
# Maximal subsemigroups of finite transformation and diagram monoids

James East, Jitender Kumar, James D. Mitchell, Wilf A. Wilson

PII: S0021-8693(18)30129-7
DOI: https://doi.org/10.1016/j.jalgebra.2018.01.048
Reference: YJABR 16589

To appear in: *Journal of Algebra*

Received date: 10 October 2017

# MAXIMAL SUBSEMIGROUPS OF FINITE TRANSFORMATION AND DIAGRAM MONOIDS

JAMES EAST, JITENDER KUMAR, JAMES D. MITCHELL, AND WILF A. WILSON

Abstract. We describe and count the maximal subsemigroups of many well-known transformation monoids, and diagram monoids, using a new unified framework that allows the treatment of several classes of monoids simultaneously. The problem of determining the maximal subsemigroups of a finite monoid of transformations has been extensively studied in the literature. To our knowledge, every existing result in the literature is a special case of the approach we present. In particular, our technique can be used to determine the maximal subsemigroups of the full spectrum of monoids of order- or orientation-preserving transformations and partial permutations considered by I. Dimitrova, V. H. Fernandes, and co-authors. We only present details for the transformation monoids whose maximal subsemigroups were not previously known; and for certain diagram monoids, such as the partition, Brauer, Jones, and Motzkin monoids.

The technique we present is based on a specialised version of an algorithm for determining the maximal subsemigroups of any finite semigroup, developed by the third and fourth authors, and available in the Semigroups package for GAP, an open source computer algebra system. This allows us to concisely present the descriptions of the maximal subsemigroups, and to clearly see their common features.

## 1. Introduction, definitions, and summary of results

A proper subsemigroup of a semigroup $S$ is *maximal* if it is contained in no other proper subsemigroup of $S$. Similarly, a proper subgroup of a group $G$ is *maximal* if it is not contained in any other proper subgroup of $G$. If $G$ is a finite group, then every non-empty subsemigroup of $G$ is a subgroup, and so these notions are not really distinct in this case. The same is not true if $G$ is an infinite group. For instance, the natural numbers form a subsemigroup, but not a subgroup, of the integers under addition.

Maximal subgroups of finite groups have been extensively studied, in part because of their relationship to primitive permutation representations, and, for example, the Frattini subgroup. The maximal subgroups of the finite symmetric groups are described, in some sense, by the O'Nan-Scott Theorem [38] and the Classification of Finite Simple Groups. Maximal subgroups of infinite groups have also been extensively investigated; see [3, 4, 7, 8, 32, 37] and the references therein.

There are also many papers in the literature relating to maximal subsemigroups of semigroups that are not groups. We describe the finite case in more detail below; for the infinite case see [14] and the references therein. Maximal subgroups of infinite groups, and maximal subsemigroups of infinite semigroups, are very different from their finite counterparts. For example, there exist infinite groups with no maximal subgroups, infinite groups with as many maximal subgroups as subsets, and subgroups that are not contained in any maximal subgroup. Analogous statements hold for semigroups.

In [23], Graham, Graham, and Rhodes showed that every maximal subsemigroup of a finite semigroup has certain features, and that every maximal subsemigroup must be one of a small number of types. As is often the case for semigroups, this classification depends on the description of maximal subgroups of certain finite groups. In [13], Donoven, Mitchell, and Wilson describe an algorithm for calculating the maximal subsemigroups of an arbitrary finite semigroup, starting from the results in [23]. In the

2010 *Mathematics Subject Classification.* (primary) 20M20 and (secondary) 20D30, 20B35.

*Key words and phrases.* maximal subsemigroups, transformation semigroup, diagram monoid, permutation groups, maximal subgroups, graph, maximal independent set, partition monoid.

current paper, we use the framework provided by this algorithm to describe and count the maximal subsemigroups of several families of finite monoids of partial transformations and monoids of partitions. The maximal subsemigroups of many families of transformation monoids have already been described or counted, principally by I. Dimitrova, V. H. Fernandes, and co-authors; see [9, 10, 11, 20, 25] and the references therein. It is possible to recover the previously known results about transformation monoids using the approach we present, and, indeed, to illustrate the usefulness of our technique, we have included full details in a longer version of this paper; see [17].

This paper is structured as follows. In Section 1.1, we describe the notation and definitions relating to semigroups in general that are used in the paper. In Sections 1.2 and 1.3, we define the monoids of transformations and partitions whose maximal subsemigroups we classify. These are monoids of order- and orientation-preserving and -reversing partial transformations; the partition, Brauer, Jones, and Motzkin monoids; and some related monoids. In Section 2, we present several results about the maximal subsemigroups of an arbitrary finite monoid. Many of the results in Section 2 follow from [13], and provide a foundation that is adapted to the specific monoids under consideration in the later sections. In Sections 3 and 4, we classify the maximal subsemigroups of the monoids defined in Sections 1.2 and 1.3, respectively. Table 1 contains the sequences of numbers of maximal subsemigroups of the considered monoids, together with references to the theorems where the maximal subsemigroups are characterised.

1.1. **Background and preliminaries for arbitrary semigroups.** A *semigroup* is a set with an associative binary operation. A *subsemigroup* of a semigroup is a subset that is also a semigroup under the same operation. A subsemigroup of $S$ is *proper* if it does not equal $S$ and it is *maximal* if it is a proper subsemigroup of $S$ that is contained in no other proper subsemigroup of $S$. A *monoid* is a semigroup $S$ with an *identity* element 1, which has the property that $1s = s1 = s$ for all $s \in S$, and a *submonoid* of a monoid $S$ is a subsemigroup that contains 1. For a subset $X$ of a semigroup $S$, the *subsemigroup of $S$ generated by $X$*, denoted by $\langle X \rangle$, is the least subsemigroup of $S$, with respect to containment, containing $X$. More generally, for a collection of subsets $X_1, \ldots, X_m$ of $S$ and a collection of elements $x_1, \ldots, x_n$ in $S$, we use the notation $\langle X_1, \ldots, X_m, x_1, \ldots, x_n \rangle$, or some reordering of this, to denote the subsemigroup of $S$ generated by $X_1 \cup \cdots \cup X_m \cup \{x_1, \ldots, x_n\}$. A *generating set* for $S$ is a subset $X$ of $S$ such that $S = \langle X \rangle$.

Let $S$ be a semigroup. A *left ideal* of $S$ is a subset $I$ of $S$ such that $SI = \{sx : s \in S,\ x \in I\} \subseteq I$. A *right ideal* is defined analogously, and an *ideal* of $S$ is a subset of $S$ that is both a left ideal and a right ideal. Let $x, y \in S$ be arbitrary. The *principal left ideal generated by $x$* is the set $Sx \cup \{x\}$, which is a left ideal of $S$, whereas the *principal ideal generated by $x$* is the set $SxS \cup Sx \cup xS \cup \{x\}$, and is an ideal. We say that $x$ and $y$ are $\mathscr{L}$-related if the principal left ideals generated by $x$ and $y$ in $S$ are equal. Clearly $\mathscr{L}$ defines an equivalence relation on $S$ — called *Green's $\mathscr{L}$-relation* on $S$. We write $x \mathscr{L} y$ to denote that $(x, y)$ belongs to $\mathscr{L}$. Green's $\mathscr{R}$-relation is defined dually to Green's $\mathscr{L}$-relation; Green's $\mathscr{H}$-relation is the meet, in the lattice of equivalence relations on $S$, of $\mathscr{L}$ and $\mathscr{R}$. Green's $\mathscr{D}$-relation is the composition $\mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L}$, and if $x, y \in S$, then $x \mathscr{J} y$ whenever the (two-sided) principal ideals generated by $x$ and $y$ are equal. In a finite semigroup $\mathscr{D} = \mathscr{J}$. We will refer to the equivalence classes of Green's $\mathscr{K}$-relation, where $\mathscr{K} \in \{\mathscr{H}, \mathscr{L}, \mathscr{R}, \mathscr{D}, \mathscr{J}\}$, as $\mathscr{K}$-classes where $\mathscr{K}$ is any of $\mathscr{R}$, $\mathscr{L}$, $\mathscr{H}$, or $\mathscr{J}$, and the $\mathscr{K}$-class of $x \in S$ will be denoted by $K_x$. We write $K_x^S$ when it is necessary to explicitly refer to the semigroup $S$ on which the relation is defined. For a $\mathscr{J}$-class $J$ of $S$ and a Green's relation $\mathscr{K} \in \{\mathscr{H}, \mathscr{L}, \mathscr{R}\}$, we denote by $J/\mathscr{K}$ the set of $\mathscr{K}$-classes of $S$ contained in $J$. A partial order on the $\mathscr{J}$-classes of $S$ is induced by containment of the corresponding principal ideals: for arbitrary elements $x, y \in S$, $J_x \leq J_y$ if and only if the principal ideal generated by $y$ contains the principal ideal generated by $x$. A semigroup $S$ is *$\mathscr{H}$-trivial* if Green's $\mathscr{H}$-relation is the equality relation on $S$.

An *idempotent* is a semigroup element $x$ such that $x^2 = x$. An $\mathscr{H}$-class of $S$ that contains an idempotent is a subgroup of $S$ [28, Corollary 2.2.6]. An element $x \in S$ is *regular* if there exists $y \in S$ such that $xyx = x$, and a semigroup is called *regular* if each of its elements is regular. A $\mathscr{D}$-class is *regular* if it contains a regular element; in this case, each of its elements is regular, and each of its $\mathscr{L}$-classes and $\mathscr{R}$-classes contains an idempotent [28, Propositions 2.3.1 and 2.3.2].

A semigroup $S$ is a *regular $*$-semigroup* [35] if it possesses a unary operation $^*$ that satisfies $(x^*)^* = x$, $(xy)^* = y^*x^*$, and $x = xx^*x$ for all $x, y \in S$. Clearly a regular $*$-semigroup is regular. Throughout this paper, for a subset $X$ of a regular $*$-semigroup, we use the notation $X^*$ to denote $\{x^* : x \in X\}$. For a regular $*$-semigroup $S$ and elements $x, y \in S$, $x\mathscr{R}y$ if and only if $x^*\mathscr{L}y^*$. An idempotent $x$ of a regular $*$-semigroup is called a *projection* if $x^* = x$. The idempotent $xx^*$ is the unique projection in the $\mathscr{R}$-class of $x$, and the idempotent $x^*x$ is the unique projection in the $\mathscr{L}$-class of $x$.

An *inverse* semigroup is a semigroup $S$ in which for each element $x \in S$, there is a unique element $x^{-1} \in S$ that satisfies $x = xx^{-1}x$ and $x^{-1} = x^{-1}xx^{-1}$. With the operation $^*$ on $S$ defined by $x^* = x^{-1}$, an inverse semigroup is a regular $*$-semigroup in which every idempotent is a projection.

An element $x$ in a monoid $S$ with identity 1 is a *unit* if there exists $x' \in S$ such that $xx' = x'x = 1$. The collection of units in a monoid is the $\mathscr{H}$-class of the identity, and is called the *group of units* of the monoid. In a finite monoid, the $\mathscr{H}$-class of the identity is also a $\mathscr{J}$-class; in this case, it is the unique maximal $\mathscr{J}$-class in the partial order of $\mathscr{J}$-classes of $S$.

A *right action* of a group $G$, with identity 1, on a set $X$ is a function $\psi : X \times G \longrightarrow X$ that satisfies $(x, 1)\psi = x$ and $((x, g)\psi, h)\psi = (x, gh)\psi$ for all $x \in X$ and $g, h \in G$. Usually, $(x, g)\psi$ is written $x \cdot g$ when $\psi$ is clear from the context. In this paper, we are concerned with a single type of right action, that of the group of units of a monoid on its set of $\mathscr{L}$-classes by right multiplication. More precisely, if $S$ is any monoid with group of units $G$, $L_s$ is the $\mathscr{L}$-class of $s \in S$, and $g \in G$, then $L_s \cdot g$ is defined to be the $\mathscr{L}$-class $L_{sg}$ of $sg$. This describes a well-defined action of $G$ on the set of $\mathscr{L}$-classes of $S$ because $(s, t) \in \mathscr{L}$ implies $(su, tu) \in \mathscr{L}$ for all $u \in S$, in other words, because $\mathscr{L}$ is a right congruence on $S$. Note that $L_{sg} = L_s g$ since $g$ is a unit. A right action of a group $G$ on a set $X$ partitions $X$ into *orbits*, where $x, y \in X$ belong to the same orbit if and only if there exist $g \in G$ such that $x \cdot g = y$. When such a right action has a unique orbit, the action is said to be *transitive*, and the group acts *transitively*. Left actions, and their orbits can be defined analogously. We are solely concerned with the left action of the group of units of a monoid on its $\mathscr{R}$-classes defined by left multiplication.

We also require the following graph theoretic notions. A *graph* $\Gamma = (V, E)$ is a pair of sets $V$ and $E$, called the *vertices* and the *edges* of $\Gamma$, respectively. An *edge* $e \in E$ is a pair $\{u, v\}$ of distinct vertices $u, v \in V$. A vertex $u$ is *adjacent* to a vertex $v$ in $\Gamma$ if $\{u, v\}$ is an edge of $\Gamma$. The *degree* of a vertex $v$ in $\Gamma$ is the number of edges in $\Gamma$ that contain $v$. An *independent subset* of $\Gamma$ is a subset $K$ of $V$ such that there are no edges in $E$ of the form $\{k, l\}$, where $k, l \in K$. A *maximal independent subset* of $\Gamma$ is an independent subset that is contained in no other independent subset of $\Gamma$. A *bipartite graph* is a graph whose vertices can be partitioned into two independent subsets. If $\Gamma = (V, E)$ is a graph, then the *induced subgraph* of $\Gamma$ on a subset $U \subseteq V$ is the graph $\big(U,\ \big\{\{u, v\} \in E : u, v \in U\big\}\big)$.

In this paper, we define $\mathbb{N} = \{1, 2, 3, \ldots\}$, and where we refer to an ordering of natural numbers, we mean the usual ordering $1 < 2 < 3 < \ldots$.

We repeatedly refer to the *Fibonacci* sequence [39, A000045], $(F_n)_{n\in\mathbb{N}}$, defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. We also define the sequence $(A_n)_{n\in\mathbb{N}}$, by $A_1 = 1$, $A_2 = A_3 = 2$, and $A_n = A_{n-2} + A_{n-3}$ for $n \geq 4$. Note that $A_n$ is the $(n+6)^{\text{th}}$ term of the Padovan sequence [39, A000931].

1.2. **Partial transformation monoids — definitions.** In this section, we introduce the cast of partial transformation monoids whose maximal subsemigroups we determine.

Let $n \in \mathbb{N}$. A *partial transformation of degree $n$* is a partial map from $\{1, \ldots, n\}$ to itself. We define $\mathcal{PT}_n$, the *partial transformation monoid of degree $n$*, to be the monoid consisting of all partial transformations of degree $n$, under composition as binary relations. Let $\alpha \in \mathcal{PT}_n$. We define

$$\operatorname{dom}(\alpha) = \big\{i \in \{1, \ldots, n\} : i\alpha \text{ is defined}\big\}, \quad \operatorname{im}(\alpha) = \big\{i\alpha : i \in \operatorname{dom}(\alpha)\big\}, \quad \text{and } \operatorname{rank}(\alpha) = |\operatorname{im}(\alpha)|,$$

which are the *domain*, *image*, and *rank* of $\alpha$, respectively, and we also define the *kernel* of $\alpha$ to be the equivalence

$$\ker(\alpha) = \big\{(i, j) \in \operatorname{dom}(\alpha) \times \operatorname{dom}(\alpha) : i\alpha = j\alpha\big\}.$$

We define the following monoids:

- $\mathcal{T}_n = \{\alpha \in \mathcal{PT}_n : \operatorname{dom}(\alpha) = \{1, \ldots, n\}\}$, the *full transformation monoid of degree* $n$;
- $\mathcal{I}_n = \{\alpha \in \mathcal{PT}_n : |\operatorname{im}(\alpha)| = |\operatorname{dom}(\alpha)|\}$, the *symmetric inverse monoid of degree* $n$; and
- $\mathcal{S}_n = \{\alpha \in \mathcal{PT}_n : \operatorname{im}(\alpha) = \{1, \ldots, n\}\}$, the *symmetric group of degree* $n$.

The elements of $\mathcal{T}_n$ are called *transformations*, those of $\mathcal{I}_n$ are called *partial permutations*, and $\mathcal{S}_n$ consists of *permutations*. Note that the symmetric group $\mathcal{S}_n$ is the group of units of $\mathcal{PT}_n$, $\mathcal{T}_n$, and $\mathcal{I}_n$.

The full transformation monoids and the symmetric inverse monoids play a role analogous to that of the symmetric group, in that every semigroup is isomorphic to a subsemigroup of some full transformation monoid [28, Theorem 1.1.2], and every inverse semigroup is isomorphic to an inverse subsemigroup of some symmetric inverse monoid [28, Theorem 5.1.7].

Let $\alpha$ be a partial transformation of degree $n$. Then $\operatorname{dom}(\alpha) = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$, for some $i_1 < \cdots < i_k$. We say that $\alpha$ is *order-preserving* if $i_1\alpha \leq \cdots \leq i_k\alpha$, and *order-reversing* if $i_1\alpha \geq \cdots \geq i_k\alpha$. We say that $\alpha$ is *orientation-preserving* if there exists at most one value $l$, where $1 \leq l \leq k-1$, such that $i_l\alpha > i_{l+1}\alpha$, and similarly, we say that $\alpha$ is *orientation-reversing* if there exists at most one value $1 \leq l \leq k-1$ such that $i_l\alpha < i_{l+1}\alpha$.

Having defined these notions, we can introduce the monoids of partial transformations whose maximal subsemigroups we will describe in Section 3. These are:

- $\mathcal{PO}_n = \{\alpha \in \mathcal{PT}_n : \alpha \text{ is order-preserving}\}$ and $\mathcal{O}_n = \mathcal{PO}_n \cap \mathcal{T}_n$;
- $\mathcal{POD}_n = \{\alpha \in \mathcal{PT}_n : \alpha \text{ is order-preserving or -reversing}\}$ and $\mathcal{OD}_n = \mathcal{POD}_n \cap \mathcal{T}_n$;
- $\mathcal{POP}_n = \{\alpha \in \mathcal{PT}_n : \alpha \text{ is orientation-preserving}\}$ and $\mathcal{POPI}_n = \mathcal{POP}_n \cap \mathcal{I}_n$; and
- $\mathcal{POR}_n = \{\alpha \in \mathcal{PT}_n : \alpha \text{ is orientation-preserving or -reversing}\}$ and $\mathcal{PORI}_n = \mathcal{POR}_n \cap \mathcal{I}_n$.

These monoids have been extensively studied; see [10, 11] and the references therein, where the notation used in this paper originates.

We require the groups of units of these monoids. Throughout this paper, we denote the permutation of degree $n$ that reverses the usual order of $\{1, \ldots, n\}$ by $(1\ n)(2\ n-1) \cdots (\lfloor n/2 \rfloor\ \lceil n/2 \rceil)$. We define $\mathcal{C}_n$ to be the cyclic group generated by the $n$-cycle $(1\ 2\ \ldots\ n)$, and

$$\mathcal{D}_n = \langle (1\ 2\ \ldots\ n),\ (1\ n)(2\ n-1) \cdots (\lfloor n/2 \rfloor\ \lceil n/2 \rceil) \rangle.$$

When $n \geq 3$, $\mathcal{D}_n$ is a dihedral group of order $2n$. Note that $\mathcal{C}_2 = \mathcal{D}_2 = \langle (1\ 2) \rangle$.

The monoids $\mathcal{PO}_n$ and $\mathcal{O}_n$ have trivial groups of units; the groups of units of $\mathcal{POD}_n$ and $\mathcal{OD}_n$ are $\langle (1\ n)(2\ n-1) \cdots (\lfloor n/2 \rfloor\ \lceil n/2 \rceil) \rangle$; the groups of units of $\mathcal{POP}_n$ and $\mathcal{POPI}_n$ are $\mathcal{C}_n$; the groups of units of $\mathcal{POR}_n$ and $\mathcal{PORI}_n$ are $\mathcal{D}_n$.

1.3. **Diagram monoids — definitions.** In this section, we define those monoids of partitions whose maximal subsemigroups we determine.

