# Computing maximal subsemigroups of a finite semigroup

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### Abstract

A proper subsemigroup of a semigroup is *maximal* if it is not contained in any other proper subsemigroup. A maximal subsemigroup of a finite semigroup has one of a small number of forms, as described in a paper of Graham, Graham, and Rhodes. Determining which of these forms arise in a given finite semigroup is difficult, and no practical mechanism for doing so appears in the literature. We present an algorithm for computing the maximal subsemigroups of a finite semigroup S given knowledge of the Green's structure of S, and the ability to determine maximal subgroups of certain subgroups of S, namely its group  $\mathcal{H}$ -classes.

In the case of a finite semigroup S represented by a generating set X, in many examples, if it is practical to compute the Green's structure of S from X, then it is also practical to find the maximal subsemigroups of S using the algorithm we present. In such examples, the time taken to determine the Green's structure of S is comparable to that taken to find the maximal subsemigroups. The generating set X for S may consist, for example, of transformations, or partial permutations, of a finite set, or of matrices over a semiring. Algorithms for computing the Green's structure of S from X include the Froidure-Pin Algorithm, and an algorithm of the second author based on the Schreier-Sims algorithm for permutation groups. The worst case complexity of these algorithms is polynomial in |S|, which for, say, transformation semigroups is exponential in the number of points on which they act.

Certain aspects of the problem of finding maximal subsemigroups reduce to other well-known computational problems, such as finding all maximal cliques in a graph and computing the maximal subgroups in a group.

The algorithm presented comprises two parts. One part relates to computing the maximal subsemigroups of a special class of semigroups, known as Rees 0-matrix semigroups. The other part involves a careful analysis of certain graphs associated to the semigroup S, which, roughly speaking, capture the essential information about the action of S on its  $\mathcal{J}$ -classes.

# 1 Introduction

A semigroup S is a set with an associative operation. A subsemigroup M of a semigroup S is just a subset that is closed under the operation of S. A maximal subsemigroup M of a semigroup S is a proper subsemigroup that is not contained in any other proper subsemigroup. Every proper subsemigroup of a finite semigroup is contained in a maximal subsemigroup, although the same is not true for infinite semigroups. For example, the multiplicative semigroup consisting of the real numbers in the interval  $(1, \infty)$  has no maximal subsemigroups, but it does have proper subsemigroups.

There are numerous papers in the literature about finding maximal subsemigroups of particular classes of semigroups; for example [5, 6, 7, 8, 9, 10, 12, 20, 21, 22, 26, 28, 34, 35, 36, 37]. Perhaps the most important paper on this topic for finite semigroups is that by Graham, Graham, and Rhodes [18]. This paper appears to have been overlooked for many years, and indeed, special cases of the results it contains have been repeatedly reproved.

The main purpose of this paper is to give algorithms that can be used to find the maximal subsemigroups of an arbitrary finite semigroup. Our algorithms are based on the paper from 1968 of Graham, Graham, and Rhodes [18]. The algorithms described in this paper are implemented in the GAP [17] package SEMIGROUPS [29]. This paper is organised as follows. In the remainder of this section, we introduce the required background material and notation. We state the main results of Graham, Graham, and Rhodes [18] in Propositions 1.4 and 1.5. In Section 2, we describe algorithms for finding the maximal subsemigroups of a finite regular Rees 0-matrix semigroup over a group. In Section 3, we use the procedures from Section 2 to describe an algorithm for finding the maximal subsemigroups of an arbitrary finite semigroup.

Henceforth we only consider finite semigroups, and we let S denote an arbitrary finite semigroup throughout. We denote by  $S^1$  the semigroup obtained from S by adjoining an identity element  $1 \notin S$ . In other words, 1s = s1 = s for all  $s \in S^1$ . If X is any subset of a semigroup S, then we denote by  $\langle X \rangle$  the least subsemigroup of S containing X. If  $\langle X \rangle = S$ , then we refer to X as a generating set for S.

Let  $x, y \in S$  be arbitrary. We say that x and y are  $\mathscr{L}$ -related if the principal left ideals generated by x and y in S are equal; in other words,  $S^1x = S^1y$ . Clearly  $\mathscr{L}$  defines an equivalence relation on S. We write  $x\mathscr{L}y$  to denote that x and y

are  $\mathscr{L}$ -related. Green's  $\mathscr{R}$ -relation is defined dually to Green's  $\mathscr{L}$ -relation; Green's  $\mathscr{H}$ -relation is the meet, in the lattice of equivalence relations on S, of  $\mathscr{L}$  and  $\mathscr{R}$ . In any semigroup,  $x \not j y$  if and only if the (2-sided) principal ideals generated by x and y are equal. However, in a finite semigroup  $\mathscr{J}$  is the join of  $\mathscr{L}$  and  $\mathscr{R}$ . We will refer to the equivalence classes as  $\mathscr{K}$ -classes where  $\mathscr{K}$  is any of  $\mathscr{R}$ ,  $\mathscr{L}$ ,  $\mathscr{H}$ , or  $\mathscr{J}$ , and the  $\mathscr{K}$ -class of  $x \in S$  will be denoted by  $K_x$ . We write  $\mathscr{K}^S$  if it is necessary to explicitly refer to the semigroup on which the relation is defined. We denote the set of  $\mathscr{K}$ -classes of a semigroup S by  $S/\mathscr{K}$ .

An *idempotent* is an element  $x \in S$  such that  $x^2 = x$ . We denote the set of idempotents in a semigroup S by E(S). A  $\mathscr{J}$ -class of a finite semigroup is *regular* if it contains an idempotent, and a finite semigroup is called *regular* if each of its  $\mathscr{J}$ -classes is regular. Containment of principal ideals induces a partial order on the  $\mathscr{J}$ -classes of S.

If J is an arbitrary  $\mathcal{J}$ -class of a finite semigroup S, then we denote the *principal factor* of J by  $J^*$ . In other words,  $J^*$  is the semigroup with elements  $J \cup \{0\}$  and multiplication \* given by setting x \* y = 0 if the product in S of  $x, y \in J$  does not belong to J, and letting it have its value in J otherwise.

A Rees 0-matrix semigroup  $\mathscr{M}^0[I, G, \Lambda; P]$  is the set  $(I \times G \times \Lambda) \cup \{0\}$  where  $I, \Lambda \neq \emptyset$ , G is a group, and P is a  $|\Lambda| \times |I|$  matrix with entries  $p_{\lambda,i}$  ( $\lambda \in \Lambda, i \in I$ ) in  $G \cup \{0\}$ , with multiplication defined by

$$0x = x0 = 0 \text{ for all } x \in \mathscr{M}^0[I, G, \Lambda; P] \quad \text{and} \quad (i, g, j)(k, h, l) = \begin{cases} (i, gp_{j,k}h, l) & \text{if } p_{j,k} \neq 0\\ 0 & \text{if } p_{j,k} = 0 \end{cases}$$

Throughout this paper we assume without loss of generality that  $I \cap \Lambda = \emptyset$ .

If J is a  $\mathcal{J}$ -class of a finite semigroup, then by the Rees Theorem [24, Theorem 3.2.3],  $J^*$  is isomorphic to either a regular Rees 0-matrix semigroup over a group, or a null semigroup.

