# A COMMUTATIVE NONCOMMUTATIVE FRACTAL GEOMETRY

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# A Commutative Noncommutative Fractal Geometry

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by

**Anthony Samuel**

*Alles sollte so einfach wie möglich gemacht sein, aber nicht einfacher.*
- Albert Einstein

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## Abstract

**A Commutative Noncommutative Fractal Geometry**

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In this thesis examples of spectral triples, which represent fractal sets, are examined and new insights into their noncommutative geometries are obtained.

Firstly, starting with Connes' spectral triple for a non-empty compact totally disconnected subset $E$ of $\mathbb{R}$ with no isolated points, we develop a noncommutative coarse multifractal formalism. Specifically, we show how multifractal properties of a measure supported on $E$ can be expressed in terms of a spectral triple and the Dixmier trace of certain operators. If $E$ satisfies a given porosity condition, then we prove that the coarse multifractal box-counting dimension can be recovered. We show that for a self-similar measure $\mu$, given by an iterated function system $S$ defined on a compact subset of $\mathbb{R}$ satisfying the strong separation condition, our noncommutative coarse multifractal formalism gives rise to a noncommutative integral which recovers the self-similar multifractal measure $\nu$ associated to $\mu$, and we establish a relationship between the noncommutative volume of such a noncommutative integral and the measure theoretical entropy of $\nu$ with respect to $S$.

Secondly, motivated by the results of Antonescu-Ivan and Christensen, we construct a family of $(1, +)$-summable spectral triples for a one-sided topologically exact subshift of finite type $(\Sigma_{\mathcal{A}}^{\mathbb{N}}, \sigma)$. These spectral triples are constructed using equilibrium measures obtained from the Perron-Frobenius-Ruelle operator, whose potential function is non-arithemetic and Hölder continuous. We show that the Connes' pseudo-metric, given by any one of these spectral triples, is a metric and that the metric topology agrees with the weak$^*$-topology on the state space $\mathcal{S}(C(\Sigma_{\mathcal{A}}^{\mathbb{N}}); \mathbb{C})$. For each equilibrium measure $\nu_\phi$ we show that the noncommuative volume of the associated spectral triple is equal to the reciprocal of the measure theoretical entropy of $\nu_\phi$ with respect to the left shift $\sigma$ (where it is assumed, without loss of generality, that the pressure of the potential function is equal to zero). We also show that the measure $\nu_\phi$ can be fully recovered from the noncommutative integration theory.

## Declaration

I, Anthony Samuel, hereby certify that this thesis, which is approximately 27,700 words, has been written by me, that it is the record of work carried out by me and that it has not been submitted in any previous application for a higher degree.

I was admitted as a research student in October 2006 and as a candidate for the degree of Doctor of Philosophy in September 2007; the higher study for which this is a record was carried out in the University of St Andrews between 2007 and 2010.

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I hereby certify that the candidate has fulfilled the conditions of the Resolution and Regulations appropriate for the degree of Doctor of Philosophy in the University of St Andrews and that the candidate is qualified to submit this thesis in application for that degree.

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# Contents

# Chapter 1: Introduction

## 1.1 Summary of Main Results

The main goal of this thesis is to develop the theory of noncommutative fractal geometry, as originally proposed by Connes [Con3] and Lapidus [Lap]. A summary of our main contributions towards this theory is as follows.

**A Noncommutative Coarse Multifractal Formalism.** We show how multifractal properties of a Borel probability measure $\mu$ supported on a non-empty compact fractal set $E$ of $\mathbb{R}$ satisfying a certain porosity condition[1] can be expressed in terms of the complementary intervals of the support of $\mu$ (by a fractal set we mean a non-empty totally disconnected space with no isolated points). This allows the development of a noncommutative analogue of a coarse multifractal formalism for Connes' spectral triple representation of the set $E$. Specifically, we prove that from this new development one can recover the coarse multifractal box-counting dimension of $\mu$. For a self-similar measure $\mu$, given by an iterated function system $S$, we then show that our noncommutative coarse multifractal formalism gives rise to a noncommutative integral which recovers the associated self-similar multifractal measure $\nu$, and we establish a relationship between the volume of such a noncommutative integral and the measure theoretical entropy of $\nu$ with respect to $S$.

**The Noncommutative Volume of a Subshift of Finite Type.** By refining the methods of Antonescu-Ivan and Christensen given in [AIC1], we derive a $(1, +)$-summable spectral triple for each one-sided topologically exact subshift of finite type $(\Sigma_A^\infty, \sigma)$ equipped with an equilibrium measure $\nu_\phi$ (where $\phi \in C(\Sigma_A^\infty; \mathbb{C})$ denotes some Hölder continuous non-arithmetic potential function). We show that a variety of geometric and measure theoretic information can be recovered form such a spectral triple. We prove that Connes' pseudo-metric, given by our spectral triple, is a metric on the state space $\mathcal{S}(C(\Sigma_A^\infty; \mathbb{C}))$ of the $C^*$-algebra of complex-valued continuous functions defined on $\Sigma_A^\infty$, and that the topology induced by this metric is equivalent to the weak$^*$-topology on $\mathcal{S}(C(\Sigma_A^\infty; \mathbb{C}))$. We show that the noncommutative integration theory of our spectral triple is capable of recovering the measure $\nu_\phi$ and that the noncommutative volume is equal to the reciprocal of the measure theoretical entropy of $\nu_\phi$ with respect to the left shift $\sigma$.

## 1.2 Motivation and History

In the 1980s Connes formalised the notion of noncommutative geometry (see for instance [Con3, Con1]) and, in doing so, showed that the tools of differential geometry can be extended to certain non-Hausdorff spaces known as "bad quotients" and to spaces of a "fractal" nature. Such spaces are abundant in nature and commonly arise from various dynamical systems.

A main idea of noncommutative geometry is to analyse geometric spaces using operator algebras, particularly $C^*$-algebras. This idea first appeared in the work of Gelfand and Naĭmark [GN], where it was shown that a $C^*$-algebra can be seen as the noncommutative analogue of the space of complex-valued continuous functions on a locally compact metric space. Also, note that for a

[1] The concept of lacunarity (derived from the Latin word *lacuna* meaning gap) or porosity (derived from the Latin word *porus* meaning pore - a minute opening in a surface) for a fractal set was introduced by Mandelbrot. His introduction of this concept in [Man4] begins with the following curious sentence. "A second skeleton rattles in the closets of most models of the distribution of galaxies."

smooth compact spin Riemannian manifold, one can recover its smooth structure, its volume and its Riemannian metric directly from its standard Dirac operator (see [Jos]). Motivated by these observations, Connes proposed the concept of a spectral triple. A *spectral triple* is a triple $(A, H, D)$ consisting of a $C^*$-algebra $A$, which acts faithfully on a separable Hilbert space $H$, and an essentially self-adjoint unbounded operator $D$ defined on $H$ with compact resolvent such that the set

$$\{a \in A : \text{the operator } [D, \pi(a)] \text{ extends to a bounded operator defined on } H\}$$

is dense in $A$. (Here $\pi : A \to B(H)$ denotes the faithful action of $A$ on $H$.) Connes showed that with such a structure one can obtain a pseudo-metric on the state space $\mathcal{S}(A)$ of $A$, analogous to how the Monge-Kantorovitch metric is defined on the space of probability measures on a compact metric space. In 1998 Rieffel [Rie2] and Pavlović [Pav] established conditions under which Connes' pseudo-metric is a metric and established conditions under which the metric topology of Connes' pseudo-metric is equivalent to the weak$^*$-topology defined on $\mathcal{S}(A)$. Also, Connes [Con3] showed that a notion of dimension (called the metric dimension) and that a theory of integration can be derived for such structures. He also proved that for an arbitrary smooth compact spin Riemannian manifold there exists a spectral triple from which the metrical information, the measure theoretical information and the smooth structure of the manifold can be recovered (see [Con3, Ren]). This illustrates that a spectral triple allows one to move beyond the limits of classical Riemannian geometry. That is to say, not only is one able to recover classical aspects of Riemannian geometry, but through the notion of a spectral triple one is able to extend the tools of Riemannian geometry to situations that present themselves at the boundary of classically defined objects, for instance, objects which "live" on the boundary of Teichmüller space (such as the noncommutative torus) or those of a "fractal" nature (such as the middle third Cantor set). Although one of the original motivations for noncommutative geometry was to be able to deal with non-Hausdorff spaces, such as foliated manifolds, which are often best represented by a noncommutative $C^*$-algebra (see [Con3, Vár, Mar, Rie3]), this new theory has scope, even when the $C^*$-algebra is commutative.

In Connes' seminal book [Con3], the concept of a noncommutative fractal geometry is introduced. Consequently, a remarkable amount of interest has developed in this subject. In Chapter IV of [Con3], Connes gives numerous examples to indicate how fractal sets can be represented by spectral triples. Connes' examples include non-empty compact totally disconnected subsets of $\mathbb{R}$ with no isolated points and limit sets of Fuchsian groups of the second kind. Subsequently, in 1997 Lapidus [Lap] proposed several ways in which the notions of a noncommutative fractal geometry could be extended, after which several important articles on the subject appeared. For instance, in [GI1] Guido and Isola analysed the spectral triple presented by Connes for limit fractals in $\mathbb{R}$ which satisfy a certain separation condition. (Note that such sets are non-empty compact totally disconnected and have no isolated points.) There, the authors investigated aspects of Connes' pseudo-metric, the metric dimension and the noncommutative integral of Connes' spectral triple. In [GI2] this construction and analysis is extended to limit fractals in $\mathbb{R}^n$, for all $n \in \mathbb{N}$. Further, Antonescu-Ivan and Christensen [AIC1] have provided a construction of a spectral triple for an AF (approximately finite) $C^*$-algebra with particular focus on aspects of Connes' pseudo-metric. In [AICL] the authors give several examples of spectral triples which represent fractal sets such as the von Koch curve and the Sierpiński gasket. There, the authors showed that for such sets the Hausdorff dimension can be recovered and that Connes' pseudo-metric induces a metric equivalent to the metric induced by the ambient space on the given set. More recently, in [BP] the authors adapt Connes' spectral triple to represent the code space $\{0,1\}^{\mathbb{N}}$ equipped with an ultra-metric $d$. It is shown that an adaptation of Connes' pseudo-metric gives rise to a metric equal to $d$. Further, they proved that the box-counting dimension can be recovered and that a noncommutative integration theory gives rise to an integral with respect to the normalised $\delta$-dimensional Hausdorff measure on the metric space $(\{0,1\}^{\mathbb{N}}, d)$. (Here, $\delta$ denotes the Hausdorff dimension of $(\{0,1\}^{\mathbb{N}}, d)$.)

In [CL, PS, Kra, IKM] the authors showed that any finite metric space can be represented by a finite spectral triple and that from such a representation one can recover the full geometric structure of the finite metric space. Within these articles a full classification of finite spectral triples is given. Further, in [Con4, GBIS] a finite spectral triple which represents the standard model in particle physics is constructed. There, neutrinos are assumed to be massless. In [Con5], investigations are carried out in which the assumption that neutrinos are massless is not made.

Attempts to build spectral triples for general (non-fractal) compact metric spaces have been made by Antonescu-Ivan and Christensen [AIC2]. There, the authors constructed spectral triples for an arbitrary compact metric space $(X, d)$ by gluing together finite spectral triples associated with a two-point set as described by Connes in Example 2a on page 563 of [Con3]. In doing so, they showed that Connes' pseudo-metric induces a metric on $X$ which is equivalent to $d$.

## 1.3 Outline and Statement of Main Results

The main contributions of this thesis are contained in Chapter 4, where our core results are contained in Theorem 4.1.9, Theorem 4.1.11, Theorem 4.2.5, Theorem 4.2.6 and Theorem 4.2.7. In Theorem 2.1.20, Theorem 2.2.10, Proposition 2.3.4, Corollary 2.3.7, Corollary 2.3.10 and Theorem 3.2.17 we give new results which are both interesting themselves and essential to the proofs of our main results. Below, we give a more detailed outline of the work carried out in this thesis.

**Chapter 2: Fractals, Dynamics and Renewal Theorems.** In this chapter, we begin by discussing some of the basic aspects of fractal geometry that will be required in the subsequent chapters. The first section, Section 2.1, is split into three main parts. A general and brief introduction to fractal measures and dimensions (Subsection 2.1.1), a brief review of the Minkowski content of a subset of $\mathbb{R}$ (Subsection 2.1.2) and finally an introduction to the notions of coarse multifractal analysis (Subsection 2.1.3). The material contained in Subsection 2.1.1 and Subsection 2.1.2 is standard in the theory of fractal geometry and these subsections are respectively based on material contained in [Fal1] and [Fal2]. In Subsection 2.1.3, we define the coarse multifractal box-counting dimension $\mathfrak{b}(q)$ at $q \in \mathbb{R}$ for a given Borel probability measure $\mu$ with compact support, where we use the extension for negative $q$ introduced by Riedi [Rie1]. We then prove that an equivalent definition of $\mathfrak{b}$ exists in terms of the complement of the support of $\mu$, provided that the support of $\mu$ is strongly porous.

**Definition.** *(Definition 2.1.10.)* A subset $E$ of $\mathbb{R}$ is defined to be *strongly porous* with *porosity constant* $\rho \in (0,1)$, if for each $x \in E$ and $r \in (0,1]$ the ball $B(x,r)$ contains a complementary interval of $E$ with diameter greater than or equal to $\rho r$.

**Theorem.** *(Theorem 2.1.20.) Let $\mu$ denote a Borel probability measure on a non-empty compact subset of $\mathbb{R}$. Assume that the support of $\mu$ is strongly porous with porosity constant $\rho > 0$, and let $\{I_k : k \in \mathbb{N}\}$ denote the set of complementary intervals of supp$(\mu)$ whose lengths are finite. If $\eta \geqslant 2\rho^{-1}$, then for each $q \in \mathbb{R}$, we have that*

$$\mathfrak{b}(q) \;=\; \inf\left\{ t \in \mathbb{R} \,:\, \sum_{k\in\mathbb{N}} \mu(\overline{I}_k^{\eta})^q |I_k|^t < \infty \right\} \;=\; \inf\left\{ t \in \mathbb{R} \,:\, \limsup_{N\to\infty} \frac{\sum_{k=1}^{N} \mu(\overline{I}_k^{\eta})^q |I_k|^t}{\ln(N)} = 0 \right\}.$$

In the above theorem, and later, for an interval $I \subset \mathbb{R}$, we let $|I|$ denote the length of $I$ and, for each $\eta > 0$, we let $\overline{I}^{\eta}$ denote the closed ball centred at the midpoint of $I$ with radius $(1+\eta)|I_k|/2$.

Although the result of the above theorem seems unusual at first within the context of standard multifractal analysis, it is useful in the formulation of a noncommutative coarse multifractal formalism.

In the next section, Section 2.2, we introduce the concept of a one-sided subshift of finite type. We describe the thermodynamic formalism for this setting, as developed by Bowen and Ruelle ([Bow1, Bow2, Rue1, Rue2]). We state the results which give the existence of a Gibbs measure and the existence and uniqueness of an equilibrium measure on a one-sided topologically exact subshift of finite type. Finally, in Theorem 2.2.10, a new notion of Haar basis for the Hilbert space $L^2(\Sigma_A^{\infty}, \mathcal{B}, \mu)$ is developed. (Here, $(\Sigma_A^{\infty}, \sigma)$ denotes a one-sided topologically exact subshift of finite type and $\mu$ denotes a Gibbs measure with support equal to $\Sigma_A^{\infty}$.) This concept enables us to describe in a natural way the filtration on $L^2(\Sigma_A^{\infty}, \mathcal{B}, \mu)$ induced by the Gelfand-Naĭmark-Segal completion and the AF-structure of the $C^*$-algebra of complex-valued continuous functions defined on $\Sigma_A^{\infty}$. Thus, we are able to refine and develop the spectral triple of Antonescu-Ivan and Christensen's for an AF $C^*$-algebra, in the setting of a one-sided topologically exact subshift of finite type.

The final section of this chapter, Section 2.3, contains a discussion of three renewal theorems for fractal sets and topologically exact subshifts of finite type. A description of the renewal theorems presented in [Fal3, Lal, GH] is given and it is shown how these results lead to various interesting counting results. Specifically, we derive the following.

1. Let $\{0,1\} \subset E \subset [0,1]$ denote a non-empty compact self-similar set whose iterated function system of similarities satisfies the strong separation condition. Set $\delta$ equal to the Hausdorff dimension of $E$ and let $\{I_k \subset [0,1] : k \in \mathbb{N}\}$ denote the set of complementary intervals of $E$. Let $\mathcal{E} : (0,\infty) \to \mathbb{R}$ be defined, for each $r \in (0,\infty)$, by

$$\mathcal{E}(r) := \sum_{\substack{k \in \mathbb{N} \\ |I_k| \geqslant r}} |I_k|^{\delta}.$$

   For $f, g : \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$, we say that $f$ is *asymptotic* to $g$ as $x$ tends to $x_0$, if $\lim_{x \to x_0} f(x)/g(x) = 1$. We write $f \sim g$ as $x$ tends to $x_0$.

   **Theorem.** *(Proposition 2.3.4.) There exists a positive constant $c$ such that, as $r$ tends to zero, $\mathcal{E}(r) \sim c\ln(r)$.*

   The precise value of the constant $c$ above is given in Proposition 2.3.4 and is related to the geometric structure of the set $E$.

2. Let $(\Sigma_A^{\infty}, \sigma)$ denote a one-sided topologically exact subshift of finite type equipped with an equilibrium measure $\nu_\phi$ for a real-valued non-arithmetric Hölder continuous potential function $\phi \in C(\Sigma_A^{\infty}; \mathbb{R})$. Let $h_{\nu_\phi}(\sigma)$ denote the measure theoretical entropy of $\nu_\phi$ with respect to the left shift $\sigma$. For each $\mathcal{V} \in \Sigma_A^* \cup \emptyset$, let $\Upsilon_{\mathcal{V}}, \Xi_{\mathcal{V}} : (0,\infty) \to [0,\infty)$ be defined, for each $r \in (0,\infty)$, by

$$\Upsilon_{\mathcal{V}}(r) := \sum_{\substack{\omega \in \Sigma_A^* \text{ with} \\ \nu_\phi([\omega]) > r \text{ and } [\omega] \subseteq [\mathcal{V}]}} 1, \qquad \Xi_{\mathcal{V}}(r) := \sum_{\substack{\omega \in \Sigma_A^* \text{ with} \\ \nu_\phi([\omega]) > r \text{ and } [\omega] \subseteq [\mathcal{V}]}} \mu_\phi([\omega]).$$

   Here, $\Sigma_A^*$ denotes the set of admissible words of finite length and $[\omega]$ denotes the cylinder set associated to $\omega \in \Sigma_A^*$. If $\mathcal{V} = \emptyset$, then we set $[\mathcal{V}] := \Sigma_A^{\infty}$.

For $f, g : \mathbb{R} \to [0, \infty)$ and for $x_0 \in \mathbb{R}$, we say that $f$ is *comparable* to $g$ as $x$ tends to $x_0$ if there exist constants $c_1, c_2 > 0$ such that for all $x$ sufficiently close to $x_0$, we have that $c_1 f(x) \leqslant g(x) \leqslant c_2 f(x)$. We write $f \asymp g$ as $x$ tends to $x_0$.

**Theorem.** *(Corollary 2.3.7 and Corollary 2.3.10) For each $\mathcal{V} \in \Sigma_A^* \cup \emptyset$, as $r$ tends to zero, we have that*

$$\Upsilon_{\mathcal{V}}(r) \asymp r^{-1}, \qquad \Xi_{\mathcal{V}}(r) \sim \frac{\nu_\phi([\mathcal{V}])r}{h_{\nu_\phi}(\sigma)}.$$

These counting results, interesting in themselves, also allow us to prove new results. In particular, they allow us to formulate a link between the notion of measure theoretical entropy and the notion of a noncommutative volume for a one-sided topologically exact subshift of finite type equipped with an equilibrium measure (Theorem 4.2.7).

**Chapter 3: $C^*$-algebras and Noncommutative Geometry.** Here, we give some of the basic concepts of noncommutative geometry which we will use and extend in Chapter 4. The work presented in this chapter is partially based on work published in [Con3, Vár, FGBV, Mar, BO, Dav] and is organised as follows.

We begin in Section 3.1 by formally defining a $C^*$-algebra and stating the seminal classification theorems of $C^*$-algebras by Gelfand and Naĭmark as presented in [GN]. We then define the notion of a noncommutative dynamical system, or, more precisely, a $C^*$-dynamical system, and describe how one can obtain from such a system the class of $C^*$-algebras called discrete cross product algebras. Such a class of $C^*$-algebras allows us to demonstrate, by way of example, how Connes' theory of noncommutative geometry can be applied to noncommutative $C^*$-algebras (see Subsection 3.3.2 and Subsection 3.3.3 for such application). A more extensive analysis of noncommutative dynamical systems can be found in the recent preprint [BMR].

In Section 3.2, following [Con3], we define the notion of a spectral triple and describe the geometric and measure theoretic information one can obtain from such an object.

1. Connes' pseudo-metric (Subsection 3.2.1).
2. The notion of a finitely summable, $(p, +)$-summable and $\theta$-summable spectral triple, where the first notion gives rise to the notion of metric dimension for spectral triples (Subsection 3.2.2).
3. The notion of a noncommutive integration theory and that of a noncommutative volume (Subsection 3.2.2).

To conclude the chapter, we give three basic examples of spectral triples, examining their noncommutative geometries. Although most of the material in this section is well-known, it is often the case that many of the finer details do not seem to appear in the literature. When this is the case we provide a full account. Specifically, we examine the noncommutative geometries of spectral triple representations of the following: the unit circle (Subsection 3.3.1), noncommutative tori (Subsection 3.3.2) and duals of countably infinite discrete groups (Subsection 3.3.3). In the case of the noncommutative torus we take a more dynamical approach than that usually presented in the literature (see [Con3, Vár, FGBV]). The material contained in Subsection 3.3.3 is based on material contained in [Con2].

**Chapter 4: A Commutative Noncommutative Fractal Geometry.** This chapter divides into two main sections: Section 4.1, a version of which has been recently published in [FS] by Falconer and Samuel and Section 4.2, which extends the results of Antonescu-Ivan and Christensen [AIC1].

In Section 4.1 we begin by describing Connes' construction of a spectral triple $(A, H, D)$ for a non-empty compact totally disconnected subset $E$ of $\mathbb{R}$ with no isolated points. Then in Subsection 4.1.1 we investigate the geometric properties of $(A, H, D)$. Specifically, we explore the relationships between the following concepts.

1. The metric dimension of $(A, H, D)$ and the Hausdorff dimension $\dim_{\mathcal{H}}(E) =: \delta$ of $E$ (Theorem 4.1.3 and Theorem 4.1.4).
2. The noncommutative volume of $(A, H, D)$ and the Minkowski content of $E$, provided that $E$ is Minkowski measurable (Theorem 4.1.5 and Corollary 4.1.8).
3. The noncommutative volume of $(A, H, D)$ and the measure theoretical entropy of the normalised $\delta$-dimensional Hausdorff measure on $E$ with respect to $S$, where $E$ is a self-similar set with associated iterated function system $S$ satisfying the strong separation condition (Theorem 4.1.6 and Corollary 4.1.8).
4. The noncommutative integral given by $(A, H, D)$ and the normalised $\delta$-dimensional Hausdorff measure on $E$ (Theorem 4.1.7 and Corollary 4.1.8).
5. Connes' pseudo metric given by $(A, H, D)$ and the Monge-Kantorovitch metric on the space of Borel probability measures on $E$ (see the concluding remarks of Subsection 4.1.1).

Although some of the relations stated above are well-known and are presented in [Con3, GI1], we provide new proofs. Further, we make explicit a point of ambiguity in Theorem 4.2 of [GI1]. In a personal communication [Bel] we were informed that this ambiguity is well-known to experts in the field of noncommutative geometry.

In Subsection 4.1.2 we develop a noncommutative analogue of a coarse multifractal formalism for Connes' spectral triple $(A, H, D)$ for a compact totally disconnected subset $E$ of $\mathbb{R}$ with no isolated points.

Let $\mathfrak{b}(q)$ denote the coarse multifractal box-counting dimension of $\mu$ at $q \in \mathbb{R}$ and let $\mathcal{L}^{1,+}(H)$ denotes the Dixmier ideal of $H$. Under the assumptions that $E$ is strongly porous and that $\mu$ is a Borel probability measure with support equal to $E$ satisfying the (mild) requirement that, as $r$ tends to zero

$$\frac{\ln(\mu(B(x,r)))}{\ln(r)} \asymp 1,$$

uniformly in $x \in E$, we prove the following.

**Theorem.** *(Theorem 4.1.9.) Let $\rho$ denote the porosity constant of $E$ and let $\eta \geqslant \rho$. Then there exists a bounded linear operator $\mathcal{Q}_{\mu,\eta} : H \to H$ (which we specify) dependent on $\mu$ and $\eta$ such that, for each $q \in \mathbb{R}$, we have that*

$$\mathfrak{b}(q) = \inf\{p \in \mathbb{R} : \mathcal{Q}_{\mu,\eta}^{p} |D|^{q} \in \mathcal{L}^{1,+}(H)\}. \tag{1.1}$$

Next, in Theorem 4.1.11, we focus on the case where $E$ denotes a self-similar subset of $\mathbb{R}$ generated by an iterated function system of similarities $S$ which satisfies the strong separation condition and where $\mu$ denotes a self-similar Borel probability measure on $E$. Here, we show that one can obtain a noncommutative integral which recovers the associated self-similar multifractal measure $\nu$. Before giving the statement of our result, we set the following notation.

1. Let $(A, H, D)$ denote Connes' spectral triple for the set $E$ where $\pi : A \to B(H)$ denotes the faithful action of $A$ on $H$ (note that $A := C(E; \mathbb{C})$).
2. Since $S$ satisfies the strong separation condition, this implies that the set $E$ is strongly porous. Letting $\rho$ denote the porosity constant of $E$, for each $\eta \geqslant \rho$, let $\mathcal{Q}_{\mu,\eta} : H \to H$ denote the bounded linear operator as in Equation (1.1).
3. For a given limiting procedure $\mathcal{W}$, let $\mathrm{Tr}_{\mathcal{W}}$ denote the Dixmier trace with respect to $\mathcal{W}$. Note that it is through the Dixmier trace that one obtains a noncommutative integral.

**Theorem.** *(Theorem 4.1.11) There exist positive constants $h_\nu, R_1$ such that for each limiting procedure $\mathcal{W}$, each $\eta \geqslant \rho$ and each $a \in A := C(E; \mathbb{C})$, we have that*

$$Tr_{\mathcal{W}}(\pi(a)\mathcal{Q}_{\mu,\eta}|D|^{-\mathfrak{b}(q)}) \;=\; 2R_1 h_\nu \int_E a \, d\nu.$$

In the above theorem the constant $h_\nu$ is equal to the measure theoretical entropy of the measure $\nu$ with respect to $S$. Further, the precise value of the constant $R_1$ is given in Proposition 2.3.4 and is related to the geometric structure of the set $E$ and the measure $\mu$.

In our final section, Section 4.2, we build on the results of Antonescu-Ivan and Christensen [AIC1]. This section divides into two main parts. In Subsection 4.2.1, we review the relevant results of [AIC1]. We also include an application of these results as given in Proposition 1.9 of [CM]. We add to the discussion presented in [CM] by considering the metric dimension, the noncommutative volume, the noncommutative integral and aspects of Connes' pseudo-metric of the given spectral triple.

In Subsection 4.2.2 we show how one can construct a spectral triple

$$(A, H, D) \;:=\; (C(\Sigma_A^\infty; \mathbb{C}), L^2(\Sigma_A^\infty, \mathcal{B}, \nu_\phi), D_{\nu_\phi})$$

for a one-sided topologically exact subshift of finite type $(\Sigma_A^\infty, \sigma)$ and an equilibrium measure $\nu_\phi$ for a Hölder continuous potential function $\phi \in C(\Sigma_A^\infty; \mathbb{R})$. This construction is motivated by the results of Antonescu-Ivan and Christensen and uses the results of Subsection 2.2.2 where we develop a notion of a Haar basis for a one-sided subshift of finite type. To justify that the spectral triple we construct is reasonable we observe and prove the following.

**Theorem.** *(Theorem 4.2.6.) Connes' pseudo-metric $d_{C(\Sigma_A^\infty;\mathbb{C})}$ given by the spectral triple $(A, H, D)$ is a metric on the state space $\mathcal{S}(A)$ of $A$. Moreover, the topology induced by Connes' pseudo-metric is equivalent to the weak$^*$-topology on $\mathcal{S}(C(\Sigma_A^\infty; \mathbb{C}))$.*

**Theorem.** *(Theorem 4.2.7.) Let $\phi \in C(\Sigma_A^\infty; \mathbb{R})$ denote a Hölder continuous non-arithmetic potential function and let $\nu_\phi$ denote the unique equilibrium measure on $\Sigma_A^\infty$ for the potential $\phi$. Then the spectral triple $(A, H, D)$ is $(1, +)$-summable with metric dimension equal to one. Moreover, for each limiting procedure $\mathcal{W}$ and each $a \in A := C(\Sigma_A^\infty; \mathbb{C})$, we have that*

$$Tr_{\mathcal{W}}(\pi(a)|D_{\nu_\phi}|^{-1}) \;=\; \frac{1}{h_{\nu_\phi}(\sigma)} \int_{\Sigma_A^\infty} a \, d\nu_\phi.$$

*In particular, the noncommutative volume of $(A, H, D)$ is equal to $1/h_{\nu_\phi}(\sigma)$.*

Very shortly before this thesis was to be submitted, the author learnt that Sharp [Sha] (motivated by our work in [FS]) developed a very similar result to that presented in Theorem 4.2.7. The main focus of [Sha] is on representing a topologically mixing sub-shift of finite type by a spectral triple and calculating the noncommutative volume using a spectral triple similar to that presented in [BP]. Although our results are similar, Sharp obtains his results using different methods.

## 1.4 Basic Notation and Definitions

In this section we set out basic terminology and notation that will frequently be encountered.

1. Let $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$ denote the sets of all natural, integer, rational, real and complex numbers, respectively. It is assumed that the natural numbers exclude zero, and so, let $\mathbb{N}_0$ denote the set of non-negative integers.

2. For a subset $E$ of $\mathbb{R}^n$ let $|E|$ denote the Euclidean diameter of $E$ and let $\overline{E}$ denote the closure of $E$, that is, the small closed subset of $\mathbb{R}^n$ containing $E$. Further, let $\partial E$ denote the closure of $E$ minus the interior of $E$, where the interior of $E$ is defined to be the largest open subset of $\mathbb{R}^n$ which is fully contained in $E$.

3. For each $z \in \mathbb{C}$, the same symbol is used for the (complex) norm of $z$, that is, $|z| := (z\overline{z})^{1/2}$.

4. Two notions which we will repeatedly use are those of comparability and asymptoticity.

   (a) For $f, g : \mathbb{R} \to [0, \infty)$ and $x_0$ belonging to the extended real numbers, we say that $f$ is *comparable* to $g$ as $x$ tends to $x_0$ if there exist constants $c_1, c_2 > 0$ such that for all $x$ sufficiently close to $x_0$ (and in the case that $x_0 = \pm\infty$, for all $x$ sufficiently large, respectively sufficiently small), we have that $c_1 f(x) \leqslant g(x) \leqslant c_2 f(x)$. We write $f \asymp g$ as $x$ tends to $x_0$.

   (b) For $f, g : \mathbb{R} \to \mathbb{R}$ and $x_0$ belonging to the extended real numbers, we say that $f$ is *asymptotic* to $g$ as $x$ tends to $x_0$ (and in the case that $x_0 = \pm\infty$, for all $x$ sufficiently large, respectively sufficiently small) if $\lim_{x \to x_0} f(x)/g(x) = 1$. We write $f \sim g$ as $x$ tends to $x_0$.

5. For a topological space $(X, \mathcal{T})$ and a continuous function $T : X \to X$, two continuous functions $g, h : X \to \mathbb{R}$ are said to be *cohomologous with respect to* $T$ if there exists a continuous function $\phi : X \to \mathbb{R}$ such that $g - h = \phi - \phi \circ T$. This difference is called the *co-boundary of* $g$ *and* $h$ *with respect to* $T$.

6. A topological space is called *totally disconnected* if and only if its connected components consist of single points. If a topological space has no open set which consists of a single point, then we say it has *no isolated points.*

7. Let $(X, \mathcal{T})$ denote a topological space. A subset $Y$ of $X$ is called *discrete* if and only if, for all $y \in Y$, there exists $U \in \mathcal{T}$ such that $Y \cap U = \{y\}$

8. For a topological space $(X, \mathcal{T})$, let $\mathcal{B}$ denote the *Borel $\sigma$-algebra*, that is, the $\sigma$-algebra generated by the open sets of $X$. Two finite measures $\mu_1$ and $\mu_2$ on $\mathcal{B}$ are said to be *equivalent*, if for each $B \in \mathcal{B}$ we have that $\mu_1(B) = 0$ if and only if $\mu_2(B) = 0$.

9. Let $\mu$ denote a finite Borel measure on a topological space $(X, \mathcal{T})$. The *support of* $\mu$, denoted by $\mathrm{supp}(\mu)$, is defined to be the set of all points $x \in X$ for which every open neighbourhood of $x$ has positive measure.

10. Let $(X_1, \mathcal{C}_1, \mu_1)$ and $(X_2, \mathcal{C}_2, \mu_2)$ be two measure spaces and let $T : X_1 \to X_2$ denote a measurable map. Then $T$ is said to be *measure preserving* if $\mu_1(T^{-1}(C)) = \mu_2(C)$ for all $C \in \mathcal{C}_2$.

11. If $(X, \mathcal{C}, \mu)$ is a measure space, $T : X \to X$ a measure preserving transformation and $\mu(X) < \infty$, then the measure preserving transformation $T$ is called *ergodic* if and only if for all $C \in \mathcal{C}$ with $T^{-1}(C) = C$ one has that $\mu(C) = 0$ or $\mu(C) = \mu(X)$.

12. For $n \in \mathbb{N}$, let $\lambda^n$ denote the $n$-dimensional Lebesgue measure.

13. For a given set $X$, we define the following.

    (a) For $Y \subseteq X$, let $\chi_Y$ denote the *characteristic function of* $Y$, that is, for each $x \in X$, define
    $$\chi_Y(x) := \begin{cases} 0 & \text{if } x \notin Y, \\ 1 & \text{if } x \in Y. \end{cases}$$

    (b) For $x \in X$, let $\delta_x$ denote the *Dirac point mass at* $x$, that is, for a subset $Y$ of $X$, define
    $$\delta_x(Y) := \begin{cases} 0 & \text{if } x \notin Y, \\ 1 & \text{if } x \in Y. \end{cases}$$

    (c) For each $x, y \in X$, let $\delta_{x,y}$ denote the *Kronecker-delta symbol*, that is,
    $$\delta_{x,y} := \begin{cases} 0 & \text{if } x \neq y, \\ 1 & \text{if } x = y. \end{cases}$$

14. For a topological space $(X, \mathcal{T})$, let $C(X; \mathbb{C})$ denote the set of complex-valued continuous functions on $X$ and let $C(X; \mathbb{R})$ denote the set of real-valued continuous functions on $X$. For a measure space $(X, \mathcal{B}, \mu)$, we let $L^2(X, \mathcal{B}, \mu)$ denote the Hilbert space of complex-valued square integrable functions on $X$ with respect to $\mu$, where the inner product is given, for all $h_1, h_2 \in L^2(X, \mathcal{B}, \mu)$, by
    $$\langle h_1, h_2 \rangle := \int_X h_1 \overline{h_2} \, d\mu.$$

15. For each $k \in \mathbb{N}$, let $\Pi_k$ denote the symmetric group on $k$ symbols.

16. For a complex separable Hilbert space $H$, let $B(H)$ denote the algebra of bounded linear operators from $H$ to $H$ and let $K(H)$ denote the ideal of $B(H)$ consisting of compact linear operators. Further, for $T \in B(H)$ denote the adjoint of $T$ by $T^*$. For a linear operator $T \in B(H)$ define the *trace* of $T$ to be $\mathrm{tr}(T) := \sum_{k \in \mathbb{N}} \langle T(e_k), e_k \rangle$, where $\{e_k\}_{k \in \mathbb{N}}$ is an arbitrary orthonormal basis for $H$. An operator $T$ is called a *trace-class operator* if and only if the trace of $|T| := (T^*T)^{\frac{1}{2}}$ is finite. Observe that set of trace-class operators form a subset of $K(H)$. Background on further relevant notions from functional analysis are included in Appendix A.

17. For a Hilbert space $H$, we let $\mathbb{1}$ denote the identity element of the set $B(H)$.

18. Following convention, we let $\otimes$ denote the tensor product of vector spaces.

19. Finally, throughout this thesis we shall assume the Axiom of Choice.

# Chapter 2: Fractals, Dynamics and Renewal Theorems

The aim of this chapter is to present necessary background material from the areas of fractal geometry (Section 2.1), symbolic dynamics (Section 2.2) and renewal theory (Section 2.3). To that end, we state various well-known results and examples, but also give some new results. With these foundations and those of Chapter 3, we will be able to achieve our overall goal of constructing and developing a theory of a noncommutative fractal geometry.

## 2.1 Fractal Geometry

Let us begin by collecting relevant results from fractal geometry. The majority of the material detailed here is well-known and so is stated here without proof, with the exception of the final two results of Subsection 2.1.3 which do not seem to appear in the current literature. For the interested reader, there is an extensive literature available, with good overviews contained in [Fal1, Fal3, Man4, Pol2].

### 2.1.1 Fractal Measures and Dimensions

In the foundational essay [Man3], Mandelbrot introduced the subject of fractal geometry. One of the main motivations was to introduce tools which would be able to deal with irregular and fragmented patterns which occur in nature and science. Often, unlike "smooth" objects whose structure becomes simpler on a shrinking scale, fractal objects tend to be irregular or fragmented on a shrinking scale. Therefore, fractal sets are too irregular to be described either locally or globally with traditional geometric tools.

Various attempts have been made to give a mathematically precise definition of a fractal, but in general such definitions have proven to be unsatisfactory. Therefore, it is often the case that a set is described as being fractal if it satisfies certain characteristics, for instance the above-described irregularity at all scales. Another characteristic is having a non-integer Hausdorff dimension, which is obtained from the Hausdorff measure, where the Hausdorff measure is defined in an analogous way to the $n$-dimensional Lebesgue measure, for $n \in \mathbb{N}$. In what follows, let $n \in \mathbb{N}$ be fix.

**Definition 2.1.1.** Let $E$ denote a subset of $\mathbb{R}^n$, let $s \geqslant 0$ and let $\eta > 0$. Define

$$\mathcal{H}^s_\eta(E) \ := \ \inf \left\{ \sum_{k \in \mathbb{N}} |E_k|^s \ : \ E \subseteq \bigcup_{k \in \mathbb{N}} E_k \text{ and } |E_k| < \eta \right\}$$

to be the $\eta$ *approximation to the $s$-dimensional Hausdorff measure* and define

$$\mathcal{H}^s(E) \ := \ \sup_{\eta > 0} \mathcal{H}^s_\eta(E)$$

to be the *$s$-dimensional Hausdorff measure* of $E$.

**Theorem 2.1.2.** *For $s \geqslant 0$, we have that $\mathcal{H}^s$ is a regular Borel measure and if $E \subseteq \mathbb{R}^n$, then there exists a unique $\delta \in \mathbb{R}$ such that, for all $\epsilon \in (0, \delta)$, we have that*

$$\mathcal{H}^{\delta - \epsilon}(E) \ = \ \infty, \qquad \mathcal{H}^{\delta + \epsilon}(E) \ = \ 0$$

*Moreover, one has that $\lambda^n = c_n \mathcal{H}^n$, where $c_n$ is the $n$-dimensional Lebesgue volume of the unit ball in $\mathbb{R}^n$.*

Proof. See page 31 of [Fal1]. ■

**Definition 2.1.3.** For $E \subset \mathbb{R}^n$, the unique $\delta$ given in Theorem 2.1.2 is defined to be the *Hausdorff dimension* of $E$ and denoted by $\dim_{\mathcal{H}}(E)$.

Another type of dimension that we shall use is the box-counting dimension. Let $E$ denote a non-empty bounded subset of $\mathbb{R}^n$. For each $\epsilon > 0$, define $N_\epsilon(E)$ to be the smallest number of subsets of $\mathbb{R}^n$ of diameter less than $\epsilon$ needed to cover $E$. The box-counting dimension of $E$ is determined by the power law relationship between $N_\epsilon(E)$ and $\epsilon$.

**Definition 2.1.4.** The *lower* and *upper box-counting dimensions* of $E \subset \mathbb{R}^n$ are respectively defined by

$$\underline{\dim}_B(E) \ := \ \liminf_{\epsilon\to 0} \frac{\ln(N_\epsilon(E))}{-\ln(\epsilon)}, \qquad \overline{\dim}_B(E) \ := \ \limsup_{\epsilon\to 0} \frac{\ln(N_\epsilon(E))}{-\ln(\epsilon)}. \tag{2.1}$$

If these are equal then the common value is referred to as the *box-counting dimension* of $E$ and is denoted by

$$\dim_B(E) \ := \ \lim_{\epsilon\to 0} \frac{\ln(N_\epsilon(E))}{-\ln(\epsilon)}.$$

It is often the case that the Hausdorff dimension and the box-counting dimension differ. However, it is well-known that the box-counting dimension gives an upper bound for the Hausdorff dimension, that is, for a subset $E$ of $\mathbb{R}^n$ we have that

$$\dim_{\mathcal{H}}(E) \ \leqslant \ \underline{\dim}_B(E) \ \leqslant \ \overline{\dim}_B(E).$$

For a proof of this result and further reading on the Hausdorff dimension and box-counting dimension we refer the reader to Chapters 2 and 3 of [Fal1].

In this thesis, a specific class of fractal sets which we will work with are self-similar sets. These sets are made up of smaller images of themselves and can be constructed by an iterated function system of similarities.

**Definition 2.1.5.** Let $K$ be a compact subset of $\mathbb{R}^n$. A *similarity* is a linear map $s : K \to K$ such that there exists an $r \in (0,1)$ with $\|s(x) - s(y)\| = r\|x - y\|$, for all $x, y \in K$. We refer to $r$ as the *contraction ratio* of the similarity $s$. Define an *iterated function system of similarities* to be a finite family of distinct similarity mappings $\{s_1, s_2, \ldots, s_m\}$ on $K$. It is assumed that $m$ is a positive integer greater than or equal to 2 in order to avoid trivial cases.

A property of an iterated function system of similarities is that it determines a unique invariant compact subset of $\mathbb{R}^n$.

**Theorem 2.1.6.** *(Hutchinson's Theorem) If $S := \{s_1, s_2, \ldots, s_m\}$ denotes an iterated function system of similarities on a compact subset $K$ of $\mathbb{R}^n$, then there exists a unique non-empty compact subset $E$ of $K$ satisfying*

$$E \ = \ \bigcup_{i=1}^{m} s_i(E).$$

*We say that $E$ is invariant under $S$ and call $E$ a self-similar set.*

Proof. See Theorem 9.1 of [Fal1]. ■

A useful condition that is commonly used in fractal geometry is the strong separation condition.

**Definition 2.1.7.** Let $S := \{s_1, s_2, \ldots, s_m\}$ denote an iterated function system of similarities on a compact subset $K$ of $\mathbb{R}^n$ and let $E$ denote the unique non-empty compact invariant set under $S$. Then $S$ is said to satisfy the *strong separation condition* if for all distinct $i, j \in \{1, 2, \ldots, m\}$ we have that $s_i(E) \cap s_j(E) = \emptyset$.

*Remark. A self-similar set satisfying the strong separation condition is necessarily a compact totally disconnected set with no isolated points.*

The following theorem gives a simple way of calculating the Hausdorff dimension of self-similar sets satisfying the strong separation condition. In fact, the theorem holds under a slightly weaker condition, namely, the open set condition (see page 129 of [Fal1]).

**Theorem 2.1.8.** *(Moran-Hutchinson Formula) Let $S := \{s_1, s_2 \ldots, s_m\}$ denote an iterated function system of similarities satisfying the strong separation condition (or more generally the open set condition) and let $E$ denote the unique non-empty compact invariant set under $S$. Let $r_i$ denote the contraction ratio of $s_i$ for $i \in \{1, 2, \ldots, m\}$. Then*

$$\dim_B(E) = \dim_{\mathcal{H}}(E) = t,$$

*where $t$ is the unique solution of the equation*

$$\sum_{i=1}^{m} r_i^t = 1.$$

Proof. See Theorem 9.3 of [Fal1]. ■

An example of a fractal set which we will repeatedly use within this thesis is the following.

**Example 2.1.9.** For $\eta \in (0, 1/2)$, consider the iterated function system of similarities $S := \{s_1, s_2\}$, where $s_1, s_2 : [0, 1] \to [0, 1]$ are defined by

$$s_1(x) := \eta x, \qquad s_2(x) := \eta x + 1 - \eta.$$

We define the *middle* $(1-2\eta)$*-Cantor* set to be the unique non-empty compact subset of $\mathbb{R}$ which is invariant under $S$ and denote it by $C_\eta$. Note that, by the Moran-Hutchinson formula, the Hausdorff dimension of $C_\eta$ is equal to $-\ln(2)/\ln(\eta)$.

It is well-known that any compact totally disconnected subset $E$ of $\mathbb{R}$ with no isolated points is homeomorphic to the middle third Cantor set (see Corollary 30.4 of [Wil]). Therefore, we can view $E$ in terms of its complement, that is, as a family of disjoint open intervals $\{I_k : k \in \mathbb{N}\}$. When viewing $E$ in this way, it will always be assumed that the *complementary intervals* $I_k$ are ordered so that their lengths are non-increasing. If, in addition, one imposes a certain porosity condition on $E$, then one obtains bounds on the rate of decrease of the lengths of the complementary intervals and that the Hausdorff dimension must be strictly positive. The porosity condition with which we shall be concerned (especially in Subsection 4.1.2) is the following.

**Definition 2.1.10.** A subset $E$ of $\mathbb{R}$ is defined to be *strongly porous* with *porosity constant* $\rho \in (0, 1)$, if for each $x \in E$ and $r \in (0, 1]$ the ball $B(x, r)$ contains a complementary interval of $E$ with diameter greater than or equal to $\rho r$.

*Remark. The standard concept of porous set, (see page 156 of [Mat]) only gives an upper bound on the dimension of the set, hence our use of the term strongly porous.*

In order to show that the Hausdorff dimension of a strongly porous set must be strictly positive we require the following lemma.

**Lemma 2.1.11.** *Let $(m_k)_{k\in\mathbb{N}_0}$ denote a sequence in $\mathbb{N}\setminus\{1\}$ and let $[0,1] =: E_0 \supset E_1 \supset E_2 \dots$, where for each $k \in \mathbb{N}_0$ the set $E_k$ is a finite union of disjoint closed intervals. Assume that every connected component of $E_k$ contains at least $m_k$ connected components of $E_{k+1}$. If the maximum length of the intervals in $E_k$ tends to zero as $k$ tends to infinity, then the set*

$$E \;:=\; \bigcap_{k\in\mathbb{N}_0} E_k$$

*is a compact totally disconnected set with no isolated points. Suppose that the intervals of $E_k$ are separated by complementary intervals of lengths at least $\epsilon_k > 0$, where $\epsilon_{k+1} < \epsilon_k$, for each $k \in \mathbb{N}_0$, then*

$$\dim_{\mathcal{H}}(E) \;\geqslant\; \liminf_{k\to\infty} \frac{\ln(m_1 m_2 \dots m_{k-1})}{-\ln(m_k \epsilon_k)}.$$

Proof. See pages 62 - 65 of [Fal1]. ∎

**Theorem 2.1.12.** *Let $E$ denote a closed strongly porous subset of the unit interval $[0,1]$, with porosity constant $\rho > 0$ and suppose that $\{0,1\} \subset E$. Then*

$$\dim_{\mathcal{H}}(E) \;\;\geqslant\;\; \frac{\ln(2)}{\ln(2) - \ln(\rho)} \tag{2.2}$$

*and*

$$-\infty \;<\; \liminf_{k\to\infty} \frac{\ln(|I_k|)}{\ln(k)} \;\leqslant\; \limsup_{k\to\infty} \frac{\ln(|I_k|)}{\ln(k)} \;\leqslant\; -1. \tag{2.3}$$

Proof. In order to calculate the lower bound of the Hausdorff dimension of $E$ we construct a subset $F$ of $E$ whose Hausdorff dimension is greater than or equal to $\ln(2)/(\ln(2)-\ln(\rho))$. We build the set $F$ inductively. Fix $x \in E\setminus\{0,1\}$ and let $r := \min\{|x|, |1-x|\}$. By the strong porosity condition there exists a complementary interval $I_x \subseteq B(x, r/2)$ of $E$, such that $\rho r/2 \leqslant |I_x| \leqslant r/2$. Let $F_0 := \partial I_x$. Suppose that the sets $F_0, F_1, \dots, F_k$ have been constructed. To obtain $F_{k+1}$, consider $y \in F_k$. By the strong porosity condition there exists a complementary interval $I_y \subseteq B(y, 2^{-k-2}\rho^{k+1} r)$ of $E$ such that $2^{-k-2}\rho^{k+2} r \leqslant |I_y| \leqslant 2^{-k-2}\rho^{k+1} r$. We then set

$$F_{k+1} \;:=\; \bigcup_{y\in F_k} \partial I_y.$$

The set $F$ is then defined by

$$F \;:=\; \overline{\bigcup_{k\in\mathbb{N}_0} F_k}.$$

Therefore, by applying Lemma 2.1.11, we have that

$$\dim_{\mathcal{H}}(F) \;\;\geqslant\;\; \frac{\ln(2)}{\ln(2) - \ln(\rho)}.$$

For the inequalities given in Equation (2.3) the reader is referred to Proposition 3.7 of [Fal3]. ∎

To conclude this subsection we give an alternative way of generating a self-similar set. This is a simplification of the construction of a cookie cutter set as described in [Fal3, Bed].

**Theorem 2.1.13.** *Let $S := \{s_1, s_2, \ldots, s_m\}$ be an iterated function system of similarities on a compact subset $K$ of $\mathbb{R}$ which satisfies the strong separation condition. Let*

$$T : \bigcup_{i=1}^{m} s_i(K) \ \to \ K$$

*be defined so that $T \circ s_i(x) := x$, for each $i \in \{1, 2, \ldots, m\}$ and each $x \in k$. Then the set*

$$\{x \in K \, : \, T^k(x) \text{ is defined for all } k \in \mathbb{N}\}$$

*is precisely the unique invariant compact subset of $K$ under $S$.*

### 2.1.2 Minkowski Content

In this subsection we define the Minkowski content of a given subset of $\mathbb{R}$. Its characterisation will be used in various calculations in Section 4.1.

For a subset $E$ of $\mathbb{R}$ and for $\epsilon > 0$ we define the *$\epsilon$-neighbourhood of $E$* by

$$E_\epsilon \ := \ \{x \in \mathbb{R} \, : \, \text{there exists } y \in E \text{ with } |x - y| \leqslant \epsilon\}.$$

Here, we want to examine the behaviour of the Lebesgue measure of the $\epsilon$-neighbourhood $E_\epsilon$ of $E$ as $\epsilon$ tends to zero from above. This quantity is linked to the Minkowski content which we now introduce.

**Definition 2.1.14.** Let $E$ denote a subset of $\mathbb{R}$ and let $s \geqslant 0$. If the limit

$$M_s(E) \ := \ \lim_{\epsilon \to 0} \frac{\lambda^1(E_\epsilon)}{\epsilon^{1-s}} \tag{2.4}$$

exists, then we define this limit to be the *$s$-dimensional Minkowski content* of $E$. In the case that $s := \dim_B(E)$ and the limit in Equation (2.4) exists we say that $E$ is *Minkowski measurable.* Note that an analogous definition exists for subsets of $\mathbb{R}^n$, for all $n \in \mathbb{N}$ (see [Fal1, Gat]).

**Theorem 2.1.15.** *Let $I \subset \mathbb{R}$ denote a compact interval, and let $\{I_k \, : \, k \in \mathbb{N}\}$ denote a countably infinite collection of disjoint open subintervals of $I$ with $\sum_{k=1}^{\infty} |I_k| := |I|$ and such that $|I_k| \geqslant |I_{k+1}|$, for each $k \in \mathbb{N}$. Let $E := I \setminus \bigcup_{k \in \mathbb{N}} I_k$. Then for all $s \in (0, 1)$ and $c > 0$, we have the following.*

1. *$|I_k| \asymp k^{-1/s}$ as $k$ tends to infinity if and only if $\lambda^1(E_\epsilon) \asymp \epsilon^{1-s}$ as $\epsilon$ tends to zero.*
2. *$\lim_{k\to\infty} |I_k| k^{1/s} = 2^{1-1/s} c^{1/s} (1-s)^{1/s}$ if and only if $M_s(E) = c$.*

Proof. See Proposition 2 of [Fal2]. ■

**Theorem 2.1.16.** *Let $E \subset \mathbb{R}$ denote a self-similar set generated by an iterated function system of similarities $\{s_1, s_2, \ldots, s_m\}$ which satisfies the strong separation condition, and let $r_i$ denote the contraction ratio of $s_i$, for $i \in \{1, 2, \ldots, m\}$. Further, let $\delta$ denote the Hausdorff dimension of $E$. Assume without loss of generality that $E$ is scaled so that $\{0, 1\} \subset E \subset [0, 1]$. Then we have the following.*

1. *The additive group generated by the set* $\{\ln(r_1^{-1}), \ln(r_2^{-1}), \ldots, \ln(r_m^{-1})\}$ *is dense in* $\mathbb{R}$*, if and only if*
$$\lim_{\epsilon \to 0} \frac{\lambda^1(E_\epsilon)}{\epsilon^{1-\delta}} = 2^{1-\delta}(1-\delta)^{-1} \frac{\sum_{i=1}^{m-1} |l_i|^\delta}{\sum_{j=1}^{m} r_j^\delta \ln(r_j^{-\delta})},$$
*where* $l_1, l_2, \ldots, l_{m-1}$ *denote the complementary intervals of* $\bigcup_{i=1}^m s_i([0,1])$ *whose lengths are finite.*

2. *If the additive group generated by* $\{\ln(r_1^{-1}), \ln(r_2^{-1}), \ldots, \ln(r_m^{-1})\}$ *is isomorphic to* $\mathbb{Z}$*, then we have that* $\lambda^1(E_\epsilon) \asymp \epsilon^{1-\delta}$ *as* $\epsilon$ *tend to zero.*

Proof. See Proposition 4 of [Fal2] for the proof of the forward implication of 1 and for the proof of 2. For the reverse implication of 1 see Theorem 6.20 and Theorem 6.21 of [FL]. ■

### 2.1.3 Coarse Multifractal Analysis

Here we describe and develop aspects of coarse multifractal analysis in such a way that allows for the introduction of an analogous notion within the theory of noncommutative geometry (see Subsection 4.1.2). The final result of this subsection (Theorem 2.1.20) will play a vital role in the formulation of this new notion. This is a new result which allows for the calculation of the coarse multifractal box-counting dimension of the support of a measure $\mu$ on $\mathbb{R}$ in terms of the complement of $\mathrm{supp}(\mu)$, provided $\mathrm{supp}(\mu)$ is compact and strongly porous.