Let $n \in \mathbb{N}$ be arbitrary. A *partition of degree* $n$ is an equivalence relation of the set $\{1, \ldots, n\} \cup \{1', \ldots, n'\}$. An equivalence class of a partition is called a *block*, and a block is *transverse* if it contains points from both $\{1, \ldots, n\}$ and $\{1', \ldots, n'\}$. A *block bijection* is a partition all of whose blocks are transverse, and a block bijection is *uniform* if each of its blocks contains an equal number of points of $\{1, \ldots, n\}$ and $\{1', \ldots, n'\}$.

Let $\alpha$ and $\beta$ be partitions of degree $n$. To calculate the product $\alpha\beta$, we require three auxiliary partitions, each being a partition of a different set. From $\alpha$ we create $\alpha^\vee$ by replacing every occurrence of each $i'$ by $i''$ in $\alpha$, so that $\alpha^\vee$ is a partition of $\{1, \ldots, n\} \cup \{1'', \ldots, n''\}$. Similarly, replacing $i$ by $i''$, we obtain from $\beta$ a partition $\beta^\wedge$ of $\{1'', \ldots, n''\} \cup \{1', \ldots, n'\}$. We define $(\alpha\beta)'$ to be the smallest equivalence on $\{1, \ldots, n\} \cup \{1', \ldots, n'\} \cup \{1'', \ldots, n''\}$ that contains the relation $\alpha^\vee \cup \beta^\wedge$, i.e. the transitive closure of $\alpha^\vee \cup \beta^\wedge$. The product $\alpha\beta$ is the intersection of $(\alpha\beta)'$ and $(\{1, \ldots, n\} \cup \{1', \ldots, n'\}) \times (\{1, \ldots, n\} \cup \{1', \ldots, n'\})$. This operation is associative, and so the collection $\mathcal{P}_n$ of all partitions of degree $n$ forms a semigroup under this operation. The partition whose blocks are $\{i, i'\}$ for all $i \in \{1, \ldots, n\}$ is the identity element of this semigroup, and is called the *identity partition of degree* $n$. Therefore $\mathcal{P}_n$ is a monoid — called the *partition monoid of degree* $n$. A *diagram monoid* is simply a submonoid of $\mathcal{P}_n$ for some $n \in \mathbb{N}$.

Let $\alpha$ be a partition of degree $n$. We define $\alpha^*$ to be the partition of $\{1, \ldots, n\} \cup \{1', \ldots, n'\}$ created from $\alpha$ by replacing the point $i$ by $i'$ in the block in which it appears, and by replacing the point $i'$ by $i$, for all $i \in \{1, \ldots, n\}$. For arbitrary partitions $\alpha, \beta \in \mathcal{P}_n$, $(\alpha^*)^* = \alpha$, $\alpha\alpha^*\alpha = \alpha$, and $(\alpha\beta)^* = \beta^*\alpha^*$. In particular, $\mathcal{P}_n$ is a regular $*$-monoid, as defined in Section 1.1.

There is a canonical embedding of the symmetric group of degree $n$ in $\mathcal{P}_n$, where a permutation $\alpha$ is mapped to the partition with blocks $\{i, (i\alpha)'\}$ for all $i \in \{1, \ldots, n\}$. Since an element of $\mathcal{P}_n$ is a unit if and only if each of its blocks has the form $\{i, j'\}$ for some $i, j \in \{1, \ldots, n\}$, it follows that the image of this embedding is the group of units of $\mathcal{P}_n$. We reuse the notation $\mathcal{S}_n$ to refer to this group.

We define a canonical ordering

$$n' < (n-1)' < \cdots < 1' < 1 < 2 < \cdots < n$$

on $\{1, \ldots, n\} \cup \{1', \ldots, n'\}$. We say that $\alpha \in \mathcal{P}_n$ is *planar* if there do not exist distinct blocks $A$ and $X$ of $\alpha$, and points $a, b \in A$ and $x, y \in X$, such that $a < x < b < y$. For a graphical description of planarity, and of partitions more generally, see [26].

In Section 4, we determine the maximal subsemigroups of $\mathcal{P}_n$ and the following submonoids:

- $\mathcal{PB}_n = \{\alpha \in \mathcal{P}_n : \text{each block of } \alpha \text{ has size at most } 2\}$, the *partial Brauer monoid of degree* $n$, introduced in [34];
- $\mathcal{B}_n = \{\alpha \in \mathcal{P}_n : \text{each block of } \alpha \text{ has size } 2\}$, the *Brauer monoid of degree* $n$, introduced in [34];
- $\mathfrak{F}_n = \{\alpha \in \mathcal{P}_n : \alpha \text{ is a uniform block bijection}\}$, the *uniform block bijection monoid of degree* $n$, or the *factorisable dual symmetric inverse monoid of degree* $n$, see [19] for more details;
- $\mathcal{PP}_n = \{\alpha \in \mathcal{P}_n : \alpha \text{ is planar}\}$, the *planar partition monoid of degree* $n$, introduced in [26];
- $\mathcal{M}_n = \{\alpha \in \mathcal{PB}_n : \alpha \text{ is planar}\}$, the *Motzkin monoid of degree* $n$, see [5] for more details; and
- $\mathcal{J}_n = \{\alpha \in \mathcal{B}_n : \alpha \text{ is planar}\}$, the *Jones monoid of degree* $n$, introduced in [29] and also known as the *Temperley-Lieb monoid*.

Each of these monoids is closed under the $^*$ operation, and is therefore a regular $*$-monoid; furthermore, $\mathfrak{F}_n$ is inverse. The group of units of $\mathcal{PB}_n$, $\mathcal{B}_n$, and $\mathfrak{F}_n$ is $\mathcal{S}_n$, and the group of units of $\mathcal{M}_n$ and $\mathcal{J}_n$ is trivial.

The factorisable dual symmetric inverse monoid $\mathfrak{F}_n$ is a submonoid of the *dual symmetric inverse monoid* $\mathcal{I}_n^*$, which consists of all block bijections of degree $n$. The maximal subsemigroups of $\mathcal{I}_n^*$ are described in [33, Theorem 19].

By [26], the planar partition monoid of degree $n$ is isomorphic to the Jones monoid of degree $2n$. Therefore, we will not determine the maximal subsemigroups of $\mathcal{PP}_n$ directly, since their description can be obtained from the results in Section 4.4.

## 2. The maximal subsemigroups of an arbitrary finite monoid

In this section, we present some results about the maximal subsemigroups of an arbitrary finite monoid, which are related to those given in [13, 23] for an arbitrary finite semigroup. Since each of the semigroups to which we apply these results is a finite monoid, we state the following results in that context. While some of the results given in this section hold for an arbitrary finite semigroup, many of them do not.

Let $S$ be a finite monoid. By [23, Proposition 1], for each maximal subsemigroup $M$ of $S$ there exists a single $\mathscr{J}$-class $J$ that contains $S \setminus M$, or equivalently $S \setminus J \subseteq M$. Throughout this paper, we call a maximal subsemigroup whose complement is contained in a $\mathscr{J}$-class $J$ a *maximal subsemigroup arising from* $J$. In the following lemma, we characterise $\mathscr{J}$-classes that do give rise to maximal subsemigroups.

| Monoid | Group of units | Number of maximal subsemigroups | OEIS [39] | Result |
|---|---|---|---|---|
| $\mathcal{PO}_n$ | Trivial | $2^n + 2n - 2$ | A131520 | [11, Theorem 1] |
| $\mathcal{M}_n$ | | $2^n + 2n - 3$ | A131898 | Theorem 4.9 |
| $\mathcal{O}_n$ | | $A_{2n-1} + 2n - 4$ | A000931 | Theorem 3.7, cf. [9, Theorem 2] |
| $\mathcal{J}_n$ | | $2F_{n-1} + 2n - 3$ | A290140 | Theorem 4.8 |
| $\mathcal{PP}_n$ | | $2F_{2n-1} + 4n - 3$ | A290140 | Theorem 4.8 |
| $\mathcal{POD}_n$ | Order 2 | $2^{\lceil n/2 \rceil} + n - 1$ | A016116 | Theorem 3.3 |
| $\mathcal{OD}_n$ | | $A_n + n - 3$ | A000931 | Theorem 3.8, cf. [25, Theorem 2] |
| $\mathcal{POPI}_n$ | $\mathcal{C}_n$ (cyclic) | $\lvert\mathbb{P}_n\rvert + \lvert\mathbb{P}_{n-1}\rvert$ | A059957 | Theorem 3.10 |
| $\mathcal{POP}_n$ | | $\lvert\mathbb{P}_n\rvert + 2$ | A083399 | Theorem 3.9 |
| $\mathcal{PORI}_n$ | $\mathcal{D}_n$ (dihedral) | $1 + \lvert\mathbb{P}_{n-1}\rvert + \sum_{p \in \mathbb{P}_n} p$ | A290289 | Theorem 3.11 |
| $\mathcal{POR}_n$ | | $3 + \sum_{p \in \mathbb{P}_n} p$ | A008472 | Theorem 3.9 |
| $\mathfrak{F}_n$ | $\mathcal{S}_n$ (symmetric) | $s_n + 1$ | A290138 | Theorem 4.5 |
| $\mathcal{B}_n$ | | $s_n + 1$ | A290138 | Theorem 4.4 |
| $\mathcal{PT}_n$ | | $s_n + 2$ | A290138 | Theorem 3.2 |
| $\mathcal{PB}_n$ | | $s_n + 3$ | A290138 | Theorem 4.3 |
| $\mathcal{P}_n$ | | $s_n + 4$ | A290138 | Theorem 4.2 |

Table 1. The number of maximal subsemigroups of the monoids from this paper, where $n$ is sufficiently large (usually $n \geq 2$ or $n \geq 3$). The maximal subsemigroups themselves are described in the referenced theorems. For $k \in \mathbb{N}$, $s_k$ is the number of maximal subgroups of the symmetric group of degree $k$ [39, A290138]; $\mathbb{P}_k$ is the set of primes that divide $k$; $A_k$ is the $k^{\text{th}}$ term of the sequence defined by $A_1 = 1$, $A_2 = A_3 = 2$, and $A_k = A_{k-2} + A_{k-3}$ for $k \geq 4$; and $F_k$ is the $k^{\text{th}}$ term of the Fibonacci sequence [39, A000045], with $F_1 = F_2 = 1$.

**Lemma 2.1.** *Let $S$ be a finite monoid, and let $J$ be a $\mathscr{J}$-class of $S$. There exist maximal subsemigroups arising from $J$ if and only if every generating set for $S$ intersects $J$ non-trivially.*

*Proof.* Let $M$ be a maximal subsemigroup of $S$ arising from $J$, so that $S \setminus J \subseteq M$. For any subset $A$ of $S$ that is disjoint from $J$, it follows that $\langle A \rangle \leq \langle S \setminus J \rangle \leq M \neq S$, and $A$ does not generate $S$. Conversely, if $J$ intersects every generating set for $S$ non-trivially, then certainly $S \setminus J$ does not generate $S$. Thus the subsemigroup $\langle S \setminus J \rangle$ of $S$ is proper, and is therefore contained in a maximal subsemigroup. □

Thus, given Lemma 2.1, in order to calculate the maximal subsemigroups of $S$, we first identify those $\mathscr{J}$-classes of $S$ that intersect every generating set of $S$ non-trivially. Then we find the maximal subsemigroups of $S$ that arise from each such $\mathscr{J}$-class.

Let $S$ be a finite monoid, let $J$ be a regular $\mathscr{J}$-class of $S$, and let $M$ be a maximal subsemigroup of $S$ arising from $J$. By [13, Section 3], the intersection $M \cap J$ has precisely one of the following forms:

(M1) $M \cap J = \varnothing$.
(M2) $M \cap J$ is a non-empty union of both $\mathscr{L}$- and $\mathscr{R}$-classes of $J$;
(M3) $M \cap J$ is a non-empty union of $\mathscr{L}$-classes of $J$;
(M4) $M \cap J$ is a non-empty union of $\mathscr{R}$-classes of $J$;
(M5) $M \cap J$ has non-empty intersection with every $\mathscr{H}$-class of $J$;

In general, the collection of maximal subsemigroups arising from a particular regular $\mathscr{J}$-class $J$ can have any combination of types (M2), (M3), (M4), and (M5). However, if $S \setminus J$ is a maximal subsemigroup of

$S$, then clearly it is the only maximal subsemigroup to arise from $J$, since a maximal subsemigroup of type (M2)–(M5) is a proper subsemigroup that properly contains $S \setminus J$. In other words, there is at most one maximal subsemigroup of type (M1) arising from $J$, and its existence precludes the occurrence of maximal subsemigroups of types (M2)–(M5); see Proposition 2.6.

It is often most difficult to calculate the maximal subsemigroups of $S$ that arise from $J$ and have type (M5) — we consider a special case in Section 2.2.2, which covers all of the instances in this paper. However, in many cases it can be easily shown that no maximal subsemigroups of type (M5) exist, such as when $S$ is $\mathscr{H}$-trivial, or when $S$ is idempotent generated. More generally, since a maximal subsemigroup of $S$ of type (M5) contains every idempotent of $S$, the following lemma holds.

**Lemma 2.2.** *Let $S$ be a finite monoid with group of units $G$, let $J$ be a $\mathscr{J}$-class of $S$ that is not equal to $G$, and let $E(S)$ be the set of idempotents of $S$. If*

(a) *each $\mathscr{H}$-class of $J$ is trivial, or*
(b) *$J \subseteq \langle G,\ E(S)\rangle$,*

*then there are no maximal subsemigroups of type* (M5) *arising from $J$.*

*Proof.* Let $T$ be an arbitrary subsemigroup of $S$ that contains $S \setminus J$ and intersects each $\mathscr{H}$-class of $J$ non-trivially. A maximal subsemigroup of $S$ of type (M5) that arises from $J$ is a *proper* subsemigroup that satisfies these conditions. Thus, it suffices (in both cases) to prove that $T = S$.

By definition, $T$ contains an element from each $\mathscr{H}$-class of $J$. Thus if each $\mathscr{H}$-class of $J$ is trivial, then $T$ contains $J$, and so $T = S$, proving part (a). To prove part (b), note in particular that $T$ contains one element from each *group* $\mathscr{H}$-class of $J$. Since $S$ is finite and $T$ is closed under multiplication, it follows that $T$ contains the identity element of each such $\mathscr{H}$-class, i.e. $T$ contains every idempotent of $J$. Since $T$ contains $S \setminus J$ and $J \neq G$, it follows that $T$ contains $G$, the idempotents of $S \setminus J$, and the idempotents of $J$, and so $\langle G,\ E(S)\rangle \subseteq T$. Therefore, if $J \subseteq \langle G,\ E(S)\rangle$, then $T$ contains $J$, and so $T = S$, as required. □

2.1. **Maximal subsemigroups arising from the group of units.** The maximal subsemigroups of a finite monoid that are easiest to describe are those that arise from the group of units. Such maximal subsemigroups exist by Lemma 2.1, since the subset of non-units in a finite monoid is an ideal, and so the group of units intersects every generating set of a finite monoid non-trivially. As shown in the following lemma, these maximal subsemigroups can be calculated from the group of units in isolation, without reference to the remainder of the semigroup.

**Lemma 2.3.** *Let $S$ be a finite monoid with group of units $G$. Then the maximal subsemigroups of $S$ arising from $G$ are the sets $(S \setminus G) \cup U$, for each maximal subsemigroup $U$ of $G$. In other words, if $G$ is trivial, then the unique maximal subsemigroup of $S$ arising from $G$ is $S \setminus G$, which has type* (M1)*; if $G$ is non-trivial, then the maximal subsemigroups of $S$ arising from $G$ are the sets $(S \setminus G) \cup U$, for each maximal subgroup $U$ of $G$, which have type* (M5).

*Proof.* Since $S \setminus G$ is an ideal of $S$, it follows that, for a subset $U$ of $G$, $(S \setminus G) \cup U$ is a subsemigroup of $S$ if and only if $U$ is a subsemigroup of $G$. Since this correspondence between subsemigroups of $S$ containing $S \setminus G$ and subsemigroups of $G$ clearly preserves inclusion, the result follows. Note that a *subsemigroup* of a finite group is a *subgroup*, unless it is empty; the only group to possess the empty semigroup as a maximal subsemigroup is the trivial group. □

Only three families of non-trivial groups appear as the group of units of a monoid in this paper: the cyclic groups, the dihedral groups, and the symmetric groups. The conjugacy classes of maximal subgroups of the finite symmetric groups are described in [30] and counted in [31]; see [39, A066115]. However, no simple formula is known for the total number of maximal subgroups. Thus we use the notation $s_k$ to denote the number of maximal subsemigroups of the symmetric group of degree $k$ [39, A290138]. For the maximal subgroups of the cyclic and dihedral groups, we present the following well-known results.

**Lemma 2.4.** *Let $n \in \mathbb{N}$, $n \geq 2$, and let $G = \langle \alpha \mid \alpha^n \rangle$ be a cyclic group of order $n$. The maximal subgroups of $G$ are the subgroups $\langle \alpha^p \rangle$, for each prime divisor $p$ of $n$. In particular, the total number of maximal subgroups is the number of prime divisors of $n$.*

**Lemma 2.5.** *Let $n \in \mathbb{N}$, $n \geq 3$, and let $G = \langle \sigma,\ \rho \mid \sigma^2,\ \rho^n,\ (\sigma\rho)^2 \rangle$ be a dihedral group of order $2n$. The maximal subgroups of $G$ are $\langle \rho \rangle$ and the subgroups $\langle \rho^p,\ \rho^{-i}\sigma\rho^i \rangle$, for each prime divisor $p$ of $n$ and for each integer $i$ with $0 \leq i \leq p-1$. In particular, the total number of maximal subgroups is one more than the sum of the prime divisors of $n$.*

Given a monoid whose group of units is trivial, cyclic, dihedral, or symmetric, the description and number of its maximal subsemigroups that arise from the group of units is an immediate consequence of the results of this section. Therefore, for the sake of brevity, in the results of Sections 3 and 4 we explicitly describe only those maximal subsemigroups that do *not* arise from the group of units.

2.2. **Maximal subsemigroups arising from a regular $\mathscr{J}$-class covered by the group of units.** Let $S$ be a finite monoid with group of units $G$, and let $J$ be a $\mathscr{J}$-class of $S$ that is not equal to $G$. The maximal subsemigroups that arise from $J$ are, in general, more complicated to describe than those that arise from $G$. This is because the elements of $S$ contained in $\mathscr{J}$-classes that are above $J$ (in the $\mathscr{J}$-class partial order) may act on, or generate, elements within $J$. Therefore it is not possible to calculate the maximal subsemigroups that arise from $J$ without considering these other $\mathscr{J}$-classes.

Certainly $G$ is a $\mathscr{J}$-class of $S$ that is strictly above $J$, since it is the unique maximal $\mathscr{J}$-class of $S$. When the group of units is the *only* $\mathscr{J}$-class strictly above $J$, the problem of finding the maximal subsemigroups that arise from $J$ is simpler than the general case. We say that such a $\mathscr{J}$-class is *covered* by the group of units. Since the elements contained in $\mathscr{J}$-classes above $J$ are units, their action on $J$ is easier than understand than the action of arbitrary semigroup elements. Additionally, maximal subsemigroups always arise from such a $\mathscr{J}$-class, as shown in the following proposition.

**Proposition 2.6** (Maximal subsemigroups of type (M1))**.** *Let $S$ be a finite monoid with group of units $G$, and let $J$ be a $\mathscr{J}$-class of $S$ that is covered by $G$. Then $S \setminus J$ is a maximal subsemigroup of $S$ if and only if no maximal subsemigroups of types* (M2)–(M5) *arise from $J$.*

*Proof.* It suffices to show that there exist maximal subsemigroups arising from $J$. Let $x, y \in S$ and suppose that $xy \in J$. By definition of the $\mathscr{J}$-class partial order, $J = J_{xy} \leq J_x$ and $J \leq J_y$. Thus $x, y \in J \cup G$, since $J$ is covered by $G$. If $x, y \in G$, then $xy \in G$, a contradiction. Thus at least one of $x$ and $y$ is contained in $J$. In other words, $S \setminus J$ is a subsemigroup of $S$. The result follows by Lemma 2.1. $\square$

For most of the monoids considered in this paper, the only $\mathscr{J}$-classes that give rise to maximal subsemigroups are the group of units, and regular $\mathscr{J}$-classes that are covered by the group of units.