A graph is a pair  $\Gamma = (V, E)$  consisting of a set of vertices V and a set of edges  $E \subseteq \{\{u, v\} : u, v \in V, u \neq v\}$ . If u is a vertex in a graph and  $\{u, v\}$  is an edge in that graph, then we say that u is incident to  $\{u, v\}$ , and we say that  $\{u, v\}$  is incident to u. If  $\Gamma = (V, E)$  is a graph and W is a subset of the vertices of  $\Gamma$ , then the subgraph induced by W is the graph with vertices W and edges  $\{\{u, v\} \in E : u, v \in W\}$ . A clique K in a graph  $\Gamma = (V, E)$  is a subset of the vertices of  $\Gamma$  such that  $\{u, v\} \in E$  for all  $u, v \in K, u \neq v$ . If  $\Gamma = (V, E)$  is a graph, then the complement of  $\Gamma$  is the graph with vertices V and edges  $\{\{u, v\} \notin E : u \neq v\}$ . An independent set in a graph is a set of vertices which is a clique in the complement. A clique is maximal if it is not properly contained in another clique. A maximal independent set is defined analogously. A graph is bipartite if its vertices can be partitioned into two independent sets. If  $\Gamma = (V, E)$  is a graph, then a path is a non-empty sequence of distinct vertices of  $\Gamma$ ,  $(v_1, \ldots, v_m)$ , where  $\{v_i, v_{i+1}\} \in E$  for all  $i \in \{1, \ldots, m-1\}$ . The connected component of a vertex v of a graph is the set consisting of all vertices u such that there is a path from u to v.

If  $R = \mathcal{M}^0[I, G, \Lambda; P]$  is a Rees 0-matrix semigroup, then we define the *Graham-Houghton graph*  $\Gamma(R)$  of R to be the bipartite graph with vertices  $I \cup \Lambda$  and edges  $\{i, \lambda\}$  whenever  $p_{\lambda,i} \neq 0$ , for  $i \in I$ ,  $\lambda \in \Lambda$ . Since we assume throughout that  $I \cap \Lambda = \emptyset$ , the Graham-Houghton graph of R has  $|I| + |\Lambda|$  vertices. Variants of this graph were introduced in [19, 23].

We will use the following well-known results repeatedly throughout this paper.

**Lemma 1.1** (Green's Lemma; Lemmas 2.2.1 and 2.2.2 in [24]). Let S be a semigroup, and let  $x, y \in S$  be such that  $x \mathscr{R} y$ . If  $s, t \in S^1$  are such that xs = y and yt = x, then the functions  $L_x \longrightarrow S$  and  $L_y \longrightarrow S$  defined by  $a \mapsto as$  and  $b \mapsto bt$ , respectively, are mutually inverse bijections from the  $\mathscr{L}$ -class  $L_x$  onto  $L_y$  and  $L_y$  onto  $L_x$ , respectively, that preserve Green's  $\mathscr{R}$ -relation.

An analogue of Lemma 1.1 holds when  $\mathscr{L}$ - and  $\mathscr{R}$ -relations are interchanged, which is also referred to as Green's Lemma.

**Lemma 1.2** (Theorem A.2.4 in [31]). Let S be a finite semigroup and let  $x, y \in S^1$ . Then  $x \not \exists xy$  if and only if  $x \not \exists xy$ , and  $x \not \exists yx$  if and only if  $x \not \exists yx$ .

An arbitrary (not necessarily finite) semigroup satisfying the conditions of Lemma 1.2 is called *stable*.

The following lemma is important to the description of the maximal subsemigroups of a finite semigroup.

**Lemma 1.3** (Proposition(1) in [18]). Let S be a finite semigroup and let M be a maximal subsemigroup of S. Then  $S \setminus M$  is contained in a single  $\mathcal{J}$ -class of S.

If S is a finite semigroup and M is a maximal subsemigroup of S, then we denote the  $\mathscr{J}$ -class of S containing  $S \setminus M$  by J(M). It is also shown in the second part of the main proposition in [18] that M is either a union of  $\mathscr{H}$ -classes of S or has non-empty intersection with every  $\mathscr{H}$ -class of S.

We include a version of the main result in [18], which is slightly reformulated for our purposes here. More specifically, Proposition 1.4(a) and (b) are reformulations of parts (3) and Case 1 of part (4) of the main proposition in [18]. Proposition 1.4(c) is similar to Case 2 of part (4) of the main proposition in [18]. Our statement follows from the proof in [18], but it is not exactly the statement in [18].

**Proposition 1.4.** Let S be a finite semigroup, let M be a maximal subsemigroup of S, let J(M) denote the  $\mathscr{J}$ -class of S such that  $S \setminus M \subseteq J(M)$ , and let  $\phi: J(M)^* \longrightarrow \mathscr{M}^0[I, G, \Lambda; P]$  be any isomorphism. Then one of the following holds:

- (a) J(M) is non-regular and  $J(M) \cap M = \emptyset$ ;
- (b) J(M) is regular, M intersects every  $\mathcal{H}$ -class of J(M) non-trivially, and  $(M \cap J(M))\phi \cup \{0\} \cong \mathcal{M}^0[I, H, \Lambda; Q]$  where H is a maximal subgroup of G and Q is a  $|\Lambda| \times |I|$  matrix with entries over  $H \cup \{0\}$ . In this case,  $(M \cap J(M))\phi \cup \{0\}$  is a maximal subsemigroup of  $\mathcal{M}^0[I, G, \Lambda; P]$ ;
- (c) J(M) is regular,  $J(M) \cap M$  is a union of  $\mathscr{H}$ -classes of J(M), and  $(M \cap J(M))\phi$  equals one of the following:
  - (i)  $(I \times G \times \Lambda) \setminus (I' \times G \times \Lambda')$  for some  $\emptyset \neq I' \subsetneq I$  and  $\emptyset \neq \Lambda' \subsetneq \Lambda$ . In this case,  $((I \times G \times \Lambda) \setminus (I' \times G \times \Lambda')) \cup \{0\}$  is a maximal subsemigroup of  $\mathscr{M}^0[I, G, \Lambda; P]$ ;
  - (ii)  $I \times G \times \Lambda'$  for some  $\emptyset \neq \Lambda' \subsetneq \Lambda$ ;
  - (iii)  $I' \times G \times \Lambda$  for some  $\emptyset \neq I' \subsetneq I$ ;
  - (iv) Ø.

The principal difference between the main proposition of [18] and Proposition 1.4 is that in the latter the isomorphism  $\phi$  is arbitrary. We will show how to effectively determine the maximal subsemigroups of each type that arise in a given finite semigroup. Certain cases reduce to other well-known problems, such as, for example, finding all of the maximal cliques in a graph, and computing the maximal subgroups of a finite group. The following result describes the maximal subsemigroups of a finite regular Rees 0-matrix semigroup over a group, and is central to the presented algorithms.

