm 1$$
   many intervals of length L^{α^k}. Also to cover any interval of length $L^{\alpha^{k+m-1}}$ we need
$$\left[\frac{L^{\alpha^{k+m-1}}}{R^{1/\theta}}\right] \pm 1$$
   many $R^{1/\theta}$-balls. Using the same tricks as in case (1), we may get rid of the integer part and the ± 1. In summary, for all sufficiently small R satisfying the conditions of case (2) we obtain
$$\begin{aligned} N(B(x,R),R^{1/\theta}) &\asymp \frac{R}{L^{\alpha^{k-1}\beta}} L^{(1-\beta)(\alpha^k+\alpha^{k+1}+\alpha^{k+2}+\cdots+\alpha^{k+m-2})} \frac{L^{\alpha^{k+m-1}}}{R^{1/\theta}} \\ &= \frac{R}{R^{1/\theta}} L^{\alpha^k \frac{1-\beta}{1-\alpha}(1-\alpha^{m-1})+\alpha^{k+m-1}-\alpha^{k-1}\beta}. \end{aligned}$$

Finally, applying the definition of k yields the desired result. □

Lemma 7.2. *For every θ' such that $\log_\alpha \frac{1}{\theta'} = m$ is an integer, there exists $\varepsilon = \varepsilon(m) > 0$ such that $\dim_{\mathrm{L}}^{\theta} F_{\alpha,\beta}$ and $\dim_{\mathrm{A}}^{\theta} F_{\alpha,\beta}$ are not constant in the interval $[\theta'(1-\varepsilon), \theta']$.*

Proof. Again, for any $R > 0$, there is an integer k such that
$$L^{\alpha^k} \leqslant R < L^{\alpha^{k-1}}.$$

The calculation may now proceed as in the proof of Lemma 7.1 but with θ very close to, but smaller than, θ'. Write $\log_\alpha \frac{1}{\theta} = m + c$ where m is the integer part and c is the fractional part which is assumed to be small. We have now four cases, which we consider in turn. In each case, we drop the integer part symbols [.] because we are only interested in the asymptotic behaviour of $N(B(x,R),R^{1/\theta})$.

1. $L^{\alpha^k} \leqslant R < L^{\alpha^{k-c}}$ and $L^{\alpha^{k+m}\beta} \leqslant R^{1/\theta} < L^{\alpha^{k+m}}$. Any ball $B(x,R)$ with $x \in F_{\alpha,\beta}$ will contain one interval of length L^{α^k} and, for any $L^{\alpha^{k+m}}$ interval, we need approximately
$$\frac{L^{\alpha^{k+m}}}{R^{1/\theta}}$$
   many $R^{1/\theta}$-balls to cover it. Therefore
$$N(B(x,R),R^{1/\theta}) \asymp \frac{L^{\alpha^{k+m}}}{R^{1/\theta}} L^{\alpha^k \frac{1-\beta}{1-\alpha}(1-\alpha^m)}.$$

2. $L^{\alpha^{k-c}} \leqslant R < L^{\alpha^{k-1}\beta}$ and $L^{\alpha^{k+m}} \leqslant R^{1/\theta} < L^{\alpha^{k+m-1}\beta}$. Any $B(x,R)$ with $x \in F_{\alpha,\beta}$ will contain one L^{α^k} interval and, on the other hand, every $L^{\alpha^{k+m}}$ interval contained inside this L^{α^k} interval needs one $R^{1/\theta}$-ball to cover it. Therefore
$$N(B(x,R), R^{1/\theta}) = L^{\alpha^k \frac{1-\beta}{1-\alpha}(1-\alpha^m)}.$$

3. $L^{\alpha^{k-1}\beta} \leqslant R < L^{\alpha^{k-1-c}\beta}$ and $L^{\alpha^{k+m}} \leqslant R^{1/\theta} < L^{\alpha^{k+m-1}\beta}$. Any $B(x,R)$ with $x \in F_{\alpha,\beta}$ contains approximately
$$\frac{R}{L^{\alpha^{k-1}\beta}}$$
many intervals of length L^{α^k}, but for each interval of length $L^{\alpha^{k+m}}$ we need one $R^{1/\theta}$-ball to cover it. Therefore
$$N(B(x,R), R^{1/\theta}) \asymp \frac{R}{L^{\alpha^{k-1}\beta}} L^{\alpha^k \frac{1-\beta}{1-\alpha}(1-\alpha^m)}.$$

4. $L^{\alpha^{k-1-c}\beta} \leqslant R < L^{\alpha^{k-1}}$ and $L^{\alpha^{k+m-1}\beta} \leqslant R^{1/\theta} < L^{\alpha^{k+m-1}}$. Any $B(x,R)$ with $x \in F_{\alpha,\beta}$ contains approximately
$$\frac{R}{L^{\alpha^{k-1}\beta}}$$
many intervals of length L^{α^k}. Also for any $L^{\alpha^{k+m-1}}$ interval we need approximately
$$\frac{L^{\alpha^{k+m-1}}}{R^{1/\theta}}$$
many $R^{1/\theta}$-balls to cover it. Therefore
$$N(B(x,R), R^{1/\theta}) \asymp \frac{R}{R^{1/\theta}} L^{\alpha^k \frac{1-\beta}{1-\alpha}(1-\alpha^{m-1}) + \alpha^{k+m-1} - \alpha^k \beta}.$$

It follows that for c sufficiently small we have the following formula for spectra
$$\dim_{\mathrm{A}}^{\theta} F_{\alpha,\beta} = \frac{\frac{\alpha}{\beta}\frac{1-\beta}{1-\alpha}\left(\alpha^c - \frac{1}{\theta}\right) - \alpha^c + 1}{1 - \frac{1}{\theta}}$$
and
$$\dim_{\mathrm{L}}^{\theta} F_{\alpha,\beta} = \frac{\frac{1-\beta}{1-\alpha}\left(\alpha^c - \frac{1}{\theta}\right)}{1 - \frac{1}{\theta}} \tag{7.1}$$
which are not constant. The above formulae are obtained by considering the 4 cases above along with estimates derived from the definition of k. This an expression for the asymptotic behaviour of $N(B(x,R), R^{1/\theta})$ which yields formulae for the spectra. These formulae only hold for c smaller than some constant $c_0 \in (0,1)$ (independent of m). This means they are valid for
$$\frac{1}{\alpha^{m+c_0}} \leqslant \theta \leqslant \frac{1}{\alpha^m}$$
for all positive integers m. □

The following corollary follows from the results (and proofs) given in this chapter. It shows that the sets we construct here exhibit some new phenomena.

Corollary 7.3. *For the sets $F_{\alpha,\beta}$ constructed in this chapter, the Assouad and lower spectra have the following properties:*

1. *they are not monotonic and, moreover, there are infinitely many disjoint intervals within which they fail to be monotonic.*
2. *they have infinitely many points of non-differentiability.*

Proof. This follows immediately by combining Lemmas 7.1 and 7.2 with the fact that the spectra are continuous, see Propositions 3.5 and 3.10. To see the non-differentiability, let $m \geqslant 3$ be an integer and consider $\theta = \alpha^{-(m+c)}$ for $c \in [0, \varepsilon]$ where ε is a small number. Then we can write (7.1) as

$$\dim_{\mathrm{L}}^{\theta} F_{\alpha,\beta} = \frac{1-\beta}{1-\alpha}\frac{1-\alpha^{-m}}{1-\theta}.$$

We write the right hand side as $f(\theta)$ which is defined on $[\alpha^{-m-\varepsilon}, \alpha^{-m}]$. The graph of f is a piece of hyperbolic curve with $(\alpha^{-m}, (1-\beta)/(1-\alpha))$ as the right end point. The left derivative if f at $\theta = \alpha^{-m}$ is not zero. However, as the lower box dimension is the maximum of the lower Assouad spectra, we see that $\dim_{\mathrm{L}}^{\theta} F_{\alpha,\beta}$ cannot be differentiable at $\theta = \alpha^{-m}$. As this holds for any $m \geqslant 3$, we have the non-differentiability for the lower Assouad spectra. The argument for the Assouad spectra is similar and we omit. □

8 Self-affine carpets

Self-affine sets are an important class of fractal subsets of Euclidean space. They are attractors of finite iterated function systems consisting of affine contractions. Due to the fact that affine maps can scale by different amounts in different directions, the dimension theory of self-affine sets is much more complicated than their self-similar counter parts; where the defining maps are similarities. Self-affine sets have been studied intensively for over 30 years and have important connections with many other fields, such as non-conformal dynamical systems. An important contribution to this area was Bedford and McMullen's independent treatments of a restricted class of self-affine sets [B84,M84]. These 'Bedford-McMullen carpets' are planar self-affine sets where the affine maps all have the same linear part; which is a simple diagonal matrix. These sets are much simpler than general self-affine sets, but still display many of the key features: the maps scale by different amounts in the two principle directions and so the constituent pieces become increasingly elongated as one iterates the construction. Due to their simple and explicit form, the dimension theory of Bedford-McMullen carpets is rather well-developed. Indeed, Bedford and McMullen computed their Hausdorff and box dimensions in the mid-1980s, Mackay computed their Assouad dimensions in 2011 [M11], and Fraser computed their lower dimensions in 2014 [F14]. In this section we give explicit formulae for the Assouad and lower spectra.

We begin by recalling the construction. Fix $m, n \in \mathbb{N}$ with $2 \leqslant m < n$ and divide the unit square $[0,1]^2$ into an $m \times n$ regular grid. Let $\mathcal{I} = \{0, \dots, m-1\}$ and $\mathcal{J} = \{0, \dots, n-1\}$ and label the mn rectangles in the regular grid by $\mathcal{I} \times \mathcal{J}$ counting from bottom left to top right. Choose a subset of the rectangles $\mathcal{D} \subseteq \mathcal{I} \times \mathcal{J}$ of size at least 2 and for each $d = (i,j) \in \mathcal{D}$, associate a contraction $S_d : [0,1]^2 \to [0,1]^2$ defined by

$$S_d(x,y) = (x/m + i/m,\ y/n + j/n).$$

Let

$$\mathcal{D}^\infty = \{\mathbf{d} = (d_1, d_2, \dots) : d_l = (i_l, j_l) \in \mathcal{D}\}$$

be the set of infinite words over $\mathcal{D}$ and let $\Pi : \mathcal{D}^\infty \to [0,1]^2$ be the canonical map from the symbolic space to the geometric space defined by

$$\Pi(\mathbf{d}) \;=\; \bigcap_{l\in\mathbb{N}} S_{d_1}\circ\cdots\circ S_{d_l}\big([0,1]^2\big).$$

The set $F = \Pi(\mathcal{D}^\infty)$ is the Bedford-McMullen carpet. We thus obtain a symbolic coding of the set F, but we note that some points may not have a unique code. The set F can be viewed in several other contexts. For example, F is the attractor of the *iterated function system* (IFS) $\{S_d\}_{d\in\mathcal{D}}$, i.e. it is the unique non-empty compact set satisfying

$$F \;=\; \bigcup_{d\in\mathcal{D}} S_d(F).$$

In the context of hyperbolic dynamical systems, Bedford-McMullen carpets are also of particular interest. Identifying $[0,1]^2$ with the 2-torus in the natural way, F is a (forward and backward) invariant repeller for the hyperbolic toral endomorphism

$$(x,y)\to(mx \mod 1,\; ny \mod 1).$$

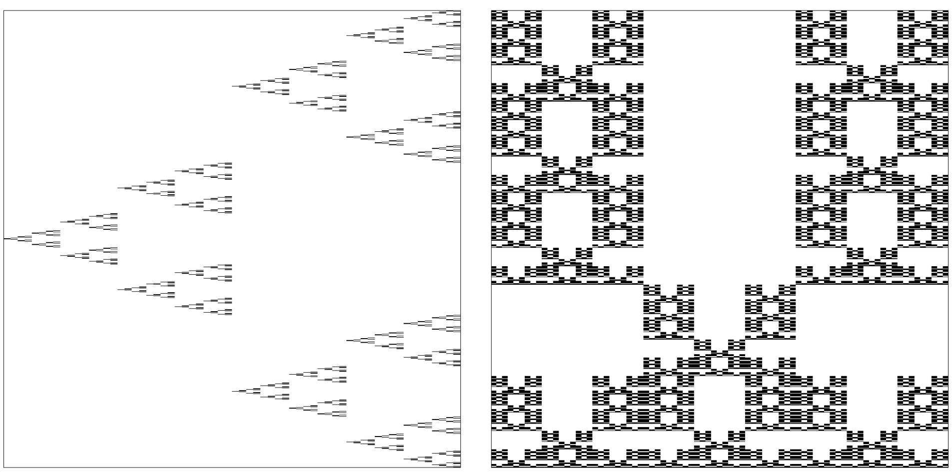

Figure 5: Two examples of self-affine carpets of the type introduced by Bedford and McMullen. On the left $m = 2$ and $n = 3$ and on the right $m = 3$ and $n = 5$.

In order to state the dimension results mentioned above we need some more notation. Let $\pi : \mathcal{I}\times\mathcal{J}\to\mathcal{I}$ be the projection onto the first coordinate. For $i\in\mathcal{I}$, let

$$C_i \;=\; |\{(i',j')\in\mathcal{D} : i' = i\}| \;=\; |\pi^{-1}(i)\cap\mathcal{D}|$$

be the number of chosen rectangles in the ith column, let $C_{\max} = \max_{i\in\mathcal{I}} C_i$ and $C_{\min} = \min_{i\in\pi\mathcal{D}} C_i$.

Theorem 8.1 (Bedford-McMullen-Mackay-Fraser)**.** *The Assouad, box, Hausdorff and lower dimensions of a Bedford-McMullen carpet F are given by*

$$\dim_{\mathrm{A}} F \;=\; \frac{\log|\pi\mathcal{D}|}{\log m} + \frac{\log C_{\max}}{\log n},$$

$$\dim_{\mathrm{B}} F \;=\; \frac{\log|\pi\mathcal{D}|}{\log m} + \frac{\log(|\mathcal{D}|/|\pi\mathcal{D}|)}{\log n}$$

$$\dim_{\mathrm{H}} F \;=\; \frac{\log \sum_{i\in\mathcal{I}} C_i^{\log m/\log n}}{\log m} = \frac{\log|\pi\mathcal{D}|}{\log m} + \frac{\log(|\pi\mathcal{D}|^{-1}\sum_{i\in\mathcal{I}} C_i^{\log m/\log n})^{\log n/\log m}}{\log n}$$

and

$$\dim_{\mathrm{L}} F \;=\; \frac{\log|\pi\mathcal{D}|}{\log m} + \frac{\log C_{\min}}{\log n}.$$

Observe that the term $\log|\pi\mathcal{D}|/\log m$ appears in four of the dimension formulae. This is the dimension of the self-similar projection of F onto the first coordinate. Where this term appears, the second term relates to the dimension of fibres: for the Assouad dimension, $\log C_{\max}/\log n$ is the maximal fibre dimension (also given by a self-similar set), for the lower dimension, $\log C_{\min}/\log n$ is the minimal fibre dimension and, for the box dimension, the second term can be interpreted as an 'average fibre dimension', observing that $|\mathcal{D}|/|\pi\mathcal{D}|$ is the arithmetic average of the C_i. The box, Hausdorff, Assouad and lower dimensions coincide if and only if C_i is the same for all $i \in \pi\mathcal{D}$, commonly referred to as the *uniform fibres case*, and otherwise they are all distinct, see [F14, M11].

Before we discuss our main results we provide a minor addition to Theorem 8.1. The lower dimension is well-known to have many strange properties which may not be seen as desirable for a 'dimension' to satisfy. For example, it is not monotone and it may take the value 0 for sets with non-empty interior, see [F14, Example 2.5]. One can modify the definition to get rid of these (perhaps) strange properties by considering the *modified lower dimension*. We recall its definition as follows.

$$\dim_{\mathrm{ML}} F \;=\; \sup\{\dim_{\mathrm{L}} E \;:\; \emptyset \neq E \subseteq F\}.$$

We mention this here simply to point out that this can be computed for self-affine carpets as a direct consequence of results of Ferguson-Jordan-Shmerkin [FJS10] and the formula for the lower dimension given by Fraser [F14]. Perhaps surprisingly the modified lower dimension is equal to the *Hausdorff* dimension and not the lower dimension.

Corollary 8.2. *The modified lower dimension of a Bedford-McMullen carpet F is given by*

$$\dim_{\mathrm{ML}} F \;=\; \dim_{\mathrm{H}} F \;=\; \frac{\log \sum_{i\in\mathcal{I}} C_i^{\log m/\log n}}{\log m}.$$

Moreover, it is also true that if F is a self-affine carpet of the type considered by Barański [B07] or Lalley-Gatzouras [GL92], then

$$\dim_{\mathrm{ML}} F \;=\; \dim_{\mathrm{H}} F.$$

We refer the readers to [B07, GL92] for the precise formulae.

Proof. Since the modified lower dimension of a compact set is always bounded above by the Hausdorff dimension we need only prove the lower bound. It suffices to prove that for any $\varepsilon > 0$ we can find a subset of F which has lower dimension within ε of the Hausdorff dimension. Via an elegant application of Stirling's formula, Ferguson-Jordan-Shmerkin [FJS10, Lemma 4.3] showed that one can always find a subset of such a carpet generated by a subsystem of an iterate of the original IFS which both has Hausdorff dimension within ε of the original Hausdorff dimension and has the key additional property that the fibres are uniform. Then it follows from [F14, Theorem 2.13] that the lower dimension and the Hausdorff dimension of the subset coincide, thus proving the result.

The result for more general carpets follows in exactly the same way and we omit the details. $\square$

Our main result on self-affine carpets is the following theorem, which gives explicit expressions for the Assouad and lower spectra.

Theorem 8.3. *Let F be a Bedford-McMullen carpet. Then for all $\theta \in (0,1)$ we have*

$$\dim_{\mathrm{A}}^{\theta} F \;=\; \frac{\dim_{\mathrm{B}} F \;-\; \theta\left(\frac{\log(|\mathcal{D}|/C_{\max})}{\log m} + \frac{\log C_{\max}}{\log n}\right)}{1-\theta} \quad \wedge \; \dim_{\mathrm{A}} F$$

and

$$\dim_{\mathrm{L}}^{\theta} F \;=\; \frac{\dim_{\mathrm{B}} F \;-\; \theta\left(\frac{\log(|\mathcal{D}|/C_{\min})}{\log m} + \frac{\log C_{\min}}{\log n}\right)}{1-\theta} \quad \vee \; \dim_{\mathrm{L}} F.$$

Moreover, both both spectra have only one phase transition which occurs at $\theta = \log m/\log n$ and both are analytic and monotonic in the interval $(0, \log m/\log n)$ and constant in $(\log m/\log n, 1)$.