By [23, Proposition 2], a maximal subsemigroup either is a union of $\mathscr{H}$-classes of the semigroup, or intersects every $\mathscr{H}$-class of the semigroup non-trivially. A maximal subsemigroup of type (M1), (M2), (M3), or (M4) is a union of $\mathscr{H}$-classes, whereas maximal subsemigroups of the second kind are those of type (M5). We consider these cases separately. We present versions of some results from [13] that are simplified to suit the current context, and that are used for calculating maximal subsemigroups of types (M2), (M3), and (M4), in Section 2.2.1. Only a few of the monoids considered in this paper exhibit maximal subsemigroups of type (M5), and we present results tailored to some of these monoids in Section 2.2.2.

2.2.1. *Maximal subsemigroups that are unions of $\mathscr{H}$-classes: types* (M2)*,* (M3)*, and* (M4)*.* Let $S$ be a finite monoid, and let $J$ be a regular $\mathscr{J}$-class of $S$ that is covered by the group of units $G$ of $S$. To find the maximal subsemigroups of $S$ arising from $J$ that have types (M2)–(M4), we construct from $J$ a bipartite graph $\Delta(S, J)$, and analyse its properties according to the forthcoming results. This bipartite graph was introduced by Donoven, Mitchell, and Wilson in [13, Section 3]. When the context unambiguously identifies the $\mathscr{J}$-class that is under consideration, i.e. when a monoid possesses only one $\mathscr{J}$-class covered by the group of units, we will often use the shorter notation $\Delta(S)$ in place of $\Delta(S, J)$.

As described in Section 1.1, right multiplication induces a right action of $G$ on the $\mathscr{L}$-classes of $J$, and left multiplication induces a left action of $G$ on the $\mathscr{R}$-classes of $J$. We define the vertices of $\Delta(S, J)$ to be the orbits of these actions. In other words, a vertex of $\Delta(S, J)$ is an orbit $\{Lg : g \in G\}$ of $\mathscr{L}$-classes of $J$ for some $L \in J/\mathscr{L}$, or an orbit $\{gR : g \in G\}$ of $\mathscr{R}$-classes of $J$ for some $R \in J/\mathscr{R}$, and every vertex of $\Delta(S, J)$ is obtained in this way. In the special case that $J$ consists of a single $\mathscr{H}$-class, we will follow the convention that the orbit of $\mathscr{L}$-classes $\{J\}$ and the orbit of $\mathscr{R}$-classes $\{J\}$ are distinct, so that $\Delta(S, J)$ contains two vertices.

There is an edge in $\Delta(S, J)$ between an orbit of $\mathscr{L}$-classes $A$ and an orbit of $\mathscr{R}$-classes $B$ if and only if there exists an $\mathscr{L}$-class $L \in A$ and an $\mathscr{R}$-class $R \in B$ such that the $\mathscr{H}$-class $L \cap R$ is a group. We define the two *bicomponents* of $\Delta(S, J)$ as follows: one bicomponent is the collection of all orbits of $\mathscr{L}$-classes of $J$, the other bicomponent is the collection of all orbits of $\mathscr{R}$-classes of $J$; the bicomponents of $\Delta(S, J)$ partition its vertices into two maximal independent subsets. Although not important for the rest of this paper, we note that $\Delta(S, J)$ is isomorphic to a quotient of the Graham-Houghton graph of the principal factor of $J$, as defined in [16, 24, 27] — in the case that the orbits of $\mathscr{L}$- and $\mathscr{R}$-classes are trivial, these graphs are isomorphic.

The following results characterize the maximal subsemigroups of $S$ of types (M2)–(M4) that arise from $J$ in terms of the graph $\Delta(S, J)$. These propositions follow from the results of [13, Section 3], having been simplified according to the assumption that $J$ is covered by the group of units of $S$. More specifically, the results of [13, Section 3] are formulated in terms of two graphs $\Delta$ and $\Theta$, and two coloured digraphs $\Gamma_{\mathscr{L}}$ and $\Gamma_{\mathscr{R}}$, that are constructed from the relevant $\mathscr{J}$-class. When the semigroup in question is a monoid and the $\mathscr{J}$-class is covered by the group of units, the graph $\Theta$ and the digraphs $\Gamma_{\mathscr{L}}$ and $\Gamma_{\mathscr{R}}$ have no edges, and each vertex of $\Gamma_{\mathscr{L}}$ and $\Gamma_{\mathscr{R}}$ has colour 0. Thus the conditions on $\Theta$, $\Gamma_{\mathscr{L}}$, and $\Gamma_{\mathscr{R}}$ are immediately satisfied. The graph $\Delta$ in [13, Section 3] is equivalent to $\Delta(S, J)$.

**Proposition 2.7** (Maximal subsemigroups of type (M2); cf. [13, Corollary 3.13])**.** *Let $T$ be a subset of $S$ such that $S \setminus T \subseteq J$. Then $T$ is a maximal subsemigroup of $S$ of type* (M2) *if and only if there exist proper non-empty subsets $A \subsetneq J/\mathscr{L}$ and $B \subsetneq J/\mathscr{R}$ such that $T \cap J$ is the union of the $\mathscr{L}$-classes in $A$ and the $\mathscr{R}$-classes in $B$, and $A$ and $B$ are unions of vertices that together form a maximal independent subset of $\Delta(S, J)$.*

By Proposition 2.7, the maximal subsemigroups of $S$ of type (M2) arising from $J$ are in bijective correspondence with the maximal independent subsets of $\Delta(S, J)$ — excluding the bicomponents of $\Delta(S, J)$. Thus we deduce the following corollary.

**Corollary 2.8.** *The number of maximal subsemigroups of $S$ of type* (M2) *arising from $J$ is two less than the number of maximal independent subsets of $\Delta(S, J)$.*

The connection between the graph $\Delta(S, J)$ and the maximal subsemigroups of $S$ of type (M3) that arise from $J$ is given in the following proposition.

**Proposition 2.9** (Maximal subsemigroups of type (M3); cf. [13, Corollary 3.15])**.** *Let $T$ be a subset of $S$ such that $S \setminus T \subseteq J$. Then $T$ is a maximal subsemigroup of $S$ of type* (M3) *if and only if there exists a proper non-empty subset $A \subsetneq J/\mathscr{L}$ such that $T \cap J$ is the union of the $\mathscr{L}$-classes in $A$, and $(J/\mathscr{L}) \setminus A$ is a vertex in $\Delta(S, J)$ that is not adjacent to a vertex of degree* 1*.*

There is a natural dual to this proposition, which describes the maximal subsemigroups of $S$ of type (M4) that arise from $J$ in terms of the graph $\Delta(S, J)$.

By Proposition 2.9, the number of maximal subsemigroups of $S$ of type (M3) is the number of orbits of $\mathscr{L}$-classes that are adjacent in $\Delta(S, J)$ only to orbits of $\mathscr{R}$-classes with degree at least 2. In the case that every orbit of $\mathscr{R}$-classes has degree 2 or more in $\Delta(S, J)$, then the number of maximal subsemigroups of type (M3) is simply the number of orbits of $\mathscr{L}$-classes. The analogous statements hold for maximal subsemigroups of type (M4). By the same token, the existence of maximal subsemigroups is restricted when there is a single orbit of $\mathscr{L}$-classes or a single orbit of $\mathscr{R}$-classes (i.e. when the group acts transitively).

**Lemma 2.10.** *If $G$ acts transitively on the $\mathscr{L}$-classes of $J$, then no maximal subsemigroups of types* (M2) *or* (M3) *arise from $J$. Similarly, if $G$ acts transitively on the $\mathscr{R}$-classes of $J$, then no maximal subsemigroups of types* (M2) *or* (M4) *arise from $J$.*

*Proof.* Suppose that $G$ acts transitively on the $\mathscr{L}$-classes of $J$. Since $J$ is regular, for every $\mathscr{L}$-class $L$ of $J$ there exists an $\mathscr{R}$-class $R$ of $J$ such that $L \cap R$ is a group, and vice versa. Therefore there are no isolated vertices in $\Delta(S, J)$, and so each vertex of $\mathscr{R}$-classes is adjacent only to the unique vertex of $\mathscr{L}$-classes. It follows that the bicomponents of $\Delta(S, J)$ are its only maximal independent subsets. By Corollary 2.8, there are no maximal subsemigroups of type (M2) arising from $J$, and by Proposition 2.9, there are no maximal subsemigroups of type (M3) either. The proof of the second statement is dual. □

When $S$ is a regular $*$-monoid, the $\mathscr{L}$-classes and $\mathscr{R}$-classes of a $\mathscr{J}$-class are in bijective correspondence via the $*$ operation, and so the graph $\Delta(S, J)$ is particularly easy to describe.

**Lemma 2.11.** *Let $S$ be a regular $*$-monoid and let $J$ be a $\mathscr{J}$-class covered by the group of units $G$ of $S$. Then a collection of $\mathscr{L}$-classes $\{L_{x_1}, \ldots, L_{x_n}\}$ is a vertex of $\Delta(S, J)$ if and only if the collection of $\mathscr{R}$-classes $\{R_{x_1^*}, \ldots, R_{x_n^*}\}$ is a vertex of $\Delta(S, J)$, and any pair of such vertices is adjacent in $\Delta(S, J)$.*

*Proof.* If $L_x$ and $L_y$ are $\mathscr{L}$-classes of $J$ in the same vertex of $\Delta(S, J)$, then there exists $g \in G$ such that $L_x g = L_y$. Therefore

$$g^* R_{x^*} = g^* L_x^* = (L_x g)^* = L_y^* = R_{y^*},$$

and so $R_{x^*}$ and $R_{y^*}$ belong to the same vertex of $\Delta(S, J)$. By symmetry, the first statement holds. The second statement holds since, for any element $x \in J$, the $\mathscr{H}$-class $L_x \cap R_{x^*}$ contains the projection $x^* x$, and is therefore a group. In particular, the vertex of $\Delta(S, J)$ containing $L_x$ is adjacent to the vertex of $\Delta(S, J)$ that contains $R_{x^*}$. □

The situation is further simplified when every idempotent of $J$ is a projection, which occurs, for instance, when $S$ is inverse.

**Corollary 2.12.** *Let $S$ be a finite regular $*$-monoid with group of units $G$, and let $J$ be a $\mathscr{J}$-class of $S$ that is covered by $G$ and whose only idempotents are projections. Suppose that $\{O_1, \ldots, O_n\}$ are the orbits of the right action of $G$ on the $\mathscr{L}$-classes of $J$. Then the maximal subsemigroups of $S$ arising from $J$ are of types* (M1), (M2), *or* (M5). *A maximal subsemigroup of type* (M2) *is the union of $S \setminus J$ and the union of the Green's classes*

$$\big\{L : L \in O_i,\ i \in A\big\} \cup \big\{L^* : L \in O_i,\ i \notin A\big\},$$

*where $A$ is any proper non-empty subset of $\{1, \ldots, n\}$. In particular, there are $2^n - 2$ maximal subsemigroups of type* (M2)*, and no maximal subsemigroups of types* (M3) *or* (M4).

*Proof.* By definition, the vertices of $\mathscr{L}$-classes of $\Delta(S, J)$ are $\{O_1, \ldots, O_n\}$, and so by Lemma 2.11, the vertices of $\mathscr{R}$-classes are $\big\{\{L^* : L \in O_i\} : i \in \{1, \ldots, n\}\big\}$. Since every idempotent of $J$ is a projection, for each $\mathscr{L}$-class $L_x$ of $J$, the only group $\mathscr{H}$-class contained in $L_x$ is $L_x \cap R_{x^*}$, and so the vertex containing $L_x$ is only adjacent to the vertex containing $R_{x^*}$. Therefore the edges of $\Delta(S, J)$ are $\{O_i,\ \{L^* : L \in O_i\}\}$ for each $i \in \{1, \ldots, n\}$. In particular, each vertex of $\Delta(S, J)$ has degree 1, and it follows from Proposition 2.9 and its dual that no maximal subsemigroups of types (M3) or (M4) arise from $J$. Furthermore, given the description of $\Delta(S, J)$, it is clear that a maximal independent subset of $\Delta(S, J)$ is formed by choosing any one vertex from each of the $n$ edges, and so there are $2^n$ maximal independent subsets. The description and number of maximal subsemigroups of type (M2) follows by Proposition 2.7 and Corollary 2.8. □

2.2.2. *Maximal subsemigroups that intersect every $\mathscr{H}$-class: type* (M5). To describe maximal subsemigroups of type (M5) — i.e. those that intersect each $\mathscr{H}$-class of $S$ non-trivially — we use a different approach from that in Section 2.2.1. Few of the monoids in this paper exhibit maximal subsemigroups of type (M5) that arise from a $\mathscr{J}$-class covered by the group of units. However, such maximal subsemigroups do occur in some instances, and in Proposition 2.16, we present a result that will be useful for these cases.

Let $S$ be a finite regular $*$-monoid with group of units $G$. To prove Proposition 2.16, we require the following definition: for a subset $A \subseteq S$, define the *setwise stabilizer of $A$ in $G$*, $\operatorname{Stab}_G(A)$, to be the subgroup $\{g \in G : Ag = A\}$ of $G$. Note that $\operatorname{Stab}_G(A)$ is defined to be the set of elements of $G$ that stabilize $A$ on the *right*. However, with $A^* = \{a^* : a \in A\}$, the set of elements of $G$ that stabilize $A$ on the *left* is equal to $\operatorname{Stab}_G(A^*)$, since

$$\big\{g \in G : gA = A\big\} = \big\{g \in G : A^*g^* = A^*\big\} = \operatorname{Stab}_G(A^*)^* = \operatorname{Stab}_G(A^*)^{-1} = \operatorname{Stab}_G(A^*).$$

Thus, for a subset $H$ of $S$ that satisfies $H^* = H$, such as for the $\mathscr{H}$-class of a projection,

$$\operatorname{Stab}_G(H) = \{g \in G : Hg = H = gH\}.$$

This observation is required in the proof of Proposition 2.16.

In Proposition 2.16, we require the set $e\operatorname{Stab}_G(H_e) = \big\{es : s \in \operatorname{Stab}_G(H_e)\big\}$, where $e$ is a projection of the regular $*$-monoid $S$, and the $\mathscr{J}$-class $J_e$ is covered by $G$. Any submonoid of $S$ that contains both $e$ and $G$ also contains $e\operatorname{Stab}_G(H_e)$. In particular, every maximal subsemigroup of type (M5) arising from $J_e$ contains $G$ and all idempotents in $J_e$, and hence contains $e\operatorname{Stab}_G(H_e)$. A stronger result, necessary for the proof of Proposition 2.16, is given by the following lemma.

**Lemma 2.13.** *Let $S$ be a finite monoid with group of units $G$, let $e$ be an idempotent of $S$, and let $T$ be a submonoid of $S$ that contains both $e$ and $G$. Then the set $e\operatorname{Stab}_G(H_e^S)$ is a subgroup of $H_e^T$.*

*Proof.* Since $e$ is an idempotent, $H_e^T = T \cap H_e^S$ [36, Proposition A.1.16]. Clearly $e\operatorname{Stab}_G(H_e^S) \subseteq eG \subseteq T$. Let $g \in \operatorname{Stab}_G(H_e^S)$. Then $eg \in H_e^S$ by definition, and so $e\operatorname{Stab}_G(H_e^S) \subseteq H_e^S$. Thus $e\operatorname{Stab}_G(H_e^S) \subseteq T \cap H_e^S = H_e^T$, and the subset is non-empty since $e = e1 \in e\operatorname{Stab}_G(H_e^S)$, where 1 is the identity of $S$. Since $S$ is finite, it remains to show that $e\operatorname{Stab}_G(H_e^S)$ is closed under multiplication. Let $g, g' \in \operatorname{Stab}_G(H_e^S)$. Since $eg \in H_e^S$ and $e$ is the identity of $H_e^S$, it follows that $(eg)e = eg$. Thus

$$(eg)(eg') = (ege)g' = (eg)g' = e(gg') \in e\operatorname{Stab}_G(H_e^S). \qquad \square$$

The following two technical lemmas are also required for the proof of Proposition 2.16.

**Lemma 2.14** ([36, Theorem A.2.4])**.** *Let $S$ be a finite semigroup and let $x, y \in S$. Then $x \mathscr{J} xy$ if and only if $x\mathscr{R}xy$, and $x \mathscr{J} yx$ if and only if $x\mathscr{L}yx$.*

**Lemma 2.15** (follows from [28, Proposition 2.3.7])**.** *Let $R$ be an $\mathscr{R}$-class of an arbitrary semigroup, and let $x, y \in R$. Then $xy \in R$ if and only if $H_x$ is a group.*

**Proposition 2.16.** *Let $S$ be a finite regular $*$-monoid with group of units $G$, let $J$ be a $\mathscr{J}$-class of $S$ that is covered by $G$, and let $H_e^S$ be the $\mathscr{H}$-class of a projection $e \in J$. Suppose that $G$ acts transitively on the $\mathscr{R}$-classes or the $\mathscr{L}$-classes of $J$, and that $J$ contains one idempotent per $\mathscr{L}$-class and one idempotent per $\mathscr{R}$-class (i.e. every idempotent of $J$ is a projection). Then the maximal subsemigroups of $S$ arising from $J$ are either:*

(a) *$(S \setminus J) \cup GUG = \langle S \setminus J,\ U\rangle$, for each maximal subgroup $U$ of $H_e^S$ that contains $e\operatorname{Stab}_G(H_e^S)$* (type (M5))*, or*
(b) *$S \setminus J$, if no maximal subsemigroups of type* (M5) *exist* (type (M1)).

*Proof.* Since $S$ is a regular $*$-monoid, $G$ acts transitively on the $\mathscr{L}$-classes of $J$ if and only if $G$ acts transitively on the $\mathscr{R}$-classes of $J$. Hence there are no maximal subsemigroups of types (M2), (M3), or (M4) arising from $J$, by Lemma 2.10. By Proposition 2.6, it remains to describe the maximal subsemigroups of type (M5).

Let $U$ be a maximal subgroup of $H_e^S$ that contains $e\operatorname{Stab}_G(H_e^S)$, and define $M_U = (S \setminus J) \cup GUG$. To prove that $M_U$ is a maximal subsemigroup of $S$, we first show that $M_U$ is a proper subset of $S$, then that it is a subsemigroup, and finally that it is maximal in $S$. Since $G$ acts transitively on the $\mathscr{L}$- and $\mathscr{R}$-classes of $J$ and $M_U$ contains $S \setminus J$, it follows that the set $M_U$ intersects every $\mathscr{H}$-class of $S$ non-trivially. Once we have shown that $M_U$ is a subsemigroup, it will obviously follows that $M_U$ is generated by $(S \setminus J) \cup U$, since $G \subseteq S \setminus J$.