**Proposition 1.5** (cf. Theorem 4 in [19]). Let  $R = \mathscr{M}^0[I, G, \Lambda; P]$  be a finite regular Rees 0-matrix semigroup over a group G, and let M be a subset of R. Then M is a maximal subsemigroup of R if and only if one of the following holds:

- (R1)  $M = \{0\}$  and  $|R| = 2;^1$
- (R2)  $M = R \setminus \{0\}$  and  $R \setminus \{0\}$  is a subsemigroup (or equivalently there are no entries in P equal to 0);
- (R3)  $|\Lambda| > 1$ ,  $M = (I \times G \times (\Lambda \setminus \{\lambda\})) \cup \{0\}$  for some  $\lambda \in \Lambda$ , and there is at least one non-zero entry in every row and every column of  $(p_{\mu,i})_{\mu \in \Lambda \setminus \{\lambda\}, i \in I}$ ;
- (R4) |I| > 1,  $M = ((I \setminus \{i\}) \times G \times \Lambda) \cup \{0\}$  for some  $i \in I$ , and there is at least one non-zero entry in every row and every column of  $(p_{\lambda,j})_{\lambda \in \Lambda, j \in I \setminus \{i\}}$ ;
- (R5)  $M = ((I \times G \times \Lambda) \setminus (I' \times G \times \Lambda')) \cup \{0\}$  where  $I' = I \setminus X$ ,  $\Lambda' = \Lambda \setminus Y$ , and X and Y are proper non-empty subsets of I and  $\Lambda$ , respectively, such that  $X \cup Y$  is a maximal independent set in the Graham-Houghton graph of R;
- (R6) *M* is a subsemigroup isomorphic to  $\mathscr{M}^0[I, H, \Lambda; Q]$  where *H* is a maximal subgroup of *G* and *Q* is a  $|\Lambda| \times |I|$  matrix over  $H \cup \{0\}$ .

The algorithms we describe require knowledge of the Green's structure of S. We briefly discuss algorithms for determining the Green's structure of a finite semigroup S. If S is a regular Rees 0-matrix semigroup  $\mathcal{M}^0[I, G, \Lambda; P]$  where G is a group, then it can be represented on a computer by a generating set for G and the matrix P. The Green's structure of such a semigroup can be obtained directly from P. For an arbitrary finite semigroup S, we suppose that S is represented by a generating set X, and that we know how to determine the value of xy for any  $x, y \in S$ . The generating set X for S may consist, for example, of transformations, or partial permutations, of a finite set, or of matrices over a semiring. Algorithms for computing the Green's structure of S from X include the Froidure-Pin Algorithm [15, 25], and the algorithms described in [11]. The worst case complexity of these algorithms is O(|S||X|), which for transformations and partial permutations is exponential in the number of points on which they act, and for matrices over a semiring, is exponential in their dimension. The Froidure-Pin Algorithm determines the right and left Cayley graphs of S with respect to X, and Green's  $\mathscr{R}$ - and  $\mathscr{L}$ -relations correspond to the strongly connected components of these graphs. There are several well-known algorithms, such as those of Tarjan or Gabow, for finding the strongly connected components of a graph. The algorithms in [11] directly enumerate the  $\mathcal{R}$ - and  $\mathcal{L}$ -classes of S, and so for examples of semigroups with relatively large  $\mathcal{J}$ -classes, unlike the Froidure-Pin Algorithm, the Green's structure of S can be found without storing every element of S in memory. In this way, we will henceforth suppose that we are able to describe the Green's structure of a finite semigroup. If computing the Green's structure of S is not practical, because, say, it requires too much time or space, then the algorithms presented here cannot be used to find the maximal subsemigroups of S. In many examples, it appears that the converse also holds; further details can be found in Section 4.

<sup>&</sup>lt;sup>1</sup>Note that in [19, Theorem 4] it is incorrectly stated that  $\{0\}$  is a maximal subsemigroup if  $R \setminus \{0\}$  is a cyclic group of prime order p. If p > 1, then the identity of  $R \setminus \{0\}$  and 0 comprise a proper subsemigroup of R strictly containing  $\{0\}$ , which is therefore not maximal.

## 2 Regular Rees 0-matrix semigroups

In this section, we discuss how to compute the maximal subsemigroups of a finite regular Rees 0-matrix semigroup over a group. This has inherent interest, but we will use the algorithms for Rees 0-matrix semigroups when computing the maximal subsemigroups of an arbitrary finite semigroup.

The possible types of maximal subsemigroups of a Rees 0-matrix semigroup are described in Proposition 1.5. We begin this section by describing how to compute the maximal subsemigroups of types (R1) to (R5). Maximal subsemigroups of type (R6) are more complicated to determine, and the remainder of this section is dedicated to describing an algorithm for finding the maximal subsemigroups of this type.

Throughout this section, we will denote by R a finite regular Rees 0-matrix semigroup  $\mathscr{M}^0[I, G, \Lambda; P]$  over a group G.

## 2.1 Maximal subsemigroups of types (R1)–(R5)

It is trivial to check whether there exists a maximal subsemigroup of types (R1) and (R2). For the former, we simply check whether |R| = 2; if it is then  $\{0\}$  is a maximal subsemigroup of R. For the latter, it suffices to check whether the matrix P contains the element 0; if it does not, then  $R \setminus \{0\}$  is a maximal subsemigroup of R.

We require the following straightforward reformulation of (R3) and (R4).

**Lemma 2.1.** Let  $R = \mathscr{M}^0[I, G, \Lambda; P]$  be a finite regular Rees 0-matrix semigroup over a group.

If  $|\Lambda| > 1$  and  $\lambda \in \Lambda$ , then  $M = (I \times G \times \Lambda \setminus \{\lambda\}) \cup \{0\}$  is a maximal subsemigroup of R if and only if every vertex in the subgraph of the Graham-Houghton graph of R induced by  $I \cup (\Lambda \setminus \{\lambda\})$  has at least one incident edge.

Likewise, if |I| > 1 and  $i \in I$ , then  $M = (I \setminus \{i\} \times G \times \Lambda) \cup \{0\}$  is a maximal subsemigroup of R if and only if every vertex in the subgraph induced by  $(I \setminus \{i\}) \cup \Lambda$  has at least one incident edge.

*Proof.* If  $M = (I \times G \times \Lambda \setminus \{\lambda\}) \cup \{0\}$  is a maximal subsemigroup of R, then it is routine to verify that M is not a maximal subsemigroup of type (R1), (R2), (R4), (R5), or (R6). Hence M is a maximal subsemigroup of type (R3), and there is at least one non-zero entry in each row and each column of  $(p_{\mu,i})_{\mu \in \Lambda \setminus \{\lambda\}, i \in I}$ , and so the subgraph of the Graham-Houghton graph of R induced by  $I \cup (\Lambda \setminus \{\lambda\})$  has no vertex without an incident edge.

If  $\lambda \in \Lambda$  is such that the subgraph of the Graham-Houghton graph of R induced by  $I \cup (\Lambda \setminus \{\lambda\})$  has no vertex without an incident edge, then it is straightforward to check that  $M = (I \times G \times \Lambda \setminus \{\lambda\}) \cup \{0\}$  satisfies (R3), and hence is maximal. The proof of the second part is dual.

It is straightforward to check whether the conditions in Lemma 2.1 hold, and so it is easy to find the maximal subsemigroups of types (R3) and (R4). Computing the maximal subsemigroups of type (R5) is equivalent to computing the maximal cliques in the complement of the Graham-Houghton graph of R. This problem is well-understood (and hard); see [2], [4], and [30].

## 2.2 Maximal subsemigroups of type (R6)

It is more complicated to determine the maximal subsemigroups of type  $(\mathbf{R6})$  than it is to determine those of other types. We require the following proposition.