Multifractal analysis originated from statistical mechanics and was later adapted to dynamical systems. It was developed by two independent groups of mathematicians and physicists. The first approach can be traced back to the work of Mandelbrot, who in [Man1, Man2] suggested that the distribution of intermittent dissipation of energy in highly turbulent fluid flows is "multifractal" in nature and studied it by calculating its moments. The second approach is due to Grassberger, Hentschel and Procaccia who in [Gra, GP, HP] generalised the work of Rényi [Rén]. These two approaches were merged in the seminal paper [HMJPS].

Multifractals represent a move from the geometry of a metric space $(X, d)$ to the geometric properties of measures supported on $X$. The distribution of the mass of such a measure $\mu$ may vary widely over $X$. By studying the local dimension of $\mu$ at each point of $X$, one obtains a family of sets referred to as "level sets". These are the intrinsic objects which multifractal analysis is predominantly concerned with. A number of approaches to multifractals have been developed. In what follows, we aim to introduce the coarse multifractal spectra for compact subsets of $\mathbb{R}$. First we introduce the Hausdorff dimension spectrum. Note that many of the ideas that follow can be extended to higher dimensions. However, as we are primarily interested in fractal subsets of $\mathbb{R}$ we state (and where necessary prove) the results for compact subsets of $\mathbb{R}$.

For a finite Borel measure $\mu$ on $\mathbb{R}$, we respectively define the *lower* and *upper local dimension* of $\mu$ at $x \in \mathrm{supp}(\mu)$ by
$$\underline{\dim}_\mu(x) := \liminf_{r \to 0} \frac{\ln(\mu(B(x,r)))}{\ln(r)}, \qquad \overline{\dim}_\mu(x) := \limsup_{r \to 0} \frac{\ln(\mu(B(x,r)))}{\ln(r)}.$$
If these coincide, we refer to the common value as the *local dimension* of $\mu$ at $x$, and denote it by $\dim_\mu(x)$. Further, we set $\dim_\mu(x) := \infty$ if $x$ lies outside the support of $\mu$ and that $\mu(x) = 0$ if $x$ is an atom of $x$. As a matter of interest, we note that the upper and lower local dimensions are measurable functions. This follows from the fact that they are upper and lower semi-continuous, respectively.

In multifractal theory one is interested in the geometric properties of the *level sets*, which are given for each $\eta \in \mathbb{R}$ by

$$X_\eta \ := \ \{x \in \operatorname{supp}(\mu) \ : \ \dim_\mu(x) = \eta\}.$$

The function $f_\mu : \mathbb{R} \to [0, \dim_{\mathcal{H}}(\operatorname{supp}(\mu))]$ defined by

$$f_\mu(\eta) \ := \ \dim_{\mathcal{H}}(X_\eta) \tag{2.5}$$

is called the *Hausdorff dimension spectrum* of $\mu$.

The coarse multifractal spectrum, in certain cases, coincides with the Hausdorff dimension spectrum and has parallel features to the box-counting dimension. A goal of coarse multifractal analysis is to study global irregularities of a measure, in particular, the asymptotic behaviour of its moment sums. There are many equivalent ways of defining the coarse spectrum. In what follows we state the intuitive definitions which are given in [Fal3] and also the equivalent definitions given by Riedi [Rie1].

Let $\mu$ denote a finite Borel measure on a compact subset of $\mathbb{R}$. For $r > 0$, let $\mathfrak{B}_r$ denote a $r$-mesh of $\mathbb{R}$, that is, a covering formed by closed intervals of length $r$ such that the interiors are pairwise disjoint. For each $r > 0$, define $\mathcal{N}_r : \mathbb{R} \to \mathbb{R}$, for each $\eta \in \mathbb{R}$, by

$$\mathcal{N}_r(\eta) \ := \ \operatorname{card}\{B \in \mathfrak{B}_r \ : \ \mu(B) \geqslant r^\eta\}.$$

Then the *lower* and *upper coarse multifractal spectra* $\overline{f}_C$, $\underline{f}_C : \mathbb{R} \to \mathbb{R}$ of $\mu$ are respectively defined, for each $\eta \in \mathbb{R}$, by

$$\begin{aligned} \underline{f}_C(\eta) \ &:= \ \liminf_{\epsilon \to 0} \liminf_{r \to 0} \frac{\ln(\mathcal{N}_r(\eta + \epsilon) - \mathcal{N}_r(\eta - \epsilon))}{-\ln(r)} \\ \overline{f}_C(\eta) \ &:= \ \limsup_{\epsilon \to 0} \limsup_{r \to 0} \frac{\ln(\mathcal{N}_r(\eta + \epsilon) - \mathcal{N}_r(\eta - \epsilon))}{-\ln(r)}. \end{aligned}$$

If, at some $\eta \in \mathbb{R}$, the lower and upper coarse multifractal spectra coincide the common value is denoted by $f_C(\eta)$. Heuristically, the coarse multifractal spectrum provides a global overview of the fluctuations of $\mu$ at an infinitesimal scale, but gives no information about the limiting behaviour of $\mu$ at a given point.

In analogy with the box-counting dimension, for a finite Borel measure $\mu$ on a compact subset of $\mathbb{R}$, we respectively define $\underline{\beta}, \overline{\beta} : \mathbb{R} \to \mathbb{R}$, for each $q \in \mathbb{R}$, by

$$\begin{aligned} \underline{\beta}(q) \ &:= \ \liminf_{r \to 0} \frac{\ln\left(\mathcal{M}^q_{\mu,r}\right)}{-\ln(r)}, \\ \overline{\beta}(q) \ &:= \ \limsup_{r \to 0} \frac{\ln\left(\mathcal{M}^q_{\mu,r}\right)}{-\ln(r)}. \end{aligned}$$

Here, $\mathcal{M}^q_{\mu,r}$ denotes the *multifractal moment sum of* $\mu$ and is defined, for each $r > 0$ and $q \in \mathbb{R}$, by

$$\mathcal{M}^q_{\mu,r} \ := \ \sum_{\substack{B \in \mathfrak{B}_r \\ \mu(B) > 0}} (\mu(B))^q.$$

For $q \in \mathbb{R}$, we refer to $\underline{\beta}(q)$ and $\overline{\beta}(q)$ as the *lower* and *upper* multifractal box-counting dimension of $\mu$ at $q$, respectively. If, for some $q \in \mathbb{R}$, we have that $\underline{\beta}(q) = \overline{\beta}(q)$, then we denote the common value by $\beta(q)$ and refer to it as the multifractal box-counting dimension of $\mu$ at $q$.

Just as we have a relationship between the box-counting dimension and the Hausdorff dimension, there exists a similar relationship between the Hausdorff dimension spectra and the coarse multifractal spectra.

**Theorem 2.1.17.** *Let $\mu$ be a finite Borel measure on a compact subset of $\mathbb{R}$. Then, for each $\eta \geqslant 0$, we have that*

$$f_\mu(\eta) \leqslant \underline{f}_C(\eta) \leqslant \overline{f}_C(\eta). \tag{2.6}$$

*Further, for each $\eta \geqslant 0$, we have*

$$\overline{f}_C(\eta) \leqslant \inf_{q\in\mathbb{R}} \left\{\overline{\beta}(q) + \eta \cdot q\right\}, \qquad \underline{f}_C(\eta) \leqslant \inf_{q\in\mathbb{R}} \left\{\underline{\beta}(q) + \eta \cdot q\right\}. \tag{2.7}$$

Proof. See [Fal3] Lemma 11.1 for a proof of Equation (2.6) and Lemma 11.2 for a proof of Equation (2.7). ■

*Remark. In many cases one has equality in Equations (2.6) and (2.7), for instance if $\mu$ is a self-similar measure (a class of measures which we will come to introduce at the end of this subsection).*

In the above definition $\beta$ is well defined for $q \geqslant 0$, whereas for $q < 0$ the multifractal moment sums may only converge on subsequences, for $r$ tending to zero. The reason for this is that for certain values of $r$ there may exist a $B \in \mathfrak{B}_r$ such that $\mu(B)$ is uncharacteristically small, and so, $\mu(B)^q$ becomes uncharacteristically large. To overcome this difficulty we add a slight modification to the definition of the multifractal box-counting dimension, as given in [Rie1]. But first, let us set some notation. For a compact interval $I := [s,t]$ of $\mathbb{R}$ and for $\eta \geqslant 0$, let $I^\eta$ denote the interval centred about $I$ of diameter $(1+\eta)(t-s)$, that is, let

$$I^\eta := [s - \eta(t-s)/2, t + \eta(t-s)/2].$$

For $r > 0$, set $\mathfrak{B}^*_r(\mu) := \{B \in \mathfrak{B}_r : \mu(B) > 0\}$. Define $\mathfrak{b} : \mathbb{R} \to \mathbb{R}$, for each $q \in \mathbb{R}$, by

$$\mathfrak{b}(q) := \inf\left\{k \in \mathbb{R} : \limsup_{r\to 0} \sum_{B\in\mathfrak{B}^*_r(\mu)} r^k \mu(B^\eta)^q = 0\right\} \tag{2.8}$$

$$= \sup\left\{k \in \mathbb{R} : \limsup_{r\to 0} \sum_{B\in\mathfrak{B}^*_r(\mu)} r^k \mu(B^\eta)^q = \infty\right\}. \tag{2.9}$$

The next proposition shows that $\mathfrak{b}(q)$ is independent of $\eta > 0$ for all $q \in \mathbb{R}$, and independent of $\eta \geqslant 0$ for $q \geqslant 0$.

**Proposition 2.1.18.** *Let $\mu$ denote a finite Borel measure on a compact subset of $\mathbb{R}$.*

1. *Given $q \geqslant 0$ and $0 \leqslant \eta_1 \leqslant \eta_2$ there exists a constant $c > 0$ such that, for sufficiently small $r$, we have that*

$$c \sum_{B\in\mathfrak{B}^*_r(\mu)} \mu(B^{\eta_2})^q \leqslant \sum_{B\in\mathfrak{B}^*_r(\mu)} \mu(B^{\eta_1})^q \leqslant \sum_{B\in\mathcal{B}^*_r(\mu)} \mu(B^{\eta_2})^q. \tag{2.10}$$

2. *Given $q < 0$ and $0 < \eta_1 \leqslant \eta_2$, for sufficiently small $r$, we have that*

$$\sum_{B\in\mathfrak{B}^*_r(\mu)} \mu(B^{\eta_2})^q \leqslant \sum_{B\in\mathfrak{B}^*_r(\mu)} \mu(B^{\eta_1})^q \leqslant 2 \sum_{B\in\mathfrak{B}^*_{(\eta_1 r)/(2(1+\eta_2))}(\mu)} \mu(B^{\eta_2})^q. \tag{2.11}$$

*In particular, $\mathfrak{b}(q)$ is independent of $\eta_1 > 0$, for all $q \in \mathbb{R}$, and independent of $\eta \geqslant 0$ for $q \geqslant 0$. Hence, we have that $\mathfrak{b}(q) = \beta(q)$, for $q \geqslant 0$.*

Proof. The upper bound of Equation (2.10) and the lower bound of Equation (2.11) are established by using the fact that $\mu(B^{\eta_1}) \leqslant \mu(B^{\eta_2})$. The lower bound of Equation (2.10) follows since for each $B \in \mathfrak{B}_r^*$, there can only exist a finite number of $\widetilde{B} \in \mathfrak{B}_r^*$ such that the set $B^{\eta_2} \cap \widetilde{B}^{\eta_1}$ is non-empty. An application of Hölders inequality (Theorem III.1(c) of [RS]) then gives the required result for $q > 1$ and an application of Minkowski's inequality (Theorem III.1(a) of [RS]) then gives the required result for $q \in [0,1]$. The upper bound of Equation (2.11) is obtained from the following observations. Firstly, to each $B \in \mathfrak{B}_r^*(\mu)$ one can associate a $\widetilde{B} \in \mathfrak{B}^*_{(\eta_1 r)/(2(1+\eta_2))}(\mu)$ such that $\widetilde{B}^{\eta_2} \subseteq B^{\eta_1}$. Secondly, each $\widetilde{B} \in \mathfrak{B}^*_{(\eta_1 r)/(2(1+\eta_2))}(\mu)$ can intersect at most 2 elements of $\mathfrak{B}_r^*(\mu)$. ■

Suppose that $\mu$ denotes a finite Borel measure on $\mathbb{R}$ with compact support. By assuming that the support of $\mu$ is strongly porous (Definition 2.1.10) and using the definition of the coarse multifractal box-counting dimension given by Riedi (Equations (2.8) and (2.9)), we develop a new result which allows one to obtain the coarse multifractal box-counting dimension of $\mu$ in terms of the complementary intervals of the support of $\mu$. Although the formula given in Theorem 2.1.20 seems unusual in the context of standard multifractal analysis, it will become clear in Subsection 4.1.2 why this particular formulation is in fact quite natural. We begin with the following technical lemma.

**Lemma 2.1.19.** *Let $\mu$ denote a Borel probability measure on a compact subset of $\mathbb{R}$. Assume that the support of $\mu$ is strongly porous with porosity constant $\rho > 0$. Let $\{I_k : k \in \mathbb{N}\}$ denote the set of complementary intervals of supp$(\mu)$ whose lengths are finite. Then, for each $q \in \mathbb{R}$, $\eta_1 \geqslant 2/\rho$ and $\eta_2 > 0$, there exist positive constants $t_1, t_2, c_1, c_2$ such that for $r > 0$ sufficiently small, we have that*

$$c_1 \sum_{\rho t_1 r \leqslant |I_k| \leqslant t_1 r} \mu(\overline{I}_k^{\eta_1})^q \leqslant \sum_{B \in \mathfrak{B}_r^*(\mu)} \mu(B^{\eta_2})^q \leqslant c_2 \sum_{\rho t_2 r \leqslant |I_k| \leqslant t_2 r} \mu(\overline{I}_k^{\eta_1})^q. \tag{2.12}$$

Proof. Without loss of generality assume that $\{0,1\} \subseteq \mu \subseteq [0,1]$. As above, assume that the complementary intervals of supp$(\mu)$ are listed so that $|I_k| \geqslant |I_{k+1}|$, for each $k \in \mathbb{N}$. For each $t \in (0,1)$, define $\mathcal{I}_t := \{I_k : \rho t \leqslant |I_k| \leqslant t\}$. Fix $r \in (0, (\eta_1+2)/\eta_2)$ and note that since the support of $\mu$ is strongly porous, the set $\mathcal{I}_{\eta_2 r/(\eta_1+2)}$ is non-empty. Now observe that for each $I_k \in \mathcal{I}_{\eta_2 r/(\eta_1+2)}$, there exists a $B \in \mathfrak{B}_r^*(\mu)$ such that $B \cap \partial I_k \neq \emptyset$ and $\overline{I}_k^{\eta_1} \subseteq B^{\eta_2}$. This follows since the boundary of $\overline{I}_k$ lies in the support of $\mu$ and since

$$|I_k| + \frac{\eta_1}{2}|I_k| \leqslant \frac{\eta_2 r}{\eta_1+2} + \frac{\eta_1}{2}\frac{\eta_2 r}{\eta_1+2} = \frac{\eta_2 r}{2}.$$

Next, observe that for each $B \in \mathfrak{B}_r^*(\mu)$ we have that

$$\operatorname{card}\left\{I_k \in \mathcal{I}_{\eta_2 r/(\eta_1+2)} : \overline{I}_k^{\eta_1} \subseteq B^{\eta_2}\right\} \leqslant (r+\eta_2 r)\frac{\eta_1+2}{\rho\eta_2 r} = \frac{(\eta_2+1)(\eta_1+2)}{\rho\eta_2}. \tag{2.13}$$

From these observations, the lower bound of Equation (2.12) follows for $q \geqslant 0$, where $t_1 = \eta_2/(\eta_1+2)$ and $c_1 = \rho\eta_2/((\eta_2+1)(\eta_1+2))$.

For $q < 0$ we argue as follows. Fix $r \in (0, (2\rho+\eta_1\rho-2)/(\eta_2+2))$ and consider an element $B \in \mathfrak{B}_r^*(\mu)$. For a fixed $x \in \operatorname{supp}(\mu) \cap B$, the strongly porous condition on the support of $\mu$ implies that there exists $I_k \in \mathcal{I}_{(\eta_2+2)r/(2\rho+\eta_1\rho-2)}$ such that $I_k \subset B(x, (\eta_2+2)r/(2\rho+\eta_1\rho-2))$ and

$\rho(\eta_2+2)r/(2\rho+\eta_1\rho-2) \leqslant |I_k|$. Therefore, we have that

$$|I_k| + \frac{\eta_1}{2}|I_k| - \frac{(\eta_2+2)r}{2\rho+\eta_1\rho-2} \;\geqslant\; \frac{\rho(\eta_2+2)r}{2\rho+\eta_1\rho-2} + \frac{\eta_1\rho}{2}\frac{(\eta_2+2)r}{2\rho+\eta_1\rho-2} - \frac{(\eta_2+2)r}{2\rho+\eta_1\rho-2} \;=\; r + \frac{1}{2}\eta_2 r.$$

From this we conclude that $B^{\eta_2} \subseteq \overline{I}_k^{\eta_1}$. Next, observe that for each $\overline{I}_k^{\eta_1}$ we have that

$$\operatorname{card}\left\{B \in \mathfrak{B}_r^*(\mu) \,:\, B^{\eta_2} \subseteq \overline{I}_k^{\eta_1}\right\} \;\leqslant\; \frac{(\eta_2+2)(2\rho+\eta_1\rho-2)^{-1}r}{r} \;=\; \frac{\eta_2+2}{2\rho+\eta_1\rho-2}. \tag{2.14}$$

From these observations, the lower bound of Equation (2.12) follows for $q<0$, where $t_1 = (\eta_2 + 2)/(2\rho+\eta_1\rho-2)$ and $c_1 = (2\rho+\eta_1\rho-2)/(\eta_2+2)$.

The upper bound follows in an analogous way. That is, for $q \geqslant 0$, fix $r \in (0, \eta_1\rho/(2+\eta_2))$. Then for each $I_k \in \mathcal{I}_{(2+\eta_2)r/(\eta_1\rho)}$, one can find $B \in \mathfrak{B}_r^*(\mu)$ such that $\overline{I}_k^{\eta_1}$ contains $B^{\eta_2}$. Moreover, as in Equation (2.14), it can be shown, independent of $r$, that each $\overline{I}_k^{\eta_1}$ can contain at most a bounded number of intervals of the set $\{B^{\eta_2} \,:\, B \in \mathfrak{B}_r^*(\mu)\}$. Similarly, for $q<0$, fix $r \in (0, (\eta_1+2)/\eta_2)$. Then using the strongly porous condition on the support of $\mu$, one has that for each $B \in \mathfrak{B}_r^*(\mu)$, there exists an $I_k \in \mathcal{I}_{\eta_2 r/(\eta_1+2)}$ such that $\overline{I}_k^{\eta_1} \subseteq B^{\eta_2}$. In addition, as in Equation (2.13), we have that each $B^{\eta_2}$ can contain at most a bounded number of intervals of the set $\{\overline{I}_k^{\eta_1} \,:\, I_k \in \mathcal{I}_{\eta_2 r/(\eta_1+2)}\}$, independent of $r$. ∎

**Theorem 2.1.20.** *Let $\mu$ denote a Borel probability measure whose support is a strongly porous compact subset of $\mathbb{R}$ with porosity constant $\rho>0$. Let $\{I_k \,:\, k \in \mathbb{N}\}$ denote the set of complementary intervals of supp($\mu$) whose lengths are finite. If $\eta \geqslant 2\rho^{-1}$, then for each $q \in \mathbb{R}$, we have that*

$$\mathfrak{b}(q) \;=\; \inf\left\{t \in \mathbb{R} \,:\, \sum_{k\in\mathbb{N}} \mu(\overline{I}_k^{\eta})^q |I_k|^t < \infty\right\} \tag{2.15}$$

$$\;=\; \inf\left\{t \in \mathbb{R} \,:\, \limsup_{N\to\infty} \frac{\sum_{k=1}^N \mu(\overline{I}_k^{\eta})^q |I_k|^t}{\ln(N)} = 0\right\}. \tag{2.16}$$

Proof. Without loss of generality, assume that $\{0,1\} \subset \operatorname{supp}(\mu) \subset [0,1]$. Let us begin by proving the equality given in Equation (2.15). By Lemma 2.1.19, for $r>0$ sufficiently small and for all $q,t \in \mathbb{R}$, there exist positive constants $t_2, c$ such that

$$\sum_{k\in\mathbb{N}} \mu(\overline{I}_k^{\eta})^q |I_k|^t \;\geqslant\; \sum_{l\in\mathbb{N}_0} \sum_{\rho^{l+1} t_2 r \leqslant |I_k| < \rho^l t_2 r} \mu(\overline{I}_k^{\eta})^q |I_k|^t \;\geqslant\; c \sum_{l\in\mathbb{N}_0} \sum_{B \in \mathfrak{B}^*_{\rho^l r}(\mu)} \mu(B^{\eta})^q (\rho^l r)^t.$$

Therefore, by using the definition of $\mathfrak{b}$ given in Equations (2.8) and (2.9), we conclude that

$$\mathfrak{b}(q) \;\leqslant\; \inf\left\{t \in \mathbb{R} \,:\, \sum_{k\in\mathbb{N}} \mu(\overline{I}_k^{\eta})^q |I_k|^t < \infty\right\}.$$

Using the lower bound in Equation (2.12), a similar argument gives that

$$\mathfrak{b}(q) \;\geqslant\; \inf\left\{t \in \mathbb{R} \,:\, \sum_{k\in\mathbb{N}} \mu(\overline{I}_k^{\eta})^q |I_k|^t < \infty\right\}.$$

Let us consider the equality given in Equation (2.16). Let $\eta, q, t$ be as above. If the series

$\sum_{k\in\mathbb{N}} \mu(\overline{I}_k^\eta)^q |I_k|^t$ is bounded, then

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^N \mu(\overline{I}_k^\eta)^q |I_k|^t}{\ln(N)} \;=\; 0.$$

Therefore, it follows that

$$\inf\left\{t\in\mathbb{R} \,:\, \limsup_{N\to\infty} \frac{\sum_{k=1}^N \mu(\overline{I}_k^\eta)^q |I_k|^t}{\ln(N)} = 0\right\} \;\leqslant\; \inf\left\{t\in\mathbb{R} \,:\, \sum_{k\in\mathbb{N}} \mu(\overline{I}_k^\eta)^q |I_k|^t < \infty\right\}.$$

To obtain the desired equality, first observe that $1 \geqslant \sum_{l=1}^k |I_l| \geqslant k|I_k|$ for each $k\in\mathbb{N}$. Secondly, if

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^N \mu(\overline{I}_k^\eta)^q |I_k|^t}{\ln(N)} \;=\; 0,$$

then there exists a positive constant $c$, such that for each $\epsilon\in(0,1)$ and each $k\in\mathbb{N}_0$, we have that

$$\sum_{l=2^k}^{2^{k+1}-1} \mu(\overline{I}_l^\eta)^q |I_l|^{t+\epsilon} \;\leqslant\; c2^{-k\epsilon}\ln(2^{k+1}).$$

Using these two observations, we conclude that there exists a positive constant $c$ such that

$$\sum_{k\in\mathbb{N}} \mu(\overline{I}_k^\eta)^q |I_k|^t \;=\; \sum_{k\in\mathbb{N}_0}\sum_{l=2^k}^{2^{k+1}-1} \mu(\overline{I}_l^\eta)^q |I_l|^t \;\leqslant\; c\sum_{k\in\mathbb{N}_0} 2^{-k\epsilon}\ln(2^{k+1}) \;<\; \infty.$$

Therefore, it follows that

$$\inf\left\{t\in\mathbb{R} \,:\, \limsup_{N\to\infty} \frac{\sum_{k=1}^N \mu(\overline{I}_k^\eta)^q |I_k|^t}{\ln(N)} = 0\right\} \;=\; \inf\left\{t\in\mathbb{R} \,:\, \sum_{k\in\mathbb{N}} \mu(\overline{I}_k^\eta)^q |I_k|^t < \infty\right\}.$$

■

To conclude this subsection, we consider the class of measures known as self-similar measures. Let $\{s_i : [0,1]\to[0,1] \,:\, i=1,2,\ldots,m\}$ denote an iterated function system of similarities and let $p := (p_1, p_2, \ldots, p_m)$ denote a probability vector. A *self-similar measure* associated to $p$ is defined to be the unique Borel measure $\mu$, given, for each Borel set $B$, by

$$\mu(B) \;:=\; \sum_{i=1}^m p_i\mu\left(s_i^{-1}(B)\right). \tag{2.17}$$

In this setting, the Hausdorff dimension spectrum of $\mu$ is obtained from the Legendre transform of the function $\text{ß} : \mathbb{R}\to\mathbb{R}$ which is given by the equation

$$\sum_{i=1}^m p_i^q r_i^{\text{ß}(q)} \;=\; 1.$$

(See Theorem 11.7 of [Fal3].) Moreover, in this setting, for all $q\in\mathbb{R}$, we have that $\mathfrak{b}(q) = \text{ß}(q)$ (see Theorem 16 of [Rie1]).

## 2.2 Symbolic Dynamics

In this section we have three main aims. Firstly, to define the concept of a one-sided subshift of finite type $(\Sigma_A^\infty, \sigma)$. Secondly, to define the notions of a Gibbs measure and equilibrium measure on the space $\Sigma_A^\infty$ and then to describe the thermodynamic formalism for the dynamical system $(\Sigma_A^\infty, \sigma)$. Thirdly, to create a new notion of a Haar Basis for the $L^2$-space of $\Sigma_A^\infty$ with respect to a Gibbs measure. Let us begin by defining the class of dynamical systems known as one-sided subshifts of finite type.

Let $M \in \mathbb{N} \setminus \{1\}$ be fixed. Let $\Sigma := \{1, 2, \dots M\}$ denote a finite alphabet and let $A := [a_{i,j}]_{i,j}$ denote an $M \times M$ matrix with entries in $\{0, 1\}$, called the *transition matrix*. Define the space $\Sigma_A^\infty$ by

$$\Sigma_A^\infty := \left\{ \omega := (\omega_1, \omega_2, \omega_3, \dots) \in \prod_{k \in \mathbb{N}} \Sigma \,:\, a_{\omega_k, \omega_{k+1}} = 1 \text{ for all } k \in \mathbb{N} \right\}. \tag{2.18}$$

In other words, $\Sigma_A^\infty$ is the space of all sequences with entries in the alphabet $\Sigma$ with transitions allowed by $A$. Define the *left shift* $\sigma : \Sigma_A^\infty \to \Sigma_A^\infty$, for each $\omega := (\omega_1, \omega_2, \omega_3, \dots) \in \Sigma_A^\infty$, by

$$\sigma(\omega_1, \omega_2, \omega_3, \dots) := (\omega_2, \omega_3, \omega_4, \dots).$$

Then the system $(\Sigma_A^\infty, \sigma)$ is called a *one-sided subshift of finite type*. If $A$ is a matrix with all entries equal to 1, we call the system $(\Sigma_A^\infty, \sigma)$ the *full shift space* on $M$ symbols and denote it by $(\Sigma^\infty, \sigma)$.

Next we introduce a topology on $\Sigma_A^\infty$. For $k \in \mathbb{N}$, define

$$\Sigma_A^k := \left\{ \omega := (\omega_1, \omega_2, \dots, \omega_k) \in \Sigma^k \,:\, A_{\omega_i, \omega_{i+1}} = 1 \right\}, \tag{2.19}$$

$$\Sigma_A^* := \bigcup_{k \in \mathbb{N}} \Sigma_A^k \tag{2.20}$$

and for $\omega \in \Sigma_A^k$ define

$$\begin{aligned} [\omega] &:= \{ \upsilon := (\upsilon_1, \upsilon_2, \dots) \in \Sigma_A^\infty \,:\, (\upsilon_1, \upsilon_2, \dots, \upsilon_k) = \omega \}, \\ |\omega| &:= k. \end{aligned}$$

We call the set $[\omega]$ a *cylinder set* and define the topology $\mathcal{T}$ on $\Sigma_A^\infty$ to be the topology generated by the family of the cylinder sets. Throughout this thesis we will always assume that the space $\Sigma_A^\infty$ is equipped with the topology $\mathcal{T}$. Observe that this topology is metrizable. We call a metric $d : \Sigma_A^\infty \to \Sigma_A^\infty$ *regular* if the following hold.

1. The topology induced by $d$ is equivalent to $\mathcal{T}$.
2. The left shift $\sigma$ is *positively expanding* with respect to $d$, that is, there exists $\epsilon > 0$ such that if $\omega, \upsilon \in \Sigma_A^\infty$ with $\omega \neq \upsilon$, then there exists $k \in \mathbb{N}_0$ with $d(\sigma^k(\omega), \sigma^k(\upsilon)) > \epsilon$.

**Definition 2.2.1.** Let $M \in \mathbb{N}$ and let $A := [a_{i,j}]_{i,j}$ denote an $M \times M$ matrix with real entries. Then $A$ is said to be *non-negative* if $a_{i,j} \geqslant 0$ for all $i, j$. For a non-negative matrix $A$ we make the following definitions.

1. $A$ is called *irreducible* if for all $i, j \in \{1, 2, \dots, M\}$ there exists some $k \in \mathbb{N}$ such that the $(i, j)$-th element of $A^k$ is strictly positive.
2. $A$ is called *irreducible and aperiodic* if there is some $k \in \mathbb{N}$ such that for all $i, j \in \{1, 2, \dots, M\}$ the $(i, j)$-th element of $A^k$ is strictly positive.

If $\Sigma$ is a finite alphabet and $A$ is an irreducible and aperiodic transition matrix for $\Sigma$, then we call $(\Sigma_A^\infty, \sigma)$ a *one-sided topologically exact subshift of finite type*.

**Example 2.2.2.** The full shift space is an example of a topologically exact subshift of finite type.

**Example 2.2.3.** Let $M \in \mathbb{N} \setminus \{1\}$ be fixed. Let $\Sigma := \{1, 2, \ldots, 2M\}$ and let $A := [a_{i,j}]_{i,j}$ denote the $2M \times 2M$ transition matrix with entries in $\{0, 1\}$ satisfying

$$a_{i,j} \;:=\; \begin{cases} 1 & \text{if } |i-j| \neq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then the subshift of finite type $(\Sigma_A^\infty, \sigma)$ is topologically exact.

### 2.2.1 Thermodynamic Formalism: The Perron-Frobenius-Ruelle Operator and Equilibrium Measures

In this subsection we introduce the notions of a Gibbs measure and an equilibrium measure defined on the space $\Sigma_A^\infty$ as well as the Perron-Frobenius-Ruelle operator for a one-sided subshift of finite type $(\Sigma_A^\infty, \sigma)$. We will see that the existence of eigenmeasures of the dual of the Perron-Frobenius-Ruelle operator proves the existence of Gibbs measures, where topological pressure appears as the logarithm of the corresponding eigenvalue. The results of this subsection were originally presented in the work of Bowen and Ruelle, see for instance [Bow1, Bow2, Rue1, Rue2]. However, as a reference for this subsection we refer the reader to [Wal1, Wal2, MU, Pol1, Pol2].

Let us begin by describing the concept of measure theoretical entropy given by Sinai and Kolmogorov. For a $\sigma$-invariant measure $\mu$ on a one-sided topologically exact subshift of finite type $(\Sigma_A^\infty, \sigma)$, we define the *measure theoretical entropy* of $\mu$ with respect to $\sigma$ by

$$h_\mu(\sigma) \;:=\; \lim_{k\to\infty} \frac{1}{k} \sum_{\omega \in \Sigma_A^k} -\mu([\omega]) \ln(\mu[\omega]).$$

This is a non-negative quantity which measures the uncertainty of $\mu$ after iterations by $\sigma$. Note that this limit exists since the sequence

$$\left( \sum_{\omega \in \Sigma_A^k} -\mu([\omega]) \ln(\mu[\omega]) \right)_{k \in \mathbb{N}}$$

is subadditive (see Corollary 4.9.1 of [Wal2]).

**Example 2.2.4.** Let $E$ denote a self-similar set satisfying the strong separation condition generated by an iterated function system of similarities $\{s_1, s_2, \ldots, s_m\}$ and let $\delta := \dim_{\mathcal{H}}(E)$ denote the Hausdorff dimension of $E$. Further, let $r_1, r_2, \ldots, r_m$ denote the associated contraction ratios of the similarities and let $T : E \to E$ denote the expanding map as defined in Theorem 2.1.13. Then the dynamical systems $(E, T)$ is topologically conjugate to a full shift space $(\Sigma^\infty, \sigma)$. That is, there exists a homeomorphism $\Phi : E \to \Sigma^\infty$ such that $\Phi T \Phi^{-1} = \sigma$. Then, letting $\widetilde{\mu}$ denote the push forward of the normalised $\delta$-dimensional Hausdorff measure under $\Phi$, we have that

$$h_{\widetilde{\mu}}(\sigma) \;=\; \sum_{i=1}^{m} r_i^\delta \ln(r_i^{-\delta}).$$

In order to introduce a Gibbs measure on a one-sided subshift of finite type let us define the Birkhoff sums of a continuous function.

**Definition 2.2.5.** For each $\phi \in C(\Sigma_A^\infty; \mathbb{R})$ and each $k \in \mathbb{N}_0$, let $S_k\phi : \Sigma_A^\infty \to \mathbb{R}$ denote the $k$-th *Birkhoff sum* of $\phi$ defined, for each $\omega \in \Sigma_A^\infty$, by

$$S_k\phi(\omega) \ := \ \phi(\omega) + \phi(\sigma(\omega)) + \cdots + \phi(\sigma^{k-1}(\omega)).$$

**Theorem 2.2.6.** *Let $(\Sigma_A^\infty, \sigma)$ denote a one-sided topologically exact subshift of finite type and let $\phi \in C(\Sigma_A^\infty; \mathbb{R})$ denote a Hölder continuous function. Then there exists a Borel probability measure $\mu_\phi$ on $\Sigma_A^\infty$ and a uniquely determined number $P(\phi, \sigma) \in [0, \infty)$ associated to $\phi$, such that for some $c > 1$, we have that, for all $k \in \mathbb{N}$ and for all $\omega := (\omega_1, \omega_2, \dots) \in \Sigma_A^\infty$, that*

$$c^{-1} \ \leqslant \ \frac{\mu_\phi[(\omega_1, \omega_2, \dots, \omega_k)]}{e^{S_k\phi(\omega) - kP(\phi,\sigma)}} \ \leqslant \ c. \tag{2.21}$$

*Moreover, to each Hölder continuous potential function $\phi \in C(\Sigma_A^\infty; \mathbb{R})$, there exists a unique $\sigma$-invariant measure satisfying the condition given in Equation (2.21).*

Proof. See Theorem 2.1.3 and Theorem 2.2.4 of [MU]. ∎

We refer to such a measure satisfying the condition given in Equation (2.21) as a *Gibbs measure* for the *potential* $\phi$. Observe that each Gibbs measure will have strictly positive entropy and that to each potential function there can exist several Gibbs measures. Moreover, the uniquely determined number $P(\phi, \sigma)$ is called the *topological pressure* of $\phi$ and is characterised by

$$P(\phi, \sigma) \ = \ \sup \left\{ h_\mu(\sigma) + \int_{\Sigma_A^\infty} \phi d\mu \ : \ \mu \in M(\Sigma_A^\infty, \sigma) \right\}.$$

Here, $M(\Sigma_A^\infty, \sigma)$ denotes the set of $\sigma$-invariant Borel probability measures on $\Sigma_A^\infty$. Further, if there exists a measure $\nu \in M(\Sigma_A^\infty, \sigma)$ such that

$$P(\phi, \sigma) \ = \ h_\nu(\sigma) + \int_{\Sigma_A^\infty} \phi d\nu, \tag{2.22}$$

then we call $\nu$ an *equilibrium measure* for the potential $\phi$. Setting $\phi = 0$, it is well-known that there exists a unique equlibrium measure associated to $\phi$ called the *measure of maximal entropy* (also called the *Parry measure*). This meeasure maximizes the measure theoretical entropy, that is, $h_\mu(\sigma) = \sup_{\nu \in M(\Sigma_{\mathcal{A}}, \sigma)} h_\nu(\sigma)$. This measure, is the combinatorial measure, namely, the measure which weights a cylindrical set $[x]$ with weighting $1/\mathrm{card}(\Sigma_{\mathcal{A}}^k)$, for $x \in \Sigma_{\mathcal{A}}^k$. We refer the reader to [Wal1, Wal2] for a more detailed description of these notions.

**Definition 2.2.7.** For $\phi \in C(\Sigma_A^\infty; \mathbb{R})$, define the *Perron-Frobenius-Ruelle operator* $\mathcal{L}_\phi$ : $C(\Sigma_\phi^\infty; \mathbb{R}) \to C(\Sigma_\phi^\infty; \mathbb{R})$, by

$$\mathcal{L}_\phi(f)(\omega) \ := \sum_{\upsilon \in \sigma^{-1}(\omega)} e^{\phi(\upsilon)} f(\upsilon).$$

Further, we denote the dual of the Perron-Frobenius-Ruelle operator by $\mathcal{L}_\phi^*$. In what follows we shall consider the restriction of $\mathcal{L}_\phi^*$ to linear functional of norm one, and hence, by the Riesz Representation Theorem (see Theorem II.4 of [RS]) we shall view $\mathcal{L}_\phi^*$ as an operator on the set $M(\Sigma_A^\infty)$ consisting of all Borel probability measures defined on $\Sigma_A^\infty$.

**Theorem 2.2.8.** *Let $(\Sigma_A^\infty, \sigma)$ denote a one-sided topologically exact subshift of finite type and let $\phi \in C(\Sigma_A^\infty; \mathbb{R})$ denote a Hölder continuous function. Then the following hold.*

1. *There exists a unique Borel probability $\mu_\phi$ on $\Sigma_A^\infty$ such that*
$$\mathcal{L}_\phi^* \mu_\phi \;=\; e^{P(\phi,\sigma)} \mu_\phi.$$
2. *The unique measure $\mu_\phi$, given in part 1, is a Gibbs measure for the potential $\phi$.*
3. *If $\psi$ is a Hölder continuous function cohomologous to $\phi$ with respect to $\sigma$, then the associated Borel probability measures, given in part 1, are equal.*
4. *There exists a unique strictly positive eigenfunction $h_\phi$ of $\mathcal{L}_\phi$ such that $\mathcal{L}_\phi(h_\phi) = e^{P(\phi,\sigma)} h_\phi$ and such that*
$$\int_{\Sigma_A^\infty} h_\phi \, d\mu_\phi \;=\; 1.$$
5. *The potential function $\phi$ has a unique equilibrium measure $\nu_\phi$. Moreover, $\nu_\phi$ is given, for each $B \in \mathcal{B}$, by*
$$\nu_\phi(B) \;:=\; \int_B h_\phi d\mu_\phi.$$
6. *The unique equilibrium measure for the potential $\phi$, given in part 5, is a Gibbs measure for the potential $\phi$.*

PROOF. See Theorem 2.16, Corollary 4.2 and Theorem 4.5 of [Wal1]. ■

*Remark. In [Pol1] the results of Theorem 2.2.8 have been extended to the case where the potential function is a complex-valued function satisfying a condition weaker than Hölder continuity. Further, the notions of entropy, pressure and that of a Gibbs and equilibrium measure also exist for more general dynamical systems, see for instance [Fal3, Pol2, Wal1, Wal2].*

### 2.2.2 Haar Basis

In this subsection, we develop an essential notion which will be required in Subsection 4.2.2. This notion is that of a Haar basis for a one-sided topologically exact subshift of finite type. As the construction is an original construction we include a full account. We begin with the following well-known example of a Haar basis for the middle third Cantor set (see for instance [Jor]).

**Example 2.2.9.** Consider the middle third Cantor set $C_{1/3}$ generated by an iterated function system of similarities $\{s_1, s_2 : [0,1] \to [0,1]\}$ and let $\delta := \dim_{\mathcal{H}}(C_{1/3}) = \ln(2)/\ln(3)$. Then, for each $k \in \mathbb{N}$ and each $(i_1, i_2, \ldots, i_k) \in \{1,2\}^k$, define $e_{i_1,i_2,\ldots,i_k} : C_{1/3} \to \mathbb{R}$ by
$$e_{i_1,i_2,\ldots,i_k}(x) \;:=\; \begin{cases} 2^{k/2} & \text{if } x \in s_{i_1}s_{i_2}\ldots s_{i_k}s_1[0,1] \cap C_{1/3} \\ -2^{k/2} & \text{if } x \in s_{i_1}s_{i_2}\ldots s_{i_k}s_2[0,1] \cap C_{1/3} \\ 0 & \text{otherwise.} \end{cases}$$

Then the set
$$\bigcup_{k\in\mathbb{N}} \left\{ e_{i_1,i_2,\ldots,i_k} \,:\, (i_1,i_2,\ldots,i_k) \in \{1,2\}^k \right\} \bigcup \left\{ \chi_{C_{1/3}}, \sqrt{2}\left(\chi_{s_1C_{1/3}} - \chi_{s_2[C_{1/3}]}\right)\right\}$$

forms an orthonormal basis for $L^2(C_{1/3}, \mathcal{B}, \mu_{\mathcal{H}^\delta(C_{1/3})})$, where $\mu_{\mathcal{H}^\delta(C_{1/3})}$ denotes the normalised $\delta$ -Hausdorff measure on $C_{1/3}$. This basis is called the *Haar basis of the middle third Cantor set.*

In what follows, our aim is to construct a basis for $L^2(\Sigma_A^\infty, \mathcal{B}, \mu_\phi)$ analogous to the Haar basis given in Example 2.2.9, where $(\Sigma_A^\infty, \sigma)$ denotes a one-sided topologically exact subshift of finite type and where $\mu_\phi$ denotes a Gibbs measure for a Hölder continuous potential $\phi \in C(\Sigma_A^\infty; \mathbb{R})$. To this end, fix $\Sigma_A^\infty$, $\phi$ and $\mu_\phi$ as described above. Define $\alpha : \Sigma_A^* \to \mathbb{N}$ by

$$\alpha(\omega) \ := \ \sum_{x \in \Sigma} a_{\omega_{|\omega|}, x}.$$

For each $\omega \in \Sigma_A^*$, fix a bijection

$$\theta_\omega : \{x \in \Sigma \, : \, a_{\omega_{|\omega|}, x} = 1\} \ \to \ \{1, 2, \ldots, \alpha(\omega)\}.$$

Then, for each $\omega \in \Sigma_A^*$, define the weighted inner product $\langle \cdot, \cdot \rangle_{\mu_\phi, \omega} : \mathbb{R}^{\alpha(\omega)} \times \mathbb{R}^{\alpha(\omega)} \to \mathbb{R}$ by

$$\langle (x_1, x_2, \ldots, x_{\alpha(\omega)}), (y_1, y_2, \ldots, y_{\alpha(\omega)}) \rangle_{\mu_\phi, \omega} \ := \ \sum_{k=1}^{\alpha(\omega)} \mu_\phi([\omega \theta_\omega^{-1}(k)]) x_k y_k$$

and observe that the set

$$\{f_{\omega, j} := (\mu_\phi([\omega \theta_\omega^{-1}(j)]))^{-1/2} \, (\underbrace{0, 0, \ldots, 0}_{j-1 \text{ times}}, 1, \underbrace{0, \ldots, 0, 0}_{\alpha(\omega) - j \text{ times}}) \, : \, j \in \{1, 2, \ldots, \alpha(\omega)\}\}$$

forms an orthonormal basis for $(\mathbb{R}^{\alpha(\omega)}, \langle \cdot, \cdot \rangle_{\mu_\phi, \omega})$. Further, for each $\omega \in \Sigma_A^*$, let $\Omega_\omega$ denote the set defined by

$$\begin{aligned} \Omega_\omega \ := \ \{U : \mathbb{R}^{\alpha(\omega)} \to \mathbb{R}^{\alpha(\omega)} : \, & U \text{is linear and has positive determinant,} \\ & \langle U(x), U(y) \rangle_{\mu_\phi, \omega} = \langle x, y \rangle_{\mu_\phi, \omega} \text{for all} \, x, y \in \mathbb{R}^{\alpha(\omega)}, \text{ and} \\ & U(f_{\omega, \alpha(\omega)}) = (\mu_\phi([\omega]))^{-1/2} (1, 1, \ldots, 1)\} \} \end{aligned}$$

and fix a sequence $(U_\omega)_{\omega \in \Sigma_A^*}$ with $U_\omega \in \Omega_\omega$. For each $(\omega, i) \in \bigcup_{\upsilon \in \Sigma_A^*} \{\upsilon\} \times \{1, 2, \ldots, \alpha(\upsilon) - 1\}$, define $e_{\omega, i} : \Sigma_A^\infty \to \mathbb{R}$ by

$$e_{\omega, i} \ := \ \sum_{k=1}^{\alpha(\omega)} (\mu_\phi([\omega(\theta_\omega^{-1}(k))]))^{-1/2} \, \langle f_{\omega, k}, U_\omega(f_{\omega, i}) \rangle_{\mu_\phi, \omega} \, \chi_{[\omega(\theta_\omega^{-1}(k))]}.$$

Following convention, for each $a \in C(\Sigma_A^\infty; \mathbb{C})$, we will let $a$ also denote, where appropriate, the equivalence class

$$\left\{ f : \Sigma_A^\infty \to \mathbb{C} \, : \, f \text{ is a measurable function and } \int_{\Sigma_A^\infty} |f - a| \, d\mu_\phi \ = \ 0 \right\}$$

of $L^2(\Sigma_A^\infty, \mathcal{B}, \mu_\phi)$.

*Remark. For each $k \in \mathbb{N} \setminus \{1\}$ and each $\omega \in \Sigma_A^k$ there exists a canonical choice for $U_\omega$. We construct this canonical choice in the following manner. Assume that $\mathbb{R}^k$ is equipped with the standard Euclidean inner product, for $k \in \{2, 3, \ldots, l\}$. We will now show how to construct a (canonical) sequence of linear transformations $(V_k)_{k=1}^l$, where $V_k : \mathbb{R}^k \to \mathbb{R}^k$ and such that each $V_k$*

*satisfies the following.*

1. $V_k$ *is orientation preserving.*
2. *For every* $x, y \in \mathbb{R}^k$*, we have that* $\langle V_k(x), V_k(y)\rangle = \langle x, y\rangle$.
3. $V_k(0,0,\dots,0,1) = n^{-1/2}(1,1,\dots,1,1)$.

*For* $k = 2$*, one has only one choice for* $V_2$*, namely*

$$V_2 \;:=\; \begin{pmatrix} 2^{-1/2} & 2^{-1/2} \\ -2^{-1/2} & 2^{-1/2} \end{pmatrix}.$$

*For* $k > 2$*, assume that* $V_{k-1} := [v_{i,j}]_{i,j} : \mathbb{R}^{k-1} \to \mathbb{R}^{k-1}$ *has been given, then define*

$$\widetilde{V}_k \;:=\; \begin{pmatrix} v_{1,1} & \dots & v_{1,k-1} & 0 \\ v_{2,1} & \dots & v_{2,k-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ v_{k-1,1} & \dots & v_{k-1,k-1} & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}, \quad O_k \;:=\; \begin{pmatrix} 1 & \dots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & 0 & 0 \\ 0 & \dots & 0 & k^{-1/2} & (1-1/k)^{1/2} \\ 0 & \dots & 0 & -(1-1/k)^{1/2} & k^{-1/2} \end{pmatrix}$$

*where* $O_k \in \mathcal{O}(k)$*, the orthogonal group of degree* $k$ *over* $\mathbb{R}$*. We then define* $V_k$ *to be the matrix* $\widetilde{V}_k O_k \widetilde{V}_k^t$*. A canonical choice for* $U_\omega$ *is then the linear transformation* $S^{-1} V_{\alpha(\omega)} S$*, where*

$$S \;:=\; diag\left(\mu_\phi([\omega\theta_\omega^{-1}(1)])^{1/2}, \dots, \mu_\phi([\omega\theta_\omega^{-1}(k)])^{1/2}\right).$$

**Theorem 2.2.10.** *The set*

$$\left\{e_{\omega,i} \,:\, (\omega,i) \in \bigcup\nolimits_{\upsilon\in\Sigma_A^*} \{\upsilon\} \times \{1,2,\dots,\alpha(\upsilon)-1\}\right\} \;\cup\; \left\{(\mu_\phi([x]))^{-1/2}\chi_{[x]} \;:\; x \in \Sigma\right\}$$

*forms an orthonormal basis for* $L^2(\Sigma_A^\infty, \mathcal{B}, \mu_\phi)$.

Proof. For each $x \in \Sigma$, we have that

$$\left\|(\mu_\phi([x]))^{-1/2}\chi_{[x]}\right\|^2 \;=\; \int_{\Sigma_A^\infty} ((\mu_\phi([x]))^{-1/2}\chi_{[x]})^2 d\mu_\phi \;=\; 1.$$

Further, for each $(\omega, i) \in \bigcup_{\upsilon\in\Sigma_A^*}\{\upsilon\} \times \{1,2,\dots,\alpha(\upsilon)-1\}$, we have that

$$\begin{aligned} \|e_{\omega,i}\|^2 \;&=\; \int_{\Sigma_A^\infty} \left(\sum_{k=1}^{\alpha(\omega)} (\mu_\phi([\omega(\theta_\omega^{-1}(k))]))^{-1/2} \langle f_{\omega,k}, U(f_{\omega,i})\rangle_{\mu_\phi,\omega}\, \chi_{[\omega(\theta_\omega^{-1}(k))]}\right)^2 d\mu_\phi \\ &=\; \int_{\Sigma_A^\infty} \sum_{k=1}^{\alpha(\omega)} (\mu_\phi([\omega(\theta_\omega^{-1}(k))]))^{-1} \langle f_{\omega,k}, U(f_{\omega,i})\rangle_{\mu_\phi,\omega}^2\, \chi_{[\omega(\theta_\omega^{-1}(k))]}\, d\mu_\phi \\ &=\; \sum_{k=1}^{\alpha(\omega)} \langle f_{\omega,k}, U_\omega(f_{\omega,i})\rangle_{\mu_\phi,\omega}^2 \\ &=\; \|U_\omega(f_{\omega,i})\|^2 \;=\; \|f_{\omega,i}\|^2 \;=\; 1. \end{aligned}$$

Next, we observe the following.

1. Let $x \in \Sigma$ and let $(\omega, i) \in \bigcup_{\upsilon \in \Sigma_A^*} \{\upsilon\} \times \{1, 2, \ldots, \alpha(\upsilon) - 1\}$. If $[\omega] \not\subseteq [x]$, then $\langle (\mu_\phi([x]))^{-1} \chi_{[x]}, e_{\omega,i} \rangle = 0$, since $\chi_{[x]}$ and $f_{\omega,i}$ are non-zero on different cylinder sets. Otherwise, if $[\omega] \subseteq [x]$, then

$$\begin{aligned}
&\langle (\mu_\phi([x]))^{-1} \chi_{[x]}, e_{\omega,i} \rangle \\
&\quad = \int_{[x]} \sum_{k=1}^{\alpha(\omega)} (\mu_\phi([x]))^{-1} (\mu_\phi([\omega(\theta_\omega^{-1}(k))]))^{-1/2} \langle f_{\omega,k}, U_\omega(f_{\omega,i}) \rangle_{\mu_\phi,\omega} \chi_{[\omega(\theta_\omega^{-1}(k))]} \, d\mu_\phi \\
&\quad = (\mu_\phi([x]))^{-1} \langle (1, 1, \ldots, 1), U_\omega(f_{\omega,i}) \rangle_{\mu_\phi,\omega} \\
&\quad = (\mu_\phi([x]))^{-1} (\mu_\phi([\omega]))^{1/2} \langle U_\omega(f_{\omega,\alpha(\omega)}), U_\omega(f_{\omega,i}) \rangle_{\mu_\phi,\omega} \\
&\quad = (\mu_\phi([x]))^{-1} (\mu_\phi([\omega]))^{1/2} \langle f_{\omega,\alpha(\omega)}, f_{\omega,i} \rangle_{\mu_\phi,\omega} \quad = \quad 0.
\end{aligned}$$

2. Let $(\omega, i), (\omega, j) \in \bigcup_{\upsilon \in \Sigma_A^*} \{\upsilon\} \times \{1, 2, \ldots, \alpha(\upsilon) - 1\}$ with $i \neq j$. Then we have that

$$\begin{aligned}
&\langle e_{\omega,i}, e_{\omega,j} \rangle \\
&= \int_{\Sigma_A^\infty} \sum_{k=1}^{\alpha(\omega)} (\mu_\phi([\omega(\theta_\omega^{-1}(k))]))^{-1} \langle f_{\omega,k}, U_\omega(f_{\omega,i}) \rangle_{\mu_\phi,\omega} \langle f_{\omega,k}, U_\omega(f_{\omega,j}) \rangle_{\mu_\phi,\omega} \chi_{[\omega(\theta_\omega^{-1}(k))]} d\mu_\phi \\
&= \sum_{k=1}^{\alpha(\omega)} \langle f_{\omega,k}, U_\omega(f_{\omega,i}) \rangle_{\mu_\phi,\omega} \langle f_{\omega,k}, U_\omega(f_{\omega,j}) \rangle_{\mu_\phi,\omega} \\
&= \sum_{k=1}^{\alpha(\omega)} \sum_{m=1}^{\alpha(\omega)} \langle \langle U_\omega(f_{\omega,i}), f_{\omega,k} \rangle_{\mu_\phi,\omega} f_{\omega,k}, \langle U_\omega(f_{\omega,j}), f_{\omega,m} \rangle_{\mu_\phi,\omega} f_{\omega,m} \rangle_{\mu_\phi,\omega} \\
&= \langle U_\omega(f_{\omega,i}), U_\omega(f_{\omega,j}) \rangle_{\mu_\phi,\omega} \\
&= \langle f_{\omega,i}, f_{\omega,j} \rangle_{\mu_\phi,\omega} \quad = \quad 0.
\end{aligned}$$

3. Let $(\omega, i), (\omega', j) \in \bigcup_{\upsilon \in \Sigma_A^*} \{\upsilon\} \times \{1, 2, \ldots, \alpha(\upsilon) - 1\} \in \Sigma_A^*$ with $\omega \neq \omega'$. Then the following hold.
   (a) If either $[\omega] \not\subseteq [\omega']$ or $[\omega'] \not\subseteq [\omega]$, then we have that $\langle e_{\omega,i}, e_{\omega',j} \rangle = 0$. This follows since $f_{\omega,i}$ and $f_{\omega',j}$ are non-zero on different cylinder sets.
   (b) If $[\omega] \subset [\omega']$, then there exists a constant $C \in \mathbb{R}$ such that

$$\begin{aligned}
\langle e_{\omega,i}, e_{\omega',j} \rangle &= C \int_{[\omega']} \sum_{k=1}^{\alpha(\omega)} (\mu_\phi([\omega(\theta_\omega^{-1}(k))]))^{-1/2} \langle f_{\omega,k}, U_\omega(f_{\omega,i}) \rangle_{\mu_\phi,\omega} \chi_{[\omega(\theta_\omega^{-1}(k))]} \, d\mu_\phi \\
&= C \sum_{k=1}^{\alpha(\omega)} (\mu_\phi([\omega(\theta_\omega^{-1}(k))]))^{1/2} \langle f_{\omega,k}, U_\omega(f_{\omega,i}) \rangle_{\mu_\phi,\omega} \\
&= (\mu([\omega]))^{1/2} \langle U_\omega(f_{\omega,\alpha(\omega)}), U_\omega(f_{\omega,i}) \rangle_{\mu_\phi,\omega} \\
&= (\mu([\omega]))^{1/2} \langle f_{\omega,\alpha(\omega)}, f_{\omega,i} \rangle_{\mu_\phi,\omega} \quad = \quad 0.
\end{aligned}$$

   (c) If $[\omega'] \subset [\omega]$, then a symmetric proof to that given in (b), implies that $\langle e_{\omega,i}, e_{\omega',j} \rangle = 0$.