An immediate consequence of Theorem 8.3 is that for any Bedford-McMullen carpet with non-uniform fibres the Assouad spectrum is *strictly smaller* than the general upper bound given by Proposition 3.1 for all $\theta \in (0, \log m/\log n)$. Also, the unique phase transition always occurs to the right of the phase transition in the general bound, i.e.

$$\frac{\log m}{\log n} > 1 - \frac{\dim_{\mathrm{B}} F}{\dim_{\mathrm{A}} F}.$$

These observations follow by simple manipulation of the various dimension formulae.

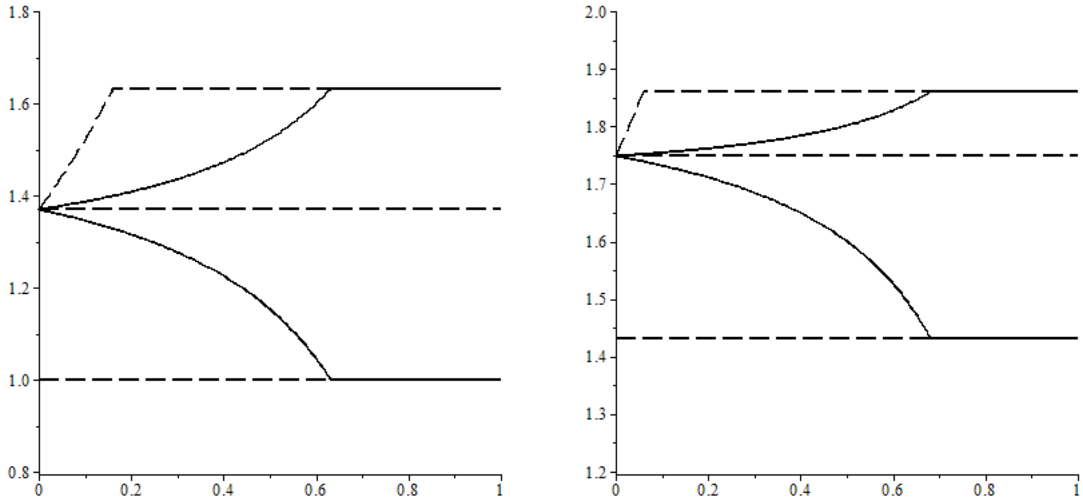

Figure 6: Plots of the Assouad and lower spectra for the two carpets depicted in Figure 5. On the left: $m = 2$, $n = 3$, $|\mathcal{D}| = 3$, $|\pi\mathcal{D}| = 2$, $C_{\max} = 2$ and $C_{\min} = 1$. On the right: $m = 3$, $n = 5$, $|\mathcal{D}| = 10$, $|\pi\mathcal{D}| = 3$, $C_{\max} = 4$ and $C_{\min} = 2$. The general bounds from Proposition 3.1 and Proposition 3.9 are shown with dashed lines.

A, perhaps surprising, corollary of Theorem 8.3 is that the lower spectra is not uniformly bounded above by the Hausdorff or modified lower dimension. This might be surprising because the lower spectrum is based on the lower dimension, which *is* bounded above by the Hausdorff and modified lower dimension for compact sets. The Assouad spectrum does not violate any of the general inequalities between the Assouad dimension and the other common dimensions since it is uniformly bounded below by the upper box dimension, see Proposition 3.1. In fact it follows that for any Bedford-McMullen carpet with non-uniform fibres there will be an open interval $(0, \theta_0)$ within which the lower spectrum is strictly larger than the Hausdorff and modified lower dimension. This follows since for such carpets the lower spectrum is continuous and converges to the box dimension as $\theta \to 0$ which is strictly larger than the Hausdorff and modified lower dimension. This goes someway to justifying the sharpness of Proposition 3.9.

Theorem 8.3 proves that the ratio $\frac{\log m}{\log n}$ is a bi-Lipschitz invariant within the class of Bedford-McMullen carpets. To the best of our knowledge this has not been observed before. Since sets are bi-Lipschitz equivalent to themselves this invariant can be useful in determining what kinds of affine IFSs give rise to a particular carpet. For example, let F be a Bedford-McMullen carpet generated by an IFS using an $m \times n$ grid for some $2 \leqslant m < n$. Assume that F does not have uniform fibres and that it can also be viewed as a Bedford-McMullen carpet generated by an IFS using an $m' \times n'$ grid for some $2 \leqslant m' < n'$. Then we may conclude that

$$\frac{\log m}{\log n} = \frac{\log m'}{\log n'}.$$

The question of when two Bedford-McMullen carpets are bi-Lipschitz equivalent has been considered by Li-Li-Miao [LLM13], where they give positive results assuming some natural conditions, including that m, n are constant. In some sense, our results compliment these results as we provide a mechanism for proving negative results. We demonstrate this by the following example.

Proposition 8.4. *We can find two topologically equivalent Bedford-McMullen carpets with the same box counting, lower and Assouad dimensions, but different spectra. Therefore E and F are not bi-Lipschitz invariant despite this not being revealed by knowledge of the other dimensions.*

Proof. We construct the first Bedford-McMullen carpet such that $m = 5$, $n = 6$, $|\pi\mathcal{D}| = 3$, $|\mathcal{D}| = 6$, $C_{\max} = 3$, $C_{\min} = 1$ and the second carpet such that $m = 5$, $n = 36$, $|\pi\mathcal{D}| = 3$, $|\mathcal{D}| = 12$, $C_{\max} = 9$, $C_{\min} = 1$. Moreover, we may choose the rectangles satisfying these parameters so that they do not touch each other, rendering both carpets totally disconnected and thus homeomorphic. It follows from Theorem 8.1 that for both carpets

$$\dim_{\mathrm{A}} F \;=\; \frac{\log 3}{\log 5} + \frac{\log 3}{\log 6},$$

$$\dim_{\mathrm{B}} F \;=\; \frac{\log 3}{\log 5} + \frac{\log 2}{\log 6}$$

and

$$\dim_{\mathrm{L}} F \;=\; \frac{\log 3}{\log 5},$$

but, since the ratio $\log m/\log n$ is not the same for both constructions, they necessarily have different spectra. $\square$

For the above examples, we guaranteed the spectra would be different by ensuring that $\frac{\log m}{\log n} \neq \frac{\log m'}{\log n'}$. This turns out to be necessary in finding such examples. The following corollary expresses this precisely and follows immediately by rearranging the formulae given in Theorems 8.1 and 8.3.

Corollary 8.5. *The dimension spectra of Bedford-McMullen carpets are completely determined by the ratio $\log m/\log n$ and the box, lower and Assouad dimensions of the carpet. More precisely, we have*

$$\dim_{\mathrm{A}}^{\theta} F \;=\; \frac{\dim_{\mathrm{B}} F \;-\; \theta\left(\dim_{\mathrm{A}} F - (\dim_{\mathrm{A}} F - \dim_{\mathrm{B}} F)\,\frac{\log n}{\log m}\right)}{1-\theta} \;\wedge\; \dim_{\mathrm{A}} F$$

and

$$\dim_{\mathrm{L}}^{\theta} F \;=\; \frac{\dim_{\mathrm{B}} F \;-\; \theta\left(\dim_{\mathrm{L}} F + (\dim_{\mathrm{B}} F - \dim_{\mathrm{L}} F)\,\frac{\log n}{\log m}\right)}{1-\theta} \;\vee\; \dim_{\mathrm{L}} F.$$

8.1 Proof of Theorem 8.3

Let $\mathbf{d} \in \mathcal{D}^{\infty}$ and $r > 0$ be small. Define $l_1(r), l_2(r)$ to be the unique natural numbers satisfying

$$m^{-l_1(r)} \leqslant r < m^{-l_1(r)+1}$$

and

$$n^{-l_2(r)} \leqslant r < n^{-l_2(r)+1}.$$

The approximate square 'centred' at $\mathbf{d} = ((i_1, j_1), (i_2, j_2), \dots) \in \mathcal{D}^\infty$ with 'radius' $r > 0$ is defined by

$$Q(\mathbf{d}, r) = \left\{\mathbf{d}' = ((i_1', j_1'), (i_2', j_2'), \dots) \in \mathcal{D}^\infty : i_l' = i_l \ \forall l \leqslant l_1(r) \text{ and } j_l' = j_l \ \forall l \leqslant l_2(r)\right\},$$

and the geometric projection of this set, $\Pi\big(Q(\mathbf{d}, r)\big)$, is a subset of F which contains $\Pi(\mathbf{d})$ and naturally sits inside a rectangle which is 'approximately a square' in that it has a base with length $m^{-l_1(r)} \in (r/m, r]$ and height $n^{-l_2(r)} \in (r/n, r]$. One obtains equivalent definitions of the Assouad and lower spectra if one replaces $B(x, R)$ by $\Pi\big(Q(\mathbf{d}, R)\big)$, i.e. we may use approximate squares instead of balls.

Our subsequent analysis breaks naturally into two cases, illustrated by the following lemma.

Lemma 8.6. *Let $R \in (0, 1)$ and $\theta \in (0, 1)$. Then*

1. *if $\theta \leqslant \log m / \log n$, then*

$$l_2(R) \leqslant l_1(R) \leqslant l_2\left(R^{1/\theta}\right) \leqslant l_1\left(R^{1/\theta}\right)$$

2. *if $\theta \geqslant \log m / \log n$, then*

$$l_2(R) \leqslant l_2\left(R^{1/\theta}\right) \leqslant l_1(R) \leqslant l_1\left(R^{1/\theta}\right).$$

Proof. This follows immediately from the definitions of $l_1(\cdot)$ and $l_2(\cdot)$. □

Proposition 8.7. *Let $\theta \in (0, \log m / \log n]$. Then*

$$\dim_{\mathrm{A}}^\theta F = \frac{\left(\frac{\log|\pi\mathcal{D}|}{\log m} + \frac{\log(|\mathcal{D}|/|\pi\mathcal{D}|)}{\log n}\right) - \theta\left(\frac{\log(|\mathcal{D}|/C_{\max})}{\log m} + \frac{\log C_{\max}}{\log n}\right)}{1 - \theta}$$

Proof. Let

$$\mathbf{d} = ((i_1, j_1), (i_2, j_2), \dots) \in \mathcal{D}^\infty$$

and $R \in (0, 1)$. When one looks at the approximate square $\Pi\big(Q(\mathbf{d}, R)\big)$ one finds that it is made up of several horizontal strips with base length the same as that of the approximate square, i.e. $m^{-l_1(R)}$, and height $n^{-l_1(R)}$ which is considerably smaller, but still larger than $R^{1/\theta}/n$ since $l_1(R) \leqslant l_2\left(R^{1/\theta}\right)$. These strips are images of F under $l_1(R)$-fold compositions of maps from $\{S_d\}_{d \in \mathcal{D}}$ and so to keep track of how many there are, one counts rectangles in the appropriate columns for each map in the composition. The total number of such strips is seen to be:

$$\prod_{l=l_2(R)+1}^{l_1(R)} C_{i_l}.$$

We want to cover these strips by sets of diameter $R^{1/\theta}$ and so we iterate the construction inside each horizontal strip until the height is $n^{-l_2(R^{1/\theta})}$, which is approximately $R^{1/\theta}$. This takes $l_2(R^{1/\theta}) - l_1(R)$ iterations and, this time, for every iteration we pick up $|\mathcal{D}|$ smaller rectangles, rather than just those inside a particular column. At this stage we are left with a large collection of rectangles with height approximately $R^{1/\theta}$ and base $m^{-l_2(R^{1/\theta})}$, which is somewhat larger. We now iterate inside each of these rectangles until we obtain a family of rectangles each with base of length $m^{-l_1(R^{1/\theta})}$. This takes a further $l_1(R^{1/\theta}) - l_2(R^{1/\theta})$ iterations. We can then cover the resulting collection by small sets of diameter $R^{1/\theta}$, observing that sets formed by this last stage of iteration can be covered simultaneously, provided they are in the same column. Therefore, for each of the last iterations we only require a factor of $|\pi\mathcal{D}|$ more covering sets. Putting all these estimates together, yields

$$\begin{aligned}
N\big(\Pi\big(Q(\mathbf{d},R)\big),R^{1/\theta}\big) &\asymp \left(\prod_{l=l_2(R)+1}^{l_1(R)} C_{i_l}\right)\left(|\mathcal{D}|^{l_2(R^{1/\theta})-l_1(R)}\right)\left(|\pi\mathcal{D}|^{l_1(R^{1/\theta})-l_2(R^{1/\theta})}\right)\\
&\leqslant \left(C_{\max}\right)^{l_1(R)-l_2(R)}\left(|\mathcal{D}|^{l_2(R^{1/\theta})-l_1(R)}\right)\left(|\pi\mathcal{D}|^{l_1(R^{1/\theta})-l_2(R^{1/\theta})}\right)\\
&\asymp (C_{\max})^{\log R/\log n-\log R/\log m}\\
&\qquad\cdot\ |\mathcal{D}|^{\log R/\log m-\log R/\theta\log n}\ |\pi\mathcal{D}|^{\log R/\theta\log n-\log R/\theta\log m}\\
&= R^{\log C_{\max}/\log n+\log(|\mathcal{D}|/C_{\max})/\log m+\log(|\pi\mathcal{D}|/|\mathcal{D}|)/\theta\log n\,-\,\log|\pi\mathcal{D}|/\theta\log m}.
\end{aligned}$$

Taking logs and dividing by $(1-1/\theta)\log R$ yields

$$\begin{aligned}
\frac{\log N\big(\Pi\big(Q(\mathbf{d},R)\big),R^{1/\theta}\big)}{(1-1/\theta)\log R} &\leqslant \frac{\frac{\log C_{\max}}{\log n}+\frac{\log(|\mathcal{D}|/C_{\max})}{\log m}+\frac{\log(|\pi\mathcal{D}|/|\mathcal{D}|)}{\theta\log n}-\frac{\log|\pi\mathcal{D}|}{\theta\log m}}{1-1/\theta}+\frac{O(1)}{\log R}\\
&= \frac{\left(\frac{\log|\pi\mathcal{D}|}{\log m}+\frac{\log(|\mathcal{D}|/|\pi\mathcal{D}|)}{\log n}\right)-\theta\left(\frac{\log(|\mathcal{D}|/C_{\max})}{\log m}+\frac{\log C_{\max}}{\log n}\right)}{1-\theta}\\
&\quad+\frac{O(1)}{\log R}.
\end{aligned}$$

Letting $R \to 0$ yields the required upper bound, noting that these estimates are uniform in $\mathbf{d} \in \mathcal{D}^\infty$. The required lower bound also follows by noting that if we choose $\mathbf{d} \in \mathcal{D}^\infty$, such that $C_{i_l} = C_{\max}$ for all l, then the only appearance of an inequality in the above argument is replaced by equality and observing that all of our covering estimates were optimal up to multiplicative constants. □

Proposition 8.8. *Let $\theta \in [\log m/\log n, 1)$. Then*

$$\dim_{\mathrm{A}}^\theta F = \frac{\log|\pi\mathcal{D}|}{\log m} + \frac{\log C_{\max}}{\log n}.$$

Proof. This proof is similar to the proof of Proposition 8.7. Let

$$\mathbf{d} = ((i_1,j_1),(i_2,j_2),\dots) \in \mathcal{D}^\infty$$

and $R \in (0,1)$. We proceed as before, but this time one obtains a family of horizontal strips with height approximately $R^{1/\theta}$ after $l_2(R^{1/\theta})$ steps, which is before the bases become smaller than R, since in this case $l_2(R^{1/\theta}) \leqslant l_1(R)$. The effect is that the 'middle term' above (concerning powers of $|\mathcal{D}|$) is not required. The horizontal strips are then covered as before yielding

$$\begin{aligned} N\big(\Pi\big(Q(\mathbf{d},R)\big),R^{1/\theta}\big) &\asymp \left(\prod_{l=l_2(R)+1}^{l_2(R^{1/\theta})} C_{i_l}\right)\left(|\pi\mathcal{D}|^{l_1(R^{1/\theta})-l_1(R)}\right) \\ &\leqslant \left(C_{\max}\right)^{l_2(R^{1/\theta})-l_2(R)} \quad |\pi\mathcal{D}|^{l_1(R^{1/\theta})-l_1(R)} \\ &\asymp (C_{\max})^{\log R/\log n - \log R/\theta\log n} \ |\pi\mathcal{D}|^{\log R/\log m - \log R/\theta \log m} \\ &= R^{(1-1/\theta)(\log C_{\max}/\log n + \log|\pi D|/\log m)}. \end{aligned}$$

Taking logs and dividing by $(1-1/\theta)\log R$ yields

$$\frac{\log N\big(\Pi\big(Q(\mathbf{d},R)\big),R^{1/\theta}\big)}{(1-1/\theta)\log R} \leqslant \frac{\log|\pi\mathcal{D}|}{\log m} + \frac{\log C_{\max}}{\log n} + \frac{O(1)}{\log R}.$$

Letting $R \to 0$ yields the required upper bound, noting that these estimates are uniform in $\mathbf{d} \in \mathcal{D}^\infty$. The required lower bound again follows by noting that if we choose $\mathbf{d} \in \mathcal{D}^\infty$, such that $C_{i_l} = C_{\max}$ for all l, then the only appearance of an inequality in the above argument is replaced by equality and observing that all of our covering estimates were optimal up to multiplicative constants. □

We now turn to the lower spectrum, which can be handled similarly and so only the key points in the proofs are given.