**Proposition 2.2** (Section 4 in [19], Theorem 4.13.34 in [31]). Let R be a regular finite Rees 0-matrix semigroup over a group G with index sets I and  $\Lambda$ , and let  $n \in \mathbb{N}$  be the number of connected components of the Graham-Houghton graph of R. Then there exist non-empty subsets  $I_1, \ldots, I_n$  of I and  $\Lambda_1, \ldots, \Lambda_n$  of  $\Lambda$  that partition I and  $\Lambda$ , respectively, such that  $I_k \cup \Lambda_k$  is a connected component of the Graham-Houghton graph for each k, and there exists a Rees 0-matrix semigroup  $R' = \mathscr{M}^0[I, G, \Lambda; P]$ , where  $P = (p_{\lambda,i})_{\lambda \in \Lambda, i \in I}$ , which is isomorphic to R and where the following hold:

- (i) there are  $i_k \in I_k$  and  $\lambda_k \in \Lambda_k$  such that  $p_{\lambda_k, i_k} = 1_G$ , for every k;
- (ii)  $\psi: R' \setminus \{0\} \longrightarrow G$  defined by  $(i, g, \lambda) \mapsto g$  is an isomorphism from  $\{i\} \times G \times \{\lambda\}$  to G whenever  $p_{\lambda,i} = 1_G$ ;
- (iii)  $G_k = (\langle E(R') \rangle \cap (\{i_k\} \times G \times \{\lambda_k\})) \psi$  is a subgroup of G, for every k;
- (iv) the matrix  $P_k = (p_{\lambda,i})_{\lambda \in \Lambda_k, i \in I_k}$  consists of elements in  $G_k \cup \{0\}$  and contains a generating set for  $G_k$ , for every k.

We refer to a Rees 0-matrix semigroup satisfying the conditions of Proposition 2.2 as being normalized. Normalizing a finite Rees 0-matrix semigroup  $R = \mathscr{M}^0[I, G, \Lambda; P]$  consists of finding the connected components of the Graham-Houghton graph of R (time complexity  $O(|I||\Lambda|)$ ), and multiplying every non-zero entry in P by some group element at most twice (complexity  $O(|I||\Lambda|)$ ). Hence such an R can be normalized in time polynomial in  $|I||\Lambda|$ . For more information about Rees

0-matrix semigroups and its connections to graph theory, and normalization, we refer the reader to [31]. For this point on, we suppose without loss of generality that  $R = \mathcal{M}^0[I, G, \Lambda; P]$  is a normalized regular finite Rees 0-matrix semigroup where G is a group, I and  $\Lambda$  are disjoint index sets, and  $P = (p_{\lambda,i})_{\lambda \in \Lambda, i \in I}$  is a  $|\Lambda| \times |I|$  matrix over  $G \cup \{0\}$ .

By [19, Theorem 2], each connected component of the Graham-Houghton graph of R corresponds to a regular Rees 0-matrix semigroup

$$(I_k \times G_k \times \Lambda_k) \cup \{0\} = \mathscr{M}^0[I_k, G_k, \Lambda_k; P_k]$$

and the idempotent generated subsemigroup  $\langle E(R) \rangle$  of R is the union

$$\bigcup_{k=1}^{n} \mathscr{M}^{0}[I_{k}, G_{k}, \Lambda_{k}; P_{k}].$$
(1)

Throughout this section we let S be a subsemigroup of R, and we use the following notation:

- for  $i \in I$  and  $\lambda \in \Lambda$ , let  $H_{i,\lambda} = S \cap (\{i\} \times G \times \{\lambda\})$  (the intersection of S with an  $\mathcal{H}$ -class of R);
- for  $k, l \in \{1, \ldots, n\}$ , let  $C_{k,l} = S \cap (I_k \times G \times \Lambda_l)$  (the intersection of S with a block of  $\mathscr{H}$ -classes of R).

Observe that with this definition  $C_{k,l} = \bigcup_{i \in I_k, \lambda \in \Lambda_l} H_{i,\lambda}$ , and  $S = \{0\} \cup \bigcup_{1 \leq k, l \leq n} C_{k,l}$ . Note that if S is regular, then  $\mathscr{H}^S = \mathscr{H}^R \cap (S \times S)$  by [24, Proposition 2.4.2]. Therefore, if S is regular, then the non-zero  $\mathscr{H}$ -classes of S are the non-empty sets of the form  $H_{i,\lambda}$ . In particular, if S intersects each  $\mathscr{H}$ -class of R non-trivially, then (since in this case S is necessarily regular) it follows that  $H_{i,\lambda}$  is an  $\mathscr{H}$ -class of S for all  $i \in I$  and  $\lambda \in \Lambda$ .

We next state three technical lemmas that describe the subsemigroups of R that intersect every  $\mathcal{H}$ -class of R nontrivially.

**Lemma 2.3.** The subsemigroup S intersects every  $\mathcal{H}$ -class of R non-trivially if and only if  $0 \in S$  and there exists a subgroup V of G and elements  $g_1 = 1_G, g_2, \ldots, g_n \in G$  such that  $G_k \leq g_k^{-1} V g_k$  and  $C_{k,l} = I_k \times g_k^{-1} V g_l \times \Lambda_l$  for all  $k, l \in \{1, \ldots, n\}.$ 

*Proof.* ( $\Leftarrow$ ) The converse implication is immediate.

 $(\Rightarrow)$  Recall that  $p_{\lambda_k,i_k} = 1_G$  for all k, and so by Green's Lemma (Lemma 1.1)

$$H_{j,\mu} = H_{j,\lambda_k}(i_k, g, \mu) = (j, g, \lambda_k) H_{i_k,\mu}, \quad \text{and so} \quad H_{j,i_k} H_{\lambda_k,\mu} = H_{j,\mu}$$

for all  $j \in I$ ,  $\mu \in \Lambda$ , and  $g \in G$ .

Since  $p_{\lambda_1,i_1} = 1_G$ , it follows that the  $\mathscr{H}$ -class  $H_{i_1,\lambda_1}$  is a group, and that  $H_{i_1,\lambda_1} = \{i_1\} \times V \times \{\lambda_1\}$  for some subgroup V of G. By assumption, for each  $k \in \{2, \ldots, n\}$  there exists an element  $(i_1, g_k, \lambda_k) \in H_{i_1, \lambda_k}$ , and so

$$H_{i_1,\lambda_k} = H_{i_1,\lambda_1}(i_1,g_k,\lambda_k) = \{i_1\} \times V p_{i_1,\lambda_1}g_k \times \{\lambda_k\} = \{i_1\} \times V g_k \times \{\lambda_k\}.$$

Similarly, by assumption, for each  $k \in \{2, ..., n\}$  the  $\mathscr{H}$ -class  $H_{i_k, \lambda_1}$  equals  $\{i_k\} \times F_k \times \{\lambda_1\}$ , where  $F_k$  is some non-empty subset of G. It follows that

$$H_{i_1,\lambda_1} = (i_1, g_k, \lambda_k) H_{i_k,\lambda_1} = \{i_1\} \times g_k F_k \times \{\lambda_1\}$$

but as shown previously,  $H_{i_1,\lambda_1} = \{i_1\} \times V \times \{\lambda_1\}$ , and so  $F_k = g_k^{-1}V$ . Hence

$$H_{i_k,\lambda_k} = H_{i_k,\lambda_1} H_{i_1,\lambda_k} = \{i_k\} \times g_k^{-1} V g_k \times \{\lambda_k\}.$$

Since S is a finite semigroup that intersects every  $\mathcal{H}$ -class of R, it contains E(R), and so it contains  $\langle E(R) \rangle$ . Therefore  $H_{i_k,\lambda_k}$  contains the elements  $\langle E(R) \rangle \cap (\{i_k\} \times G \times \{\lambda_k\})$ , and so

$$G_k = \left( \langle E(R) \rangle \cap \left( \{ i_k \} \times G \times \{ \lambda_k \} \right) \right) \psi \le (H_{i_k, \lambda_k}) \psi = g_k^{-1} V g_k$$

for each k, as required.