By construction, every characteristic function of a cylinder set can be generated by a finite sum of elements of the set

$$\left\{ e_{\omega,i} \,:\, (\omega, i) \in \bigcup_{\upsilon \in \Sigma_A^*} \{\upsilon\} \times \{1, 2, \ldots, \alpha(\upsilon) - 1\} \right\} \,\cup\, \left\{ (\mu_\phi([x]))^{-1/2} \chi_{[x]} \,:\, x \in \Sigma \right\}.$$

The result then follows from the Stone-Weierstrass Theorem (stated below) and the fact that $C(\Sigma_A^\infty;\mathbb{C})$ is $L^2$-norm-dense in $L^2(\Sigma_A^\infty,\mathcal{B},\mu_\phi)$. ■

**Definition 2.2.11.** Let $X$ denote a topological space and let $R$ denote a subset of $C(X;\mathbb{C})$. Then $R$ is said to *separates points*, if for all distinct $x, y \in X$ there exists a function $f \in R$ with $f(x) \neq f(y)$.

**Theorem 2.2.12.** *(Stone-Weierstrass Theorem for Complex Functions) Let $X$ be a compact Hausdorff space and recall that $C(X;\mathbb{C})$ denotes the set of complex-valued continuous functions on $X$. Let $\mathcal{A}$ be a complex sub-algebra of $C(X;\mathbb{C})$ with the property that if $a \in \mathcal{A}$, then the complex conjugate $\overline{a}$ belongs to $\mathcal{A}$. If $\mathcal{A}$ separates points and contains the set of constant functions, then $\mathcal{A}$ is norm-dense in $C(X;\mathbb{C})$ with respect to the supremum norm.*

**Definition 2.2.13.** We refer to the basis given in Theroem 2.2.10 as a *Haar basis for* $L^2(\Sigma_A^\infty,\mathcal{B},\mu_\phi)$.

## 2.3 Renewal Theorems

The renewal theorem is a major theorem from probabilistic analysis. It guarantees convergence to a steady state for a large class of stochastic processes. In this section three different formulations of the renewal theorem (Theorems 2.3.2, 2.3.6 and 2.3.8) are given. These allow us to obtain counting results for self-similar sets and Birkhoff sums of Hölder continuous functions on topologically exact subshifts of finite type, some of which, to the best of our knowledge, do not appear within the current literature. When building various noncommutative representations for such sets, these counting results will allow us to broaden already existing links between various noncommutative quantities and invariants of dynamical systems (see Chapter 4). The results of this section are based on those given in [Fal3, Lal, GH]. Let us begin by describing the formulation given in [Fal3].

**Definition 2.3.1.** Let $\mu$ be a measure on $\mathbb{R}$. Then $\mu$ is called *arithmetic* if there exists a positive real number $\tau$ such that the support of $\mu$ is contained in the additive group $\tau\mathbb{Z}$. If $\tau$ is the greatest positive number such that this holds, then $\mu$ is called *$\tau$-arithmetric*. If there exists no such number $\tau$, then we say that the measure $\mu$ is *non-arithmetic*. Similarly, for $m \in \mathbb{N}$, we call a set $Y := \{y_1, y_2, \ldots, y_m\} \subset \mathbb{R}$ *arithmetic* if there exists a positive real number $\tau$ such that $y_i \in \tau\mathbb{Z}$, for each $i \in \{1, 2, \ldots, m\}$. If $\tau$ is the greatest positive number such that this holds, then we say that $Y$ is *$\tau$-arithemetic*. If there does not exist such a number $\tau$, then we say that $Y$ is *non-arithmetic*.

Let $m \geqslant 2$ denote a natural number, let $(p_1, p_2, \ldots, p_m)$ denote a probability vector and let $\{y_i : i \in \{1, 2, \ldots, m\}\}$ denote a set of positive real numbers. Let $\mu$ denote the measure on $\mathbb{R}$ with support $\{y_i : i \in \{1, 2, \ldots, m\}\}$ such that $\mu(\{y_i\}) := p_i$, for $i \in \{1, 2, \ldots, m\}$. Then a function $f : \mathbb{R} \to \mathbb{R}$ is said to satisfy a *renewal equation* if there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that for each $t \in \mathbb{R}$, one has that

$$f(t) = \sum_{j=1}^{m} p_j f(t - y_j) + g(t). \tag{2.23}$$

**Theorem 2.3.2.** *Let $\mu$ be as above and let $f, g : \mathbb{R} \to \mathbb{R}$ satisfy the renewal equation given in Equation (2.23). Additionally, assume the following.*

1. *The function $g$ has a discrete set of discontinuities.*
2. *There exist constants $c_1, c_2 > 0$, such that $|g(t)| \leqslant c_1 e^{-c_2|t|}$, for each $t \in \mathbb{R}$.*
3. *The function $f$ is a Borel measurable function such that $f$ is bounded on the half-line $(-\infty, \eta)$, for each $\eta \in \mathbb{R}$. Further, we have that $\lim_{t\to-\infty} f(t) = 0$.*

*If $\mu$ is non-arithmetic, then*

$$\lim_{t\to\infty} f(t) \;=\; \frac{\int_{-\infty}^{\infty} g(y)\, dy}{\sum_{i=1}^{m} y_i p_i}.$$

*If there exists a $\tau \in \mathbb{R}$ such that $\mu$ is $\tau$-arithmetic, then for all $y \in [0,\tau)$, we have that*

$$\lim_{k\to\infty} f(k\tau + y) \;=\; \frac{\sum_{k=-\infty}^{\infty} g(k\tau + y)}{\sum_{i=1}^{m} y_i p_i}.$$

Proof. See Corollary 7.3 of [Fal3]. ∎

This allows us to compute the first of our counting results. For a compact subset $E$ of $\mathbb{R}$ with Hausdorff dimension $\delta$, we make the following definitions.

1. Define $\mathcal{G} := \mathcal{G}_E : (0, +\infty) \to \mathbb{N}$ by letting $\mathcal{G}(r)$ equal the number of complementary intervals of $E$ of length greater than or equal to $r$, ignoring the two infinite components.

2. Define $\mathcal{E} := \mathcal{E}_E : (0, +\infty) \to \mathbb{R}$ by

$$\mathcal{E}(r) \;:=\; \sum_{k\in\mathbb{N} \text{ with } |I_k|\geqslant r} |I_k|^{\delta},$$

where the set $\{I_k \,:\, k \in \mathbb{N}\}$ denotes the set of complementary intervals of $E$ with finite length.

The following two propositions give the asymptotic behaviour of these functions as $r$ tends to zero. In particular, we consider a self-similar set $\{0,1\} \subseteq E \subset [0,1]$, which is generated by the iterated function system of similarities $\{s_1, s_2, \ldots, s_m\}$ satisfying the strong separation condition. We begin with the following proposition which gives the asymptotic behaviour of the function $\mathcal{G}$.

**Proposition 2.3.3.** *Let $\delta := \dim_{\mathcal{H}}(E)$ and let $r_i$ denote the contraction ratio of $s_i$, for each $i \in \{1, 2, \ldots, m\}$. If the set $\{\ln(r_1^{-1}), \ln(r_2^{-1}), \ldots, \ln(r_m^{-1})\}$ is non-arithmetic, then there exists a $c > 0$ such that as $r$ tends to zero we have that*

$$\mathcal{G}(r) \;\sim\; c r^{-\delta}.$$

*If $\{\ln(r_1^{-1}), \ln(r_2^{-1}), \ldots, \ln(r_m^{-1})\}$ is a $\tau$-arithmetic set, for some positive real number $\tau$, then there exists a bounded periodic function $P : \mathbb{R} \to \mathbb{R}$ with period $\tau$, such that for $y \in [0,\tau)$, as $k$ tends to positive infinity we have that*

$$\mathcal{G}(e^{-k\tau + y}) \;\sim\; P(y) e^{\delta(k\tau - y)}.$$

Proof. See Proposition 7.5 of [Fal3]. ∎

In the next proposition we obtain a new result which gives the asymptotic behaviour of the function $\mathcal{E}$. This result, along with that given above, will prove useful in Section 4.1.

**Proposition 2.3.4.** *Let $\{l_i \,:\, i \in \{1, 2, \ldots, m-1\}\}$ denote the set of complementary intervals of the set $\bigcup_{i=1}^{m} s_i([0,1])$ whose lengths are finite, let $r_i$ denote the contraction ratio of $s_i$ and set $\delta := \dim_{\mathcal{H}}(E)$. Then, as $r$ tends to zero, we have that*

$$\mathcal{E}(r) \;\sim\; \frac{\sum_{i=1}^{m-1} |l_i|^{\delta}}{\sum_{i=1}^{m} r_i^{\delta} \ln(r_i^{-1})} \ln(r^{-1}).$$

Proof. Observe that for each $i \in \{1, 2, \ldots, m\}$ the lengths of the complementary intervals of $s_i(E)$ are exactly the lengths of the complementary intervals of $E$ multiplied by $r_i$. Hence,

$$\mathcal{E}(r) \;=\; \sum_{i=1}^{m} r_i^\delta \mathcal{E}\left(\frac{r}{r_i}\right) \;+ \sum_{\substack{i \in \{1,2,\ldots,m-1\} \\ \text{with } |l_i| \geqslant r}} |l_i|^\delta. \tag{2.24}$$

Substituting $r = e^{-t}$ and $\psi(t) = \mathcal{E}(e^{-t})$ into Equation (2.24) gives

$$\psi(t) \;=\; \sum_{i=1}^{m} r_i^\delta \psi(t - \ln(r_i^{-1})) \;+ \sum_{\substack{i \in \{1,2,\ldots,m-1\} \\ \text{with } |l_i| \geqslant e^{-t}}} |l_i|^\delta. \tag{2.25}$$

Although this is a renewal equation, it is not in a form which allows for the application of the renewal theorem (Theorem 2.3.2). Therefore, with the aim of applying Theorem 2.3.2, the following definitions and substitutions are made. Let

$$c \;:=\; \frac{\sum_{i=1}^{m-1} |l_i|^\delta}{\sum_{i=1}^{m} r_i^\delta \ln(r_i^{-1})}.$$

Define $\psi_1 : \mathbb{R} \to \mathbb{R}$ by

$$\psi_1(t) \;:=\; \begin{cases} \psi(t) - ct & \text{if } \;\; t > 0, \\ \psi(t) & \text{if } \;\; t \leqslant 0 \end{cases}$$

and define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(t) \;=\; \begin{cases} \displaystyle\sum_{\substack{i \in \{1,2,\ldots,m-1\} \\ \text{with } |l_i| \geqslant e^{-t}}} |l_i|^\delta - c \sum_{i=1}^{m} r_i^\delta \ln(r_i^{-1}) & \\ \displaystyle\qquad - c \sum_{i=1}^{m} r_i^\delta \Big(t - \ln(r_i^{-1})) \chi_{(-\infty,0)}(t - \ln(r_i^{-1}))\Big) & \text{if } t > 0, \\ 0 & \text{if } t \leqslant 0. \end{cases}$$

Let us now show that $\psi_1$ and $g$ satisfy the renewal equation (Equation (2.23)) with $p_i := r_i^\delta$ and $y_i := \ln(r_i^{-1})$ for $i \in \{1, 2, \ldots, m\}$. First, let us consider the case $t \leqslant 0$. Since in this case we have that $g(t) = 0$, that $\psi(t) = 0$ and that $t - \ln(r_i^{-1}) \leqslant 0$, for $i \in \{1, 2, \ldots, m\}$, it follows that

$$\sum_{i=1}^{m} r_i^\delta \psi_1(t - \ln(r_i^{-1})) + g(t) \;=\; \sum_{i=1}^{m} r_i^\delta \psi(t - \ln(r_i^{-1})) \;=\; 0 \;=\; \psi(t) \;=\; \psi_1(t).$$

Now, let us consider the case $t > 0$. In this case we have that

$$\begin{aligned} &\sum_{i=1}^{m} r_i^\delta \psi_1(t - \ln(r_i^{-1})) \\ &\quad= \sum_{i=1}^{m} r_i^\delta \Big(\psi(t - \ln(r_i^{-1})) - c(t - \ln(r_i^{-1})) + c(t - \ln(r_i^{-1}))\chi_{(-\infty,0)}(t - \ln(r_i^{-1}))\Big) \\ &\quad= \psi(t) - \sum_{\substack{i \in \{1,2,\ldots,m-1\} \\ \text{with } |l_i| \geqslant e^{-t}}} |l_i|^\delta - ct + c\sum_{i=1}^{m} r_i^\delta \ln(r_i^{-1}) + c\sum_{i=1}^{m} r_i^\delta (t - \ln(r_i^{-1}))\chi_{(-\infty,0)}(t - \ln(r_i^{-1})) \end{aligned}$$

$$
\begin{aligned}
&= \psi(t) - ct - g(t) \\
&= \psi_1(t) - g(t).
\end{aligned}
$$

Moreover, it is clear from the definition of $g$ that $g$ has a discrete set of discontinuities and that $g(t) = 0$ for $t \geqslant \max\{\ln(r_i^{-1}) : i \in \{1, 2, \ldots, m\}\}$. It is easy to verify that $\psi_1$ is Borel measurable, that $\psi_1$ is bounded on the half-line $(-\infty, \eta)$, for each $\eta \in \mathbb{R}$, and that $\psi(t)$ converges to zero as $t$ tends to negative infinity. Therefore, Theorem 2.3.2 can be applied and we conclude that

$$
\lim_{t\to\infty} \frac{\psi(t)}{ct} = \lim_{t\to\infty} \frac{\psi_1(t) + ct}{ct} = 1.
$$

Hence, it follows that $\mathcal{E}(r) \sim c\ln(r^{-1})$ as $r$ tends to zero. ■

Next, we consider such counting problems for a one-sided subshift of finite type $(\Sigma_A^\infty, \sigma)$. Specifically, for a Gibbs measure $\mu_\phi$ for a Hölder continuous potential function $\phi \in C(\Sigma_A^\infty; \mathbb{R})$ and for $\mathcal{V} \in \Sigma_A^* \cup \emptyset$, we are interested in the asymptotic behaviour, as $r$ tends to zero, of the sums

$$
\Upsilon_{\mathcal{V}}(r) := \sum_{\substack{\omega\in\Sigma_A^* \text{ with} \\ \mu_\phi([\omega])>r \text{ and } [\omega]\subseteq[\mathcal{V}]}} 1, \tag{2.26}
$$

$$
\Xi_{\mathcal{V}}(r) := \sum_{\substack{\omega\in\Sigma_A^* \text{ with} \\ \mu_\phi([\omega])>r \text{ and } [\omega]\subseteq[\mathcal{V}]}} \mu_\phi([\omega]). \tag{2.27}
$$

Here, when $\mathcal{V} = \emptyset$, we set $[\mathcal{V}] := \Sigma_A^\infty$. To calculate the behaviour of such sums, we use renewal theorems presented in [Lal, GH]. Let us begin with the renewal theorem which is stated in [Lal]. This enables the calculation of the asymptotic behaviour of the sum given in Equation (2.26).

**Definition 2.3.5.** A real-valued function $\phi$ is called *arithmetic* if it's cohomologous, with respect to $\sigma$, to a function taking values in a discrete subgroup of $\mathbb{R}$. Otherwise $\phi$ is called *non-arithmetic.*

Let $(\Sigma_A^\infty, \sigma)$ denote a one-sided topologically exact subshift of finite type, let $\phi_1 : \Sigma_A^\infty \to \mathbb{R}$ denote a non-arithmetic Hölder continuous function and let $\phi_2 : \Sigma_A^\infty \to \mathbb{R}$ denote a non-negative Hölder continuous function that is not identically equal to zero. Define $N_{\phi_1,\phi_2}(r, \omega)$, for each $r \in \mathbb{R}$ and $\omega \in \Sigma_A^\infty$, by

$$
N_{\phi_1,\phi_2}(r, \omega) := \sum_{k\in\mathbb{N}_0} \sum_{\upsilon\in\sigma^{-k}(\omega)} \phi_2(\upsilon)\, \chi_{(-\infty,r]}(S_k\phi_1(\upsilon)).
$$

Observe that $N_{\phi_1,\psi_2}(r, \omega)$ satisfies a renewal equation, that is,

$$
N_{\phi_1,\phi_2}(r, \omega) = \sum_{\upsilon\in\sigma^{-1}(\omega)} N_{\phi_1,\phi_2}(r - \phi_1(\upsilon), \upsilon) + \phi_2(\omega)\chi_{[-\infty,r)}(0).
$$

**Theorem 2.3.6.** *Assume that $P(\phi_1, \sigma) = 0$, let $\mu_{\phi_1}$ denote the unique fixed point of $\mathcal{L}_{\phi_1}^*$ and let $h_{\phi_1}$ denote the unique fixed point of $\mathcal{L}_{\phi_1}$ such that*

$$
\int_{\Sigma_A^\infty} h_{\phi_1}\, d\mu_{\phi_1} = 1.
$$

*Define the continuous function $C : \Sigma_A^\infty \to \mathbb{R}$ by*

$$C(\omega) := \frac{\int \phi_2 d\mu_{\phi_1}}{\int \phi_1 h_{\phi_1} d\mu_{\phi_1}} h_{\phi_1}(\omega).$$

*Then, $N_{\phi_1,\phi_2}(r,\omega) \sim C(\omega)e^r$ as $r \to \infty$, uniformly for each $\omega \in \Sigma_A^\infty$.*

Proof. See Theorem 1 of [Lal]. ∎

**Corollary 2.3.7.** *Under the conditions of Theorem 2.3.6, for each $\mathcal{V} \in \Sigma_A^* \cup \emptyset$, we have that $\Upsilon_{\mathcal{V}}(t) \asymp t^{-1}$ as $t$ tends to zero from above. (Note that the constants in the comparability statment may depend on $|\mathcal{V}|$.)*

Proof. Recall from Theorem 2.2.8 that the eigenfunction $h_\phi$ of $\mathcal{L}_\phi$ is strictly positive and since $\Sigma_A^\infty$ is compact, $h_\phi$ is bounded. Hence, for each $\omega \in \Sigma_A^\infty$, each $\mathcal{V} \in \Sigma^*$, by Theorem 2.3.6, as $r$ tends to positive infinity, we have that

$$\begin{aligned} e^r &\asymp N_{\phi,\chi_{[\mathcal{V}]}}(r,\omega) \\ &\asymp \operatorname{card}\{\upsilon := (\upsilon_1,\upsilon_2,\dots) \in \sigma^{-k}(\omega) \,:\, k \in \mathbb{N},\ [(\upsilon_1,\upsilon_2,\dots,\upsilon_k)] \subseteq [\mathcal{V}] \text{ and} \\ &\qquad\qquad - \ln(\mu_\phi([(\upsilon_1,\upsilon_2,\dots,\upsilon_k)])) \leqslant r\} \\ &\asymp \operatorname{card}\{\upsilon \in \Sigma_A^* \,:\, \mu_\phi([\upsilon]) \geqslant e^{-r} \text{ and } [\upsilon] \subset [\mathcal{V}]\} \\ &= \Upsilon_w(e^{-r}). \end{aligned}$$

∎

The next result is a modification of a renewal theorem for subshifts of finite type as given in [GH] and allows for the calculation of the asymptotic behaviour of the sum in Equation (2.27).

**Theorem 2.3.8.** *Let $(\Sigma_A^\infty,\sigma)$ denote a one-sided topologically exact subshift of finite type, let $\nu_\phi$ denote an Equilibrium measure on $\Sigma_A^\infty$ for a Hölder continuous potential function $\phi \in C(\Sigma_A^\infty;\mathbb{R})$ and let $\psi : \Sigma_A^\infty \to \mathbb{R}$ denote a non-arithmetic Hölder continuous function with positive moment with respect to $\nu_\phi$, that is,*

$$\gamma := \int_{\Sigma_A^\infty} \psi d\nu_\phi > 0.$$

*For each $\omega \in \Sigma_A^*$, define $v_\omega$ on the set $\{I \subset \mathbb{R} : I \text{ is bounded and connected}\}$ by*

$$v_\omega(I) := \int_{[\omega]} \sum_{k\in\mathbb{N}_0} \chi_I(S_k\psi(x))\, d\nu(x).$$

*Then $v_\omega$ can be extended to a Radon Borel measure on $\mathbb{R}$ and we have that*

$$\begin{aligned} \lim_{t\to-\infty} v_\omega(I+t) &= 0, \\ \lim_{t\to+\infty} v_\omega(I+t) &= \gamma^{-1}\lambda^1(I)\nu_\phi([\omega]). \end{aligned}$$

Proof. For each $k \in \mathbb{N}_0$, define $\mathcal{F}_k$ to be the Borel measure on the product space $\Sigma_A^\infty \times \mathbb{R} \times \Sigma_A^\infty$ which is given, for each Borel set $X \times B \times Y$, by

$$\mathcal{F}_k(X \times B \times Y) := \nu_\phi(\{x : (x, S_k\psi(x), \sigma^k(x)) \in X \times B \times Y\}).$$

For each $t \in \mathbb{R}$, let $\mathcal{F}_k * \delta_t$ denote the translate of $\mathcal{F}_k$ given by $(x, a, y) \mapsto (x, a+t, y)$. In the proof of Theorem 4 (Section B, page 95) of [GH], Guivarc'h and Hardy show that, provided $\psi$ is a non-arithmetic Hölder continuous function with positive moment with respect to $\nu_\phi$, one has that the following hold

$$\begin{aligned} \lim_{t\to-\infty} \sum_{k\in\mathbb{N}_0} \mathcal{F}_k * \delta_t &= 0, \\ \lim_{t\to+\infty} \sum_{k\in\mathbb{N}_0} \mathcal{F}_k * \delta_t &= \gamma^{-1} \cdot \nu_\phi \times \lambda^1 \times \nu_\phi. \end{aligned}$$

(Here, the limits are taken with respect to the vague topology and that within the proof of this result one requires the property that $\nu_\phi$ is $\sigma$-invariant.) Therefore, for each $\omega \in \Sigma_A^*$ and each bounded connected interval $I$, we deduce that

$$\begin{aligned} \lim_{t\to-\infty} v_{[\omega]}(I+t) &= \lim_{t\to-\infty} \int_{[\omega]} \sum_{k\in\mathbb{N}_0} \chi_{I+t}(S_k\psi(x)) d\nu_\phi(x) \\ &= \lim_{t\to-\infty} \int_{\Sigma_A^\infty\times\mathbb{R}\times\Sigma_A^\infty} \chi_{[\omega]}(x)\chi_{I+t}(b)\chi_{\Sigma_A^\infty}(y)\, d\sum_{k\in\mathbb{N}_0} \mathcal{F}_k(x,b,y) \\ &= \lim_{t\to-\infty} \int_{\Sigma_A^\infty\times\mathbb{R}\times\Sigma_A^\infty} \chi_{[\omega]}(x)\chi_{I}(b)\chi_{\Sigma_A^\infty}(y)\, d\sum_{k\in\mathbb{N}_0} \mathcal{F}_k(x,b,y) * \delta_t \\ &= 0. \end{aligned}$$

Further, we conclude that

$$\begin{aligned} \lim_{t\to+\infty} v_{[\omega]}(I+t) &= \lim_{t\to+\infty} \int_{[\omega]} \sum_{k\in\mathbb{N}_0} \chi_{I+t}(S_k\psi(x)) d\nu_\phi(x) \\ &= \lim_{t\to+\infty} \int_{\Sigma_A^\infty\times\mathbb{R}\times\Sigma_A^\infty} \chi_{[\omega]}(x)\chi_{I+t}(b)\chi_{\Sigma_A^\infty}(y)\, d\sum_{k\in\mathbb{N}_0} \mathcal{F}_k(x,b,y) \\ &= \lim_{t\to+\infty} \int_{\Sigma_A^\infty\times\mathbb{R}\times\Sigma_A^\infty} \chi_{[\omega]}(x)\chi_{I}(b)\chi_{\Sigma_A^\infty}(y)\, d\sum_{k\in\mathbb{N}_0} \mathcal{F}_k(x,b,y) * \delta_t \\ &= \gamma^{-1} \int_{\Sigma_A^\infty\times\mathbb{R}\times\Sigma_A^\infty} \chi_{[\omega]}(x)\chi_{I}(b)\chi_{\Sigma_A^\infty}(y)\, d(\nu_\phi \times \lambda^1 \times \nu_\phi)(x,b,y) \\ &= \gamma^{-1}\lambda^1(I)\nu_\phi([\omega]). \end{aligned}$$

■

**Corollary 2.3.9.** *Let $(\Sigma_A^\infty, \sigma)$ denote a one-sided topologically exact subshift of finite type and let $\nu_\phi$ denote the unique equilibrium measure on $\Sigma_A^\infty$ for a non-arithmetic Hölder continuous potential $\phi \in C(\Sigma_A^\infty; \mathbb{R})$. Further, assume that $P(\phi, \sigma)$ is equal to zero. Then, for each $\mathcal{V} \in \Sigma_A^* \cup \emptyset$, as $r$ tends to positive infinity, we have that*

$$\Xi_{\mathcal{V}}(e^{-r}) \sim \frac{\nu_\phi([\mathcal{V}])r}{h_{\nu_\phi}(\sigma)}.$$

Proof. Since $P(\sigma, \phi)$ is defined to equal zero and since the measure theoretical entropy of a Gibbs measure is strictly positive, we have that $-\int_{\Sigma_A^\infty} \phi \, d\nu_\phi > 0$. Next, since $\nu_\phi$ is a Gibbs measure there exists $c > 1$ such that for each $r > 0$, we have that

$$\Xi_{\mathcal{V}}(e^{-r}) \;=\; \int_{[\mathcal{V}]} \sum_{\omega \in \Sigma_A^*} \chi_{[0,r]} \left(-\ln(\nu_\phi[\omega])\right) \cdot \chi_{[\omega]}(x)\, d\nu_\phi(x) \;\leqslant\; \int_{[\mathcal{V}]} \sum_{k \in \mathbb{N}_0} \chi_{[0,r+\ln(c)]} \left(-S_k\phi(x)\right)\right)\, d\nu_\phi(x)$$

and that

$$\Xi_{\mathcal{V}}(e^{-r}) \;=\; \int_{[\mathcal{V}]} \sum_{\omega \in \Sigma_A^*} \chi_{[0,r]} \left(-\ln(\nu_\phi[\omega])\right) \cdot \chi_{[\omega]}(x)\, d\nu_\phi(x) \;\geqslant\; \int_{[\mathcal{V}]} \sum_{k \in \mathbb{N}_0} \chi_{[0,r-\ln(c)]} \left(-S_k\phi(x)\right)\right)\, d\nu_\phi(x).$$

Then, by Theorem 2.3.8, given $\epsilon > 0$ there exists $R \in \mathbb{N}$ such that, for all $r \gg \lceil R + \ln(c) \rceil$, we have that

$$\begin{aligned}
&\Xi_{\mathcal{V}}(e^{-r}) \\
&\leqslant \int_{[\mathcal{V}]} \sum_{k \in \mathbb{N}_0} \chi_{[0,\lceil R+\ln(c)\rceil)} \left(-S_k\phi(x)\right) d\nu_\phi(x) + \sum_{m=\lceil R+\ln(c)\rceil}^{\lceil r+\ln(c)\rceil -1} \int_{[\mathcal{V}]} \sum_{k \in \mathbb{N}_0} \chi_{[m,m+1]} \left(-S_k\phi(x)\right)\right) d\nu_\phi(x) \\
&\leqslant \int_{[\mathcal{V}]} \sum_{k \in \mathbb{N}_0} \chi_{[0,\lceil R+\ln(c)\rceil)} \left(-S_k\phi(x)\right) d\nu_\phi(x) \;+\; \frac{(r-R)(1+\epsilon)\nu_\phi([\mathcal{V}])}{-\int_{\Sigma_A^\infty} \phi\, d\nu_\phi}
\end{aligned}$$

and

$$\begin{aligned}
&\Xi(e^{-r}) \\
&\geqslant \int_{[\mathcal{V}]} \sum_{k \in \mathbb{N}_0} \chi_{[0,\lceil R-\ln(c)\rceil)} \left(-S_k\phi(x)\right) d\nu_\phi(x) + \sum_{m=\lceil R-\ln(c)\rceil}^{\lfloor r-\ln(c)\rfloor -1} \int_{[\mathcal{V}]} \sum_{k \in \mathbb{N}_0} \chi_{[m,m+1]} \left(-S_k\phi(x)\right)\right) d\nu_\phi(x) \\
&\geqslant \int_{[\mathcal{V}]} \sum_{k \in \mathbb{N}_0} \chi_{[0,\lceil R-\ln(c)\rceil)} \left(-S_k\phi(x)\right) d\nu_\phi(x) \;+\; \frac{(r-R-2)(1-\epsilon)\nu_\phi([\mathcal{V}])}{-\int_{\Sigma_A^\infty} \phi\, d\nu_\phi}.
\end{aligned}$$

Since $\nu_\phi$ is an equilibrium measure for the potential, by the charicterisation of the pressure function given in Equation (2.22) $\phi$, the result follows. ■

**Corollary 2.3.10.** *Let $(\Sigma_A^\infty, \sigma)$ denote a one-sided topologically exact subshift of finite type and let $\nu_\phi$ denote the unique equilibrium measure for a given non-arithmetic Hölder continuous potential $\phi \in C(\Sigma_A; \mathbb{R})$. If*

$$\int_{\Sigma_A^\infty} \phi\, d\mu_\phi \;\neq\; 0,$$

*then, for each $\mathcal{V} \in \Sigma_A^* \cup \emptyset$, we have that*

$$\lim_{\epsilon \to 0} \frac{\Xi_{\mathcal{V}}(\epsilon)}{\ln(\Upsilon_{\mathcal{V}}(\epsilon))} \;=\; \frac{\mu_\phi([\mathcal{V}])}{h_{\nu_\phi}(\sigma)}.$$

Proof. This result is an immediate consequence of Theorem 2.2.8, Corollary 2.3.7 and Corollary 2.3.9. ■

# Chapter 3: $C^*$-Algebras and Noncommutative Geometry

In the present work, our aim is to add to the ongoing attempts to define noncommutative representations of fractal sets as introduced by Connes in [Con3]. To this end, we introduce Connes' theory of noncommutative geometry. The fundamental idea behind noncommutative geometry is that of viewing geometric structures in terms of operator algebras. The approach of representing such structures by algebraic objects has its origins in the work of Gelfand and Naĭmark on locally compact Hausdorff spaces and $C^*$-algebras (see [GN]). In [Con2, Con3, Con1] Connes showed that it is possible to generalise classical Riemannian geometry in terms of operator algebras, specifically $C^*$-algebras. This generalisation takes the form of a spectral triple.

The aim of this chapter is to give some of the basic ideas of noncommutative geometry, which we will use and extend in Chapter 4. The work in this chapter is organised as follows. In Section 3.1 we introduce $C^*$-algebras and $C^*$-dynamical systems. In Section 3.2 we define the notion of a spectral triple and include a description of some of the geometric properties of this object. To conclude, in Section 3.3 we include three examples of spectral triples describing, in detail, aspects of their noncommutative geometries.

## 3.1 $C^*$-algebras

### 3.1.1 $C^*$-Algebras and The Gelfand-Naĭmark Theorems

In this subsection we begin by giving the definition of a $C^*$-algebra (Definition 3.1.2). We discuss the Gelfand-Naĭmark-Segal completion of a $C^*$-algebra and state the seminal classification theorems of $C^*$-algebras by Gelfand and Naĭmark (Theorems 3.1.14 and 3.1.11). To conclude this subsection, we define two classes of $C^*$-algebras, which we use within of this thesis. Let us begin with the following definition.

**Definition 3.1.1.** A *complex algebra* is a vector space $V$ over $\mathbb{C}$ with an associative operation $\cdot$ (multiplication), which satisfies, for all $v_1, v_2, v_3 \in V$ and $z \in \mathbb{C}$, the following three conditions.

1. $v_1 \cdot (v_2 + v_3) = v_1 \cdot v_2 + v_1 \cdot v_3$.
2. $(v_1 + v_2) \cdot v_3 = v_1 \cdot v_3 + v_2 \cdot v_3$.
3. $z(v_1 \cdot v_2) = (zv_1) \cdot v_2 = v_1 \cdot (zv_2)$.

An *involution* on $V$ is an operation $^*$ such that, for all $v_1, v_2 \in V$ and all $z_1, z_2 \in \mathbb{C}$, the following hold.

1. $(v_1^*)^* = v_1$.
2. $(v_1 \cdot v_2)^* = v_2^* \cdot v_1^*$.
3. $(z_1 v_1 + z_2 v_2)^* = \overline{z_1} v_1^* + \overline{z_2} v_2^*$.

A complex algebra equipped with an involution is called a *complex $*$-algebra*.

**Definition 3.1.2.** A *$C^*$-algebra* $A$ is a complex $*$-algebra equipped with a norm $\|\cdot\|$ such that the following hold.

1. $A$ is complete with respect to the norm $\|\cdot\|$.
2. $\|a_1 \cdot a_2\| \leqslant \|a_1\| \|a_2\|$, for all $a_1, a_2 \in A$.

3. $\|a^* \cdot a\| = \|a\|^2$, for all $a \in A$.

A norm satisfying these three conditions is referred to as a $C^*$*-norm*.

*Remark. Any finite dimensional $C^*$-algebra is isometrically isomorphic to the product of a finite collection of closed subalgebras of matrix algebras.*

**Example 3.1.3.** The primary example of a commutative $C^*$-algebra is the set $C_c(X)$ of continuous complex-valued functions with compact support on a locally compact Hausdorff space $X$. Here, the associative operation $\cdot$ is pointwise multiplication, the involution is pointwise complex conjugation and the $C^*$-norm is the supremum norm.

**Example 3.1.4.** An example of a noncommutative $C^*$-algebra is the set $B(H)$ of bounded operators on a complex separable Hilbert space $H$. In this case, the associative operation $\cdot$ is composition, the involution is given by taking adjoints and the $C^*$-norm is the operator norm. In fact, any subalgebra of $B(H)$ closed under involution and closed under the strong operator norm topology is also a $C^*$-algebra.

*Remark. It is not necessary for a $C^*$-algebra to have a unit. However, throughout this thesis we will assume that a $C^*$-algebra is unital and that the norm of the unit is equal to* 1*. We will denote the unit of a $C^*$-algebra by the symbol $\mathbb{I}$.*

Let us now describe the celebrated Gelfand-Naĭmark-Segal completion of a $C^*$-algebra. This construction allows one to obtain a complex Hilbert space $H$ from a $C^*$-algebra $A$, such that there exists an injective map from $A$ to $B(H)$. This naturally leads to the classification theorems of Gelfand and Naĭmark. In order to describe the Gelfand-Naĭmark-Segal completion of a $C^*$-algebra and to state the classification theorems we require the following definitions.

**Definition 3.1.5.** A $*$-*homomorphism* is a homomorphism between two complex $*$-algebras, which preserves involution. A $*$-homomorphism is said to be *faithful* if for all $a \in A$ one has that $\phi(a^*a) = 0$ if and only if $a = 0$. Further, define a $*$-*automorphism* to be an isomorphic $*$-homomorphism from a $C^*$-algebra onto itself.

It is important to note that a $*$-homomorphism between two $C^*$-algebras is *non-expansive*, that is, bounded with operator norm less than or equal to 1. If, in addition, the $*$-homomorphism is faithful, then it also preserves the $C^*$-norm (see Appendix A.4).

**Definition 3.1.6.** A $*$-*representation* of a complex $*$-algebra $V$ is a tuple $(\pi, H)$ consisting of a complex Hilbert space $H$ and a linear $*$-homomorphism $\pi : V \to B(H)$. A $*$-representation $(\pi, H)$ of a complex $*$-algebra is said to be *faithful* if $\pi$ is faithful. Further, a $*$-representation $(\pi, H)$ of a complex $*$-algebra $V$ is *irreducible* if and only if the only sets which are invariant under the action of $\pi(V)$ are $H$ and the trivial subspace.

**Definition 3.1.7.** An element $a$ of a $C^*$-algebra $A$ is said to be *non-negative*, denoted by $a \geqslant 0$, if and only if there exists $b \in A$ such that $a = b^*b$. If, in addition, $a \neq 0$, then we say that $a$ is *positive* and write $a > 0$. Further, a linear functional $\psi : A \to \mathbb{C}$ is said to be *non-negative* if for each non-negative $a \in A$, we have that $\psi(a) \geqslant 0$. Similarly, a functional $\psi : A \to \mathbb{C}$ is said to be *positive* if for each positive $a \in A$, we have that $\psi(a) > 0$.

**Definition 3.1.8.** A positive linear functional $\psi$ of norm one acting on a $C^*$-algebra $A$ is called a *state*. Further, we call a state *tracial* if $\psi(ab) = \psi(ba)$, for all $a, b \in A$ and we let $\mathcal{S}(A)$ denote the *state space* of $A$, that is, the set of all states on $A$.

For a $C^*$-algebra $A$, observe that the state space $\mathcal{S}(A)$ is a convex set. Recall that an *extremal point* of a convex set $S$ is a point which is not an internal point of any closed line segment contained in $S$.

**Definition 3.1.9.** The extremal points of $\mathcal{S}(A)$ are referred to as *pure states*.

Observe that if $(X, d)$ is a compact metric space, then the state space $\mathcal{S}(C(X;\mathbb{C}))$ of the $C^*$-algebra $C(X;\mathbb{C})$ of complex-valued continuous functions is equivalent to the space $M(X)$ of Borel probability measures on $X$. This follows from the Riesz Representation Theorem (see Theorem II.4 of [RS]).

**Definition 3.1.10.** Let $A$ denote a $C^*$-algebra and let $(\pi, H)$ denote a $*$-representation. Then a vector $h \in H$ is said to be *cyclic* for $A$ if the set $\{\pi(a)(h) : a \in A\}$ is norm-dense in $H$, with respect to the Hilbert space norm. Further, a vector $h \in H$ is said to be *separating* for $A$ if and only if whenever $a, b \in A$ and $\pi(a)(h) = \pi(b)(h)$ then $a = b$.

We are now in a position to define the Gelfand-Naĭmark-Segal completion of a $C^*$-algebra. Let $A$ denote a $C^*$-algebra and let $\psi \in \mathcal{S}(A)$. Observe that $\langle a, b\rangle_\psi := \psi(b^*a)$ defines a positive sesquilinear form on $A$ which satisfies the Cauchy-Schwarz inequality in the form $|\psi(b^*a)|^2 \leqslant \psi(a^*a)\psi(b^*b)$. Therefore,

$$N_\psi \;:=\; \{b \in A \,:\, \psi(b^*b) = 0\} \;=\; \{b \in A \,:\, \psi(a^*b) = 0 \,\text{for all}\, a \in A\}$$

is a closed ideal in $A$. Note that $N_\psi = \{0\}$ if and only if $\psi$ is a faithful state. The quotient space $A/N_\psi$ is then a pre-Hilbert space under the positive sesquilinear form $\langle a + N_\psi, b + N_\psi\rangle_\psi := \psi(b^*a)$. The norm completion of $A/N_\psi$ under this form is called the *Gelfand-Naĭmark-Segal completion* of $A$ with respect to $\psi$ and is denoted by $H_\psi$. Observe that $A$ can be embedded into $B(H_\psi)$. More precisely, let $B(A/N_\psi)$ denote the set of bounded linear operators on the pre-Hilbert space $A/N_\psi$ and define $\pi_\psi : A \to B(H_\psi)$ to be the continuous linear extension of the map $\pi : A \to B(A/N_\psi)$ given by $\pi(a)(b + N_\psi) := ab + N_\psi$. It can then be shown that the following hold.

1. $(\pi_\psi, H_\psi)$ is a $*$-representation of $A$.

2. $\mathbb{I} + N_\psi$ is a cyclic vector for the representation $\pi_\psi$, where $\mathbb{I}$ denote the unit of the $A$.

3. If $\psi$ is faithful, then $\mathbb{I} + N_\psi$ is a separating vector.

4. The representation $(\pi_\psi, H_\psi)$ is irreducible if and only if the state $\psi$ is a pure state.

These results are proved in a number of texts, see for instance [Dav, BR, FGBV, Dix2]. This naturally leads us to the Gelfand-Naĭmark classification theorems of $C^*$-algebras.

**Theorem 3.1.11.** *(The Gelfand-Naĭmark Classification Theorem) To each $C^*$-algebra $A$ there exists a complex Hilbert space $H$ and a faithful $*$-homomorphism which maps $A$ onto a closed sub-$*$-algebra of $B(H)$.*

Proof. See Theorem 1.17 [FGBV] or [GN] for the original proof. ■

Observe that for an arbitrary $C^*$-algebra, the Hilbert space given by Theorem 3.1.11 need not be separable. This leads to the following definition.

**Definition 3.1.12.** A $C^*$-algebra $A$ is said to be *separable* if and only if there exists a $*$-representation $(\pi, H)$ where $H$ is a separable complex Hilbert space.

**Example 3.1.13.** Examples of a commutative separable $C^*$-algebra include the set of continuous functions on a compact metric space equipped with a finite Borel measure. An example of a non-separable commutative $C^*$-algebra is the algebra of bounded complex valued functions on $\mathbb{R}$ equipped with the supremum norm.

**Theorem 3.1.14.** *(Gelfand-Naĭmark classification theorem for commutative $C^*$-algebras) Given a commutative $C^*$-algebra $A$, there exists a compact Hausdorff space $X$ (unique up to homeomorphism) such that there exists a bijective $*$-homomorphism from $A$ onto $C(X;\mathbb{C})$.*

Proof. See Lemma 1 of [GN]. ■

*Remark. Similar statements to those given in Theorem 3.1.14 and Theorem 3.1.11 also exist for non-unital $C^*$-algebras (see [GN] or [FGBV]).*

To conclude this subsection we define two classes of $C^*$-algebras, which will be used within this thesis.

**Definition 3.1.15.** Let $\vartheta^1$ denote the one-dimensional spherical measure with $\vartheta^1(\mathbb{S}^1) = 1$. Let $\theta \in (0,1)$ be an irrational number and let $\mathcal{R}_\theta$ denote the sub-$*$-algebra of $B(L^2(\mathbb{S}^1, \mathcal{B}, \vartheta^1))$ generated by the operators $U$ and $V$ given, for each $f \in L^2(\mathbb{S}^1, \mathcal{B}, \vartheta^1)$ and $z \in \mathbb{S}^1$, by

$$U(f)(z) \;:=\; z \cdot f(z), \qquad V(f)(z) \;:=\; f(z \cdot e^{-2\pi i\theta}). \tag{3.1}$$

The *irrational rotation algebra* $A_\theta$ is then defined to be the completion of $\mathcal{R}_\theta$ with respect to the *universal norm*, that is, the norm given by

$$\|a\|_{\mathrm{u}} \;:=\; \sup\left\{\|\pi(a)\| \,:\, (\pi, H) \,\text{is a $*$-representation of}\, R_\theta\right\}.$$

*Remark. It is well-known that the irrational rotation algebra $A_\theta$ ($\theta \in (0,1) \cap \mathbb{R} \setminus \mathbb{Q}$) is* simple, *that is, it contains no proper ideals (see Theorem $VI$.1.4 of [Dav]). Moreover, the irrational rotation algebra is universal, that is, any $C^*$-algebra which is minimally generated by two distinct elements which satisfy the following,*

$$UU^* \;=\; U^*U \;=\; VV^* \;=\; V^*V \;=\; 1, \qquad VU \;=\; e^{-2\pi i\theta}UV. \tag{3.2}$$

*is necessarily isometric $*$-homomorphic to $A_\theta$ (see Theorem $VI$.1.4 of [Dav]).*

**Definition 3.1.16.** If a $C^*$-algebra $A$ is the normed closure of an increasing sequence of finite dimensional $C^*$-algebras, then $A$ is called an $AF$ (*approximately finite*) *$C^*$-algebra.*

**Example 3.1.17.** The algebra of continuous complex-valued functions on a homeomorphic image of the middle third Cantor set is an AF $C^*$-algebra. An example of a noncommutative AF $C^*$-algebra is a *uniformly hyperfinite $C^*$-algebra*, that is, the operator norm closure of an increasing sequence of full matrix algebras.

### 3.1.2 $C^*$-Dynamical Systems and The Discrete Cross Product Algebra

The main aim of this subsection is to define the notion of a noncommutative dynamical system, or, more precisely, a $C^*$-dynamical system, and to show how to obtain from such a system the class of $C^*$-algebras called discrete cross product algebras. Note that such a class of $C^*$-algebras allows us to demonstrate, by way of example, how Connes' theory of noncommutative geometry can be applied to noncommutative $C^*$-algebras, which we consider in Subsection 3.3.2 and Subsection 3.3.3. For a general reference on $C^*$-dynamical systems and the discrete cross product algebra we refer the reader to [BO, Dav].

**Definition 3.1.18.** A *$C^*$-dynamical system* is a triple $(A, \alpha, G)$, consisting of a separable $C^*$-algebra $A$, a countable discrete group $G$ and a homomorphism $\alpha$ from $G$ into the group $\mathrm{Aut}(A)$ consisting of all $*$-automorphisms on $A$.

Given a $C^*$-dynamical system $(A, \alpha, G)$, our goal is to construct a single $C^*$-algebra which encodes the $C^*$-algebra $A$ and the group action of $G$ on $A$. In group theory, the analogue of this procedure is called the semi-direct product. We will adapt this idea to create a $C^*$-algebra $A \rtimes_\alpha G$, called the discrete cross product algebra. This $C^*$-algebra is constructed using the *group-algebra* $AG$, which is defined by

$$AG \ := \ \left\{ \sum_{g \in G} a_g \delta_g \ : \ a_g \in A \text{ for all } g \in G \text{ and } a_g = 0 \text{ for all but a finite number of } g \in G \right\}.$$

(Recall that $\delta_g : G \to \{0,1\}$ denotes the Dirac point mass at $g \in G$.) In other words, $AG$ is the space of continuous $A$-valued functions on $G$ with compact support. A multiplication on $AG$ is given by a *twisted convolution product* $*$ on $AG$, which, for all $a_1 := \sum_{g \in G} a_{1,g} \delta_g, a_2 := \sum_{g \in G} a_{2,g} \delta_g \in AG$, is defined by

$$a_1 * a_2 \ := \ \sum_{g_1 \in G} \left( \sum_{g_2 \in G} a_{1,g_2} \cdot \alpha(g_2) \left( a_{2,g_2^{-1} g_1} \right) \right) \delta_{g_1}. \tag{3.3}$$

Define an involution $^*$ on $AG$ given, for each $a := \sum_{g \in G} a_g \delta_g \in AG$, by

$$a^* \ := \ \sum_{g \in G} \alpha(g)(a_{g^{-1}}{}^*) \delta_g. \tag{3.4}$$

The group-algebra $AG$ with multiplication given by the convolution product defined in Equation (3.3) and the involution as given in Equation (3.4), is a complex $*$-algebra. This algebra encodes both the $C^*$-algebra $A$ and the group action of $G$ on $A$. In order to form a $C^*$-algebra from the complex $*$-algebra $AG$ it must be completed with respect to a $C^*$-norm. There are two commonly used methods to complete the complex $*$-algebra $AG$, namely, the universal completion and the reduced completion. To define these we introduce the notion of a regular covariant representation of a $C^*$-dynamical system and to guarantee that these completions are well defined we require Proposition 3.1.20 and Proposition 3.1.22.

**Definition 3.1.19.** A *unitary representation* of a group $G$ is a pair $(U, H)$ consisting of a complex Hilbert space $H$ and a homomorphism $U : G \to B(H)$ such that $U(g)$ is a unitary operator and $U(g)^* = U(g^{-1})$, for all $g \in G$. A *covariant representation* of a $C^*$-dynamical system $(A, \alpha, G)$ is a triple $(\pi, H, U)$ consisting of a $*$-representation $(\pi, H)$ of $A$ and a unitary representation $(U, H)$ of $G$ such that $U(g)\pi(a)U(g)^* = \pi(\alpha(g)(a))$, for all $a \in A$ and $g \in G$.

The following proposition is well-known, but since the proof is important and rather constructive, we include it here for completeness.

**Proposition 3.1.20.** *For any $C^*$-dynamical system there exists a covariant representation.*

Proof. Let $(A, \alpha, G)$ denote a $C^*$-dynamical system and let $(\pi, H)$ denote a faithful $*$-representation of $A$. Such a representation exists by the Gelfand-Naĭmark Classification Theorem

(Theorem 3.1.11). Consider the Hilbert space

$$l^2(G) \;:=\; \left\{\gamma : G \to \mathbb{C} \,:\, \sum_{g\in G} |\gamma(g)|^2 < \infty\right\},$$

with the inner product $\langle\cdot,\cdot\rangle : l^2(G)\times l^2(G) \to \mathbb{C}$ defined by $\langle\gamma_1,\gamma_2\rangle := \sum_{g\in G}\gamma_1(g)\overline{\gamma_2(g)}$, for all $\gamma_1,\gamma_2 \in l^2(G)$. Note that the set $\{\delta_g \,:\, g\in G\}$ forms an orthonormal basis for $l^2(G)$. To define a covariant representation of $A$ we define a $*$-representation $(\widetilde{\pi}, H\otimes l^2(G))$ of $A$ and a unitary representation $(U, H\otimes l^2(G))$ of $G$ as follows.

1. Define $\widetilde{\pi} : A \to B(H\otimes l^2(G))$ as follows. For all $a\in A$, $h\in H$ and $\gamma\in l^2(G)$ define

$$\widetilde{\pi}(a)(h\otimes\gamma) \;:=\; \sum_{g\in G}\pi(\alpha(g^{-1})(a))(h)\otimes\gamma(g)\delta_g. \tag{3.5}$$

   To define $\widetilde{\pi}(a)$ on $H\otimes l^2(G)$ we extend the definition given in Equation 3.5 using linearity. Since $\pi$ and $\alpha(g)$, for all $g\in G$, are linear $*$-homomorphisms, it follows that $\widetilde{\pi}$ is a linear $*$-homomorphism. Hence, $(\widetilde{\pi}, H\otimes l^2(G))$ is a $*$-representation of $A$.

2. Define $U : G \to B(H\otimes l^2(G))$ as follows. For all $g\in G$, $h\in H$ and $\gamma\in l^2(G)$ define

$$U(g)(h\otimes\gamma) \;:=\; \sum_{s\in G} h\otimes\gamma(s)\delta_{gs}. \tag{3.6}$$

   To define $U(g)$ on $H\otimes l^2(G)$ we extend the definition given in Equation 3.6 using linearity. Since for all $g_1,g_2\in G$, $h\in H$ and $\gamma\in l^2(G)$, we have that

$$U(g_1)U(g_2)(h\otimes\gamma) \;=\; U(g_1)\left(\sum_{s\in G} h\otimes\gamma(s)\delta_{g_2 s}\right) \;=\; \sum_{s\in G} h\otimes\gamma(s)\delta_{g_1 g_2 s} \;=\; U(g_1g_2)(h\otimes\gamma)$$

   and

$$U(g_1)^*(h\otimes\gamma) \;=\; \sum_{s\in G} h\otimes\gamma(s)\delta_{g_1^{-1}s},$$

   it follows that $(U, H\otimes l^2(G))$ is a unitary representation of $G$.

We conclude that $(\widetilde{\pi}, H\otimes l^2(G), U)$ is a covariant representation of $(A,\alpha,G)$, by observing that

$$\begin{aligned}
U(g)\widetilde{\pi}(a)U(g)^*(h\otimes\gamma) &= \sum_{s\in G} U(g)\widetilde{\pi}(a)(h\otimes\gamma(s)\delta_{g^{-1}s}) \\
&= \sum_{s\in G} U(g)\left(\pi(\alpha\left(s^{-1}g\right)(a))(h)\otimes\gamma(s)\delta_{g^{-1}s}\right) \\
&= \sum_{s\in G} \left(\pi(\alpha\left(s^{-1}g\right)(a))(h)\otimes\gamma(s)\delta_{s}\right) \\
&= \widetilde{\pi}(\alpha(g)a)(h\otimes\gamma).
\end{aligned}$$

■

**Definition 3.1.21.** The representation $(\widetilde{\pi}, H\otimes l^2(G), U)$ given in the above proof is called a *left regular covariant representation*.

**Proposition 3.1.22.** *Every covariant representation of a $C^*$-dynamical system $(A, \alpha, G)$ gives rise to a $*$-representation of the complex $*$-algebra $AG$ and conversely, every non-trivial $*$-representation of the complex $*$-algebra $AG$ arises in this way.*

PROOF. See page 117 of [BO]. ∎

The above two propositions ensure that the supremum in the following definition is not taken over the empty set.

**Definition 3.1.23.** The *full cross product $C^*$-algebra $A \rtimes_\alpha G$* is the completion of the complex $*$-algebra $AG$ with respect to the *universal norm* given, for each $a \in AG$, by

$$\|a\|_{\mathrm{u}} := \sup\{\|\pi(a)\| : (\pi, H) \text{ is a } *\text{-representation of } AG\}.$$

**Definition 3.1.24.** Let $(\widetilde{\pi}, H, U)$ be a left regular covariant representation for the $C^*$-dynamical system $(A, \alpha, G)$ such that the induced $*$-representation $(\pi, H)$ of the complex $*$-algebra $AG$ is faithful. Then the *reduced cross product algebra $A \rtimes_\alpha G$* is the norm closure of $AG$ under the reduced norm given, for each $a \in AG$, by $\|a\|_{\mathrm{red}} := \|\pi(a)\|$.

**Proposition 3.1.25.** *The reduced cross product algebra does not depend on the choice of the left regular covariant representation.*

PROOF. See Proposition 4.1.5 of [BO]. ∎

The following theorem concerning amenable groups will be useful in Example 3.1.28. For more information on amenable groups, the interested reader is referred to [Kes].

**Theorem 3.1.26.** *Let $(A, \alpha, G)$ be a $C^*$-dynamical system. If $G$ is an amenable group, then the reduced cross product algebra and the full cross product algebra are equivalent.*

PROOF. See Theorem VII.2.8 of [Dav]. ∎

A particular class of reduced cross product algebras, which we will make use of in Subection 3.3.3 and Subection 4.2.2, is given by the reduced discrete group algebra and is defined as follows.