Proposition 8.9. *Let $\theta \in (0, \log m/\log n]$. Then*

$$\dim_{\mathrm{L}}^\theta F = \frac{\left(\frac{\log|\pi\mathcal{D}|}{\log m} + \frac{\log(|\mathcal{D}|/|\pi\mathcal{D}|)}{\log n}\right) - \theta\left(\frac{\log(|\mathcal{D}|/C_{\min})}{\log m} + \frac{\log C_{\min}}{\log n}\right)}{1-\theta}$$

Proof. Let $\mathbf{d} = ((i_1,j_1),(i_2,j_2),\dots) \in \mathcal{D}^\infty$ and $R \in (0,1)$. Proceeding as above, one obtains

$$N\big(\Pi\big(Q(\mathbf{d},R)\big),R^{1/\theta}\big) \asymp \left(\prod_{l=l_2(R)+1}^{l_1(R)} C_{i_l}\right)\left(|\mathcal{D}|^{l_2(R^{1/\theta})-l_1(R)}\right)\left(|\pi\mathcal{D}|^{l_1(R^{1/\theta})-l_2(R^{1/\theta})}\right)$$

but this time we continue by considering uniform lower bounds:

$$N\big(\Pi\big(Q(\mathbf{d},R)\big),R^{1/\theta}\big) \geqslant \left(C_{\min}\right)^{l_1(R)-l_2(R)} \left(|\mathcal{D}|^{l_2(R^{1/\theta})-l_1(R)}\right)\left(|\pi\mathcal{D}|^{l_1(R^{1/\theta})-l_2(R^{1/\theta})}\right)$$

$$
\begin{aligned}
&\asymp (C_{\min})^{\log R/\log n - \log R/\log m} \\
&\qquad\qquad \cdot\ |\mathcal{D}|^{\log R/\log m - \log R/\theta\log n}\ |\pi\mathcal{D}|^{\log R/\theta\log n - \log R/\theta \log m} \\
&= R^{\log C_{\min}/\log n + \log(|\mathcal{D}|/C_{\min})/\log m + \log(|\pi\mathcal{D}|/|\mathcal{D}|)/\theta\log n\ -\ \log|\pi\mathcal{D}|/\theta\log m}.
\end{aligned}
$$

Taking logs and dividing by $(1-1/\theta)\log R$ yields

$$
\begin{aligned}
\frac{\log N\big(\Pi\big(Q(\mathbf{d},R)\big),R^{1/\theta}\big)}{(1-1/\theta)\log R} &\geqslant \frac{\frac{\log C_{\min}}{\log n} + \frac{\log(|\mathcal{D}|/C_{\min})}{\log m} + \frac{\log(|\pi\mathcal{D}|/|\mathcal{D}|)}{\theta\log n} - \frac{\log|\pi\mathcal{D}|}{\theta\log m}}{1-1/\theta} + \frac{O(1)}{\log R} \\
&= \frac{\left(\frac{\log|\pi\mathcal{D}|}{\log m} + \frac{\log(|\mathcal{D}|/|\pi\mathcal{D}|)}{\log n}\right) - \theta\left(\frac{\log(|\mathcal{D}|/C_{\min})}{\log m} + \frac{\log C_{\min}}{\log n}\right)}{1-\theta} \\
&\quad + \frac{O(1)}{\log R}.
\end{aligned}
$$

Letting $R \to 0$ yields the required lower bound. The required upper bound also follows by noting that if we choose $\mathbf{d} \in \mathcal{D}^\infty$, such that $C_{i_l} = C_{\min}$ for all l, then the only appearance of an inequality in the above argument is replaced by equality and observing that all of our covering estimates were optimal up to multiplicative constants. □

Proposition 8.10. *Let $\theta \in [\log m/\log n, 1)$. Then*

$$
\dim_{\mathrm{L}}^{\theta} F = \frac{\log|\pi\mathcal{D}|}{\log m} + \frac{\log C_{\min}}{\log n}.
$$

Proof. Let $\mathbf{d} = ((i_1,j_1),(i_2,j_2),\dots) \in \mathcal{D}^\infty$ and $R \in (0,1)$. Following the strategy of the proof of Lemma 8.8 but this time considering uniform lower bounds we get:

$$
\begin{aligned}
N\big(\Pi\big(Q(\mathbf{d},R)\big),R^{1/\theta}\big) &\asymp \left(\prod_{l=l_2(R)+1}^{l_2(R^{1/\theta})} C_{i_l}\right)\left(|\pi\mathcal{D}|^{l_1(R^{1/\theta})-l_1(R)}\right) \\
&\geqslant \Big(C_{\min}\Big)^{l_2(R^{1/\theta})-l_2(R)}\ |\pi\mathcal{D}|^{l_1(R^{1/\theta})-l_1(R)} \\
&\asymp (C_{\min})^{\log R/\log n - \log R/\theta\log n}\ |\pi\mathcal{D}|^{\log R/\log m - \log R/\theta\log m} \\
&= R^{(1-1/\theta)(\log C_{\min}/\log n + \log|\pi D|/\log m)}
\end{aligned}
$$

which yields

$$
\frac{\log N\big(\Pi\big(Q(\mathbf{d},R)\big),R^{1/\theta}\big)}{(1-1/\theta)\log R} \geqslant \frac{\log|\pi\mathcal{D}|}{\log m} + \frac{\log C_{\min}}{\log n} + \frac{O(1)}{\log R}.
$$

Letting $R \to 0$ yields the required lower bound. The required upper bound again follows by noting that if we choose $\mathbf{d} \in \mathcal{D}^\infty$, such that $C_{i_l} = C_{\min}$ for all l, then the only appearance of an inequality in the above argument is replaced by equality and observing that all of our covering estimates were optimal. □

Theorem 8.3 follows immediately by combining Propositions 8.7, 8.8, 8.9, 8.10 and the known formulae for the box, Assouad and lower dimensions.

9 Self-similar and self-conformal sets with overlaps

Self-similar and self-conformal sets are another important class of fractals generated by iterated function systems. They are simpler than the self-affine sets considered in the previous chapter in that the defining maps locally scale by the same amount in every direction. If the constituent pieces making up the set do not overlap too much, then the situation is quite straightforward and it follows from standard results that the lower and Assouad dimensions coincide, see for example [F14, Corollary 2.11], rendering the dimension spectra not interesting. However, if there are overlaps in the construction then Fraser proved that the Assouad dimension can be strictly larger than the upper box dimension, but the lower dimension always equals the upper box dimension, see [F14, Section 3.1 and Corollary 2.11]. Therefore, the interesting problem is to consider the Assouad spectrum in the case of complicated overlaps. This is not straightforward and we do not obtain precise results, but we show how to obtain non-trivial bounds in terms of the local dimensions of related Gibbs measures. This technique was used by Fraser and Jordan to study the Assouad dimension of self-affine carpets where the self-similar set in the projection has overlaps, see [FJ16].

Let $U \subset \mathbb{R}^d$ be a bounded simply connected non-empty open set. A map $S : U \to \mathbb{R}^d$ is called a *conformal map* if it is differentiable and the Jacobian $S'(x)$ satisfies $|S'(x)y| = |S'(x)||y| > 0$ for all $x \in U$ and $y \in \mathbb{R}^d \setminus \{0\}$. Let $\{S_i\}_{i\in\mathcal{I}}$ be a finite collection of conformal maps on a common domain $U \subseteq \mathbb{R}^d$, and assume that each maps U into itself. For convenience we will extend each of the maps to the boundary of U by continuity. Furthermore, assume that each map is a bi-Lipschitz contraction, i.e.

$$0 < \inf_{x\in U} |S'(x)| \leqslant \sup_{x\in U} |S'(x)| < 1.$$

There is a unique non-empty compact set F satisfying

$$F = \bigcup_{i \in \mathcal{I}} S_i(F)$$

which is called the *self-conformal set* associated with the iterated function system (IFS) $\{S_i\}_{i \in \mathcal{I}}$. See [F03, Chapter 9] for more details on the theory of attractors of IFSs. If for each map the Jacobian $|S'(x)|$ is independent of x, then the maps S are *similarities* and the set F is called *self-similar*. Here the situation is slightly simpler, but is subsumed into the following analysis of the self-conformal setting, see [F97, Chapters 4,5] for more details.

Let

$$\mathcal{I}^\infty = \{\mathbf{i} = (i_1, i_2, \dots) : i_l \in \mathcal{I}\}$$

be the set of infinite words over $\mathcal{I}$ and let $\Pi : \mathcal{I}^\infty \to F$ be the canonical map from the symbolic space to the geometric space defined by

$$\Pi(\mathbf{i}) = \bigcap_{l \in \mathbb{N}} S_{i_1} \circ \cdots \circ S_{i_l}(\overline{U}).$$

Evidently $F = \Pi(\mathcal{I}^\infty)$ and we thus obtain a symbolic coding of the set F, but we note that some points may have many codes. Write $\mathcal{I}^* = \cup_{k \in \mathbb{N}} \mathcal{I}^k$ for the set of all finite words over $\mathcal{I}$. For $\mathbf{i} = (i_1, i_2, \dots, i_k) \in \mathcal{I}^*$ write

$$[\mathbf{i}] = \{\mathbf{ij} : \mathbf{j} \in \mathcal{I}^\infty\}$$

for the cylinder consisting of all infinite words beginning with the finite word $\mathbf{i}$ and

$$S_\mathbf{i} = S_{i_1} \circ \cdots \circ S_{i_k}.$$

If $d \geqslant 2$, then it is well-known that conformal maps satisfy the *bounded distortion principle*, see [F97, Proposition 4.2]. This means that for $\mathbf{i} \in \mathcal{I}^k$ and $x, y \in U$

$$|S'_\mathbf{i}(x)| \asymp |S'_\mathbf{i}(y)|$$

with the implied constants independent of x, y and k. This means that for each $\mathbf{i} \in \mathcal{I}^k$, there is a constant $\mathrm{Lip}(\mathbf{i}) \in (0,1)$ such that

$$|S'_\mathbf{i}(x)|^{1/d} \asymp |S_\mathbf{i}(\overline{U})| \asymp \mathrm{Lip}(\mathbf{i})$$

where we abuse notation slightly by writing $|S'_\mathbf{i}(x)|$ for the determinant of the Jacobian $S'_\mathbf{i}(x)$ and $|S_\mathbf{i}(\overline{U})|$ for the diameter of the set $S_\mathbf{i}(\overline{U})$. The constant $\mathrm{Lip}(\mathbf{i})$ could be the determinant of the Jacobian derivative of $S_\mathbf{i}$ at the fixed point of $S_\mathbf{i}$, or the upper (or lower) Lipschitz constant of $S_\mathbf{i}$. For definiteness let

$$\mathrm{Lip}(\mathbf{i}) = \sup_{x,y \in \overline{U}} \frac{|S_\mathbf{i}(x) - S_\mathbf{i}(y)|}{|x-y|}$$

be the upper Lipschitz constant of $S_\mathbf{i}$. To get the bounded distortion principle to hold in $\mathbb{R}$ (which we will need), we assume Hölder continuity of the derivatives S'_i. We emphasise that this extra assumption is not required for $d \geqslant 2$.

The *topological pressure* of the IFS is defined by

$$P(s) \;=\; \lim_{k\to\infty} \frac{1}{k} \log \sum_{\mathbf{i}\in\mathcal{I}^k} \mathrm{Lip}(\mathbf{i})^s$$

where the limit is easily seen to exist by submultiplicativity of the upper Lipschitz constant for example. Actually we have quasi-multiplicativity in the sense that for $\mathbf{i},\mathbf{j} \in \mathcal{I}^*$

$$\mathrm{Lip}(\mathbf{ij}) \asymp \mathrm{Lip}(\mathbf{i})\mathrm{Lip}(\mathbf{j}).$$

The topological pressure is a continuous decreasing function of $s \geqslant 0$ and there is a unique zero, i.e. a unique $s \geqslant 0$ such that $P(s) = 0$. From now on we will fix s to be the unique zero of the topological pressure. A classical result in dimension theory states that

$$\dim_{\mathrm{L}} F = \dim_{\mathrm{H}} F = \overline{\dim}_{\mathrm{B}} F \leqslant \min\{d, s\}$$

and if the *open set condition* is satisfied, then one has equality. This dimension formula is often referred to as *Bowen's formula* goes back to Bowen [B75] and Ruelle [R82]. We will be focusing on the case where the open set condition fails, but the general upper bound given by s is still significant. We recall that the Assouad dimension may exceed s. Another classical result, this time from the thermodynamic formalism, states that there exists a unique Borel probability measure m on $\mathcal{I}^\infty$ (equipped with the product topology) satisfying

$$m([\mathbf{i}]) \asymp \mathrm{Lip}(\mathbf{i})^s.$$

This measure is the *Gibbs measure* corresponding to the potential $\phi : \mathbf{i} \mapsto s\log(\mathrm{Lip}(\mathbf{i}))$ and is a fundamental object in the thermodynamical formalism and related problems from statistical physics, see [F97, R82].

Let $\mu = m \circ \Pi^{-1}$ be the push forward of m onto F. The measure μ is a Borel probability measure and is intimately related to the dimension theory of F. The number

$$t \;:=\; \sup\left\{t' \geqslant 0 \;:\; (\exists C > 0)\,(\forall x \in F)\,\mu\,(B(x,r)) \leqslant Cr^{t'}\right\}$$

will play a central role in our analysis. This number is related to the fine structure of μ and can be expressed in terms of more familiar dimensional quantities, such as the local dimension or L^q-spectrum. We record these connections to put our results in context. The lower local dimension of a Borel probability measure ν at a point x in its support is defined by

$$\underline{\dim}_{\mathrm{loc}}(\nu, x) \;=\; \liminf_{r\to 0} \frac{\log \nu\big(B(x,r)\big)}{\log r}$$

with the upper local dimension defined similarly with lim sup in place of lim inf. It is clear that

$$t \;\leqslant\; \inf_{x\in F} \underline{\dim}_{\mathrm{loc}}(\mu, x) \;\leqslant\; \overline{\dim}_{\mathrm{B}} F \;\leqslant\; \min\{s,\, d\}$$

The L^q-spectrum gives a coarse indication of the global fluctuations of a measure and are a standard tool in multifractal analysis and information theory. For $q > 0$ and

$r > 0$ let

$$M_r^q(\mu) \;=\; \sup\left\{ \sum_i \mu(U_i)^q \;:\; \{U_i\}_i \text{ is a centered packing of } \operatorname{supp}\mu \text{ by balls of radius } r \right\}.$$

The upper L^q-spectrum of μ is defined by

$$\tau_\mu(q) = \limsup_{r\to 0} \frac{\log M_r^q(\mu)}{\log r}.$$

It is straightforward to see that $\tau_\mu(q)$ is non-decreasing and concave in q. By following the argument in [FJ16, Lemma 2.1], one may prove that t is actually equal to the slope of the asymptote as $q \to \infty$, i.e.,

$$t \;=\; \inf\left\{t_0 \geqslant 0 \;:\; (\forall q > 0)\, \tau_\mu(q) < t_0 q\right\}.$$

Moreover, if the *multifractal formalism* holds for a sequence of $q \to \infty$, then

$$t \;=\; \inf\left\{t_0 \geqslant 0 \;:\; (\forall q > 0)\, \tau_\mu(q) < t_0 q\right\} \;=\; \inf_{x\in F} \underline{\dim}_{\mathrm{loc}}(\mu, x).$$

See [FL09] for more details on the multifractal formalism. If F is self-similar, then the Gibbs measure μ is a self-similar measure and in this case we know more. For example, it was shown by Peres and Solomyak [PS00] that the L^q-spectrum exists, and Feng [F07] proved that the multifractal formalism holds for q whenever $\tau_\mu(.)$ is differentiable at q. This must happen for a sequence of $q \to \infty$ since $\tau_\mu(q)$ is non-decreasing and concave. It seems highly likely to us that these results extend to Gibbs measure on self-conformal sets, but we omit further discussion.

We can now state our main result, which gives a non-trivial upper bound for the Assouad spectrum of F.

Theorem 9.1. *Let F be a self-conformal set in $\mathbb{R}^d$ satisfying the bounded distortion principle and let s and t be as above. Then*

$$\dim_{\mathrm{A}}^\theta F \;\leqslant\; \frac{s - t\theta}{1-\theta}.$$

In certain situations we can say more. For simplicity we will specialise to $d = 1$ and the self-similar setting, although some results also hold in higher dimensions and for self-conformal sets. The main result of [FHOR15] was that for a self-similar subset of the line, either the Assouad and box dimensions coincide, or the Assouad dimension is 1. This was generalised to the self-conformal setting in [AT16]; see also [KR16]. The precise condition determining this dichotomy is also known to be the *weak separation condition*: if the weak separation condition is satisfied, then the Assouad, Hausdorff and box dimensions of F coincide, and otherwise the Assouad dimension is 1. The weak separation condition was introduced by Lau and Ngai [L99] and developed by Zerner [Z96]. For the formal definition and basic properties we refer the reader to [Z96] but, roughly speaking, it means the overlaps are 'simple' in the sense that either small cylinders overlap exactly or are somewhat separated.

A famous folklore conjecture on self-similar sets is that if the semigroup generated by the defining maps is free (i.e., there are no exact overlaps), then the Hausdorff and box dimension of the associated self-similar set should be given by $\min\{1, s\}$. Significant progress has been made on this conjecture in recent years, in particular through the work of Hochman [H14]. Hochman proved that if the dimension is not given by the expected value, then there must be 'super-exponential concentration of cylinders'. The only known mechanism for such concentration is exact overlaps and if the parameters defining the maps in the IFS are algebraic then the two phenomena are equivalent.

Shmerkin [S16] recently extended Hochman's work to include the L^q-spectrum, which in turn allows us to obtain a precise result in an important and general setting. For self-similar sets the measure μ is self-similar and it therefore follows from [S16, Theorem 6.6] that the L^q-spectrum of μ is affine and given by $\tau_\mu(q) = s(q-1)$, provided the IFS does not have 'super-exponential concentration of cylinders'. In particular, this implies that $t = s$. For the precise definition of 'super-exponential concentration of cylinders' we refer the reader to [H14, S16]. Roughly speaking it means that sequences of construction cylinders of level n cluster towards each other super-exponentially fast as $n \to \infty$. We can glean the following corollary by combining our Theorem 9.1 with the work of Hochman [H14] and Shmerkin [S16] and the main result of [FHOR15] (for a self-similar subset of the line, either the Assouad and box dimensions coincide, or the Assouad dimension is 1).

