SUBSTITUTES FOR THE NON-EXISTENT SQUARE LATTICE DESIGNS FOR 36 VARIETIES

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1 Background

Trials of new crop varieties typically have a large number of varieties. Even at a well-run testing centre, inhomogeneity among the plots (experimental units) makes it desirable to group the plots into homogeneous blocks, usually too small to contain all the varieties. For management reasons, it is often convenient if the blocks can themselves be grouped into replicates, in such a way that each variety occurs exactly once in each replicate. Such a block design is called *resolvable*.

Yates (1936, 1937) introduced *square lattice designs* for this purpose. The number of varieties has the form n^2 for some integer n, and each replicate consists of n blocks of n plots. Imagine the varieties listed in an abstract $n \times n$ square array. The rows of this array form the blocks of the first replicate, and the columns of this array form the blocks of the second replicate.

Let r be the number of replicates. If $r > 2$ then $r-2$ mutually orthogonal Latin squares of order n are needed. For each of these Latin squares, each letter determines a block of size n.

Square lattice designs have (at least) three good properties. One is the fact that adding or removing a replicate gives another square lattice design, which can permit last-minute changes in the number of replicates used. More importantly, Cheng and Bailey (1991) showed that square lattice designs are *optimal* in the sense that they minimize the average variance of the estimators of pairwise variety differences over all designs for n^2 varieties in rn blocks of size n when $r \leq n+1$. Thus the aforementioned addition or removal of a replicate does not result in a poor design.

The *concurrence* of two varieties is the number of blocks in which they both occur. It is widely believed that optimal designs have all concurrences as equal as possible, and so this condition is often used as a proxy for optimality in the search for good designs. In square lattice designs, all concurrences are equal to 0 or 1. If $r = n+1$ then all concurrences are equal to 1 and so the design is balanced.

Unfortunately, there is no pair of mutually orthogonal Latin squares of order six. Hence there are no square lattice designs for 36 varieties in r replicates if $r \geq 4$. However, resolvable designs for such parameters are still needed. Using a computer search, limited to designs in which all concurrences are in $\{0,1,2\}$, Patterson and Williams (1976) gave a good design for 36 varieties in four replicates of six blocks of size six.

2 The new designs

Recently, some good resolvable designs for 36 varieties in blocks of size six have been found as an accidental byproduct of two other pieces of research.

In their search for row–column designs with certain properties, Bailey et al. (2018) worked out a construction using the Sylvester graph. This is a graph with 36 vertices, which may be imagined to be laid out in a 6×6 square array. Each vertex is joined by an edge to five other vertices, one in each row not containing it and one in each column not containing it. So this collection of six vertices is like a letter in a Latin square.

Using properties of the Sylvester graph, which is distance-regular (see Brouwer et al., 1989) and has many symmetries, Bailey et al. (2018) found a set of 6×6 Latin squares which, together with rows

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and columns, can be used to construct resolvable designs in up to eight replicates in just the way that square lattice designs are made. Of course, no pair of the Latin squares are orthogonal to each other, but all concurrences are in $\{0,1,2\}$ and the efficiency factors of the the new designs are very close to the unachievable (if $r > 3$) upper bound given by the non-existent square lattice designs.

Independently of this, Soicher (2013) looked for good $(6 \times 6)/s$ *semi-Latin squares* with $2 \leq s \leq 6$. These are arrangements of $6s$ letters in a 6×6 square array, with s letters per cell, in such a way that each letter occurs once in each row and once in each column. He found a sequence of 6×6 Latin squares L_1, …, L_6 such that, for $2 \leq s \leq 6$, the superimposition of L_1, …, L_s gives a semi-Latin square which is efficient when considered as a block design for $6s$ varieties in blocks of size s. These Latin squares can also be used to copy the square lattice construction.

Inspired by these developments, Emlyn Williams (personal communication) has run a more up-to-date version of his search program on a more up-to-date computer, and thus found resolvable designs for 36 varieties in up to eight replicates of blocks of size six. All concurrences are in $\{0,1,2\}$.

3 Comparison of designs

Two block designs are *isomorphic* if one can be converted into the other by a permutation of varieties and a permutation of blocks. If two designs are isomorphic then their efficiency factors are the same, but the converse may not be true.

For $r = 2$ and $r = 3$ the designs in all three of the new series are square lattice designs. For $4 \leq r \leq 7$ the designs in all three series have efficiency factors not far below the unachievable upper bound. For $r = 8$, they all do better than a balanced square lattice design with one replicate duplicated.

However, for most values of r greater than three, the efficiency factors of the three new designs differ slightly, so no pair of the new designs are isomorphic. For $r = 8$, all three new designs have the same efficiency factor. However, no pair are isomorphic, even though there are permutations of the varieties that convert any one concurrence matrix into either of the other two.

References

Bailey R. A., Cameron P. J., Nilson T. (2018). Sesqui-arrays, a generalisation of triple arrays. *Australasian Journal of Combinatorics*, to appear.

Brouwer A. E., Cohen A. M., Neumaier A. (1989). *Distance-Regular Graphs*, Ergebnisse der Mathematik **318**, Springer: Heidelberg.

Cheng C.-S., Bailey R. A. (1991). Optimality of some two-associate-class partially balanced incomplete-block designs. *Annals of Statistics*, **19**, 1667–1671.

Patterson H. D., Williams E. R. (1976). A new class of resolvable incomplete block designs. *Biometrika*, **63**, 83–92.

Soicher L. H. (2013). Optimal and efficient semi-Latins squares. *Journal of Statistical Planning and Inference*, **143**, 573–582.

Yates F. (1936). A new method of arranging variety trials involving a large number of varieties. *Journal of Agricultural Science*, **226**, 424–455.

Yates F. (1937) A further note on the arrangement of variety trials: quasi-Latin squares. *Annals of Eugenics*, **7**, 319–332.