To prove that $M_U$ is a proper subset of $S$, it suffices to show that $GUG \cap H_e^S \subseteq U$. Let $x \in GUG \cap H_e^S$. Since $x \in GUG$, we may write $x = \alpha u \beta$ for some $\alpha, \beta \in G$ and $u \in U$. Since $u, \alpha u \beta \in H_e^S$, it is straightforward to show that $\alpha u,\ u\beta \in H_e^S$. Thus

$$\alpha H_e^S = \alpha(uH_e^S) = (\alpha u)H_e^S = H_e^S, \quad \text{and} \quad H_e^S\beta = (H_e^S u)\beta = H_e^S(u\beta) = H_e^S.$$

In other words, $\alpha$ and $\beta$ stabilize $H_e^S$ on the left and right, respectively. Thus $\alpha, \beta \in \operatorname{Stab}_G(H_e^S)$, and

$$x = ex = e\alpha u\beta = e\alpha ue\beta \in \big(e\operatorname{Stab}_G(H_e^S)\big)U\big(e\operatorname{Stab}_G(H_e^S)\big) \subseteq U^3 = U.$$

In order to show that $M_U$ is a subsemigroup, it suffices to show that $xy \in M_U$ whenever $x, y \in G \cup GUG$, because $S \setminus (G \cup J)$ is an ideal of $S$. If $x \in G$ and $y \in G$, then certainly $xy \in G$. If $x \in G$ and $y \in GUG$, then $xy \in G^2UG = GUG$ and $yx \in GUG^2 = GUG$. For the final case, assume that $x, y \in GUG$ and that $xy \in J$. By definition, $x = \alpha u\beta$ and $y = \sigma v\tau$ for some $\alpha, \beta, \sigma, \tau \in G$ and $u, v \in U$. It suffices to show that $\beta\sigma \in \operatorname{Stab}_G(H_e^S)$, because then

$$xy = \alpha u\beta\sigma v\tau = \alpha(ue)\beta\sigma v\tau = \alpha u(e\beta\sigma)v\tau \in GU\left(e\operatorname{Stab}_G(H_e^S)\right)UG \subseteq GU^3G = GUG.$$

Since $H_e^S$ is a group containing $u$ and $v$, it follows that $u^*u = vv^* = e$. Thus

$$e\beta\sigma e = u^*u\beta\sigma vv^* = u^*\alpha^{-1}(\alpha u\beta\sigma v\tau)\tau^{-1}v^* = u^*\alpha^{-1}(xy)\tau^{-1}v^*.$$

Together with $xy = \alpha u(e\beta\sigma e)v\tau$, it follows that $e\beta\sigma e \in J$. By Lemma 2.14, $e\beta\sigma e \in R_e^S$. Since the elements $e\beta\sigma$ and $e$, and their product $e\beta\sigma e$, are all contained in $R_e^S$, Lemma 2.15 implies that $H_{e\beta\sigma}^S$ is a group. By assumption, $R_e^S$ contains only one group $\mathscr{H}$-class, which is $H_e^S$. Thus $e\beta\sigma \in H_e^S$, and so $H_e^S\beta\sigma = (H_e^S e)\beta\sigma = H_e^S(e\beta\sigma) = H_e^S$, i.e. $\beta\sigma \in \operatorname{Stab}_G(H_e^S)$, as required.

Let $M$ be a maximal subsemigroup of $S$ that contains $M_U$. By [23, Proposition 4], $M \cap H_e^S$ is a maximal subgroup of $H_e^S$, and the intersection of $M$ with any $\mathscr{H}$-class of $J$ contains exactly $|M \cap H_e^S|$ elements. Since $M \cap H_e^S$ contains $U$, the maximality of $U$ in $H_e^S$ implies that $U = M \cap H_e^S$. Since the group $G$ acts transitively on the $\mathscr{L}$- and $\mathscr{R}$-classes of $J$, the intersection of $GUG$ with any $\mathscr{H}$-class of $J$ contains at least $|U|$ elements. Thus $|M| \leq |M_U|$, and so $M = M_U$.

Conversely, suppose that $M$ is a maximal subsemigroup of $S$ of type (M5) arising from $J$. By [23, Proposition 4], the intersection $U = M \cap H_e^S = H_e^M$ is a maximal subgroup of $H_e^S$, and it contains $e\operatorname{Stab}_G(H_e^S)$ by Lemma 2.13. Since $M$ contains $G$, $U$, and $S \setminus J$, it contains the maximal subsemigroup $M_U = (S \setminus J) \cup GUG$. But $M$ is a proper subsemigroup, which implies that $M = M_U$. □

2.3. **Maximal subsemigroups arising from other $\mathscr{J}$-classes.** The following lemma can be used to find the maximal subsemigroups that arise from an arbitrary $\mathscr{J}$-class of a finite semigroup. In the later sections, for conciseness, we will sometimes use this lemma to find the maximal subsemigroups that arise from a $\mathscr{J}$-class of a monoid that is covered by the group of units. Additionally, a small number of the diagram monoids in Section 4 exhibit maximal subsemigroups arising from a $\mathscr{J}$-class that is neither equal to nor covered by the group of units. The following lemma will be particularly useful when we determine the maximal subsemigroups that arise in this case. Although the results of [13] are, in their full generality, applicable to such cases, the few examples in this paper do not warrant their use.

**Lemma 2.17.** *Let $S$ be a finite semigroup, and let $J$ be a $\mathscr{J}$-class of $S$. Suppose that there distinct subsets $X_1, \ldots, X_k \subseteq J$ such that for all $A \subseteq J$, $S = \langle S \setminus J,\ A\rangle$ if and only if $A \cap X_i \neq \varnothing$ for all $i \in \{1, \ldots, k\}$. Then the maximal subsemigroups of $S$ arising from $J$ are precisely the sets $S \setminus X_i$ for each $i \in \{1, \ldots, k\}$.*

*Proof.* Note that, by the definition of the sets $X_i$ and the assumption that they are distinct, no set $X_i$ is contained in a different set $X_j$. Let $i \in \{1, \ldots, k\}$. We show that $S \setminus X_i$ is a subsemigroup of $S$; its maximality is then obvious. Let $x, y \in S \setminus X_i$. Since $S \setminus X_i$ does not generate $S$, but it contains $S \setminus J$ and an element $x_j \in X_j$ for each $j \in \{1, \ldots, n\} \setminus \{i\}$, it follows that $xy \notin X_i$. Conversely, let $M$ be a maximal subsemigroup of $S$ arising from $J$. If $M \cap X_i \neq \varnothing$ for each $i$ then, by assumption, $S = \langle M \rangle = M$, a contradiction. Thus $M \cap X_i = \varnothing$ for some $i$. In other words, $M \subseteq S \setminus X_i$. By the maximality of $M$ in $S$, it follows that $M = S \setminus X_i$. □

We also prove the following corollary, which will be useful in Section 4.3.

**Corollary 2.18.** *Let $S$ be a finite monoid with group of units $G$, and suppose there exists a non-empty subset $X$ of $S \setminus G$ with the property that $S = \langle G,\ x \rangle$ if and only if $x \in X$. Then the maximal subsemigroups of $S$ are those that arise from the group of units and $S \setminus X$.*

*Proof.* Let $x \in X$. Since $S = \langle G,\ x \rangle$ and $G$ is closed under multiplication, the principal ideal generated by $x$ is $S \setminus G$. Since $x$ was arbitrary, every element of $X$ generates the same principal ideal, and so $X$ is contained in some $\mathscr{J}$-class $J$ of $S$. Therefore the maximal subsemigroups of $S$ are those that arise from its group of units, and $S \setminus X$, which can be found by applying Lemma 2.17 with $k = 1$ and $X_1 = X$. □

## 3. Partial transformation monoids

In this section, we find the maximal subsemigroups of the families of monoids of partial transformations defined in Section 1.2. Recall that we will only explicitly describe those maximal subsemigroups that do not arise from the group of units: in each case, given the group of units, the description of these maximal subsemigroups, and their number, follows immediately from the results of Section 2.1.

There are several families of partial transformation semigroups related to those we consider here, and whose maximal subsemigroups have been described in the literature; see [10, 20] and the references therein. The maximal subsemigroups of the singular ideal of $\mathcal{OD}_n$ were described in [9, 25]. However, since the group of units of $\mathcal{OD}_n$ is non-trivial, this is a fundamentally different problem than finding the maximal subsemigroups of $\mathcal{OD}_n$. Furthermore, the maximal subsemigroups of $\mathcal{PT}_n$ and $\mathcal{O}_n$ are known: those of $\mathcal{PT}_n$ are well-known folklore, and those of $\mathcal{O}_n$ were found in [9]. We find that the descriptions of the maximal subsemigroups of $\mathcal{O}_n$ and $\mathcal{OD}_n$ are closely linked. Thus it is instructive to reprove the known result about $\mathcal{O}_n$ alongside the new result about $\mathcal{OD}_n$, using the results of Section 2. Our techniques allow us to correct an incorrect result in the literature about the number of maximal subsemigroups of $\mathcal{O}_n$. To our knowledge, no description of the maximal subsemigroups of any of the remaining monoids that we consider in this section has appeared in the literature.

Let $n \in \mathbb{N}$, $n \geq 2$. We require some facts and notation that are common to the submonoids of $\mathcal{PT}_n$ defined in Section 1.2; let $S$ be such a monoid. Green's relations on $\mathcal{PT}_n$ are characterised by:

- $\alpha \mathscr{L} \beta$ if and only if $\operatorname{im}(\alpha) = \operatorname{im}(\beta)$,
- $\alpha \mathscr{R} \beta$ if and only if $\ker(\alpha) = \ker(\beta)$, and
- $\alpha \mathscr{J} \beta$ if and only if $\operatorname{rank}(\alpha) = \operatorname{rank}(\beta)$,

for $\alpha, \beta \in \mathcal{PT}_n$ — see [21, Theorem 4.5.1]. Note that $\ker(\alpha) = \ker(\beta)$ implies that $\operatorname{dom}(\alpha) = \operatorname{dom}(\beta)$, by definition. Since $S$ is a regular submonoid of $\mathcal{PT}_n$, it is straightforward to see that this also characterises the Green's relations on $S$. Furthermore, any generating set for $S$ contains elements of ranks $n$ and $n-1$, but needs not contain elements of smaller rank.

Therefore, by Lemma 2.1, to describe the maximal subsemigroups of $S$, we must find those maximal subsemigroups that arise from the group of units, and those that arise from the $\mathscr{J}$-class containing elements of rank $n-1$. The results of Section 2.1 apply in the former case, and the results of Section 2.2 apply in the latter case.

Notation for the groups of units that appear in this section was defined in Section 1.2. In order to describe the remaining maximal subsemigroups, we require the following notation for the Green's classes that contain partial transformations of rank $n-1$. Define

$$J_{n-1} = \{\alpha \in \mathcal{PT}_n : \operatorname{rank}(\alpha) = n-1\}$$

to be the $\mathscr{J}$-class of $\mathcal{PT}_n$ consisting of partial transformations of rank $n-1$. A partial transformation of rank $n-1$ lacks exactly one element from its image, and is either a partial permutation that lacks one element from its domain, or is a transformation with a unique non-trivial kernel class, which contains two points. Thus for distinct $i, j \in \{1, \ldots, n\}$, we define the Green's classes

- $L_i = \{\alpha \in J_{n-1} : i \notin \operatorname{im}(\alpha)\}$, which is an $\mathscr{L}$-class;
- $R_i = \{\alpha \in J_{n-1} : i \notin \operatorname{dom}(\alpha)\}$, which is an $\mathscr{R}$-class consisting of partial permutations; and
- $R_{\{i,j\}} = \{\alpha \in J_{n-1} : (i,j) \in \ker(\alpha)\}$, which is an $\mathscr{R}$-class consisting of transformations.

An $\mathscr{H}$-class of the form $L_i \cap R_j$ is a group if and only if $i = j$, and an $\mathscr{H}$-class of the form $L_i \cap R_{\{j,k\}}$ is a group if and only if $i \in \{j, k\}$.

Let $S$ be one of the submonoids of $\mathcal{PT}_n$ defined in Section 1.2. It follows that the set $J_{n-1} \cap S$ is a regular $\mathscr{J}$-class of $S$, that the $\mathscr{L}$-classes of $J_{n-1} \cap S$ are the sets of the form $L_i \cap S$, and that the $\mathscr{R}$-classes of $J_{n-1} \cap S$ are those non-empty sets of the form $R_i \cap S$ and $R_{\{i,j\}} \cap S$, for distinct $i, j \in \{1, \ldots, n\}$. Whenever we present a picture of the graph $\Delta(S) = \Delta(S, J_{n-1} \cap S)$, such as the picture given in Figure 2, we label an $\mathscr{L}$-class as $L_i$ rather than as $L_i \cap S$, and so on, in order to avoid cluttering the image. This approach also has the advantage of emphasizing the similarities between the graphs of related monoids — indeed, some graphs may be obtained as induced subgraphs of others.

Note that the non-trivial kernel class of an order-preserving or -reversing transformation of rank $n-1$ has the form $\{i, i+1\}$ for some $i \in \{1, \ldots, n-1\}$, and that the non-trivial kernel class of an orientation-preserving or -reversing transformation of rank $n-1$ has the same form, or is equal to $\{1, n\}$. Any non-empty subset of $\{1, \ldots, n\}$ appears as the image of some partial transformation in each of the monoids defined in Section 1.2.

Often, the principal obstacle to describing the maximal subsemigroups of $S$ is to determine the maximal independent subsets of $\Delta(S)$. To do this, we must calculate the left and right actions of the group of units $G$ of $S$ on the $\mathscr{R}$-classes and on the $\mathscr{L}$-classes of $J_{n-1} \cap S$, respectively. The following lemma show that these actions correspond to natural right actions of $G$ on points and pairs in $\{1, \ldots, n\}$.

**Lemma 3.1.** *Let $S$ be one of the submonoids of $\mathcal{PT}_n$ defined in Section 1.2, with group of units $G$.*

(a) *Let $\Omega \subseteq \{1, \ldots, n\}$. Then $\Omega$ is an orbit of $G$ on $\{1, \ldots, n\}$ if and only if $\{L_i \cap S : i \in \Omega\}$ is an orbit of the right action of $G$ on $(J_{n-1} \cap S)/\mathscr{L}$.*

(b) *Let $\Omega \subseteq X = \{i : R_i \cap S \neq \varnothing\}$. Then $\Omega$ is an orbit of $G$ on $X$ if and only if $\{R_i \cap S : i \in \Omega\}$ is an orbit of the left action of $G$ on the $\mathscr{R}$-classes of $J_{n-1} \cap S$ that contain partial permutations.*

(c) *Let $\Omega \subseteq Y = \{\{i, j\} : i \neq j,\ R_{\{i,j\}} \cap S \neq \varnothing\}$. Then $\Omega$ is an orbit of $G$ on $Y$ if and only if $\{R_{\{i,j\}} : \{i, j\} \in \Omega\}$ is an orbit of the left action of $G$ on the $\mathscr{R}$-classes of $J_{n-1} \cap S$ that contain transformations.*

*Proof.* To prove part (a), let $i, j \in \{1, \ldots, n\}$ and let $\alpha \in L_i \cap S$. Then, with respect to the right action of $G$ on $\{1, \ldots, n\}$,

$$\begin{aligned} i \text{ and } j \text{ belong to the same orbit} \quad &\Leftrightarrow \quad i\sigma = j \text{ for some } \sigma \in G \\ &\Leftrightarrow \quad j \notin \operatorname{im}(\alpha\sigma) \text{ for some } \sigma \in G \\ &\Leftrightarrow \quad \alpha\sigma \in L_j \cap S \text{ for some } \sigma \in G \\ &\Leftrightarrow \quad (L_i \cap S)\sigma = L_j \cap S \text{ for some } \sigma \in G. \end{aligned}$$

The proofs of the remaining parts are similar. □

The right actions of the trivial group, $\langle (1\ n)(2\ n-1) \cdots (\lfloor n/2 \rfloor\ \lceil n/2 \rceil) \rangle$, $\mathcal{C}_n$, $\mathcal{D}_n$, and $\mathcal{S}_n$ (as defined in Section 1.2) on the sets of points, and pairs of points, are easy to understand, since a permutation of degree $n$ is defined in terms of its action on $\{1, \ldots, n\}$. Therefore, if $S$ is one of the monoids defined in

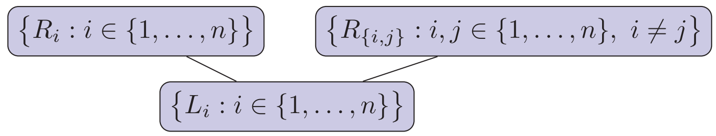


FIGURE 1. The graph $\Delta(\mathcal{PT}_n)$.

Section 1.2, then by Lemma 3.1, the right and left actions of its group of units on its $\mathscr{L}$- and $\mathscr{R}$-classes of $J_{n-1} \cap S$ may be readily understood.

3.1. $\mathcal{PT}_n$**.** First we find the maximal subsemigroups of the partial transformation monoid $\mathcal{PT}_n$. This result is well-known folklore, but we include the following result as a gentle introduction to the application of the results of Section 2.

**Theorem 3.2.** *Let $n \in \mathbb{N}$, $n \geq 2$, and let $s_n$ be the number of maximal subgroups of $\mathcal{S}_n$. Then the maximal subsemigroups of $\mathcal{PT}_n$ are those $s_n$ arising from the group of units, and:*

(a) $\mathcal{PT}_n \setminus \{\alpha \in \mathcal{T}_n : \operatorname{rank}(\alpha) = n-1\}$ (type (M4))*; and*
(b) $\mathcal{PT}_n \setminus \{\alpha \in \mathcal{I}_n : \operatorname{rank}(\alpha) = n-1\}$ (type (M4)).

*In particular, there are $s_n + 2$ maximal subsemigroups of $\mathcal{PT}_n$.*

*Proof.* It is well-known that $\mathcal{PT}_n$ is generated by its group of units $\mathcal{S}_n$, along with *any* partial permutation of rank $n-1$ and *any* transformation of rank $n-1$. Since $\mathcal{T}_n$ and $\mathcal{I}_n$ are subsemigroups of $\mathcal{PT}_n$, any generating set for $\mathcal{PT}_n$ contains *both* a transformation and a partial permutation of rank $n-1$. Thus, using Lemma 2.17 with $k = 2$, $X_1 = J_{n-1} \cap \mathcal{T}_n$, and $X_2 = J_{n-1} \cap \mathcal{I}_n$, the result follows. □

The description of the maximal subsemigroups of $\mathcal{PT}_n$ that arise from its $\mathscr{J}$-class $J_{n-1}$ can also be obtained by using the graph $\Delta(\mathcal{PT}_n)$ and the results of Section 2.2.1. Since $\mathcal{PT}_n$ is generated by its units and its idempotents of rank $n-1$, Lemma 2.2 implies that no maximal subsemigroups of type (M5) arise from $J_{n-1}$. The right action of $\mathcal{S}_n$ on the $\mathscr{L}$-classes of $J_{n-1}$ by right multiplication contains a single orbit (i.e. the right action is transitive), and so by Lemma 2.10 there are no maximal subsemigroups of types (M2) or (M3). However, there are two orbits under the left action of $\mathcal{S}_n$ on the $\mathscr{R}$-classes of $J_{n-1}$: one contains the $\mathscr{R}$-classes of transformations, the other contains the $\mathscr{R}$-classes of partial permutations. These orbits are adjacent in $\Delta(\mathcal{PT}_n)$ to the unique orbit of $\mathscr{L}$-classes; a picture of $\Delta(\mathcal{PT}_n)$ is shown in Figure 1. By the dual of Proposition 2.9, there are two maximal subsemigroups of type (M4) arising from $J_{n-1}$, formed by removing the $\mathscr{R}$-classes from each of these orbits in turn.

3.2. $\mathcal{POD}_n$**.** The maximal subsemigroups of $\mathcal{PO}_n$ were described in [11]. To our knowledge, the maximal subsemigroups of $\mathcal{POD}_n$ have not been described in the literature. Using our approach, we find that the maximal subsemigroups of $\mathcal{POD}_n$ are closely linked to those of $\mathcal{PO}_n$.

The main result of this section is the following theorem; to state it, we require the following notation. Let $n \in \mathbb{N}$, $n \geq 2$. Then $J_{n-1} \cap \mathcal{POD}_n$ is a regular $\mathscr{J}$-class of $\mathcal{POD}_n$, the $\mathscr{L}$-classes of $J_{n-1} \cap \mathcal{POD}_n$ are $\{L_i \cap \mathcal{POD}_n : i \in \{1, \ldots, n\}\}$, and the $\mathscr{R}$-classes are $\{R_i \cap \mathcal{POD}_n : i \in \{1, \ldots, n\}\}$ and $\{R_{\{i,i+1\}} \cap \mathcal{POD}_n : i \in \{1, \ldots, n-1\}\}$. Note that $\mathcal{PO}_n$ is idempotent generated [22, Theorem 3.13], and that $\mathcal{POD}_n$ is generated by $\mathcal{PO}_n$ and the permutation $(1\ n)(2\ n-1)\cdots(\lfloor n/2 \rfloor\ \lceil n/2 \rceil)$.