Let  $k \in \{1, 2, ..., n\}$  and let  $i \in I_k \setminus \{i_k\}$ . Since the indices i and  $i_k$  are in the same connected component of the Graham-Houghton graph of R, there exists a path in the graph from i to  $i_k$ . That is, there exists an alternating sequence  $(i = a_1, b_1, a_2, b_2, \dots, a_m = i_k)$  of indices from  $I_k$  and  $\Lambda_k$ , respectively, such that  $p_{b_i, a_i} \neq 0$  and  $p_{b_i, a_{i+1}} \neq 0$  for all possible *j*. Therefore  $x = \prod_{j=1}^{m-1} (a_j, p_{b_j, a_j}^{-1}, b_j) \in \langle E(R) \rangle \leq S$ . It follows that

$$xH_{i_k,\lambda_k} = \{i\} \times p_{b_1,a_1}^{-1} p_{b_1,a_2} \cdots p_{b_{m-1},a_{m-1}}^{-1} p_{b_{m-1},a_m} g_k^{-1} V g_k \times \{\lambda_k\} \subseteq H_{i,\lambda_k}.$$

By Proposition 2.2(iv), this product of matrix entries is contained in the subgroup  $G_k \leq g_k^{-1} V g_k$ . Therefore the expression simplifies to  $xH_{i_k,\lambda_k} = \{i\} \times g_k^{-1}Vg_k \times \{\lambda_k\} \subseteq H_{i,\lambda_k}$ . It follows that  $(i, 1_G, \lambda_k) \in H_{i,\lambda_k}$  and so

$$H_{i,\lambda_k} = (i, 1_G, \lambda_k) H_{i_k,\lambda_k} = \{i\} \times g_k^{-1} V g_k \times \{\lambda_k\}.$$

By a similar argument  $H_{i_k,\lambda} = \{i_k\} \times g_k^{-1} V g_k \times \{\lambda\}$  for all  $\lambda \in \Lambda_k$ . Hence for any  $i \in I_k, \lambda \in \Lambda_k$ 

$$H_{i,\lambda} = H_{i,\lambda_k} H_{i_k,\lambda} = \{i\} \times g_k^{-1} V g_k p_{\lambda_k,i_k} g_k^{-1} V g_k \times \{\lambda\} = \{i\} \times g_k^{-1} V g_k \times \{\lambda\},$$

i.e.  $C_{k,k} = I_k \times g_k^{-1} V g_k \times \Lambda_k$ . To conclude the proof let  $k, l \in \{1, \ldots, n\}$  be arbitrary and let  $H_{i,\lambda}$  be an  $\mathscr{H}$ -class of S in  $C_{k,l}$  (i.e.  $i \in I_k$  and  $\lambda \in \Lambda_l$ ). Then we see that

$$H_{i,\lambda} = H_{i,\lambda_k} H_{i_k,\lambda_l} H_{i_l,\lambda_l} H_{i_l,\lambda_l}$$

$$= (\{i\} \times g_k^{-1} V g_k \times \{\lambda_k\}) (\{i_k\} \times g_k^{-1} V \times \{\lambda_l\}) (\{i_l\} \times V g_l \times \{\lambda_l\}) (\{i_l\} \times g_l^{-1} V g_l \times \{\lambda\})$$

$$= \{i\} \times g_k^{-1} V g_k p_{\lambda_k,i_k} g_k^{-1} V p_{\lambda_l,i_l} V g_l p_{\lambda_l,i_l} g_l^{-1} V g_l \times \{\lambda\}$$

$$= \{i\} \times g_k^{-1} V g_l \times \{\lambda\},$$

$$(2)$$

and so  $C_{k,l} = I_k \times g_k^{-1} V g_l \times \Lambda_l$ .

**Lemma 2.4.** Let V be a subgroup of G and let  $g_1 = 1_G, g_2, \ldots, g_n \in G$  be elements such that  $G_k \leq g_k^{-1} V g_k$ . Then  $0 \in S$ and  $C_{k,l} = I_k \times g_k^{-1} V g_l \times \Lambda_l$  for all  $k, l \in \{1, \ldots, n\}$  if and only if  $S = \langle E(R), \{i_1\} \times V \times \{\lambda_1\}, x_2, \ldots, x_n, y_2, \ldots, y_n \rangle$ , where  $x_k = (i_1, g_k, \lambda_k)$  and  $y_k = (i_k, g_k^{-1}, \lambda_1)$  for all k.

*Proof.*  $(\Rightarrow)$  Let  $x_1 = y_1 = (i_1, 1_G, \lambda_1)$ . Then since  $x_1, y_1 \in \{i_1\} \times V \times \{\lambda_1\}$ , and since  $\{i_1\} \times V \times \{\lambda_1\} = H_{i_1, \lambda_1}$  by assumption, it suffices to show that the set  $X = \{E(R), H_{i_1,\lambda_1}, x_1, \dots, x_n, y_1, \dots, y_n\}$  generates S. Clearly  $\langle X \rangle \leq S$ .

Since  $0 \in E(R) \leq \langle X \rangle$ , to prove that  $S \leq \langle X \rangle$ , let  $k, l \in \{1, \ldots, n\}$  be arbitrary and let  $H_{i,\lambda}$  be an  $\mathscr{H}$ -class of S in  $C_{k,l}$  (i.e.  $i \in I_k$  and  $\lambda \in \Lambda_l$ ). We must show that  $H_{i,\lambda} \subseteq \langle X \rangle$ . By Green's Lemma [Lemma 1.1],  $H_{i_k,\lambda_1} = y_k H_{i_1,\lambda_1} \subseteq \langle X \rangle$ and  $H_{i_1,\lambda_l} = H_{i_1,\lambda_1} x_l \subseteq \langle X \rangle$ . By (1),  $(i, 1_G, \lambda_k)$  and  $(i_l, 1_G, \lambda)$  are elements of  $\langle E(R) \rangle$ , and are therefore contained in  $\langle X \rangle$ . Hence by Green's Lemma,  $H_{i,\lambda_k} = (i, 1_G, \lambda_k) H_{i_k,\lambda_k} = (i, 1_G, \lambda_k) y_k H_{i_1,\lambda_1} x_k \subseteq \langle X \rangle$  and  $H_{i_l,\lambda} = H_{i_l,\lambda_l}(i_l, 1_G, \lambda) = H_{i_l,\lambda_l}(i_l, 1_G, \lambda)$  $y_l H_{i_1,\lambda_1} x_l(i_l, 1_G, \lambda) \subseteq \langle X \rangle$ . By (2),  $H_{i,\lambda} = H_{i,\lambda_k} H_{i_k,\lambda_1} H_{i_1,\lambda_l} H_{i_l,\lambda} \subseteq \langle X \rangle$ , and it follows that  $S \leq \langle X \rangle$ .

( $\Leftarrow$ ) That S contains 0, follows since  $0 \in E(R)$ .

To prove that  $C_{k,l} \subseteq I_k \times g_k^{-1} V g_l \times \Lambda_l$  for all k, l, we must show that if  $(i, g, \lambda) \in C_{k,l}$ , for some k, l, then  $g \in g_k^{-1} V g_l$ . We proceed by induction on the length of a product in the generators equalling any given element. Certainly, if  $(i, g, \lambda)$ is one of the generators of S, then there exist k, l such that  $(i, g, \lambda) \in C_{k,l}$ , and  $g \in g_k^{-1} V g_l$  by definition.