**Definition 3.1.27.** Let $\mathbb{C}$ be the $C^*$-algebra of complex numbers, let $G$ be a countable discrete group and let $\alpha : G \to \mathbb{C}$ be the trivial action, that is, $\alpha(g) := 1$, for all $g \in G$. For each $a := \sum_{g \in G} z_g \delta_g \in \mathbb{C}G$ and each $\gamma \in l^2(G)$, define the convolution of $a$ with $\gamma$ by

$$a * \gamma := \sum_{g_1 \in G} \left( \sum_{g_2 \in G} z_{g_2} \cdot \gamma(g_2^{-1} g_1) \right) \delta_{g_1} \in l^2(G). \tag{3.7}$$

The *reduced discrete group algebra* is then defined to be the closure of the complex $*$-algebra $\mathbb{C}G$ with respect to the norm given, for each $a \in \mathbb{C}G$, by

$$\|a\|_{\mathrm{red}} := \sup\left\{\|a * \gamma\|_2 : \gamma \in l^2(G) \text{ with } \|\gamma\|_2 = 1\right\}.$$

We conclude this subsection by showing that the irrational rotation algebra can be expressed as a cross product algebra.

**Example 3.1.28.** Let $A$ denote the set of $2\pi$-periodic complex-valued functions on $\mathbb{R}$, fix an irrational number $\theta \in (0,1)$ and let $\alpha_\theta : \mathbb{Z} \to \mathrm{Aut}(A)$ be defined, for all $k \in \mathbb{Z}$, $a \in A$ and $x \in \mathbb{R}$, by

$$(\alpha_\theta(k))a(x) \ := \ a(x - 2\pi k\theta).$$

Observe that $(A, \alpha_\theta, \mathbb{Z})$ forms a $C^*$-dynamical system. Also, note the following.

1. The additive group $\mathbb{Z}$ is amenable.

2. The $*$-algebra $A\mathbb{Z}$ is generated by $\{U := \phi_1\delta_0, V := \chi_{\mathbb{R}}\delta_1\}$, where $\delta_0, \delta_1 : G \to \{0,1\}$ denotes the Dirac point masses on $G$ at 0 and 1, respectively, and $\phi_1(x) := (2\pi)^{-1/2}e^{2\pi i x}$, for all $x \in \mathbb{R}$.

3. The elements $U$ and $V$ satisfy the following relations

$$UU^* \ = \ U^*U \ = \ VV^* \ = \ V^*V \ = \ 1, \qquad VU \ = \ e^{-2\pi i\theta}UV.$$

Therefore, by the remark which immediately follows Defintion 3.1.15 and by Theorem 3.1.26, we have that the reduced and the universal norm completions of the complex $*$-algebra $A\mathbb{Z}$ coincide and that the resulting cross product algebra is isometric $*$-homomorphic to the irrational rotation algebra $A_\theta$.

## 3.2 The Geometric Side of Noncommutative Geometry

The main aims of this section are to give the definition of a spectral triple, which represents the noncommuative analogue of a compact metric space, to describe the geometric information one can obtain from this noncommutative object and to give some basic examples. The reader who is not familiar with the notions of functional analysis, which are in use, is referred to Section 3.1 and Appendix A.1.

**Definition 3.2.1.** A *spectral triple* is a triple $(A, H, D)$ consists of a $C^*$-algebra $A$ acting faithfully on a complex separable Hilbert space $H$ and an operator $D$ with the following properties. The operator $D$ is an essentially self-adjoint unbounded linear operator with a compact resolvent, such that the set

$$\{a \in A \,:\, \text{the operator } [D, \pi(a)] \text{ is densely defined and extends to a bounded operator on } H\}$$

is $C^*$-norm-dense in $A$. Here, $\pi$ represents the faithful action of $A$ on $H$ and $[D, \pi(a)]$ denotes the commutator of $D$ with $\pi(a)$. The operator $D$ is called a *Dirac operator*.

*Remark. The compact resolvent property of D in Definition 3.2.1 can be regarded as a generalisation of the ellipticity property of the standard Dirac operator defined on a compact smooth Riemannian manifold (see [Mar]). The condition that the closure of* $[D, \pi(a)]$ *is densely defined and extends to a bounded operator is analogous to a Lipschitz condition (see [BMR, Mar]).*

**Definition 3.2.2.** A $\mathbb{Z}_2$*-graded complex vector space* $V$ is a complex vector space which decomposes into a direct sum of two vector spaces $V_0$ and $V_1$, that is, $V = V_0 \oplus V_1$. If there exists an operator $\Gamma : V \to V$ such that $\Gamma^2 = \mathbb{1}$ and such that $\Gamma(V_k) = V_{k+1\,(mod\,2)}$, for $k \in \{0,1\}$, then we call $\Gamma$ a $\mathbb{Z}_2$*-grading operator*.

**Definition 3.2.3.** A spectral triple $(A, H, D)$ is called *even* if there exists a $\mathbb{Z}_2$-grading operator $\Gamma$ on $H$ such that $\Gamma$ commutes with each representative of $A$ in $B(H)$ and anti-commutes with the Dirac operator on its domain of definition.

*Remark. Observe that in this setting the grading operator corresponds to the chirality operator defined for a Riemannian spin manifold (see pages* 26 - 27 *of [Vár]).*

In what follows we describe some of the geometric aspects of a spectral triple $(A, H, D)$. Specifically, in Subsection 3.2.1 we introduce Connes' pseudo-metric on the state space $\mathcal{S}(A)$ of $A$ and in Subsection 3.2.2 we define the operator algebraic analogy of a measurable function, the metric dimension of a spectral triple and the noncommutative integral. We will see that in order to derive these concepts the Dirac operator, in particular its singular values, will play a crucial role.

### 3.2.1 Connes' Pseudo-Metric

Let us begin by recalling the definition of the weak$^*$-topology defined on the state space of a $C^*$-algebra.

**Definition 3.2.4.** Let $A$ denote a $C^*$-algebra. For each $a \in A$, let $\widehat{a} : \mathcal{S}(a) \to \mathbb{C}$ denote the *Gelfand transform* of $a$ given, for each $\psi \in \mathcal{S}(A)$, by

$$\widehat{a}(\psi) \ := \ \psi(a).$$

The *weak$^*$-topology* on the state space of $A$ is then defined to be the weakest topology on $\mathcal{S}(A)$ such that, for each $a \in A$, the Gelfand transform $\widehat{a}$ is continuous.

Observe that a compact metric space $X$ naturally embeds into the state space $\mathcal{S}(C(X;\mathbb{C}))$ of $C(X;\mathbb{C})$ and recall that $\mathcal{S}(C(X;\mathbb{C}))$ coincides with $M(X)$, the space of Borel probability measures on $X$. Therefore, by the Banach-Alaoglu Theorem (see Theorem IV.21 of [RS]) this space is weak$^*$-compact. Moreover, the *Monge-Kantorovitch metric* given, for all $\mu, \nu \in M(X)$, by

$$d_{MK}(\mu,\nu) \ := \ \sup\left\{ \int_X f\,d\mu - \int_X f\,d\nu \ : \ f \in C(X;\mathbb{C}) \text{ and is Lipschitz continuous with} \right.$$
$$\left. \text{Lipschitz constant less than or equal to one} \right\},$$

defines a metric on $M(X)$ whose topology coincides with the weak$^*$-topology (see Theorem 2.5.17 of [Edg]).

Within the theory of noncommutative geometry, given a spectral triple $(A, H, D)$, the analogue of a Lipschitz function is an element $a \in A$ such that the commutator $[D, \pi(a)]$ is densely defined and extends to a bounded operator. Further, the analogue of the Monge-Kantorovitch metric is the pseudo-metric on the space $\mathcal{S}(A)$ known as Connes' pseudo-metric, which is defined as follows.

**Definition 3.2.5.** Let $(A, H, D)$ denote a spectral triple, where the $*$-representation is denoted by $(\pi, H)$. Let $\mathcal{A}$ denote a $C^*$-norm-dense complex sub-$*$-algebra of $A$, where for all $a \in \mathcal{A}$, the commutator $[D, \pi(a)]$ extends to a bounded linear operator on $H$. Then, for each such $*$-algebra $\mathcal{A}$, define the psudeo-metric $d_{\mathcal{A}} : \mathcal{S}(A) \times \mathcal{S}(A) \to \mathbb{R}$ by

$$d_{\mathcal{A}}(\psi_1,\psi_2) \ := \ \sup\{|\psi_1(a) - \psi_2(a)| \ : \ a \in \mathcal{A} \text{ and the operator } [D,\pi(a)] \text{ extends to a bounded}$$
$$\text{operator with norm less than or equal to one}\}.$$

We refer to this pseudo-metric as *Connes' pseudo-metric.*

The term *pseudo-metric* is used because it is not clear that $d(\psi_1, \psi_2)$ is finite for all pairs $\psi_1, \psi_2 \in \mathcal{S}(A)$. However, the other axioms of a metric are fulfilled. Therefore, the question which naturally arises is, when is $d_{\mathcal{A}}$ a metric and if $d_{\mathcal{A}}$ is a metric, then when does the topology induced by $d_{\mathcal{A}}$

agree with the weak$^*$-topology on $\mathcal{S}(A)$? The following theorem by Rieffel gives a characterisation which addresses this question.

**Theorem 3.2.6.** *Let $(A, H, D)$ denote a spectral triple, where the $*$-representation is denoted by $(\pi, H)$. Let $\mathcal{A}$ denote a $C^*$-norm-dense sub-$*$-algebra of $A$, where for all $a \in \mathcal{A}$, the commutator $[D, \pi(a)]$ extends to a bounded linear operator on $H$. Then the following hold.*

1. *The pseudo-metric $d_{\mathcal{A}}$ is a metric if and only if the set*

$$\{a \in \mathcal{A} : \text{the operator } [D, \pi(a)] \text{ is densely defined and extends to a bounded operator with norm less than or equal to one}\} \tag{3.8}$$

*has a bounded image in the quotient space $A/\{z\mathbb{I} : z \in \mathbb{C}\}$, where $\mathbb{I}$ denotes the identity element of $A$.*

2. *The topology induced by the pseudo-metric $d_{\mathcal{A}}$ coincides with the weak$^*$-topology if and only if the set given in Equation (3.8) has a totally bounded image in the quotient space $A/\{z\mathbb{I} : z \in \mathbb{C}\}$.*

PROOF. See Theorem 2.1 of [Rie3]. ■

*Remark. The essence of the above theorem is that the set given in Equation (3.8) is large enough that it separates the states of $A$ and, at the same time (by definition), small enough that it has a bounded image in the quotient space $A/\{z\mathbb{I} : z \in \mathbb{C}\}$.*

### 3.2.2 Infinitesimals, Measurability and Dimension

As we have seen in Chapter 2 the Hausdorff dimension of a subset $E$ of $\mathbb{R}^n$ is given by

$$\inf\{s > 0 : \mathcal{H}^s(E) = 0\}$$

(see Theorem 2.1.2). Note that such a relationship also exists for other fractal measures and fractal dimensions, for instance the Patterson measure and the Poincaré exponent of convergence (see [Nic, Pat]) and the packing measure and the packing dimension (see [Fal1]). Likewise, in the noncommutative setting one has an analogous relationship. Therefore, in this subsection, we first introduce the expectation of a compact operator which arises within the theory of operator algebras and then we present the definition of the metric dimension of a spectral triple. Having developed these notions, we are then able to define the noncommutative integral which arises from a spectral triple.

Within the theory of operator algebras, for a complex separable Hilbert space $H$, the notion of an expectation of an operator $T \in K(H)$ is given by the coefficient of logarithmic divergence of the eigenvalues of $T$. In particular, the ideal $K(H)$ of $B(H)$ provides the "infinitesimal" of noncommutative geometry. Heuristically, in the commutative setting, an infinitesimal is an "object" smaller than any feasible measurement and not zero in "size", but so small that it cannot be distinguished from zero by any available means. As a matter of interest, we remark that the founders of calculus, Euler, Leibniz and Newton initially formulated the theory of calculus using infinitesimals. However, the notion and definition was foreshadowed in Archimedes' script *The Method of Mechanical Theorems*.

Returning to the noncommutative setting, we define an infinitesimal operator as follows.

**Definition 3.2.7.** Let $H$ denote a complex separable Hilbert space and let $T \in K(H)$. For each $k \in \mathbb{N}$, let $\sigma_k(T)$ denote the $k$-th largest *singular value* (including multiplicities) of $T$, that is, the $k$-th largest eigenvalue (including multiplicities) of $|T| := (TT^*)^{1/2}$. We say that $T$ is an *infinitesimal of order* $s > 0$, if $\sigma_k(T) \asymp k^{-s}$ as $k$ tends to infinity.

Early attempts to define an expectation within the theory of operator algebras (see [Seg]) used ordinary traces of Hilbert space operators, where trace-class operators play the role of integrable functions. However, it soon became apparent that this is not sufficient. In 1966 Dixmier [Dix1] found other tracial states that are more suitable. He noted that to appropriately define an expectation within the theory of operator algebras, one must suppress infinitesimals of order higher than one. More precisely, one wants to find the coefficient of the divergence rate of the singular values of an infinitesimal operator of order one. In order to obtain this coefficient, we require the following definitions.

**Definition 3.2.8.** A *limiting procedure* is a positive linear functional $\mathcal{W}$ defined on the set

$$l^\infty(\mathbb{R}) \;:=\; \{(x_1, x_2, \dots) : x_k \in \mathbb{R} \text{ for all } k \in \mathbb{N} \text{ and } \sup\{|x_k| : k \in \mathbb{N}\} < \infty\},$$

where for each $(x_1, x_2, \dots) \in l^\infty(\mathbb{R})$, one has that

$$\liminf_{k\to\infty} x_k \;\leqslant\; \mathcal{W}(x_1, x_2, \dots) \;\leqslant\; \limsup_{k\to\infty} x_k$$

Following convention, for $(x_1, x_2, \dots) \in l^\infty(\mathbb{R})$ and for a limiting procedure $\mathcal{W}$, we set

$$\mathrm{Lim}_{\mathcal{W}}(x_1, x_2, \dots) \;:=\; \mathcal{W}(x_1, x_2, \dots).$$

*Remark. The existence of a liniting procedure follows immediately from the Hahn-Banach Theorem (see Chapter* 3 *of [Bol]).*

**Example 3.2.9.** A generalised limit is an example of a limiting procedure (see [Bol] page 59).

**Definition 3.2.10.** The *Dixmier ideal* of a separable Hilbert space $H$ is denoted by $\mathcal{L}^{1,+}(H)$ and is defined by

$$\mathcal{L}^{1,+}(H) \;:=\; \left\{ T \in K(H) : \limsup_{N\to\infty} \frac{\sum_{k=1}^N \sigma_k(T)}{\ln(N)} \;<\; \infty \right\}.$$

For a limiting procedure $\mathcal{W}$, we define a *Dixmier trace* of a positive linear operator $T \in \mathcal{L}^{1,+}(H)$ by

$$\mathrm{Tr}_{\mathcal{W}}(T) \;:=\; \mathrm{Lim}_{\mathcal{W}} \left( \frac{\sum_{k=1}^N \sigma_k(T)}{\ln(N)} \right)_{N\in\mathbb{N}}. \tag{3.9}$$

For a general operator in $\mathcal{L}^{1,+}(H)$ the Dixmier trace is defined to be the natural complex linear extension of $\mathrm{Tr}_{\mathcal{W}}$.

**Definition 3.2.11.** Let $H$ denote a complex Hilbert space and let $I$ denote an ideal of $B(H)$. Then, a *singular trace on* $I$ is a linear functional $\mathcal{T}$ of norm one with domain $I$ such that the following hold.

1. $\mathcal{T}$ vanishes on operators with finite dimensional range.
2. If $T_1, T_2 \in I$ are such that $\lim_{k\to\infty} \sigma_k(T_1)/\sigma_k(T_2) = 1$, then $\mathcal{T}(T_1) = \mathcal{T}(T_2)$.

3. If $T_1, T_2 \in I$ have the property that $\sigma_k(T_1) \leqslant \sigma_k(T_2)$ for all but a finite number of $k \in \mathbb{N}$, then $\mathcal{T}(T_1) \leqslant \mathcal{T}(T_2)$.
4. For $T_1, T_2 \in I$, we have that $\mathcal{T}(T_1 T_2) = \mathcal{T}(T_2 T_1)$.

**Theorem 3.2.12.** *Let $H$ denote a complex separable Hilbert space and let $\mathcal{W}$ denote a limiting procedure. Then the Dixmier ideal $\mathcal{L}^{1,+}(H)$ is an ideal of $B(H)$ and the functional $Tr_{\mathcal{W}}$ is a singular trace.*

PROOF. See Appendix A.3, where we given an independent complete proof. ∎

In the following definition we define the notions of a measurable operator and the noncommutative analogue of an expectation.

**Definition 3.2.13.** If $T \in \mathcal{L}^{1,+}(H)$ and if $\mathrm{Tr}_{\mathcal{W}}(T)$ is independent of the limiting procedure $\mathcal{W}$, meaning that the limit

$$\lim_{N\to\infty} \frac{\sum_{k=1}^{N} \sigma_k(T)}{\ln(N)}$$

exists, then we call $T$ *measurable*. The *noncommutative expectation* of a measurable operator $T \in \mathcal{L}^{1,+}(H)$ is denoted by $⨍\, T$ and given by

$$⨍ T \ := \ \lim_{N\to\infty} \frac{\sum_{k=1}^{N} \sigma_k(T)}{\ln(N)}.$$

*Remark. If $c_1, c_2 \in \mathbb{C}$ and if $T_1, T_2$ are measurable operators defined on some complex separable Hilbert space, then one has the following*

$$⨍ (c_1 T_1 + c_2 T_2) \ = \ c_1 ⨍ T_1 + c_2 ⨍ T_2.$$

*This follows since the Dixmier trace is a linear functional (see Theorem 3.2.12). Moreover, the subset of measurable operators in $\mathcal{L}^{1,+}(H)$ is a closed subset of $\mathcal{L}^{1,+}(H)$ (see Proposition* 7.15 *of [FGBV]).*

Having introduced the Dixmier ideal of a complex separable Hilbert space, we now discuss the metric dimension of a spectral triple. The metric dimension of a spectral triple $(A, H, \pi, D)$ is given by the non-negative positive integer $\delta$ to which the singular values of $|1 + D^2|^{-\delta/2}$ form a logarithmically divergent series. Loosely speaking, this value is given by the exponent to which the operator $(\mathbb{1} + D^2)^{-1/2}$ is an infinitesimal of order 1. However, such a number does not necessarily have to exist. Therefore, we introduce the following summability conditions on a spectral triple.

**Definition 3.2.14.** Let $(A, H, D)$ be a spectral triple.

1. If for some $p > 0$
$$\mathrm{tr}((\mathbb{1} + D^2)^{-p/2}) \ < \ \infty, \tag{3.10}$$
then $(A, H, D)$ is called a *finitely summable spectral triple*.
2. For $p > 1$, if
$$\limsup_{N\to\infty} \frac{\sum_{k=1}^{N} \sigma_k((\mathbb{1} + D^2)^{-1/2})}{N^{1-1/p}} \ < \ \infty, \tag{3.11}$$
then $(A, H, D)$ is called a *$(p,+)$-summable spectral triple*. If $p = 1$, then we say that $(A, H, D)$ is $(1,+)$-summable if and only if $|D|^{-1} \in \mathcal{L}^{1,+}(\ker(D)^{\perp})$.

3. If for all $t > 0$
$$\operatorname{tr}\left(e^{-tD^2}\right) \;<\; \infty, \tag{3.12}$$
then $(A, H, D)$ is called a *$\theta$-summable spectral triple*.

*Remark. Let $(A, H, D)$ denote a spectral triple. If $D$ is an invertible operator, then $(A, H, D)$ is finitely summable if and only if for some $p > 0$ we have that tr$(|D|^{-p}) < \infty$. Similarly, $(A, H, D)$ is $(p, +)$-summable, for $p > 1$, if and only if we have that*
$$\limsup_{N\to\infty} \frac{\sum_{k=1}^{N} \sigma_k(|D|^{-p})}{\ln(N)} \;<\; \infty.$$

*Remark. In the above definition of $\theta$-summability we have stated the original definition given by Connes. This notion was introduced in order to deal with noncommutative representations of reduced group $C^*$-algebras of non-amenable groups. We refer the reader to Section 7 of [Con2] for a more in-depth discussion.*

Observe that the condition of a spectral triple being finitely summable can be written in terms of a $(1, +)$-summability condtion. Although it seems that this result is known to experts in the field of noncommutative geometry, we could not find it explicitly stated within the literature, and so we include a complete proof. Our proof will require the following definition and proposition.

**Definition 3.2.15.** Let $H$ denote a complex separable Hilbert space. An operator $T \in B(H)$ is said to be a *Hilbert-Schmidt operator* if and only if $T$ is a compact operator and $\operatorname{tr}(TT^*) < \infty$.

**Proposition 3.2.16.** *The class of Hilbert-Schmidt operators on a complex separable Hilbert space $H$ is an ideal of $B(H)$.*

Proof. See Theorem VI.22 of [RS]. ■

**Theorem 3.2.17.** *A spectral triple $(A, H, D)$ is finitely summable if and only if it there exists a $p > 0$ such that $|D|^{-p} \in \mathcal{L}^{1,+}(\ker(D)^{\perp})$. Moreover, a finitely summable spaectral is always $\theta$-summable.*

Proof. The first part follows directly from Lemma 3.2.18 given below. For the second part, we have the following equality
$$e^{-tD^2} \;=\; (\mathbb{1} + D^2)^{p/2} e^{-tD^2} (\mathbb{1} + D^2)^{-p/2}.$$
Since $(A, H, D)$ is finitely summable, there exists an $N > 0$ such that for all $p > N$, we have that $(\mathbb{1} + D^2)^{-p/2}$ is a trace-class operator, and so a Hilbert-Schmidt operator. Further, since the function $f : \mathbb{R} \to \mathbb{R}$ given, for each $x \in \mathbb{R}$, by
$$f(x) \;:=\; (1 + x^2)^{p/2} e^{-tx^2},$$
has supremum $(p/(2t))^{p/2}\, e^{t-p/2}$, we have that $(\mathbb{1} + D^2)^{p/2} e^{-tD^2}$ is a bounded operator. The result then follows by an application of Proposition 3.2.16. ■

**Lemma 3.2.18.** *Let $(x_k)_{k\in\mathbb{N}}$ denote an increasing unbounded sequence of positive real numbers and* *let*

$$\begin{aligned}
d_1 &:= \sup\left\{\alpha \geqslant 0 \,:\, \limsup_{N\to\infty} \frac{\sum_{k=1}^{N} {x_k}^{-\alpha}}{\ln(N)} = \infty\right\},\\
d_2 &:= \inf\left\{\alpha \geqslant 0 \,:\, \limsup_{N\to\infty} \frac{\sum_{k=1}^{N} {x_k}^{-\alpha}}{\ln(N)} = 0\right\},\\
d_3 &:= \inf\left\{\alpha \geqslant 0 \,:\, \sum_{k=1}^{\infty} {x_k}^{-\alpha} < \infty\right\} \quad = \quad \sup\left\{\alpha \geqslant 0 \,:\, \sum_{k=1}^{\infty} {x_k}^{-\alpha} = \infty\right\},\\
d_4 &:= \left(\liminf_{k\to\infty} \frac{\ln(x_k)}{\ln(k)}\right)^{-1}
\end{aligned}$$

*Then if any of the above are positive and finite, they are all equal.*

Proof. To see that $d_1 = d_2$, fix a $p > 0$ such that

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^{N} {x_k}^{-p}}{\ln(N)} \;>\; 0.$$

Then for $0 < q < p$ and for each $m \in \mathbb{N}$, we have that

$$\frac{\sum_{k=m}^{N} {x_k}^{-p}}{\ln(N)} \;=\; \frac{\sum_{k=m}^{N} {x_k}^{-(p-q)}{x_k}^{-q}}{\ln(N)} \;\leqslant\; {x_m}^{-(p-q)}\frac{\sum_{k=m}^{N} {x_k}^{-q}}{\ln(N)} \;\leqslant\; {x_m}^{-(p-q)}\frac{\sum_{k=1}^{N} {x_k}^{-q}}{\ln(N)}.$$

Therefore, for each $m \in \mathbb{N}$, we have that

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^{N} {x_k}^{-p}}{\ln(N)} \;=\; \limsup_{N\to\infty} \frac{\sum_{k=m}^{N} {x_k}^{-p}}{\ln(N)} \;\leqslant\; {x_m}^{-(p-q)} \limsup_{N\to\infty} \frac{\sum_{k=1}^{N} {x_k}^{-q}}{\ln(N)}.$$

Letting $m$ tend to infinity, then gives

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^{N} {x_k}^{-q}}{\ln(N)} \;=\; \infty.$$

Hence, we have that $d_1 = d_2$.

It is clear that $d_2 \leqslant d_3$, since, if $\sum_{k=1}^{\infty} {x_k}^{-p} < \infty$, then

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^{N} {x_k}^{-p}}{\ln(N)} \leqslant \limsup_{N\to\infty} \frac{\sum_{k=1}^{\infty} {x_k}^{-p}}{\ln(N)} \;=\; 0.$$

To see that $d_3 \leqslant d_4$, for each $k \in \mathbb{N}$ set

$$y_k := \frac{\ln(x_k)}{\ln(k)} \tag{3.13}$$

and assume that $d_4$ is positive and finite. Since, $\liminf_{k\to\infty} y_k = d_4^{-1}$, for each $a > d_4$ there exists a $c > 1$ such that for sufficiently large $m$, we have that $a \cdot y_m \geqslant c$. Therefore, there exists a constant $C > 0$ such that

$$\sum_{k=1}^{\infty} {x_k}^{-a} \;=\; \sum_{k=1}^{\infty} k^{-a\cdot y_k} \leqslant C + \sum_{k=1}^{\infty} k^{-c}.$$

Hence, if $a > d_4$, then $a \in \left\{s > 0 \,:\, \sum_{k=1}^{\infty} {x_k}^{-s} < \infty\right\}$, and so, we have that $d_3 \leqslant d_4$.

Finally, we need to show that $d_4 \leqslant d_1$. Note that if the sequence $(y_k)_{k\in\mathbb{N}}$ does not have an accumalation point then, give a positive $M \in \mathbb{R}$, there exists a $N \in \mathbb{N}$ such that for all $k \geqslant M$, we have that $y_k \geqslant M$. Therefore, for $p > 0$, it follows that

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^N x_k^p}{\ln(N)} \;\leqslant\; \limsup_{N\to\infty} \frac{\sum_{k=1}^N k^{-pM}}{\ln(N)}.$$

Hence, $d_1 \leqslant 1/M$ for all positive $M \in \mathbb{R}$, and so, assuming that $d_1$ is positive, we have that there exists an accumulation point $y > 0$ of the sequence $(y_k)_{k\in\mathbb{N}}$. Thus, there exists a strictly monotonically increasing sequence $(k_m)_{m\in\mathbb{N}}$ in $\mathbb{N}$ such that $y_{k_m}$ tends to $y$ as $k$ tends to infinity. Fix $a < y^{-1}$ and fix $\epsilon \in (0,1)$ such that there exists a $K \in \mathbb{N}$ with $a \cdot y_{k_m} < 1-\epsilon$, for all $m > K$. Then, for each $m > K$, we have that

$$\sum_{i=1}^{k_m} {x_i}^{-a} \;\geqslant\; k_m \cdot {x_{k_m}}^{-a} \;=\; k_m \cdot k_m^{-a\cdot y_{k_m}} \;\geqslant\; k_m^{\epsilon}.$$

Hence,

$$\frac{\sum_{i=1}^{k_m} {x_i}^{-a}}{\ln(k_m)}$$

tends to infinity as $m$ tends to infinity. Therefore, for all $a < d_4$, we have that

$$a \;\in\; \left\{ s \geqslant 0 \,:\, \limsup_{N\to\infty} \frac{\sum_{k=1}^N {x_k}^{-s}}{\ln(N)} = \infty \right\}.$$

This then concludes the proof. ■

*Remark. In general, if for a positive compact operator $T \in K(H)$ there exists $p > 0$ such that*

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^N \sigma_k(T)}{N^{1-1/p}} \;<\; \infty,$$

*then $T^p \in \mathcal{L}^{1,+}(H)$. However, the converse is not necessarilly true, see pages $316-317$ of [FGBV] for further details.*

In the following definition, we introduce the metric dimension of a finitely summable spectral triple. Note that the metric dimension is only defined for finitely summable spectral triples.

**Definition 3.2.19.** Let $(A,H,D)$ denote a finitely summable spectral triple. Then the *metric dimension* of $(A,H,D)$ is defined to be the non-negative real number

$$\delta \;=\; \delta(A,H,D) \;\;:=\;\; \inf\left\{p \geqslant 0 \,:\, \mathrm{tr}((\mathbb{1}+D^2)^{-p/2}) < \infty\right\} \tag{3.14}$$

$$=\;\; \sup\left\{p \geqslant 0 \,:\, \mathrm{tr}((\mathbb{1}+D^2)^{-p/2}) = \infty\right\} \tag{3.15}$$

$$=\;\; \sup\left\{p \geqslant 0 \,:\, \limsup_{N\to\infty} \frac{1}{\ln(N)} \sum_{k=1}^N \sigma_k((\mathbb{1}+D^2)^{-p/2}) = \infty\right\} \tag{3.16}$$

$$=\;\; \inf\left\{p \geqslant 0 \,:\, \limsup_{N\to\infty} \frac{1}{\ln(N)} \sum_{k=1}^N \sigma_k((\mathbb{1}+D^2)^{-p/2}) = 0\right\}. \tag{3.17}$$

If $(A,H,D)$ is a $\theta$-summable spectral triple which is not finitely summable, then we say that $(A,H,D)$ has infinite metric dimension.

*Remark.* *By the definition of the metric dimension, the operator $|D|^{-\delta}$ belongs to the Dixmier ideal $\mathcal{L}^{1,+}(ker(D)^\perp)$.*

*Remark.* *The dimension of a spectral triple can take the value zero. The two circumstances under which the dimension is equal to zero are the following.*

1. *The singular values of $(\mathbb{1}+D^2)^{-\delta/2}$ converge to zero exponentially fast, see [AIC2] for examples of this case.*

2. *The algebra and the Hilbert space are finite dimensional. Such spectral triples have been fully classified and the classification can be found in [PS, Kra, IKM].*

Note that for a finitely summable spectral triple $(A, H, D)$, the Dixmier trace of the operator $|D|^{-\delta}$ generalises the notion of a volume.

**Definition 3.2.20.** Let $(A, H, D)$ denote a finitely summable spectral triple with non-zero metric dimension $\delta$ and let $\mathcal{W}$ denote a limiting procedure. Then the *volume* of $(A, H, D)$ with respect to $\mathcal{W}$ is defined by

$$V_{\mathcal{W}} \ = \ V_{\mathcal{W}}(A, H, D) \ := \ \mathrm{Tr}_{\mathcal{W}}(|D|^{-\delta}).$$

Here the Dixmier trace is taken over the ideal $\mathcal{L}^{1,+}(\ker(D)^\perp)$ of the orthogonal complement of the kernel of $D$. If $|D|^{-\delta}$ is a measurable operator, then we denote the *volume* of $(A, H, D)$ by

$$V \ = \ V(A, H, D) \ := \ \fint |D|^{-\delta} \, .$$

Here the noncommutative integral is taken over the ideal $\mathcal{L}^{1,+}(\ker(D)^\perp)$ of the orthogonal complement of the kernel of $D$.

More generally, one can define a noncommutative integral with respect to a spectral triple.

**Definition 3.2.21.** Let $(A, H, D)$ denote a finitely summable spectral triple with non-zero metric dimension $\delta$ and let $\mathcal{W}$ denote a limiting procedure. Then the *$\mathcal{W}$-noncommutative integral* of an element $a \in A$ with respect to the spectral triple $(A, H, D)$ is given by

$$\mathrm{Tr}_{\mathcal{W}}(\pi(a)|D|^{-\delta}) \tag{3.18}$$

Here, $(\pi, H)$ denotes the faithful $*$-representation of $A$ on $H$ associated to $(A, H, D)$. Further, the Dixmier trace is taken over the ideal $\mathcal{L}^{1,+}(\ker(D)^\perp)$ of the orthogonal complement of the kernel of $D$. If $\pi(a)|D|^{-\delta}$ is measurable then we refer to the common values of the Dixmier traces as the *noncommutative integral* of $a$ with respect to the spectral triple $(A, H, D)$ and denote the common value by

$$\fint \pi(a)|D|^{-\delta}.$$

*Remark.* *Let $(A, H, D)$ denote a finitely summable spectral triple with non-zero metric dimension $\delta$ and let $(\pi, H)$ denote the $*$-representation associated to $(A, H, D)$. If $|D|^{-\delta}$ is a measurable operator, then it is not necessarily the case that $\pi(a)|D|^{-\delta}$ will be a measurable operator, for $a \in A$. However, in the examples which follow, we shall see that for all $a \in A$, the operator $\pi(a)|D|^{-\delta}$ is a measurable operator.*

Let us end this section with the following theorem, which provides us with an example of a spectral triple and is presented by Connes [Con3, Con2] as a prototype of such an object. We refer the interested reader to Chapters 1 and 2 of [Jos] for background on the notions of differential geometry, which are in use.

**Theorem 3.2.22.** *Let $n \in \mathbb{N}$ and let $M$ denote a smooth compact orientable complex manifold of (real) dimension $2n$ equipped with a spin$^c$ structure. Let $A$ denote the $C^*$-algebra of continuous complex-valued functions on $M$ acting by multiplication on the complex Hilbert space $H$, which is generated by spinor fields. Further, let $D$ denote the Dirac operator determined by the spin structure. Then $(A, H, D)$ is an even finitely summable spectral triple where the grading operator is given by the chirality operator. Moreover, one recovers the theory of Riemannian geometry from the noncommutative setting. For instance one has the following.*

1. *The metric dimension is $2n$.*
2. *Connes' pseudo-metric $d_{C^\infty(M;\mathbb{C})}$ induces a metric on $M$ which is bi-Lipschitz equivalent to the Riemannian metric.*
3. *If $n \neq 1$, then for each $a \in C^\infty(M;\mathbb{C})$, we have that*

$$⨍ \pi(a)|D|^{-2n} \;=\; c_n \int_M a \star (1).$$

*Here, $(\pi, H)$ denotes the faithful $*$-representation of $A$ and $c_n$ denotes a constant dependent on $n$. Further, following convention, we let $\star$ denote the Hodge star operator which acts on the $p$-forms of $M$.*

Proof. See for instance Chapter 11 of [FGBV] or Theorem 9 of [Ren]. ■

For a simple and fundamental example of this theorem, namely that of the spin geometry of the Riemann sphere, we refer the reader to Section 9.A of [FGBV].

## 3.3 Examples of Spectral Triples

To conclude this chapter, we present three basic examples of spectral triples, examining their noncommutative geometries. Although most of the material in this section is well-known, it is often the case that many of the finer details are omitted, and so, where this is the case we provide a full account. Specifically, we consider and examine the noncommutative geometries of spectral triples which represent the unit circle (Subsection 3.3.1), noncommutative tori (Subsection 3.3.2) and duals of countably infinite discrete groups (Subsection 3.3.3). In the case of the noncommutative torus we take a different approach to that usually presented in the literature. Namely, by noting that the irrational rotation algebra gives an appropriate representation of the noncommutative torus and using the representation of the irrational rotation algebra given in Example 3.1.28, we show how the spectral triple for the unit circle can be extended to obtain a spectral triple for the noncommutative torus. Before beginning, we recall the following results from functional analysis and measure theory.

**Theorem 3.3.1.** *(Riesz Representation Theorem) Let $X$ denote a locally compact Hausdorff space. If $T$ is a positive bounded linear functional on $C_c(X)$, then there exists a unique finite Borel measure $\mu$ on $X$ such that $T(a) = \int_X a \, d\mu$, for all $a \in C_c(X)$. Moreover, the norm of $T$ is equal to $\mu(X)$.*

Proof. See Theorem II.4 of [RS]. ■

**Theorem 3.3.2.** *If $T$ is a densely defined symmetric operator on a Hilbert space $H$, then $T$ is essentially self-adjoint if and only if the range of the operators $T \pm i\mathbb{1}$ are norm-dense in $H$.*

Proof. See Corollary to Theorem VIII.3 in [RS]. ■

**Definition 3.3.3.** Let $X$ denote a given space and let $R$ denote a subset of the power set of $X$. Then $R$ is called a *semi-ring* if it has the following properties.

1. $\emptyset \in R$.

2. If $Y_1, Y_2 \in R$, then $Y_1 \cap Y_2 \in R$.

3. If $Y_1, Y_2 \in R$ and $Y_1 \subset Y_2$, then there exist $Z_1, Z_2, \ldots, Z_n \in R$ which are pairwise disjoint such that
$$Y_2 \setminus Y_1 \;=\; \bigcup_{k=1}^{n} Z_k.$$

**Definition 3.3.4.** For a semi-ring $R$ and for a set function $\Lambda : R \to [0, \infty]$, we make the following definitions.

1. The set function $\Lambda$ is said to be *additive* if for a finite collection of pairwise disjoint sets $W_1, W_2, \ldots, W_m \in R$ for which $\bigcup_{k=1}^{m} W_k \in R$, we have that
$$\Lambda\left(\bigcup_{k=1}^{m} W_k\right) \;=\; \sum_{k=1}^{m} \Lambda(W_k).$$

2. The set function $\Lambda$ is said to be *$\sigma$-subadditive* if for a countable collection of sets $W_1, W_2, \cdots \in R$, we have, for each $W \in R$ with
$$W \;\subset\; \bigcup_{k\in\mathbb{N}} W_k,$$
that
$$\Lambda(W) \;\leqslant\; \sum_{k\in\mathbb{N}} \Lambda(W_k).$$

3. The set function $\Lambda$ is said to be *$\sigma$-finite* on $R$ if there exists a nested sequence of sets $X_1 \subset X_2 \subset \ldots$ such that
$$X \;=\; \bigcup_{k\in\mathbb{N}} X_k$$
and such that $\Lambda(X_k)$ is finite for each $k \in \mathbb{N}$.

**Theorem 3.3.5.** *(Hahn-Kolmogorov Theorem) Let $R$ be a semi-ring and $\Lambda : R \to [0, \infty]$ an additive $\sigma$-subadditive set function with $\Lambda(\emptyset) = 0$. Then $\Lambda$ can be extended to a measure on the $\sigma$-algebra generated by $R$. Moreover, if $\Lambda$ is $\sigma$-finite on $R$, then this extension is unique.*

Proof. See either [Keß] or Theorem 9.8 of [Bar]. ■

Let us begin by describing the standard spectral triple representation of the unit circle in $\mathbb{S}^1$ as given in [Con3, AIC1, AICL, Pal].

### 3.3.1 Circles

Let $A$ denote the set of continuous $2\pi$-periodic complex-valued functions on $\mathbb{R}$ and let $\tau$ denote a state on $A$ given, for each $a \in A$, by
$$\tau(a) \;:=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} a\, d\lambda^1. \tag{3.19}$$

Let $H_\tau$ denote the Gelfand-Naĭmark-Segal completion of $A$ with respect to $\tau$ and observe that $H_\tau \cong L^2(\mathbb{S}^1, \mathcal{B}, \vartheta^1)$. (Here, $\vartheta^1$ denotes the one-dimensional spherical measure on $\mathbb{S}^1$ with $\vartheta^1(\mathbb{S}^1)$ equal to one.) Let $\pi : A \to B(H_\tau)$ denote the faithful $*$-homomorphism given by $\pi(a)(h) := a \cdot h$. Next, observe that the Hilbert space $H_\tau$ has a canonical orthonormal basis $\{\phi_k : \mathbb{R} \to \mathbb{C} \,:\, k \in \mathbb{Z}\}$, where $\phi_k(x) := (2\pi)^{-1/2} e^{ikx}$ for each $k \in \mathbb{Z}$ and each $x \in \mathbb{R}$. (Note that we follow convention, in that we do not distinguish between a measurable function $f : \mathbb{R} \to \mathbb{C}$ and its equivalence class

$$\left\{ g : \mathbb{R} \to \mathbb{C} \,:\, g \text{ is a } 2\pi\text{-periodic measurable function with } \int_{\mathbb{R}} |f - g| \, d\lambda^1 \;=\; 0 \right\}$$

belonging to $H_\tau$). Next, set

$$D \;:=\; -i\frac{d}{dt} + \frac{1}{2}\mathbb{1}.$$

Observe that $D$ is a linear unbounded operator on $H_\tau$ and that $\phi_k$ is an eigenfunction of $D$ with eigenvalue $k + 1/2$, for each $k \in \mathbb{Z}$. It is clear that $D$ is an unbounded symmetric operator and that the domain of $D$ is given by

$$\operatorname{Dom}(D) \;:=\; \left\{ h \in H_\tau \,:\, \sum_{k \in \mathbb{Z}} (k + 1/2)^2 |\langle h, \phi_k \rangle|^2 < \infty \right\}.$$

**Proposition 3.3.6.** *The triple $(A, H_\tau, D)$ is a spectral triple, where the $*$-representation is given by $(\pi, H_\tau)$.*

Proof. The images of the domain of $D$ under $D \pm i\mathbb{1}$ are both norm-dense in $H_\tau$. Therefore, by an application of Theorem 3.3.2, it follows that $D$ is an essentially self-adjoint operator. Further, since zero does not belong to the spectrum of $D$, the inverse of $D$ is well defined. Moreover, we have that $D^{-1}(\phi_k) = 2(2k+1)^{-1}\phi_k$, for each $k \in \mathbb{Z}$. Hence, we conclude that $D^{-1}$ is of trace-class, and so, $D$ has a compact resolvent. By noting that

$$\begin{aligned} &C^\infty(\mathbb{S}^1; \mathbb{C}) \\ &\subset \{a \in A \,:\, \text{the operator } [D, \pi(a)] \text{ is densely defined and extends to a bounded operator on } H_\tau\}, \end{aligned}$$

we obtain that $(A, H_\tau, D)$ is a spectral triple, where the $*$-representation is given by $(\pi, H_\tau)$. ■

**Theorem 3.3.7.** *The metric dimension of the spectral triple $(A, H_\tau, D)$ is equal to one. Moreover, the $D^{-1}$ is a measurable operator.*

Proof. Let $p > 0$ and observe that

$$\operatorname{tr}(|D|^{-p}) \;=\; \sum_{k \in \mathbb{N}_0} 2^{1+p}(2k+1)^{-p}.$$

Since this sum diverges at a logarithmic rate for $p = 1$, the result follows. ■

**Theorem 3.3.8.** *For all $a \in A$, we have that*

$$⨍ \pi(a)\, |D|^{-1} \;=\; \pi^{-1} \int_{-\pi}^{\pi} a \, d\lambda^1. \tag{3.20}$$

Proof. We note that it is enough to show that the result holds for each non-negative real valued function $a \in A$. We begin by making the following obsevations and fixing the following notations.

1. For each $c \in \mathbb{R}$ and for each $k \in \mathbb{N}$, we have that

$$\sigma_{2k}(\pi(c\chi_{\mathbb{R}})|D|^{-1}) \;=\; \frac{2|c|}{2k-1},$$

and for each $k \in \mathbb{N}_0$, we have that

$$\sigma_{2k+1}(\pi(c\chi_{\mathbb{R}})|D|^{-1}) \;=\; \frac{2|c|}{2(k+1)-1}.$$

Hence, it follows that $\sum_{k=1}^{N} \sigma_k(\pi(c\pi(\chi_{\mathbb{R}}))|D|^{-1}) \sim 2|c|\ln(N)$.

2. If $T \in B(H_\tau)$ is a positive compact operator, then for each $N \in \mathbb{N}$, we have that

$$\sum_{k=1}^{N} \sigma_k(T) \;=\; \sup\left\{\mathrm{tr}(TP) \,:\, P = P^2 = P^* \text{ and } \dim(P(H_\tau)) = N\right\}.$$

(See Lemma A.3.5 of Appendix A.3.)

3. For each Borel set $B \subseteq \mathbb{R}$, let $\widetilde{B}$ denote the Borel set

$$\{x \in \mathbb{R} \,:\, \text{there exists } y \in B \text{ and } k \in \mathbb{Z} \text{ with } x = y + 2k\pi\}.$$

4. For each $k \in \mathbb{N} \setminus \{1\}$, let $K_k := [-\lfloor (k-2)/2 \rfloor, \lfloor (k-1)/2 \rfloor] \cap \mathbb{Z}$ and let $Q_k : H_\tau \to H_\tau$ denote the projection given, for each $h \in H_\tau$, by

$$Q_k(h) \;:=\; \sum_{k_1 \in K_k} \langle h, \phi_{k_1} \rangle \phi_{k_1}.$$

Then, for each Borel set $B \subseteq \mathbb{R}$ and for each natural number $N \geqslant 4$, since $\pi(\chi_{\widetilde{B}})|D|^{-1}Q_N$ is a positive operator, we have that

$$\begin{aligned}
\sum_{k=1}^{N} \sigma_k(\pi(\chi_{\widetilde{B}})|D|^{-1}) \;&=\; \sup\{\,\mathrm{tr}(\pi(\chi_{\widetilde{B}})|D|^{-1}P) \,:\, P = P^2 = P^* \text{ and } \dim(P(H_\tau)) = N\} \\
&\geqslant\; \mathrm{tr}(\pi(\chi_{\widetilde{B}})|D|^{-1}Q_N) \\
&=\; \sum_{k \in \mathbb{N}} \langle \pi(\chi_{\widetilde{B}})|D|^{-1}Q_N\phi_k, \phi_k \rangle \\
&=\; \sum_{k \in K_N} 2|2k+1|^{-1}(2\pi)^{-1}\lambda^1(\widetilde{B} \cap [-\pi,\pi]) \\
&\geqslant\; (2\pi)^{-1}\lambda^1(\widetilde{B} \cap [-\pi,\pi]) \sum_{k=2}^{\lfloor N/2 \rfloor} 2k^{-1}.
\end{aligned}$$

Hence, for each real valued function $a \in A$, we have that

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^{N} \sigma_k(\pi(a)|D|^{-1})}{\ln(N)} \;\leqslant\; 2\|a\|_\infty, \tag{3.21}$$

with equality holding if $a$ is a constant function. Further, for each Borel set $B \subset \mathbb{R}$ we have that the limit

$$\lim_{N\to\infty} \frac{\sum_{k=1}^{N} \sigma_k(\pi(\chi_{\widetilde{B}})|D|^{-1})}{\ln(N)} \tag{3.22}$$

exists and is bounded below by $\pi^{-1}\lambda^1(\widetilde{B} \cap [-\pi, \pi])$. Letting $\mathcal{W}$ denote an arbitrary limiting procedure, the map $a \mapsto \mathrm{Tr}_{\mathcal{W}}(a|D|^{-1})$ defined on $A$ is a bounded linear functional. By the Riesz Representation Theorem there exists a unique finite Borel measure $\mu$ such that, for each $a \in A$, we have that

$$\mathrm{Tr}_{\mathcal{W}}(\pi(a)|D|^{-1}) \;=\; \int_{-\pi}^{\pi} a \, d\mu.$$

Moreover, from the lower bound on the limit given in Equation (3.22), for each non-negative real valued $a \in A$ one can deduce that

$$\mathrm{Tr}_{\mathcal{W}}(\pi(a)|D|^{-1}) \;=\; \int_{-\pi}^{\pi} a \, d\mu \;\geqslant\; \pi^{-1} \int_{-\pi}^{\pi} a \, d\lambda^1. \tag{3.23}$$

Suppose that equality does hold, then we can assume without loss of generality, that there exists a real valued function $a \in A$ with $0 \leqslant a(x) \leqslant 1$ for all $x \in \mathbb{R}$, such that

$$\mathrm{Tr}_{\mathcal{W}}(\pi(a)|D|^{-1}) \;>\; \pi^{-1} \int_{-\pi}^{\pi} a \, d\lambda^1.$$

If this is the case, then

$$\mathrm{Tr}_{\mathcal{W}}(\pi(1-a)|D|^{-1}) \;=\; 2 - \mathrm{Tr}_{\mathcal{W}}(\pi(a)|D|^{-1}) \;<\; 2 - \pi^{-1} \int_{-\pi}^{\pi} a \, d\lambda^1 \;=\; \pi^{-1} \int_{-\pi}^{\pi} 1 - a \, d\lambda^1.$$

This gives a contradiction to Equation (3.23). Therefore, equality holds in Equation (3.23) and so concludes the proof. ∎

Since the operator $D$ is not a closed operator, it is not a self-adjoint operator. However, as we have seen above, it is an essentially self-adjoint operator. Therefore, when taking the closure of $D$ one needs to be cautious, since there exists a non-constant $2\pi$-periodic continuous function $\widetilde{\theta}_\nu$ with the following properties.

1. $\widetilde{\theta}_\nu$ has derivative equal to zero almost everywhere with respect to the Lebesgue measure.
2. The element $(\widetilde{\theta}_\nu, \widetilde{\theta}_\nu/2) \in H_\tau \times H_\tau$ belongs to the closure of the graph of $D$.

This leads us to the following theorem.

**Theorem 3.3.9.** *Let $(A, H_\tau, D)$ denote the spectral triple as given in Proposition 3.3.6 and let $\mathcal{A}$ denote the set of Lipschitz continuous $2\pi$-periodic complex-valued functions. Then we have the following.*

1. *Connes' pseudo-metric $d_A$ is not bounded.*
2. *Connes' pseudo-metric $d_{\mathcal{A}}$ is bounded and is equal to the Monge-Kantorovitch metric.*

Proof. The second part of the theorem follows from the fact that for any Lipschitz continuous function $a \in \mathcal{A}$, the operator $[D, \pi(a)]$ extends to a bounded linear operator with norm equal to the Lipschitz constant of $a$.

For the first part of theorem, consider the middle third Cantor set $C_{1/3}$. Let $\delta := \ln(2)/\ln(3)$ and let $\mu_{\mathcal{H}^\delta(C_{1/3})}$ denote the normalised $\delta$-dimensional Hausdorff measure on $C_{1/3}$. Further, let $\Theta : [0,1] \to \mathbb{R}$ be given, for each $x \in C_{1/3}$, by

$$\Theta(x) \;:=\; \mu_{\mathcal{H}^\delta(C_{1/3})}([0,x] \cap C_{1/3}).$$

Let $\widetilde{\Theta} : \mathbb{R} \to \mathbb{R}$ denote the $2\pi$-periodic extension of $\Theta$ to $\mathbb{R}$ given, for each $x \in \mathbb{R}$, by

$$\widetilde{\Theta}(x) \; := \; \sum_{k \in \mathbb{Z}} \chi_{[2k\pi,(2k+1)\pi)}(x)\Theta\left(\tfrac{x-2k\pi}{\pi}\right) + \chi_{[(2k+1)\pi,(2k+2)\pi)}(x)\Theta\left(1 - \tfrac{x-(2k+1)\pi}{\pi}\right).$$

Then, for each $k \in \mathbb{N}$, we have the following.

1. The operator $[D, \pi(k\widetilde{\Theta})]$ is densely defined and extends to a bounded operator whose norm is equal to zero.
2. $|k\widetilde{\Theta}_\nu(0) - k\widetilde{\Theta}_\nu(\pi)| = k$.

Hence, we have that $d_A(\delta_0, \delta_\pi) = \infty$. ∎

In the following two subsections we turn our attention to two classes of algebras that are not commutative. We begin with the irrational rotation algebra which gives a representation of the object known as the noncommutative torus.

### 3.3.2 The Noncommutative Torus

The noncommutative torus is an example of a space known as a "bad quotient". In particular, it is a quotient space which is non-Hausdorff. It arises from an irrational rotation of the unit circle $\mathbb{S}^1 \cong \mathbb{R}/2\pi\mathbb{Z}$. More precisely, fix an irrational number $\theta \in (0,1)$ and let $T_\theta : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}/2\pi\mathbb{Z}$ be defined, for each equivalence class $[x] \in \mathbb{R}/2\pi\mathbb{Z}$, by

$$T_\theta([x]) \; := \; [x - 2\pi\theta].$$

Define the equivalence relation $\sim_\theta$ on $\mathbb{R}/2\pi\mathbb{Z}$ by, $[x_1] \sim_\theta [x_2]$ if and only if there exists $k \in \mathbb{Z}$ such that $T_\theta^k([x_1]) = [x_2]$, for each $[x_1], [x_2] \in \mathbb{R}/2\pi\mathbb{Z}$. Then the space of equivalence classes $(\mathbb{R}/2\pi\mathbb{Z})/\sim_\theta$, equipped with the quotient topology, is a non-Hausdorff space and is called the *noncommutative torus*. Note that the set of complex-valued continuous functions on $(\mathbb{R}/2\pi\mathbb{Z})/\sim_\theta$, equipped witht the quotient topology, is isomorphic to $\mathbb{C}$. Therefore, if one wants to study the space $(\mathbb{R}/2\pi\mathbb{Z})/\sim_\theta$ on an algebraic level, one is required to consider a more complex algebra. For instance, an algebra which encodes the $C^*$-algebra $A$ of $2\pi$-periodic complex-valued continuous functions defined on $\mathbb{R}$ and the group action of $\mathbb{Z}$ given by $\alpha_\theta$. (Recall, that $\alpha_\theta : \mathbb{Z} \to \mathrm{Aut}(A)$ is given by $\alpha_\theta(k)f(x) := f(x - 2\pi k\theta)$, for each $f \in A$.) In other words, the appropriate algebra to study the space $(\mathbb{R}/2\pi\mathbb{Z})/\sim_\theta$ is the irrational rotation algebra $A_\theta := A \rtimes_{\alpha_\theta} \mathbb{Z}$.

To construct a spectral triple for $A_\theta$, we first need a suitable Hilbert space. If there exists a tracial state $\tau$ on $A_\theta$, then the GNS completion $H_\tau$ of $A_\theta$ would provide such an object. In fact it is well-known that there exists a unique tracial state on $A_\theta$, see for instance Proposition VI.1.3. of [Dav]. With the representation of $A_\theta$ as presented in Example 3.1.28, we observe that the unique tracial state $\tau$ is given, for each $\sum_{k\in\mathbb{Z}} a_k\delta_k$ belonging to the complex $*$-algebra $A\mathbb{Z}$, by

$$\tau\left(\sum_{k\in\mathbb{Z}} a_k \delta_k\right) := \int_{-\pi}^{\pi} a_0 \, d\lambda^1. \tag{3.24}$$

To define $\tau$ on the reduced completion $A \rtimes_{\alpha_\theta} \mathbb{Z}$ of $A\mathbb{Z}$ we extend the definition given in Equation (3.24) by continuity. The result that $\tau$ is a unique tracial state follows from an application of the Riesz Representation Theorem (Theorem 3.3.1) and the following observation. The only $T_\theta$-invariant ergodic Borel probability measure on $\mathbb{R}/2\pi\mathbb{Z}$ is the push forward of the Lebesgue measure on $\mathbb{R}$ to the quotient space $\mathbb{R}/2\pi\mathbb{Z}$ (see Theorem 6.18 of [Wal2]). Due to the fact that the irrational rotation algebra is universal, this provides an alternative proof for the existence of a unique tracial state on the irrational rotation algebra to the standard proof as given in [Dav], Proposition VI.1.3.