**Theorem 3.3.** *Let $n \in \mathbb{N}$, $n \geq 2$. Then the maximal subsemigroups of $\mathcal{POD}_n$ are the unique maximal subsemigroup arising from the group of units, and:*

(a) *the union of $\mathcal{POD}_n \setminus J_{n-1}$ and the union of the sets in*

$$\{(L_i \cup L_{n-i+1}) \cap \mathcal{POD}_n : i \in A\} \cup \{(R_i \cup R_{n-i+1}) \cap \mathcal{POD}_n : i \notin A\} \cup \{(R_{\{i,i+1\}} \cup R_{\{n-i,n-i+1\}}) \cap \mathcal{POD}_n : i, i+1 \notin A\},$$

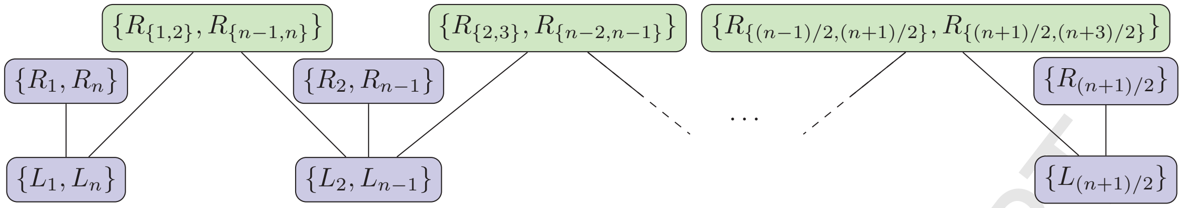


FIGURE 2. The graph $\Delta(\mathcal{POD}_n)$, when $n$ is odd.

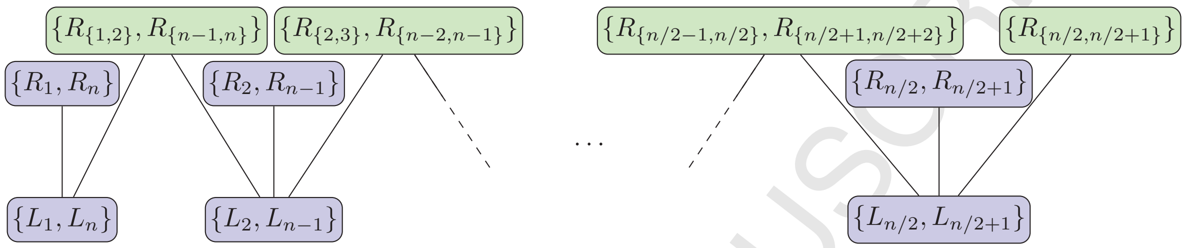


FIGURE 3. The graph $\Delta(\mathcal{POD}_n)$, when $n$ is even.

*where $A$ is any non-empty proper subset of* $\{1, \dots, \lceil n/2\rceil\}$ (type (M2))*;*
(b) $\mathcal{POD}_n \setminus (R_i \cup R_{n-i+1})$*, for* $i \in \{1, \dots, \lceil n/2\rceil\}$ (type (M4))*; and*
(c) $\mathcal{POD}_n \setminus (R_{\{i,i+1\}} \cup R_{\{n-i,n-i+1\}})$*, for* $i \in \{1, \dots, \lfloor n/2\rfloor\}$ (type (M4)).

*In particular, there are* $2^{\lceil n/2\rceil} + n - 1$ *maximal subsemigroups of* $\mathcal{POD}_n$.

The most substantial part of the proof of Theorem 3.3 is the description of the maximal independent subsets of $\Delta(\mathcal{POD}_n)$. The group of units of $\mathcal{POD}_n$ is $\langle(1\ n)(2\ n-1)\cdots(\lfloor n/2\rfloor\ \lceil n/2\rceil)\rangle$; this consists of $(1\ n)(2\ n-1)\cdots(\lfloor n/2\rfloor\ \lceil n/2\rceil)$ and the identity transformation of degree $n$, which fixes each point in $\{1, \dots, n\}$. By Lemma 3.1, since $\langle(1\ n)(2\ n-1)\cdots(\lfloor n/2\rfloor\ \lceil n/2\rceil)\rangle$ has $\lceil n/2\rceil$ orbits on the set $\{1, \dots, n\}$, there are $\lceil n/2\rceil$ corresponding orbits of $\mathscr{L}$-classes and $\lceil n/2\rceil$ orbits of $\mathscr{R}$-classes of partial permutations. Furthermore, there are $\lfloor n/2\rfloor$ orbits of $\langle(1\ n)(2\ n-1)\cdots(\lfloor n/2\rfloor\ \lceil n/2\rceil)\rangle$ on the set $\big\{\{i, i+1\} : i \in \{1, \dots, n-1\}\big\}$, and these orbits correspond to $\lfloor n/2\rfloor$ orbits of $\mathscr{R}$-classes of transformations. A picture of $\Delta(\mathcal{POD}_n)$ is shown in Figure 2 for odd $n$, and in Figure 3 for even $n$; see these pictures for a description of the edges of this graph. Given this description of $\Delta(\mathcal{POD}_n)$, we establish the following lemma.

**Lemma 3.4.** *Let $K$ be any collection of vertices of the graph $\Delta(\mathcal{POD}_n)$. Then $K$ is a maximal independent subset of $\Delta(\mathcal{POD}_n)$ if and only if $K$ is equal to*

$$\big\{\{L_i \cap \mathcal{POD}_n, L_{n-i+1} \cap \mathcal{POD}_n\} : i \in A\big\} \cup \big\{\{R_i \cap \mathcal{POD}_n, R_{n-i+1} \cap \mathcal{POD}_n\} : i \notin A\big\} \\ \cup \big\{\{R_{\{i,i+1\}} \cap \mathcal{POD}_n, R_{\{n-i,n-i+1\}} \cap \mathcal{POD}_n\} : i, i+1 \notin A\big\},$$

*for some subset $A$ of* $\{1, \dots, \lceil n/2\rceil\}$.

*Proof.* ($\Rightarrow$) Suppose that $K$ is a maximal independent subset of $\Delta(\mathcal{POD}_n)$. There exists a set $A \subseteq \{1, \dots, \lceil n/2\rceil\}$ of indices such that $\big\{\{L_i \cap \mathcal{POD}_n, L_{n-i+1} \cap \mathcal{POD}_n\} : i \in A\big\}$ is the collection of $\mathscr{L}$-class vertices in $K$. Since a vertex of the form $\{R_i \cap \mathcal{POD}_n, R_{n-i+1} \cap \mathcal{POD}_n\}$ is adjacent in $\Delta(\mathcal{POD}_n)$ only to the vertex $\{L_i \cap \mathcal{POD}_n, L_{n-i+1} \cap \mathcal{POD}_n\}$, it follows by the maximality of $K$ that $\{R_i \cap \mathcal{POD}_n, R_{n-i+1} \cap \mathcal{POD}_n\} \in K$ if and only if $i \notin A$. Similarly, since an orbit of the form $\{R_{\{i,i+1\}} \cap \mathcal{POD}_n, R_{\{n-i,n-i+1\}} \cap \mathcal{POD}_n\}$ is adjacent in $\Delta(\mathcal{POD}_n)$ only to the orbits $\{L_i \cap \mathcal{POD}_n, L_{n-i+1} \cap \mathcal{POD}_n\}$ and $\{L_{i+1} \cap \mathcal{POD}_n, L_{n-i} \cap \mathcal{POD}_n\}$, it follows that $\{R_{\{i,i+1\}} \cap \mathcal{POD}_n, R_{\{n-i,n-i+1\}} \cap \mathcal{POD}_n\} \in K$ if

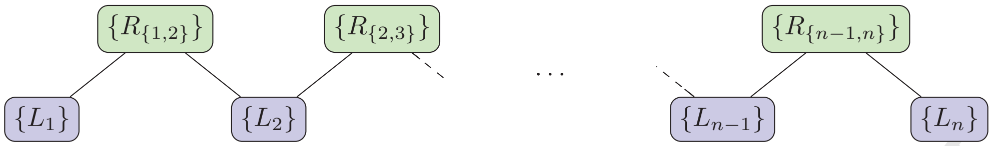


FIGURE 4. The graph $\Delta(\mathcal{O}_n)$.

and only if $i \notin A$ and $i+1 \notin A$. Since we have considered all vertices of $\Delta(\mathcal{POD}_n)$, it follows that $K$ has the required form.

($\Leftarrow$) It is easy to verify that $K$ is a maximal independent subset of $\Delta(\mathcal{POD}_n)$. $\square$

*Proof of Theorem* 3.3. The group of units $\langle (1\ n)(2\ n-1)\cdots(\lfloor n/2 \rfloor\ \lceil n/2 \rceil)\rangle$ of $\mathcal{POD}_n$ has order 2, which gives rise to one maximal subsemigroup by Lemma 2.3. Since $\mathcal{POD}_n$ is generated by $\mathcal{PO}_n$ and the permutation $(1\ n)(2\ n-1)\cdots(\lfloor n/2 \rfloor\ \lceil n/2 \rceil)$, and since $\mathcal{PO}_n$ is idempotent generated, it follows by Lemma 2.2 that there are no maximal subsemigroups of type (M5) arising from $J_{n-1} \cap \mathcal{POD}_n$. It follows directly from Proposition 2.7, and Lemma 3.4, that the maximal subsemigroups of type (M2) are those described in the theorem. There exist vertices of degree 1 in $\Delta(\mathcal{POD}_n)$: the orbits of $\mathscr{R}$-classes of partial permutations. Each orbit of $\mathscr{L}$-classes is adjacent to such a vertex. Thus by Proposition 2.9 and its dual, there are no maximal subsemigroups of type (M3) arising from $\mathcal{POD}_n$, but each orbit of $\mathscr{R}$-classes can be removed to provide a maximal subsemigroup of type (M4); there are $n$ maximal subsemigroups of this type. By Proposition 2.6, there is no maximal subsemigroup of type (M1). $\square$

3.3. **$\mathcal{O}_n$ and $\mathcal{OD}_n$.** The maximal subsemigroups of the singular ideal of $\mathcal{O}_n$ were incorrectly described and counted in [41]: the given formula for the number of maximal subsemigroups of the singular ideal of $\mathcal{O}_n$ is correct for $2 \leq n \leq 5$, but gives only a lower bound when $n \geq 6$. A correct description, although no number, was later given in [9]. The maximal subsemigroups of the singular ideal of $\mathcal{OD}_n$ were described in [25]. The group of units of $\mathcal{O}_n$ is trivial, and so the maximal subsemigroups of its singular ideal correspond in an obvious way to the maximal subsemigroups of $\mathcal{O}_n$. However, the group of units of $\mathcal{OD}_n$ is $\langle (1\ n)(2\ n-1)\cdots(\lfloor n/2 \rfloor\ \lceil n/2 \rceil)\rangle$, which acts on $\mathcal{OD}_n$ in such a way as to break the correspondence between the maximal subsemigroups of the singular part, and the maximal subsemigroups of $\mathcal{OD}_n$ itself. Thus [25] solves an essentially different problem than the description of the maximal subsemigroups of $\mathcal{OD}_n$.

Recall that $\mathcal{O}_n = \mathcal{PO}_n \cap \mathcal{T}_n$, and $\mathcal{OD}_n = \mathcal{POD}_n \cap \mathcal{T}_n$. Let $S \in \{\mathcal{O}_n, \mathcal{OD}_n\}$. Then $S$ is a regular monoid, the subset $J_{n-1} \cap S$ is a regular $\mathscr{J}$-class of $S$, the set of $\mathscr{L}$-classes of $J_{n-1} \cap S$ is $\big\{L_i \cap S : i \in \{1, \ldots, n\}\big\}$, and the set of $\mathscr{R}$-classes of $J_{n-1} \cap S$ is $\big\{R_{\{i,i+1\}} \cap S : i \in \{1, \ldots, n-1\}\big\}$.

Since $\mathcal{OD}_n = \mathcal{POD}_n \cap \mathcal{T}_n$, we may identify a Green's class of $J_{n-1} \cap \mathcal{OD}_n$ with the corresponding Green's class of $J_{n-1} \cap \mathcal{POD}_n$ that contains it, so that $L_i \cap \mathcal{OD}_n$ corresponds with $L_i \cap \mathcal{POD}_n$, and $R_{\{i,i+1\}} \cap \mathcal{OD}_n$ corresponds with $R_{\{i,i+1\}} \cap \mathcal{POD}_n$. In this way, we obtain $\Delta(\mathcal{OD}_n)$ as the induced subgraph of $\Delta(\mathcal{POD}_n)$ on those orbits of Green's classes that contain transformations — thus the definition of $\Delta(\mathcal{OD}_n)$ is contained in that of $\Delta(\mathcal{POD}_n)$. The graph $\Delta(\mathcal{O}_n)$ contains $n$ singleton orbits of $\mathscr{L}$-classes and $n-1$ singleton orbits of $\mathscr{R}$-classes. A picture of $\Delta(\mathcal{O}_n)$ is shown in Figure 4, which gives a description of its edges.

For $k \in \mathbb{N}$, we define the path graph of order $k$ to be the graph with vertices $\{1, \ldots, k\}$ and edges

$$\big\{\{i, i+1\} : i \in \{1, \ldots, k-1\}\big\}.$$

The vertices of degree 1 in the path graph of order $k$ are the end-points, 1 and $k$. It is easy to see that $\Delta(\mathcal{O}_n)$ is isomorphic to the path graph of order $2n-1$, via the isomorphism that maps the orbit $\{L_i \cap \mathcal{O}_n\}$ to the vertex $2i-1$, and maps the orbit $\{R_{\{i,i+1\}} \cap \mathcal{O}_n\}$ to the vertex $2i$. Similarly, $\Delta(\mathcal{OD}_n)$ is isomorphic to the path graph of order $n$. We can describe and count the number of maximal independent subsets of a path graph, and hence of $\Delta(\mathcal{O}_n)$ and $\Delta(\mathcal{OD}_n)$, by using the following results.

**Lemma 3.5.** *Let $n \in \mathbb{N}$ be arbitrary, let $\Gamma$ be the path graph of order $n$, and let $U$ be a subset of the vertices of $\Gamma$. Then $U$ is a maximal independent subset of $\Gamma$ if and only if the following conditions hold:*

(a) *the least vertex in $U$ is either 1 or 2; and*
(b) *for each $i \in U \cap \{1, \ldots, n-1\}$, $i+1 \notin U$; and*
(c) *for each $i \in U \cap \{1, \ldots, n-2\}$, exactly one of $i+2$ and $i+3$ is contained in $U$.*

*Proof.* Since vertices in $\Gamma$ are adjacent if and only if they are consecutive, $U$ is an independent subset of $\Gamma$ if and only if (b) holds. It is easy to verify that an independent subset $U$ of $\Gamma$ is maximal when conditions (a) and (c) hold. Conversely, if $U$ satisfies (b) but contains neither 1 nor 2, then $U \cup \{1\}$ is an independent subset properly containing $U$, and $U$ is not maximal. Similarly, suppose that $U$ satisfies (b) and contains some $i \in \{1, \ldots, n-2\}$, but contains neither $i+2$ nor $i+3$. Then since $U$ also does not contain $i+1$, it follows that $U \cup \{i+2\}$ is an independent subset properly containing $U$, and $U$ is not maximal. Thus, if $U$ is a maximal independent subset of $\Gamma$, then conditions (a) and (c) hold. $\square$

There are two special maximal independent subsets of a path graph: the subset of all even vertices, and the subset of all odd vertices. These maximal independent subsets correspond to the bicomponents of $\Delta(\mathcal{O}_n)$ and $\Delta(\mathcal{OD}_n)$, and so they are the unique maximal independent subsets that do not give rise to maximal subsemigroups of $\mathcal{O}_n$ and $\mathcal{OD}_n$ of type (M2) — see Proposition 2.7 and Corollary 2.8.

**Corollary 3.6.** *The number of maximal independent subsets of the path graph is counted by the sequence $(A_n)_{n\in\mathbb{N}}$, as defined in Section 1.1, which satisfies the recurrence relation $A_1 = 1$, $A_2 = A_3 = 2$, and $A_n = A_{n-2} + A_{n-3}$ for $n \geq 4$.*

*Proof.* For $n \in \mathbb{N}$, define $\Gamma_n$ to be the path graph of order $n$, and let $f(n)$ be the number of maximal independent subsets of $\Gamma_n$. It is straightforward to verify that $f(1) = 1$, and $f(2) = f(3) = 2$, so suppose that $n \geq 4$. By Lemma 3.5, if $U$ is a maximal independent subset of $\Gamma_{n-3}$, then $U \cup \{n-1\}$ is a maximal independent subset of $\Gamma_n$, and if $U$ is a maximal independent subset of $\Gamma_{n-2}$, then $U \cup \{n\}$ is a maximal independent subset of $\Gamma_n$. Thus distinct maximal independent subsets of $\Gamma_{n-3}$ and $\Gamma_{n-2}$ give rise to distinct maximal independent subsets of $\Gamma_n$, and $f(n) \geq f(n-2) + f(n-3)$. Conversely, if $U$ is a maximal independent subset of $\Gamma_n$, then by Lemma 3.5, $U$ contains either $n-1$ or $n$. If $n-1 \in U$, then $n-2 \notin U$, which implies that $U \setminus \{n-1\}$ is a maximal independent subset of $\Gamma_{n-3}$. Otherwise, $U \setminus \{n\}$ is a maximal independent subset of $\Gamma_{n-2}$. Thus $f(n) \leq f(n-2) + f(n-3)$, and $f(n) = A(n)$. $\square$

We may now prove the main results of this section.

**Theorem 3.7.** *Let $n \in \mathbb{N}$, $n \geq 2$. The maximal subsemigroups of $\mathcal{O}_n$ are the unique maximal subsemigroup arising from the group of units, and:*

(a) *the union of $\mathcal{O}_n \setminus J_{n-1}$ and the union of the Green's classes in*
$$\big\{L_{(i+1)/2} \cap \mathcal{O}_n : i \in A,\ i \text{ is odd}\big\} \cup \big\{R_{\{i/2,(i/2)+1\}} \cap \mathcal{O}_n : i \in A,\ i \text{ is even}\big\},$$
*where $A$ is a maximal independent subset of the path graph of order $2n-1$ that contains both odd and even numbers, as described in Lemma 3.5* (type (M2))*;*
(b) *$\mathcal{O}_n \setminus L$, where $L$ is any $\mathscr{L}$-class in $J_{n-1} \cap \mathcal{O}_n$* (type (M3))*; and*
(c) *$\mathcal{O}_n \setminus R_{\{i,i+1\}}$, where $i \in \{2, \ldots, n-2\}$* (type (M4)).

*In particular, for $n \geq 3$ there are $A_{2n-1} + 2n - 4$ maximal subsemigroups of $\mathcal{O}_n$, where $A_{2n-1}$ is as defined at the end of Section 1.1.*

**Theorem 3.8.** *Let $n \in \mathbb{N}$, $n \geq 3$. The maximal subsemigroups of $\mathcal{OD}_n$ are the unique maximal subsemigroup arising from the group of units, and:*

(a) *the union of $\mathcal{OD}_n \setminus J_{n-1}$ and the union of the sets in*
$$\big\{(L_{(i+1)/2} \cup L_{n+1-(i+1)/2}) \cap \mathcal{OD}_n : i \in A,\ i \text{ is odd}\big\}$$
$$\cup \big\{(R_{\{i/2,(i/2)+1\}} \cup R_{\{n-(i/2),n+1-(i/2)\}}) \cap \mathcal{OD}_n : i \in A,\ i \text{ is even}\big\},$$

*where $A$ is a maximal independent subset of the path graph of order $n$ that contains both odd and even numbers, as described in Lemma 3.5* (type (M2))*;*

(b) $\mathcal{OD}_n \setminus (L_i \cup L_{n-i+1})$, *where* $\begin{cases} i \in \{1, \ldots, (n+1)/2\} & \text{if } n \text{ is odd,} \\ i \in \{1, \ldots, n/2 - 1\} & \text{if } n \text{ is even} \end{cases}$ (type (M3))*; and*

(c) $\mathcal{OD}_n \setminus (R_{\{i,i+1\}} \cup R_{\{n-i,n-i+1\}})$, *where* $\begin{cases} i \in \{2, \ldots, (n-3)/2\} & \text{if } n \text{ is odd,} \\ i \in \{2, \ldots, n/2\} & \text{if } n \text{ is even} \end{cases}$ (type (M4)).