Assume that if  $(i, g, \lambda) \in S$  can be expressed as a product of length at most m in the generators of S, then  $g \in g_k^{-1} V g_l$ , where k, l are such that  $(i, g, \lambda) \in C_{k,l}$ . Let  $(i, g, \lambda) \in S$  be any element that can be given as a product in X of length m+1. Then there exist k, l such that  $(i, g, \lambda) \in C_{k,l}$ , and there exist  $(i, h_1, \lambda'), (i', h_2, \lambda) \in S$  that can both be expressed as product of length at most m over X and  $(i, h_1, \lambda')(i', h_2, \lambda) = (i, g, \lambda)$ . Since this product is non-zero, it follows that  $p_{\lambda',i'} \neq 0$  and so  $i' \in I_r$  and  $\lambda' \in \Lambda_r$  for some r. By induction,  $h_1 \in g_k^{-1} V g_r, h_2 \in g_r^{-1} V g_l$ , and so

$$g = h_1 p_{\lambda',i'} h_2 \in g_k^{-1} V g_r p_{\lambda',i'} g_r^{-1} V g_l$$

Since  $p_{\lambda',i'} \in G_r \leq g_r^{-1}Vg_r$  it follows that  $g_r p_{\lambda',i'} g_r^{-1} \in V$ . Therefore  $g \in g_k^{-1}Vg_l$ , as required. Let  $k, l \in \{1, \dots, n\}$ . To prove that  $I_k \times g_k^{-1}Vg_l \times \Lambda_l \subseteq C_{k,l}$  we will show that  $\{i\} \times g_k^{-1}Vg_l \times \{\lambda\} \subseteq S$  for  $i \in I_k$  and  $\lambda \in \Lambda_l$ . By (1),  $(i, 1_G, \lambda_k)$  and  $(i_l, 1_G, \lambda)$  are elements of  $\langle E(R) \rangle$  and so

$$\{i\} \times g_k^{-1} V g_l \times \{\lambda\} = (i, 1_G, \lambda_k) y_k (\{i_1\} \times V \times \{\lambda_1\}) x_l (i_l, 1_G, \lambda) \subseteq S.$$

**Lemma 2.5.** Let V and  $\overline{V}$  be subgroups of G such that there exist  $g_1 = 1_G, g_2, \ldots, g_n, \overline{g_1} = 1_G, \overline{g_2}, \ldots, \overline{g_n} \in G$  where  $G_k \leq g_k^{-1} V g_k \text{ and } G_k \leq \overline{g_k}^{-1} \overline{V} \overline{g_k} \text{ for all } k. \text{ Let } X = \{E(R), \{i_1\} \times V \times \{\lambda_1\}, x_2, \dots, x_n, y_2, \dots, y_n\} \text{ and } \overline{X} = \{E(R), \{i_1\} \times V \times \{\lambda_1\}, x_2, \dots, x_n, y_2, \dots, y_n\} \text{ and } \overline{X} = \{E(R), \{i_1\} \times V \times \{\lambda_1\}, x_2, \dots, x_n, y_2, \dots, y_n\} \text{ and } \overline{X} = \{E(R), \{i_1\} \times V \times \{\lambda_1\}, x_2, \dots, x_n, y_2, \dots, y_n\} \text{ and } \overline{X} = \{E(R), \{i_1\} \times V \times \{\lambda_1\}, x_2, \dots, x_n, y_2, \dots, y_n\} \text{ and } \overline{X} = \{E(R), \{i_1\} \times V \times \{\lambda_1\}, x_2, \dots, x_n, y_2, \dots, y_n\} \text{ and } \overline{X} = \{E(R), \{i_1\} \times V \times \{\lambda_1\}, x_2, \dots, x_n, y_2, \dots, y_n\} \text{ and } \overline{X} = \{E(R), \{i_1\} \times V \times \{\lambda_1\}, x_2, \dots, x_n, y_2, \dots, y_n\} \text{ and } \overline{X} = \{E(R), \{i_1\} \times V \times \{\lambda_1\}, x_2, \dots, x_n, y_2, \dots, y_n\} \text{ and } \overline{X} = \{E(R), \{i_1\} \times V \times \{\lambda_1\}, x_2, \dots, x_n, y_2, \dots, y_n\} \text{ and } \overline{X} = \{E(R), \{i_1\} \times V \times \{\lambda_1\}, x_2, \dots, x_n, y_2, \dots, y_n\} \text{ and } \overline{X} = \{E(R), \{i_1\} \times V \times \{\lambda_1\}, x_2, \dots, x_n, y_2, \dots, y_n\} \text{ and } \overline{X} = \{E(R), \{i_1\} \times V \times \{\lambda_1\}, x_2, \dots, x_n, y_2, \dots, y_n\} \text{ and } \overline{X} = \{E(R), \{i_1\} \times V \times \{\lambda_1\}, x_2, \dots, x_n, y_n\} \text{ and } \overline{X} = \{E(R), \{i_1\} \times V \times \{\lambda_1\}, x_2, \dots, x_n, y_n\} \text{ and } \overline{X} = \{E(R), \{i_1\}, x_2, \dots, x_n, y_n\} \text{ and } \overline{X} = \{E(R), \{i_1\}, x_2, \dots, x_n, y_n\} \text{ and } \overline{X} = \{E(R), \{i_1\}, i_2\}, \dots, i_n\} \text{ and } \overline{X} = \{E(R), \{i_1\}, i_2\}, \dots, i_n\} \text{ and } \overline{X} = \{E(R), \{i_1\}, i_2\}, \dots, i_n\} \text{ and } \overline{X} = \{E(R), \{i_1\}, i_2\}, \dots, i_n\} \text{ and } \overline{X} = \{E(R), \{i_1\}, i_n\} \text{ and } \overline{X} = \{E(R), \{i_1\}, i_n\}, \dots, i_n\} \text{ and } \overline{X} = \{E(R), \{i_1\}, i_n\}, \dots, i_n\} \text{ and } \overline{X} = \{E(R), \{i_1\}, i_n\}, \dots, i_n\} \text{ and } \overline{X} = \{E(R), \{i_1\}, i_n\}, \dots, i_n\} \text{ and } \overline{X} = \{E(R), \{i_1\}, i_n\}, \dots, i_n\} \text{ and } \overline{X} = \{E(R), \{i_1\}, i_n\}, \dots, i_n\} \text{ and } \overline{X} = \{E(R), \{i_1\}, i_n\}, \dots, i_n\} \text{ and } \overline{X} = \{E(R), \{i_1\}, i_n\}, \dots, i_n\} \text{ and } \overline{X} = \{E(R), \{i_1\}, i_n\}, \dots, i_n\} \text{ and } \overline{X} = \{E(R), \{i_n\}, \dots, i_n\} \text{ and } \overline{X} = \{E(R), \{i_n\}, \dots, i_n\} \text{ and } \overline{X} = \{E(R), \{i_n\}, \dots, i_n\} \text{ and } \overline{X} = \{E(R), \{i_n\}, \dots, i_n\} \text{ and } \overline{X} = \{E(R), \{i_n\}, \dots, i_n\} \text{ and } \overline{X} = \{E(R), \{i_n\}, \dots, i_n\} \text{ and } \overline{X} = \{E(R), \{i_n\}, \dots, i_n$  $\overline{V} \times \{\lambda_1\}, \overline{x_2}, \dots, \overline{x_n}, \overline{y_2}, \dots, \overline{y_n}\}, \text{ where } x_k = (i_1, g_k, \lambda_k), \ \overline{x_k} = (i_1, \overline{g_k}, \lambda_k), \ y_k = (i_k, g_k^{-1}, \lambda_1), \text{ and } \overline{y_k} = (i_k, \overline{g_k^{-1}}, \lambda_1) \text{ for all } k.$ 