Corollary 9.2. *Let $F \subset \mathbb{R}$ be a self-similar set. If the IFS does not have 'super-exponential concentration of cylinders', then for all $\theta \in (0,1)$*

$$\dim_{\mathrm{A}}^{\theta} F \;=\; \overline{\dim}_{\mathrm{B}} F.$$

In particular, this is satisfied if the semigroup generated by the defining maps is free and the parameters defining the maps are algebraic. If the semigroup is not free, but the weak separation property is satisfied, then for all $\theta \in (0,1)$

$$\dim_{\mathrm{A}}^{\theta} F \;=\; \overline{\dim}_{\mathrm{B}} F \;=\; \dim_{\mathrm{A}} F.$$

Finally we note that if a self-similar measure can be written as an infinite iterated convolution of a fixed probability measure, then one may write it as the convolution of: a self-similar measure with no overlaps, and another measure. Then, using the fact that taking convolutions does not lower the value of t, one can conclude that $t > 0$. This means that we get a genuinely better upper bound than that given by Proposition 3.1 in the region:

$$\frac{s - \overline{\dim}_{\mathrm{B}} F}{t} < \theta < \frac{1-s}{1-t}$$

which is non-empty provided

$$t > \frac{s - \overline{\dim}_{\mathrm{B}} F}{1 - \overline{\dim}_{\mathrm{B}} F}.$$

In particular, if $s = \overline{\dim}_{\mathrm{B}} F$ then $t > 0$ guarantees an improved bound. For more details of the convolution argument, see [FJ16, Lemma 2.4]. We remark here that in [GH17], Corollary 9.2 can be improved by dropping the 'super-exponential concentration' condition.

It would clearly be valuable to find non-trivial *lower* bounds (if they exist) or to compute the spectrum explicitly for specific examples. Two specific examples come to mind: the example from [F14, Section 3.1] where the overlaps are complicated by the fact that two of the maps share a fixed point; and the example considered by Bandt and Graf [B92, Section 2 (5)] where the translations are chosen carefully to ensure that the weak separation condition fails despite all the contraction ratios being the same.

Question 9.3. *What is the Assouad spectrum for general self-similar and self-conformal sets?*

Given Corollary 9.2 and the various folklore conjectures on self-similar sets and measures with overlaps, we conjecture that in fact for any self-similar and self-conformal set the Assouad spectrum coincides with the upper box dimension.

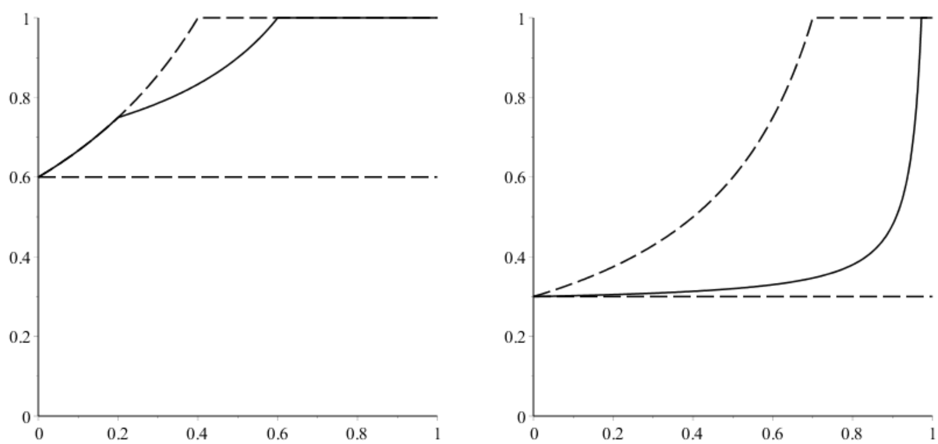

Figure 7: Two plots showing improvements on the general upper bound from Proposition 3.1. In both cases the weak separation property is assumed to fail, rendering the Assouad dimension equal to 1. On the left we have chosen $t = 0.5$, $\overline{\dim}_{\mathrm{B}} F = 0.6$, $s = 0.7$ and on the right we have chosen $t = 0.28$, $\overline{\dim}_{\mathrm{B}} F = s = 0.3$. The familiar upper and lower bounds on the Assouad spectrum from Proposition 3.1 are shown with dashed lines.

9.1 Proof of Theorem 9.1

Let $x \in F$, $R \in (0,1)$ and $\theta \in (0,1)$. For $\delta \in (0,1)$, let $\mathcal{I}(\delta) \subset \mathcal{I}^*$ be the δ-*stopping*, defined by

$$\mathcal{I}(\delta) \;=\; \left\{ \mathbf{i} \in \mathcal{I}^* \;:\; \mathrm{Lip}(\mathbf{i}) \leqslant \delta < \mathrm{Lip}(\mathbf{i}^\dagger) \right\}$$

where $\mathbf{i}^\dagger$ is $\mathbf{i}$ with the final letter removed. Let

$$M(x,R) \;=\; \#\left\{ \mathbf{i} \in \mathcal{I}(R) \;:\; S_{\mathbf{i}}(F) \cap B(x,R) \neq \emptyset \right\}.$$

All of the sets $S_{\mathbf{i}}(F)$ contributing to $M(x,R)$ lie completely inside the ball $B(x, aR)$ for some fixed constant $a > 1$ which is independent of R and x. This follows from the bounded distortion principle. Furthermore, the sets $S_{\mathbf{i}}(F)$ contributing to $M(x,R)$ all carry (symbolic) weight $m([\mathbf{i}]) \asymp \mathrm{Lip}(\mathbf{i})^s \asymp R^s$ and so

$$\mu(B(x,aR)) \gtrsim M(x,R)\, R^s.$$

It also follows from the definition of t that

$$\mu(B(x,aR)) \lesssim (aR)^{t^-} \lesssim R^{t^-}$$

where, crucially, the implied constants are independent of x, and therefore

$$M(x,R) \lesssim R^{t^- - s}.$$

We wish to cover $B(x,R)$ by sets of diameter less than or equal to $R^{1/\theta}$ and to do this we iterate the construction inside each of the cylinders $S_{\mathbf{i}}(F)$ contributing to $M(x,R)$. More precisely, we consider the stopping $\mathcal{I}(R^{1/\theta-1})$ and then decompose each set $S_{\mathbf{i}}(F)$ individually into the sets

$$\left\{S_{\mathbf{ij}}(F) \ : \ \mathbf{j} \in \mathcal{I}\left(R^{1/\theta-1}\right)\right\}.$$

All of these sets have diameters $\lesssim R^{1/\theta}$ and so can themselves be covered by $\lesssim 1$ many sets of diameter less than or equal to $R^{1/\theta}$. We claim that

$$\left|\mathcal{I}\left(R^{1/\theta-1}\right)\right| \lesssim R^{-s^+(1/\theta-1)}.$$

To see this note that

$$\left|\mathcal{I}\left(R^{1/\theta-1}\right)\right| R^{s^+(1/\theta-1)} \asymp \sum_{\mathbf{j}\in\mathcal{I}\left(R^{1/\theta-1}\right)} \mathrm{Lip}(\mathbf{j})^{s^+} \leqslant \sum_{k=1}^{\infty}\sum_{\mathbf{j}\in\mathcal{I}^k} \mathrm{Lip}(\mathbf{j})^{s^+} < \infty.$$

The fact that the final term is finite follows since

$$\limsup_{k\to\infty} \left(\sum_{\mathbf{j}\in\mathcal{I}^k} \mathrm{Lip}(\mathbf{j})^{s^+}\right)^{1/k} = \exp(P(s^+)) < 1$$

by the definition of s. Thus

$$\begin{aligned} N\left(B(x,R)\cap F,\ R^{1/\theta}\right) \lesssim M(x,R)\left|\mathcal{I}\left(R^{1/\theta-1}\right)\right| &\lesssim R^{t^- - s}\, R^{-s^+(1/\theta-1)} \\ &\leqslant R^{-(s^+/\theta - t^-)} \end{aligned}$$

and so directly from the definition and then letting $t^- \nearrow t$ and $s^+ \searrow s$ we conclude that

$$\dim_{\mathrm{A}}^{\theta} F \leqslant \frac{s/\theta - t}{1/\theta - 1} = \frac{s - t\theta}{1-\theta}$$

as required.

10 Mandelbrot percolation

Mandelbrot percolation, first appearing in the works of Mandelbrot in the 1970s as a model for intermittent turbulence [M74], is one of the most well-studied and famous random fractal constructions and can be defined as follows. Fix $n \geqslant 2$, $d \in \mathbb{N}$ and $p \in (0,1)$. Let $D_0 = [0,1]^d$ and divide D_0 into n^d identical closed smaller cubes by dividing each side into n equal length closed intervals. For each of the n^d small cubes either 'select it' with probability p or 'reject it' with probability $(1-p)$. Do this independently for each cube and let the union of all the selected cubes be denoted by D_1. Assuming D_1 is not empty, then we repeat the 'dividing and selecting' process for each of the cubes making up D_1 *independently*. This process is repeated inductively and, assuming non-extinction, we obtain a nested sequence of compact sets D_n. The intersection of all D_n is denoted by

$$F = \bigcap_{n=0}^{\infty} D_n$$

and is a randomly generated compact subset of the unit cube. The random set F is the set of interest and is the known as the limit set of Mandelbrot percolation.

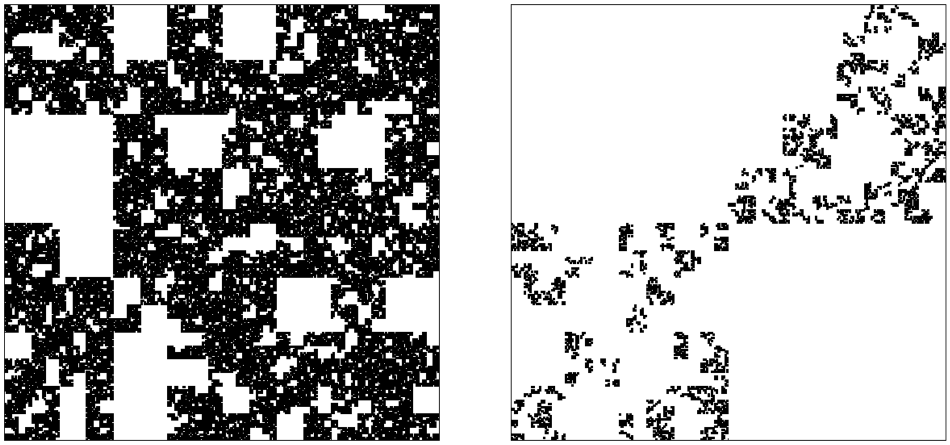

Figure 8: Two examples of the limit sets of Mandelbrot percolation. On the left $n = 2$ and $p = 0.9$ and on the right $n = 2$ and $p = 0.7$.

It is well known that if $p > \frac{1}{n^d}$, then there is a positive probability of non-extinction and, conditioned on non-extinction, almost surely

$$\dim_{\mathrm{B}} F = \dim_{\mathrm{H}} F = \frac{\log pn^d}{\log n}.$$

Using the work of Järvenpää-Järvenpää-Mauldin [JJM02] on upper and lower porosities, it can be shown that if $p > \frac{1}{n^d}$ then, conditioned on non-extinction, almost surely

$$\dim_{\mathrm{A}} F = d \qquad \text{and} \qquad \dim_{\mathrm{L}} F = 0$$

which are notably independent of p. See also Fraser-Miao-Troscheit [FMT14]. Our main result in this chapter gives the almost sure Assouad spectrum of Mandelbrot percolation.

Theorem 10.1. *Suppose $p > \frac{1}{n^d}$. Then, conditioned on non-extinction, almost surely*

$$\dim_{\mathrm{A}}^{\theta} F = \dim_{\mathrm{B}} F = \frac{\log pn^d}{\log n}$$

for all $\theta \in (0, 1)$.

Surprisingly, the Assouad spectrum *does* depend on p and does not reach the Assouad dimension. Since the spectrum is constantly equal to the (upper) box dimension, Mandelbrot percolation provides another family of examples showing that the lower bound in Proposition 3.1 is sharp.

We will provide the proof which appeared originally in [FY16b]. We note here that in [T17], a shorter proof is presented.

10.1 Branching processes

Let X be a non-negative integer-valued random variable with finite k-th moment for any $k \geqslant 0$, that is, $\mathbb{E}[X^k] < \infty$ for any $k \geqslant 0$. We can define a stochastic process Y_n $(n \geqslant 0)$ by letting

$$Y_0 = 1$$

and for any $n \geqslant 1$, we define

$$Y_{n+1} = \sum_{i=1}^{Y_n} X_i,$$

where X_i are independent identically distributed random variables with the same distribution as X. We adopt the convention that $\sum_{i=1}^{0} X_i = 0$. The process Y_n is known as a *Galton-Watson process*, see [K06, Chapter 3]. We will need the following result, which expresses the kth moments of Y_n as an explicit polynomial. Although the result for the first moment is well-known, we could not find a reference for the kth moment and so include the details for completeness.

Theorem 10.2. *For any positive integers n, k we have:*

$$\mathbb{E}[Y_n^k] = \sum_{i=1}^{k} a_{ki} \mu^{in}$$

where $\mu = \mathbb{E}[X]$ and the coefficients a_{ki} can be computed explicitly.

Proof. We will proceed by induction, first noting the well-known case when $k = 1$. For any $n \geqslant 0$ we have

$$\mathbb{E}[Y_{n+1}] = \mathbb{E}\left[\mathbb{E}\left[\sum_{i=1}^{Y_n} X_i \middle| Y_n\right]\right] = \mathbb{E}[X]\mathbb{E}[Y_n] = \mu\mathbb{E}[Y_n] = \cdots = \mu^{n+1}\mathbb{E}[Y_0] = \mu^{n+1}.$$

Suppose that for any n we have

$$\mathbb{E}[Y_n^k] = \sum_{i=1}^{k} a_{ki} \mu^{in} \tag{10.1}$$

for $k < N$ with $N > 0$ an integer. Then

$$\mathbb{E}[Y_{n+1}^{k+1}] = \mathbb{E}\left[\mathbb{E}\left[\left(\sum_{i=1}^{Y_n} X_i\right)^{k+1} \middle| Y_n\right]\right]$$

and we can expand the

$$\left(\sum_{i=1}^{Y_n} X_i\right)^{k+1}$$

into polynomial terms of the following form:

$$X_{i_1}^{k_1} X_{i_2}^{k_2} \dots X_{i_s}^{k_s}$$

where $s \leqslant k+1$ is an integer, $i_1, \ldots, i_s$ are s different integers from 1 to Y_n, and $k_1, \ldots, k_s$ are positive integers which sum to $k+1$. If we write $\mathbb{E}[X^m] = \mu_m$, then by independence we get

$$\mathbb{E}[X_{i_1}^{k_1} X_{i_2}^{k_2} \ldots X_{i_s}^{k_s}] = \mu_{k_1}\mu_{k_2}\ldots\mu_{k_s}$$

and so the expectations do not really depend on the i-s but rather on the k-s. The expansion of

$$\mathbb{E}\left[\left(\sum_{i=1}^{Y_n} X_i\right)^{k+1} \middle| Y_n\right]$$

can be written as the sum of terms of the form:

$$\left(\binom{Y_n}{s} s!\right) \mu_{k_1}\mu_{k_2}\ldots\mu_{k_s}$$

where the binomial coefficients take the value 0 when $s > Y_n$. We can arrange this sum with respect to the powers of Y_n since the binomial coefficients are polynomials in Y_n. We obtain an expression of the form

$$\mathbb{E}\left[\left(\sum_{i=1}^{Y_n} X_i\right)^{k+1} \middle| Y_n\right] = \sum_{i=0}^{k+1} b_i Y_n^i$$

and it is clear that $b_{k+1} = \mu^{k+1}$ since there is only one way to write $k+1$ as the sum of $k+1$ positive integers, and the corresponding value of $\mu_{k_1}\ldots\mu_{k_s}$ is μ^{k+1}. We then have

$$\mathbb{E}\left[\left(\sum_{i=1}^{Y_n} X_i\right)^{N} \middle| Y_n\right] = \sum_{i=0}^{N} b_i Y_n^i = \sum_{i=0}^{N-1} b_i Y_n^i + \mu^N Y_n^N.$$

We will write the expectation of the first term on the right side as

$$K(n) = \mathbb{E}\left[\sum_{i=1}^{N-1} b_i Y_n^i\right] = \sum_{i=1}^{N-1} b_i \mathbb{E}[Y_n^i].$$

Then

$$\begin{aligned}
\mathbb{E}[Y_n^N] &= \mathbb{E}\left[\mathbb{E}\left[\left(\sum_{i=1}^{Y_{n-1}} X_i\right)^{N} \middle| Y_{n-1}\right]\right] \\
&= \sum_{i=0}^{N} b_i \mathbb{E}[Y_{n-1}^i] \\
&= K(n-1) + \mu^N \mathbb{E}[Y_{n-1}^N] \\
&= K(n-1) + \mu^N K(n-2) + \mu^{2N} K(n-3) + \cdots + \mu^{N(n-1)} \mathbb{E}[Y_1^N] \\
&= K(n-1) + \mu^N K(n-2) + \mu^{2N} K(n-3) + \cdots + \mu^{N(n-1)} \mu_N
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{N-1} b_i \left(\mathbb{E}[Y_{n-1}^i] + \mu^N \mathbb{E}[Y_{n-2}^i] + \cdots + \mu^{N(n-2)} \mathbb{E}[Y_1^i]\right) + \mu^{N(n-1)} \mu_N \\
&= \sum_{i=1}^{N-1} \sum_{j=1}^{i} b_i a_{ij} \left(\mu^{j(n-1)} + \mu^{j(n-2)+N} + \cdots + \mu^{j+N(n-2)}\right) + \mu^{N(n-1)} \mu_N \\
&= \sum_{i=1}^{N-1} \sum_{j=1}^{i} b_i a_{ij} \frac{\mu^{j(n-1)} - \mu^{N(n-1)}}{1 - \mu^{N-j}} + \mu^{N(n-1)} \mu_N \\
&= \sum_{j=1}^{N-1} \sum_{i=j}^{N-1} \frac{b_i a_{ij}}{1 - \mu^{N-j}} \mu^{j(n-1)} + \left(\mu_N - \sum_{i=1}^{N-1} \sum_{j=1}^{i} \frac{b_i a_{ij}}{1 - \mu^{N-j}}\right) \mu^{N(n-1)}
\end{aligned}
$$

By our inductive assumption (10.1), all the coefficients of μ^{jn} do not depend on n, therefore we can of course write

$$
\mathbb{E}[Y_n^N] = \sum_{i=1}^{N} a_{Ni} \mu^{in}
$$

for a_{Ni} which do not depend on n, completing the proof. $\square$

In particular, this theorem shows that $\mathbb{E}[Y_n^N]$ can be written as a polynomial in μ^n with degree (at most) N and the coefficients of the polynomial depend on N but not n.