We now want to construct a Dirac operator. For reasons which will become clear (see Theorem 3.3.12) fix $z \in \mathbb{C}$ with $\Im m(z) \neq 0$. Define $\nabla_z$ on the subset

$$\mathrm{Dom}(\nabla_z) \;:=\; \left\{ \sum_{k\in\mathbb{Z}} h_k \delta_k \in A \rtimes_{\alpha_\theta} \mathbb{Z} \,:\, h_k \in A \cap \mathrm{Dom}\left(-i\frac{d}{dx}\right) \text{ for each } k \in \mathbb{Z} \right\}$$

of $H_\tau$ by

$$\nabla_z \left( \sum_{k\in\mathbb{Z}} h_k \delta_k \right) \;:=\; \sum_{k\in\mathbb{Z}} \left( \left( -i\frac{d}{dx} + z\left(k + \frac{1}{2}\right) \mathbb{1} \right) h_k \right) \delta_k .$$

Since to each element $\sum_{k\in\mathbb{Z}} a_k \delta_k$ belonging the complex $*$-algebra $A\mathbb{Z}$, there exists a unique equivalence class in $H_\tau$ which contains $\sum_{k\in\mathbb{Z}} a_k \delta_k$, following convention, we do not distinguish between the element $\sum_{k\in\mathbb{Z}} a_k \delta_k$ and its equivalence class.

**Lemma 3.3.10.** *The operator $\nabla_z$ is a densely defined unbounded operator with a compact resolvent such that $\nabla_z^* = \nabla_{\overline{z}}$ on Dom$(\nabla_z)$.*

Proof. By the Stone-Weierstrass Theorem (Theorem 2.2.12) and the GNS construction of $H_\tau$, the set $\mathrm{Dom}(\nabla_z)$ is norm-dense in $H_\tau$. Linearity of the operator $\nabla_z$ follows since $-i\frac{d}{dx}$ is a linear operator and multiplication by a constant is a linear operator. Similarly, since $-i\frac{d}{dx}$ is an unbounded operator, it follows that $\nabla_z$ is an unbounded operator. Moreover, for all $\sum_{k_1\in\mathbb{Z}} h_{k_1}\delta_{k_1}, \sum_{k_2\in\mathbb{Z}} g_{k_2}\delta_{k_2} \in \mathrm{Dom}(\nabla_z)$, we have that

$$\begin{aligned}
&\tau\left(\nabla_z\left(\textstyle\sum_{k_1\in\mathbb{Z}} h_{k_1}\delta_{k_1}\right) * \left(\textstyle\sum_{k_2\in\mathbb{Z}} g_{k_2}\delta_{k_2}\right)^*\right) \\
&\quad= \tau\Big(\textstyle\sum_{k_1\in\mathbb{Z}} -i\frac{d}{dx} h_{k_1}\delta_{k_1} * \sum_{k_2\in\mathbb{Z}} \alpha_\theta(k_2)(\overline{g}_{-k_2})\delta_{k_2} + \\
&\qquad\quad \textstyle\sum_{k_1\in\mathbb{Z}} z(k_1+\frac{1}{2})h_{k_1} * \sum_{k_2\in\mathbb{Z}} \alpha_\theta(k_2)(\overline{g}_{-k_2})\delta_{k_2}\Big) \\
&\quad= \tau\left(\textstyle\sum_{k_1\in\mathbb{N}}(-i\frac{d}{dx} h_{k_1}\alpha_\theta(k_1)(\overline{g}_{k_1}))\delta_0 + \sum_{k_1\in\mathbb{Z}} z(k_1+\frac{1}{2})h_{k_1}\alpha_\theta(k_1)(\overline{g}_{k_1})\delta_0\right) \\
&\quad= \tau\left(\textstyle\sum_{k_1\in\mathbb{N}} h_{k_1}\alpha_\theta(k_1)(-i\frac{d}{dx}\overline{g}_{k_1})\delta_0 + \sum_{k_1\in\mathbb{Z}} h_{k_1}\alpha_\theta(k_1)(\overline{\overline{z}(k_1+\frac{1}{2})g_{k_1}})\delta_0\right) \\
&\quad= \tau\Big(\textstyle\sum_{k_1\in\mathbb{Z}} h_{k_1}\delta_{k_1} * \sum_{k_2\in\mathbb{Z}} \alpha_\theta(k_2)(\overline{-i\frac{d}{dx} g_{-k_2}})\delta_{k_2} + \\
&\qquad\quad \textstyle\sum_{k_1\in\mathbb{Z}} h_{k_1} * \sum_{k_2\in\mathbb{Z}} \alpha_\theta(k_2)(\overline{\overline{z}\left(k_1+\frac{1}{2}\right) g_{-k_2}})\delta_{k_2}\Big) \\
&\quad= \tau\left(\textstyle\sum_{k_1\in\mathbb{Z}} h_{k_1}\delta_{k_1} * \left(\nabla_{\overline{z}}\left(\sum_{k_2\in\mathbb{Z}} g_{k_2}\delta_{k_2}\right)\right)^*\right).
\end{aligned}$$

Hence, it follows that

$$\left\langle \nabla_z\left(\textstyle\sum_{k_1\in\mathbb{Z}} h_{k_1}\delta_{k_1}\right), \sum_{k_2\in\mathbb{Z}} g_{k_2}\delta_{k_2}\right\rangle \;=\; \left\langle \textstyle\sum_{k_1\in\mathbb{Z}} h_{k_1}\delta_{k_1}, \nabla_{\overline{z}}\left(\sum_{k_2\in\mathbb{Z}} g_{k_2}\delta_{k_2}\right)\right\rangle.$$

Next, note that ${\nabla_z}^{-1}$ is defined, for each $\sum_{k\in\mathbb{Z}} h_k\delta_k \in \mathrm{Dom}(\nabla_z)$, by

$${\nabla_z}^{-1}\left(\sum_{k\in\mathbb{Z}} h_k\delta_k\right) \;:=\; \sum_{k_1\in\mathbb{Z}} \left( \sum_{k_2\in\mathbb{Z}} \frac{1}{k_2 + z\,(k_1+1/2)} \langle h_{k_1}, \phi_{k_2}\rangle\, \phi_{k_2} \right) \delta_{k_1}. \tag{3.25}$$

(Recall that $\phi_k(x) := (2\pi)^{-1/2}e^{ikx}$, for each $k \in \mathbb{Z}$ and each $x \in \mathbb{R}$.) Observe that ${\nabla_z}^{-1}$ has a unique extension to a bounded operator on the GNS completion $H_\tau$ of $A \rtimes_{\alpha_\theta} \mathbb{Z}$. It follows that

${\nabla_z}^{-1}$ is compact, since ${\nabla_z}^{-1}$ can be approximated by a sequence of bounded operators with finite dimensional range. Such a sequence is given by the closure of $\nabla_{z,k}^{-1}$ defined, for each $\sum_{k\in\mathbb{Z}} h_k\delta_k$ in the $*$-algebra $A\mathbb{Z}$, by

$$\nabla_{z,k}{}^{-1}\left(\sum_{k\in\mathbb{Z}} h_k\delta_k\right) \;:=\; \sum_{k_1\in\mathbb{Z}}\left(\sum_{k_2\in\mathbb{Z}} \frac{\chi_{[-k,k]}(k_1)\chi_{[-k,k]}(k_2)}{k_2+z\,(k_1+1/2)}\,\langle h_{k_1},\phi_{k_2}\rangle\,\phi_{k_2}\right)\delta_{k_1}.$$

Therefore, $\nabla_z$ has a compact resolvent. ■

As $\nabla_z$ is not symmetric we cannot consider it as a Dirac operator. However, the operator $D_z$ define on $\mathrm{Dom}(\nabla_z)\oplus\mathrm{Dom}(\nabla_z)\subset H_\tau\oplus H_\tau$ and given by

$$D_z \;:=\; \begin{pmatrix} 0 & \nabla_{\overline{z}} \\ \nabla_z & 0\end{pmatrix},$$

is a symmetric operator and a more suitable candidate for the Dirac operator. Finally, we require a representation of $A$ on $H_\tau\oplus H_\tau$. Recall that the GNS completion provides a faithful $*$-representation $(\pi_\tau,H_\tau)$ of $A\rtimes_{\alpha_\theta}\mathbb{Z}$, and so, define the $*$-representation $(\widetilde{\pi}_\tau, H_\tau\oplus H_\tau)$ of $A\rtimes_{\alpha_\theta}\mathbb{Z}$, by $\widetilde{\pi}_\tau(a):=\pi_\tau(a)\oplus\pi_\tau(a)$, for each $a\in A\rtimes_{\alpha_\theta}\mathbb{Z}$.

**Proposition 3.3.11.** *Let $z\in\mathbb{C}$ with $\Im m(z)\neq 0$. Then the triple $(A\rtimes_{\alpha_\theta}\mathbb{Z}, H_\tau\oplus H_\tau, D_z)$ with the $*$-representation, $(\widetilde{\pi}_\tau, H_\tau\oplus H_\tau)$ is a spectral triple.*

Proof. By the properties of $\nabla_z$ given in Lemma 3.3.10, it follows that $D_z$ is a densely defined unbounded symmetric linear operator with compact resolvent on $H_\tau\oplus H_\tau$. In order to conclude that $D$ is an essentially self-adjoint operator we apply Theorem 3.3.2. Therefore, we need to show that $\mathrm{Ran}(D_z\pm i\mathbb{1})$ are norm-dense in $H_\tau\oplus H_\tau$, with respect to the Hilbert space norm. Recall that the irrational rotation algebra is generated by the two unitary operators $U:=\phi_1\delta_0$ and $V:=\chi_{\mathbb{R}}\delta_1$. (Here we remind the reader that $\phi_1(x):=(2\pi)^{-1/2}e^{ix}$ and that $\chi_{\mathbb{R}}$ denote the characteristic function of $\mathbb{R}$.) Then for all $k_1,k_2\in\mathbb{N}$, we have that

$$\begin{aligned}
(D_z\pm i\mathbb{1})\left(\frac{1}{k_1+z(k_2+1/2)\pm i}(U^{k_1}*V^{k_2}\oplus 0)\right) &= 0\oplus U^{k_1}*V^{k_2},\\
(D_z\pm i\mathbb{1})\left(\frac{1}{k_1+z(k_2+1/2)\pm i}(0\oplus U^{k_1}*V^{k_2})\right) &= U^{k_1}*V^{k_2}\oplus 0.
\end{aligned}$$

Since the set of polynomials in $U$ and $V$ is norm-dense in $H_\tau$ we have that the ranges of $D_z\pm i\mathbb{1}$ are norm-dense in $H_\tau\oplus H_\tau$. Therefore, $D$ is an essentially self-adjoint operator.

Next, we aim to show that the set

$$\{a\in C(\mathbb{S}^1)\rtimes_{\tau_\theta}\mathbb{Z} \,:\, [D,\widetilde{\pi}_\tau(\alpha)]\,\text{extends to a bounded operator}\}$$

is a dense subset of the irrational rotation algebra. Indeed this follows by observing that, for all

$k_1, k_2, q_1, q_2, t_1, t_2 \in \mathbb{Z}$, we have that

$$\begin{aligned}
&\left\| \left[ D_z, \widetilde{\pi}_\tau \left( U^{k_1} * V^{k_2} \right) \right] \begin{pmatrix} U^{q_1} * V^{t_1} \\ U^{q_2} * V^{t_2} \end{pmatrix} \right\| \\
&\qquad = \left\| \left[ \begin{pmatrix} 0 & \nabla_{\overline{z}} \\ \nabla_z & 0 \end{pmatrix}, \begin{pmatrix} \pi_\tau(U^{k_1} * V^{k_2}) & 0 \\ 0 & \pi_\tau(U^{k_1} * V^{k_2}) \end{pmatrix} \right] \begin{pmatrix} U^{q_1} * V^{t_1} \\ U^{q_2} * V^{t_2} \end{pmatrix} \right\| \\
&\qquad = \left\| \begin{pmatrix} \nabla_{\overline{z}} \left( U^{k_1} * V^{k_2} * U^{q_2} * V^{t_2} \right) - (U^{k_1} * V^{k_2}) * (\nabla_{\overline{z}} \left( U^{q_2} * V^{t_2} \right)) \\ \nabla_z \left( U^{k_1} * V^{k_2} * U^{q_1} * V^{t_1} \right) - (U^{k_1} * V^{k_2}) * (\nabla_z \left( U^{q_1} * V^{t_1} \right)) \end{pmatrix} \right\|
\end{aligned}$$

$$
\begin{aligned}
&= \left\| e^{-i\theta k_2 q_2} \left(k_1 + z\left(k_2 + 1/2\right)\right) U^{k_2+q_2} * V^{k_1+t_2} \right\| + \\
&\quad \left\| e^{-i\theta k_2 q_1} \left(k_1 + \overline{z}\left(k_2 + 1/2\right)\right) U^{k_2+q_1} * V^{k_1+t_1} \right\| \\
&= \left\| k_1 + z\left(k_2 + 1/2\right) \right\| + \left\| k_1 + \overline{z}\left(k_2 + 1/2\right) \right\|.
\end{aligned}
$$

■

*Remark. Observe that the operator $\Gamma \in B(H_\tau \oplus H_\tau)$ defined by*

$$
\Gamma := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
$$

*provides a grading of the Hilbert space $H_\tau \oplus H_\tau$. Moreover, for each $z \in \mathbb{C}$ with $\Im m(z) \neq 0$, the data set $(A \rtimes_{\alpha_\theta} \mathbb{Z}, H_\tau \oplus H_\tau, D_z)$ with the $*$-representation, $(\widetilde{\pi}_\tau, H_\tau \oplus H_\tau)$ and grading operator $\Gamma$ is an even spectral triple.*

**Theorem 3.3.12.** *Let $z \in \mathbb{C}$ with $\Im m(z) \neq 0$. Then the metric dimension of the spectral triple $(A \rtimes_{\alpha_\theta} \mathbb{Z}, H_\tau \oplus H_\tau, D_z)$, as given in Proposition 3.3.11, is equal to 2. Moreover, $|D_z|^{-2}$ is a measurable operator with $V(A \rtimes_{\alpha_\theta} \mathbb{Z}, H_\tau \oplus H_\tau, D_z)$ equal to $2\pi/\Im m(z)$.*

PROOF. Recall that $\phi_1 : \mathbb{R} \to \mathbb{C}$ is defined, for each $x \in \mathbb{R}$, by $\phi_1(x) := (2\pi)^{-1} e^{ix}$ and that $\chi_{\mathbb{R}}$ denotes the characteristic function of $\mathbb{R}$. Further, recall that $U := \phi_1 \delta_0$ and $V := \chi_{\mathbb{R}} \delta_1$ form a generating set for the reduced cross product algebra $A \rtimes_{\alpha_\theta} \mathbb{Z}$. Next, observe that the following hold.

1. The set
$$
\left\{ U^{k_1} V^{k_2} \oplus 0 \,:\, k_1, k_2 \in \mathbb{Z} \right\} \cup \left\{ 0 \oplus U^{k_1} V^{k_2} \,:\, k_1, k_2 \in \mathbb{Z} \right\}
$$
forms an orthonormal basis for $H_\tau \oplus H_\tau$.

2. For $k_1, k_2 \in \mathbb{Z}$ we have that
$$
\begin{aligned}
(\mathbb{1} + D^2)(U^{k_1} V^{k_2} \oplus 0) &= (\|k_1 + z\left(k_2 + 1/2\right)\|^2 + 1)(U^{k_1} V^{k_2} \oplus 0), \\
(\mathbb{1} + D^2)(0 \oplus U^{k_1} V^{k_2}) &= \left(\|k_1 + z\left(k_2 + 1/2\right)\|^2 + 1\right)(0 \oplus U^{k_1} V^{k_2}).
\end{aligned}
$$

Next, observe that as $R$ tends to positive infinity, we have the following

$$
\begin{aligned}
\frac{1}{\ln(2R^2)} \sum_{k=1}^{2R^2} \sigma_k(|D_z|^{-2}) &\sim \frac{1}{2\ln(R)} \sum_{\substack{k_1, k_2 \in \mathbb{Z} \\ k_1^2 + k_2^2 \leqslant R}} \frac{2}{\|k_1 + z(k_2 + \frac{1}{2})\|^2} && (3.26) \\
&\sim \frac{1}{\ln(R)} \sum_{\substack{k_1, k_2 \in \mathbb{Z} \\ k_1^2 + k_2^2 \leqslant R}} \frac{1}{\|k_1 + k_2 z\|^2} && (3.27) \\
&\sim \frac{1}{\ln(R)} \int_1^R \int_{-\pi}^{\pi} \frac{1}{\|r\cos(\theta) + zr\sin(\theta)\|^2} \, r \, dr \, d\theta && (3.28) \\
&\sim \int_{-\pi}^{\pi} \frac{1}{\|\cos(\theta) + z\sin(\theta)\|^2} \, d\theta. && (3.29)
\end{aligned}
$$

Applying the change of variables $u = \tan(\theta)$, we observe that

$$\begin{aligned}
\|\cos(\theta) + z\sin(\theta)\|^2 &= (\cos(\theta) + \Re e(z)\sin(\theta))^2 + \Im m(z)^2 \sin^2(\theta) \\
&= \cos^2(\theta) + 2\Re e(z)\sin(2\theta) + \Re e(z)^2 \sin^2(\theta) + \Im m(z)^2 \sin(z)^2 \\
&= \frac{1+\cos(2\theta)}{2} + \frac{\Re e(z)^2}{2}(1-\cos(2\theta)) + \frac{\Im m(z)^2}{2}(1-\cos(2\theta)) + \Re e(z)\sin(2)\theta \\
&= \frac{1+\Re e(z)^2 + \Im m(z)^2}{2} + \frac{1-\Re e(z)^2 - \Im m(z)^2}{2}\cos(2\theta) + \Re e(z)\sin(2\theta) \\
&= \frac{(1+\Re e(z)^2+\Im m(z)^2)(1+u^2) + (1-\Re e(z)^2 - \Im m(z)^2)(1-u^2) + 4u\Re e(z)}{2(1+u^2)} \\
&= \frac{1+2\Re e(z)u + (\Re e(z)^2 + \Im m(z)^2)u^2}{1+u^2}.
\end{aligned}$$

Here we have used the following trigonometric identities.

1. $\cos(2\theta) = 2\cos^2(\theta) - 1$.

2. $\sin(2\theta) = \dfrac{2\tan(\theta)}{1+\tan^2(\theta)}$.

3. $\cos(2\theta) = \dfrac{1-\tan^2(\theta)}{1+\tan^2(\theta)}$.

Therefore, since

$$\frac{du}{d\theta} = \sec^2\theta = 1 + \tan^2(\theta) = 1 + u^2$$

we have that

$$\begin{aligned}
&\int_{-\pi}^{\pi} \frac{1}{\|\cos(\theta) + z\sin(\theta)\|^2}\, d\theta \\
&\quad = 2\int_{-\pi/2}^{\pi/2} \frac{1}{\|\cos(\theta) + z\sin(\theta)\|^2}\, d\theta \\
&\quad = 2\int_{-\infty}^{\infty} \frac{1}{1+2\Re e(z)u + (\Re e(z)^2 + \Im m(z)^2)u^2}\, du \\
&\quad = \frac{2}{\Re e(z)^2 + \Im m(z)^2}\int_{\infty}^{\infty} \left( \left( u + \frac{\Re e(z)}{\Re e(z)^2 + \Im m(z)^2}\right)^2 + \frac{\Im m(z)^2}{(\Re e(z)^2 + \Im m(z)^2)^2} \right)^{-1} du \\
&\quad = \frac{2}{\Re e(z)^2 + \Im m(z)^2}\left[ \frac{\Re e(z)^2 + \Im m(z)^2}{\Im m(z)^2} \tan^{-1}\left( u + \frac{\Re e(z)}{\Re e(z)^2 + \Im m(z)^2}\right)\right]_{-\infty}^{\infty} \\
&\quad = \frac{2\pi}{\Im m(z)}.
\end{aligned}$$

By an application of Lemma 3.2.18 the result then follows. ■

We now include a brief discussion on the metric aspects of the spectral triple $(A \rtimes_{\alpha_\theta} \mathbb{Z}, H_\tau \oplus H_\tau, D_z)$ as given in Proposition 3.3.11. Observe that an analogous argument to that given in part 1 of Theorem 3.3.9 can be constructed to show that Connes' pseudo-metric $d_{A \rtimes_{\alpha_\theta} \mathbb{Z}}$ not a metric. However, by considering an appropriate complex sub-$*$-algebra $\mathcal{A}$ of the irrational rotation algebra $A_\theta \cong A \rtimes_{\alpha_\theta} \mathbb{Z}$, a consequence of Theorem 4.2 of [Rie2] shows that one obtains that $d_{\mathcal{A}}$ is a metric. An appropriate choice for the complex sub-$*$-algebra is the complex $*$-algebra generated by all polynomials in the two generators of $A_\theta$.

### 3.3.3 Discrete Groups

Finally, let us conclude this section with the spectral triple which Connes investigates in [Con2], that is, a spectral triple which represents the reduced discrete crossed product algebra of a countably infinite discrete group. Here we only outline Connes' construction and refer the reader to [Con2] for various geometric and algebraic results of the spectral triple.

**Definition 3.3.13.** For a group $G$, define a *length function* of $G$ to be a map $L : G \to [0, \infty)$ such that the following hold.

1. $L(g_1 g_2) \leqslant L(g_1) + L(g_2)$, for all $g_1, g_2 \in G$.
2. $L(g_1^{-1}) = L(g_1)$, for all $g \in G$.
3. $L(e) = 0$, where $e$ is the group identity.

**Example 3.3.14.** For a finitely generated group $G$, the reduced word length with respect to a fixed symmetric generating set is an example of a length function.

*Remark. If $L$ is a length function on a group $G$ such that $L(g) = 0$ if and only if $g = e$, then $L$ gives rise to a metric on $G$. For instance, the map $d_L : G \times G \to [0, \infty)$ defined, for each $g_1, g_2 \in G$, by $d_L(g_1, g_2) := L(g_1^{-1} g_2)$.*

Let $G$ denote a countably infinite discrete group $G$ and let $L : G \to [0, \infty)$ denote a given length function. Define $D_L$ on $l^2(G)$ by

$$D_L(\gamma)(g) \;:=\; L(g)\gamma(g).$$

**Lemma 3.3.15.** *The operator $D_L$ is an essentially self-adjoint operator.*

Proof. For $\gamma_1, \gamma_2 \in \mathrm{Dom}(D_L)$ we observe that

$$\langle D_L(\gamma_1), \gamma_2\rangle \;=\; \sum_{g \in G} L(g)\gamma_1(g)\overline{\gamma_2(g)} \;=\; \sum_{g \in G} \gamma_1(g)\overline{L(g)\gamma_2(g)} \;=\; \langle \gamma_1, D_L(\gamma_2)\rangle.$$

Thus, $D_L$ is symmetric on its domain. If $D_L$ is a bounded operator, then the result immediately follows. If $D_L$ is an unbounded operator, then since the set

$$\{\delta \in l^2(G) \,:\, \delta(g) = 0 \text{ for all but a finite number of } g \in G\}$$

is norm-dense in $l^2(G)$, one can conclude the following.

1. The operator $D_L$ is densely defined.
2. The ranges of the operator $D_L \pm i\mathbb{1}$ are norm-dense in $l^2(G)$.

Hence, the result then follows by an application of Theorem 3.3.2. ■

**Proposition 3.3.16.** *Let $G$ denote a countably infinite discrete group, let $L$ denote a length function of $G$ and let $A$ denote the reduced discrete group $C^*$-algebra $\mathbb{C} \rtimes_\alpha G$. Further, let $(\pi, l^2(G))$ denote the $*$-representation of $A$, where for each $a \in A$ we define $\pi(a)(\gamma) := a * \gamma$ (see Equation 3.7). If for each $k \in \mathbb{N}_0$ the cardinality of the set $L^{-1}(k)$ is finite and the sequence $(\mathrm{card}(L^{-1}(k)))_{k \in \mathbb{N}_0}$ contains only finitely many zeros, then $(A, l^2(G), D_L)$ is a spectral triple. Moreover, we have that $\|[D, \pi(\delta_g)]\| = L(g)$ for all $g \in G$.*

PROOF. The following proof is similar to the proof given by Connes [Con2], however we include some further details.

By Lemma 3.3.15 we have that the operator $D_L$ is an essentially self-adjoint operator. Further, since for each $k \in \mathbb{N}_0$ the cardinality of the set $L^{-1}(k)$ is finite and non-zero for all but a finite number of $k \in \mathbb{N}_0$, we conclude that the operator $(\mathbb{1} + D_L^2)^{-1}$ can be written as the limit of a sequence of operators with finite dimensional range. Hence, $(\mathbb{1} + D_L^2)$ is a compact operator and $D$ is an unbounded operator.

Letting $e$ denote the group identity of $G$, observe that, for each $g \in G$, one has that

$$\|[D_L, \pi(\delta_g)]\| \geqslant \|[D_L, \pi(\delta_g)]\delta_e\| = \|D_L(\delta_e * \delta_g) - \delta_g * D_L(\delta_e)\| = \|D_L(\delta_g)\| = \|L(g)\delta_g\| = L(g).$$

Moreover, for each $\gamma \in l^2(G)$ with $\|\gamma\|_2 \leqslant 1$, observe that

$$\begin{aligned}
\|[D_L, \pi(\delta_g)]\gamma\|_2 &= \left\| D_L\left(\sum_{g_1 \in G} \gamma(g^{-1}g_1)\delta_{g_1}\right) - \delta_g * \sum_{g_2 \in G} L(g_2)\gamma(g_2)\delta_{g_2} \right\|_2 \\
&= \left\| \sum_{g_1 \in G} L(g_1)\gamma(g^{-1}g_1)\delta_{g_1} - \sum_{g_2 \in G} L(g^{-1}g_2)\gamma(g^{-1}g_2)\delta_{g_2} \right\|_2 \\
&= \left\| \sum_{g_1 \in G} (L(g_1) - L(g^{-1}g_1)\gamma(g^{-1}g_1)\delta_{g_1} \right\|_2 \\
&\leqslant L(g)\|\gamma\|_2 \leqslant L(g).
\end{aligned}$$

Since the set $\{\delta \in l^2(G) : \delta(g) = 0 \text{ for all but a finite number of} g \in G\}$ is norm-dense in $l^2(G)$, the result follows. ■

**Theorem 3.3.17.** *Let $G$ denote a finitely generated countably infinite discrete group, and fix a finite generating set. Suppose that $L$ is the length function given by the reduced word length of a group element with respect to the fixed generating set. Then the following hold.*

1. *If $G$ has polynomial growth, then the spectral triple $(A, l^2(G), D_L)$, as given in Proposition 3.3.16, is finitlely summable.*

2. *If the spectral triple $(A, H, D_L)$ is finitely summable then $G$ has polynomial growth.*

PROOF. The following proof is similar to the proof given by Connes [Con2], however we include some further details.

For the first part, for each $k \in \mathbb{N}_0$, let $B_k := \{g \in G : L(g) \leqslant k\}$. Since $G$ is of polynomial growth there exist constants $c, r > 0$, such that for $k \in \mathbb{N}_0$, the cardinality of $B_k$ is less than or equal to $c(1+k)^r$. Subsequently, if $p > r+1$, then we have that

$$\begin{aligned}
\operatorname{tr}\left((\mathbb{1} + D_L^2)^{-p/2}\right) = \sum_{g \in G}(L(g)^2 + 1)^{-p/2} &= \sum_{k \in \mathbb{N}} (\operatorname{card}(B_k) - \operatorname{card}(B_{k-1}))\left(1 + k^2\right)^{-p/2} \\
&\leqslant \sum_{k \in \mathbb{N}} (\operatorname{card}(B_k))(1+k)^{-p} \\
&\leqslant \sum_{k \in \mathbb{N}} c(1+k)^r(1+k)^{-p} \\
&\leqslant c\sum_{k \in \mathbb{N}} k^{-p+r} < \infty.
\end{aligned}$$

Hence, the spectral triple $(A, l^2(G), D_L)$ is is finitely summable.

For the second part, if the spectral triple $(A, H, D_L)$ is finitely summable, then for some $p > 0$, we have that

$$\sum_{k\in\mathbb{N}} (\mathrm{card}(B_k) - \mathrm{card}(B_{k-1})) \left(1 + k^2\right)^{-p/2} \;=\; \sum_{g\in G} \left(1 + L(g)^2\right)^{-p/2} \;=\; \mathrm{tr}\left((\mathbb{1} + D_L^2)^{-p/2}\right) \;<\; \infty.$$

Therefore, there exists $N \in \mathbb{N}$, where for each $k \in \mathbb{N}$, we have that

$$\mathrm{card}(B_{N+k}) - \mathrm{card}(B_{N+k-1}) \;<\; (N + k + 1)^p.$$

Hence, it follows that there exists a constant $c$ so that

$$\mathrm{card}(B_{N+k}) \;<\; c(N + k + 1)^{1+p}.$$

This finishes the proof. ■

# Chapter 4: A Commutative Noncommutative Fractal Geometry

In the past three decades it has gradually emerged from problems in pure mathematics that the class of Riemannian manifolds is too narrow to encompass all interesting spaces. To rectify this, Connes suggested that one should work on a $C^*$-algebraic level and developed the theory of noncommutative geometry (see Chapter 3). Although one of the original motivations for noncommutative geometry was to be able to deal with non-Hausdorff spaces, such as foliated manifolds, which are often best represented by a noncommutative $C^*$-algebra (see for instance [Con3, Vár, Mar, Rie3], it has been shown that the theory has a far wider scope even when the $C^*$-algebra is commutative. For instance, in [CL, PS, Kra, IKM] the authors show that any finite metric space can be represented by a finite spectral triple, from which one can recover the geometric structure. In [Con4, GBIS] spectral triples which represent the standard model in particle physics are considered. Note that, within these articles neutrinos are assumed to be massless. However, in [Con5] Connes constructs a noncommutative representation of the standard model where this assumption is not made. Further, in [Con3, Con2] Connes has shown that from a spectral triple representation of a spin manifold $M$, one can recover much of the geometric information of $M$ (see Theorem 3.2.22). Also, attempts to build spectral triples for an arbitrary compact metric space have been made by Christensen and Ivan in [AIC2]. There the authors construct spectral triples for an arbitrary compact metric space $(X, d)$ by gluing together spectral triples associated with pairs of points. More recently the work of Palmer [Pal] continues along this research thread.

In what follows, we consider how one can represent a compact totally disconnected set with no isolated points via a spectral triple and give new insight into the geometric aspects of such a spectral triple. This illustrates that the tools of noncommutative geometry are capable of bridging the gap between the continuum and the discrete.

The work in this section is split into two parts. Firstly, in Section 4.1 we briefly review Connes' method for constructing a spectral triple on a compact fractal subset of $\mathbb{R}$ and then we consider geometric aspects of this spectral triple. In particular, it is proven that from this representation of a compact "fractal" subset $E$ of $\mathbb{R}$ the multifractal box-counting dimension $\mathfrak{b}$ can be recovered if $E$ is strongly porous. Moreover, if one has a self-similarity condition, it is shown that the noncommutative integration theory is able to recover the associated multifractal auxiliary measures.

Secondly, in Section 4.2 we consider the construction of a spectral triple given by Antonescu-Ivan and Christensen in [AIC1]. Motivated by this construction, given a one-sided topologically exact subshift of finite type $(\Sigma_A^\infty, \sigma)$ and a Gibbs measure $\mu_\phi$ for a Hölder continuous potential function $\phi \in C(\Sigma_A^\infty; \mathbb{R})$, it is shown that there exists a spectral triple which represents the measure space $(\Sigma_A^\infty, \mathcal{B}, \mu)$. Moreover, it is proved that the topology arising from Connes metric $d_{C(\Sigma_A^\infty;\mathbb{C})}$ agrees with the weak$^*$-topology on the state space of $C(\Sigma_A^\infty; \mathbb{C})$. Further, if in addition $\phi$ is non-arithmetic and $\mu_\phi$ is the unique equilibrium measure for the potential $\phi$, then it is shown that the spectral triple is $(1, +)$-summable with metric dimension equal to one, that the noncommutative volume coincides with the reciprocal of the measure theoretical entropy of $\mu_\phi$ and that the noncommutative integral coincides with the integral with respect to $\mu_\phi$.

## 4.1 A Spectral Triple for Homeomorphic Images of the Middle Third Cantor Set

Throughout this section, unless otherwise stated let $E$ denote a compact totally disconnected subset of $\mathbb{R}$ with no isolated points. We will assume, without loss of generality, that $\{0,1\} \subset E \subset [0,1]$. Recall from Section 2.1 that such a set can be viewed as the complement of a family $\{I_k : k \in \mathbb{N}\}$ of countably many pairwise disjoint open intervals, ignoring the two infinite connected components. Here, we assume that the complementary intervals are ordered so that their lengths are non-increasing. Further, for each $k \in \mathbb{N}$, we denote the boundary of $I_k$ by $\partial I_k := \{b_k^-, b_k^+\}$, where $b_k^-$ and $b_k^+$ respectively denote the left and right end points of $\overline{I}_k$.

Our present aim is to describe the spectral triple presented by Connes in [Con3], which represents a fractal set $E$. To this end let $A$ denote the $C^*$-algebra of complex-valued continuous functions on $E$ equipped with the supremum norm and let $H := l^2 \oplus l^2$. Here, $l^2$ denotes the Hilbert space of sequences in $\mathbb{C}$ whose sum is absolutely convergent, where the inner product is given, for all $(z_1, z_2, \dots), (w_1, w_2, \dots) \in l^2$, by

$$\langle (z_1, z_2, \dots), (w_1, w_2, \dots) \rangle := \sum_{k=1}^{\infty} z_k \overline{w}_k.$$

Let $(\pi, H)$ denote the $*$-representation of $A$, where $\pi : A \to B(H)$ is defined by

$$\pi(a)\left((x_k, y_k)_{k\in\mathbb{N}}\right) := \left(x_k \cdot a(b_k^-), y_k \cdot a(b_k^+)\right)_{k\in\mathbb{N}}. \tag{4.1}$$

Since the set $\{b_k^+, b_k^- : k \in \mathbb{N}\}$ is dense in $E$, it follows that the $*$-representation $(\pi, H)$ of $A$ is faithful. Let $D$ denote the operator on $H$ with domain

$$\mathrm{Dom}(D) := \left\{ (x_k, y_k)_{k\in\mathbb{N}} \in H : \sum_{k\in\mathbb{N}} |I_k|^{-2}\left(x_k^2 + y_k^2\right) < \infty \right\} \subset H,$$

and defined, for each $(x_k, y_k)_{k\in\mathbb{N}} \in \mathrm{Dom}(D)$, by

$$D\left((x_k, y_k)_{k\in\mathbb{N}}\right) := \left(|I_k|^{-1}(y_k, x_k)\right)_{k\in\mathbb{N}}. \tag{4.2}$$

Note that

$$\mathrm{Dom}(D) \supset \left\{ (x_k, y_k)_{k\in\mathbb{N}} \in H : \text{there exist at most finitely many } k \in \mathbb{N} \text{ with } (x_k,\, y_k) \neq (0,\, 0) \right\},$$

which implies that $D$ is a densely defined operator on $H$. Further, since the lengths of the complementary intervals are decreasing, we have that $D$ is a well defined unbounded operator. For the remainder of this section, let $A$, $H$ and $D$ be fixed as above.

**Proposition 4.1.1.** *The operator $D$ is an essentially self-adjoint operator with a compact resolvent.*

Proof. It is clear that $D$ is a symmetric operator. From Theorem 3.3.2 and the fact that

$$\mathrm{Ran}(D \pm i\mathbb{1}) \supset \{ (x_k, y_k)_{k\in\mathbb{N}} : \text{there exist at most finitely many } k \in \mathbb{N} \text{ with } (x_k, y_k) \neq (0,\, 0) \},$$

it follows that $D$ is an essentially self-adjoint operator. Moreover, as for each $(x_k, y_k)_{k\in\mathbb{N}} \in H$ we have that

$$D^{-1}\left((x_k, y_k)_{k\in\mathbb{N}}\right) = \left(|I_k|\,(y_k, x_k)\right)_{k\in\mathbb{N}},$$

one can construct a sequence of operators with finite dimensional range which converges to $D^{-1}$. This implies that $D$ has a compact resolvent. ■

**Proposition 4.1.2.** *The triple $(A, H, D)$ is an even spectral triple with the action of $A$ on $H$ given by the $*$-representation $(\pi, H)$ and grading operator $\Gamma \in B(H)$ given, for each $(x_k, y_k)_{k\in\mathbb{N}} \in H$, by*

$$\Gamma\left((x_k, y_k)_{k\in\mathbb{N}}\right) := (x_k, -y_k)_{k\in\mathbb{N}}.$$

Proof. From Proposition 4.1.1, we have that $D$ is an unbounded essentially self-adjoint operator with a compact resolvent. Next, observe that for each $(x_k, y_k)_{k\in\mathbb{N}} \in H$ and $a \in A$, the following hold.

1. $D\Gamma\left((x_k, y_k)_{k\in\mathbb{N}}\right) = (|I_k|^{-1}(-y_k, x_k))_{k\in\mathbb{N}} = -(|I_k|^{-1}(y_k, -x_k))_{k\in\mathbb{N}} = -\Gamma D\left((x_k, y_k)_{k\in\mathbb{N}}\right)$.
2. $\Gamma\pi(a)\left((x_k, y_k)_{k\in\mathbb{N}}\right) = (x_k a(b_k^-), -y_k a(b_k^+))_{k\in\mathbb{N}} = \pi(a)\Gamma\left((x_k, y_k)_{k\in\mathbb{N}}\right)$.

Therefore, all that remains, is to show that the set

$$\{a \in \mathcal{A} : \text{the operator } [D, \pi(a)] \text{ is densely defined and extends to a bounded operator}\}$$

is $C^*$-norm-dense in $A$. Indeed, if $a \in C(E; \mathbb{C})$ belongs to the set $\mathrm{Lip}(E)$ of Lipschitz continuous functions on $E$, then, for each $(x_k, y_k)_{k\in\mathbb{N}} \in H$, we have that

$$\begin{aligned}
\|[D, \pi(a)](x_k, y_k)_{k\in\mathbb{N}}\| &= \|D\pi(a)(x_k, y_k)_{k\in\mathbb{N}} - \pi(a)D(x_k, y_k)_{k\in\mathbb{N}}\| \\
&= \|D(x_k \cdot a(b_k^-), y_k \cdot a(b_k^+))_{k\in\mathbb{N}} - \pi(a)(|I_k|^{-1}(y_k, x_k))_{k\in\mathbb{N}}\| \\
&= \|(|I_k|^{-1}(y_k \cdot a(b_k^+), x_k \cdot a(b_k^-)))_{k\in\mathbb{N}} - (|I_k|^{-1}(y_k \cdot a(b_k^-), x_k \cdot a(b_k^+)))_{k\in\mathbb{N}}\| \\
&= \|((a(b_k^+) - a(b_k^-))\, |I_k|^{-1}\, (y_k, -x_k))_{k\in\mathbb{N}}\| \\
&\leqslant \mathrm{Lip}(a).
\end{aligned}$$

Here, $\mathrm{Lip}(a)$ denotes the Lipschitz constant of $a$ and is defined by

$$\mathrm{Lip}(a) := \inf\{r > 0 : \|a(x) - a(y)\| < r|x - y| \text{ for all } x, y \in E\}.$$

An application of the Stone-Weierstrass Theorem (Theorem 2.2.12) then finishes the proof. ■

### 4.1.1 Geometric Properties of Connes' Spectral Triple

In this subsection, the metric dimension, the noncommutative volume, the noncommutative integral and aspects of Connes' pseudo-metric of the spectral triple $(A, H, D)$ are discussed. Although most of the results in this subection are stated in [Con3, GI1], we obtain these results by using different methods. In contrast to the results given in Theorem 4.4 of [GI1], we show that Connes' pseudo-metric $d_{C(E;\mathbb{C})}$ induced by this spectral triple, is not a metric. On personal communication with Bellissard [Bel] we learnt that the ambiguity in Theorem 4.2 of [GI1] is apparently well-known to experts in the field of noncommutative geometry. Let us begin by discussing the metric dimension.

**Theorem 4.1.3.** *Let $E$ denote a self-similar set satisfying the strong separation condition and assume, without loss of generality, that $\{0, 1\} \subset E \subset [0, 1]$. Further, let $(A, H, D)$ denote the spectral triple representation of $E$ as given in Proposition 4.1.2. Then the metric dimension of $(A, H, D)$ is equal to the Hausdorff dimension of $E$.*

Proof. Let $E$ be generated by the iterated function system of similarities $S := \{s_1, s_2, \ldots, s_m\}$, which satisfies the strong separation condition and let $p \geqslant 0$ be fixed. Denote the contraction ratios of the similarities of $S$ by $r_1, r_2, \ldots, r_m$ respectively. Applying $S$ to the unit interval $[0,1]$ gives a set consisting of $m$ connected components which are separated by $m-1$ open connected components of the complement of $S([0,1])$. We denote these separating components by $l_1, \ldots l_{m-1}$. Further, let

$$K := \sum_{i=1}^{m-1} |l_i|^p.$$

Then we have that

$$\mathrm{tr}((\mathbb{1} + D^2)^{-p/2}) = \sum_{k\in\mathbb{N}} \frac{2|I_k|^p}{(1+|I_k|^2)^{p/2}} < 2K \sum_{k\in\mathbb{N}_0} (r_1^p + r_2^p + \cdots + r_m^p)^k,$$

$$\mathrm{tr}((\mathbb{1} + D^2)^{-p/2}) = \sum_{k\in\mathbb{N}} \frac{2|I_k|^p}{(1+|I_k|^2)^{p/2}} > 2^{1-p/2} K \sum_{k\in\mathbb{N}_0} (r_1^p + r_2^p + \cdots + r_m^p)^k.$$

This shows that $\mathrm{tr}((\mathbb{1} + D^2)^{-p/2})$ is finite if and only if $r_1^p + r_2^p + \cdots + r_m^p < 1$. Hence, by the Moran-Hutchinson Formula (Theorem 2.1.8), this holds if and only if $p > \dim_{\mathcal{H}}(E)$. ∎

**Theorem 4.1.4.** *Let $(A, H, D)$ be as in Proposition 4.1.2. Suppose that for some $s > 0$, we have that $|I_k| \asymp k^{-1/s}$ as $k$ tends to positive infinity. Then the metric dimension of $(A, H, D)$ is equal to $s$ and coincides with $\dim_B(E)$.*

Proof. Let $p > 0$ be fixed. Then there exist positive constants $c_1, c_2$ such that for sufficiently large $k \in \mathbb{N}$ we have that

$$c_1 k^{-1/s} < |I_k| < c_2 k^{-1/s}.$$

It then follows, by using Hölder's inequality and Minkowski's inequality that

$$\mathrm{tr}\left(\left(\mathbb{1} + D^2\right)^{-p/2}\right) = \sum_{k\in\mathbb{N}} \frac{2|I_k|^p}{(1+|I_k|^2)^{p/2}} \leqslant \sum_{k\in\mathbb{N}} \frac{2c_2^p}{(k^{2/s}+c_1^2)^{p/2}} \leqslant \frac{2c_2^p}{\min\{1, 2^{1-p/2}\}} \sum_{k\in\mathbb{N}} k^{-p/s},$$

$$\mathrm{tr}\left(\left(\mathbb{1} + D^2\right)^{-p/2}\right) = \sum_{k\in\mathbb{N}} \frac{2|I_k|^p}{(1+|I_k|^2)^{p/2}} \geqslant \sum_{k\in\mathbb{N}} \frac{2c_1^p}{(k^{2/s}+c_2^2)^{p/2}} \geqslant \frac{c_1^p}{\max\{1, 2^{1-p/2}\}} \sum_{k=\lceil c_2\rceil}^{\infty} k^{-p/s}.$$

Hence, $\mathrm{tr}((\mathbb{1} + D^2)^{-p/2}$ is positive and finite if and only if $p > s$. Thus, $\delta(A,H,D)$ is equal to $s$. It then follows from Theorem 2.1.15 that $\delta(A,H,D)$ coincides with the box-counting dimension of $E$. ∎

The results of Theorem 4.1.3 and Theorem 4.1.4 now allow us to consider the noncommutative volume and the noncommutative integral.

**Theorem 4.1.5.** *Let $(A, H, D)$ be as in Proposition 4.1.2, and assume that $E$ is Minkowski measurable with $d$-dimensional Minkowski content $M_d(E)$, where $d := \dim_B(E)$. Then the volume $V(A,H,D)$ is equal to $2^d(1-d)M_d(E)$.*

Proof. For ease of notation let $c := 2^{d-1}(1-d)M_d(E)$. Since $E$ is Minkowski measurable, applying Theorem 2.1.15 gives that for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that, for all $k \geqslant N$, the following inequities hold

$$\frac{2c-\epsilon}{2k} \leqslant |I_k|^d \leqslant \frac{2c+\epsilon}{2k}. \tag{4.3}$$

Now let $\epsilon > 0$ be fixed and let $N \in \mathbb{N}$ be such that for all $k > N$ the inequalities in Equation (4.3) are satisfied for this $\epsilon$. Then, for each $M > 2N$, we have that

$$\sum_{k=1}^{\lfloor M/2 \rfloor} \frac{2c-\epsilon}{k} + \sum_{k=1}^{N} \left( 2|I_k|^d - \frac{2c-\epsilon}{k} \right) \leqslant \sum_{k=1}^{M} \sigma_k(|D|^{-1}) \leqslant \sum_{k=1}^{\lceil M/2 \rceil} \frac{2c+\epsilon}{k} + \sum_{k=1}^{N} \left( 2|I_k|^d - \frac{2c+\epsilon}{k} \right)$$

Therefore, it follows that

$$\liminf_{M\to\infty} \frac{\sum_{k=1}^{M} \sigma_k(|D|^{-1})}{\ln(M)} \geqslant \liminf_{M\to\infty} (2c-\epsilon) \frac{\ln(M-1)-\ln(2)}{\ln(M)} = 2c - \epsilon,$$

$$\limsup_{M\to\infty} \frac{\sum_{k=1}^{M} \sigma_k(|D|^{-1})}{\ln(M)} \leqslant \limsup_{M\to\infty} (2c+\epsilon) \frac{1+\ln(M+1)-\ln(2)}{\ln(M)} = 2c + \epsilon.$$

The result then follows by letting $\epsilon$ tend to zero. ■

In [Con3] Connes showed that a similar result can also be obtained for certian self-similar sets which are not Minkowski measurable. Namely, self-similar sets for which the contraction ratios are all equal and the covering intervals are all equally spaced. Indeed, these sets are not Minkowski measurable by Theorem 2.1.16. Further, in [GI1] Connes' result was extended to any self-similar set. In the following proposition we state this result and present an alternative proof.

**Theorem 4.1.6.** *Let $(A, H, D)$ be as in Proposition 4.1.2. Additionally, assume that $E$ is a self-similar set satisfying the strong separation condition which is generated by an iterated function system of similarities $\{s_1, s_2, \ldots, s_m\}$. Denote by $r_1, r_2, \ldots, r_m$ the associated contraction ratios of the similarities and let $l_1, l_2 \ldots, l_{m-1}$ denote the complementary intervals of $\bigcup_{i=1}^{m} s_i[0,1]$ whose lengths are finite. Further, let $\delta$ denote the Hausdorff dimension of $E$. Then we have that*

$$V(A, H, D) = \frac{2\sum_{i=1}^{m-1} |l_i|^\delta}{\sum_{i=1}^{m} r_i^\delta \ln(r_i^{-\delta})}. \tag{4.4}$$

Proof. In order to prove the equality given in Equation (4.4) we will use Proposition 2.3.3 and Proposition 2.3.4. Recall that for each $r > 0$ we let $\mathcal{G}(r)$ denote the number of complementary intervals (ignoring the two infinite complementary intervals) with length greater than or equal to $r$. Further, recall that, for $r > 0$, we set

$$\mathcal{E}(r) := \sum_{k\in\mathbb{N} \text{ with } |I_k| \geqslant r} |I_k|^\delta.$$

For ease of notation let

$$c := \frac{\sum_{i=1}^{m-1} |l_i|^\delta}{\sum_{i=1}^{m} r_i^\delta \ln(r_i^{-1})}.$$

Next fix $\epsilon > 0$ and observe that we have the following.

1. By Proposition 2.3.4, there exists $\eta_1 > 0$ such that if $0 < r \leqslant \eta_1$, then

$$(1-\epsilon)c\ln(r^{-1}) \leqslant \mathcal{E}(r) \leqslant (1+\epsilon)c\ln(r^{-1}).$$

2. By Proposition 2.3.3, there exists $\eta_2 > 0$ such that for each $0 < r \leqslant \eta_2$ the following hold.

   (a) If the set $\{\ln(r_1), \ln(r_2), \ldots, \ln(r_m)\}$ is non-arithmetic, then there exists a positive constant $c_1$, independent of $r$ and $\eta_2$, with

$$c_1(1-\epsilon)r^{-\delta} \leqslant \mathcal{G}(r) \leqslant c_1(1+\epsilon)r^{-\delta}.$$

(b) If the set $\{\ln(r_1), \ln(r_2), \ldots, \ln(r_m)\}$ is $\tau$-arithmetic, for some positive real number $\tau$, then, for each $k \in \mathbb{N}_0$ and each $y \in [0,\tau)$ such that $e^{-k\tau+y} < \eta_2$, we have that

$$(1-\epsilon)P(y)e^{\delta(k\tau-y)} \leqslant \mathcal{G}(e^{-k\tau+y}) \leqslant (1+\epsilon)P(y)e^{\delta(k\tau-y)}.$$

Here, $P$ is a positive bounded function on the interval $[0,\tau)$ which is also bounded away from zero.

Therefore, there exist constants $K_1, K_2 > 0$ such that

$$\ln(r^{-\delta}) + \ln(K_1) + \ln(1-\epsilon) \leqslant \ln(\mathcal{G}(r)) \leqslant \ln(r^{-\delta}) + \ln(K_2) + \ln(1+\epsilon),$$

and hence, we have that

$$\begin{aligned}
\limsup_{r\to 0} \frac{\mathcal{E}(r)}{\ln(\mathcal{G}(r))} &\leqslant \limsup_{r\to 0} \frac{(1+\epsilon)c\ln(r^{-1})}{\ln(r^{-\delta}) + \ln(K_1) + \ln(1-\epsilon)} \\
&= \frac{(1+\epsilon)c}{\delta} \limsup_{r\to 0} \frac{\ln(r^{-1})}{\ln(r^{-1}) + \ln((K_1(1-\epsilon))^{1/\delta})} \\
&= (1+\epsilon)\frac{c}{\delta}.
\end{aligned}$$

As the left hand side of this inequality is not dependent on $\epsilon$, we can let $\epsilon$ tend to zero, and obtain

$$\limsup_{r\to 0} \frac{\mathcal{E}(r)}{\ln(\mathcal{G}(r))} \leqslant \frac{c}{\delta} = \frac{\sum_{i=1}^{m-1} |l_i|^\delta}{\sum_{i=1}^m r_i^\delta \ln(r_i^{-\delta})}.$$

A similar argument shows that

$$\liminf_{r\to 0} \frac{\mathcal{E}(r)}{\ln(\mathcal{G}(r))} \geqslant \frac{c}{\delta} = \frac{\sum_{i=1}^{m-1} |l_i|^\delta}{\sum_{i=1}^m r_i^\delta \ln(r_i^{-\delta})}.$$

Therefore, we can conclude that

$$\lim_{r\to 0} \frac{\mathcal{E}(r)}{\ln(\mathcal{G}(r))} = \frac{\sum_{i=1}^{m-1} |l_i|^\delta}{\sum_{i=1}^m r_i^\delta \ln(r_i^{-\delta})}.$$

Thus, for all $r \in (0,1)$, it follows that

$$\lim_{k\to\infty} \frac{\mathcal{E}(r^k)}{\ln(\mathcal{G}(r^k))} = \frac{\sum_{i=1}^{m-1} |l_i|^\delta}{\sum_{i=1}^m r_i^\delta \ln(r_i^{-\delta})}.$$

Next, let $r \in (0,1)$ be fixed. Since $\sigma_k(|D|^{-\delta})$ converges to zero as $k$ tends to infinity, given $N \in \mathbb{N}$ there exists $\eta_N \in \mathbb{N}$ such that $r^{\eta_N+1} < \sigma_N(|D|^{-\delta}) \leqslant r^{\eta_N}$. Observe that as $N$ tends to infinity we have that $\eta_N$ tends to infinity. Therefore, it follows that

$$\begin{array}{ccccc} 2\mathcal{E}(r^{\eta_N - 1}) & \leqslant & \sum_{k=1}^{N} \sigma_k(|D|^{-\delta}) & \leqslant & 2\mathcal{E}(r^{\eta_N + 1}), \\ \ln(2\mathcal{G}(r^{\eta_N - 1})) & \leqslant & \ln(N) & \leqslant & \ln(2\mathcal{G}(r^{\eta_N + 1})). \end{array}$$

The result then follows since

$$\begin{array}{ccccc} \limsup_{N\to\infty} \frac{\sum_{k=1}^{N} \sigma_k(|D|^{-\delta})}{\ln(N)} & \leqslant & \limsup_{N\to\infty} \frac{2\mathcal{E}(r^{\eta_N+1})}{\ln(2\mathcal{G}(r^{\eta_N-1}))} & = & \frac{2\sum_{i=1}^{m-1}|l_i|^{\delta}}{\sum_{j=1}^{m} r_j^{\delta}\ln(r_j^{-\delta})}, \\ \liminf_{N\to\infty} \frac{\sum_{k=1}^{N} \sigma_k(|D|^{-\delta})}{\ln(N)} & \geqslant & \liminf_{N\to\infty} \frac{2\mathcal{E}(r^{\eta_N-1})}{\ln(2\mathcal{G}(r^{\eta_N+1}))} & = & \frac{2\sum_{i=1}^{m-1}|l_i|^{\delta}}{\sum_{j=1}^{m} r_j^{\delta}\ln(r_j^{-\delta})}. \end{array}$$

■

**Theorem 4.1.7.** *Assume that we are in the setting of Theorem 4.1.6 and let $\mathcal{T}$ denote a singular trace whose domain contains the operator $|D|^{-\delta}$. Then there exists a positive constant $c$ such that for all $a \in A$, the following equality holds*

$$\mathcal{T}(\pi(a)|D|^{-\delta}) \;=\; c \cdot \int_E a \, d\mu_{\mathcal{H}^\delta(E)}. \tag{4.5}$$

*Here, $\mu_{\mathcal{H}^\delta(E)}$ denotes the normalised $\delta$-dimensional Hausdorff measure on $E$.*

Proof. Let $\mathcal{T}$ denote a singular trace defined on an ideal $I$ of $B(H)$ such that $|D|^{-\delta} \in I \subset K(H)$ and fix an iterated function system of similarities $\{s_1, s_2, \ldots, s_m\}$ with unique invariant non-empty set $E$. For $k \in \mathbb{N}$, let $\Sigma^k := \{1, 2, \ldots, m\}^k$ and let $\Sigma^* := \bigcup_{k\in\mathbb{N}} \Sigma^k$. For each $k, l \in \mathbb{N}$, $i := (i_1, i_2, \ldots, i_k) \in \Sigma^k$, we have that

$$\begin{aligned} \sigma_l(\pi(\chi_{s_{i_1}\cdots s_{i_k}(0,1)\cap E})|D|^{-\delta}) &= \sigma_l(r_{i_1}^{\delta}\pi(\chi_{s_{i_2}\cdots s_{i_k}(0,1)\cap E})|D|^{-\delta}) \\ &= r_{i_1}^{\delta}\sigma_l(\pi(\chi_{s_{i_1}\cdots s_{i_k}(0,1)\cap E})|D|^{-\delta}). \end{aligned}$$

(Recall that for a subset $F$ of $E$, we let $\chi_F$ denote the characteristic function on $F$.) Therefore, since $\mathcal{T}$ is a singular trace, it follows that

$$\begin{aligned} \mathcal{T}(\pi(\chi_{s_{i_1}\cdots s_{i_k}(E)})\,|D|^{-\delta}) &= r_{i_1}^{\delta}\ldots r_{i_k}^{\delta}\mathcal{T}(|D|^{-\delta}) \\ &= \mu_{\mathcal{H}^\delta(E)}(s_{i_1}\ldots s_{i_k}(E))\mathcal{T}(|D|^{-\delta}). \end{aligned}$$

Moreover, since $\mathcal{T}(|D|^{-\delta})$ is finite and since the domain of $\mathcal{T}$ is an ideal of $B(H)$ to which the operator $|D|^{-\delta}$ belongs, it is immediate that the map defined, for all $a \in A$, by

$$a \;\mapsto\; \mathcal{T}(\pi(a)|D|^{-\delta}),$$

is a bounded linear functional. Hence, by the Riesz Representation Theorem (Theorem 3.3.1), we have that there exists a unique finite Borel measure $\mu$ such that, for each $a \in A$, the following equality holds

$$\mathcal{T}(\pi(a)|D|^{-\delta}) \;=\; \int_E a \, d\mu.$$

Next, observe that the set

$$R \;:=\; \{s_{i_1}s_{i_2}\dots s_{i_k}(E) \,:\, k\in\mathbb{N} \text{ and } i_1,i_2,\dots,i_k\in\Sigma^*\},$$

forms a semi-ring on which the set function $\Lambda : R\to[0,\infty)$ given, for each $I\in R$, by

$$\Lambda(I) \;:=\; \mathcal{T}(\chi_I|D|^{-\delta}),$$

defines an additive $\sigma$-additive set function. Therefore, since $\Lambda$ is also $\sigma$-finite by the Hahn-Kolmogorov Theorem (Theorem 3.3.5), we have that

$$\mathcal{T}(\pi(a)|D|^{-\delta}) \;=\; \mathcal{T}(|D|^{-1})\int_E a\,d\mu_{\mathcal{H}^\delta(E)}.$$

■

**Corollary 4.1.8.** *Assume we are in the setting of Theorem 4.1.6. Then we have that*

$$⨍\,\pi(a)|D|^{-\delta} \;=\; \frac{2\sum_{i=1}^{m-1}|l_i|^\delta}{-\sum_{j=1}^m r_j^\delta\ln(r_j^\delta)}\int_E a\,d\mu_{\mathcal{H}^\delta(E)}.$$

*Additionally, if $E$ is Minkowski measurable, then*

$$⨍\,\pi(a)|D|^{-\delta} \;=\; 2^\delta(1-\delta)M_\delta(E)\int_E a\,d\mu_{\mathcal{H}^\delta(E)}.$$

*In particular, we have that*

$$M_\delta(E) \;=\; \frac{2^{1-\delta}\sum_{i=1}^{m-1}|l_i|^\delta}{(\delta-1)\sum_{j=1}^m r_j^\delta\ln(r_j^\delta)}.$$

Proof. The results follow from Theorems 4.1.5, 4.1.6 and 4.1.7. ■

Let us now consider the metric aspects of the spectral triple $(A,H,D)$ given in Proposition 4.1.2. By considering the charicteristic function $\chi_E\in A$, one has that

$$|x-y| \;\leqslant\; \sup\{|a(x)-a(y)| \,:\, a\in A \text{ and the operator } [D,a] \text{ extends to a bounded operator whose norm is less than or equal to one}\}.$$

Observe that equality does not necessarily hold in the above equation. This follows, since there can exist a non-constant function $a\in A$ such that $\|[D,\pi(a)]\|=0$. For instance, consider the middle third Cantor set $C_{1/3}\subset[0,1]$ and let $\delta:=\dim_{\mathcal{H}}(C_{1/3})=\ln(2)/\ln(3)$. Further, let $\Theta : C_{1/3}\to\mathbb{R}$ denote the continuous function whose graph is the *Devil's staircase* of the middle third Cantor set, that is, for each $x\in C_{1/3}$, we define

$$\Theta(x) \;:=\; \mu_{\mathcal{H}^\delta(C_{1/3})}([0,x]\cap C_{1/3}). \tag{4.6}$$

Then, for each $h\in H$ and each $k\in\mathbb{N}$, we have that

$$[D,\pi(k\Theta)]h \;=\; 0. \tag{4.7}$$

Hence, Connes' pseudo-metric $d_A : \mathcal{S}(A)\times\mathcal{S}(A)\to[0,\infty]$ is unbounded (see Definition 3.2.5).