*In particular, for $n \geq 4$, there are $A_n + n - 3$ maximal subsemigroups of $\mathcal{OD}_n$, where $A_n$ is as defined at the end of Section 1.1.*

*Proof of Theorems* 3.7 *and* 3.8. In each case, the group of units has a single maximal subsemigroup, and so by Lemma 2.3, there is a unique maximal subsemigroup rising from the group of units.

Since $\mathcal{O}_n$ is generated by its idempotents of rank $n-1$ [1], and since $\mathcal{OD}_n$ is generated by $\mathcal{O}_n$ and the permutation $(1\ n)(2\ n-1)\cdots(\lfloor n/2 \rfloor\ \lceil n/2 \rceil)$ [18], it follows by Lemma 2.2 there are no maximal subsemigroups of type (M5) arising from $J_{n-1} \cap S$. We have already noted that $\Delta(\mathcal{O}_n)$ and $\Delta(\mathcal{OD}_n)$ are paths of length $2n-1$ and $n$, respectively. It follows by Proposition 2.7 and Lemma 3.5 that the maximal subsemigroups of type (M2) are those described in the theorems. By Corollary 2.8 and Corollary 3.6, the number of maximal subsemigroups of type (M2) is $A_{2n-1} - 2$ for $\mathcal{O}_n$, and $A_n - 2$ for $\mathcal{OD}_n$.

To describe the maximal subsemigroups of types (M3) and (M4), it suffices to identify the two vertices of $\Delta(S)$ that are adjacent to the end-points of $\Delta(S)$. From this, the description of the maximal subsemigroups of types (M3) and (M4) follows from Proposition 2.9 and its dual. In particular, the total number of both types of maximal subsemigroups is two less than the number of vertices of $\Delta(S)$.

By Proposition 2.6, and since $n \geq 4$, there is no maximal subsemigroup of $S$ of type (M1). □

3.4. $\mathcal{POP}_n$ **and** $\mathcal{POR}_n$**.** Let $n \in \mathbb{N}$, $n \geq 2$. To state the results of this section, we require the following notation. Let $S \in \{\mathcal{POP}_n, \mathcal{POR}_n\}$. Then $J_{n-1} \cap S$ is a regular $\mathscr{J}$-class of $S$. The $\mathscr{L}$-classes of $J_{n-1} \cap S$ are the sets $L_i \cap S$ for each $i \in \{1, \ldots, n\}$, and the $\mathscr{R}$-classes of $J_{n-1} \cap S$ are the sets $R_i \cap S$ for each $i \in \{1, \ldots, n\}$ and $R_{\{i,i+1\}} \cap S$ for each $i \in \{1, \ldots, n-1\}$, along with the set $R_{\{1,n\}} \cap S$. The group of units of $\mathcal{POP}_n$ is $\mathcal{C}_n$, and the group of units of $\mathcal{POR}_n$ is $\mathcal{D}_n$ — see Section 1.2 for the definitions of these groups.

The following theorem is the main result of this section.

**Theorem 3.9.** *Let $n \in \mathbb{N}$, $n \geq 3$, and let $S \in \{\mathcal{POP}_n, \mathcal{POR}_n\}$. The maximal subsemigroups of $S$ are those arising from the group of units, and:*

(a) $S \setminus \{\alpha \in S \cap \mathcal{T}_n : \operatorname{rank}(\alpha) = n-1\}$ (type (M4))*; and*

(b) $S \setminus \{\alpha \in S \cap \mathcal{I}_n : \operatorname{rank}(\alpha) = n-1\}$ (type (M4)).

*In particular, there are $|\mathbb{P}_n| + 2$ maximal subsemigroups of $\mathcal{POP}_n$, and there are $3 + \sum_{p \in \mathbb{P}_n} p$ maximal subsemigroups of $\mathcal{POR}_n$ for $n \geq 3$, where $\mathbb{P}_n$ is the set of primes that divide $n$.*

*Proof.* By Lemmas 2.3, 2.4, and 2.5, there are $|\mathbb{P}_n|$ maximal subsemigroups arising from the group of units $\mathcal{C}_n$ of $\mathcal{POP}_n$, and $1 + \sum_{p \in \mathbb{P}_n} p$ that arise from the group of units $\mathcal{D}_n$ of $\mathcal{POR}_n$.

Let $S \in \{\mathcal{POP}_n, \mathcal{POR}_n\}$, and let $G$ be the group of units of $S$. Since $\mathcal{PO}_n$ is idempotent generated [22, Theorem 3.13], and $S = \langle \mathcal{PO}_n,\ G \rangle$, it follows by Lemma 2.2 that there are no maximal subsemigroups of type (M5) arising from $J_{n-1} \cap S$.

The remainder of the proof is similar to the discussion in Section 3.1 after the proof of Theorem 3.2. The group of units $G$ of $S$ acts transitively on the $\mathscr{L}$-classes of $J_{n-1} \cap S$, and so there are no maximal subsemigroups of types (M2) and (M3) by Lemma 2.10. On the other hand, $G$ has two orbits on the set of $\mathscr{R}$-classes of $J_{n-1} \cap S$: it transitively permutes the $\mathscr{R}$-classes of transformations, and it transitively permutes the $\mathscr{R}$-classes of partial permutations. By the dual of Proposition 2.9, the two maximal subsemigroups of $S$ of type (M4) are found by removing either the partial permutations, or the transformations, of rank $n-1$. By Proposition 2.6, there is no maximal subsemigroup of type (M1). □

3.5. $\mathcal{POPI}_n$ **and** $\mathcal{PORI}_n$**.** The maximal subsemigroups of the inverse monoids $\mathcal{POPI}_n$ and $\mathcal{PORI}_n$ exhibit maximal subsemigroups of type (M5) arising from a $\mathscr{J}$-class covered by the group of units, and to which we can apply the results of Section 2.2.2, and Proposition 2.16 in particular.

Let $n \in \mathbb{N}$, and let $S \in \{\mathcal{POPI}_n, \mathcal{PORI}_n\}$. Then $J_{n-1} \cap S$ is a regular $\mathscr{J}$-class of $S$ consisting of partial permutations. By definition, $\mathcal{POPI}_n = \mathcal{POP}_n \cap \mathcal{I}_n$ and $\mathcal{PORI}_n = \mathcal{POR}_n \cap \mathcal{I}_n$. Therefore the group of units of $\mathcal{POPI}_n$ is $\mathcal{C}_n$ and the group of units of $\mathcal{PORI}_n$ is $\mathcal{D}_n$, and given the description of the Green's classes of $\mathcal{POP}_n$ and $\mathcal{POR}_n$ in Section 3.4, it follows that the $\mathscr{L}$-classes and $\mathscr{R}$-classes of $J_{n-1} \cap S$ are $\big\{L_i \cap S : i \in \{1, \ldots, n\}\big\}$ and $\big\{R_i \cap S : i \in \{1, \ldots, n\}\big\}$, respectively.

In the following theorems, we describe the maximal subsemigroups of $\mathcal{POPI}_n$ and $\mathcal{PORI}_n$.

**Theorem 3.10.** *Let $n \in \mathbb{N}$, $n \geq 3$, and define $\zeta_n$ to be the partial permutation*

$$\begin{pmatrix} 1 & 2 & \cdots & n-2 & n-1 & n \\ 2 & 3 & \cdots & n-1 & 1 & - \end{pmatrix}.$$

*For $k \in \mathbb{N}$, let $\mathbb{P}_k$ denote the set of all primes that divide $k$. Then the maximal subsemigroups of $\mathcal{POPI}_n$ are those $|\mathbb{P}_n|$ arising from the group of units, and the subsemigroups $\langle \mathcal{POPI}_n \setminus J_{n-1},\ \zeta_n^p \rangle$ for each $p \in \mathbb{P}_{n-1}$, which have type* (M5)*. In particular, there are $|\mathbb{P}_n| + |\mathbb{P}_{n-1}|$ maximal subsemigroups of $\mathcal{POPI}_n$.*

*Proof.* The group of units of $\mathcal{POPI}_n$ is $\mathcal{C}_n$, which, by Lemma 2.4, gives rise to $|\mathbb{P}_n|$ maximal subsemigroups. Since $\mathcal{POPI}_n$ is inverse, each of its idempotents is a projection. Clearly the group of units $\mathcal{C}_n$ acts transitively on the $\mathscr{L}$-classes and $\mathscr{R}$-classes of $J_{n-1} \cap \mathcal{POPI}_n$. Define

$$H = \big\{\alpha \in \mathcal{POPI}_n : \operatorname{dom}(\alpha) = \operatorname{im}(\alpha) = \{1, \ldots, n-1\}\big\}.$$

Then $H$ is a group $\mathscr{H}$-class in the $\mathscr{J}$-class $J_{n-1} \cap \mathcal{POPI}_n$. Note that $H$ is isomorphic to the cyclic group of order $n-1$, and is generated by $\zeta_n$. Since the conditions of Proposition 2.16 are satisfied, we may apply its results. Therefore the maximal subsemigroups that arise from $J_{n-1} \cap \mathcal{POPI}_n$ are the subsemigroups $\langle \mathcal{POPI}_n \setminus J_{n-1},\ U \rangle$, for each maximal subgroup $U$ of $H$ that contains $\operatorname{Stab}_{\mathcal{C}_n}(H)$ (defined in Section 2.2.2), or $\mathcal{POPI}_n \setminus J_{n-1}$, if no such maximal subgroups exist. The setwise stabilizer $\operatorname{Stab}_{\mathcal{C}_n}(H)$ is equal to the pointwise stabilizer $\{\sigma \in \mathcal{C}_n : n\sigma = n\}$ of $n$ in $\mathcal{C}_n$, which is trivial. Therefore any maximal subgroup of $H$ gives rise to a maximal subsemigroup of $\mathcal{POPI}_n$; by Lemma 2.4, the maximal subgroups of $H$ are $\langle \zeta_n^p \rangle$ for each $p \in \mathbb{P}_{n-1}$, and as $n \geq 3$, the result follows. ☐

**Theorem 3.11.** *Let $n \in \mathbb{N}$, $n \geq 4$, and define partial permutations $\zeta_n$ and $\tau_n$ of degree $n$ by*

$$\zeta_n = \begin{pmatrix} 1 & 2 & \cdots & n-2 & n-1 & n \\ 2 & 3 & \cdots & n-1 & 1 & - \end{pmatrix}, \quad \text{and} \quad \tau_n = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n-1 & n-2 & \cdots & 1 & - \end{pmatrix}.$$

*For $k \in \mathbb{N}$, let $\mathbb{P}_k$ denote the set of all primes that divide $k$. Then the maximal subsemigroups of $\mathcal{PORI}_n$ are those $1 + \sum_{p \in \mathbb{P}_n} p$ that arise from the group of units, and the subsemigroups $\langle \mathcal{PORI}_n \setminus J_{n-1},\ \zeta_n^p,\ \tau_n \rangle$ for each $p \in \mathbb{P}_{n-1}$, which have type* (M5)*. In particular, there are $1 + |\mathbb{P}_{n-1}| + \sum_{p \in \mathbb{P}_n} p$ maximal subsemigroups of $\mathcal{PORI}_n$.*

*Proof.* The number of maximal subsemigroups arising from the group of units $\mathcal{D}_n$ follows by Lemma 2.3.

Each idempotent of $\mathcal{PORI}_n$ is a projection since it is an inverse semigroup, and $\mathcal{D}_n$ acts transitively on the $\mathscr{L}$- and $\mathscr{R}$-classes of $J_{n-1} \cap \mathcal{PORI}_n$. Therefore we may use Proposition 2.16 to describe the maximal subsemigroups that arise from $J_{n-1} \cap \mathcal{PORI}_n$. Define $\mathrm{id}_{n-1}$ to be the partial transformation with domain $\{1, \ldots, n-1\}$ that fixes each point in its domain, and define

$$H = H_{\mathrm{id}_{n-1}}^{\mathcal{PORI}_n} = \big\{\alpha \in \mathcal{PORI}_n : \operatorname{dom}(\alpha) = \operatorname{im}(\alpha) = \{1, \ldots, n-1\}\big\}.$$

Then $H$ is a group $\mathscr{H}$-class contained in $J_{n-1} \cap \mathcal{PORI}_n$. Note that since $n \geq 4$, $H$ is a dihedral group of order $2(n-1)$, and it is generated by the partial permutations $\zeta_n$ and $\tau_n$. An element of $\mathcal{D}_n$ belongs to the setwise stabilizer $\operatorname{Stab}_{\mathcal{D}_n}(H)$ if and only if it fixes the point $n$. Thus $\operatorname{Stab}_{\mathcal{D}_n}(H)$ contains only the identity permutation, which fixes every point in $\{1, \ldots, n\}$, and the permutation that fixes $n$ and reverses the order of $\{1, \ldots, n-1\}$. In particular,

$$\mathrm{id}_{n-1} \operatorname{Stab}_{\mathcal{D}_n}(H) = \left\{\mathrm{id}_{n-1} \cdot h : h \in \operatorname{Stab}_{\mathcal{D}_n}(H)\right\} = \{\mathrm{id}_{n-1}, \tau_n\}.$$

Since any subgroup of $H$ contains $\mathrm{id}_{n-1}$, it follows from Proposition 2.16 that the maximal subsemigroups arising from $\mathcal{PORI}_n \cap J_{n-1}$ are $\langle \mathcal{PORI}_n \setminus J_{n-1},\ U\rangle$, for each maximal subgroup $U$ of $H$ that contains $\tau_n$. By Lemma 2.5, the maximal subgroups of $H$ are $\langle \zeta_n \rangle$ and the subgroups $\langle \zeta_n^p,\ \zeta_n^{-i}\tau_n\zeta_n^i\rangle$, where $p \in \mathbb{P}_{n-1}$ and $0 \leq i \leq p-1$. Thus the maximal subgroups of $H$ that contain $\tau_n$ are those in the latter form where $i = 0$. It follows that the maximal subsemigroups arising from $\mathcal{PORI}_n \cap J_{n-1}$ are those stated in the theorem, and that there are $|\mathbb{P}_{n-1}|$ such maximal subsemigroups. □

## 4. Diagram monoids

In this section, we determine the maximal subsemigroups of the monoids of partitions defined in Section 1.3. Note that, again, we do not explicitly describe those maximal subsemigroups arising from the group of units, because their description can be immediately deduced from the results of Section 2.1.

There is a natural injective function $\phi : \mathcal{PT}_n \longrightarrow \mathcal{P}_n$ such that if $\alpha \in \mathcal{PT}_n$, then the non-singleton blocks of $\alpha\phi$ are $\{i'\} \cup i\alpha^{-1}$, for each $i \in \mathrm{im}(\alpha)$. Although $\phi$ is not a homomorphism, partitions can nevertheless be thought of as generalisations of transformations via $\phi$. Furthermore, it is easy to see that for any $\alpha \in \mathcal{PT}_n$, $\alpha$ is order-preserving if and only if $\alpha\phi$ is planar, and so we may consider planar partitions to be generalisations of order-preserving partial transformations.

Let $\alpha \in \mathcal{P}_n$. The *rank* of $\alpha$, denoted $\mathrm{rank}(\alpha)$, is the number of transverse blocks that it contains. We define $\ker(\alpha)$, the *kernel* of $\alpha$, to be the restriction of the equivalence $\alpha$ to $\{1, \ldots, n\}$. We also define $\mathrm{dom}(\alpha)$, the *domain* of $\alpha$, to be the subset of $\{1, \ldots, n\}$ comprising those points that are contained in a transverse block of $\alpha$. Given these definitions, we define $\mathrm{coker}(\alpha) = \ker(\alpha^*)$ and $\mathrm{codom}(\alpha) = \mathrm{dom}(\alpha^*)$, the *cokernel* and *codomain* of $\alpha$, respectively. For the majority of the monoids defined in Section 1.3, the Green's relations are completely determined by domain, kernel, and rank.

**Lemma 4.1.** *Let $S \in \{\mathcal{P}_n, \mathcal{PB}_n, \mathcal{B}_n, \mathcal{M}_n, \mathcal{J}_n\}$ and let $\alpha, \beta \in S$. Then:*
(a) *$\alpha \mathscr{R} \beta$ if and only if $\mathrm{dom}(\alpha) = \mathrm{dom}(\beta)$ and $\ker(\alpha) = \ker(\beta)$;*
(b) *$\alpha \mathscr{L} \beta$ if and only if $\mathrm{codom}(\alpha) = \mathrm{codom}(\beta)$ and $\mathrm{coker}(\alpha) = \mathrm{coker}(\beta)$; and*
(c) *$\alpha \mathscr{J} \beta$ if and only if $\mathrm{rank}(\alpha) = \mathrm{rank}(\beta)$.*

See [34], [40, Theorem 17], and [12, Theorem 2.4] for the proof of this lemma. For the uniform block bijection monoid $\mathfrak{F}_n$, parts (a) and (b) of Lemma 4.1 hold, since it is a regular submonoid of $\mathcal{P}_n$ [28, Proposition 2.4.2]. However, while part (c) does not hold for $\mathfrak{F}_n$ in general, it does hold for uniform block bijections of ranks $n$ or $n-1$. For $n \in \mathbb{N}$ and $k \in \{0, 1, \ldots, n\}$, we define $J_k = \{\alpha \in \mathcal{P}_n : \mathrm{rank}(\alpha) = k\}$ to be the $\mathscr{J}$-class of $\mathcal{P}_n$ that comprises the partitions of rank $k$.

In general, if $S$ is any of the diagram monoids defined in Section 1.3, then $S$ has a unique $\mathscr{J}$-class $J$ that is covered by the group of units of $S$. In several cases, to determine the maximal subsemigroups of $S$ that arise from $J$, we require the graph $\Delta(S, J)$, as defined in Section 2.2.1. Given a description of the $\mathscr{L}$-classes and $\mathscr{R}$-classes of $J$, to describe $\Delta(S, J)$ it remains to describe the action of the group of units on the $\mathscr{R}$-classes of $J$. Since $S$ is a regular $*$-monoid, a description of the action of the group of units on the $\mathscr{L}$-classes of $J$ is obtained as a consequence. We observe that, for $\alpha \in \mathcal{P}_n$ and $\sigma \in \mathcal{S}_n$,

(1) $$\mathrm{dom}(\sigma\alpha) = \left\{i\sigma^{-1} : i \in \mathrm{dom}(\alpha)\right\} \quad \text{and} \quad \ker(\sigma\alpha) = \left\{\left(i\sigma^{-1}, j\sigma^{-1}\right) : (i, j) \in \ker(\alpha)\right\}.$$

Given this description and Lemma 4.1, the action of a subgroup of $\mathcal{S}_n$ on the $\mathscr{R}$-classes of a particular $\mathscr{J}$-class is straightforward to determine.

4.1. **The partition monoid $\mathcal{P}_n$.** Let $n \in \mathbb{N}$, $n \geq 2$. We require the following information about the Green's classes of $\mathcal{P}_n$ in the $\mathscr{J}$-class $J_{n-1}$. Let $\alpha \in J_{n-1}$. By definition, $\alpha$ contains $n-1$ transverse blocks. Since each transverse block contains at least two points, and there are only $2n$ points in

$\{1,\ldots,n\}\cup\{1',\ldots,n'\}$, there are few possible combinations of kernel and domain for $\alpha$. In particular, either $\ker(\alpha)$ is trivial and $\mathrm{dom}(\alpha)=\{1,\ldots,n\}\setminus\{i\}$ for some $i\in\{1,\ldots,n\}$, or $\mathrm{dom}(\alpha)=\{1,\ldots,n\}$ and $\{i,j\}$ is the unique non-trivial kernel class of $\alpha$, for some distinct $i,j\in\{1,\ldots,n\}$. By Lemma 4.1, these properties describe the $\mathscr{R}$-classes of $J_{n-1}$. Since the $\mathscr{L}$-classes and $\mathscr{R}$-classes of a regular $*$-semigroup correspond via the $^*$ operation, analogous statements hold for the $\mathscr{L}$-classes of $J_{n-1}$. Thus, for distinct $i,j\in\{1,\ldots,n\}$, we make the following definitions:

- $R_i=\big\{\alpha\in J_{n-1}:\mathrm{dom}(\alpha)=\{1,\ldots,n\}\setminus\{i\}\big\}$, an $\mathscr{R}$-class;
- $R_{\{i,j\}}=\big\{\alpha\in J_{n-1}:(i,j)\in\ker(\alpha)\big\}$, an $\mathscr{R}$-class;
- $L_i=R_i^*=\big\{\alpha\in J_{n-1}:\mathrm{codom}(\alpha)=\{1,\ldots,n\}\setminus\{i\}\big\}$, an $\mathscr{L}$-class;
- $L_{\{i,j\}}=R_{\{i,j\}}^*=\big\{\alpha\in J_{n-1}:(i,j)\in\mathrm{coker}(\alpha))\big\}$, an $\mathscr{L}$-class.