10.2 Proof of Theorem 10.1

Write

$$
B = \frac{\log pn^d}{\log n}
$$

to denote the unique value of the box dimension which occurs with positive probability, and fix $\theta \in (0,1)$. Let $N(s)$ be the number of cubes selected at step $s \in \mathbb{N}$. We call the collection of selected cubes at level s the *s-layer cubes*. They are of side length n^{-s}.

Let $s \in \mathbb{N}$ and $k, M > 0$. We refer to an s-layer cube as a *$G(s,k,M)$-cube* if there are at least M cubes selected at layer $s + [k]$ which lie inside it.

The selected cubes can be considered independently and so the probability of at least one $G(s,k,M)$-cube existing is

$$
1 - (1 - P(s,k,M))^{N(s)},
$$

where $P(s,k,M)$ is the probability that a particular s-layer cube is a $G(s,k,M)$-cube which, by the statistical self-similarity of the process, is the same as the probability that F itself has more than M many k-layer cubes, i.e. $P(N(k) \geqslant M)$.

We can view the construction of the Mandelbrot percolation in the context of the Galton-Watson process discussed in the previous section. In this case the branching process Y_n is constructed from the Binomial random variable $X \sim B(n^d, p)$, where we note that $\mu = n^B$. By the Markov inequality,

$$P(s,k,M) \;=\; P(Y_{[k]} \geqslant M) \;\leqslant\; \frac{\mathbb{E}[Y_{[k]}^N]}{M^N}$$

for any $N \in \mathbb{N}$. Therefore we have the following bound on the probability that there exists at least one $G(s,k,M)$-cube:

$$1-(1-P(s,k,M))^{N(s)} \;\leqslant\; N(s)P(s,k,M) \;\leqslant\; N(s)\frac{\mathbb{E}[Y_{[k]}^N]}{M^N}.$$

However, what we really want to estimate is the probability that there exists at least one $G(s,k,M)$-cube without knowledge of (conditioning on) the value of $N(s)$. This is easily overcome, however, and we obtain the probability

$$\begin{aligned}\sum_{z=0}^{n^{sd}}(1-(1-P(s,k,M))^z)P(N(s)=z) \;&\leqslant\; \sum_{z=0}^{n^{sd}} z\frac{\mathbb{E}[Y_{[k]}^N]}{M^N}P(N(s)=z)\\ &\leqslant\; \frac{\mathbb{E}[Y_{[k]}^N]}{M^N}\mathbb{E}[N(s)]\\ &=\; \frac{\mathbb{E}[Y_{[k]}^N]}{M^N}n^{Bs}.\end{aligned}$$

Therefore, for any layer s, the probability that there exists at least one $G(s, s/\theta - s, n^{sB_1(1/\theta-1)})$-cube can be bounded from above by

$$n^{Bs}\frac{\mathbb{E}[Y_{[s(1/\theta-1)]}^N]}{n^{NB_1 s(1/\theta-1)}}.$$

Here $B_1 > 0$ is a non-specified constant. For large enough s, we have from Theorem 10.2 that

$$\mathbb{E}[Y_{[s(1/\theta-1)]}^N]$$

is a polynomial in $n^{Bs(1/\theta-1)}$ with degree at most N. Therefore we can bound the whole term from above by

$$Ln^{NBs(1/\theta-1)}$$

where L is a positive number depending only on N. We have

$$n^{Bs}\frac{\mathbb{E}[Y_{[s(1/\theta-1)]}^N]}{n^{NB_1 s(1/\theta-1)}} \;\leqslant\; Ln^{s(B+BN(1/\theta-1)-NB_1(1/\theta-1))}$$

which means that if

$$B + BN(1/\theta-1) - NB_1(1/\theta-1) < 0 \tag{10.2}$$

then

$$\sum_{s=1}^{\infty} n^{Bs} \frac{\mathbb{E}[Y^N_{[s(1/\theta-1)]}]}{n^{NB_1 s(1/\theta-1)}} \; < \; \infty.$$

The Borel-Cantelli Lemma then implies that almost surely there are only finitely many layers s such that there exists at least one $G(s, s/\theta - s, n^{sB_1(1/\theta-1)})$-cube. This implies that almost surely $\dim_{\mathrm{A}}^{\theta} F \leqslant B_1^+$, provided (10.2) holds. Therefore, almost surely, we have

$$\dim_{\mathrm{A}}^{\theta} F \leqslant \frac{B\theta}{N(1-\theta)} + B$$

and since N can be chosen arbitrarily large, it follows that almost surely, $\dim_{\mathrm{A}}^{\theta} F \leqslant B$. The reverse inequality holds almost surely, conditioned on non-extinction, due to the general bounds from Proposition 3.1 and the classical dimension result for box dimension. Therefore, for our fixed choice of θ, almost surely, conditioned on non-extinction, we have

$$\dim_{\mathrm{A}}^{\theta} F = B.$$

Finally, since $\theta \in (0,1)$ was arbitrary we can obtain this result simultaneously for all $\theta \in (0,1) \cap \mathbb{Q}$ and since we know the Assouad spectrum is continuous in θ (Corollary 3.5), this is sufficient to move the quantifier on θ inside the almost sure result, which proves the theorem.

11 Moran constructions

Moran constructions are an important and well-studied class of fractals which can be thought of in a similar way to attractors of IFSs but where the rigidity between construction levels has been relaxed in a number of different ways. Similar to IFS attractors, one defines a set as the intersection of a nested sequence of compact sets made up of increasingly small construction cylinders. However, the difference is that the relative position, size and number of 'child' cylinders within their 'parent' can vary. This leads to a richer theory, less rigid construction rules and, as we shall see, more complicated dimension results.

A natural way to code the nested construction cylinders is via an infinite rooted tree. Let T be such a tree with root vertex v_0 on level 0 and such that every vertex on level $n \geqslant 0$ has at least one 'child' vertex at level $n+1$. This guarantees that there are no 'degenerate' paths, i.e., paths starting from v_0 and terminating at some finite level. In general, the 'level' of a vertex is defined uniquely as the graph distance of the vertex to the root v_0. Let E be the set of edges, V be the set of vertices, $l(v) \in \mathbb{N}$ be the level of vertex $v \in V$, and $V(k)$ be the set of level k vertices.

Let $M(v_0) = M_0 \subseteq \mathbb{R}^d$ be a compact convex set which is equal to the closure of its interior, for example we could take M_0 to be a ball or a cube. For each $v \in V \setminus \{v_0\}$ we assign $c(v) \in (0,1)$ and a set $M(v) \subseteq \mathbb{R}^d$ with diameter $|M(v)|$ such that

1. For all $v \in V$, $M(v)$ is similar to M_0, i.e., the image of M_0 under a similarity map.
2. If $v' \in V$ is a child of $v \in V$, then $M(v') \subseteq M(v)$
3. If $v' \in V$ is a child of $v \in V$, then $|M(v')| = c(v')|M(v)|$
4. If $v', v'' \in V$ are distinct children of $v \in V$, then the interiors of $M(v')$ and $M(v'')$ are disjoint

5. There is a uniform constant $c > 0$ such that $c(v) \geqslant c$ for all $v \in \setminus\{v_0\}$.

6. Let $v_0 v_1 \ldots$ be an infinite path starting from the root vertex. Then $\lim_{i\to\infty} |M(v_i)| = 0$.

By construction the sequence $\cup_{v\in V(k)} M(v)$ is a decreasing sequence of non-empty compact sets and so we may define

$$M = \bigcap_{k=1}^{\infty} \bigcup_{v\in V(k)} M(v)$$

which is itself a non-empty compact, typically fractal, set and is the main object of study in this chapter. Such sets are often called *Moran sets* or *Moran constructions.* For convenience, we make a further homogeneity assumption that the contraction constants $c(v)$ are uniform across a given level. More precisely, we assume that for each $k \in \mathbb{N}$, there is a constant $c(k) \in (0,1)$ such that for all $v \in V(k)$ we have $c(v) = c(k)$. In this situation we call the set M a *homogeneous Moran set.*

Before we can state our main result, we need some more notation. For $v \in V(k)$ and $l \geqslant 1$, let $N(v,l)$ be the number of level $k+l$ children of v, i.e., the number of vertices one can get to from v by going a further l levels down the tree. Given $\theta \in (0,1)$ and $k \in \mathbb{N}$ let $l(\theta,k) \in \mathbb{N}$ be defined by

$$l(\theta,k) = \max\left\{ l \in \mathbb{N} \ : \ \prod_{i=1}^{l} c(i) \geqslant \prod_{i=1}^{k} c(i)^{1/\theta} \right\}$$

and observe that $l(\theta,k) \geqslant k$.

Theorem 11.1. *Let M be a homogeneous Moran set. Then*

$$\dim_{\mathrm{A}}^{\theta} M = \limsup_{k\to\infty} \max_{v\in V(k)} \frac{\log N(v, l(\theta,k)-k)}{(1-1/\theta)\log \prod_{i=1}^{k} c(i)}$$

and

$$\dim_{\mathrm{L}}^{\theta} M = \liminf_{k\to\infty} \min_{v\in V(k)} \frac{\log N(v, l(\theta,k)-k)}{(1-1/\theta)\log \prod_{i=1}^{k} c(i)}.$$

Since the dimension spectra are local quantities, it is instructive to make the further simplifying assumption that the growth is also uniform across level. More precisely, suppose that for each $k \in \mathbb{N}$, there is a constant $N(k) \geqslant 1$ such that for all $v \in V(k)$ we have $N(v,1) = N(k)$. In this setting we call M a *uniformly homogeneous Moran set.* This makes describing examples with particular properties more straightforward as one only has to specify the sequences $c(k)$ and $N(k)$.

Corollary 11.2. *Let M be a uniformly homogeneous Moran set. Then*

$$\dim_{\mathrm{A}}^{\theta} M = \limsup_{k\to\infty} \frac{\log \prod_{i=k+1}^{l(\theta,k)} N(i)}{(1-1/\theta)\log \prod_{i=1}^{k} c(i)}$$

and

$$\dim_{\mathrm{L}}^{\theta} M \;=\; \liminf_{k\to\infty} \frac{\log \prod_{i=k+1}^{l(\theta,k)} N(i)}{(1-1/\theta)\log \prod_{i=1}^{k} c(i)}.$$

Moreover, if $c(i) = c \in (0,1)$ *for all* $i \geqslant 1$, *then*

$$\dim_{\mathrm{A}}^{\theta} M \;=\; \frac{\theta}{(\theta-1)\log c} \limsup_{k\to\infty} \frac{1}{k} \log \prod_{i=k+1}^{\lfloor k/\theta \rfloor} N(i)$$

and

$$\dim_{\mathrm{L}}^{\theta} M \;=\; \frac{\theta}{(\theta-1)\log c} \liminf_{k\to\infty} \frac{1}{k} \log \prod_{i=k+1}^{\lfloor k/\theta \rfloor} N(i).$$

11.1 Proof of Theorem 11.1

Let $R, \theta \in (0,1)$ and define $k \in \mathbb{N}$ by

$$k \;=\; \max\left\{ l \in \mathbb{N} \;:\; \prod_{i=1}^{l} c(i) \geqslant R \right\}.$$

Observe that level k cylinders $M(v)$ are images of M_0 scaled down by similarities with contraction ratio $\asymp R$, where the implied constant depends only on the constant c appearing in condition (5) above. This guarantees that any R-ball centered in M will intersect $\lesssim 1$ many level k cylinders. This uses the fact that each level k cylinder contains a ball with radius $\gtrsim R$ where the implied constant depends only on M_0 and by construction these balls may be taken to be disjoint. The conclusion then follows from the doubling property of Euclidean space. Furthermore, a particular level k cylinder $M(v)$ may be broken down into precisely $N(v, l(\theta,k)-k)$ many level $l(\theta,k)$ cylinders. Each of these cylinders is a scaled down copy of M_0 by a factor of

$$\prod_{i=1}^{l(\theta,k)} c(i) \asymp \prod_{i=1}^{k} c(i)^{1/\theta} \asymp R^{1/\theta}$$

and may therefore be covered by $\lesssim 1$ many open balls of diameter $R^{1/\theta}$. Thus we may cover any R-ball with

$$\lesssim \max_{v\in V(k)} N(v, l(\theta,k)-k)$$

many $R^{1/\theta}$-balls. This shows that

$$\begin{aligned}
\dim_{\mathrm{A}}^{\theta} F \;&=\; \limsup_{R\to 0} \sup_{x\in M} \frac{\log N\big(B(x,R)\cap M, R^{1/\theta}\big)}{(1-1/\theta)\log R} \\
&\leqslant\; \limsup_{k\to\infty} \frac{\log\left(\max_{v\in V(k)} N(v, l(\theta,k)-k)\right)}{(1-1/\theta)\log\left(\prod_{i=1}^{k} c(i)\right)}
\end{aligned}$$

which is the desired upper bound. To prove the lower bound, choose a sequence of $k \to \infty$ and corresponding maximising $v \in V(k)$ realising the lim sup above. Observe that the cover constructed above is optimal up to a constant since any ball of diameter $R^{1/\theta}$ used in the cover can cover at most $\lesssim 1$ many level $l(\theta, k)$ cylinders. This again uses the fact that each level $l(\theta, k)$ cylinder contains a ball with radius $\gtrsim R^{1/\theta}$ where the implied constant depends only on M_0 and by construction these balls may be taken to be disjoint. Therefore choosing $x \in M(v) \cap M$ we have

$$N\big(B(x, R) \cap M, R^{1/\theta}\big) \ \gtrsim \ N(v, l(\theta, k) - k) \ = \ \max_{v \in V(k)} N(v, l(\theta, k) - k).$$

This proves the lower bound and completes the proof for the Assouad spectrum. The analogous result for the lower spectrum is proved similarly and we omit the details.

11.2 Examples and applications

Let $M_0 = [0, 1]$, $c(k) \equiv 1/2$ and $N(k) \in \{1, 2\}$ for all $v \in V$. For example, we can think of the construction cylinders $M(v)$ as just the closed dyadic intervals. Let

$$T(k, \theta) = \# \{ i = k + 1, \dots, \lfloor k/\theta \rfloor \ : \ N(i) = 2 \}.$$

It follows from Corollary 11.2 that

$$\dim_{\mathrm{A}}^{\theta} M \ = \ \limsup_{k \to \infty} \frac{T(k, \theta)}{k/\theta - k}$$

and

$$\dim_{\mathrm{L}}^{\theta} M \ = \ \liminf_{k \to \infty} \frac{T(k, \theta)}{k/\theta - k}.$$

This gives us an easy way of building interesting examples simply by specifying the sequence $N(i)$.

Simple computations show that the upper and lower box dimensions of M are given by the, more familiar, upper and lower *asymptotic densities*:

$$\overline{\dim}_{\mathrm{B}} M \ = \ \limsup_{k \to \infty} \frac{\# \{ i = 1, \dots, k \ : \ N(i) = 2 \}}{k}$$

and

$$\underline{\dim}_{\mathrm{B}} M \ = \ \liminf_{k \to \infty} \frac{\# \{ i = 1, \dots, k \ : \ N(i) = 2 \}}{k}$$

and the Assouad and lower dimensions are given by the upper and lower *Banach densities*:

$$\dim_{\mathrm{A}} M \ = \ \limsup_{n \to \infty} \sup_{l \in \mathbb{N}} \frac{\# \{ i = l, \dots, l + n - 1 \ : \ N(i) = 2 \}}{n}$$

and

$$\dim_{\mathrm{L}} M \ = \ \liminf_{n \to \infty} \inf_{l \in \mathbb{N}} \frac{\# \{ i = l, \dots, l + n - 1 \ : \ N(i) = 2 \}}{n}$$

The density functions we study are clearly related to the asymptotic and Banach densities but seem not to be so well-studied. Generally, for $\lambda > 1$ one might define the *upper λ-tail density* of a set $X \subset \mathbb{N}$ as

$$\overline{D}(X, \lambda) \; = \; \limsup_{k\to\infty} \frac{\# X \cap [k, \lambda k]}{\lambda k - k}$$

and the *lower λ-tail density* as

$$\underline{D}(X, \lambda) \; = \; \liminf_{k\to\infty} \frac{\# X \cap [k, \lambda k]}{\lambda k - k}.$$

In the same way that our spectra give finer information regarding the geometric scaling and homogeneity of the fractal set in question, the λ-tail densities would give finer information (via upper and lower spectra) regarding the density of the set X within $\mathbb{N}$. Due to the relationships between the λ-tail densities and our dimension spectra, we can deduce several basic properties of the tail densities immediately from the basic properties of the dimension spectra.

Proposition 11.3. *Let $X \subseteq \mathbb{N}$ and write $\underline{D}(X)$, $\overline{D}(X)$, $\underline{B}(X)$, $\overline{B}(X)$, for the lower and upper asymptotic densities and the lower and upper Banach densities respectively.*

1. *The upper and lower λ-tail densities are continuous as functions of $\lambda > 1$*
2. *For any $\lambda > 1$ we have*

$$\overline{D}(X) \; \leqslant \; \overline{D}(X, \lambda) \; \leqslant \; \frac{\lambda \, \overline{D}(X)}{\lambda - 1} \wedge \overline{B}(X)$$

3. *For any $\lambda > 1$ we have*

$$\frac{\lambda \, \underline{D}(X) - 1}{\lambda - 1} \vee \underline{B}(X) \; \leqslant \; \underline{D}(X, \lambda) \; \leqslant \; \underline{D}(X)$$

Proof. (1) and (2) follow immediately from Corollary 3.5 and Proposition 3.1 and setting $\lambda = 1/\theta$. If either were false, there would exist a set X exhibiting their falsehood and choosing the sequence $N(i)$ to be 2 on elements indexed by X and 1 otherwise would yield a Moran construction with spectra contradicting the basic results we have derived. (3) follows by observing that

$$\underline{D}(X, \lambda) \; = \; 1 - \overline{D}(\mathbb{N} \setminus X, \lambda)$$

and applying (2). □

Similar to Proposition 3.3, we see that if the upper asymptotic density is 0 then the upper λ-tail density is identically 0 and, furthermore, if the lower asymptotic density is 1 then the lower λ-tail density is identically 1.