Indeed, for all $k \in \mathbb{N}$ and $x, y \in C_{1/3}$, we have that

$$d_A(\delta_x, \delta_y) \geqslant k \, \mu_{\mathcal{H}^\delta(C_{1/3})}([x, y] \cap C_{1/3}).$$

This observation is also reflected in the fact that the set $\{k\Theta : k \in \mathbb{N}\}$ is a subset of

$$\{a \in A : \text{the operator } [D, \pi(a)] \text{ is densely defined and extends to a bounded operator}\}$$

which does not have a bounded image in the quotient normed space $A/\{z\chi_E : z \in \mathbb{C}\}$ (see Theorem 3.2.6). Note that this result is in contrast to Section 4.2 of [GI1] as remarked at the beginning of this subsection.

Let us now return to the general case, where $\{0, 1\} \subset E \subset [0, 1]$ is a compact totally disconnected set and let $\mathrm{Lip}_d(E)$ denote the dense $*$-sub-algebra of $A := C(E; \mathbb{C})$ consisting of complex-valued Lipschitz continuous functions with respect to the metric $d$ induced by the Euclidean distance on $E$. Then Connes' pseudo-metric $d_{\mathrm{Lip}_d(E)} : \mathcal{S}(A) \times \mathcal{S}(A) \to \mathbb{R}$ is a metric and is equivalent to the Monge-Kantorovitch metric $d_{MK}$. Indeed this follows since we have that

$$\begin{aligned}
&\{a \in \mathrm{Lip}_d(E) : \mathrm{Lip}_d(a) \leqslant 1\} \\
&\quad = \{a \in \mathrm{Lip}_d(E) : \text{the operator } [D, \pi(a)] \text{ is densely defined and extends to a bounded} \\
&\qquad\qquad \text{operator with norm less than or equal to } 1\}.
\end{aligned}$$

(Here, $\mathrm{Lip}_d(a)$ denotes the Lipschitz constant of $a \in \mathcal{A}$ with respect to $d$.) However, it is important to observe the following. There can exist a metric $\mathfrak{d} : E \times E \to [0, \infty)$, which is equivalent to the metric $d$ given by the Euclidean distance on $E$, such that Connes' pseudo-metric $d_{\mathrm{Lip}_\mathfrak{d}(E)}$ is not a metric. (Here, $\mathrm{Lip}_\mathfrak{d}(E)$ denotes the dense sub-$*$-algebra of $A := C(E; \mathbb{C})$ consisting of complex-valued Lipschitz continuous functions with respect to $\mathfrak{d}$.) For instance, consider the middle third Cantor set $C_{1/3}$ and let $\mathfrak{d} : C_{1/3} \times C_{1/3} \to [0, \infty)$ be defined, for each $x, y \in C_{1/3}$, by

$$\begin{aligned}
\mathfrak{d}(x, y) := \inf\{2^{-k} &: k \in \mathbb{N} \text{ and such that there exists a sequence } (i_1, i_2, \dots i_k) \in \{1, 2\}^k \\
&\qquad \text{with the property that } x, y \in s_{i_1} s_{i_2} \dots s_{i_k}(C_{1/3})\}.
\end{aligned}$$

(Here, $s_1, s_2$ are the similarity mappings as given in Example 2.1.9.) Then the map $\Theta$, as defined in Equation (4.6), is a Lipschitz continuous map with respect to the metric $\mathfrak{d}$. Since, for all $k \in \mathbb{N}$ and $h \in H$, we have that $[D, k\Theta]h = 0$, it follows that the pseudo-metric $d_{\mathrm{Lip}_\mathfrak{d}(C_{1/3})}$ is not a metric on $\mathcal{S}(C(C_{1/3}; \mathbb{C}))$.

*Remark. The spectral triple presented by Connes to represent a compact "fractal" set $E \subset [0, 1]$ is an "atomic" representation of $E$. With this in mind, consider the map $T_\eta : [0, 1] \to [0, 1]$ defined, for each $\eta \in (0, 1/2]$, by*

$$T_\eta(x) := \begin{cases} \frac{x}{\eta} & \text{if } 0 \leqslant x \leqslant \eta \\ \frac{x-\eta}{1-\eta} & \text{if } \eta < x \leqslant 1. \end{cases}$$

*In this case, the spectral triple given in Proposition 4.1.2 can be extended to represent the dynamical system $([0, 1], T_\eta)$. Moreover, this can be done such that the metric dimension is equal to $1$ and such that, for each $a \in C([0, 1]; \mathbb{C})$, we have that*

$$⨍ \pi(a)|D|^{-1} = \frac{1}{\ln(2)} \int_{[0,1]} a d\nu.$$

*Here, $\nu$ denotes the Borel probability measure defined as follows. Let $s_1 : [0,1] \to [0,\eta]$ and $s_2 : [0,1] \to [\eta,1]$ denote the contractions given, for each $x \in [0,1]$, by*

$$s_1(x) := \eta x, \qquad s_2(x) := (1-\eta)x + \eta.$$

*Let $R$ denote the semi-ring $\{s_{i_1}s_{i_2}\dots s_{i_k}([0,1]) : k \in \mathbb{N}_0 \text{ and } (i_1,i_2,\dots,i_k) \in \{1,2\}^k\}$ and let $\Lambda : R \to [0,\infty)$ denote the set function given, for each $k \in \mathbb{N}$ and $(i_1,i_2,\dots,i_k) \in \{1,2\}^k$, by*

$$\Lambda(s_{i_1}s_{i_2}\dots s_{i_k}([0,1])) := \frac{1}{2^k}.$$

*We then let $\nu$ denote the unique extension of $\Lambda$ to a Borel probability measure with support $[0,1]$. Indeed, the existence and uniqueness of such a measure follows from the Hahn-Kolmogorov Theorem (Theorem 3.3.5).*

### 4.1.2 Multifractal Analysis of Connes' Dirac Operator

The material contained in this subsection forms the final section of the paper by Falconer and Samuel [FS]. Our main aim is to show how certain coarse multifractal information of a measure supported on a compact "fractal" subset of $[0,1]$ satisfying a porosity condition can be rediscovered through Connes' spectral triple, as given in Proposition 4.1.2.

Recall that we let $E$ denote a strongly porous compact totally disconnected subset of $\mathbb{R}$ with no isolated points, where we assume, without loss of generality, that $\{0,1\} \subset E \subset [0,1]$. Further, recall that we let $\{I_k := (b_k^-, b_k^+) : k \in \mathbb{N}\}$ denote the set of complementary intervals of $E$ of finite length, ordered so that $|I_k| \geqslant |I_{k+1}|$.

Let us now introduce a multifractal component for the spectral triple $(A,H,D)$, which gives a representation of a non-empty compact totally disconnected subset $E$ of $\mathbb{R}$ with no isolated points, as introduced in Proposition 4.1.2. Specifically, we define an operator $\mathcal{Q}$ which encodes the multifractal behaviour of a measure whose support is equal to $E$. Fix $\eta > 0$ and fix a probability measure $\mu$ whose support is equal to $E$. Define

$$\mathcal{Q} := \mathcal{Q}_{\eta,\mu} : \bigcup_{k\in\mathbb{N}} \{b_k^-, b_k^+\} \to [0,\infty)$$

to be the non-zero positive function given, for each $k \in \mathbb{N}$, by

$$\mathcal{Q}(b_k^-) := \mathcal{Q}(b_k^+) := \mu(\overline{I}_k^\eta). \tag{4.8}$$

(Recall that $\overline{I}_k^\eta$ denotes the closed interval $[b_k^- - \eta|I_k|/2, b_k^- + \eta|I_k|/2]$.) To express the function $\mathcal{Q}$ as an operator on $H := l^2 \oplus l^2$, we let

$$\mathcal{Q} := \mathcal{Q}_{\eta,\mu} : H \to H,$$

be given, for each $((x_k,y_k))_{k\in\mathbb{N}} \in H$, by

$$\mathcal{Q}\left((x_k,y_k)_{k\in\mathbb{N}}\right) := \pi(\mathcal{Q})\left((x_k,y_k)_{k\in\mathbb{N}}\right) = (\mathcal{Q}(b_k^-)x_k, \mathcal{Q}(b_k^+)y_k)_{k\in\mathbb{N}}. \tag{4.9}$$

Recall that $(\pi, H)$ denotes the $*$-representation as given in Equation (4.1).

The next theorem (Theorem 4.1.9) is a multifractal analogue of Theorem 4.1.3. It shows that for a given $q \in \mathbb{R}$ the critical value

$$\inf\{p \in \mathbb{R} : \mathcal{Q}^q |D|^{-p} \in \mathcal{L}^{1,+}(H)\}$$

reflects the behaviour of the multifractal moment sums of $E$, given that $E$ is strongly porous and that $\mu$ satisfies the mild density condition given in Equation (4.10). Moreover, from this result, in Corollary 4.1.10, we show that the noncommutative integral gives rise to a non-degenerate integral with respect to the underlying measure.

**Theorem 4.1.9.** *Let $E$ denote a non-empty compact totally disconnected strongly porous subset of $[0,1]$ with no isolated points and porosity constant $\rho$. Suppose that there exists a probability measure $\mu$ whose support is equal to $E$ and where, for each $x \in E$, we have that as $r$ tends to zero*

$$\frac{\ln(\mu(B(x,r)))}{\ln(r)} \asymp 1, \tag{4.10}$$

*uniformly in $x$. Then, for each $q \in \mathbb{R}$ and each $\eta \geqslant 2\rho^{-1}$, setting $\mathcal{Q} := \mathcal{Q}_{\eta,\mu}$, we have that*

$$\mathfrak{b}(q) = \inf\{p \in \mathbb{R} : \mathcal{Q}^q |D|^{-p} \in \mathcal{L}^{1,+}(H)\}. \tag{4.11}$$

*Here, $\mathfrak{b}$ denotes the coarse multifractal box-counting dimension as introduced in Equation (2.9).*

Proof. By Theorem 2.1.12 and Equation (4.10), we have that as $k$ tends to infinity

$$\frac{\ln(\mu(\overline{I}_k^\eta))}{-\ln(k)} \asymp 1. \tag{4.12}$$

Indeed, this follows since, for each $k \in \mathbb{N}$, we have that

$$(B(b_k^-, \eta|I_k|/2) \cup B(b_k^+, \eta|I_k|/2)) \cap E = \overline{I}_k^\eta \cap E.$$

For ease of notation, let $\mathfrak{N}(r) : (0,\infty) \to \mathscr{P}(\mathbb{N})$ be defined, for each $r \in (0,\infty)$, by

$$\mathfrak{N}(r) := \{k \in \mathbb{N} : \mu(\overline{I}_k^\eta)^q |I_k|^p \geqslant r\}.$$

(Here, $\mathscr{P}(\mathbb{N})$ denotes the power set of the set of natural numbers.) Observe that by Theorem 2.1.12 and Equation(4.12), we have the following.

1. There exist positive constants $t_1, t_2$ such that, for each $r > 0$ sufficiently small, we have that

$$\{k \in \mathbb{N} : |I_k| \geqslant r^{t_1}\} \subseteq \mathfrak{N}(r) \subseteq \{k \in \mathbb{N} : |I_k| \geqslant r^{t_2}\}, \tag{4.13}$$

and so,

$$\sum_{|I_k| \geqslant r^{t_1}} \mu(\overline{I}_k^\eta)^q |I_k|^p \leqslant \sum_{k \in \mathfrak{N}(r)} \mu(\overline{I}_k^\eta)^q |I_k|^p \leqslant \sum_{|I_k| \geqslant r^{t_2}} \mu(\overline{I}_k^\eta)^q |I_k|^p.$$

2. Since, as $k$ tends to positive infinity, we have that $\ln(|I_k|) \asymp -\ln(k)$, it follows that, as $r$ tends to zero

$$-\ln(r) \asymp \ln(\mathrm{card}\{k \in \mathbb{N} : |I_k| \geqslant r\}) \asymp \ln(\mathrm{card}(\mathfrak{N}(r)))$$

We now aim to show that

$$\mathfrak{b}(q) \;\geqslant\; \inf\left\{ p \in \mathbb{R} \,:\, \limsup_{N\to\infty} \frac{\sum_{k=1}^{N} \sigma_k(\mathcal{Q}^q|D|^{-p})}{\ln(N)} < \infty \right\}.$$

Observe that each element of the set $\{\mu(\overline{I}_k^{\eta})^q|I_k|^p \,:\, k \in \mathbb{N}\}$ is a singular value of the operator $\mathcal{Q}^q|D|^{-p}$ with multiplicity two and that by Theorem 2.1.20, we have that

$$\mathfrak{b}(q) \;=\; \inf\left\{ t \in \mathbb{R} \,:\, \limsup_{N\to\infty} \frac{\sum_{k=1}^{N} \mu(\overline{I}_k^{\eta})^q|I_k|^t}{\ln(N)} = 0 \right\}.$$

Fix $q \in \mathbb{R}$ and let $p \in \mathbb{R}$ be such that

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^{N} \mu(\overline{I}_k^{\eta})^q|I_k|^p}{\ln(N)} \;=\; 0.$$

By definition the sequence $(\sigma_k(\mathcal{Q}^q|D|^{-p}))_{k\in\mathbb{N}}$ forms a decreasing sequence which converges to zero. Also, by Theorem 2.1.12 and Equation (4.12), to each sufficiently large $N \in \mathbb{N}$ there exists an $\widetilde{N} \in \mathbb{N}$ such that $|I_{\widetilde{N}}| \geqslant \sigma_N(\mathcal{Q}^q|D|^{-p})^{t_2}$ and such that $\ln(\widetilde{N}) \asymp -\ln(\sigma_N(\mathcal{Q}^q|D|^{-p}))$, as $N$ tends to infinity. Hence, there exist positive constants $c_1, c_2, c_3$ so that for sufficiently large $N \in \mathbb{N}$, we have that

$$\begin{aligned}
\frac{\sum_{k=1}^{N} \sigma_k(\mathcal{Q}^q|D|^{-p})}{\ln(N)} \;&\leqslant\; \frac{2\sum_{k\in\mathfrak{N}(\sigma_N(\mathcal{Q}^q|D|^{-p}))} \mu(\overline{I}_k^{\eta})^q|I_k|^p}{\ln(\mathrm{card}(\mathfrak{N}(\frac{\sigma_N(\mathcal{Q}^q|D|^{-p})}{1-\sigma_N(\mathcal{Q}^q|D|^{-p})})))} \\
&\leqslant\; c_1 \frac{\sum_{|I_k|\geqslant\sigma_N(\mathcal{Q}^q|D|^{-p})^{t_2}} \mu(\overline{I}_k^{\eta})^q|I_k|^p}{\ln(\mathrm{card}\{k \in \mathbb{N} \,:\, |I_k| \geqslant \frac{\sigma_N(\mathcal{Q}^q|D|^{-p})}{1-\sigma_N(\mathcal{Q}^q|D|^{-p})}\})} \\
&\leqslant\; c_2 \frac{\sum_{k=1}^{\widetilde{N}} \mu(\overline{I}_k^{\eta})^q|I_k|^p}{-\ln(\sigma_N(\mathcal{Q}^q|D|^{-p}))} \\
&\leqslant\; c_3 \frac{\sum_{k=1}^{\widetilde{N}} \mu(\overline{I}_k^{\eta})^q|I_k|^p}{\ln(\widetilde{N})}.
\end{aligned}$$

This then implies that

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^{N} \sigma_k(\mathcal{Q}^q|D|^{-p})}{\ln(N)} \;=\; 0.$$

Therefore, we have that

$$\begin{aligned}
\mathfrak{b}(q) \;&=\; \inf\left\{ t \in \mathbb{R} \,:\, \limsup_{N\to\infty} \frac{\sum_{k=1}^{N} \mu(\overline{I}_k^{\eta})^q|I_k|^t}{\ln(N)} = 0 \right\} \\
&\geqslant\; \inf\left\{ p \in \mathbb{R} \,:\, \limsup_{N\to\infty} \frac{\sum_{k=1}^{N} \sigma_k(\mathcal{Q}^q|D|^{-p})}{\ln(N)} < \infty \right\}.
\end{aligned}$$

Next, we aim to show that

$$\mathfrak{b}(q) \;\leqslant\; \inf\left\{ p \in \mathbb{R} \,:\, \limsup_{N\to\infty} \frac{\sum_{k=1}^{N} \sigma_k(\mathcal{Q}^q|D|^{-p})}{\ln(N)} < \infty \right\}.$$

To this end, fix $q \in \mathbb{R}$ and let $p \in \mathbb{R}$ be such that

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^{N} \sigma_k(\mathcal{Q}^q|D|^{-p})}{\ln(N)} \;<\; \infty.$$

By Theorem 2.1.12, we have that

$$\mathfrak{b}(q) \;=\; \inf\left\{ t \in \mathbb{R} \,:\, \limsup_{N\to\infty} \frac{\sum_{k=1}^{N} \mu(\overline{I}_k^{\eta})^q |I_k|^t}{\ln(N)} < \infty \right\}.$$

Further, by Theorem 2.1.12 and Equation (4.12), for each sufficiently large $N \in \mathbb{N}$, there exists $r_{N} > 0$ such that $|I_N| \geqslant r_N^{t_1}$ and such that $\ln(N) \asymp -\ln(r_{N})$, as $N$ tends to infinity. Hence, there exist positive constants $c_4, c_5, c_6$ so that for sufficiently large $N \in \mathbb{N}$, we have that

$$\begin{aligned} \frac{\sum_{k=1}^{N} \mu(\overline{I}_k^{\eta})^q |I_k|^p}{\ln(N)} &\leqslant c_4 \frac{\sum_{|I_k|\geqslant r_N^{t_1}} \mu(\overline{I}_k^{\eta})^q |I_k|^p}{\ln(r_{N})} \\ &\leqslant c_5 \frac{\sum_{k\in\mathfrak{N}(r_N)} \mu(\overline{I}_k^{\eta})^q |I_k|^p}{\ln(\mathrm{card}\{k \in \mathbb{N} \,:\, |I_k| \geqslant r_{N}\})} \\ &\leqslant c_6 \frac{\sum_{k\in\mathfrak{N}(r_N)} \mu(\overline{I}_k^{\eta})^q |I_k|^p}{\ln(\mathrm{card}(\mathfrak{N}(r_{N})))} \end{aligned}$$

Therefore, we have that

$$\begin{aligned} \mathfrak{b}(q) &= \inf\left\{ t \in \mathbb{R} \,:\, \limsup_{N\to\infty} \frac{\sum_{k=1}^{N} \mu(\overline{I}_k^{\eta})^q |I_k|^t}{\ln(N)} < \infty \right\} \\ &\leqslant \inf\left\{ p \in \mathbb{R} \,:\, \limsup_{N\to\infty} \frac{\sum_{k=1}^{N} \sigma_k(\mathcal{Q}^q|D|^{-p})}{\ln(N)} < \infty \right\}. \end{aligned}$$

This then completes the proof. ■

**Corollary 4.1.10.** *Assume that we are in the setting of Theorem 4.1.9 and let $q \in \mathbb{R}$. If for some $p \in \mathbb{R}$ and some $\eta > 2\rho^{-1}$, the measure $\mu$ satisfies the condition*

$$\sum_{|I_k|\geqslant r} \mu(\overline{I}_k^{\eta})^q |I_k|^p \;\asymp\; -\ln(r), \tag{4.14}$$

*as $r$ tends to zero, then $p = \mathfrak{b}(q)$. Moreover, as $N$ tends to infinity, we have that*

$$\frac{\sum_{k=1}^{N} \sigma_k(\mathcal{Q}^q|D|^{-p})}{\ln(N)} \;\asymp\; 1. \tag{4.15}$$

*In particular, for any limiting procedure $\mathcal{W}$, we have, for each $a \in A := C(E;\mathbb{C})$, that the linear functional*

$$a \;\mapsto\; Tr_{\mathcal{W}}\left(\pi(a)\mathcal{Q}^q|D|^{-p}\right) \tag{4.16}$$

*gives rise to a non-trivial integral.*

Proof. The result follows from Theorem 2.1.12, the Riesz Representation Theorem (Theorem 3.3.1) and Theorem 4.1.9. ■

*Remark.* *By Lemma 2.1.19, if for some $q \in \mathbb{R}$ and some $\eta > \rho/2$ a measure $\mu$ satisfies the moment condition*

$$\sum_{B \in \mathfrak{B}_r^*(\mu)} \mu(B^\eta)^q r^{\mathfrak{b}(q)} \asymp 1, \tag{4.17}$$

*then the condition given in Equation (4.14) is satisfied.*

In the following theorem (Theorem 4.1.11) we specialise to the case where $E$ is a self-similar set satisfying the strong separation condition and where $\mu$ is a self-similar measure. The result is a multifractal analogue of Theorem 4.1.7 and Corollary 4.1.8. Indeed, by setting $q = 0$, one recovers the results of Theorem 4.1.7 and Corollary 4.1.8.

**Theorem 4.1.11.** *Let $E \subset [0,1]$ denote the unique non-empty compact invariant subset of an iterated function system of similarities $S := \{s_i : [0,1] \to [0,1] \,:\, i \in \{1,2,\dots,m\}\}$ with associated contraction ratios $r_1, r_2, \dots r_m$. Further, assume that $S$ satisfies the strong separation condition (Definition 2.1.7) and let $\mu$ denote the associated self-similar probability measure given by the probability vector $p := (p_1, p_2, \dots, p_m)$. Then, for each $\eta > 0$, each $q \in \mathbb{R}$ and each $a \in A := C(E; \mathbb{C})$, the operator*

$$\pi(a)\mathcal{Q}^q_{\eta,\mu}|D|^{-\mathfrak{b}(q)}$$

*is a measurable operator. Moreover, if $\nu$ is the unique probability measure on $E$ given, for each Borel set $B \in \mathcal{B}$, by*

$$\nu(B) \;:=\; \sum_{i=1}^{m} p_i^q r_i^{\mathfrak{b}(q)} \nu(s_i^{-1}(B)),$$

*then, for each $a \in A$, we have that*

$$⨍ \pi(a)\mathcal{Q}^q_{\eta,\mu}|D|^{-\mathfrak{b}(q)} \;=\; 2R_1 {h_\nu}^{-1} \int_E a \, d\nu. \tag{4.18}$$

*Here,*

$$R_1 \;:=\; \sum_{i=1}^{m} \sum_{I_k \subseteq s_i([0,1])} |I_k|^{\mathfrak{b}(q)} \left(\mu(\overline{I}_k^\eta)^q - p_i^q \mu(s_i^{-1}(\overline{I}_k^\eta))^q\right) + \sum_{I_k \subset [0,1] \setminus \bigcup_{i=1}^m s_i([0,1])} |I_i|^{\mathfrak{b}(q)} \mu(\overline{I}_i^\eta)^q \;<\; \infty,$$

*and*

$$h_\nu \;:=\; \sum_{i=1}^{m} p_i^q r_i^{\mathfrak{b}(q)} \ln(p_i^{-q} r_i^{-\mathfrak{b}(q)}).$$

*Remark.* *Each of the sums in the definition of $R_1$ is over a finite set, and so, $R_1$ is well defined. Further, the notation $h_\nu$ is purposefully chosen, as it is the measure theoretical entropy of the measure $\nu$ with respect to the expanding map defined on $E$ whose inverse branches are given by the similarities of $S$ (see Example 2.2.4).*

Proof. Let $\mathscr{E} : (0,\infty) \to [0,\infty)$ denote the function given, for each $r > 0$, by

$$\mathscr{E}(r) \;:=\; \sum_{\mu(\overline{I}_k^\eta)^q |I_k|^{\mathfrak{b}(q)} \geqslant r} \mu(\overline{I}_k^\eta)^q |I_k|^{\mathfrak{b}(q)}$$

and let $\mathcal{R}_1 : (0, \infty) \to [0, \infty)$ be given, for each $r > 0$, by

$$\mathcal{R}_1(r) \;:=\; \sum_{i=1}^{m} \sum_{I_k \subseteq s_i[0,1], |I_k| \geqslant r} |I_k|^{\mathfrak{b}(q)} \left( \mu(\overline{I}_k^{\eta})^q - p_i^q \mu(s_i^{-1}(\overline{I}_k^{\eta}))^q \right) \;+ \sum_{\substack{|I_k| \geqslant r \\ I_k \subset [0,1] \setminus \bigcup_{i=1}^m s_i([0,1])}} |I_i|^{\mathfrak{b}(q)} \mu(\overline{I}_i^{\eta})^q.$$

Note that, for each $r > 0$, the sums in the definition of $\mathcal{R}_1(r)$ are over a finite set, and so, $\mathcal{R}_1(r)$ is well defined. Since $\mu$ is a self-similar measure, for $r > 0$, we obtain the following scaling relation

$$\mathscr{E}(r) \;=\; \sum_{i=1}^{m} p_i^q r_i^{\mathfrak{b}(q)} \mathscr{E}\left( r p_i^{-q} r_i^{-\mathfrak{b}(q)} \right) + \mathcal{R}_1(r).$$

By applying the transformations $t = -\ln(r)$ and $\psi_1(t) = \mathscr{E}(e^{-t})$, we obtain that

$$\psi_1(t) \;=\; \sum_{i=1}^{m} p_i^q r_i^{\mathfrak{b}(q)} \psi_1\left( t - \ln\left( p_i^{-q} r_i^{-\mathfrak{b}(q)} \right) \right) + \mathcal{R}_1(e^{-t}). \tag{4.19}$$

Although this is a renewal equation (see Equation (2.23)), since $\mathcal{R}_1(e^{-t})$ is equal to $R_1 > 0$ for $t$ sufficiently large, Equation (4.19) is not in a form which allows for the application of the renewal theorem, as given in Theorem 2.3.2. Therefore, with the aim of applying Theorem 2.3.2, the following definitions and substitutions are made. Let $c := R_1 {h_\nu}^{-1}$ and let $\phi : \mathbb{R} \to \mathbb{R}$ be defined, for each $t \in \mathbb{R}$, by

$$\phi(t) \quad := \quad \begin{cases} \psi_1(t) - ct & \text{if } t > 0, \\ 0 & \text{if } t \leqslant 0. \end{cases}$$

Further, let $g : \mathbb{R} \to \mathbb{R}$ be defined, for each $t \in \mathbb{R}$, by

$$g_1(t) \;:=\; \begin{cases} \mathcal{R}_1(e^{-t}) - c \sum_{i=1}^{m} p_i^q r_i^{\mathfrak{b}(q)} \ln(p_i^{-q} r_i^{-\mathfrak{b}(q)}) & \\ \qquad - c \sum_{i=1}^{m} p_i^q r_i^{\mathfrak{b}(q)} (t - \ln(p_i^{-q} r_i^{-\mathfrak{b}(q)})) \chi_{(-\infty,0]}(t - \ln(p_i^{-q} r_i^{-\mathfrak{b}(q)})) & \\ \qquad + \sum_{i=1}^{m} p_i^q r_i^{\mathfrak{b}(q)} \psi_1(t - \ln(p_i^{-q} r_i^{-\mathfrak{b}(q)})) \chi_{(-\infty,0]}(t - \ln(p_i^{-q} r_i^{-\mathfrak{b}(q)})) & \text{if } t > 0, \\ 0 & \text{if } t \leqslant 0. \end{cases}$$

Let us now show that $\phi$ and $g_1$ satisfy the renewal equation (Equation (2.23)). First let us consider the case $t \leqslant 0$. Since in this case we have that $g_1(t) = 0$, that $\phi(t) = 0$ and that $t - \ln(p_i^{-q} r_i^{-\mathfrak{b}(q)}) \leqslant 0$, for $i \in \{1, 2, \ldots, m\}$, it follows that

$$\sum_{i=1}^{m} p_i^q r_i^{\mathfrak{b}(q)} \phi(t - \ln(p_i^{-q} r_i^{-\mathfrak{b}(q)}) + g_1(t) \;=\; 0 \;=\; \phi(t).$$

Now, let us consider the case $t > 0$. In this case we have that

$$\phi(t) \;=\; \psi_1(t) - ct \;=\; \sum_{i=1}^{m} p_i^q r_i^{\mathfrak{b}(q)} \psi_1(t - \ln(p_i^{-q} r_i^{-\mathfrak{b}(q)})) + \mathcal{R}_1(e^{-t}) - ct =$$

$$\begin{aligned}
&= \sum_{i=1}^{m} p_i^q r_i^{\mathfrak{b}(q)} \phi(t - \ln(p_i^{-q} r_i^{-\mathfrak{b}(q)})) + \mathcal{R}_1(e^{-t}) - c \sum_{i=1}^{m} p_i^q r_i^{\mathfrak{b}(q)} \ln(p_i^{-q} r_i^{-\mathfrak{b}(q)}) \\
&\quad - c \sum_{i=1}^{m} p_i^q r_i^{\mathfrak{b}(q)} (t - \ln(p_i^{-q} r_i^{-\mathfrak{b}(q)})) \chi_{(-\infty,0]}(t - \ln(p_i^{-q} r_i^{-\mathfrak{b}(q)})) \\
&\quad + \sum_{i=1}^{m} p_i^q r_i^{\mathfrak{b}(q)} \psi_1(t - \ln(p_i^{-q} r_i^{-\mathfrak{b}(q)})) \chi_{(-\infty,0]}(t - \ln(p_i^{-q} r_i^{-\mathfrak{b}(q)})) \\
&= \sum_{i=1}^{m} p_i^q r_i^{\mathfrak{b}(q)} \phi(t - \ln(p_i^{-q} r_i^{-\mathfrak{b}(q)})) + g_1(t).
\end{aligned}$$

Moreover, it is clear from the definition of $g_1$ that it has a discrete set of discontinuities and that $g_1(t) = 0$ for $t$ sufficiently large. It is easy to verify that $\phi$ is Borel measurable, that $\phi$ is bounded on the half-line $(-\infty, t)$, for each $t \in \mathbb{R}$, and that $\lim_{t \to -\infty} \phi(t) = 0$. Therefore, Theorem 2.3.2 can be applied and we conclude that

$$\lim_{r \to 0} \frac{\mathscr{E}(r)}{-\ln(r)} = \lim_{t \to \infty} \frac{\psi_1(t)}{t} = \lim_{t \to \infty} \frac{\phi(t) + ct}{t} = c. \tag{4.20}$$

Next, let us consider the function $\mathscr{G}(r) : (0, \infty) \to [0, \infty)$ given, for each $r > 0$, by

$$\mathscr{G}(r) := \operatorname{card}\{k : \mu(\overline{I}_k^\eta)^q |I_k|^{\mathfrak{b}(q)} \geqslant r\}.$$

Let $\mathcal{R}_2 : (0, \infty) \to [0, \infty)$ be given, for each $r > 0$, by

$$\begin{aligned}
\mathcal{R}_2(r) \; := \; & \sum_{i=1}^{m} \operatorname{card}\left\{ k \in \mathbb{N} : I_k \subseteq s_i([0,1]) \text{ with } r_i^{\mathfrak{b}(q)} |I_k|^{\mathfrak{b}(q)} p_i^q \mu(s_i^{-1}(\overline{I}_k^\eta))^q < r \leqslant |I_k|^{\mathfrak{b}(q)} \mu(\overline{I}_k^\eta)^q \right\} \\
& + \operatorname{card}\left\{ k \in \mathbb{N} : |I_k|^{\mathfrak{b}(q)} \mu(\overline{I}_k^\eta)^q \geqslant r \text{ and } I_k \subset [0,1] \setminus \textstyle\bigcup_{i=1}^m s_i([0,1]) \right\}.
\end{aligned}$$

Observe that $\mathcal{R}_2(r)$ is decreasing with $\mathcal{R}_2(r)$ equal to zero for $r$ sufficiently large, and that there exists a positive constant $R_2 > 0$ such that $\mathcal{R}_2(r) = R_2$, for sufficiently small $r$. Further, since $\mu$ is a self-similar measure, for $r > 0$, we obtain the following scaling relation

$$\mathscr{G}(r) = \sum_{i=1}^{m} \mathscr{G}\left( r p_i^{-q} r_i^{-\mathfrak{b}(q)} \right) + \mathcal{R}_2(r). \tag{4.21}$$

As above, we apply the following transformations. For $r > 0$, let $t := -\ln(r)$ and let $\psi_2 : \mathbb{R} \to \mathbb{R}$, be given, for each $t \in \mathbb{R}$, by

$$\psi_2(t) := e^{-t} \mathscr{G}(e^{-t}).$$

Further, let $g_2 : \mathbb{R} \to \mathbb{R}$ be given, for each $t \in \mathbb{R}$, by

$$g_2(t) := e^{-t} \mathcal{R}_2(e^{-t}).$$

Applying these transformations to Equation (4.21), then gives the renewal equation

$$\psi_2(t) = \sum_{i=1}^{m} p_i^q r_i^{\mathfrak{b}(q)} \psi_2\left( t - \ln\left( p_i^{-q} r_i^{-\mathfrak{b}(q)} \right) \right) + g_2(t).$$

Next, note that the following hold.

1. By the definition of $\mathcal{R}_2$, we have that $g_2$ has a discrete set of discontinuities.

2. For sufficiently large $t \in \mathbb{R}$, we have that $g_2(t) \leqslant R_2 e^{-t}$.
3. The function $\psi_2$ is Borel measurable and bounded on the half-line $(-\infty, t)$, for each $t \in \mathbb{R}$.
4. By the definition of $\psi_2$, we have that $\psi_2(t)$ tends to zero as $t$ tends to minus infinity.

Therefore, we can apply Theorem 2.3.2, and so, we obtain that $\psi_2(t) \asymp 1$ as $t \to \infty$, or, more precisely, we have $\psi_2(t)$ either converges to a constant or is asymptotic to a periodic function which is bounded from above and bounded away from zero. Hence, it follows that

$$\lim_{r\to 0} \frac{\ln(\mathrm{card}\{k : \mu(\overline{I}_k^{\eta})^q |I_k|^{\mathfrak{b}(q)} \geqslant r\})}{-\ln(r)} = \lim_{r\to 0} \frac{\ln(\mathscr{G}(r))}{-\ln(r)} = 1. \tag{4.22}$$

By combining Equation (4.22) and Equation (4.20), we can conclude that

$$\lim_{N\to\infty} \frac{\sum_{k=0}^{N-1} \sigma_k(\mathcal{Q}^q_{\eta,\mu}|D|^{-\mathfrak{b}(q)})}{\ln(N)} = \lim_{r\to 0} \frac{2\mathscr{E}(r)}{\ln(\mathrm{card}\{k : \mu(\overline{I}_k^{\eta})^q |I_k|^{\mathfrak{b}(q)} \geqslant r\})} = 2c = 2R_1 {h_\nu}^{-1}.$$

Therefore, the operator $\mathcal{Q}^q_{\eta,\mu}|D|^{-\mathfrak{b}(q)}$ is measurable and we have that

$$⨍ \mathcal{Q}^q_{\eta,\mu}|D|^{-\mathfrak{b}(q)} = 2R_1 {h_\nu}^{-1}. \tag{4.23}$$

Hence, the map defined, for each $a \in A$, by

$$a \mapsto \mathrm{Tr}_{\mathscr{W}}\left(\pi(a)\mathcal{Q}^q_{\eta,\mu}|D|^{-\mathfrak{b}(q)}\right)$$

is a bounded linear functional. The Riesz Representation Theorem (Theorem 3.3.1) and Equation (4.23) together imply that there exists a unique Borel probability measure $\mathcal{U}$ such that, for each $a \in A$, the following equality holds

$$\mathrm{Tr}_{\mathscr{W}}\left(\pi(a)\mathcal{Q}^q_{\eta,\mu}|D|^{-\mathfrak{b}(q)}\right) = 2R_1 {h_\nu}^{-1} \int_E a \, d\mathcal{U}.$$

In order to show that $\mathcal{U} = \nu$, we use a scaling argument to show that $\mathcal{U}$ and $\nu$ agree on a semi-ring which generates the Borel $\sigma$-algebra. Then by an application of the Hahn-Kolmogorov Theorem (Theorem 3.3.5) the result follows. To this end, consider $(i_1, i_2, \ldots, i_k) \in \{1, 2, \ldots, m\}^k$ and let $I := s_{i_1} s_{i_2} \ldots s_{i_k}(E)$. Then the singular values of $\pi(\chi_I)\mathcal{Q}^q_{\eta,\mu}|D|^{-\mathfrak{b}(q)}$ are precisely those corresponding to the complementary intervals contained in $I$ and those corresponding to the complementary intervals whose closure intersects the boundary of $I$. Next, note that the following hold.

1. The mapping $s_{i_1} s_{i_2} \ldots s_{i_k}$ gives a bijection between the sets $\{I_k : k \in \mathbb{N}\}$ and $\{I_k : k \in \mathbb{N} \text{ and } I_k \subset I\}$.
2. For any interval $J \subseteq [0,1]$, we have $|s_{i_1} s_{i_2} \ldots s_{i_k}(J)| = r_{i_1} r_{i_2} \ldots r_{i_k} |J|$.
3. For any interval $J \subseteq [0,1]$ of sufficiently small diameter, we have that $\mu(s_{i_1} s_{i_2} \ldots s_{i_k}(J)) = p_{i_1} p_{i_2} \ldots p_{i_k} \mu(J)$.
4. The Dixmier trace is linear and vanishes on operators with finite dimensional range.

Together, these observations then give, for each $(i_1, i_2, \ldots, i_k) \in \{1, 2, \ldots, m\}^k$, that

$$\begin{aligned} \mathrm{Tr}_{\mathscr{W}}\left(\pi(\chi_I)\mathcal{Q}^q_{\eta,\mu}|D|^{-\mathfrak{b}(q)}\right) &= (r_{i_1} r_{i_2} \ldots r_{i_k})^{\mathfrak{b}(q)} (p_{i_1} p_{i_2} \ldots p_{i_k})^q \mathrm{Tr}_{\mathscr{W}}\left(\mathcal{Q}^q_{\eta,\mu}|D|^{-\mathfrak{b}(q)}\right) \\ &= 2R_1 {h_\nu}^{-1} \nu(I). \end{aligned}$$

Therefore, since the set

$$\{s_{i_1} s_{i_2} \ldots, s_{i_k}(E) \,:\, k \in \mathbb{N} \text{ and } (i_1, i_2, \ldots, i_k) \in \{1, 2, \ldots, m\}^k\}$$

forms a semi-ring and generates the Borel $\sigma$-algebra on which $\mathcal{U}$ and $\nu$ are defined and since for each

$$I \in \{s_{i_1} s_{i_2} \ldots, s_{i_k}(E) \,:\, k \in \mathbb{N} \text{ and } (i_1, i_2, \ldots, i_k) \in \{1, 2, \ldots, m\}^k\}$$

we have that $\mathcal{U}(I) = \nu(I)$, the result follows. ∎

## 4.2 Noncommutative Geometry and Subshifts of Finite Type

In this section, the construction of a spectral triple for an AF $C^*$-algebra, given by Antonescu-Ivan and Christensen in [AIC1], is adapted to give a representation of the measure space $(\Sigma_A^\infty, \mathcal{B}, \mu_\phi)$. Here $(\Sigma_A^\infty, \sigma)$ denotes a one-sided topologically exact subshift of finite type and $\mu_\phi$ denotes a Gibbs measure for a Hölder continuous potential function $\phi \in C(\Sigma_A^\infty; \mathbb{R})$. Further, various geometric aspects of such a spectral triple are investigated. It is within this section that the results of Subsection 2.2.2 will play a major role. The main results of this section are Theorem 4.2.5, Theorem 4.2.6 and Theorem 4.2.7. Let us begin by giving a brief overview of relevant results which are presented in [AIC1].

### 4.2.1 A Review of Antonescu-Ivan and Christensen's Spectral Triple on AF $C^*$-Algebras

Let $A$ denote an AF $C^*$-algebra, given by the inductive limit of a sequence of finite dimensional $C^*$-algebras $\{A_k\}_{k\in\mathbb{N}_0}$, where $A_0 := \mathbb{C}$. Further, suppose that there exists a faithful state $\psi$ on $A$. Recall that $H_\psi$ denotes the Gelfand-Naĭmark-Segal completion of $A$ with respect to $\psi$ and that there exists a natural $*$-representation $(\pi, H_\psi)$ of $A$ (see Section 3.1). Further, since $\psi$ is faithful, we know that there exists a separating vector $\mathfrak{h} \in H_\psi$ with norm equal to one. Therefore, the map $\Psi : A \to H_\psi$ defined, for each $a \in A$, by

$$\Psi(a) \;:=\; \pi(a)\mathfrak{h},$$

induces a bijective linear homomorphism of the finite dimensional algebra $A_k$ onto the finite dimensional subspace $H_{\psi,k} := \Psi(A_k)$ of $H_\psi$, for each $k \in \mathbb{N}$. Next, for each $k \in \mathbb{N}$, fix a projection $P_k$ mapping $H_\psi$ onto $H_{\psi,k}$. In the following theorem, it is shown that there exists a sequence of positive real numbers $(\alpha_k)_{k\in\mathbb{N}}$ such that the operator $D_{(\alpha_k)_{k\in\mathbb{N}}}$, defined by

$$D_{(\alpha_k)_{k\in\mathbb{N}}} \;:=\; \sum_{k\in\mathbb{N}} \alpha_k (P_k - P_{k-1}), \tag{4.24}$$

can serve as an appropriate operator so that the triple $(A, H_\psi, D_{(\alpha_k)_{k\in\mathbb{N}}})$ is a spectral triple.

**Theorem 4.2.1.** *Let $A$, $H_\psi$ and $\pi$ be as described above. Then the following hold.*

1. *There exists a sequence $(\alpha_k)_{k\in\mathbb{N}}$ such that the triple $(A, H_\psi, D_{(\alpha_k)_{k\in\mathbb{N}}})$ is a spectral triple. Moreover, Connes' pseudo-metric $d_A$ induces a metric on the state space of $A$ whose topology coincides with the weak$^*$-topology.*
2. *Given any $p > 0$, there exists a sequence $(\alpha_k)_{k\in\mathbb{N}}$ such that the triple $(A, H_\psi, D_{(\alpha_k)_{k\in\mathbb{N}}})$ has metric dimension equal to $p$.*

Proof. See Theorem 2.1 of [AIC1]. ■

Antonescu-Ivan and Christensen apply this result to obtain a spectral triple representation of the middle $(1-2\eta)$-Cantor set $C_\eta$ (as introduced in Example 2.1.9), where $\eta \in (0, 1/2)$. More precisely, consider the representation of $C_\eta$ given by the full shift space $(\Sigma^\infty, \sigma)$, where $\Sigma := \{0, 1\}$. Recall that $(\Sigma^\infty, d_\Sigma)$ is a compact metric space, where the metric $d_\Sigma : \Sigma^\infty \times \Sigma^\infty \to [0, \infty)$ is defined, for each $\omega := (\omega_1, \omega_2, \dots), \upsilon := (\upsilon_1, \upsilon_2, \dots) \in \Sigma^\infty$, by

$$d_\Sigma(\omega, \upsilon) \;:=\; 2^{-\omega \wedge \upsilon},$$

where $\omega \wedge \upsilon := \max\{\sup\{n \in \mathbb{N} \,:\, \omega, \upsilon \in [x], \text{ for some } x \in \Sigma_{\mathcal{A}}^n\}, 0\}$. As in Section 2.2, for each $k \in \mathbb{N}$ and each $\omega \in \Sigma^k$, let

$$[\omega] \;:=\; \{\upsilon := (\upsilon_1, \upsilon_2, \dots) \in \Sigma^\infty \,:\, (\upsilon_1, \upsilon_2, \dots, \upsilon_k) = \omega\}$$

and let $\chi_{[\omega]} : \Sigma^\infty \to \mathbb{R}$ denote the characteristic function on $[\omega]$. Further, set

$$A_0 \;:=\; \{z\chi_{\Sigma^\infty} \,:\, z \in \mathbb{C}\} \cong \mathbb{C}$$

and, for each $k \in \mathbb{N}$, set

$$A_k \;:=\; \{z\chi_{[\omega]} \,:\, z \in \mathbb{C} \text{ and } \omega \in \Sigma^k\} \;\cong\; \mathbb{C}^{2^k}.$$

Then the $C^*$-algebra $A := C(\Sigma^\infty; \mathbb{C})$ of continuous complex-valued functions on $\Sigma^\infty$ is the norm completion of the inductive limit of the sequence $(A_k)_{k \in \mathbb{N}_0}$. Here, the norm completion is taken with respect to the supremum norm. Letting $\nu$ denote the measure of maximal entropy of the dynamical system $(\Sigma^\infty, \sigma)$, define the faithful tracial state $\tau$ on $A$ by

$$\tau(a) \;:=\; \int_{\Sigma^\infty} a \, d\nu.$$

As before, let $H_\tau$ denote the Gelfand-Naĭmark-Segal completion of $A$ with respect to $\tau$. Note that $H_\tau$ is precisely the Hilbert space $L^2(\Sigma^\infty, \mathcal{B}, \nu)$, where $\mathcal{B}$ denotes the Borel sigma algebra on $\Sigma^\infty$ generated by the ring $\{[\omega] \,:\, \omega \in \Sigma^*\}$.

**Theorem 4.2.2.** *Let $A := C(\Sigma^\infty; \mathbb{C})$, let $H := H_\tau$ and let $\pi : A \to B(H)$ denote the linear $*$-homomorphism given, for each $a \in A$ and each $h \in H$, by*

$$\pi(a)h \;:=\; a \cdot h.$$

*Then the following hold.*

1. *Let $(\alpha_k)_{k \in \mathbb{N}_0}$ denote a sequence of real numbers such that $\alpha_0 := 0$ and such that*

$$\sum_{k=1}^{\infty} \sup\{|\alpha_k - \alpha_i|^{-1} \,:\, 0 \leqslant i \leqslant k-1\} \;<\; \infty.$$

*Then the triple $(A, H, D_{(\alpha_k)_{k \in \mathbb{N}}})$ is a spectral triple, where the $*$-representation of $A$ is given by $(\pi, H)$. Further, Connes' pseudo-metric $d_A : \mathcal{S}(A) \times \mathcal{S}(A) \to [0, \infty)$ is a metric and the topology induced by $d_A$ is equivalent to the weak$^*$-topology defined on $\mathcal{S}(A)$. Moreover, $d_A$ induces a metric on $\Sigma^\infty$ which is equivalent to the metric $d_\Sigma$.*

2. *For a given $\eta \in (0, 1/2)$, consider the sequence $(\alpha_k)_{k\in\mathbb{N}_0}$ given by $\alpha_0 := 0$ and $\alpha_k := \eta^{-k+1}$, for each $k \in \mathbb{N}$. Then the triple $(A, H, D_{(\eta^{-k+1})_{k\in\mathbb{N}}})$ is a finitely summable spectral triple with metric dimension equal to $-\ln(2)/\ln(\eta)$, where the $*$-representation of $A$ is given by $(\pi, H)$. Further, the metric $d$ induced by Connes' pseudo-metric $d_A$ on $\Sigma^\infty$ satisfies, for all $x, y \in \Sigma^\infty$ with $x \neq y$, the inequalities*

$$2\,\eta^{\min\{k\in\mathbb{N}\,:\,x_k\neq y_k\}-1} \;\leqslant\; d(x,y) \;\leqslant\; 2\,\frac{\eta^{\min\{k\in\mathbb{N}\,:\,x_k\neq y_k\}-1}}{(1-\eta)^2}.$$

Proof. See Theorem 4.1 of [AIC1] ∎

Recall that $\Theta : C_{1/3} \to \mathbb{R}$ denotes the continuous function whose graph is the Devil's staircase of the middle third Cantor set $C_{1/3}$, as defined in Equation (4.6). Further, let $\Sigma$ denote the alphabet $\{0, 1\}$, as above. Consider the continuous function which is the lift of $\Theta$ from $C_{1/3}$ onto $\Sigma^\infty$ and let it also be denoted by $\Theta$. The following proposition shows that for the spectral triple presented in the second part of Theorem 4.2.2, the operator $[D_{(\eta^{-k+1})_{k\in\mathbb{N}}}, \pi(\Theta)]$ is not a densely defined operator on $H$, for any $\eta \in (0, 2^{-2/3})$. This is in contrast to the result of Equation (4.7) for Connes' spectral triple.