*Proof.* By Lemma 2.4,  $\langle X \rangle = \langle \overline{X} \rangle$ if and only if  $(I_k \times g_k^{-1} V g_l \times \Lambda_l) = (I_k \times \overline{g_k}^{-1} \overline{V} \overline{g_l} \times \Lambda_l)$  for all  $k, l \in \{1, \dots, n\}$ if and only if  $g_k^{-1}Vg_l = (\overline{g_k})^{-1}\overline{V}\overline{g_l}$  for all  $k, l \in \{1, \dots, n\}$ if and only if  $V = \overline{V}$  and  $g_k^{-1}Vg_l = (\overline{g_k})^{-1}V\overline{g_l}$  for all  $k, l \in \{1, \dots, n\}$ if and only if  $V = \overline{V}, g_k^{-1}Vg_l = (\overline{g_k})^{-1}V\overline{g_l}, g_k^{-1}V = (\overline{g_k})^{-1}V$ , and  $Vg_k = V\overline{g_k}$  for all  $k, l \in \{2, \dots, n\}$ if and only if  $V = \overline{V}$  and  $Vg_k = V\overline{g_k}$  for all  $k \in \{2, \ldots, n\}$ .

Suppose that V is a subgroup of G such that  $G_1 \leq V$ . By Lemmas 2.3, 2.4, and 2.5, in order to find the subsemigroups S of R that intersect every  $\mathscr{H}$ -class of R non-trivially, and such that  $S \cap (\{i_1\} \times G \times \{\lambda_1\}) = \{i_1\} \times V \times \{\lambda_1\}$ , it suffices to find an arbitrary set  $\mathfrak{T}$  of representatives (called a *transversal*) of the right cosets of V in G, and the sets

$$\{g \in \mathfrak{T} : G_k \leq g^{-1}Vg\}$$

for all  $k \ge 2$ . More explicitly, given any transversal  $\mathfrak{T}$  of the right cosets of V in G, by Lemma 2.5 such subsemigroups of R are in 1-1 correspondence with the Cartesian product

$$\prod_{k=2}^{n} \{g \in \mathfrak{T} : G_k \le g^{-1} V g\}.$$
(3)

Thus, if  $t \in G$  is such that  $G_1 \leq t^{-1}Vt$  and  $\mathfrak{U}$  is an arbitrary transversal of the right cosets of  $t^{-1}Vt$  in G, then the subsemigroups T of R that intersect every  $\mathscr{H}$ -class of R non-trivially, and such that  $T \cap (\{i_1\} \times G \times \{\lambda_1\}) = \{i_1\} \times t^{-1}Vt \times \{\lambda_1\}$ , are in 1-1 correspondence with the Cartesian product

$$\prod_{k=2}^{n} \left\{ g \in \mathfrak{U} : G_k \le g^{-1} \left( t^{-1} V t \right) g \right\}$$

Note that we call any such subsemigroup T a subsemigroup arising from a conjugate of V. However, if  $\mathfrak{T}$  is a transversal of the right cosets of V in G, then the set  $\{t^{-1}g : g \in \mathfrak{T}\}$  is a transversal of the right cosets of  $t^{-1}Vt$  in G. In particular, given any transversal  $\mathfrak{T}$  of the right cosets of V in G, the collection of all such subsemigroups T is in 1-1 correspondence with the Cartesian product

$$\prod_{k=2}^{n} \left\{ t^{-1}g : g \in \mathfrak{T} \text{ and } G_k \le (t^{-1}g)^{-1} \left( t^{-1}Vt \right) (t^{-1}g) \right\} = \prod_{k=2}^{n} \left\{ t^{-1}g : g \in \mathfrak{T} \text{ and } G_k \le g^{-1}Vg \right\}.$$

Therefore, to find all subsemigroups arising from conjugates of V, it suffices to find the conjugates of V, as well as the set in (3).

We require the following lemma for the proof of Proposition 2.7.

**Lemma 2.6.** If S intersects every  $\mathcal{H}$ -class of R non-trivially, then every non-zero  $\mathcal{H}$ -class of S contains the same number of elements. In particular, if S intersects every  $\mathcal{H}$ -class of R non-trivially and contains a non-zero  $\mathcal{H}$ -class of R, then S = R.

*Proof.* By Lemma 2.3, there exists a subgroup V of G such that every non-zero  $\mathscr{H}$ -class of S has the form  $\{i\} \times g_k^{-1} V g_l \times \{\lambda\}$  for some indices  $i, \lambda$  and for some elements  $g_k, g_l \in G$ . The result follows.

The following proposition will allow us to find all of the maximal subsemigroups in R of type (R6) arising from conjugates of a given maximal subgroup of the group G.

**Proposition 2.7.** Let  $R = \mathscr{M}^0[I, G, \Lambda; P]$  be a normalized regular finite Rees 0-matrix semigroup over a group G, where I and  $\Lambda$  are disjoint index sets, and  $P = (p_{\lambda,i})_{\lambda \in \Lambda, i \in I}$  is a  $|\Lambda| \times |I|$  matrix over  $G \cup \{0\}$ , and let S be a subsemigroup of R. Suppose that the number of connected components of the Graham-Houghton graph of R is n, and let  $G_1, \ldots, G_n$  and  $i_1, \ldots, i_n \in I$  and  $\lambda_1, \ldots, \lambda_n \in \Lambda$  be as defined in Proposition 2.2. Then S is a maximal subsemigroup of R that intersects every  $\mathscr{H}$ -class of R non-trivially if and only if there exists a maximal subgroup V of G and elements  $g_1 = 1_G, g_2, \ldots, g_n \in G$  such that  $G_k \leq g_k^{-1}Vg_k$  for all k, and

$$S = \langle E(R), \{i_1\} \times V \times \{\lambda_1\}, x_2, \dots, x_n, y_2, \dots, y_n \rangle$$

where  $x_k = (i_1, g_k, \lambda_k)$  and  $y_k = (i_k, g_k^{-1}, \lambda_1)$  for all k.

*Proof.* ( $\Rightarrow$ ) After applying the direct implications of Lemmas 2.3 and 2.4, it remains to prove that V is a maximal subgroup of G. Let K be a subgroup of G with  $V \leq K \leq G$ . Then  $T = \langle E(R), \{i_1\} \times K \times \{\lambda_1\}, x_2, \ldots, x_n, y_2, \ldots, y_n \rangle$  is a subsemigroup of R, and  $S \leq T$ . Since S is maximal, either T = S or T = R. In the former case, K = V by Lemma 2.5; in the latter case, by the converse implication of Lemma 2.4, it follows that  $\{i_1\} \times K \times \{\lambda_1\} = \{i_1\} \times G \times \{\lambda_1\}$ , i.e. K = G.

( $\Leftarrow$ ) By the converse implication of Lemma 2.4, it follows that  $0 \in S$  and  $C_{k,l} = I_k \times g_k^{-1} V g_l \times \Lambda_l$  for all k, l; in particular, S is a proper subsemigroup of R that intersects every  $\mathscr{H}$ -class of R non-trivially.