Returning to the issue of examples, first we give an example illustrating the sharpness of Proposition 3.1, 3.9.

Let $N(i)$ be a sequence of 1s and 2s where the 2s are placed with λ-tail density equal to 1/2 for all $\lambda > 1$, but that there are arbitrarily long sequences of 1s and 2s (thus making the upper and lower Banach densities 1 and 0 respectively). One way to build such a sequence would be to begin with the sequence $1, 2, 1, 2, 1, 2, \ldots$ and then for each $n \in \mathbb{N}$ insert a run of n 1s followed by n 2s starting at position $f(n)$ where $f(n)$ is some sequence which grows very fast, for example $n^{(n^n)}$. It follows that

$$0 = \dim_{\mathrm{L}} M < \dim_{\mathrm{L}}^{\theta} M = \dim_{\mathrm{B}} M = \dim_{\mathrm{A}}^{\theta} M = 1/2 < \dim_{\mathrm{A}} M = 1$$

and so this gives a simple demonstration that the lower (respectively upper) bounds from Proposition 3.1,3.9 are sharp.

In a different direction, if we let M^* be a set produced by following the construction in the previous paragraph of M but with the placement of 1s and 2s reversed, then we get

$$\dim_{\mathrm{A}}^{\theta} M^* \;=\; 1 - \dim_{\mathrm{L}}^{\theta} M$$

and

$$\dim_{\mathrm{L}}^{\theta} M^* \;=\; 1 - \dim_{\mathrm{A}}^{\theta} M$$

with similar formulae relating the Assouad and lower dimensions and the upper and lower box dimensions. This is a very simple observation, but it allows one to build examples with lower spectrum equal to

$$\dim_{\mathrm{L}}^{\theta} M^* \;=\; \frac{\underline{\dim}_{\mathrm{B}} M^* - \theta}{1-\theta} \;\vee\; \dim_{\mathrm{L}} M^*$$

which has not been seen yet and tends not to turn up as naturally as the analogous formula for the Assouad spectrum. Also it means one can produce examples simultaneously for the Assouad and lower spectrum whilst only worrying about the upper densities.

We end this section with a simple recipe for building Moran constructions with interesting spectra, but we leave the details of the calculations to the interested reader. Let $t \in [0, 1]$ and $\lambda > 1$ and choose the sequence $N(i)$ as follows. Every entry is a 1, apart from for a sequence of blocks $f(k), \ldots, [\lambda f(k)]$ where $f(k)$ is a sequence of integers which grows very quickly. Within each block $f(k), \ldots, [\lambda f(k)]$, distribute 2s with density t and as uniformly as possible, i.e. choose $[([\lambda f(k)] - f(k) + 1)t]$ many of the $N(i)$ to be 2, rather than 1. Simple calculations reveal that for the corresponding Moran construction M we have

$$\underline{\dim}_{\mathrm{B}} M = \dim_{\mathrm{L}} M = 0, \qquad \overline{\dim}_{\mathrm{B}} M = \frac{t}{\lambda}(\lambda - 1), \qquad \dim_{\mathrm{A}} M = t$$

and for all $\theta \in (0, 1)$

$$\dim_{\mathrm{A}}^{\theta} M = \frac{\frac{t}{\lambda}(\lambda - 1)}{1-\theta} \wedge t.$$

This simple construction allows us to build a variety of Moran constructions with interesting spectra. For example, we may vary the parameters t, λ above to create finitely many Moran constructions, M_i, with different spectra of the above form. We can then build another Moran construction, M, which is the disjoint union of scaled down copies of the M_i. This can be done by choosing as many first level vertices as there are sets M_i and then mimicking the constructions of each M_i independently within each first level child. Finally, using the fact that the Assouad spectrum is stable under taking finite unions, see Proposition 4.1, the Assouad spectrum of M is given by the maximum of the Assouad spectra of the M_i. Finally, by 'inverting' the construction of each M_i to M_i^* as above, we get a corresponding example with lower spectrum mirroring the Assouad spectrum of the initial example. Observe that this time the lower spectrum of M^* is given by *minimum* of the lower spectra of the M_i^*, see Proposition 4.1. Some examples of this type are given in Figure 9, with the details below. In particular, this recipe allows us to produce examples with the following properties, which we have not observed in any of our previous examples:

1. An arbitrarily large (but finite) number of phase transitions.
2. An arbitrarily large (but finite) number of disjoint intervals where the spectrum is constant (with different constants).
3. As θ approaches 1, the spectra approach intermediate values, not equal to any of the familiar dimensions.

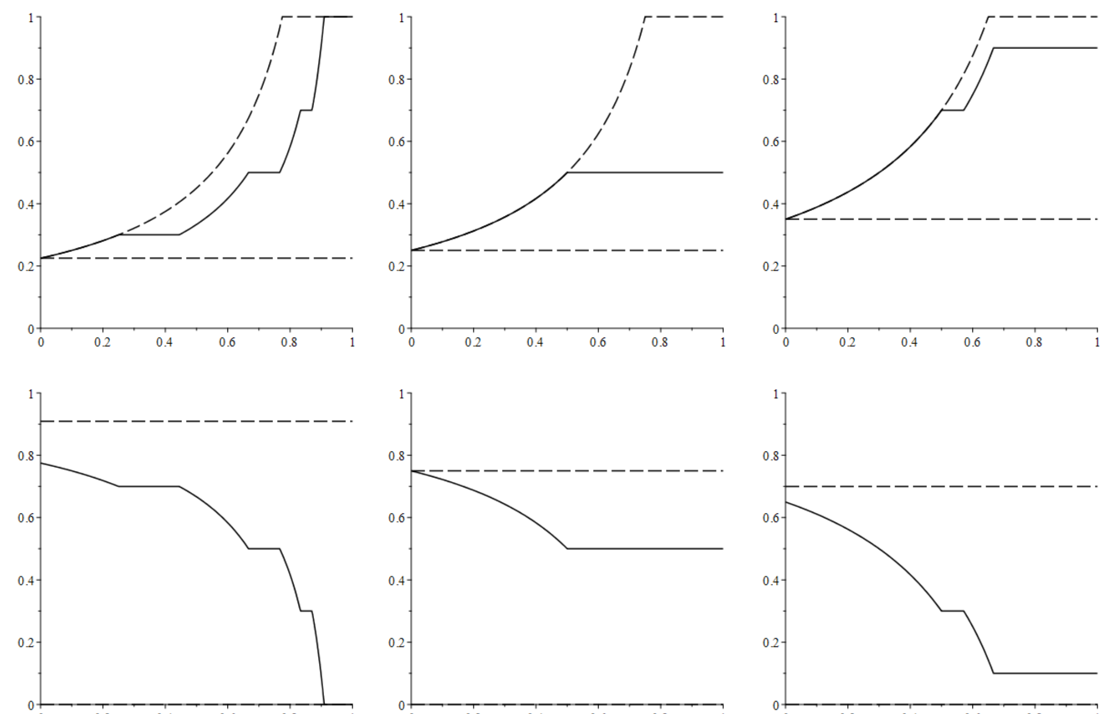

Figure 9: Top row: the Assouad spectra of three different homogeneous Moran constructions. The general bounds from Proposition 3.1 are shown as dashed lines. Bottom row: the lower spectra of the corresponding Moran construction M^*. The general bounds from Proposition 3.9 are shown as dashed lines.

For completeness we provide the precise construction data for the above examples. The first example is produced with four basic Moran constructions with: $t = 1$ and $\lambda = 1.1$, $t = 0.7$ and $\lambda = 1.2$, $t = 0.5$ and $\lambda = 1.5$, and $t = 0.3$ and $\lambda = 4$, respectively. The second example is produced with two basic Moran constructions. The first has $t = 0.5$ and $\lambda = 2$ and the second one is produced by a sequence $N(i)$ which has all 1s apart from arbitrarily long runs of 2s inserted at positions which increase very rapidly. This guarantees that the upper box dimension is 0, but the Assouad dimension is 1. The third example is produced with three basic Moran constructions. The first two have $t = 0.7$ and $\lambda = 2$, and $t = 0.9$ and $\lambda = 1.5$, respectively, and the third one is again the construction with Assouad dimension 1 and upper box dimension 0.

12 Arithmetic progressions and the Assouad dimension

12.1 Arithmetic patches and progressions

An *arithmetic progression* is a finite subset of $\mathbb{R}$ of the form

$$P = \{t + \delta x : x = 0, \ldots, k - 1\}$$

for some $t, \delta > 0$ and some $k \in \mathbb{N}$. Here we say P is an arithmetic progression of length k and gap length δ. Finding arithmetic progressions inside subsets of $\mathbb{R}$ is a topic of great interest in additive combinatorics, number theory and geometry and, in particular, giving conditions which either guarantee the existence of arithmetic progressions or forbid them has attracted much attention. In the discrete setting, Szemerédi's celebrated theorem states that any subset of the natural numbers with positive density necessarily contains arbitrarily long arithmetic progressions. The conclusion of Szemerédi's theorem is known to hold for some sets with zero density, however, such as the primes. This is the content of the Green-Tao Theorem, see [GT08]. In the continuous setting Łaba and Pramanik [LP09, Theorem 1.2] showed that a set contains an arithmetic progression of length 3 if it supports a regular measure with certain Fourier decay properties. Chan, Łaba and Pramanik also considered some higher dimensional analogues in [CLP14]. In the negative direction, Shmerkin [S17] provided examples of compact Salem sets of any dimension $s \in [0, 1]$ which do not contain any arithmetic progressions of length 3, which is in stark contrast to the discrete setting.

The starting point for this work, however, was a recent and elegant observation of Dyatlov and Zahl, which states that any Ahlfors-David regular set of dimension $s < 1$ cannot contain arbitrarily long arithmetic progressions, [DZ16, Proposition 6.13]. Recall that an Ahfors-David regular set of dimension s is a set F such that there exists a constant $C \geqslant 1$ such that for all $x \in F$ and sufficiently small $r > 0$ we have

$$C^{-1}r^s \leqslant \mathcal{H}^s(B(x,r) \cap F) \leqslant Cr^s$$

where $\mathcal{H}^s$ is the s-dimensional Hausdorff measure. If a set is Ahfors-David regular of dimension s, then most of the familiar notions of dimension used to describe fractal sets coincide and equal s. In particular, the Hausdorff, packing, upper box, lower box, lower, and Assouad dimensions are all necessarily equal to s. For a review of dimension theory and the basic relationships between these dimensions, see [F03, M95, R11]. In particular, the Assouad dimension is at least as big as any of the other dimensions listed above.

Our first result, Theorem 12.3, refines the observation of Dyatlov and Zahl by showing that if the Assouad dimension of a set $F \subseteq \mathbb{R}$ is strictly less than 1, then it cannot contain arbitrarily long arithmetic progressions. This is sharp in the sense that the Assouad dimension cannot generally be replaced by any of the smaller dimensions mentioned above. We construct examples to show that the converse of this result is not true, but if one relaxes the definition of arithmetic progressions slightly, then one can obtain an 'if and only if' statement.

Our results also hold in arbitrary finite dimensional real Banach spaces, where the notion of arithmetic progression will be replaced by the appropriate analogue: the arithmetic patch. Let X be a finite dimensional real Banach space with basis $\mathbf{e} = \{e_1, \dots, e_d\}$ for some $d \in \mathbb{N}$. We denote the associated norm by $\| \cdot \|$ and use the norm to induce a metric, a Borel topology, and Lebesgue measure on X, all in the natural way. For $k \in \mathbb{N}$ and $\delta > 0$ we say that a set $P \subset X$ is an *arithmetic patch* of size k and scale δ (with respect to the basis $\mathbf{e}$) if

$$P = \left\{ t \, + \, \delta \sum_{i=1}^{d} x_i e_i \ : \ x_1 = 0, \dots, k-1; \dots ; x_d = 0, \dots, k-1 \right\}$$

for some $t \in X$. In particular, P is a discrete set of cardinality $|P| = k^d$ and, in $\mathbb{R}$, arithmetic patches of size k and scale δ are precisely the arithmetic progressions of length k and gap length δ.

We say that a set $F \subseteq X$ *contains arbitrarily large arithmetic patches* (with respect to the basis $\mathbf{e}$) if for all $k \in \mathbb{N}$ there exists a $\delta = \delta(k)$ and an arithmetic patch $P = P(k, \delta)$ of size k and scale δ (with respect to the basis $\mathbf{e}$) such that $P \subseteq X$.

Since arithmetic patches are finite sets, containing arbitrarily large arithmetic patches is a (strictly) weaker property than the Steinhaus property, which says that a set contains a scaled copy of every finite configuration of points. More precisely, we say that a set $F \subseteq X$ satisfies the *Steinhaus property* if for any finite set $P \subseteq X$, there exists $\delta > 0$ and $t \in X$ such that $t + \delta P \subseteq F$. It is a consequence of the Lebesgue density theorem that any Lebesgue measurable set in $\mathbb{R}^n$ with positive Lebesgue measure satisfies the Steinhaus property.

12.2 Weak tangents

Weak tangents are tools for capturing the local structure of a metric space. First we need a suitable notion of convergence for compact sets, which will be given by the Hausdorff metric. Let $\mathcal{K}(X)$ denote the set of all non-empty compact subsets of X, which is a complete metric space when equipped with the Hausdorff metric $d_{\mathcal{H}}$ defined by

$$d_{\mathcal{H}}(A, B) = \inf\{\delta \ : \ A \subseteq B_\delta \text{ and } B \subseteq A_\delta\}$$

where, for any $C \in \mathcal{K}(X)$,

$$C_\delta \ = \ \{x \in X \ : \ \|x - y\| < \delta \text{ for some } y \in C\}$$

denotes the open δ-neighbourhood of C. We will write $B(0, 1) \subset X$ to denote the closed unit ball.

Definition 12.1. *Let $F \in \mathcal{K}(X)$ and E be a compact subset of the unit ball. Suppose there exists a sequence of similarity maps $T_k : X \to X$ such that $d_{\mathcal{H}}(E, T_k(F) \cap B(0, 1)) \to 0$ as $k \to \infty$. Then E is called a* weak tangent *to F.*

Recall that a similarity map on a metric space is a bi-Lipschitz map from the space to itself where the upper and lower Lipschitz constants are equal. In particular, there is a fixed positive and finite constant depending only on the map such that the map scales all distances uniformly by this constant.

12.3 Weak tangent

One of the most effective ways to bound the Assouad dimension and conformal Assouad dimension of a set from below is to use the weak tangents considered in the previous section. This approach was pioneered by Mackay and Tyson [MT10].

Proposition 12.2. [MT10, Proposition 6.1.5]. *Let $F, E \in \mathcal{K}(X)$ and suppose E is a weak tangent to F. Then $\dim_{\mathrm{A}} F \geqslant \dim_{\mathrm{A}} E$.*

12.4 Results

Our first result gives a necessary condition for a set to contain arbitrarily large arithmetic patches.

Theorem 12.3. *Let F be a non-empty subset of a d-dimensional real Banach space X. If $\dim_{\mathrm{A}} F < d$, then F does not contain arbitrarily large arithmetic patches.*

In fact, Theorem 12.3 will follow from the stronger Theorem 12.6 below. This result is sharp in the sense that the Assouad dimension cannot be replaced by any of the other standard dimension functions. In particular, consider the countable union of arithmetic patches,

$$E = \bigcup_{n\in\mathbb{N}} \left\{ \delta_n \sum_{i=1}^{d} x_i e_i \ : \ x_1 = 0, \ldots, n-1; \ldots; x_d = 0, \ldots, n-1 \right\}$$

where $\delta_n \to 0$ very quickly (exponentially will do). Then it is easy to see that the upper box dimension (which is the second largest of the standard dimensions) of E is equal to 0, despite the set containing arbitrarily large arithmetic patches.

The converse of Theorem 12.3 does not hold, as the following example demonstrates. Let

$$E_p = \{1/n^p : n \in \mathbb{N}\} \subseteq \mathbb{R}$$

for $p > 0$. It is well-known that $\dim_{\mathrm{A}} E_p = 1$ for any $p > 0$, see for example [GHM15]. However, E_p contains no arithmetic progressions of length 3 if $p \geqslant 3$ is an integer. Suppose to the contrary that there exists $a, b, c \in \mathbb{N}$ such that

$$1/a^p - 1/b^p = 1/b^p - 1/c^p.$$

It follows that $(bc)^p - 2(ac)^p + (ab)^p = 0$, which is not possible. This was proved by Darmon and Merel [DM97], and was originally the content of Dénes' Conjecture. Arithmetic progressions of length 3 *are* possible within the squares (and hence within the reciprocals of the squares), but arithmetic progressions of length 4 are not possible. This last fact was proved by Euler; see [R97] for a lucid summary of such results.

Essentially, the above examples highlight that the presence of arithmetic patches is more rigid than having maximal dimension. If we relax the condition slightly, then we can obtain a partial converse.

Definition 12.4. *We say that F* asymptotically contains arbitrarily large arithmetic patches *if, for all $k \in \mathbb{N}$ and $\varepsilon > 0$, there exists $\delta > 0$ and an arithmetic patch P of size k and scale δ and a set $E \subseteq F$ such that*

$$d_{\mathcal{H}}(E, P) \leqslant \varepsilon\delta.$$

In the above definition, $\varepsilon > 0$ can be chosen to depend on k if one so wishes. We also consider an 'asymptotic' or 'approximate' version of the Steinhaus property.

Definition 12.5. *We say that F satisfies the* asymptotic Steinhaus property *if, for all finite sets $P \subset X$ and all $\varepsilon > 0$, there exists $\delta > 0$, $t \in X$ and $E \subseteq F$ such that*

$$d_{\mathcal{H}}(E, t + \delta P) \leqslant \varepsilon\delta.$$

This time $\varepsilon > 0$ can be chosen to depend on P if one so wishes. Satisfying the asymptotic Steinhaus property clearly implies that a set asymptotically contains arbitrarily large arithmetic patches, but we shall see that these two properties are actually equivalent. We can now state our main result.