**Proposition 4.2.3.** *Let $\eta \in (0, 2^{-2/3})$ and let $(A, H, D_{(\eta^{-k+1})_{k\in\mathbb{N}}})$ denote the spectral triple as given in the second part of Theorem 4.2.2. Then the operator $[D_{(\eta^{-k+1})_{k\in\mathbb{N}}}, \pi(\Theta)]$ is not a densely defined operator on $H$.*

Proof. Let $\eta \in (0, 2^{-2/3})$ be fixed and let $\nu$ denote the measure of maximal entropy of $(\Sigma^\infty, \sigma)$. Further, let $e_\omega : \Sigma^\infty \to \mathbb{R}$ be given, for each $k \in \mathbb{N}$ and each $\omega \in \Sigma^*$, by

$$e_\omega(x) \;:=\; \begin{cases} 2^{k/2} & \text{if } x \in [\omega 0] \\ -2^{k/2} & \text{if } x \in [\omega 1] \\ 0 & \text{if } x \in \Sigma^\infty \setminus [\omega]. \end{cases}$$

Then $\{e_\omega : \Sigma^\infty \to \mathbb{R}\}_{\omega\in\Sigma^*} \cup \{\chi_{\Sigma^\infty}, \chi_{[0]} - \chi_{[1]}\}$ is a Haar basis for the Hilbert space $L^2(\Sigma^\infty, \mathcal{B}, \nu)$, as described in Subsection 2.2.2. Recall that we do not make a distinction between a measurable function $f : \Sigma^\infty \to \mathbb{C}$ and its equivalence class

$$\left\{ g : \Sigma^\infty \to \mathbb{C} \,:\, g \text{ is a measurable function and } \int_{\Sigma^\infty} |f - g|\, d\nu = 0 \right\}.$$

Then, for each $h \in L^2(\Sigma^\infty, \mathcal{B}, \nu)$, we have that

$$D_{(\eta^{-k+1})_{k\in\mathbb{N}}}(h) \;=\; \langle h, \chi_{[0]} - \chi_{[1]}\rangle(\chi_{[0]} - \chi_{[1]}) + \sum_{k\in\mathbb{N}} \eta^{-k} \sum_{\omega\in\Sigma^k} \langle h, e_\omega\rangle e_\omega.$$

Next, note that

$$\langle \Theta, \chi_{[0]} - \chi_{[1]}\rangle \;=\; \int_{[0]} \Theta\, d\nu - \int_{[1]} \Theta\, d\nu \;=\; -\frac{1}{4}$$

and that, for each $k \in \mathbb{N}$ and each $\omega \in \Sigma^k$, we have that

$$\langle \Theta, e_\omega\rangle \;=\; \int_{[\omega 0]} 2^{k/2}\, \Theta\, d\nu - \int_{[\omega 1]} 2^{k/2}\, \Theta\, d\nu \;=\; -2^{k/2}\, \frac{1}{2^{2(k+1)}} \;=\; -\frac{1}{2^{(3k/2)+2}}.$$

Therefore, it follows that

$$[D_{(\eta^{-k+1})_{k\in\mathbb{N}}}, \Theta]\chi_{\Sigma^\infty} \;=\; D_{(\eta^{-k+1})_{k\in\mathbb{N}}}(\Theta) \;=\; -\frac{1}{4}(\chi_{[0]} - \chi_{[1]}) - \sum_{k\in\mathbb{N}} \eta^{-k} 2^{-(3k/2)-2} \sum_{\omega\in\Sigma^k} e_\omega. \quad (4.25)$$

Since $\eta^{-1} > 2^{2/3}$, we have that the right-hand side of Equation (4.25) does not belong to $H$. Therefore, $[D_{(\eta^{-k+1})_{k\in\mathbb{N}}}, \pi(\Theta)]$ is not defined on constant functions. ∎

*Remark. For each $\eta \in (0, 1/2)$, another representation of $C_\eta$ is given by the compact Abelian group $\prod_{k\in\mathbb{N}} \mathbb{Z}_2$ equipped with the product topology. Observe that the dual of this group is the infinite product group $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}_2$, and that there exists a norm preserving bijection between the reduced discrete group algebra of $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}_2$ and the $C^*$-algebra of continuous functions defined on $C_\eta$ equipped with the supremum norm. Next, consider the length function $L : \bigoplus_{k\in\mathbb{N}} \mathbb{Z}_2 \to \mathbb{R}_0^+$ defined, for each $g := (g_1, g_2, \dots) \in \bigoplus_{n\in\mathbb{N}} \mathbb{Z}_2$, by*

$$L(g) \;:=\; \begin{cases} \eta^{-\max\{k\in\mathbb{N}\,:\, g_k = 1\}} & \text{if } g \neq (0,0,\dots), \\ 0 & \text{if } g = (0,0,\dots). \end{cases}$$

*Then the spectral triple constructed in [Con2], and as described in Subsection 3.3.3, coincides with the spectral triple given in the second part of Theorem 4.2.2.*

Let us conclude this subsection with the following application of Theorem 4.2.1. Here, we construct a spectral triple which represents the set of continuous functions on the boundary of a free group generated by $k \in \mathbb{N}$ elements. (Note that the term "boundary of a group" is used in the sense of Gromov, see for instance [Gro, ABC$^+$].) Although this construction is partially discussed in Proposition 1.9 of [CM], we add to the discussion by considering the metric dimension, the volume, the noncommutative integral and aspects of Connes' pseudo-metric of the given spectral triple.

**Example 4.2.4.** Let $\mathbb{F}_M$ denote the free group generated by $M \in \mathbb{N}\setminus\{1\}$ elements. Observe that the boundary $\partial\mathbb{F}_M$ of $\mathbb{F}_M$ (in the sense of Gromov) has a representation as a one-sided topologically exact subshift of finite type. Namely, let $\Sigma := \{1, 2, \dots, 2M\}$ and let $A := [a_{i,j}]_{i,j}$ denote the $2M \times 2M$ transition matrix with entries in $\{0, 1\}$ satisfying

$$a_{i,j} \;:=\; \begin{cases} 1 & \text{if } |i-j| \neq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then there exists a homeomorphism between $\partial\mathbb{F}_M$ and $\Sigma_A^\infty$. For the remainder of this example we shall work with this representation.

Consider the $C^*$-algebra $C(\Sigma_A^\infty)$ of continuous functions on $\Sigma_A^\infty$ and observe that $C(\Sigma_A^\infty; \mathbb{C})$ is an AF-$C^*$-algebra. Indeed, the inductive limit of finite dimensional algebras

$$A_k \;:=\; \{z\chi_{[\omega]} \,:\, z \in \mathbb{C} \text{ and } \omega \in \Sigma_A^k\} \;=\; \mathbb{C}^{2M(2M-1)^k},$$

for $k \in \mathbb{N}$, with $A_0 := \{z\chi_{\Sigma_A^\infty} \,:\, z \in \mathbb{C}\}$, is isometrically $*$-homomorphic to $C(\Sigma_A^\infty; \mathbb{C})$. (Here, the inductive limit is taken with respect to the supremum norm.) Let $\mu$ denote a Gibbs measure and let $\tau_\mu : C(\Sigma_A^\infty; \mathbb{C}) \to \mathbb{C}$ denote the state given, for each $a \in C(\Sigma_A^\infty; \mathbb{C})$, by

$$\tau_\mu(a) \;:=\; \int_{\Sigma_A^\infty} a\, d\mu.$$

The Gelfand-Naĭmark-Segal completion of $A$ with respect to the tracial state $\tau_\mu$ is precisely the Hilbert space $H := L^2(\Sigma_A^\infty, \mathcal{B}, \mu)$. Recall that we do not make a distinction between $a \in A$ and its equivalence class

$$\left\{ f : \Sigma_A^\infty \to \mathbb{C} \,:\, f \text{ is a measurable function and } \int_{\Sigma_A^\infty} |f - a| \, d\mu = 0 \right\}.$$

Now let $\pi : A \to B(H)$ denote the natural $*$-homomorphism of $A$ given, for each $a \in A$ and $h \in H$, by $\pi(a)h := a \cdot h$. Since the characteristic function $\chi_{\Sigma^\infty}$ is a separating and cyclic vector for the sub-$*$-algebra $\pi(A) \subset B(H)$ and since $A$ is an AF-$C^*$-algebra, we obtain a natural filtration $(H_k)_{k \in \mathbb{N}_0}$ of $H$, where, for each $k \in \mathbb{N}_0$, we define $H_k := \{\pi(a)\chi_{\Sigma^\infty} \,:\, a \in A_k\}$. Let $P_k : H \to H_k$ denote the orthogonal projection from $H$ onto $H_k$, for each $k \in \mathbb{N}_0$. Motivated by Theorem 4.2.2, Cornelissen and Marcolli in [CM] define a Dirac operator $D$ on $H$ by

$$D \;:=\; \sum_{k \in \mathbb{N}} \dim(A_k)(P_k - P_{k-1}) \;=\; \sum_{k \in \mathbb{N}} 2M(2M-1)^{k-1}(P_k - P_{k-1}).$$

In Proposition 1.9 of [CM] it is then shown that the triple $(A, H, D)$ is a finitely summable spectral triple, where the $*$-representation is given by $(\pi, H)$. By Theorem 4.2.1, we also have that Connes' pseudo-metric $d_A$ is a metric on the state space $\mathcal{S}(A)$ whose topology is equivalent to the weak$^*$-topology on $\mathcal{S}(A)$.

Let us now consider the metric dimension of the spectral triple $(A, H, D)$. It is easy to see that $\sigma_1(D) = \sigma_2(D) = \cdots = \sigma_{2M}(D) = 2M$. Further, for each $k \in \mathbb{N}$ and each integer $l \in [2M(2M-1)^{k-1} + 1, 2M(2M-1)^k]$, we have that $\sigma_l(D) = 2M(2M-1)^{k-1}$. Therefore, for each $p > 0$, it follows that

$$\begin{aligned} &\lim_{N \to \infty} \frac{\sum_{k=1}^{N} \sigma_k((\mathbb{1} + D^2)^{-p/2})}{\ln(N)} \\ &\qquad = \lim_{N' \to \infty} \frac{\sum_{k=1}^{N'} (2M(2M-2)(2M-1)^{k-1})(1 + (2M)^2(2M-1)^{2\cdot(k-1)})^{-p/2}}{\ln(2M(2M-1)^{N'})}. \end{aligned}$$

Hence, the metric dimension is equal to 1. Moreover, the noncommutative volume is give by

$$V(A, H, D) \;=\; \frac{2(M-1)}{\ln(2M-1)}.$$

Next we shall consider the noncommutative integral given by $(A, H, D)$. For each $k \in \mathbb{N}$ and for each $\omega, \upsilon \in \Sigma_A^k$, we have that $\sigma_k(\pi(\chi_{[\omega]})|D|^{-1}) = \sigma_k(\pi(\chi_{[\upsilon]})|D|^{-1})$ and hence,

$$\mathrm{Tr}_{\mathcal{W}}(\pi(\chi_{[\omega]})|D|^{-1}) \;=\; \mathrm{Tr}_{\mathcal{W}}(\pi(\chi_{[\upsilon]})|D|^{-1}). \tag{4.26}$$

Here, we recall that, by definition, the Dixmier trace is taken over the ideal $\mathcal{L}^{1,+}(\ker(D)^\perp)$. Since the map given, for each $a \in C(\Sigma_A^\infty)$, by

$$a \;\mapsto\; \mathrm{Tr}_{\mathcal{W}}(\pi(a)|D|^{-1})$$

is a linear functional, by the Riesz Representation Theorem (Theorem 3.3.1), there exists a Borel probability measure $\mu$ so that, for each $a \in A$, we have that

$$\mathrm{Tr}_{\mathcal{W}}(\pi(a)|D|^{-1}) \;=\; \frac{2(M-1)}{\ln(2M-1)} \int_{\Sigma_A^\infty} a \, d\mu.$$

Therefore, by Equation (4.26) and applying the Hahn-Kolmogorov Theorem (Theorem 3.3.5), it follows that $\mu$ is the unique measure given, for each $k \in \mathbb{N}$ and each $\omega \in \Sigma_A^k$, by

$$\mu([\omega]) \;=\; 2M(2M-1)^{-k+1}.$$

This is precisely the measure of maximal entropy for the dynamical system $(\Sigma_A^\infty, \sigma)$. Further, since this holds for all limiting procedures, for each $a \in A$, we have that

$$⨍ \pi(a)|D|^{-1} \;=\; \frac{2(M-1)}{\ln(2M-1)} \int_{\Sigma_A^\infty} a \, d\mu.$$

### 4.2.2 The Noncommutative Volume of a Subshift of Finite Type

The construction of a spectral triple, given in [AIC1] and reviewed in Subsection 4.2.1, gives a noncommutative representation of a one-sided topologically exact subshift of finite type $(\Sigma_A^\infty, \sigma)$. This follows, since the $C^*$-algebra of complex-valued continuous functions on $\Sigma_A^\infty$ is an AF $C^*$-algebra. However, the measure theoretical properties of such a spectral triple essentially encodes the measure of maximal entropy. In what follows, we consider a one-sided topologically exact subshift of finite type $(\Sigma_A^\infty, \sigma)$ and an equilibrium measure $\nu_\phi$ for a non-arithmetic Hölder continuous potential function $\phi \in C(\Sigma_A^\infty; \mathbb{C})$. By refining Antonescu-Ivan's and Christensen's construction, we provide a spectral triple which represents $\Sigma_A^\infty$ whose measure theoretical properties encode the measure $\nu_\phi$. Indeed, by breaking down the projections used in Theorem 4.2.1 and by relating the singular values of the Dirac operator to the $\nu_\phi$-measure of the cylinder sets of $\Sigma_A^\infty$, we prove that a spectral triple $(A, H, D) := (C(\Sigma_A^\infty; \mathbb{C}), L^2(\Sigma_A^\infty, \mathcal{B}, \nu_\phi), D_{\nu_\phi})$ can be constructed so that the following hold.

1. Connes' pseudo-metric $d_{C(\Sigma_A^\infty;\mathbb{C})}$ is a metric on the state space $\mathcal{S}(C(\Sigma_A; \mathbb{C}))$ and the topology induced by $d_{C(\Sigma_A^\infty;\mathbb{C})}$ on $\mathcal{S}(C(\Sigma_A; \mathbb{C}))$ is equivalent to the weak$^*$-topology defined on $\mathcal{S}(A)$.

2. The spectral triple $(A, H, D)$ is $(1, +)$-summable with metric dimension equal to one.

3. The noncommutative integral given by $(A, H, D)$ agrees with the integral with respect to $\nu_\phi$.

4. The noncommuative volume of $(A, H, D)$ is equal to $1/h_{\nu_\phi}(\sigma)$. Recall that $h_{\nu_\phi}(\sigma)$ denotes the measure theoretical entropy of $\nu_\phi$ with respect to the left shift map $\sigma$.

Let us begin by setting up some notation. Let $\Sigma$ denote a finite alphabet with $\mathrm{card}(\Sigma) =: M \in \mathbb{N}$ and let $A := [a_{i,j}]_{i,j}$ denote an irreducible aperiodic $M \times M$ transition matrix for $\Sigma$. Let $\mu_\phi$ denote a Gibbs measure for some Hölder continuous potential function $\phi \in C(\Sigma_A^\infty; \mathbb{R})$ and let

$$\left\{ (\mu_\phi([x]))^{-1/2} \chi_{[x]} \,:\, x \in \Sigma \right\} \cup \left\{ e_{\omega,i} \,:\, (\omega, i) \in \textstyle\bigcup_{v \in \Sigma_A^*} \{v\} \times \{1, 2, \ldots, \alpha(v) - 1\} \right\}$$

denote a Haar basis for $L^2(\Sigma_A^\infty, \mathcal{B}, \mu_\phi)$, as described in Subsection 2.2.2). Recall that $\alpha : \Sigma_A^* \to \mathbb{N}$ denotes the set function given, for each $k \in \mathbb{N}$ and each $\omega := (\omega_1, \omega_2, \ldots, \omega_k) \in \Sigma_A^k$, by

$$\alpha(\omega) \;:=\; \sum_{x \in \Sigma} a_{\omega_k, x}.$$

Let $\tau_{\mu_\phi} : C(\Sigma_A^\infty;\mathbb{C}) \to \mathbb{C}$ denote the tracial state defined, for each $a \in C(\Sigma_A^\infty;\mathbb{C})$, by

$$\tau_{\mu_\phi}(a) := \int_{\Sigma_A^\infty} a\, d\mu_\phi.$$

Then the Gelfand-Naĭmark-Segal completion of $C(\Sigma_A^\infty;\mathbb{C})$ with respect to $\tau_{\mu_\phi}$ is precisely the Hilbert space $L^2(\Sigma_A^\infty, \mathcal{B}, \mu_\phi)$. Next, define the Dirac operator $D_{\mu_\phi}$ on $L^2(\Sigma_A^\infty, \mathcal{B}, \mu_\phi)$ by

$$D_{\mu_\phi} := \sum_{x\in\Sigma} \frac{1}{\mu_\phi([x])}\langle \cdot, \chi_{[x]}\rangle \chi_{[x]} - \langle \cdot, \chi_{\Sigma_A^\infty}\rangle \chi_{\Sigma_A^\infty} + \sum_{\omega\in\Sigma_A^*} \frac{\alpha(\omega)-1}{\mu_\phi([\omega])} \sum_{i=1}^{\alpha(\omega)-1} \langle \cdot, e_{\omega,i}\rangle e_{\omega,i}. \tag{4.27}$$

Observe that the operator $D_{\mu_\phi}$ is a well defined unbounded operator since the following hold.

1. The measure $\mu_\phi$ is a Gibbs measure and hence, $\mu_\phi$ is non-atomic and $\mu_\phi([\omega]) \neq 0$, for each $\omega \in \Sigma_A^*$.

2. The domain of $D_{\mu_\phi}$ contains the set of locally constant functions, which is $L^2$-norm-dense in $L^2(\Sigma_A^\infty, \mathcal{B}, \mu_\phi)$.

To compare this with the Antonescu-Ivan and Christensen Dirac operator, given in Equation (4.24), we observe that the projections used are given, for each $k \in \mathbb{N}$, by

$$P_{k+1} := \sum_{x\in\Sigma} \frac{1}{\mu_\phi([x])}\langle \cdot, \chi_{[x]}\rangle\, \chi_{[x]} + \sum_{l=1}^{k} \sum_{\omega\in\Sigma_A^l} \sum_{i=1}^{\alpha(\omega)-1} \langle \cdot, e_{\omega,i}\rangle e_{\omega,i},$$

where

$$P_0 := \langle \cdot, \chi_{\Sigma_A^\infty}\rangle \chi_{\Sigma_A^\infty}, \qquad P_1 := \sum_{x\in\Sigma} \frac{1}{\mu_\phi([x])}\langle \cdot, \chi_{[x]}\rangle\, \chi_{[x]}.$$

Recall that we do not make a distinction between a measurable function $f : \Sigma_A^\infty \to \mathbb{C}$ and its equivalence class

$$\left\{ g : \Sigma_A^\infty \to \mathbb{C} \,:\, g \text{ is a measurable function and } \int_{\Sigma_A^\infty} |g-f|\, d\mu_\phi = 0 \right\}.$$

**Theorem 4.2.5.** *The triple $(C(\Sigma_A^\infty;\mathbb{C}), L^2(\Sigma_A^\infty, \mathcal{B}, \mu_\phi), D_{\mu_\phi})$ is a spectral triple, where the $*$-representation $(\pi, L^2(\Sigma_A^\infty, \mathcal{B}, \mu_\phi))$ is given by $\pi(a)h := a \cdot h$, for each $a \in C(\Sigma_A^\infty;\mathbb{C})$ and each $h \in L^2(\Sigma_A^\infty, \mathcal{B}, \mu_\phi)$.*

Proof. Observe that the set $C(\Sigma_A^\infty;\mathbb{C})$ equipped with the supremum norm is a $C^*$-algebra, that $L^2(\Sigma_A^\infty, \mathcal{B}, \mu_\phi)$ is a complex Hilbert space and that $(\pi, L^2(\Sigma_A^\infty, \mathcal{B}, \mu_\phi))$ is a faithful $*$-representation of $C(\Sigma_A^\infty;\mathbb{C})$. Further, we have seen that $D_{\mu_\phi}$ is a well defined unbounded operator. Next, observe that the kernel of $D_{\mu_\phi}$ consists of all equivalence classes of $L^2(\Sigma_A^\infty, \mathcal{B}, \mu_\phi)$ which contain some constant function on $\Sigma_A^\infty$. Moreover, by the properties of a Gibbs measure, we have that $D_{\mu_\phi}^{-1}$ is a bounded operator on the complex Hilbert space $\ker(D_{\mu_\phi})^\perp \subset L^2(\Sigma_A^\infty, \mathcal{B}, \mu_\phi)$. Hence, the operator $(\mathbb{1}+D_{\mu_\phi}^2)^{-1/2}$ is a bounded operator which can be approximated by operators in $B(\ker(D_{\mu_\phi})^\perp)$ with finite dimensional range. Therefore, $D_{\mu_\phi}$ has a compact resolvent. Moreover, the sets $\mathrm{Ran}(D_{\mu_\phi} \pm i\mathbb{1})$ are $L^2$-norm-dense in $L^2(\Sigma_A^\infty, \mathcal{B}, \mu_\phi)$. This follows, since the set of locally constant functions is $L^2$-norm-dense in $L^2(\Sigma_A^\infty, \mathcal{B}, \nu_\phi)$, since the operator $(D_{\mu_\phi} \pm i\mathbb{1})$ is linear and since we have the following.

1. For each $x \in \Sigma$, we have that

$$(D_{\mu_\phi} \pm i\mathbb{1}) \left( \frac{1}{1 \pm i}(\chi_{[x]}) \mp i\mu_\phi([x])\chi_{\Sigma_A^\infty} \right) = \chi_{[x]}.$$

2. For each $(\omega, j) \in \bigcup_{\upsilon \in \Sigma_A^*} \{\upsilon\} \times \{1, 2, \ldots, \alpha(\upsilon) - 1\}$, we have that

$$(D_{\mu_\phi} \pm i\mathbb{1}) \left( \frac{\mu_\phi([\omega])}{\alpha(\omega) - 1 \pm i} e_{\omega,j} \right) = e_{\omega,j}.$$

Moreover, we have that $D_{\mu_\phi}$ is symmetric on its domain. Indeed, for each $h_1, h_2 \in \mathrm{Dom}(D_{\mu_\phi}) \subset L^2(\Sigma_A^\infty, \mathcal{B}, \mu_\phi)$, we have that

$$\begin{aligned}
\langle D_{\mu_\phi}(h_1), h_2 \rangle &= \Bigg\langle \sum_{x \in \Sigma} \frac{1}{\mu_\phi([x])} \langle h_1, \chi_{[x]} \rangle \chi_{[x]} - \langle h_1, \chi_{\Sigma_A^\infty} \rangle \chi_{\Sigma_A^\infty} \\
&\quad + \sum_{\omega \in \Sigma_A^*} \frac{\alpha(\omega) - 1}{\mu_\phi([\omega])} \sum_{i=1}^{\alpha(\omega)-1} \langle h_1, e_{\omega,i} \rangle e_{\omega,i}, \\
&\quad \sum_{y \in \Sigma} \frac{1}{\mu_\phi([y])} \langle h_2, \chi_{[y]} \rangle \chi_{[y]} + \sum_{\upsilon \in \Sigma_A^*} \sum_{j=1}^{\alpha(\upsilon)-1} \langle h_2, e_{\upsilon,j} \rangle e_{\upsilon,j} \Bigg\rangle \\
&= \sum_{x \in \Sigma} \frac{1}{\mu_\phi([x])} \langle h_1, \chi_{[x]} \rangle \overline{\langle h_2, \chi_{[x]} \rangle} \\
&\quad + \sum_{\omega \in \Sigma_A^*} \frac{\alpha(\omega) - 1}{\mu_\phi([\omega])} \sum_{i=1}^{\alpha(\omega)-1} \langle h_1, e_{\omega,i} \rangle \overline{\langle h_2, e_{\omega,i} \rangle} \\
&\quad - \sum_{y \in \Sigma} \frac{1}{\mu_\phi([y])} \langle h_1, \chi_{\Sigma_A^\infty} \rangle \overline{\langle h_2, \chi_{[y]} \rangle} \langle \chi_{\Sigma_A^\infty}, \chi_{[y]} \rangle \\
&= \sum_{x \in \Sigma} \frac{1}{\mu_\phi([x])} \langle h_1, \chi_{[x]} \rangle \overline{\langle h_2, \chi_{[x]} \rangle} \\
&\quad + \sum_{\omega \in \Sigma_A^*} \frac{\alpha(\omega) - 1}{\mu_\phi([\omega])} \sum_{i=1}^{\alpha(\omega)-1} \langle h_1, e_{\omega,i} \rangle \overline{\langle h_2, e_{\omega,i} \rangle} \\
&\quad - \sum_{\mathcal{V} \in \Sigma} \frac{1}{\mu_\phi([\mathcal{V}])} \langle h_1, \chi_{[\mathcal{V}]} \rangle \langle \chi_{\Sigma_A^\infty}, \chi_{[\mathcal{V}]} \rangle \sum_{y \in \Sigma} \frac{1}{\mu_\phi([y])} \overline{\langle h_2, \chi_{[y]} \rangle} \langle \chi_{\Sigma_A^\infty}, \chi_{[y]} \rangle \\
&= \sum_{x \in \Sigma} \frac{1}{\mu_\phi([x])} \langle h_1, \chi_{[x]} \rangle \overline{\langle h_2, \chi_{[x]} \rangle} \\
&\quad + \sum_{\omega \in \Sigma_A^*} \frac{\alpha(\omega) - 1}{\mu_\phi([\omega])} \sum_{i=1}^{\alpha(\omega)-1} \langle h_1, e_{\omega,i} \rangle \overline{\langle h_2, e_{\omega,i} \rangle} \\
&\quad - \sum_{\mathcal{V} \in \Sigma} \frac{1}{\mu_\phi([\mathcal{V}])} \langle h_1, \chi_{[\mathcal{V}]} \rangle \langle \chi_{\Sigma_A^\infty}, \chi_{[\mathcal{V}]} \rangle \overline{\langle h_2, \chi_{\Sigma_A^\infty} \rangle} \\
&= \langle h_1, D_{\mu_\phi}(h_2) \rangle.
\end{aligned}$$

Hence, by Theorem 3.3.2, it follows that $D_{\mu_\phi}$ is an essentially self-adjoint operator. In order to show that $(C(\Sigma_A^\infty; \mathbb{C}), L^2(\Sigma_A^\infty, \mathcal{B}, \mu_\phi), D_{\mu_\phi})$ is a spectral triple, it remains to show that the set

$$\{a \in C(\Sigma_A^\infty; \mathbb{C}) : \text{the operator } [D_{\mu_\phi}, \pi(a)] \text{ is densely defined and extends to a bounded operator}\}$$

is norm-dense in $C(\Sigma_A^\infty;\mathbb{C})$ with respect to the supremum norm. To show this, first note that by the Stone-Weierstrass Theorem (Theorem 2.2.12), the set of locally constant functions is a norm-dense subset of $C(\Sigma_A^\infty;\mathbb{C})$ with respect to the supremum norm. Secondly, note that for each $h \in L^2(\Sigma_A^\infty, \mathcal{B}, \mu_\phi)$ and for each $y \in \Sigma$, we have that

$$\begin{aligned}
[D_{\mu_\phi}, \pi(\chi_{[y]})]h &= D_{\mu_\phi}(\chi_{[y]} \cdot h) - \pi(\chi_{[y]})D_{\mu_\phi}(h) \\
&= \sum_{x\in\Sigma} \frac{1}{\mu_\phi([x])}\langle \chi_{[y]} \cdot h, \chi_{[x]}\rangle \chi_{[x]} - \langle \chi_{[y]} \cdot h, \chi_{\Sigma_A^\infty}\rangle \chi_{\Sigma_A^\infty} \\
&\quad + \sum_{\omega\in\Sigma_A^*} \frac{\alpha(\omega)-1}{\mu_\phi([\omega])} \sum_{i=1}^{\alpha(\omega)-1} \langle \chi_{[y]} \cdot h, e_{\omega,i}\rangle e_{\omega,i} \\
&\quad - \sum_{\mathcal{V}\in\Sigma} \frac{1}{\mu_\phi([\mathcal{V}])}\langle h, \chi_{[\mathcal{V}]}\rangle \chi_{[y]} \cdot \chi_{[x]} + \langle h, \chi_{\Sigma_A^\infty}\rangle \chi_{[y]} \\
&\quad - \sum_{\upsilon\in\Sigma_A^*} \frac{\alpha(\upsilon)-1}{\mu_\phi([\upsilon])} \sum_{j=1}^{\alpha(\upsilon)-1} \langle h, e_{\upsilon,j}\rangle \chi_{[y]} \cdot e_{\upsilon,j} \\
&= \frac{1}{\mu_\phi([y])}\langle h, \chi_{[y]}\rangle \chi_{[y]} - \langle \chi_{[y]} \cdot h, \chi_{\Sigma_A^\infty}\rangle \chi_{\Sigma_A^\infty} \\
&\quad + \sum_{\substack{\omega\in\Sigma_A^* \\ [\omega]\subseteq[y]}} \frac{\alpha(\omega)-1}{\mu_\phi([\omega])} \sum_{i=1}^{\alpha(\omega)-1} \langle h, e_{\omega,i}\rangle e_{\omega,i} \\
&\quad - \frac{1}{\mu_\phi([y])}\langle h, \chi_{[y]}\rangle \chi_{[y]} + \langle h, \chi_{\Sigma_A^\infty}\rangle \chi_{[y]} \\
&\quad - \sum_{\substack{\upsilon\in\Sigma_A^* \\ [\upsilon]\subseteq[y]}} \frac{\alpha(\upsilon)-1}{\mu_\phi([\upsilon])} \sum_{j=1}^{\alpha(\upsilon)-1} \langle h, e_{\upsilon,j}\rangle e_{\upsilon,j} \\
&= -\langle \chi_{[y]} \cdot h, \chi_{\Sigma_A^\infty}\rangle \chi_{\Sigma_A^\infty} + \langle h, \chi_{\Sigma_A^\infty}\rangle \chi_{[y]}.
\end{aligned}$$

By taking norms and applying the triangle inequality and applying Parseval's identity (Theorem II.6 of [RS]), since this is a finite sum, one can deduce that the operator-norm of $[D_{\mu_\phi}, \pi(\chi_{[y]})]$ is finite. Further, for each $k \in \mathbb{N}$, each $(\omega, i) := ((\omega_1, \ldots, \omega_k), i) \in \bigcup_{\upsilon\in\Sigma_A^*}\{\upsilon\} \times \{1, \ldots, \alpha(\upsilon)-1\}$ and each $h \in L^2(\Sigma_A^\infty, \mathcal{B}, \mu_\phi)$ we have that

$$\begin{aligned}
[D_{\mu_\phi}, \pi(e_{\omega,i})]h &= D_{\mu_\phi}(e_{\omega,i} \cdot h) - \pi(e_{\omega,i})D(h) \\
&= \sum_{x\in\Sigma} \frac{1}{\mu_\phi([x])}\langle e_{\omega,i} \cdot h, \chi_{[x]}\rangle \chi_{[x]} - \langle e_{\omega,i} \cdot h, \chi_{\Sigma_A^\infty}\rangle \chi_{\Sigma_A^\infty} \\
&\quad + \sum_{\upsilon\in\Sigma_A^*} \frac{\alpha(\upsilon)-1}{\mu_\phi([\upsilon])} \sum_{j=1}^{\alpha(\upsilon)-1} \langle e_{\omega,i} \cdot h, e_{\upsilon,j}\rangle e_{\upsilon,j} \\
&\quad - \sum_{y\in\Sigma} \frac{1}{\mu_\phi([y])}\langle h, \chi_{[y]}\rangle \cdot e_{\omega,i} \cdot \chi_{[y]} + \langle h, \chi_{\Sigma_A^\infty}\rangle e_{\omega,i} \\
&\quad - \sum_{\mathcal{V}\in\Sigma_A^*} \frac{\alpha(\mathcal{V})-1}{\mu_\phi([\mathcal{V}])} \sum_{l=1}^{\alpha(\mathcal{V})-1} \langle h, e_{\mathcal{V},l}\rangle e_{\omega,i} \cdot e_{\mathcal{V},l} \quad =
\end{aligned}$$

$$
\begin{aligned}
= \ & \frac{1}{\mu_\phi([\omega_1])}\langle e_{\omega,i}\cdot h, \chi_{[\omega_1]}\rangle \chi_{[\omega_1]} - \langle e_{\omega,i}\cdot h, \chi_{\Sigma_A^\infty}\rangle \chi_{\Sigma_A^\infty} \\
& + \sum_{\substack{\upsilon\in\Sigma_A^* \\ [\omega]\subseteq[\upsilon]}} \frac{\alpha(\upsilon)-1}{\mu_\phi([\upsilon])} \sum_{j=1}^{\alpha(\upsilon)-1} \langle e_{\omega,i}\cdot h, e_{\upsilon,j}\rangle e_{\upsilon,j} \\
& - \frac{1}{\mu_\phi([\omega_1])}\langle h, \chi_{[\omega_1]}\rangle e_{\omega,i} + \langle h, \chi_{\Sigma_A^\infty}\rangle e_{\omega,i} \\
& - \sum_{\substack{\mathcal{V}\in\Sigma_A^* \\ [\omega]\subseteq[\mathcal{V}]}} \frac{\alpha(\mathcal{V})-1}{\mu_\phi([\mathcal{V}])} \sum_{j=1}^{\alpha(\mathcal{V})-1} \langle h, e_{\mathcal{V},j}\rangle e_{\omega,i}\cdot e_{\mathcal{V},j}.
\end{aligned}
$$

By taking norms and applying the triangle inequality and applying Parseval's identity (Theorem II.6 of [RS]), since this is a finite sum, one can deduce that the operator norm of the operator $[D_{\mu_\phi}, \pi(e_{\omega,i})]$ is finite. By these observations, it follows that the set

$$\{a \in C(\Sigma_A^\infty;\mathbb{C}) \,:\, \text{the operator } [D_{\mu_\phi}, \pi(a)] \text{ is densely defined and extends to a bounded operator}\}$$

forms a norm-dense subset of $C(\Sigma_A^\infty;\mathbb{C})$, with respect to the supremum norm. ∎

In the following theorem we consider the metric aspects of the spectral triple given in Theorem 4.2.5. Specifically, we verify that Connes' pseudo-metric $d_{C(\Sigma_A^\infty;\mathbb{C})}$ associated to this spectral triple is a metric and that the topology induced by $d_{C(\Sigma_A^\infty;\mathbb{C})}$ is equivalent to the weak$^*$-topology on the state space of $C(\Sigma_A^\infty;\mathbb{C})$.

**Theorem 4.2.6.** *The pseudo-metric $d_{C(\Sigma_A^\infty;\mathbb{C})}$ is a metric on the state space $\mathcal{S}(C(\Sigma_A^\infty;\mathbb{C}))$ of $C(\Sigma_A^\infty;\mathbb{C})$. Moreover, the topology induced by the metric $d_{C(\Sigma_A^\infty;\mathbb{C})}$ on $\mathcal{S}(C(\Sigma_A^\infty;\mathbb{C}))$ is equivalent to the weak$^*$-topology on $\mathcal{S}(C(\Sigma_A^\infty;\mathbb{C}))$.*

Proof. The proof of this result is motivated by the proof of Theorem 2.1 of [AIC1]. For ease of notation, set $\text{card}(\Sigma) =: M \in \mathbb{N}$ and set

$$
\begin{aligned}
A_{D_{\mu_\phi}} \ := \ \{a \in C(\Sigma_A^\infty;\mathbb{C}) \,:\, & \text{ the operator } [D_{\mu_\phi}, \pi(a)] \text{ is densely defined and extends to} \\
& \text{a bounded operator with norm less than or equal to one}\}.
\end{aligned}
$$

For each $k \in \mathbb{N}$, set

$$C_k \ := \ \sup\{\sqrt{\mu_\phi([\omega])} \,:\, \omega \in \Sigma_A^k\}.$$

Since $\mu_\phi$ is a Gibbs measure, it follows that $C_k$ converges to zero as $k$ tends to infinity.

Observe that the characteristic function $\chi_{\Sigma_A^\infty} \in L^2(\Sigma_A^\infty, \mathcal{B}, \mu_\phi)$ is a separating and cyclic vector for the subalgeba $\pi(C(\Sigma_A^\infty;\mathbb{C}))$ of $B(L^2(\Sigma_A^\infty, \mathcal{B}, \mu_\phi))$ and define the following mappings.

1. Let $P_0, P_1 : L^2(\Sigma_A^\infty, \mathcal{B}, \mu_\phi) \to L^2(\Sigma_A^\infty, \mathcal{B}, \mu_\phi)$ denote the projections given by

$$P_0 \ := \ \langle \cdot, \chi_{\Sigma_A^\infty}\rangle \chi_{\Sigma_A^\infty}, \qquad P_1 \ := \ \sum_{x\in\Sigma} \frac{1}{\mu_\phi([x])}\langle \cdot, \chi_{[x]}\rangle \chi_{[x]}.$$

Further, for each $k \in \mathbb{N}$, let $P_{k+1} : L^2(\Sigma_A^\infty, \mathcal{B}, \mu_\phi) \to L^2(\Sigma_A^\infty, \mathcal{B}, \mu_\phi)$ denote the projections given by

$$P_{k+1} \ := \sum_{\omega\in\Sigma_A^k} \sum_{i=1}^{\alpha(\omega)-1} \langle \cdot, e_{\omega,i}\rangle e_{\omega,i}.$$

2. Let $\pi_0, \pi_1 : C(\Sigma_A^\infty;\mathbb{C}) \to C(\Sigma_A^\infty;\mathbb{C})$ denote the projections given, for each $a \in C(\Sigma_A^\infty;\mathbb{C})$, by

$$\pi_0(a) := \left(\int_{\Sigma_A^\infty} a \cdot \chi_{\Sigma_A^\infty}\, d\mu_\phi\right) \chi_{\Sigma_A^\infty}, \qquad \pi_1(a) := \sum_{x\in\Sigma} \frac{1}{\mu_\phi([x])} \left(\int_{[x]} a\, d\mu_\phi\right) \chi_{[x]}.$$

Further, for each $k \in \mathbb{N}$, let $\pi_{k+1} : C(\Sigma_A^\infty;\mathbb{C}) \to C(\Sigma_A^\infty;\mathbb{C})$ denote the projection given, for each $a \in C(\Sigma_A^\infty;\mathbb{C})$ by

$$\pi_{k+1}(a) := \sum_{x\in\Sigma} \frac{1}{\mu_\phi([x])} \left(\int_{[x]} a\, d\mu_\phi\right) \chi_{[x]} + \sum_{l=1}^{k} \sum_{\omega\in\Sigma_A^l} \sum_{i=1}^{\alpha(\omega)-1} \left(\int_{\Sigma_A^\infty} a \cdot e_{\omega,i}\, d\mu_\phi\right) e_{\omega,i}.$$

Since $\mu_\phi$ is a Gibbs measure and since we have that $D_{\mu_\phi}(\chi_{\Sigma_A^\infty}) = 0$, by the triangle inequality and by Parseval's identity (Theorem II.6 of [RS]), there exists a positive constant $C$ dependant on $\phi$ such that for each $k, m \in \mathbb{N}$ and $a \in A$, we have that

$$\|\pi_k(a) - \pi_{k+m}(a)\|_\infty = \sup_{x\in\Sigma_A^\infty} \left\{ \left| \sum_{l=k}^{k+m} \sum_{\omega\in\Sigma_A^l} \sum_{i=1}^{\alpha(\omega)-1} \left(\int_{\Sigma_A^\infty} a \cdot e_{\omega,i}\, d\mu_\phi\right) e_{\omega,i}(x) \right| \right\} \tag{4.28}$$

$$\leqslant \sum_{l=k}^{k+m} \sum_{\omega\in\Sigma_A^l} \frac{C}{\sqrt{\mu_\phi([\omega])}} \sum_{i=1}^{\alpha(\omega)-1} \left| \int_{\Sigma_A^\infty} a \cdot e_{\omega,i}\, d\mu_\phi \right| \tag{4.29}$$

$$\leqslant \sum_{l=k}^{k+m} \sum_{\omega\in\Sigma_A^l} \frac{M^{1/2} C}{\sqrt{\mu_\phi([\omega])}} \left( \sum_{i=1}^{\alpha(\omega)-1} \left| \int_{\Sigma_A^\infty} a \cdot e_{\omega,i}\, d\mu_\phi \right|^2 \right)^{1/2} \tag{4.30}$$

$$\leqslant \sum_{l=k}^{k+m} \sum_{\omega\in\Sigma_A^l} \frac{C \cdot M^{1/2} \cdot C_k(\alpha(\omega)-1)}{\mu_\phi([\omega])} \left( \sum_{i=1}^{\alpha(\omega)-1} \left| \int_{\Sigma_A^\infty} a \cdot e_{\omega,i}\, d\mu_\phi \right|^2 \right)^{1/2} \tag{4.31}$$

$$= C \cdot M^{1/2} \cdot C_k \cdot \sum_{l=k}^{k+m} \left\| P_l[D_{\mu_\phi}, \pi(a)]\chi_{\Sigma_A^\infty} \right\|_{L^2} \tag{4.32}$$

$$= C \cdot M^{1/2} \cdot C_k \cdot \left\| \sum_{l=k}^{k+m} P_l[D_{\mu_\phi}, \pi(a)]\chi_{\Sigma_A^\infty} \right\|_{L^2} \tag{4.33}$$

$$\leqslant C \cdot M^{1/2} \cdot C_k \cdot \left\| [D_{\mu_\phi}, \pi(a)]\chi_{\Sigma_A^\infty} \right\|_{L^2} \tag{4.34}$$

$$\leqslant C \cdot M^{1/2} \cdot C_k \cdot \|[D_{\mu_\phi}, \pi(a)]\|. \tag{4.35}$$

Further, by using the fact that $D_{\mu_\phi}(\chi_{\Sigma_A^\infty}) = 0$, applying the triangle inequality and applying Parseval's identity (Theorem II.6 of [RS]), for each $k \in \mathbb{N}$ and $a \in C(\Sigma_A^\infty;\mathbb{C})$, we have that

$$\begin{aligned}
\|\pi_0(a) - \pi_k(a)\|_\infty &= \|\pi_0(a) - \pi_1(a) + \pi_1(a) - \pi_k(a)\|_\infty \\
&\leqslant \|\pi_0(a) - \pi_1(a)\|_\infty + \|\pi_1(a) - \pi_k(a)\|_\infty \\
&\leqslant \|[D_{\mu_\phi}, \pi(a)]\chi_{\Sigma_A^\infty}\|_{L^2} + C \cdot M^{1/2} \|[D_{\mu_\phi}, \pi(a)]\| \\
&\leqslant (C \cdot M^{1/2} + 1)\|[D_{\mu_\phi}, \pi(a)]\|.
\end{aligned}$$

Therefore, for each $k, m \in \mathbb{N}_0$ and $a \in A_{D_{\mu_\phi}}$, we have that

$$\|\pi_k(a) - \pi_{k+m}(a)\|_\infty \leqslant C \cdot M^{1/2} + 1. \tag{4.36}$$

Moreover, from the Equations (4.28) - (4.35), it follows that, for each $a \in A_{D_{\mu_\phi}}$, the sequence $(\pi_k(a))_{k\in\mathbb{N}}$ is a Cauchy sequence in the $C^*$-algebra $C(\Sigma_A^\infty;\mathbb{C})$, with respect to the supremum norm $\|\cdot\|_\infty$. Hence, the sequence $(\pi_k(a))_{k\in\mathbb{N}}$ is a convergent sequence. Let $b \in C(\Sigma_A^\infty;\mathbb{C})$ denote the limit of this sequence. Then we have that

$$\pi(b)\chi_{\Sigma_A^\infty} \;=\; \lim_{k\to\infty} \pi(\pi_k(a))\chi_{\Sigma_A^\infty} \;=\; \lim_{k\to\infty} \sum_{m=1}^{k} P_m \pi(a)\chi_{\Sigma_A^\infty} \;=\; \pi(a)\chi_{\Sigma_A^\infty}.$$

Since $\chi_{\Sigma_A^\infty}$ is a separating vector, it follows that $b = a$. Namely, we have that $\pi_k(a)$ converges to $a$ as $k$ tends to infinity, for each $a \in A_{D_{\mu_\phi}}$. Therefore, by Equation (4.36), for each $a \in A_{D_{\mu_\phi}}$, it follows that

$$\|\pi_0(a) - a\|_\infty \;\leqslant\; C \cdot M^{1/2} + 1.$$

This shows that for an arbitrary $a \in A_{D_{\mu_\phi}}$, the quotient norm of $a + \{\pi_0(a) \,:\, a \in A\}$ in the quotient space $A_{D_{\mu_\phi}}/\{\pi_0(a) \,:\, a \in A\}$, is bounded by $C \cdot M^{1/2} + 1$. Therefore, by the first part of Theorem 3.2.6, Connes' pseudo-metric $d_{C(\Sigma_A^\infty;\mathbb{C})}$ is a metric.

In order to show that the topology induced by the metric $d_{C(\Sigma_A^\infty;\mathbb{C})}$ on $\mathcal{S}(C(\Sigma_A^\infty;\mathbb{C}))$ is equivalent to the weak$^*$-topology on $\mathcal{S}(C(\Sigma_A^\infty;\mathbb{C}))$, we observe the following. Firstly, by Equations (4.28) - (4.35) and since $\pi_k(a)$ converges to $a$ as $k$ tends to infinity with respect to the supremum norm, for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all natural numbers $k \geqslant N$ and for each $a \in A_{D_{\mu_\phi}}$, we have that

$$\|a - \pi_k(a)\|_\infty \;<\; \frac{\epsilon}{2}.$$

Secondly, for each $k \in \mathbb{N}$, the space $\pi_k(A)$ is finite dimensional, and so, the closed ball of radius $C \cdot M^{1/2} + 1$ in $\pi_k(A)$ is norm compact. By these observations, it follows that given $\epsilon > 0$, the set

$$\{a - \pi_0(a) \,:\, a \in A_{D_{\mu_\phi}}\}$$

can be covered by a finite number of sets of diameter less than $\epsilon/2$. In other words, the set $\{a - \pi_0(a) \,:\, a \in A_{D_{\mu_\phi}}\}$ is a totally bounded subset of $C(\Sigma_A^\infty;\mathbb{C})$. Consequently, it follows that under the quotient map $C(\Sigma_A^\infty;\mathbb{C}) \to C(\Sigma_A^\infty;\mathbb{C})/\{\pi_0(a) \,:\, a \in A\}$, the image of $A_{D_{\mu_\phi}}$ is totally bounded. The result then follows from the second part of Theorem 3.2.6. ■

Let us now consider the metric dimension and measure theoretical aspects of the spectral triple $(C(\Sigma_A^\infty;\mathbb{C}), L^2(\Sigma_A^\infty, \mathcal{B}, \mu_\phi), D_{\mu_\phi})$, as given in Theorem 4.2.5.

**Theorem 4.2.7.** *Let $\phi \in C(\Sigma_A^\infty;\mathbb{R})$ denote a Hölder continuous non-arithmetic potential function and let $\nu_\phi$ denote the unique equilibrium measure for the potential $\phi$. Then the spectral triple $(C(\Sigma_A^\infty;\mathbb{C}), L^2(\Sigma_A^\infty, \mathcal{B}, \nu_\phi), D_{\nu_\phi})$ is $(1,+)$-summable with metric dimension equal to one. Moreover, for each $a \in C(\Sigma_A^\infty;\mathbb{C})$, we have that*

$$⨍ \pi(a)|D_{\nu_\phi}|^{-1} \;=\; \frac{1}{h_{\nu_\phi}(\sigma)} \int_{\Sigma_A^\infty} a \, d\nu_\phi. \tag{4.37}$$

*In particular, we have that*

$$V(C(\Sigma_A^\infty;\mathbb{C}), L^2(\Sigma_A^\infty, \mathcal{B}, \nu_\phi), D_{\nu_\phi}) \;=\; \frac{1}{h_{\nu_\phi}(\sigma)}. \tag{4.38}$$

PROOF. For each $\mathcal{V} \in \Sigma_A^* \cup \emptyset$, let $\Upsilon_{\mathcal{V}}, \Xi_{\mathcal{V}} : (0,\infty) \to [0,\infty)$ denote the functions that are respectively defined in Equation (2.26) and Equation (2.27). Let $r \in (0,1)$ be fixed and let $\mathrm{card}(\Sigma) =: M \in \mathbb{N}$.

For each $k \in \mathbb{N}$ and $a \in C(\Sigma_A^\infty;\mathbb{C})$, recall that $\sigma_k(\pi(a)|D_{\nu_\phi}|^{-1})$ denotes the $k$-th largest singular value of the operator $\pi(a)|D_{\nu_\phi}|^{-1} \in B(\ker(D_{\nu_\phi})^\perp)$. Further, since $D_{\nu_\phi}$ has a compact resolvent, the sequence of singular values $(\sigma_k(|D_{\nu_\phi}|^{-1}))_{k\in\mathbb{N}}$ of the operator $|D_{\nu_\phi}|^{-1} \in B(\ker(D_{\nu_\phi})^\perp)$ converges to zero as $k$ tends to infinity. Therefore, for each $k \in \mathbb{N}$, there exists $\eta_k \in \mathbb{N}$ such that $r^{\eta_k} \leqslant \sigma_k(|D_{\nu_\phi}|^{-1}) < r^{\eta_k - 1}$ and such that $\eta_k$ tends to infinity as $k$ tends to infinity. Hence, for each $\mathcal{V} \in \Sigma_A^* \cup \emptyset$ there exists a positive constant $c$ such that for $N \in \mathbb{N}$ sufficiently large, we have that

$$\Xi_{\mathcal{V}}\left(Mr^{\eta_N - 1}\right) \;\leqslant\; \sum_{k=1}^{N} \sigma_k(\pi(\chi_{[\mathcal{V}]})|D_{\nu_\phi}|^{-1}) \;\leqslant\; c + \Xi_{\mathcal{V}}(r^{\eta_N}), \tag{4.39}$$

$$\ln\left(\Upsilon_{\mathcal{V}}\left(Mr^{\eta_N - 1}\right)\right) \;\leqslant\; \ln(N) \;\leqslant\; \ln\left(c + M\Upsilon_{\mathcal{V}}(r^{\eta_N})\right). \tag{4.40}$$

The pressure of the potential function $\phi - P(\phi,\sigma)$ is equal to zero and that by Theorem 2.2.8, we have that the unique equilibrium measure $\nu_{\phi - P(\phi,\sigma)}$ for the potential function $\phi - P(\phi,\sigma)$ is equal to $\nu_\phi$. Then, using the inequalities given in Equation (4.39) and Equation (4.40) and by the results of Corollary 2.3.7 and Corollary 2.3.9, for each $\mathcal{V} \in \Sigma_A^* \cup \emptyset$, we have that

$$\begin{aligned}
\liminf_{N\to\infty} \frac{\sum_{k=1}^{N} \sigma_k(\pi(\chi_{[\mathcal{V}]})|D_{\nu_\phi}|^{-1})}{\ln(N)} \quad &\leqslant \quad \liminf_{N\to\infty} \frac{c + \Xi_{\mathcal{V}}(r^{\eta_N})}{\ln(\Upsilon_{\mathcal{V}}(Mr^{\eta_N - 1}))} \\
&= \quad \liminf_{N\to\infty} \frac{c + \ln(r^{\eta_N})\,\nu_\phi([\mathcal{V}])\left(\int_{\Sigma_A^\infty} \phi\, d\nu_\phi\right)^{-1}}{\ln(M^{-1} r^{-\eta_N + 1})} \\
&= \quad \frac{\nu_\phi([\mathcal{V}])}{h_{\nu_\phi}(\sigma)}.
\end{aligned}$$

Moreover, by a similar argument, one can deduce that

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^{N} \sigma_k(\pi(\chi_{[\mathcal{V}]})|D_{\nu_\phi}|^{-1})}{\ln(N)} \;\geqslant\; \frac{\nu_\phi([\mathcal{V}])}{h_{\nu_\phi}(\sigma)}.$$

Therefore, for each $\mathcal{V} \in \Sigma_A^\infty \cup \emptyset$, we have that the Dixmier trace of the operator $\pi(\chi_{[\mathcal{V}]})|D_{\nu_\phi}|^{-1}$ is independent of the limiting procedure. Moreover, it follows that

$$⨍ \pi(\chi_{[\mathcal{V}]})|D_{\nu_\phi}|^{-1} \;=\; \frac{\nu_\phi([\mathcal{V}])}{h_{\nu_\phi}(\sigma)}.$$

In particular, we have that the noncommutative volume of the spectral triple $(C(\Sigma_A^\infty;\mathbb{C}), L^2(\Sigma_A^\infty, \mathcal{B}, \nu_\phi), D_{\nu_\phi})$ is equal to $1/h_{\nu_\phi}(\sigma)$. Subsequently, by Definition 3.2.19, it follows that the metric dimension of the spectral triple $(C(\Sigma_A^\infty;\mathbb{C}), L^2(\Sigma_A^\infty, \mathcal{B}, \nu_\phi), D_{\nu_\phi})$ is equal to 1. Next, note that for each limiting procedure $\mathcal{W}$, the operator defined, for each $a \in C(\Sigma_A^\infty;\mathbb{C})$, by

$$a \;\mapsto\; \mathrm{Tr}_{\mathcal{W}}(\pi(a)|D_{\nu_\phi}|^{-1}),$$

is a bounded linear functional on $C(\Sigma_A^\infty;\mathbb{C})$. Hence, by the Riesz Representation Theorem (Theorem 3.3.1), there exists a finite Borel measure $\nu$ such that, for each $a \in C(\Sigma_A^\infty;\mathbb{C})$, we have that

$$\mathrm{Tr}_{\mathcal{W}}(\pi(a)|D_{\nu_\phi}|^{-1}) \;=\; \int_{\Sigma_A^\infty} a\, d\nu.$$

Further, the set $R := \{[\omega] \,:\, \omega \in \Sigma_A^\infty\} \cup \{\emptyset, \Sigma_A^\infty\}$ forms a semi-ring on which the set function $\Lambda : R \to [0,\infty)$ given, for each $I \in R$, by

$$\Lambda(I) \;:=\; \mathrm{Tr}_{\mathscr{W}}(\pi(\chi_I)|D_{\nu_\phi}|^{-1}),$$

is an additive $\sigma$-additive set function (see Definition 3.3.4). Therefore, since $\Lambda$ is also $\sigma$-finite, by the Hahn-Kolmogorov Theorem (Theorem 3.3.5), for an arbitary limiting procedure $\mathscr{W}$ and for each $a \in C(\Sigma_A^\infty;\mathbb{C})$, we have that

$$\mathrm{Tr}_{\mathscr{W}}(\pi(a)|D_{\nu_\phi}|^{-1}) \;=\; \frac{1}{h_{\nu_\phi}(\sigma)} \int_{\Sigma_A^\infty} a\, d\nu_\phi.$$

Moreover, for each $a \in C(\Sigma_A^\infty;\mathbb{C})$, we have that

$$⨍ \pi(a)|D_{\nu_\phi}|^{-1} \;=\; \frac{1}{h_{\nu_\phi}(\sigma)} \int_{\Sigma_A^\infty} a\; d\nu_\phi.$$

■

*Remark. In certain cases, the condition of $\phi$ being non-arithmetic can be weakened. This can be done, for instance, when $(\Sigma_A^\infty, \sigma)$ is the full shift space. More specifically, for $M \in \mathbb{N}$, consider the case when $\Sigma := \{1, 2, \ldots M\}$ and when the potential function $\phi$ is given as follows. Let $p := (p_1, p_2, \ldots, p_M)$ denote a probability vector with $p_i$ non-zero for all $i \in \{1, 2, \ldots, M\}$. Then let $\phi : \Sigma_A^\infty \to \mathbb{R}$ be given, for each $\omega := (\omega_1, \omega_2, \ldots,) \in \Sigma_A^\infty$, by*

$$\phi(\omega) \;:=\; -\ln(p_{\omega_1}).$$

*Note that if the set $\{\ln(p_1), \ln(p_2), \ldots, \ln(p_M)\}$ is arithmetic then the potential function $\phi$ will be arithmetic, and if the set $\{\ln(p_1), \ln(p_2), \ldots, \ln(p_M)\}$ is non-arithmetic then the potential function $\phi$ will be non-arithmetic. Let $\nu_\phi$ denote the unique equilibrium measure for the potential $\phi$ and observe that, for each $k \in \mathbb{N}$ and each $\omega := (\omega_1, \omega_2, \ldots, \omega_k) \in \Sigma^k$, we have that*

$$\nu_\phi([\omega]) \;:=\; p_{\omega_1} p_{\omega_2} \ldots p_{\omega_k}.$$

*Then by Theorem 4.2.5, the triple $(C(\Sigma_A^\infty;\mathbb{C}), L^2(\Sigma_A^\infty, \mathcal{B}, \nu_\phi), D_{\nu_\phi})$ is a spectral triple. Moreover, by an argument synonymous to that used in the proof of Proposition 4.1.6, one can deduce that the spectral triple $(C(\Sigma_A^\infty;\mathbb{C}), L^2(\Sigma_A^\infty, \mathcal{B}, \nu_\phi), D_{\nu_\phi})$ has metric dimension equal to one and that the noncommutative volume is equal to $1/h_{\nu_\phi}(\sigma)$. Further, the measure $\nu_\phi$ can be recovered from the noncommutative integration theory of the spectral triple $(C(\Sigma_A^\infty;\mathbb{C}), L^2(\Sigma_A^\infty, \mathcal{B}, \nu_\phi), D_{\nu_\phi})$.*

*Remark. In [BP], Bellissard and Pearson presents an alternative spectral triple to that considered here, which represents the full shift space $\Sigma$ on two symbols equipped with an ultra-metric. An example of such an ultra metric is given, for $\omega, \upsilon \in \Sigma$ by*

$$d_{\nu_\phi}(\omega, \upsilon) \;:=\; \inf\{\nu_\phi([x]) \,:\, x \in \Sigma^* \cup \{\emptyset\} \textit{ and } \omega, \upsilon \in [x]\},$$

*where $\nu_\phi$ is an equilibrium measure for a Hölder continuous potential function $\phi \in C(\Sigma;\mathbb{R})$. For such a metric, our results give that the noncommutative volume constant of Bellissard and Pearson's spectral triple is equal to $2/h_{\nu_\phi}(\sigma)$.*

# Appendix A: Operator Theory and $C^*$-algebras

## A.1 Basic Definitions

In this section we define some basic objects and discuss some fundamental results from functional analysis which are in use within this thesis. There is an extensive literature available on the subject, with good overviews found in [Rud, RS]. For the proofs of the results stated in this section we refer the reader to [Rud, RS].