An $\mathscr{H}$-class of the form $L_i\cap R_j$ is a group if and only if $i=j$, an $\mathscr{H}$-class of the form $L_i\cap R_{\{j,k\}}$ or $R_i\cap L_{\{j,k\}}$ is a group if and only if $i\in\{j,k\}$, and an $\mathscr{H}$-class of the form $L_{\{i,j\}}\cap R_{\{k,l\}}$ is a group if and only if $\{i,j\}=\{k,l\}$.

The main result of this section is the following theorem.

**Theorem 4.2.** *Let $n\in\mathbb{N}$, $n\geq 2$, and let $s_n$ denote the number of maximal subgroups of the symmetric group $\mathcal{S}_n$. Then the maximal subsemigroups of $\mathcal{P}_n$ are those $s_n$ that arise from the group of units, and:*

(a) $\mathcal{P}_n\setminus\big\{\alpha\in\mathcal{P}_n:\mathrm{rank}(\alpha)=n-1$ *and* $\ker(\alpha)$ *is trivial*$\big\}$ (type (M4))*;*
(b) $\mathcal{P}_n\setminus\big\{\alpha\in\mathcal{P}_n:\mathrm{rank}(\alpha)=n-1$ *and* $\mathrm{dom}(\alpha)=\{1,\ldots,n\}\big\}$ (type (M4))*;*
(c) $\mathcal{P}_n\setminus\big\{\alpha\in\mathcal{P}_n:\mathrm{rank}(\alpha)=n-1$ *and* $\mathrm{coker}(\alpha)$ *is trivial*$\big\}$ (type (M3))*; and*
(d) $\mathcal{P}_n\setminus\big\{\alpha\in\mathcal{P}_n:\mathrm{rank}(\alpha)=n-1$ *and* $\mathrm{codom}(\alpha)=\{1,\ldots,n\}\big\}$ (type (M3)).

*In particular, there are $s_n+4$ maximal subsemigroups of $\mathcal{P}_n$.*

*Proof.* By [15, Section 6], the ideal $\mathcal{P}_n\setminus\mathcal{S}_n$ is generated by its idempotents of rank $n-1$. Thus, since $\mathscr{J}$-equivalence in $\mathcal{P}_n$ is determined by rank, the maximal subsemigroups of $\mathcal{P}_n$ arise from its group of units $\mathcal{S}_n$, and the $\mathscr{J}$-class $J_{n-1}$. By Lemma 2.3, there are $s_n$ maximal subsemigroups of the first kind.

By Lemma 2.2, there are no maximal subsemigroups arising from $J_{n-1}$ of type (M5). It is clear from (1) that $\mathcal{S}_n$ transitively permutes the $\mathscr{R}$-classes of $J_{n-1}$ with trivial kernel, and it transitively permutes the $\mathscr{R}$-classes of $J_{n-1}$ with domain $\{1,\ldots,n\}$. Thus there are two orbits of $\mathscr{R}$-classes of $J_{n-1}$ under the action of $\mathcal{S}_n$; therefore there are two corresponding orbits of $\mathscr{L}$-classes.

Since the bicomponents are the only maximal independent subsets of $\Delta(\mathcal{P}_n)=\Delta(\mathcal{P}_n,J_{n-1})$, Corollary 2.8 implies that there are no maximal subsemigroups of type (M2). Each vertex of $\Delta(\mathcal{P}_n)$ has degree 2, and so by Proposition 2.9, there are two maximal subsemigroups of type (M3), formed by removing each orbit of $\mathscr{L}$-classes in turn. Similarly, there are two maximal subsemigroups of type (M4). □

4.2. **The partial Brauer monoid $\mathcal{PB}_n$.** The restriction to $\mathcal{I}_n$ of the injective map $\phi:\mathcal{PT}_n\to\mathcal{P}_n$, defined at the start of Section 4, is an injective homomorphism from $\mathcal{I}_n$ into $\mathcal{P}_n$. We will identify $\mathcal{I}_n$ with its image under $\phi$. In this way, $\mathcal{I}_n$ is clearly a submonoid of $\mathcal{PB}_n$.

To describe the maximal subsemigroups of $\mathcal{PB}_n$, we require a description of the elements of $\mathcal{PB}_n$ whose rank is at least $n-2$. Partitions of degree $n$ that have rank $n$ are units, and the group of units of $\mathcal{PB}_n$ is $\mathcal{S}_n$.

Let $\alpha\in J_{n-1}\cap\mathcal{PB}_n$. By definition, $\alpha$ contains precisely $n-1$ transverse blocks of size two, and two singleton blocks $\{i\}$ and $\{j'\}$, for some $i,j\in\{1,\ldots,n\}$. Therefore $\alpha$ is the image of some partial permutation of rank $n-1$ under the embedding $\phi$. Since $\alpha$ was arbitrary, and $\mathcal{I}_n\subseteq\mathcal{PB}_n$, it follows that $J_{n-1}\cap\mathcal{PB}_n=J_{n-1}\cap\mathcal{I}_n$.

Let $\alpha\in\mathcal{PB}_n$, and suppose that $\mathrm{rank}(\alpha)=n-2$. Then $\alpha$ contains $n-2$ transverse blocks, which leaves a pair of points of $\{1,\ldots,n\}$ and a pair of points of $\{1',\ldots,n'\}$ that are not contained in transverse blocks. Each of these pairs forms either a block of size 2, or two singleton blocks. In particular, $\mathrm{dom}(\alpha)$ lacks some two points $i$ and $j$, and either $\ker(\alpha)$ is trivial, or $\{i,j\}$ is the unique non-trivial kernel class of $\alpha$. A similar statement holds for the codomain and cokernel of $\alpha$.

**Theorem 4.3.** *Let $n \in \mathbb{N}$, $n \geq 2$, and let $s_n$ denote the number of maximal subgroups of $\mathcal{S}_n$. The maximal subsemigroups of $\mathcal{PB}_n$ are those $s_n$ arising from the group of units, and:*

(a) $\mathcal{PB}_n \setminus \{\alpha \in \mathcal{PB}_n : \operatorname{rank}(\alpha) = n-1\}$ (type (M1));
(b) $\mathcal{PB}_n \setminus \{\alpha \in \mathcal{PB}_n : \operatorname{rank}(\alpha) = n-2$ *and* $\ker(\alpha)$ *is non-trivial*$\}$ (type (M4)); *and*
(c) $\mathcal{PB}_n \setminus \{\alpha \in \mathcal{PB}_n : \operatorname{rank}(\alpha) = n-2$ *and* $\operatorname{coker}(\alpha)$ *is non-trivial*$\}$ (type (M3)).

*In particular, there are $s_n + 3$ maximal subsemigroups of $\mathcal{PB}_n$.*

*Proof.* By [12], $\mathcal{PB}_n$ is generated by its elements with rank at least $n-2$, and any generating set contains elements of ranks $n$, $n-1$, and $n-2$. By Lemma 4.1, Green's $\mathscr{J}$-relation in $\mathcal{PB}_n$ is determined by rank. Thus, the $\mathscr{J}$-classes of $\mathcal{PB}_n$ from which there arise maximal subsemigroups are its group of units, $J_{n-1} \cap \mathcal{PB}_n$, and $J_{n-2} \cap \mathcal{PB}_n$. By Lemma 2.3, $s_n$ maximal subsemigroups arise from the group of units.

Since the $\mathscr{J}$-class $J_{n-1} \cap \mathcal{PB}_n$ is covered by the group of units, $\mathcal{PB}_n \setminus J_{n-1}$ is a subsemigroup of $\mathcal{PB}_n$. Let $\alpha \in J_{n-1} \cap \mathcal{PB}_n = J_{n-1} \cap \mathcal{I}_n$ be arbitrary. It is well-known that $\mathcal{I}_n$ is generated by its group of units $\mathcal{S}_n$ along with any element of rank $n-1$. Thus $\langle \mathcal{PB}_n \setminus J_{n-1},\ \alpha \rangle \supseteq \langle \mathcal{PB}_n \setminus J_{n-1},\ \mathcal{I}_n \rangle = \mathcal{PB}_n$. This shows that $\mathcal{PB}_n \setminus J_{n-1}$ (type (M1)) is the unique maximal subsemigroup to arise from $J_{n-1} \cap \mathcal{PB}_n$.

In order to determine the maximal subsemigroups of $\mathcal{PB}_n$ that arise from its $\mathscr{J}$-class of rank $n-2$, we define the subsets

$$\begin{aligned} X &= \{\alpha \in \mathcal{PB}_n : \operatorname{rank}(\alpha) = n-2 \text{ and } \ker(\alpha) \text{ is non-trivial}\}, \text{ and} \\ X^* = \{\alpha^* : \alpha \in X\} &= \{\alpha \in \mathcal{PB}_n : \operatorname{rank}(\alpha) = n-2 \text{ and } \operatorname{coker}(\alpha) \text{ is non-trivial}\}. \end{aligned}$$

Note that $X$ is a union of $\mathscr{R}$-classes of $\mathcal{PB}_n$, and $X^*$ is a union of $\mathscr{L}$-classes. Let $A$ be a subset of $J_{n-2} \cap \mathcal{PB}_n$ such that $(\mathcal{PB}_n \setminus J_{n-2}) \cup A$ generates $\mathcal{PB}_n$. Let $\alpha \in X$ be arbitrary. Then $\alpha$ can the written as a product $\alpha = \beta_1 \cdots \beta_k$ of some of these generators. Clearly the generators $\beta_1, \ldots, \beta_k$ have rank at least $n-2$. Every element in $\mathcal{PB}_n$ of rank $n$ and $n-1$ has a trivial kernel, and the subset of partitions with trivial kernel in $\mathcal{P}_n$ forms a subsemigroup. Thus there exists some $r \in \{1, \ldots, k\}$ such that $\operatorname{rank}(\beta_r) = n-2$ and $\ker(\beta_r)$ is non-trivial — in other words, $\beta_r \in X$. A dual argument shows that $A \cap X^* \neq \varnothing$. Conversely, for any subset $A$ of $J_{n-2} \cap \mathcal{PB}_n$ that intersects $X$ and $X^*$ non-trivially, we have $\mathcal{PB}_n = \langle \mathcal{PB}_n \setminus J_{n-2},\ A \rangle$. By Lemma 2.17, the maximal subsemigroups of $\mathcal{PB}_n$ arising from $J_{n-2} \cap \mathcal{PB}_n$ are $\mathcal{PB}_n \setminus X$ and $\mathcal{PB}_n \setminus X^*$; these maximal subsemigroups have types (M4) and (M3), respectively. □

4.3. **The Brauer monoid $\mathcal{B}_n$ and the uniform block bijection monoid $\mathfrak{F}_n$.** Let $n \in \mathbb{N}$, $n \geq 2$. The main results of this section are the following theorems, which describe the maximal subsemigroups of $\mathcal{B}_n$ and $\mathfrak{F}_n$. Recall that the group of units of $\mathcal{B}_n$ and $\mathfrak{F}_n$ is $\mathcal{S}_n$, and that we denote the number of maximal subgroups of $\mathcal{S}_n$ by $s_n$.

**Theorem 4.4.** *Let $n \in \mathbb{N}$, $n \geq 2$. The maximal subsemigroups of $\mathcal{B}_n$ are those $s_n$ that arise from the group of units, and $\mathcal{B}_n \setminus \{\alpha \in \mathcal{B}_n : \operatorname{rank}(\alpha) = n-2\}$, which has type* (M1). *In particular, there are $s_n + 1$ maximal subsemigroups of $\mathcal{B}_n$.*

**Theorem 4.5.** *Let $n \in \mathbb{N}$, $n \geq 2$. The maximal subsemigroups of $\mathfrak{F}_n$ are those $s_n$ arising from the group of units, and $\mathfrak{F}_n \setminus \{\alpha \in \mathfrak{F}_n : \operatorname{rank}(\alpha) = n-1\}$, which has type* (M1). *In particular, there are $s_n + 1$ maximal subsemigroups of $\mathfrak{F}_n$.*

Recall that an element $\alpha$ of a regular $*$-semigroup is a projection if $\alpha = \alpha^2 = \alpha^{-1}$. By [2], $\mathcal{B}_n$ is generated by $\mathcal{S}_n$ and any projection of rank $n-2$, and by [33, Section 5], $\mathfrak{F}_n$ is generated by $\mathcal{S}_n$ and any projection of rank $n-1$. These facts are used in the following proof.

*Proof of Theorems* 4.4 *and* 4.5. The number of maximal subsemigroups arising from the group of units follows by Lemma 2.3. Let $\alpha \in J_{n-2} \cap \mathcal{B}_n$. The non-transverse blocks of $\alpha$ are $\{i, j\}$ and $\{k', l'\}$ for some $i, j, k, l \in \{1, \ldots, n\}$ with $i \neq j$ and $k \neq l$. Let $\tau \in \mathcal{S}_n$ be a permutation that contains the blocks $\{k, i'\}$ and $\{l, j'\}$. Therefore the non-transverse blocks of $\alpha\tau$ are $\{i, j\}$ and $\{i', j'\}$, and so $(\alpha\tau)^m$ is a

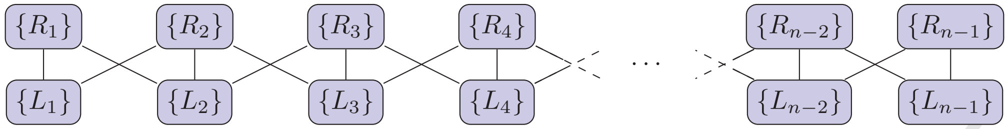


FIGURE 5. The graph $\Delta(\mathcal{J}_n, \mathcal{J}_n \cap J_{n-2})$.

projection of rank $n-2$ for some $m \in \mathbb{N}$. Thus $\langle \mathcal{S}_n,\ \alpha\rangle \supseteq \langle \mathcal{S}_n,\ (\alpha\tau)^m\rangle = \mathcal{B}_n$, and so $\mathcal{B}_n = \langle \mathcal{S}_n,\ \alpha\rangle$. By a similar argument, $\mathfrak{F}_n = \langle \mathcal{S}_n,\ \beta\rangle$ for any uniform block bijection of rank $n-1$. By Corollary 2.18, the remaining maximal subsemigroups are those stated in the theorems. $\square$

4.4. **The Jones monoid $\mathcal{J}_n$.** Let $n \in \mathbb{N}$. In this section, we find the maximal subsemigroups of the Jones monoid $\mathcal{J}_n$ (also known as the Temperley-Lieb monoid). Since the planar partition monoid of degree $n$ is isomorphic to the Jones monoid of degree $2n$ [26], by determining the maximal subsemigroups of $\mathcal{J}_n$ we obtain those of $\mathcal{PP}_n$.

Suppose that $n \geq 2$. By [6], $\mathcal{J}_n$ is generated by the identity partition and its projections of rank $n-2$. By Lemma 4.1, the set $J_{n-2} \cap \mathcal{J}_n$ is a $\mathscr{J}$-class of $\mathcal{J}_n$, and since there are no elements of $\mathcal{J}_n$ with rank $n-1$, it follows that this $\mathscr{J}$-class is covered by the group of units. Note that $\mathcal{J}_n$ is $\mathscr{H}$-trivial, since it consists of planar partitions.

To describe the maximal subsemigroups of $\mathcal{J}_n$ that arise from its $\mathscr{J}$-class of rank $n-2$ partitions, we require the graph $\Delta(\mathcal{J}_n) = \Delta(\mathcal{J}_n, J_{n-2} \cap \mathcal{J}_n)$. Thus we require a description of the Green's classes of $J_{n-2} \cap \mathcal{J}_n$. Let $\alpha \in J_{n-2} \cap \mathcal{J}_n$. Then $\alpha$ has $n-2$ transverse blocks, and these contain two points. By planarity, the remaining blocks are of the form $\{i, i+1\}$ and $\{j', (j+1)'\}$ for some $i, j \in \{1, \ldots, n-1\}$. By Lemma 4.1, the $\mathscr{J}$-class $J_{n-2} \cap \mathcal{J}_n$ contains $n-1$ $\mathscr{R}$-classes and $n-1$ $\mathscr{L}$-classes. For $i \in \{1, \ldots, n-1\}$, we define

- $R_i = \big\{\alpha \in \mathcal{J}_n : \operatorname{rank}(\alpha) = n-2 \text{ and } \{i, i+1\} \text{ is a block of } \alpha\big\}$, an $\mathscr{R}$-class; and
- $L_i = \big\{\alpha \in \mathcal{J}_n : \operatorname{rank}(\alpha) = n-2 \text{ and } \{i', (i+1)'\} \text{ is a block of } \alpha\big\}$, an $\mathscr{L}$-class.

The intersection of the $\mathscr{L}$-class $L_i$ and the $\mathscr{R}$-class $R_j$ is a group if and only if $|i-j| \leq 1$. Since the group of units of $\mathcal{J}_n$ is trivial, its action on the $\mathscr{L}$-classes and $\mathscr{R}$-classes of $J_{n-2} \cap \mathcal{J}_n$ is trivial. A picture of $\Delta(\mathcal{J}_n)$ is shown in Figure 5.

The maximal independent subsets of $\Delta(\mathcal{J}_n)$ are described and counted in the following results.

**Lemma 4.6.** *Let $n \in \mathbb{N}$, $n \geq 2$, be arbitrary and let $U$ be a subset of the vertices of $\Delta(\mathcal{J}_n)$. Then $U$ is a maximal independent subset of $\Delta(\mathcal{J}_n)$ if and only if the following conditions hold:*

(a) *$U$ contains either $\{L_1\}$ or $\{R_1\}$, but not both; and*
(b) *$U$ contains either $\{L_{n-1}\}$ or $\{R_{n-1}\}$, but not both; and*
(c) *if $\{L_i\}$ is in $U$ for some $i \in \{1, \ldots, n-2\}$, then the vertex in $U \setminus \big\{\{L_i\}\big\}$ with smallest index greater than or equal to $i$ is either $\{L_{i+1}\}$ or $\{R_{i+2}\}$; and*
(d) *if $\{R_i\}$ is in $U$ for some $i \in \{1, \ldots, n-2\}$, then the vertex in $U \setminus \big\{\{R_i\}\big\}$ with smallest index greater than or equal to $i$ is either $\{R_{i+1}\}$ or $\{L_{i+2}\}$.*

*Proof.* It is straightforward to verify from the definition of $\Delta(\mathcal{J}_n)$ that a subset of the vertices satisfying the conditions of the lemma is a maximal indepedent subset.

Conversely, let $U$ be a maximal independent subset of $\Delta(\mathcal{J}_n)$. We first show that condition (a) holds. Since $\{L_1\}$ and $\{R_1\}$ are adjacent in $\Delta(\mathcal{J}_n)$, they are not both contained in $U$. Similarly, at least one of $\{L_2\}$ and $\{R_2\}$ is not contained in $U$. If $\{L_2\} \notin U$, then either $\{L_1\} \in U$, or the maximality of $U$ implies that $\{R_1\} \in U$. If instead $\{R_2\} \notin U$, then it follows similarly that $\{L_1\} \in U$ or $\{R_1\} \in U$. By a similar argument, $U$ contains precisely one of $\{L_{n-1}\}$ and $\{R_{n-1}\}$, i.e. condition (b) holds.