Theorem 12.6. *Let F be a non-empty subset of a d-dimensional real Banach space X. Then the following are equivalent:*

1. *F asymptotically contains arbitrarily large arithmetic patches,*
2. *F satisfies the asymptotic Steinhaus property,*
3. *F has maximal Assouad dimension, i.e. $\dim_A F = d$,*
4. *F has a weak tangent with non-empty interior,*
5. *$B(0, 1)$ is a weak tangent to F.*

The hardest part of proving this theorem is establishing the implication 3. ⇒ 5., which we will do in Section 12.6.1. We will prove that 5. ⇒ 2. in Section 12.6.2. Recalling that 2. ⇒ 1. is trivial, we will prove that 1. ⇒ 4. in Section 12.6.3. The implication 4. ⇒ 5. ⇒ 3. is given by Proposition 12.2 and the basic fact that sets with non-empty interior have full Assouad dimension, see [MT10].

Note that Theorem 12.3 follows immediately from Theorem 12.6 since 'asymptotically containing arbitrarily large arithmetic patches' is a weaker property than 'containing arbitrarily large arithmetic patches'. Indeed the following strengthening of Theorem 12.3 follows immediately from Theorem 12.6.

Corollary 12.7. *Let F be a non-empty subset of a d-dimensional real Banach space X. If $\dim_A F < d$, then F does not asymptotically contain arbitrarily large arithmetic patches.*

Up until now, we have been working with a fixed Banach space with a fixed basis. It is straightforward to show, however, that this is not necessary in Theorem 12.6.

Corollary 12.8. *Let F be a non-empty subset of a d-dimensional real vector space X. If the statements in Theorem 12.6 hold for F for a particular choice of basis and norm, then the statements hold for F simultaneously for* any *choice of basis and norm.*

Proof. Property 3. (having full Assouad dimension) only depends on the choice of norm, but, since every norm on a finite dimensional real vector space is equivalent, this property is clearly independent of basis *and* norm. Moreover, since for a given choice of basis and norm, 1. is equivalent to the other statements, this independence passes to the other statements too. □

In particular, the property 'asymptotically containing arbitrarily large arithmetic patches' is both base and norm independent. This is not generally true of the more rigid property 'containing arbitrarily large arithmetic patches' provided $d \geqslant 2$ as the following example shows. (It *is* clearly independent of base and norm if $d = 1$.) Let $X = \mathbb{R}^2$ and consider the standard basis $\{(0, 1), (1, 0)\}$. Build a set $E \subset [0, 1]^2$ by, for every $k \in \mathbb{N}$, adding an arithmetic patch (in this case a discrete $k \times k$ grid) such that all points land on dyadic rationals. By definition E contains arbitrarily large arithmetic patches. Observe that the 'direction set' generated by E is countable, i.e.

$$\mathrm{Dir}(E) \;=\; \left\{ \frac{x-y}{\|x-y\|} \;:\; x, y \in E \right\} \;\subseteq\; S^1$$

is a countable subset of the circle. Now consider the basis $\{(0,1), e_2\}$, where $e_2 \in S^1 \setminus \mathrm{Dir}(E) \neq \emptyset$. By the choice of e_2, E cannot contain an arithmetic patch of size 2 with respect to this new basis.

12.5 Applications to sets of integers

In this sections we present two applications of our results to sets of integers. In particular we prove that prime powers asymptotically contain arbitrarily long arithmetic progressions (Corollary 12.11) and we also prove a weak version of the Erdös-Turán conjecture on arithmetic progressions (Theorem 12.13).

First of all, the following formulation of Theorem 12.6 for sets of integers follows immediately from Theorem 12.6 and could be viewed as an asymptotic or approximate version of Szemerédi's Theorem.

Corollary 12.9. *Let $F \subseteq \mathbb{Z}$. Then the following are equivalent:*

1. $\dim_{\mathrm{A}} F = 1$,
2. $\dim_{\mathrm{A}}(1/F) = 1$, *where* $1/F = \{1/n : n \in F \setminus \{0\}\}$,
3. *for all $k \in \mathbb{N}$ and $\varepsilon > 0$, there exists $\delta \in \mathbb{N}$ such that one may form an arithmetic progression of length k and gap length δ by moving elements in F by less than $\varepsilon\delta$.*

Proof. Equivalence of 1. and 2. follows by the simple fact that Assouad dimension is preserved under the Möbius transformation $x \mapsto 1/x$, see [L98, Theorem A.10]. The equivalence of 1. and 3. follows immediately from Theorem 12.6. □

In particular, $F \subseteq \mathbb{Z}$ may have full Assouad dimension, but zero upper Banach density (and so Szemerédi's Theorem does not apply directly). Such examples include: the set $\{n^m : n \in \mathbb{N}\}$ for any $m > 1$, the primes, the set of prime powers $\{p^m : p \text{ prime}\}$ for any $m \geqslant 1$, and 'large' sets in the sense of Erdös-Turán. These last examples are particularly interesting because it is known that there do not exist arbitrarily long arithmetic progressions inside powers of integers, let alone powers of primes, and the strict Erdös-Turan conjecture is a wide open problem in number theory. As such we include the details.

We first require the following technical lemma.

Lemma 12.10. *For any $m \geqslant 1$, we have*

$$\dim_{\mathrm{A}} \{1/p^m : p \text{ prime}\} = 1.$$

We will prove Lemma 12.10 in Section 12.6.4. We actually prove a stronger result. Specifically, we show that there is an infinite subset of the primes which grows polynomially and whose reciprocals form a sequence with decreasing gaps. This will use recent work on gaps in the primes by Baker, Harman and Pintz [BHP01].

Corollary 12.11. *Let $m \geqslant 1$ and consider the set of mth powers of primes $\mathbb{P}^m = \{p^m : p \text{ prime}\}$. For all $k \in \mathbb{N}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that one may form an arithmetic progression of length k and gap length δ by moving elements in $\mathbb{P}^m$ by less than $\varepsilon\delta$.*

This result follows immediately from Corollary 12.9 and Lemma 12.10. One may also obtain higher dimensional analogues of Theorem 12.11 for sets such as $\mathbb{P}^2 \times \mathbb{P}^2 \subseteq \mathbb{Z}^2$ or even

$$\prod_{i=1}^{d} \mathbb{P}^{m_i} \subseteq \mathbb{R}^d$$

for any set of reals $m_i \geqslant 1$, but we leave the precise formulations to the reader.

The Erdös-Turán conjecture on arithmetic progressions is a famous open problem in number theory dating back to 1936 [ET36]. It states that if $F = \{a_n\}_{n\in\mathbb{N}} \subseteq \mathbb{N}$ where a_n is a strictly increasing sequence of positive integers such that

$$\sum_{n=1}^{\infty} 1/a_n \; = \; \infty, \tag{12.1}$$

then F should contain arbitrarily long arithmetic progressions. Sets of integers which satisfy (12.1) are called *large*. We again begin with a technical lemma concerning Assouad dimension.

Lemma 12.12. *If $F \subseteq \mathbb{N}$ is large, then $\dim_{\mathrm{A}} \{1/x : x \in F\} = 1$.*

We will prove Lemma 12.12 in Section 12.6.5. As an immediate consequence of Corollary 12.9 and Lemma 12.12 we obtain the following weak solution to the Erdös-Turán conjecture:

Theorem 12.13. *If $F \subseteq \mathbb{N}$ is large, then for all $k \in \mathbb{N}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that one may form an arithmetic progression of length k and gap length δ by moving elements in F by less than $\varepsilon\delta$.*

12.6 Remaining proofs

12.6.1 Full dimension guarantees the unit ball is a weak tangent

In this section we will prove that 3. $\Rightarrow$ 5. in the statement of Theorem 12.6. Let F be a set with $\dim_{\mathrm{A}} F = d$. Let X be a real Banach space with basis $\{e_1, \dots, e_d\}$. For a given $R > 0$, let $\mathcal{Q}(R)$ be the natural tiling of X consisting of the basic set

$$\left\{ R \sum_{i=1}^{d} x_i e_i \; : \; x_1 \in [0,1], \dots, x_d \in [0,1] \right\}$$

and translations thereof by elements of the subgroup $\langle Re_i : i = 1, \dots, d\rangle$ of $(X, +)$. By *tiling* we mean that the union over all $Q \in \mathcal{Q}(R)$ is the whole of X and distinct members of $\mathcal{Q}(R)$ intersect in a set of measure zero (a face of dimension $\leqslant d-1$). For $r > 0$ and a given $Q \in \mathcal{Q}(R)$, let

$$M_r(Q) \;=\; \#\left\{Q' \in \mathcal{Q}(r) : Q' \subseteq Q \text{ and } Q' \cap F \neq \emptyset\right\}.$$

If we replace the term $N(B(x,R) \cap F, r)$ in the definition of Assouad dimension with $M_r(Q)$ and then take supremum over $Q \in \mathcal{Q}(R)$, rather than $x \in F$, then it is easily seen that one obtains an equivalent definition. This is essentially because basic sets $Q \in \mathcal{Q}(R)$ are both contained in a ball of radius comparable to R and contain a ball of radius comparable to R. It is also sufficient to only consider scales R and r which are dyadic rationals. This will help to simplify our subsequent calculations because the grids formed by the dyadic rationals fit perfectly inside each other.

Let $F \subseteq X$ be such that $\dim_{\mathrm{A}} F = d = \dim X$. It follows that for all $n \in \mathbb{N}$ there exists dyadic rationals r_n, R_n satisfying $0 < r_n < R_n$ and $\delta(n) := r_n/R_n \to 0$ as $n \to \infty$ and $Q_n \in \mathcal{Q}(R_n)$ such that

$$M_{r_n}(Q_n) \geqslant \delta(n)^{-(d-1/n)}. \tag{12.2}$$

In order to reach a contradiction, assume that $B(0,1)$ is *not* a weak tangent to F. This means that there exists $\varepsilon' > 0$ such that for all balls B we have $d_{\mathcal{H}}(F \cap B, B) > \varepsilon'|B|$, where $|B|$ denotes the diameter of B. This means that we can find a dyadic rational $\varepsilon > 0$ such that for all dyadic rationals $R > 0$ and all $Q \in \mathcal{Q}(R)$, there exists a $Q' \in \mathcal{Q}(\varepsilon R)$ such that $Q' \subset Q$ and $F \cap Q' = \emptyset$. In particular, for all $Q \in \mathcal{Q}(R)$ and all dyadic rationals $r < \varepsilon R$, we can guarantee that

$$\begin{aligned} M_r(Q) \;&\leqslant\; \#\{Q'' \in \mathcal{Q}(r) : Q'' \subseteq Q\} \;-\; \#\{Q'' \in \mathcal{Q}(r) : Q'' \subseteq Q'\} \\ &=\; \left(\frac{R}{r}\right)^d - \left(\frac{\varepsilon R}{r}\right)^d. \end{aligned}$$

Consider $Q_n \in \mathcal{Q}(R_n)$ above and note that by the previous statement we know that

$$M_{r_n}(Q_n) \;\leqslant\; \left(\frac{R_n}{r_n}\right)^d - \left(\frac{\varepsilon R_n}{r_n}\right)^d \;=\; \delta(n)^{-d}\left(1-\varepsilon^d\right)$$

provided $r_n < \varepsilon R_n$. This does not contradict (12.2) alone, which is why we now need to cut out more basic tiling sets of increasingly smaller size. Consider the $(\varepsilon^{-d} - 1)$ tiling sets from $\mathcal{Q}(\varepsilon R_n)$ which we did not cut out from Q_n. Within each of these, we may cut out one tiling set from $\mathcal{Q}(\varepsilon^2 R_n)$ and, provided $r_n < \varepsilon^2 R_n$, this provides the following improved estimate for $M_{r_n}(Q_n)$:

$$\begin{aligned} M_{r_n}(Q_n) \;&\leqslant\; \delta(n)^{-d}\left(1-\varepsilon^d\right) - \left(\varepsilon^{-d}-1\right)\left(\frac{\varepsilon^2 R_n}{r_n}\right)^d \\ &=\; \delta(n)^{-d}\left(1-\varepsilon^d - \varepsilon^{2d}(\varepsilon^{-d}-1)\right). \end{aligned}$$

We can continue this process of 'cutting and reducing' as long as $r_n < \varepsilon^k R_n$. Therefore, if we choose $m = m(n) \in \mathbb{N}$ such that $\varepsilon^{m+1} R_n \leqslant r_n < \varepsilon^m R_n$, then we finally obtain

$$\begin{aligned} M_{r_n}(Q_n) &\leqslant \delta(n)^{-d} \left(1 - \sum_{k=1}^{m} \varepsilon^{kd} (\varepsilon^{-d} - 1)^{k-1}\right) \\ &= \delta(n)^{-d} \left(1 - \frac{1}{\varepsilon^{-d} - 1} \sum_{k=1}^{m} (1 - \varepsilon^d)^k\right) \\ &= \delta(n)^{-d} \left(1 - \varepsilon^d\right)^m \\ &\leqslant \delta(n)^{-d} \left(1 - \varepsilon^d\right)^{\log \delta(n)/\log \varepsilon - 1} \\ &= \frac{\delta(n)^{-d} \delta(n)^{\log(1-\varepsilon^d)/\log \varepsilon}}{1 - \varepsilon^d} \end{aligned}$$

Combining this estimate with (12.2) yields that for all n we have

$$\delta(n)^{1/n} \leqslant \frac{\delta(n)^{\log(1-\varepsilon^d)/\log \varepsilon}}{1 - \varepsilon^d}.$$

This is a contradiction since $\delta(n) \to 0$ as $n \to \infty$.

12.6.2 The unit ball being a weak tangent implies the asymptotic Steinhaus property

In this section we will prove that 5. $\Rightarrow$ 2. in the statement of Theorem 12.6. Fix a finite set $P \subseteq X$ with at least 2 points (otherwise the result is trivial) and $\varepsilon > 0$. Suppose that $F \subseteq X$ is such that the unit ball $B(0,1)$ is a weak tangent to F. This means that for any $n \in \mathbb{N}$ we can find a ball B_n such that $d_{\mathcal{H}}(F \cap B_n, B_n) \leqslant |B_n|/n$. Choose

$$n > \frac{|P|}{\varepsilon}$$

where $|P|$ is the (necessarily positive and finite) diameter of P and let

$$\delta = \frac{|B_n|}{|P|} > 0.$$

This choice of δ guarantees that we can find $t \in X$ such that $t + \delta P \subseteq B_n$. One then observes that

$$\inf_{E \subseteq F} d_{\mathcal{H}}(E, t + \delta P) \leqslant d_{\mathcal{H}}(F \cap B_n, B_n) \leqslant \frac{|B_n|}{n} < \frac{\varepsilon |B_n|}{|P|} = \varepsilon \delta$$

which completes the proof.

12.6.3 Arbitrarily large patches asymptotically implies weak tangent with interior

In this section we will prove that 1. $\Rightarrow$ 4. in the statement of Theorem 12.6. More specifically, we will prove that

$$C \;=\; \left\{ \frac{\sum_{i=1}^{d} x_i e_i}{2\sum_{i=1}^{d} \|e_i\|} \;:\; x_1 \in [0,1], \ldots, x_d \in [0,1] \right\} \;\subseteq\; B(0,1)$$

is contained in some weak tangent of F. Since F asymptotically contains arbitrarily large arithmetic patches we know that for all $k \in \mathbb{N}$ ($k \geqslant 2$), there exists $\delta > 0$ and an arithmetic patch P_k of size k and scale δ and a subset $E_k \subseteq F$ such that

$$d_{\mathcal{H}}(E_k, P_k) \leqslant \delta.$$

Let T_k be the rotation and reflection free similarity which maps the convex hull of P_k to C and consider the sequence

$$T_k(F) \cap B(0,1) \in \mathcal{K}(B(0,1)).$$

Since $(\mathcal{K}(B(0,1)), d_{\mathcal{H}})$ is compact we may extract a convergent subsequence with limit $A \subseteq B(0,1)$. We claim that $C \subseteq A$, which is sufficient to complete the proof. Indeed, since $T_k(E_k) \subseteq T_k(F) \cap B(0,1)$ and

$$\begin{aligned} d_{\mathcal{H}}(T_k(E_k), C) &= \frac{d_{\mathcal{H}}(E_k, T_k^{-1}(C))}{2\delta(k-1)\sum_{i=1}^{d}\|e_i\|} \\ &\leqslant \frac{d_{\mathcal{H}}(E_k, P_k) + d_{\mathcal{H}}(P_k, T_k^{-1}(C))}{2\delta(k-1)\sum_{i=1}^{d}\|e_i\|} \\ &\leqslant \frac{\delta + \delta\sum_{i=1}^{d}\|e_i\|}{2\delta(k-1)\sum_{i=1}^{d}\|e_i\|} \\ &= \frac{1 + \sum_{i=1}^{d}\|e_i\|}{2(k-1)\sum_{i=1}^{d}\|e_i\|} \\ &\to 0 \end{aligned}$$

as $k \to \infty$, the desired inclusion follows.

12.6.4 The dimension of the primes

In this section we will prove Lemma 12.10. Computing the Assouad dimension of decreasing sequences $\{x_n : x_n \searrow 0\}$ is an interesting problem which has recently been considered in detail by Garcıa, Hare and Mendivil [GHM15]. The problem is greatly simplified if the sequence has *decreasing gaps*, i.e. $x_n - x_{n+1}$ decreases as $n \to \infty$

(or at least eventually decreases). In fact in this case there is a dichotomy: either the sequence decays subexponentially and the Assouad dimension is 1; or the sequence decays at least exponentially and the Assouad dimension is 0, see [GHM15, Proposition 4] and Chapter 5 of this thesis. We will show that there is a subset of the primes whose reciprocals both 'carry the dimension' and have decreasing gaps.