Let $H$ denote a complex separable Hilbert space and let $B(H)$ denote the set of bounded linear operators $T : H \to H$. The *spectrum* of such an operator $T$ is defined to be the set

$$\sigma(T) \;:=\; \{z \in \mathbb{C} \,:\, T - z\mathbb{1} \text{ is not invertible}\}.$$

The *resolvent set* of an operator $T \in B(H)$ is defined to be the complement of $\sigma(T)$ in $\mathbb{C}$.

The *adjoint* of an operator $T \in B(H)$ is denoted by $T^*$ and is defined to be the unique operator, which, for all $h_1, h_2 \in H$, satisfies the following equality

$$\langle T(h_1), h_2\rangle \;=\; \langle h_1, T^*(h_2)\rangle.$$

We say that $T \in B(H)$ is *self-adjoint* if and only if $T = T^*$. Note that the spectrum of a self-adjoint operator is fully contained in $\mathbb{R}$.

*Remark. The notions of the spectrum, the resolvent set and the adjoint for an unbounded densely defined operator on a complex separable Hilbert space also exist. For this case we refer the reader to Appendix A.2.*

An operator $T \in B(H)$ defined on a complex separable Hilbert space $H$ is said to be *compact* if one of the following equivalent conditions hold.

1. The closure of the image of the closed unit ball in $H$ under $T$ is compact.

2. For any bounded sequence $(h_k)_{k\in\mathbb{N}}$ in $H$, the sequence $(T(h_k))_{k\in\mathbb{N}}$ contains a convergent subsequence.

Observe that for a bounded operator $T \in B(H)$, defined on a complex separable Hilbert space $H$, the following are equivalent.

1. The operator $T$ is compact.

2. The operator $T^*$ is compact.

3. The operator $TT^*$ is compact

Further, note that the set $K(H)$ of compact operators on a complex separable Hilbert space $H$ is a two sided ideal of $B(H)$. Moreover, given a compact operator $T \in B(H)$, one has that for each $\epsilon > 0$ there exists a finite dimensional subspace $K \subset H$ such that the norm of $T$ restricted to $K^\perp \subset H$ is smaller then $\epsilon$. (Here, $K^\perp$ denotes the orthogonal complement of $K$ in $H$.) In other words, the ideal of compact operators is the operator-norm closure of the set of operators with finite dimensional range. Also, note that the spectrum $\sigma(T)$ of a compact operator $T$ has the following properties.

1. The spectrum $\sigma(T)$ of $T$ is compact and non-empty.

2. The cardinality of $\sigma(T)$ is at most countable.
3. The spectrum $\sigma(T)$ of $T$ contains at most one limit point and in the case that there exists a limit point the limit point is equal to zero.

Moreover, one has the following result.

**Theorem A.1.1.** *(Spectral Theorem for Self-adjoint Compact Opeators) Let $H$ denote a complex separable Hilbert space. For each self-adjoint $T \in K(H)$ there exists a finite or infinite set $\{\phi_k\}_{k\in\mathbb{N}}$ of eigenvectors of $T$, with corresponding real non-zero eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$, such that, for every $h \in H$,*

$$T(h) \;=\; \sum_{n\in\mathbb{N}} \lambda_n \langle h, \phi_n\rangle \phi_n.$$

*Assuming that the eigenvalues are listed in a non-increasing order, if the set $\{\lambda_k\}_{k\in\mathbb{N}}$ is infinite, then $\lambda_k$ tends to zero as $k$ tends to infity.*

An operator $T \in B(H)$ defined on a complex separable Hilbert space $H$ is said to have a *compact resolvent* if for all $z \notin \sigma(T)$ the operator $(T - z\mathbb{1})^{-1}$ is compact.

**Proposition A.1.2.** *Let $H$ denote a complex separable Hilbert space. Then an operator $T \in B(H)$ has compact resolvent if and only if there exists a complex number $z \notin \sigma(T)$ such that $(T - z\mathbb{1})^{-1}$ is compact.*

Proof. The forward implication follows trivially. For the reverse implication let $z_1, z_2 \notin \sigma(T)$ and observe that

$$(T - z_1\mathbb{1})((T - z_1\mathbb{1})^{-1} - (T - z_2\mathbb{1})^{-1})(T - z_2\mathbb{1}) \;=\; (T - z_2\mathbb{1}) - (T - z_1\mathbb{1}) \;=\; z_1\mathbb{1} - z_2\mathbb{1}.$$

This implies that

$$(T - z_1\mathbb{1})^{-1} - (T - z_2\mathbb{1})^{-1} \;=\; (z_1 - z_2)(T - z_1\mathbb{1})^{-1}(T - z_2\mathbb{1})^{-1}.$$

Therefore, $(T - z_1\mathbb{1})^{-1}$ is compact if and only if $(T - z_2\mathbb{1})^{-1}$ is compact. ■

**Proposition A.1.3.** *If $T \in B(H)$ denotes a self-adjoint operator defined on a complex separable Hilbert space $H$, then $T$ has a compact resolvent if and only if $(T^2 + \mathbb{1})^{-1}$ is a compact operator.*

Proof. Since $T$ is a self-adjoint operator, we have that $\pm i \notin \sigma(T)$. Therefore, $T$ has a compact resolvent if and only if the operators $(T \pm i\mathbb{1})^{-1}$ are compact, which is if and only if the operator

$$(T^* - i\mathbb{1})^{-1}(T + i\mathbb{1})^{-1} \;=\; (T^2 + \mathbb{1})^{-1}$$

is compact. ■

To conclude this section, we define the operator theoretical analogue of a positive number and state the Square Root Lemma.

**Definition A.1.4.** Let $H$ denote a complex separable Hilbert space. An operator $T \in B(H)$ is called *positive* if $\langle T(h), h\rangle \in [0, +\infty)$, for all $h \in H$. We write $T \geqslant 0$ if $T$ is a positive operator and $T_1 \leqslant T_2$ if $T_2 - T_1 \geqslant 0$, where $T_1, T_2 \in B(H)$.

Note that every bounded positive operator on a complex Hilbert space $H$ is self-adjoint. Moreover, for any $T \in B(H)$, we have that the operator $T^*T$ is positive.

**Theorem A.1.5.** *(Square Root Lemma) Let $H$ denote a complex separable Hilbert space and let $T \in B(H)$ denote a positive operator. Then there exists a unique $S \in B(H)$ with $S \geqslant 0$ and $T = S^2$. Furthermore, $S$ commutes with every bounded operator which commutes with $T$.*

**Definition A.1.6.** In the setting of Theorem A.1.5, we call $S$ the square root of $T$ and write $T^{1/2}$ $(:= S)$.

**Definition A.1.7.** Let $H$ denote a complex separable Hilbert space. The *modulus of an operator* $T \in B(H)$ is denoted by $|T|$ and is defined by $|T| := (T^*T)^{1/2}$. Further, $x \in \mathbb{R}$ is said to be a *singular value of $T$ with multiplicity* $k \in \mathbb{N}$ if and only if $x$ is an eigenvalue of $|T|$ of multiplicity $k$.

## A.2 Symmetric and Self-Adjoint Unbounded Operators

In this section we review some of the basic definitions for unbounded operators. There are several texts on functional analysis which deal with unbounded operators, for instance [Rud, RS, BR]. In this section we closely follow Chapter VIII of [RS] and refer the reader to [RS] for the proofs of the stated results.

Let $H$ denote a complex Hilbert space, let $T$ denote a densely defined linear operator on $H$ and let $\mathrm{Dom}(T)$ denote the domain of $T$. The *graph* of $T$ is then defined to be the set

$$\Gamma(T) \;:=\; \{(h, T(h)) \,:\, h \in \mathrm{Dom}(T)\} \;\subseteq\; H \times H.$$

If $\Gamma(T)$ is a closed subset of $H \times H$, then $T$ is said to be *closed*. An operator $T$ is called *closable* if it has a closed extension. Note that every closable operator has a smallest closed extension called its closure which will be denoted by $\overline{T}$.

**Definition A.2.1.** Let $T$ denote a densely defined linear operator on a complex Hilbert space $H$. Define $\mathrm{Dom}(T^*)$ to be the set of $h_1 \in H$ for which there exists a $h_2 \in H$ such that, for all $h_3 \in \mathrm{Dom}(T)$, one has that

$$\langle T(h_3), h_1\rangle \;=\; (h_3, h_2). \tag{A.1}$$

The *adjoint* of $T$ is then defined to be the linear operator $T^* : H \to H$ given by $T^*(h_1) := h_2$. Note that $T^*$ is well defined, since $\mathrm{Dom}(T)$ is a dense subset of $H$.

*Remark. Given a densely defined unbounded operator $T$, unlike for bounded operators, the domain of $T^*$ may not be dense in $H$.*

**Example A.2.2.** Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a bounded measurable function which is not square integrable with respect to the Lebesgue measure. Let $h_0 \in L^2(\mathbb{R}, \mathcal{B}, \lambda^1)$ be fixed and define the linear operator $T$ on $L^2(\mathbb{R}, \mathcal{B}, \lambda^1)$ with domain

$$\mathrm{Dom}(T) \;:=\; \left\{ h \in L^2(\mathbb{R}, \mathcal{B}, \lambda^1) \,:\, \int |f \cdot h| \, d\lambda^1 < \infty \right\}$$

by

$$T(h) \;:=\; \langle h, f\rangle h_0.$$

Since the domain of $T$ contains all square integrable functions with compact support, $T$ is a densely defined operator. Now, for each $h_1 \in \mathrm{Dom}(T^*)$ and each $h_2 \in \mathrm{Dom}(T)$, we have that

$$\langle h_2, T^*(h_1)\rangle \;=\; \langle T(h_2), h_1\rangle \;=\; \langle\langle h_2, f\rangle h_0, h_1\rangle \;=\; \langle h_2, f\rangle\langle h_0, h_1\rangle \;=\; \langle h_2, \overline{\langle h_0, h_1\rangle} f\rangle.$$

Therefore, for each $h_1 \in \mathrm{Dom}(T^*)$, it follows that

$$T^*(h_1) \;=\; \langle h_1, h_0\rangle f.$$

Hence, for all $h_1 \in \mathrm{Dom}(T^*)$, we have that $\langle h_1, h_0\rangle = 0$. Thus, the domain of $T^*$ consists of the orthogonal complement of the vector $h_0$. In particular, we have that the domain of $T^*$ is not dense in $H$ and that it vanishes on its domain of definition.

**Definition A.2.3.** Let $T$ denote a densely defined linear operator on a complex Hilbert space $H$. The *resolvent set* of $T$ is defined to be the set of all $z \in \mathbb{C}$ such that the operator $T - z\mathbb{1}$ is a bijective mapping from $\mathrm{Dom}(T)$ into $H$ whose inverse belongs to $B(H)$. Further, the *spectrum* of an unbounded operator is defined to be the complement of the resolvent set.

**Definition A.2.4.** A densely defined linear operator $T$ on a complex Hilbert space is called *symmetric* if $\mathrm{Dom}(T) \subseteq \mathrm{Dom}(T^*)$ and $T(h) = T^*(h)$, for all $h \in \mathrm{Dom}(T)$. If, in addition, $\mathrm{Dom}(T) = \mathrm{Dom}(T^*)$, then we call $T$ *self-adjoint*.

**Theorem A.2.5.** *Let $T$ denote a densely defined linear operator on a complex Hilbert space $H$. Then we have the following.*

1. *The operator $T^*$ is closed.*
2. *The operator $T$ is closable if and only if Dom($T^*$) is dense, in which case $\overline{T} = T^{**}$.*
3. *If $T$ is closable then $\left(\overline{T}\right)^* = T^*$.*

**Corollary A.2.6.** *Let $T$ denote a densely defined linear operator on a complex Hilbert space $H$. Then we have the following.*

1. *If $T$ is symmetric then Dom($T$) $\subseteq$ Dom($T^{**}$) $\subseteq$ Dom($T^*$).*
2. *If $T$ is closed and symmetric, then $T = T^{**}$ and Dom($T$) $\subseteq$ Dom($T^{**}$) $\subseteq$ Dom($T^*$).*
3. *If $T$ is self-adjoint, then $T = T^{**} = T^*$ and Dom($T$) $=$ Dom($T^{**}$) $=$ Dom($T^*$).*

*Remark. The distinction between closed symmetric operators and self-adjoint operators is important, as it is only for self-adjoint operators that one can formulate a spectral theorem.*

Another important type of operator is an essentially self-adjoint operator. We note that most differential operators, although they are symmetric, are not self-adjoint. However, they are often essentially self-adjoint.

**Definition A.2.7.** A symmetric operator is called *essentially self-adjoint* if its closure is self-adjoint.

Observe that if $T$ is an essentially self-adjoint operator on a complex Hilbert space, then it has a unique closed self-adjoint extension. This follows since if $S$ is a self-adjoint extension of $T$, then $S$ is closed, and so, since $\mathrm{Dom}(T) \subseteq \mathrm{Dom}(S)$, we have that $\mathrm{Dom}(T^{**}) \subseteq \mathrm{Dom}(S)$. Thus, $\mathrm{Dom}(S) = \mathrm{Dom}(S^*) \subseteq \mathrm{Dom}((T^{**})^*) = \mathrm{Dom}(T^{**})$ and so $S = T^{**}$.

The following theorem gives the basic criteria for a densely defined symmetric operator to be self-adjoint and/or essentially self-adjoint.

**Theorem A.2.8.** *Let $T$ be a densely defined symmetric operator on a Hilbert space $H$. Then, $T$ is self-adjoint if and only if Ran($T \pm i\mathbb{1}$) $= H$, where Ran($T \pm i\mathbb{1}$) detnotes the range of the operator $T \pm i\mathbb{1}$. Moreover, $T$ is esentially self-adjoint if and only if Ran($T \pm i\mathbb{1}$) are dense in $H$.*

Proof. See Theorem VIII.3 and the corollary that follows of [RS]. ■

## A.3 The Dixmier Ideal and The Dixmier Trace

Here we give a complete proof (of our own design) of the fact that for a complex separable Hilbert space $H$, the Dixmier ideal $\mathcal{L}^{1,+}(H)$ is an ideal of $B(H)$ and that the Dixmier trace is a singular trace defined on $\mathcal{L}^{1,+}(H)$. The original proof can be found in [Dix1]. Before proving the main results we give several eigenvalue inequalities which will be required. Further, throughout this section, we let $H$ denote a complex separable Hilbert space.

**Lemma A.3.1.** *For each $T \in K(H)$ and each $k \in \mathbb{N}$, we have that $\sigma_k(T) = \sigma_k(T^*)$. (Recall that $\sigma_k(T)$ denotes the $k$-th largest singular value of $T$, where $k \in \mathbb{N}$.)*

Proof. If $h_1 \in H$ is a non-zero eigenvector of $T^*T$ with eigenvalue $z_1$, then $T^*T(h_1)-z_1 h_1 = 0$, and so $TT^*T(h_1) - z_1 T(h_1) = 0$. Therefore, $T(h_1)$ is a non-zero eigenvector of $TT^*$ with the eigenvalue $z_1$. Similarly, if $h_2$ is an eigenvector of $TT^*$ with eigenvalue $z_2$, then $T^*(h_2)$ is an eigenvector of $T^*T$ with eigenvalue $z_2$. Further, if $h_3$ and $h_4$ are two non-zero orthogonal eigenvectors of $T^*T$ with non-zero eigenvalue $z_3$, then we have that

$$\langle T(h_3), T(h_4)\rangle \;=\; \langle T^*T(h_3), h_4\rangle \;=\; z_3\langle h_3, h_4\rangle \;=\; 0.$$

Thus, $T^*T$ and $TT^*$ have the same eigenvalues with the same multiplicity. Note that we have implicitly used the assumption that $T$ is compact, since we have used the fact that the eigenspace of an eigenvalue of a compact operator is finite dimensional. ■

**Lemma A.3.2.** *For each positive $T \in K(H)$ and each $k \in \mathbb{N}$, we have that*

$$\sigma_k(T) = \inf\left\{ \|T(\mathbb{1} - P)\| \,:\, P \in B(H),\, P = P^2 = P^* \text{ and } \dim(P(H)) = k-1 \right\}.$$

Proof. Assume, without loss of generality that the dimension of the range of $T$ is countably infinite, since, in the case that the dimension of range of $T$ is finite, a simplification of the following argument will give the required result. Further, since $T$ is a positive compact operator there exists a set of orthonormal vectors $\{\phi_k \in H \,:\, k \in \mathbb{N}\}$ such that, for each $h \in H$, we have that

$$T(h) \;=\; \sum_{k\in\mathbb{N}} \sigma_k(T)\langle h, \phi_k\rangle\phi_k. \tag{A.2}$$

Now, for each $k \in \mathbb{N}$, let $H_k$ denote the complex linear span of the set $\{\phi_m\}_{m=1}^{k-1}$ and let $P_k : H \to H$ denote the orthogonal projection from $H$ onto the subspace $H_k \subset H$. Next, for each $k \in \mathbb{N}$ and each $h \in H$, observe that

$$\begin{aligned} T(\mathbb{1} - P_k)(h) &= \sum_{m\in\mathbb{N}} \sigma_m(T)\left\langle \sum_{l\geqslant k}\langle h, \phi_l\rangle\phi_l, \phi_m\right\rangle \phi_m \\ &= \sum_{m\in\mathbb{N}} \sigma_m(T)\sum_{l\geqslant k}\langle h, \phi_l\rangle\langle\phi_l, \phi_m\rangle\phi_m \\ &= \sum_{l\geqslant k} \sigma_l(T)\langle h, \phi_l\rangle\phi_l. \end{aligned}$$

Therefore, $\sigma_l(T)$ is an eigenvalue of $T(\mathbb{1} - P_k)$ with eigenvector $\phi_l$, for each $k \in \mathbb{N}$ and each natural number $l \geqslant k$. Moreover, for each $k \in \mathbb{N}$, we claim that

$$\sigma(T(1 - P_k)) \;=\; \{\sigma_l \,:\, l \in \mathbb{N}\,\text{and}\, l \geqslant k\}.$$

To see this, fix a $k \in \mathbb{N}$ and suppose that there exists a non-zero eigenvalue

$$z \in \mathbb{C} \setminus \{\sigma_l(T) : l \in \mathbb{N} \text{ and } l \geqslant k\}$$

of $T(\mathbb{1} - P_k)$ with eigenvector $\mathfrak{h}$. Since, for each natural number $l \geqslant k$, we have that $\phi_l$ is an eigenvalue of $T(\mathbb{1} - P_k)$, it follows that $\langle \mathfrak{h}, \phi_l \rangle = 0$, for all natural numbers $l \geqslant k$. Therefore, we have that $T(\mathbb{1} - P_k)(\mathfrak{h}) = 0$. This is a contradiction to the assumption that $z$ is non-zero. Therefore, we conclude that $\|T(\mathbb{1} - P_k)\| \leqslant \sigma_k(T)$, for each $k \in \mathbb{N}$. Hence, for each $k \in \mathbb{N}$, we have that

$$\inf\left\{\|T(\mathbb{1} - P)\| : P = P^2 = P^* \text{ and } \dim(P(H)) = k-1\right\} \leqslant \sigma_k(T).$$

In order to complete the proof we are required to show that for each $k \in \mathbb{N}$, we have that

$$\inf\left\{\|T(\mathbb{1} - P)\| : P = P^2 = P^* \text{ and } \dim(P(H)) = k-1\right\} \geqslant \sigma_k(T).$$

However, this is an immediate consequence of the following inequalities.

$$\sigma_k(T) \leqslant \inf_{\substack{W \subset H \\ \dim(W)=k-1}} \sup_{\substack{h \in W^{\perp} \\ \|h\| \leqslant 1}} \langle T(h), h \rangle, \tag{A.3}$$

$$\|T\| \geqslant \sup_{\substack{h \in H \\ \|h\| \leqslant 1}} |\langle T(h), h \rangle|. \tag{A.4}$$

The inequality given in Equation (A.4) follows directly from the Cauchy-Schwarz inequality. Indeed we have that

$$\|T\| = \sup_{\|x\| \leqslant 1} \|T(x)\| \geqslant \sup_{\|x\| \leqslant 1} \|T(x)\| \|x\| \geqslant \sup_{\|x\| \leqslant 1} |\langle T(x), x \rangle|.$$

Let us now prove the inequality given in Equation (A.3). Let $k \in \mathbb{N}$ be fixed and consider a $(k-1)$-dimensional subspace $W$ of $H$. Let $\{e_m\}_{m=1}^{k-1}$ denote an orthonormal basis for $W$ and let $\{e_m\}_{m \in \mathbb{N} \setminus \{1,2,\ldots k-1\}}$ denote a set of orthonormal vectors of $H$, such that the set $\{e_m\}_{m \in \mathbb{N}}$ is an orthonormal basis for $H$. Let $\{\phi_m\}_{m \in \mathbb{N}}$ denote the orthonormal basis as given in Equation (A.2) and let $V$ denote the closed linear span of $\{\phi_m : m \in \{1, 2, \ldots, k\}\}$. Suppose that $V \cap W^{\perp} = \{0\}$, then for all $h \in V$, we have that

$$h = \sum_{m=1}^{k-1} \langle h, e_m \rangle e_m.$$

This implies that $k = \dim(V) \leqslant k - 1$, and so, provides a contradiction to the assumption that $V \cap W^{\perp} = \{0\}$. Therefore, $V \cap W^{\perp}$ contains a non-zero vector. Let $h \in V \cap W^{\perp}$ with norm equal to one and observe that

$$\begin{aligned}
\langle T(h), h \rangle &= \left\langle \sum_{m=1}^{k} \langle h, \phi_m \rangle T(\phi_m), \sum_{l=1}^{k} \langle h, \phi_l \rangle \phi_l \right\rangle \\
&= \sum_{m=1}^{k} \sum_{l=1}^{k} \langle h, \phi_m \rangle \overline{\langle h, \phi_l \rangle} \sigma_m(T) \langle \phi_m, \phi_l \rangle \\
&\geqslant \sigma_k(T) \sum_{m=1}^{k} |\langle h, \phi_m \rangle|^2 \\
&= \sigma_k(T).
\end{aligned}$$

(Note that the final inequality follows from Parseval's identity (Theorem II.6 of [RS])). Hence, we have that

$$\sup_{h \in W^{\perp} \text{ with } \|h\| \leqslant 1} \langle T(h), h \rangle \;\geqslant\; \sigma_k(T).$$

Since the right-hand side of this inequality is independent of the chosen subspace $W$, the result follows. ■

**Corollary A.3.3.** *For each positive operator $T \in K(H)$ and each $k \in \mathbb{N}$, we have that*

$$\sigma_k(T) \;=\; \inf\{\|T(\mathbb{1} - P)\| \,:\, P = P^2 = P^* \text{ and } \dim(P(H)) \leqslant k-1\}.$$

**Definition A.3.4.** For each $k \in \mathbb{N}$, let $S_k : B(H) \to [0, \infty)$ be defined, for each $T \in B(H)$, by

$$S_k(T) \;:=\; \sum_{m=1}^{k} \sigma_m(T).$$

**Lemma A.3.5.** *Let $T \in B(H)$ denote a positive compact operator. Then for each $k \in \mathbb{N}$, we have that*

$$\begin{aligned} S_k(T) \;&=\; \sup\left\{ tr(TP) \,:\, P = P^2 = P^* \text{ and } \dim(P(H)) = k \right\} && \text{(A.5)} \\ &=\; \sup\left\{ tr(PTP) \,:\, P = P^2 = P^* \text{ and } \dim(P(H)) = k \right\}. && \text{(A.6)} \end{aligned}$$

Proof. Let us begin by showing the equality given in Equation (A.5). Let $k \in \mathbb{N}$ be fixed. Then, for each projection $P \in B(H)$ such that $\dim(P(H)) = k$, it is clear to see that $\sigma_k(T) \geqslant \sigma_k(TP)$. Therefore, since $T$ is positive, it follows that $S_k(T) \geqslant \operatorname{tr}(TP)$. Hence, we have that

$$S_k(T) \;\geqslant\; \sup\left\{ \operatorname{tr}(TP) \,:\, P = P^2 = P^* \text{ and } \dim(P(H)) = k \right\}.$$

On the other hand, let $\{\phi_m\}_{m=1}^k$ denote the set of eigenvectors of $T$ with corresponding eigenvalues $\sigma_m(T)$. Let $P \in B(H)$ denote the projection given by

$$P \;:=\; \sum_{m=1}^{k} \langle \cdot, \phi_m \rangle \phi_m.$$

Then, it immediately follows that $S_k(T) \leqslant \operatorname{tr}(TP)$, and so, we have that

$$S_k(T) \;=\; \sup\left\{ \operatorname{tr}(TP) \,:\, P = P^2 = P^* \text{ and } \dim(P(H)) = k \right\}.$$

Let us now show the equality given in Equation (A.6). Observe that for any two traceclass operators $T_1, T_2 \in B(H)$, we have that $\operatorname{tr}(T_1 T_2) = \operatorname{tr}(T_2 T_1)$. Therefore, if $T \in B(H)$ is a compact operator and $P \in B(H)$ is a projection from H onto a finite dimensional subspace of $H$, it follows that

$$\operatorname{tr}(PTP) \;=\; \operatorname{tr}(TPP) \;=\; \operatorname{tr}(TP).$$

This then completes the proof. ■

**Lemma A.3.6.** *For any two positive compact operators $T_1, T_2 \in B(H)$ and for each $k \in \mathbb{N}$, we have that*

$$S_k(T_1 + T_2) \;\leqslant\; S_k(T_1) + S_k(T_2).$$

PROOF. Let $T_1, T_2 \in B(H)$ denote two positive compact operators and let $k \in \mathbb{N}$ be fixed. Then, by Theorem A.3.5, we have that

$$\begin{aligned} S_k(T_1+T_2) &= \sup\left\{\operatorname{tr}((T_1+T_2)P) \,:\, P=P^2=P^* \text{ and } \dim(P(H))=k\right\} \\ &= \sup\left\{\operatorname{tr}(T_1P)+\operatorname{tr}(T_2P) \,:\, P=P^2=P^* \text{ and } \dim(P(H))=k\right\} \\ &\leqslant \sup\left\{\operatorname{tr}(T_1P) \,:\, P=P^2=P^* \text{ and } \dim(P(H))=k\right\} + \\ &\qquad \sup\left\{\operatorname{tr}(T_2P) \,:\, P=P^2=P^* \text{ and } \dim(P(H))=k\right\} \\ &= S_k(T_1)+S_k(T_2). \end{aligned}$$

■

**Lemma A.3.7.** *For any two positive compact operators $T_1, T_2 \in B(H)$ and for all $k_1, k_2 \in \mathbb{N}$, we have that*

$$S_{k_1+k_2}(T_1+T_2) \;\geqslant\; S_{k_1}(T_1)+S_{k_2}(T_2).$$

PROOF. Let $P_1, P_2$ respectively denote two projections from $H$ onto a $k_1$-dimensional and $k_2$-dimensional subspace of $H$. Further, let $P \in B(H)$ denote a projection with $\dim(P(H)) = k_1 + k_2$ and such that $P(H) \supseteq P_1(H) \cup P_2(H)$. Observe that $\operatorname{tr}(T_1P_1) \leqslant \operatorname{tr}(T_1P)$ and $\operatorname{tr}(T_2P_2) \leqslant \operatorname{tr}(T_2P)$. Hence, since $\operatorname{tr} : K(H) \to \mathbb{R}$ is a linear functional on the set of compact operators, we have that

$$\operatorname{tr}(T_1P_1)+\operatorname{tr}(T_2P_2) \;\leqslant\; \operatorname{tr}(T_1P)+\operatorname{tr}(T_2P) \;=\; \operatorname{tr}((T_1+T_2)P).$$

The result then follows from Lemma A.3.5. ■

**Corollary A.3.8.** *If $T_1, T_2 \in K(H)$ are both positive, then for each $k \in \mathbb{N}$, we have that*

$$S_k(T_1+T_2) \;\leqslant\; S_k(T_1)+S_k(T_2) \;\leqslant\; S_{2k}(T_1+T_2).$$

PROOF. This is an immediate consequence of Lemma A.3.6 and Lemma A.3.7. ■

Recall that $\mathcal{L}^{1,+}(H)$ denotes the Dixmier ideal of $B(H)$ and is defined as follows

$$\mathcal{L}^{1,+}(H) \;:=\; \left\{T \in K(H) \,:\, \limsup_{k\to\infty} \frac{\sum_{m=1}^{k}\sigma_m(T)}{\ln(k)} \;<\; \infty\right\}.$$

Further, let $\mathcal{L}^{1,+}_{+}(H)$ denote the *positive cone* of $\mathcal{L}^{1,+}(H)$, that is, the set

$$\mathcal{L}^{1,+}_{+}(H) \;:=\; \left\{T \in K(H) \,:\, \limsup_{k\to\infty} \frac{\sum_{m=1}^{k}\sigma_m(T)}{\ln(k)} \;<\; \infty \text{ and } T \geqslant 0\right\}.$$

**Lemma A.3.9.** *For each limiting procedure $\mathcal{W}$ (see Defintion 3.2.8), we have that*

$$Tr_{\mathcal{W}} : \mathcal{L}^{1,+}_{+}(H) \;\to\; [0,\infty)$$

*defined, for each $T \in \mathcal{L}^{1,+}_{+}(H)$, by*

$$Tr_{\mathcal{W}}(T) \;:=\; \operatorname{Lim}_{\mathcal{W}}\left(\frac{\sum_{k=1}^{N}\sigma_k(T)}{\ln(N)}\right)_{N\in\mathbb{N}},$$

*is a positive linear functional on* $\mathcal{L}_+^{1,+}(H)$.

PROOF. For each $T \in \mathcal{L}_+^{1,+}(H)$, it is clear to see that, for each $N \in \mathbb{N}$, we have that

$$\frac{\sum_{k=1}^{N} \sigma_k(T)}{\ln(N)} \;\geqslant\; 0.$$

Therefore, it follows that the mapping $\mathrm{Tr}_{\mathcal{W}} : \mathcal{L}_+^{1,+}(H) \to \mathbb{R}$ is a positive functional on the positive cone of the Dixmier ideal $\mathcal{L}^{1,+}(H)$.

Next, let $T_1, T_2 \in B(H)$ denote two positive compact operators. By Corollary A.3.8, for each $k \in \mathbb{N}$, we have that

$$\frac{S_k(T_1+T_2)}{\ln(k)} \;\leqslant\; \frac{S_k(T_1)}{\ln(k)} + \frac{S_k(T_2)}{\ln(k)} \;\leqslant\; \frac{\ln(2k)}{\ln(k)}\frac{S_{2k}(T_1+T_2)}{\ln(2k)}.$$

Therefore, since $\lim_{k\to\infty} \ln(2k)/\ln(k) = 1$, for each $T_1, T_2 \in \mathcal{L}_+^{1,+}(H)$, it follows that

$$\mathrm{Tr}_{\mathcal{W}}(T_1+T_2) \;=\; \mathrm{Tr}_{\mathcal{W}}(T_1) + \mathrm{Tr}_{\mathcal{W}}(T_2).$$

Further, for each $T \in \mathcal{L}_+^{1,+}(H)$, each $k \in \mathbb{N}$ and each $\eta \in [0,\infty)$, we have that

$$\sigma_k(\eta T) \;=\; \eta\sigma_k(T),$$

Hence, it follows that

$$\mathrm{Tr}_{\mathcal{W}}(\eta T) \;=\; \eta \mathrm{Tr}_{\mathcal{W}}(T).$$

This completes the proof. ■

**Theorem A.3.10.** *The set* $\mathcal{L}^{1,+}(H)$ *is an ideal of* $B(H)$.

PROOF. Let $H$ denote a complex separable Hilbert space. Let $T \in B(H)$ denote a compact operator. Here, we assume that $T(H)$ is not finite-dimensional, since, if not, then the result follows trivially. By the inequality given in Equation (A.3), for each $k \in \mathbb{N}$, we have that

$$(\sigma_k(T))^2 \;=\; \sigma_k(T^*T) \;\leqslant\; \inf_{\substack{W\subset H\\ \dim(W)\leqslant k-1}} \sup_{\substack{h\in W^\perp\\ \|h\|=1}} |\langle T^*T(h), h\rangle| \;=\; \inf_{\substack{W\subset H\\ \dim(W)\leqslant k-1}} \sup_{\substack{h\in W^\perp\\ \|h\|=1}} \|T(h)\|^2.$$

Therefore, for each $k \in \mathbb{N}$, each $S \in B(H)$ and $T \in \mathcal{L}^{1,+}(H)$, we have that

$$\sigma_n(ST)^2 \;=\; \inf_{\substack{W\subset H\\ \dim(W)\leqslant k-1}} \sup_{\substack{h\in W^\perp\\ \|h\|=1}} \|ST(h)\|^2 \;\leqslant\; \|S\|^2 \inf_{\substack{W\subset H\\ \dim(W)\leqslant k-1}} \sup_{\substack{h\in W^\perp\\ \|h\|=1}} \|T(h)\|^2 \;=\; \|S\|^2\,\sigma_n(T)^2.$$

This implies that $\mathcal{L}^{1,+}(H)$ is a left ideal in $B(H)$.

Let us now show that $\mathcal{L}^{1,+}(H)$ is a right ideal in $B(H)$. By Lemma A.3.1, for each $k \in \mathbb{N}$, we have that

$$\sigma_k(TB) \;=\; \sigma_k((TB)^*) \;=\; \sigma_k(B^*T^*) \;\leqslant\; \|B^*\|\sigma_k(T^*) \;=\; \|B\|\sigma_k(T).$$

This implies that $\mathcal{L}^{1,+}(H)$ is a left ideal, and so, completes the proof. ■

Let us recall the definition of the Dixmier trace $\mathrm{Tr}_{\mathcal{W}}$ on the Dixmier ideal $\mathcal{L}^{1,+}(H)$, for $\mathcal{W}$ an arbitrary limiting procedure. Fix a limiting procedure $\mathcal{W}$. Then given a self-adjoint compact operator $T \in \mathcal{L}^{1,+}(H)$, observe that there exist positive compact operators $T_1, T_2 \in \mathcal{L}^{1,+}(H)$ such that $T = T_1 - T_2$. We then set

$$\mathrm{Tr}_{\mathcal{W}}(T) \;:=\; \mathrm{Tr}_{\mathcal{W}}(T_1) - \mathrm{Tr}_{\mathcal{W}}(T_2).$$

This is well-defined, since if $S_1, S_2, T_1, T_2$ are positive operators such that $T = S_1 - S_2 = T_1 - T_2$, then we have that

$$\mathrm{Tr}_{\mathcal{W}}(T_1) + \mathrm{Tr}_{\mathcal{W}}(S_2) \;=\; \mathrm{Tr}_{\mathcal{W}}(T_1 + S_2) \;=\; \mathrm{Tr}_{\mathcal{W}}(T_2 + S_1) \;=\; \mathrm{Tr}_{\mathcal{W}}(T_2) + \mathrm{Tr}_{\mathcal{W}}(S_1).$$

Therefore, it follows that

$$\mathrm{Tr}_{\mathcal{W}}(T_1) - \mathrm{Tr}_{\mathcal{W}}(T_2) \;=\; \mathrm{Tr}_{\mathcal{W}}(S_1) - \mathrm{Tr}_{\mathcal{W}}(S_2).$$

Next, observe that for each $T \in \mathcal{L}^{1,+}(H)$, there exist unique self-adjoint compact operators $T_3, T_4 \in \mathcal{L}^{1,+}(H)$ such that $T = T_3 + iT_4$. We then set

$$\mathrm{Tr}_{\mathcal{W}}(T) \;:=\; \mathrm{Tr}_{\mathcal{W}}(T_3) + i\mathrm{Tr}_{\mathcal{W}}(T_4).$$

**Definition A.3.11.** Let $H$ denote a complex Hilbert space and let $I$ denote an ideal of $B(H)$. Then, a *singular trace on* $I$ is a bounded linear functional $\mathcal{T}$ with domain $I$ such that the following hold.

1. $\mathcal{T}$ vanishes on operators with finite dimensional range.
2. If $T_1, T_2 \in I$ are such that $\lim_{k\to\infty} \sigma_k(T_1)/\sigma_k(T_2) = 1$, then $\mathcal{T}(T_1) = \mathcal{T}(T_2)$.
3. If $T_1, T_2 \in I$ have the property that $\sigma_k(T_1) \leqslant \sigma_k(T_2)$ for all but a finite number of $k \in \mathbb{N}$, then $\mathcal{T}(T_1) \leqslant \mathcal{T}(T_2)$.
4. For $T_1, T_2 \in I$, we have that $\mathcal{T}(T_1T_2) = \mathcal{T}(T_2T_1)$.

**Theorem A.3.12.** *For each limiting procedure $\mathcal{W}$, we have that the Dixmier trace $\mathrm{Tr}_{\mathcal{W}}$ is a singular trace on $\mathcal{L}^{1,+}(H)$.*

Proof. By Lemma A.3.9 and its extension to $\mathcal{L}^{1,+}(H)$ it follows that $\mathrm{Tr}_{\mathcal{W}}$ is a linear positive functional on $\mathcal{L}^{1,+}(H)$. Further, $\mathrm{Tr}_{\mathcal{W}}$ vanishes on operators with finite dimensional range. If $T_1, T_2 \in \mathcal{L}^{1,+}(H)$ are such that $\lim_{k\to\infty} \sigma_k(T_1)/\sigma_k(T_2) = 1$, then for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that, for all $k \geqslant N$, we have that

$$\begin{aligned}
(1-\epsilon)\frac{\sum_{m=1}^{k}\sigma_m(T_2)}{\ln(k)} &\leqslant \frac{\sum_{m=1}^{k}\sigma_m(T_1)}{\ln(k)} + \frac{\sum_{l=1}^{N}\sigma_l(T_2)}{\ln(k)}\\
\frac{\sum_{m=1}^{k}\sigma_m(T_1)}{\ln(k)} &\leqslant \frac{\sum_{l=1}^{N}\sigma_l(T_1)}{\ln(k)} + (1+\epsilon)\frac{\sum_{m=1}^{k}\sigma_m(T_2)}{\ln(k)}.
\end{aligned}$$

Hence it follows that

$$\mathrm{Tr}_{\mathcal{W}}(T_1) \;=\; \mathrm{Tr}_{\mathcal{W}}(T_2).$$

Further, it immediately follows from Lemma A.3.5 that $\mathrm{Tr}_{\mathcal{W}}(T_1T_2) = \mathrm{Tr}_{\mathcal{W}}(T_2T_1)$, for all positive $T_1, T_2 \in \mathcal{L}^{1,+}(H)$. Thus by linearity of the Dixmier trace, we have that condition 4 of Definition A.3.11 holds. Finally, if for all but a finite number of $k \in \mathbb{N}$, $\sigma_k(T_1) \leqslant \sigma_k(T_2)$, by the behaviour of the logarithm, we have that $\mathrm{Tr}_{\mathcal{W}}(T_1) \leqslant \mathrm{Tr}_{\mathcal{W}}(T_2)$. ■

## A.4 Representations of $C^*$-Algebras

The aim of this section is to prove the following result.

**Theorem.** *Let $A, B$ denote two unital $C^*$-algebras and let $\pi : A \to B$ denote a $*$-homomorphism. Then the following hold.*

1. *The $*$-homomorphism $\pi$ is a non-negative map.*

2. *For each $a \in A$, we have that $\|\pi(a)\| \leqslant \|a\|$.*

*Moreover, the following statements are equivalent.*

1. *The $*$-homomorphism $\pi$ is faithful.*

2. *The $*$-homomorphism $\pi$ is an isometry.*

3. *The $*$-homomorphism $\pi$ is a positive map.*

The material in this section is largely based on the material contained in Chapter 2 of [BR]. However, other sources where the results can be found are [Rud, RS].

**Definition A.4.1.** Let $A$ denote a unital $C^*$-algebra. Then, for each $a \in A$ we define the following.

1. The *spectrum* of $a$, denoted by $\sigma(a)$, is defined to be the set

$$\{z \in \mathbb{C} \,:\, a - z\mathbb{I} \text{ does not have an inverse in } A\}.$$

   (Here $\mathbb{I}$ denotes the unit of $A$.)

2. The *spectral radius* $\rho(a)$ of $a \in A$ is defined by

$$\rho(a) \;:=\; \sup\{|\lambda| \,:\, \lambda \in \sigma(a)\}.$$

**Definition A.4.2.** Let $A$ denote a unital $C^*$-algebra. Then an element $a \in A$ is called *normal* if and only if $aa^* = a^*a$.

Recall that an element $a$ of a unital $C^*$-algebra $A$ is said to be non-negative, written $a \geqslant 0$, if and only if there exists $b \in A$ such that $a = b^*b$. Equivalently, $a$ is said to be non-negative if and only if $a$ is self-adjoint and its spectrum $\sigma(a)$ is a subset of $[0, \infty)$ (see Theorem 2.2.12 of [BR]). Further, recall that an element $a \in A$ is said to be positive, written $a > 0$, if and only if $a \neq 0$ and $a \geqslant 0$. We also define the relations $<$ and $\leqslant$ between positive elements $a, b \in A$ by $a < b$ if $b - a > 0$ and $a \leqslant b$ if $b - a \geqslant 0$.

**Lemma A.4.3.** *Let $A$ denote a unital $C^*$-algebra. Then*

$$\rho(a) \;=\; \lim_{n\to\infty} \|a^n\|^{1/n}. \tag{A.7}$$

*Moreover, if $a$ is normal, then the spectral radius of $a$ is equal to $\|a\|$. In particular, for an arbitrary $a \in A$, we have that*

$$\|a\| \;=\; \|a^*a\|^{1/2} \;=\; \rho(a^*a)^{1/2}.$$

PROOF. The equality given in Equation (A.7) is known as the *spectral radius formula*, and, in fact, holds for all Banach algebras. There are several texts which give detailed proofs of this result, see for instance Proposition 2.2.2 of [BR].

If $a$ denotes a normal element of $A$, then, for each $k \in \mathbb{N}$, we have that

$$\|a^{2^k}\|^2 = \|(a^*)^{2^k} a^{2^k}\| = \|(a^*a)^{2^k}\| = \|(a^*a)^{2^{k-1}}\|^2 = \ldots = \|a^*a\|^{2^k} = \|a\|^{2^{k+1}}.$$

Therefore, by the spectral radius formula we have that

$$\rho(a) = \lim_{k\to\infty} \|a^{2^k}\|^{2^{-k}} = \|a\|.$$

■

In the following Lemma we use the following notation, for $\Lambda$ a subset of $\mathbb{C}$, we set

$$\overline{\Lambda} := \{z \in \mathbb{C} : \overline{z} \in \Lambda\}.$$

**Lemma A.4.4.** *Let $A$ denote a unital $C^*$-algebra. Then we have the following.*

1. *For each $a \in A$, we have that $\sigma(a^*) = \overline{\sigma(a)}$.*
2. *For each $a$ such that $a$ is invertible, we have that $\sigma\left(a^{-1}\right) = (\sigma(a))^{-1}$.*
3. *If $a \in A$ is unitary, then $\sigma(a) \subset \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.*

PROOF. Let $\mathbb{I}$ denote the unit of $A$. For each $z \in \mathbb{C}$, we have $a^* - z\mathbb{I} = (a - \overline{z}\mathbb{I})^*$. Therefore, it follows that

$$\sigma(a^*) = \overline{\sigma(a)}.$$

If $a \in A$ is invertible, then since $(a - z\mathbb{I}) = za(-a^{-1} + z^{-1}\mathbb{I})$, it follows that

$$(\sigma(a))^{-1} = \sigma\left(a^{-1}\right).$$

Finally, if $a$ is unitary, then by Lemma A.4.3 we have that $\sigma(a) \subseteq \{z \in \mathbb{C} : |z| \leqslant 1\}$. Further, we have that

$$\sigma(a) = \overline{\sigma(a^*)} = \overline{\sigma(a^{-1})} = \overline{\sigma(a)}^{-1}.$$

This then completes the proof. ■

**Lemma A.4.5.** *Let $A$ denote a $C^*$-algebra and let $a \in A$ denote a self-adjoint element. Then we have that*

$$\sigma(a) \subseteq [-\|a\|, \|a\|].$$

*Moreover, we have that*

$$\sigma(a^2) \subseteq \left[0, \|a\|^2\right].$$

PROOF. Observe that every self-adjoint element is normal. Therefore, by Lemma A.4.3, for each self-adjoint $a \in A$, we have that $\rho(a) = \|a\|$. Hence, it follows that

$$\sigma(a) \subseteq \{z \in \mathbb{C} : |z| \leqslant \|a\|\}.$$

Thus, if $z \in \mathbb{C}$ with $|z^{-1}| > \|a\|$, we have that $\mathbb{I} + i|\lambda|a$ has an inverse in $A$. Let $U \in A$ denote the unitary element given by $U := (\mathbb{I} + i|\lambda|a)(\mathbb{I} - i|\lambda|a)^{-1}$ and, for each $\xi \in \mathbb{C}$ with $\Im m(\xi) \neq 0$, observe

that
$$\frac{1-i|\lambda|\xi}{1+i|\lambda|\xi} \not\in \{z \in \mathbb{C} : |z| = 1\}.$$
Hence, by Lemma A.4.3, for each $\xi \in \mathbb{C}$ with $\Im m(\xi) \neq 0$, the element
$$U - \frac{1-i|\lambda|\xi}{1+i|\lambda|\xi}\,\mathbb{I}$$
has an inverse in $A$. Next, observe that
$$U - \frac{1-i|\lambda|\xi}{1+i|\lambda|\xi}\,\mathbb{I} \;=\; \frac{2i|\lambda|(a+\xi 1)(1-i|\lambda|a)^{-1}}{1+i|\lambda|\xi}.$$
Therefore $a - \xi\mathbb{I}$ has an inverse in $A$, for all $\xi \in \mathbb{C}$ with $\Im m(\xi) \neq 0$. This then implies that
$$\sigma(a) \;\subset\; \{z \in \mathbb{C} : |z| < \|a\|\} \cap \mathbb{R} \;=\; [-\|a\|, \|a\|].$$
Finally, since $a$ is self-adjoint, it follows that $a^2 = a^*a$. This implies that $a^2 \geqslant 0$, and so, we have that
$$\sigma(a) \;\subset\; \{z \in \mathbb{C} : |z| < \|a\|\} \cap \mathbb{R} \subseteq [0, \|a\|].$$
■

**Lemma A.4.6.** *Let $A$ denote a unital $C^*$-algebra and let $a, b \in A$. If $a \geqslant b \geqslant 0$, then $\|a\| \geqslant \|b\|$ and $a\|a\| \geqslant a^2$.*

PROOF. Recall that we let $\mathbb{I}$ denote the unit of $A$. By Lemma A.4.3 we have $a \leqslant \|a\|\mathbb{I}$. Hence, it follows that $0 \leqslant b \leqslant \|a\|\mathbb{I}$, and so, $0 \leqslant \|a\|\mathbb{I} - b$. This implies that the spectrum of $\|a\|\mathbb{I} - b$ is a subset of $[0, \infty)$. Hence, for all $z \in \sigma(b)$ we have that $\|a\| - z \geqslant 0$. Therefore, by Lemma A.4.3, since $a, b$ are positive they are self-adjoint, and so, it follows that $\|a\| \geqslant \rho(b) = \|b\|$.

Let us now show that $a\|a\| \geqslant a^2$. By Lemma A.4.5 we have that
$$\sigma\left(\left(a - \frac{\|a\|}{2}\,\mathbb{I}\right)^2\right) \;\subset\; \left[0, \frac{\|a\|^2}{4}\right].$$
Hence, it follows that
$$0 \;\leqslant\; \left(a - \frac{\|a\|}{2}\,\mathbb{I}\right)^2 \;\leqslant\; \frac{\|a\|^2}{4}\,\mathbb{I}.$$
This then implies that $a^2 \leqslant \|a\|a$. ■

**Theorem A.4.7.** *Let $A, B$ denote two unital $C^*$-algebras and let $\pi : A \to B$ denote a $*$-homomorphism. Then we have the following.*

1. *The $*$-homomorphism $\pi$ is a non-negative map.*

2. *For each $a \in A$, we have that $\|\pi(a)\| \leqslant \|a\|$.*

PROOF. Part 1 is immediate, since if $a$ is a positive element of $A$, then there exists a $b \in A$ such that $a = b^*b$. Therefore, since $\pi$ is a $*$-homomorphism, we have that

$$\pi(a) \;=\; \pi(b^*b) \;=\; \pi(b^*)\pi(b) \;=\; (\pi(b))^*\pi(b) \geqslant 0.$$

Let us now prove the second part of the theorem. Note that if $\|a\| = 0$ then the result follows trivially. Therefore, we assume that $\|a\| > 0$. By Lemma A.4.6 we have that

$$0 \;\leqslant\; (a^*a)^2 \;\leqslant\; a^*a\|a^*a\|.$$

Hence, by part 1 it follows that

$$0 \;\leqslant\; \pi(a^*a)^2 \;\leqslant\; \pi(a^*a)\|a^*a\|.$$

Then, by Lemma A.4.3 and Lemma A.4.6, we have that

$$\|\pi(a)\|^4 \;=\; \|\pi(a^*a)\|^2 \;=\; \|\pi(a^*a)^2\| \;\leqslant\; \|\pi(a^*a)\|\|a^*a\| \;=\; \|\pi(a)\|^2\|a\|^2.$$

■

**Theorem A.4.8.** *Let $A, B$ denote two unital $C^*$-algebras and let $\pi : A \to B$ denote a $*$-homomorphism. Then the following are equivalent.*

1. *The $*$-homomorphism $\pi$ is faithful.*
2. *The $*$-homomorphism $\pi$ is an isometry.*
3. *The $*$-homomorphism $\pi$ is a positive map.*

Proof. Assume that $\pi$ is faithful, then there exists a $*$-homomorphism $\pi^{-1} : \pi(A) \to A$ such that $\pi^{-1}(\pi(a)) = a$, for each $a \in A$. Then by Theorem A.4.7 and since $\pi(A)$ is a complete sub-$*$-algebra of $B$, for all $a \in A$, we have that

$$\|a\| \;=\; \|\pi^{-1}(\pi(a))\| \;\leqslant\; \|\pi(a)\| \;\leqslant\; \|a\|$$

Hence, $\pi$ is an isometry.

Assume that $\pi$ is an isometry and let $a \in A$ denote a positive element. Then $\|\pi(a)\| = \|a\| > 0$, and so, $\pi(a) \neq 0$. Since by Theorem A.4.7 we have that $\pi$ is non-negative, we conclude that $\pi(a) > 0$.

Finally, we show that 3 implies 1. In order to do so we use a contra-positive argument. Assume that $\pi$ is a positive map which is not faithful. Then, there exists a non-zero element $a \in \ker(\pi)$. Since $\pi$ is a $*$-homomorphism it follows that $\pi(a^*a) = 0$. However, since $a$ is non-zero we have that $\|a^*a\| > 0$, which implies that $a^*a > 0$. This provides a contradiction to the assumption that $\pi$ is positive. ■

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# List of Symbols

| | |
|---|---|
| $V(A, H, D)$ | noncommutative volume for of a spectral triple $(A, H, D)$, 51 |
| $\mathbb{Z}$ | set of integers, 8 |
| | |
| $\widehat{a}$ | Gelfand transform of an element $a$ of a $C^*$-algebra, 43 |
| $\mathfrak{b}$ | Riedi's multifractal box-counting dimension, 17 |
| $d_{\mathcal{A}}$ | Connes' metric, 43 |
| $d_{MK}$ | Monge-Kantorovitch metric, 43 |
| $\dim_B$ | box counting dimension, 11 |
| $\dim_{\mathcal{H}}$ | Hausdorff dimension, 11 |
| $\dim_\mu$ | local dimension of a measure $\mu$, 15 |
| $h_\mu(\sigma)$ | measure theoretical entropy of a measure $\mu$ with respect to left shift $\sigma$, 22 |
| $l^\infty(\mathbb{R})$ | the set of all bounded sequences with real valued entries, indexed by $\mathbb{N}$, 45 |
| ß | self-similar multifractal box-counting dimension, 20 |
| $\mathrm{tr}$ | trace, 9 |
| | |
| $\Gamma(T)$ | Graph of an operator $T$, 98 |
| | |
| $\beta$ | multifractal box-counting dimension, 16 |
| $\delta(A, H, D)$ | metric dimension of a spectral triple $(A, H, D)$, 50 |
| $\vartheta^1$ | one-dimensional spherical measure, 38 |
| $\lambda^k$ | $k$-dimensional Lebesgue measure, 9 |
| $\mu_\phi$ | Gibbs measure for the potential $\phi$, 23 |
| $\nu_\phi$ | equilibrium measure for the potential $\phi$, 24 |
| $\pi$ | $*$-homomorphism, 36 |
| $\rho$ | porosity constant, 12 |
| $\sigma$ | left shift map, 21 |
| $\sigma_k(T)$ | the $k$-th largest singular value of $T$ (including multiplicities), 45 |
| $\phi$ | Hölder continuous potential function, 23 |
| | |
| $\sim$ | asymptotic, 8 |
| $\asymp$ | comparable, 8 |
| $\mathbb{1}$ | identity element of $B(H)$, 9 |
| $^*$ | involution, 35 |
| $*$ | twisted convolution, 39 |
| $\fint$ | noncommutative expectation, 46 |

# Index