To prove that condition (c) holds, let $i \in \{1, \ldots, n-2\}$, and suppose that $U$ contains the vertex $\{L_i\}$. Consider the vertex in $U \setminus \big\{\{L_i\}\big\}$ with smallest index greater than or equal to $i$; such a vertex exists, since $U$ contains $\{L_{n-1}\}$ or $\{R_{n-1}\}$. Certainly this vertex is not $\{R_i\}$ or $\{R_{i+1}\}$, since these are adjacent in $\Delta(\mathcal{J}_n)$ to $\{L_i\}$. If $U$ does not contain $\{R_{i+2}\}$, then the maximality of $U$ implies that $U$ contains $\{L_{i+1}\}$. Noting that $U$ does not contain both $\{L_{i+2}\}$ and $\{R_{i+2}\}$, it follows that the vertex in $U \setminus \{\{L_i\}\}$ with smallest index greater than or equal to $i$ is either $\{L_{i+1}\}$ or $\{R_{i+2}\}$, as required. An analogous argument show that condition (d) holds. ☐

Recall that $(F_n)_{n\in\mathbb{N}}$ is the Fibonacci sequence, which is defined at the end of Section 1.1.

**Corollary 4.7.** *The number of maximal independent subsets of $\Delta(\mathcal{J}_n)$ is $2F_{n-1}$.*

*Proof.* The result may be verified directly for $n \in \{2, 3\}$, so suppose that $n \geq 4$. By the symmetry of $\Delta(\mathcal{J}_n)$, the maximal independent subsets that contain $\{L_1\}$ are in bijective correspondence with the maximal independent subsets that contain $\{R_1\}$. Therefore we shall count $a(n)$, the number of maximal independent subsets of $\Delta(\mathcal{J}_n)$ that contain $\{L_1\}$; the total number of maximal independent subsets is $2a(n)$.

For $i \in \{1, 2\}$, define $\Lambda_{n-i}$ be the induced subgraph of $\Delta(\mathcal{J}_n)$ on the vertices

$$\big\{\{L_{i+1}\}, \ldots, \{L_{n-1}\}, \{R_{i+1}\}, \ldots, \{R_{n-1}\}\big\}.$$

Clearly $\Lambda_{n-i}$ is isomorphic to $\Delta(\mathcal{J}_{n-i})$, and so the number of maximal independent subsets of $\Lambda_{n-1}$ that contain $\{L_2\}$ is $a(n-1)$, and the number of maximal independent subsets of $\Lambda_{n-2}$ that contain $\{R_3\}$ is $a(n-2)$.

Let $U$ be a maximal independent subset of $\Delta(\mathcal{J}_n)$ containing $\{L_1\}$. By Lemma 4.6, $U$ contains precisely one of $\{L_2\}$ or $\{R_3\}$. If $U$ contains $\{L_2\}$, then $U \setminus \{\{L_1\}\}$ is a maximal independent subset of $\Lambda_{n-1}$ that contains $\{L_2\}$, while if $U$ contains $\{R_3\}$, then $U \setminus \{\{L_1\}\}$ is a maximal independent subset of $\Lambda_{n-2}$ that contains $\{R_3\}$. Conversely, maximal independent subsets of $\Lambda_{n-1}$ containing $\{L_2\}$, and maximal independent subsets of $\Lambda_{n-2}$ containing $\{R_3\}$, give rise to distinct maximal independent subsets of $\Delta(\mathcal{J}_n)$ that contain $\{L_1\}$, via the addition of $\{L_1\}$. It follows that $a(n) = a(n-1) + a(n-2)$. By this recurrence, and since $a(2) = F_1$ and $a(3) = F_2$, it follows that $a(n) = F_{n-1}$. ☐

We may now describe and count the maximal subsemigroups of $\mathcal{J}_n$.

**Theorem 4.8.** *Let $n \in \mathbb{N}$, $n \geq 3$. The maximal subsemigroups of $\mathcal{J}_n$ are the unique maximal subsemigroup arising from the group of units, and:*

(a) *The union of $\mathcal{J}_n \setminus J_{n-2}$ and the union of the Green's classes contained in a maximal independent subset of $\Delta(\mathcal{J}_n)$ that is not a bicomponent of $\Delta(\mathcal{J}_n)$, as described in Lemma 4.6* (type (M2))*;*
(b) *$\mathcal{J}_n \setminus L$, where $L$ is any $\mathscr{L}$-class in $\mathcal{J}_n$ of rank $n-2$* (type (M3))*; and*
(c) *$\mathcal{J}_n \setminus R$, where $R$ is any $\mathscr{R}$-class in $\mathcal{J}_n$ of rank $n-2$* (type (M4))*.*

*In particular, there are $2F_{n-1} + 2n - 3$ maximal subsemigroups of $\mathcal{J}_n$, where $F_{n-1}$ is the $(n-1)^{th}$ term of the Fibonacci sequence, as defined at the end of Section 1.1.*

*Proof.* Since the Jones monoid $\mathcal{J}_n$ is $\mathscr{H}$-trivial, there are no maximal subsemigroups of type (M5) by Lemma 2.2, and there is one maximal subsemigroup arising from the group of units by Lemma 2.3. By Lemma 4.6 and Proposition 2.7, the maximal subsemigroups of type (M2) arising from the $\mathscr{J}$-class of rank $n-2$ are those described in the theorem, and by Corollary 2.8 and Corollary 4.7, there are $2F_{n-1} - 2$ such maximal subsemigroups. Since $n \geq 3$, each vertex of $\Delta(\mathcal{J}_n)$ has degree at least 2, and so it follows by Proposition 2.9 that any $\mathscr{L}$-class of rank $n-2$ can be removed to form a maximal subsemigroup of type (M3). Similarly, any $\mathscr{R}$-class of rank $n-2$ can be removed to form a maximal subsemigroup of type (M4). Thus there are $n-1$ maximal subsemigroups of each of these types. By Proposition 2.6, there is no maximal subsemigroup of type (M1). ☐

4.5. **The Motzkin monoid $\mathcal{M}_n$.** Finally, in this section, we describe and count the maximal subsemigroups of the Motzkin monoid $\mathcal{M}_n$. Let $n \in \mathbb{N}$, $n \geq 2$. By [12, Proposition 4.2], $\mathcal{M}_n$ is generated by its elements of rank at least $n-2$, and any generating set for $\mathcal{M}_n$ contains elements of ranks $n$, $n-1$, and $n-2$. By Lemma 4.1, Green's $\mathscr{J}$-relation on $\mathcal{M}_n$ is determined by rank, and so the maximal subsemigroups of $\mathcal{M}_n$ arise from the $\mathscr{J}$-classes that correspond to these ranks. To describe the maximal subsemigroups of $\mathcal{M}_n$, we therefore require a description of its elements that have rank at least $n-2$.

Clearly the unique element of $\mathcal{M}_n$ of rank $n$ is the identity partition of degree $n$.

An arbitrary element of rank $n-1$ in $\mathcal{M}_n$ has trivial kernel and cokernel, and is uniquely determined by the point $i$ that it lacks from its domain and the point $j$ that it lacks from its codomain. By Lemma 4.1, this determines the $\mathscr{L}$- and $\mathscr{R}$-classes of $J_{n-1} \cap \mathcal{M}_n$. An element of $J_{n-1} \cap \mathcal{M}_n$ is an idempotent if its domain and codomain are equal, and so every idempotent $\varepsilon$ in $J_{n-1} \cap \mathcal{M}_n$ satisfies $\varepsilon^* = \varepsilon$; in other words, every idempotent in $J_{n-1} \cap \mathcal{M}_n$ is a projection, as defined in Section 1.1.

We will use Lemma 2.17 to describe the maximal subsemigroups of $\mathcal{M}_n$ that arise from $J_{n-2} \cap \mathcal{M}_n$. To apply this lemma, we require [12, Lemma 4.11], which, in the case that $r = n-1$, states that $\mathcal{M}_n$ is generated by its elements of ranks $n$ and $n-1$ along with its projections of rank $n-2$ that have non-trivial kernel and cokernel.

The main result of this section is the following theorem.

**Theorem 4.9.** *Let $n \in \mathbb{N}$, $n \geq 2$. The maximal subsemigroups of $\mathcal{M}_n$ are the unique maximal subsemigroup that arises from the group of units, and:*

(a) *The union*

$$\big(\mathcal{M}_n \setminus J_{n-1}\big) \cup \bigcup_{i \in A} \big\{\alpha \in \mathcal{M}_n : \operatorname{rank}(\alpha) = n-1 \textit{ and } \{i\} \textit{ is a block of } \alpha\big\} \cup \bigcup_{i \notin A} \big\{\alpha \in \mathcal{M}_n : \operatorname{rank}(\alpha) = n-1 \textit{ and } \{i'\} \textit{ is a block of } \alpha\big\},$$

*where $A$ is any non-empty proper subset of $\{1,\ldots,n\}$* (type (M2))*;*

(b) $\mathcal{M}_n \setminus \big\{\alpha \in J_{n-2} : \{i, i+1\}$ *is a block of* $\alpha\big\}$ *for* $i \in \{1,\ldots,n-1\}$ (type (M4))*; and*

(c) $\mathcal{M}_n \setminus \big\{\alpha \in J_{n-2} : \{i', (i+1)'\}$ *is a block of* $\alpha\big\}$ *for* $i \in \{1,\ldots,n-1\}$ (type (M3)).

*In particular, there are $2^n + 2n - 3$ maximal subsemigroups of $\mathcal{M}_n$.*

*Proof.* The Motzkin monoid has a trivial group of units, and so by Lemma 2.3, it gives rise to a single maximal subsemigroup. Given the above description of the $\mathscr{J}$-class $J_{n-1} \cap \mathcal{M}_n$, it follows from Corollary 2.12 that the maximal subsemigroups that arise from this $\mathscr{J}$-class are those described in the theorem of type (M2), and that there are $2^n - 2$ of them; there are no maximal subsemigroups of type (M5) since $\mathcal{M}_n$ is $\mathscr{H}$-trivial. It remains to describe the maximal subsemigroups that arise from the $\mathscr{J}$-class of rank $n-2$.

For $i \in \{1,\ldots,n-1\}$, we define the subsets

$$X_i = \{\alpha \in \mathcal{M}_n : \operatorname{rank}(\alpha) = n-2 \text{ and } \{i, i+1\} \text{ is a block of } \alpha\}, \text{ and}$$
$$X_i^* = \{\alpha^* : \alpha \in X_i\} = \{\alpha \in \mathcal{M}_n : \operatorname{rank}(\alpha) = n-2 \text{ and } \{i', (i+1)'\} \text{ is a block of } \alpha\}$$

of the $\mathscr{J}$-class $J_{n-2} \cap \mathcal{M}_n$. Note that $X_i$ is an $\mathscr{R}$-class of $\mathcal{M}_n$ and $X_i^*$ is an $\mathscr{L}$-class of $\mathcal{M}_n$.

Let $A$ be a subset of $J_{n-2} \cap \mathcal{M}_n$ such that $(\mathcal{M}_n \setminus J_{n-2}) \cup A$ generates $\mathcal{M}_n$. Let $i \in \{1,\ldots,n-1\}$ and $\alpha \in X_i$ be arbitrary. Then $\alpha$ can be written as a product $\alpha = \beta_1 \cdots \beta_k$ of the generators that have rank $n-1$ or $n-2$. If $\operatorname{rank}(\beta_1) = n-1$, then $\beta_1$, and each of its right-multiples, contains a singleton block of the form $\{j\}$ for some $j \in \{1,\ldots,n\}$. However, $\alpha$ is a right-multiple of $\beta_1$ and $\alpha$ contains no such block; thus $\operatorname{rank}(\beta_1) = n-2 = \operatorname{rank}(\alpha)$. Lemmas 2.14 and 4.1 imply that $\ker(\alpha) = \ker(\beta_1)$, i.e. $A \cap X_i \neq \varnothing$. A dual argument shows that $A \cap X_i^* \neq \varnothing$.

Conversely, for any subset $A$ of $J_{n-2} \cap \mathcal{M}_n$ that intersects $X_i$ and $X_i^*$ non-trivially for all $i \in \{1,\ldots,n-1\}$, it is straightforward to see that $\langle \mathcal{M}_n \setminus J_{n-2},\ A\rangle$ contains every projection in $J_{n-2} \cap \mathcal{M}_n$, and hence is equal to $\mathcal{M}_n$. By Lemma 2.17, the maximal subsemigroups of $\mathcal{M}_n$ arising from its $\mathscr{J}$-class of rank $n-2$ are the sets $\mathcal{M}_n \setminus X_i$ and $\mathcal{M}_n \setminus X_i^*$ for $i \in \{1, \ldots, n-1\}$; these maximal subsemigroups have types (M4) and (M3), respectively. □

## Acknowledgements

The first author gratefully acknowledges the support of the Glasgow Learning, Teaching, and Research Fund in partially funding his visit to the third author in July, 2014. The second author wishes to acknowledge the support of research initiation grant [0076 | 2016] provided by BITS Pilani, Pilani. The fourth author wishes to acknowledge the support of his Carnegie Ph.D. Scholarship from the Carnegie Trust for the Universities of Scotland. The authors also thank the anonymous referee for their helpful comments.

## References

[1] A. Ya. Aĭzenštat, The defining relations of the endomorphism semigroup of a finite linearly ordered set, *Sibirsk. Mat. Z.* **3** (1962) 161–169.
[2] Karl Auinger, Krohn–Rhodes complexity of Brauer type semigroups, *Port. Math.* **69** (2012) 341–360.
[3] Ralph W. Ball, Maximal subgroups of symmetric groups, *Trans. Amer. Math. Soc.* **121** (1966) 393–407.
[4] James E. Baumgartner, Saharon Shelah, and Simon Thomas, Maximal subgroups of infinite symmetric groups, *Notre Dame J. Formal Logic* **34** (1993) 1–11.
[5] Georgia Benkart and Tom Halverson, Motzkin algebras, *European J. Combin.* **36** (2014) 473–502.
[6] Mirjana Borisavljević, Kosta Došen, and Zoran Petrić, Kauffman monoids, *J. Knot Theory Ramifications* **11** (2002) 127–143.
[7] Marcus Brazil, Jacinta Covington, Tim Penttila, Cheryl E. Praeger, and Alan R. Woods, Maximal subgroups of infinite symmetric groups, *Proc. London Math. Soc.* **68** (1994) 77–111.
[8] Jacinta Covington, Dugald Macpherson, and Alan Mekler, Some maximal subgroups of infinite symmetric groups, *Quart. J. Math. Oxford Ser.* **47** (1996) 297–311.
[9] I. Dimitrova and J. Koppitz, On the maximal subsemigroups of some transformation semigroups, *Asian-Eur. J. Math.* **1** (2008) 189–202.
[10] Ilinka Dimitrova, Vítor H Fernandes, and Jörg Koppitz, The maximal subsemigroups of semigroups of transformations preserving or reversing the orientation on a finite chain, *Publ. Math. Debrecen* **81** (2012) 11–29.
[11] Ilinka Dimitrova and Tsvetelina Mladenova, Classification of the maximal subsemigroups of the semigroup of all partial order-preserving transformations, *Union of Bulgarian Mathematicians* **41** (2012) 158–162.
[12] Igor Dolinka, James East, and Robert D. Gray, Motzkin monoids and partial Brauer monoids, *J. Algebra* **471** (2017) 251–298.
[13] C. Donoven, J. D. Mitchell, and W. A. Wilson, Computing maximal subsemigroups of a finite semigroup, preprint `https://arxiv.org/abs/1606.05583`
[14] J. East, J. D. Mitchell, and Y. Péresse, Maximal subsemigroups of the semigroup of all mappings on an infinite set, *Trans. Amer. Math. Soc.* **367** (2015) 1911–1944.
[15] James East, On the singular part of the partition monoid, *Internat. J. Algebra Comput.* **21** (2011) 147–178.
[16] James East and Robert D. Gray, Diagram monoids and Graham–Houghton graphs: Idempotents and generating sets of ideals, *J. Combin. Theory Ser. A* **146** (2017) 63–128.
[17] James East, Jitender Kumar, James D. Mitchell, and Wilf A. Wilson, Maximal subsemigroups of finite transformation and diagram monoids, preprint `https://arxiv.org/abs/1706.04967`.
[18] Vítor H. Fernandes, Gracinda M. S. Gomes, and Manuel M. Jesus, Presentations for some monoids of partial transformations on a finite chain, *Comm. Algebra* **33** (2005) 587–604.
[19] Desmond G. FitzGerald, A presentation for the monoid of uniform block permutations, *Bull. Austral. Math. Soc.* **68** (2003) 317–324.
[20] Olexandr Ganyushkin and Volodymyr Mazorchuk, On the structure of $IO_n$, *Semigroup Forum* **66** (2003) 455–483.
[21] Olexandr Ganyushkin and Volodymyr Mazorchuk, *Classical finite transformation semigroups*, volume 9 of *Algebra and Applications*, Springer-Verlag, London, 2009
[22] Gracinda M. S. Gomes and John M. Howie, On the ranks of certain semigroups of order-preserving transformations, *Semigroup Forum* **45** (1992) 272–282.
[23] N. Graham, R. Graham, and J. Rhodes, Maximal subsemigroups of finite semigroups, *J. Combinatorial Theory* **4** (1968) 203–209.
[24] R. L. Graham, On finite 0-simple semigroups and graph theory, *Math. Systems Theory* **2** (1968) 325–339.
[25] Iliya Gyudzhenov and Ilinka Dimitrova, On the maximal subsemigroups of the semigroup of all monotone transformations, *Discuss. Math. Gen. Algebra Appl.* **26** (2006) 199–217.

[26] Tom Halverson and Arun Ram, Partition algebras, *European J. Combin.* **26** (2005) 869–921.
[27] C. H. Houghton, Completely 0-simple semigroups and their associated graphs and groups, *Semigroup Forum* **14** (1977) 41–67.
[28] John M. Howie, *Fundamentals of semigroup theory*, volume 12 of *London Mathematical Society Monographs*, The Clarendon Press Oxford University Press, New York, 1995.
[29] Kwok Wai Lau and D. G. FitzGerald, Ideal structure of the Kauffman and related monoids, *Comm. Algebra* **34** (2006) 2617–2629.
[30] Martin W. Liebeck, Cheryl E. Praeger, and Jan Saxl, A classification of the maximal subgroups of the finite alternating and symmetric groups, *J. Algebra* **111** (1987) 365–383.
[31] Martin W. Liebeck and Aner Shalev, Maximal subgroups of symmetric groups, *J. Combin. Theory Ser. A* **75** (1996) 341–352.
[32] H. D. Macpherson and Peter M. Neumann, Subgroups of infinite symmetric groups, *J. London Math. Soc.* **42** (1990) 64–84.
[33] Victor Maltcev, On a new approach to the dual symmetric inverse monoid $\mathcal{I}_X^*$, *Internat. J. Algebra Comput.* **17** (2007) 567–591.
[34] Volodymyr Mazorchuk, On the structure of Brauer semigroup and its partial analogue, *Problems in Algebra* **13** (1998) 29–45.
[35] T. E. Nordahl and H. E. Scheiblich, Regular $*$ semigroups, *Semigroup Forum* **16** (1978) 369–377.
[36] John Rhodes and Benjamin Steinberg, *The q-theory of finite semigroups*, Springer Monographs in Mathematics. Springer, New York, 2009.
[37] Fred Richman, Maximal subgroups of infinite symmetric groups, *Canad. Math. Bull.* **10** (1967) 375–381.
[38] Leonard L. Scott, Representations in characteristic $p$, in *The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979)*, volume 37 of *Proc. Sympos. Pure Math.*, pages 319–331. Amer. Math. Soc., Providence, R.I., 1980.
[39] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, `http://oeis.org`.
[40] Stewart Wilcox, Cellularity of diagram algebras as twisted semigroup algebras, *J. Algebra* **309** (2007) 10–31.
[41] Yang Xiuliang, A classification of maximal subsemigroups of finite order-preserving transformation semigroups, *Comm. Algebra* **28** (2000) 1503–1513.



(James East) Centre for Research in Mathematics, School of Computing, Engineering and Mathematics, Western Sydney University, Sydney, Australia
*E-mail address*: `J.East@westernsydney.edu.au`

(Jitender Kumar) Department of Mathematics, Birla Institute of Technology and Science (BITS) Pilani, Pilani Campus-333031, Rajasthan, India
*E-mail address*: `jitenderarora09@gmail.com`

(James D. Mitchell) Mathematical Institute, North Haugh, St Andrews, Fife, KY16 9SS, Scotland
*E-mail address*: `jdm3@st-andrews.ac.uk`

(Wilf A. Wilson) Mathematical Institute, North Haugh, St Andrews, Fife, KY16 9SS, Scotland
*E-mail address*: `waw7@st-andrews.ac.uk`