The main result in [BHP01] is that for all $n \geqslant n_0$ there is at least one prime p_n satisfying

$$n \leqslant p_n \leqslant n + O(n^{21/40})$$

for some effective constant n_0. Relying on this result, we may choose an increasing sequence of primes p_k satisfying:

$$k^5 \leqslant p_k \leqslant k^5 + Ck^{21/8}$$

for some absolute constant $C > 0$. We wish to show that the difference between successive gaps

$$G(k) \;:=\; \left(\frac{1}{p_k} - \frac{1}{p_{k+1}}\right) - \left(\frac{1}{p_{k+1}} - \frac{1}{p_{k+2}}\right)$$

is positive for sufficiently large k. We have

$$\begin{aligned}
&p_k p_{k+1} p_{k+2} G(k)\\
&\quad= \; p_{k+1}p_{k+2} + p_k p_{k+1} - 2p_k p_{k+2}\\
&\quad\geqslant \; (k+1)^5(k+2)^5 + k^5(k+1)^5 - 2\left(k^5 + Ck^{21/8}\right)\left((k+2)^5 + C(k+2)^{21/8}\right)\\
&\quad\geqslant \; 30k^8 - O(k^{61/8}) \; > \; 0
\end{aligned}$$

for large k. It follows that $P = \{1/p_k\}$ is a decreasing sequence with eventually decreasing gaps, which makes the Assouad dimension straightforward to calculate. Indeed, since the elements of P have a polynomial lower bound ($1/p_k \geqslant c/k^5$ for some constant c) it follows from [GHM15, Proposition 4] that $\dim_{\mathrm{A}} P = 1$. Moreover, it follows that $P^m = \{p^m : p \in P\} \subseteq [0,1]$ is also a decreasing sequence with eventually decreasing gaps and the terms have a polynomial lower bound (this time $\geqslant c'/k^{5m}$). We may conclude that for any $m \geqslant 1$ we have

$$1 \;\geqslant\; \dim_{\mathrm{A}} \{1/p^m : p \text{ prime}\} \;\geqslant\; \dim_{\mathrm{A}} P^m \;=\; 1$$

which proves the lemma.

12.6.5 The dimension of large sets: proof of Lemma 12.12

Let $F = \{a_n\}_{n\in\mathbb{N}} \subseteq \mathbb{N}$ be large with a_n strictly increasing and let $1/F = \{1/x : x \in F\}$. In order to reach a contradiction, assume that $\dim_{\mathrm{A}}(1/F) < 1$, recalling that F and

$1/F$ necessarily have equal Assouad dimensions. It follows that there exists $s \in (0,1)$ and $C > 0$ such that for all $k \in \mathbb{N}$ and $r \in (0, 2^{-k})$

$$N_r\left(B(0,2^{-k}) \cap 1/F\right) \leqslant C\left(\frac{2^{-k}}{r}\right)^s.$$

If $a_n, a_{n+1} \leqslant 2^{k+1}$ for some $k \in \mathbb{N}$ we have

$$1/a_n - 1/a_{n+1} \geqslant 4^{-(k+1)}$$

and so no open set of diameter $4^{-(k+1)}$ can cover the reciprocals of any two distinct points in $F_k := F \cap [2^k, 2^{k+1}]$. It follows that

$$|F_k| \leqslant N_{4^{-(k+1)}}\left(B(0,2^{-k}) \cap 1/F\right) \leqslant C\left(\frac{2^{-k}}{4^{-(k+1)}}\right)^s = C\,2^{s(k+2)}.$$

Since F is large, we have

$$\infty = \sum_{n=1}^{\infty} 1/a_n \leqslant \sum_{k=1}^{\infty} \sum_{n\,:\,a_n \in F_k} 1/a_n \leqslant \sum_{k=1}^{\infty} |F_k| 2^{-k} \leqslant C2^{2s} \sum_{k=1}^{\infty} 2^{(s-1)k} < \infty$$

since $s < 1$, which is the desired contradiction. Observe that the argument in this section directly shows that the Assouad dimension of the primes is 1, however, it does not say anything directly about sets of prime powers. To deal with these sets, in the previous section we proved that the primes contained a decreasing sequence with decreasing gaps and a polynomial lower bound, which is stronger than having full Assouad dimension.

13 Further discussions

Most of the work related to this thesis was done during the first two years of the author's PhD study. Since then, there was a lot of research focusing on this topic. In this chapter we discuss some further problems as well as some recent results.

Corollary 3.5 and Theorem 3.10 show that all of the spectra we consider are continuous in θ, but many of our examples exhibit phase transitions, preventing the spectra from being any more regular globally. However, in all of our examples the spectra are piecewise analytic and it is natural to ask whether this is always the case, either with finitely or countably many phase transitions. By a recent result in [FHHTY18] we know that this is not true.

In Corollary 3.5 and Theorem 3.10 we proved that the Assouad and lower spectra are Lipschitz when restricted to any closed subinterval of $(0, 1)$. However, we have not ruled out the possibility of examples where the spectra exhibit less regularity on the whole domain.

Question 13.1. *Is it possible for the Assouad and lower spectra to fail to be Lipschitz, or even Hölder, on the whole interval* $(0, 1)$*?*

All the examples we have in this thesis have the property that their Assouad spectra are piecewise convex functions. It is natural to ask whether this should always be the case. Recently, this question was answered negatively in [FHHTY18] by explicitly constructing an example with piecewise concave Assouad spectrum. For lower spectrum, analogous results can be found in [CWC18].

In Proposition 4.3 we proved that the modified lower dimension is stable under finite unions, provided the sets are 'properly separated'. Note that this property does not hold for the lower dimension. We were unable to determine if the 'properly separated' condition can be dropped.

Question 13.2. *Is it true that for subsets E, F of a common metric space, we always have*

$$\dim_{\mathrm{ML}} E \cup F \leqslant \dim_{\mathrm{ML}} E \vee \dim_{\mathrm{ML}} F?$$

The opposite inequality is a trivial consequence of monotonicity.

Recall that in our study of spirals with sub-exponential and monotonic winding we needed to make the additional assumption that the upper box dimension was strictly larger than 1. At first this might seem like a strange assumption, but it is the analogue of assuming that a decreasing sequence has positive box dimension. Indeed, if a set has box dimension 0, then Corollary 3.3 tells us that the Assouad spectrum is constantly equal to 0, thus hiding any strange properties which may occur in that case. There is no analogous result here and it remains an interesting problem to investigate what can happen when the box dimension of a spiral is 1. We suspect that Theorem 6.2 no longer holds and that other phenomena are possible.

Question 13.3. *What can one say in general about the dimension spectra of spirals with sub-exponential and monotonic winding in the case where the box dimension of the spiral is 1?*

Also on the topic of spirals, we proved that if a straight line segment is mapped to a spiral with sub-exponential and monotonic winding, then the Hölder exponent of that map must satisfy certain restrictions based on the upper box dimension of the spiral. It would be interesting to investigate the sharpness of this result. A first step in this investigation could be the following question.

Question 13.4. *Can a spiral with sub-exponential and monotonic winding and with upper box dimension strictly larger than 1 be mapped to line segment by a bi-Hölder map? If so, what are the sharp bounds on the Hölder exponents?*

Once one has a reasonable notion of metric dimension, one may wish to consider how this dimension behaves under canonical geometric operations, such as orthogonal projections or sections (intersections with hyperplanes). This theory is very well-developed for the Hausdorff dimension, starting with the classical paper of Marstrand [M54], see also [M75] for the higher dimensional analogue and the survey papers [FFJ15, M14] for more details and up-to-date references. Roughly speaking, the philosophy behind Marstrand's Theorem and later developments is that if $F \subseteq \mathbb{R}^d$ has 'dimension' $s \in [0, d]$, then the 'dimension' of the projection of F onto hyperplanes of dimension $k < d$ should be almost surely constant, with respect to the natural measure on the Grassmanian manifold. In the case of the Hausdorff dimension, the almost sure value is the largest possible, namely $s \wedge k$. However, for box dimension the almost sure constant is more subtle and given by a *dimension profile*, introduced by Falconer and Howroyd see [FH96, FH97]. Recently Fraser and Orponen [FO17] proved that the Assouad dimension does not follow in the spirit of Marstrand's Theorem in that it can attain multiple values with positive probability (under projection). It would be interesting to consider these results for the Assouad spectrum, since it can be viewed as an interpolation between the Assouad and upper box dimension.

Question 13.5. *For a given $\theta \in (0,1)$, is the Assouad spectrum of given set almost surely constant under projection onto hyperplanes?*

Independent of the answer to the above question, it seems likely that a spectrum of dimension profiles would play a role in the study of how the Assouad spectrum behaves under projection.

A key theme of this thesis has been what happens when one fixes the relationship between the two scales R and r used in the definition of the Assouad dimension. Of course, there are many ways to fix this relationship. Indeed, let $\phi : [0,1] \to [0,1]$ be a decreasing continuous function such that for all $\phi(x) \leqslant x$ for all $x \in [0,1]$. Then one may define the *ϕ-Assouad dimension* to be the analogue where the relationship between the two scales is fixed by always choosing $r = \phi(R)$. We have studied the continuously parameterised family of functions $\phi(x) = x^{1/\theta}$ and it turns out that this really is the 'correct' family to consider in order to develop a rich theory. Indeed it follows from our results that if

$$\frac{\log x}{\log \phi(x)} \to 0 \qquad (x \to 0)$$

then the ϕ-Assouad dimension coincides with the upper box dimension for any totally bounded set. Moreover, if

$$\frac{\log x}{\log \phi(x)} \to 1 \qquad (x \to 0)$$

then the ϕ-Assouad dimension coincides with the Assouad dimension for any set where the Assouad dimension is 'witnessed' by the Assouad spectrum (i.e. the spectrum reaches the Assouad dimension for some $\theta \in (0,1)$). Therefore one will (usually) only obtain a rich theory for functions ϕ which have an intermediate behaviour, which leads one directly to our functions $\phi(x) = x^{1/\theta}$. However, sets for which the Assouad dimension is *not* 'witnessed' by the Assouad spectrum fall through the net in some sense. We propose the following programme to deal with such examples. For functions ϕ defined above, let

$$\dim_{\mathrm{A}}^{\phi} F \;=\; \inf \Bigg\{ \alpha \;:\; (\exists C > 0)\,(\exists \rho > 0)\,(\forall 0 < r \leqslant \phi(R) \leqslant R \leqslant \rho) \\ \sup_{x \in F} N\big(B(x,R) \cap F, r\big) \;\leqslant\; C \left(\frac{R}{r}\right)^{\alpha} \Bigg\}.$$

Notice that this is not quite the definition we alluded to above because we only require $r \leqslant \phi(R)$, and not $r = \phi(R)$. However, this seems more natural for what follows. One now asks the question: how difficult is it to witness the Assouad dimension? More precisely, the problem is to classify for which functions ϕ we have $\dim_{\mathrm{A}}^{\phi} F = \dim_{\mathrm{A}} F$.

In Chapter 8 we computed Assouad type spectra for Bedford-McMullen carpets, which are the simplest, and probably most studied, family of self-affine sets. However the Assouad and lower dimensions are known for considerably more general families, in particular, self-affine classes where the matrices do not need to be constant, see for example the Lalley-Gatzouras class [GL92] or Barański class [B07]. For more details on the Assouad dimensions of these carpets, we refer the reader to [F14, M11]. Also see [FJ16] for results on the Assouad dimension of self-affine carpets with no underlying grid structure. It would be interesting to also compute the Assouad and lower spectra for these more general families, but this seems to be considerably more complicated. We will attempt to explain this extra complication in this section via a simple heuristic, and also try to indicate what different phenomena we expect to find.

We will consider Lalley-Gatzouras carpets, which are similar to the Bedford-McMullen carpets in that they are generated by affine maps based on diagonal matrices and the matrices all contract more strongly in the vertical direction than in the horizontal direction. Moreover, the projection onto the first coordinate is a self-similar set satisfying the open set condition and again this set plays a key role in the dimension theory of the self-affine set. The difference is that the family of matrices need not be constant and so the columns can have varying widths and within each column the rectangles can have varying heights and be distributed less rigidly.

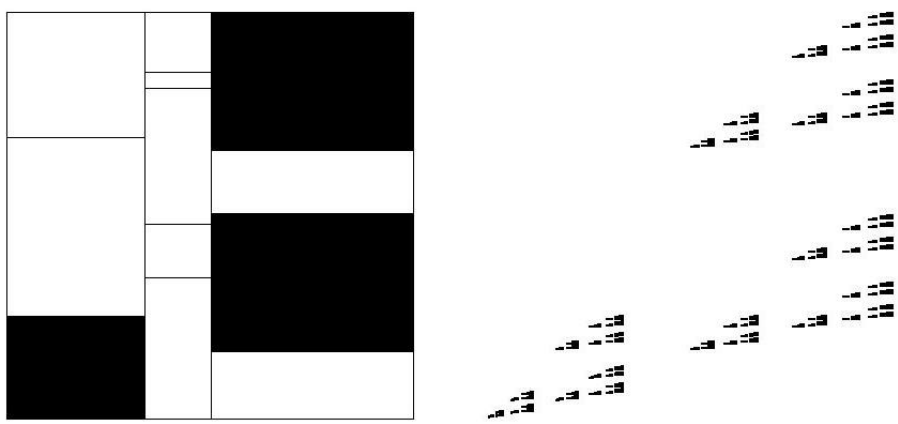

Figure 10: An example of a Lalley-Gatzouras carpet.

The Assouad dimension of a Lalley-Gatzouras carpet is given by a formula similar to that used in the Bedford-McMullen case. Indeed, it is equal to the dimension of the projection onto the first coordinate plus the largest dimension corresponding to a particular column. This is related to 'tangent sets' having a 'product structure', see [M11]. This is also seen when computing the Assouad spectrum of Bedford-McMullen carpets. When $\theta \geqslant \log m/\log n$, the approximate squares have a 'discretised product structure' viewed at the pair of scales R and $R^{1/\theta}$. However, if $\theta < \log m/\log n$, then one has to iterate the construction for a longer time before the height of the basic sets is roughly $R^{1/\theta}$. This means the approximate squares lose their discrete product structure and the box dimension starts to contribute to the formula for the spectrum. We would expect a similar phenomena to occur for Lalley-Gatzouras sets, but different columns may have a different *threshold* for witnessing a product structure. For example, suppose the column with the largest dimension, i.e., the one which contributes to the Assouad dimension consisted only of $1/4 \times 1/8$ rectangles. This would give rise to a discrete product structure (and so the Assouad spectrum would be equal to the Assouad dimension) for $\theta > \log 4/\log 8 = 2/3$, but for smaller values of θ the spectrum would decay towards the box dimension. However, if there was another column which corresponded to a smaller dimension but which consisted only of $1/3 \times 1/9$ rectangles, then this column would allow a discrete product structure to appear on small scales for $\theta > \log 3/\log 9 = 1/2$. This would mean that the Assouad spectrum was at least equal to the dimension of the projection plus the dimension corresponding to the second column for $\theta > 1/2$. We expect that this could lead to phase transitions at $\theta = 1/2$ and $\theta = 2/3$; recall that the Assouad spectrum in the Bedford-McMullen case always has precisely one phase transition. Moreover, by carefully arranging many columns which have smaller dimension as the log-log eccentricity increases could lead to the same phenomena many times, perhaps leading to arbitrarily many phase transitions.

Computing the dimension spectra in this setting seems like a delicate and interesting problem. Even in the simpler subcase (which we allude to here) where the matrices are constant within each column seems challenging and probably a good place to start. The advantage of this case is that each column has a well-defined log-log eccentricity.

Question 13.6. *What are the dimension spectra for the more general self-affine carpets considered by Lalley-Gatzouras and Barański?*

Another possible generalisation of our results on self-affine sets would be to consider the higher dimensional analogue of the Bedford-McMullen carpets; the so-called *self-affine sponges*. These were first considered by Kenyon and Peres [KP96] and recently the Assouad and lower dimensions were computed by Fraser and Howroyd [FH15]. Here the maps act on d-dimensional space (instead of the plane) and the matrices associated to the affine maps are constantly equal to

$$\begin{pmatrix} 1/n_1 & 0 & \dots & 0 \\ 0 & 1/n_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1/n_d \end{pmatrix}$$

for some fixed integers $2 \leqslant n_1 < n_2 < \cdots < n_d$. Again we believe that more complicated spectra are possible, with phase transitions possibly occurring at the $d-1$ (possibly distinct) ratios $\log n_k/\log n_{k+1}$ ($k = 1, \dots, d-1$). Recall that in the (non-uniform fibres) planar case precisely one phase transition occurs at the ratio $\log n_1/\log n_2 = \log m/\log n$.

For random carpets, one can compute the Assouad spectrum generically, see [FT18].

Question 13.7. *What are the dimension spectra for self-affine sponges?*

In Chapter 10 we considered the Mandelbrot percolation. There are many other natural and important methods for producing random fractals and it would be interesting to see if randomness tends to lead to similar behaviour more generally; i.e., the spectrum not reaching the Assouad dimension or even being constantly equal to the box dimension.

One possible interpretation of this is as follows. Once you introduce randomness, the Assouad dimension is generically very large because almost surely you eventually see extremal behaviour. However, for the Assouad spectrum to be large, one needs to control (in some precise sense) how long one has to wait to see the extremal behaviour.

Question 13.8. *What is the Assouad spectrum for natural 'random fractals', for example, the graph of fractional Brownian motion?*

We note here that the Assouad dimension of the graph of fractional Brownian motion was considered in [HY17]. For example, the Assouad dimension of a typical graph of the Wiener process is equal to 2.

Thus it is natural to consider the Assouad type spectra for graphs of another famous class of functions, namely, the Weierstrass type functions. The most general way of constructing such functions is to first choose a 'seed function' $T : \mathbb{R} \to \mathbb{R}$ which is periodic with period 1. Then for a pair of positive numbers a, b we define

$$T_{a,b}(x) = \sum_{i=0}^{\infty} a^i T(b^i x).$$

In general we require that $a < 1, b > 1$. When $T(x) = \sin(2\pi x)$ the above function is the Weierstrass function with parameters (a, b). Now we consider the function T such that $T(x) = 2x$ for $x \leqslant 0.5$ and $1 - 2x$ when $0.5 < x \leqslant 1$ and we extend T periodically to $\mathbb{R}$. In this case $T_{a,b}$ is called the Takagi function with parameters (a, b). We ask here the following question.

Question 13.9. *What is the Assouad spectrum for graphs of Weierstrass type functions?*

In this direction, some partial results are known, for example, in [Y18] the author showed that for some Takagi functions $T_{a,b}$ the Assouad spectrum at $\theta = 1/(2 + \ln a/\ln b)$ of their graphs is strictly larger than their upper box dimensions.

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