It remains to show that S is maximal. Let  $x = (i, g, \lambda) \in R \setminus S$  be arbitrary. Then  $\langle S, x \rangle$  is a subsemigroup of R that intersects every  $\mathscr{H}$ -class of R non-trivially. Since  $\langle S, x \rangle \cap (\{i\} \times G \times \{\lambda\})$  contains at least |V| + 1 elements, so does the group  $\langle S, x \rangle \cap (\{i\} \times G \times \{\lambda_1\})$  by Lemma 2.6. Therefore

$$V = \left(S \cap \left(\{i_1\} \times G \times \{\lambda_1\}\right)\right)\psi \leq \left(\langle S, x \rangle \cap \left(\{i_1\} \times G \times \{\lambda_1\}\right)\right)\psi \leq G.$$

Since V is a maximal subgroup of G,  $(\langle S, x \rangle \cap (\{i_1\} \times G \times \{\lambda_1\}))\psi = G$ . The result follows by Lemma 2.6.

Let V be a representative of a conjugacy class of maximal subgroups of G. By Lemma 2.5, Proposition 2.7, and the previous arguments, to find all of the maximal subsemigroups in R of type (R6) that arise from conjugates of V, we first require any transversal  $\mathfrak{T}$  of the right cosets of V in G, and any transversal  $\mathfrak{U}$  of the right cosets of  $N_G(V)$  in G, where  $N_G(V)$  is the normalizer of V in G. Note that since  $N_G(V)$  is subgroup of G that contains the maximal subgroup V, it follows that either  $N_G(V) = V$ , in which case we may choose  $\mathfrak{U} = \mathfrak{T}$ , or  $N_G(V) = G$ , in which case we may choose  $\mathfrak{U} = \{1_G\}$ . We then require all coset representatives  $g \in \mathfrak{T}$  such that  $G_k \leq g^{-1}Vg$ , for all  $k \in \{2, \ldots, n\}$ , and all coset representatives  $t \in \mathfrak{U}$  such that  $G_1 \leq t^{-1}Vt$ . By performing this process for each conjugacy class of maximal subgroups of G, we find all maximal subsemigroups of R of type (R6).

**Corollary 2.8.** Let V be a maximal subgroup of G, let  $\mathfrak{T}$  be an arbitrary transversal of the right cosets of V in G and let  $\mathfrak{U}$  be an arbitrary transversal of the right cosets of  $N_G(V)$  in G. Then the number of maximal subsemigroups in R of type (R6) that arise from conjugates of V is

$$M = |\{t \in \mathfrak{U} : G_1 \le t^{-1}Vt\}| \cdot \prod_{k=2}^n |\{g \in \mathfrak{T} : G_k \le g^{-1}Vg\}|.$$

If  $N_G(V) = G$ , then  $M \leq [G:V]^{n-1}$ . Otherwise,  $N_G(V) = V$ , and so  $M \leq [G:V]^n$ .

The upper bounds in Corollary 2.8 are tight. Let B(G,m) denote the Brandt semigroup  $\mathscr{M}^0[I,G,I;P]$  where P is the identity matrix and |I| = m. For example, suppose that  $G = S_3$ , the symmetric group of degree 3. The number of maximal subsemigroups of B(G,m) of type (R6) arising from conjugates of  $V = A_3$ , the alternating group of degree 3, is  $[G:V]^{m-1} = 2^{m-1}$ , whilst the number arising from  $V = \langle (1 \ 2) \rangle$ , a cyclic group of order 2, is  $[G:V]^m = 3^m$ .

On the other hand, suppose that  $G = \{1_G, x\} = \langle x \rangle$  is a cyclic group of order 2, that  $|I| = |\Lambda| = 2$ , and that

$$P = \begin{pmatrix} 1_G & 1_G \\ 1_G & x \end{pmatrix}.$$

Then in  $R = \mathscr{M}^0[I, G, \Lambda; P]$  the subgroup  $G_1$  is equal to G, and so there are no maximal subsemigroups of type (R6), since no maximal subgroup of G contains  $G_1$ .

A method for finding the maximal subsemigroups of type (R6) is given in Algorithm 1.

## 2.3 An example

In this section, we give an example of a Rees 0-matrix semigroup and show how our algorithm can be applied to calculate its maximal subsemigroups.

Let R be the regular Rees 0-matrix semigroup  $\mathscr{M}^0[I, S_4, \Lambda; P]$ , where  $I = \{1, \ldots, 6\}$ ,  $\Lambda = \{-6, \ldots, -1\}$ ,  $S_4$  is the symmetric group of degree 4, and P is the  $6 \times 6$  matrix

$$P = \begin{pmatrix} (3\ 4) & (1\ 3\ 2\ 4) & (1\ 4)(2\ 3) & 0 & 0 & 0 \\ (2\ 4) & 0 & (1\ 3\ 2) & 0 & 0 & 0 \\ 0 & (3\ 4) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (1\ 4\ 3) & (1\ 3)(2\ 4) & 0 \\ 0 & 0 & 0 & (1\ 4) & (1\ 4\ 2) & 0 \\ 0 & 0 & 0 & 0 & 0 & (1\ 4\ 2) \end{pmatrix}$$

A diagram of the Graham-Houghton graph of R is shown in Figure 1.

Since  $|R| = 6^2 \cdot 4! + 1 = 865$ , R has no maximal subsemigroup of type (R1). Since P contains 0,  $R \setminus \{0\}$  is not a subsemigroup and so there are no maximal subsemigroups of type (R2) either.

The removal of any of the vertices 2, 6, or -6 from the Graham-Houghton graph  $\Gamma(R)$ , results in an induced subgraph containing a vertex without any incident edges. Hence the only maximal subsemigroups of R of types (R3) and (R4) arise

Algorithm 1 Maximal subsemigroups of type (R6)

**Input:** *R*, a finite regular Rees 0-matrix semigroup over a group. **Output:** the maximal subsemigroups  $\mathfrak{M}$  of R of type (R6). 1: find an isomorphism  $\Psi: R \longrightarrow \mathscr{M}^0[I, G, \Lambda; P] = R'$  such that R' is normalized [Section 4 of [19]] 2: find partitions  $I_1, \ldots, I_n$  and  $\Lambda_1, \ldots, \Lambda_n$  of I and  $\Lambda$  such that  $I_k \cup \Lambda_k$  is a connected component of the Graham-Houghton graph of R' for all k[Standard breadth or depth first search [32, Section 4.1]3: fix  $i_1 \in I_1, \ldots, i_n \in I_n$ , and  $\lambda_1 \in \Lambda_1, \ldots, \lambda_n \in \Lambda_n$  such that  $p_{\lambda_k, i_k} = 1_G$  for all k 4:  $E(R') := \{(i, p_{\lambda,i}^{-1}, \lambda) : i \in I, \lambda \in \Lambda, p_{\lambda i} \neq 0\} \cup \{0\}$ [The idempotents of R'] 5: set  $G_k := \langle \{p_{\lambda,i} : i \in I_k, \ \lambda \in \Lambda_k, p_{\lambda,i} \neq 0\} \rangle$  for all k [Proposition 2.2(iv)] 6: set  $\mathcal{C}$  to be a set of conjugacy class representatives of the maximal subgroups of G[Standard group theoretic algorithm [3, 14]7:  $\mathfrak{M} := \varnothing$ 8: for  $V \in \mathcal{C}$  do compute  $\mathfrak{U}$ , a transversal of the right cosets of  $N_G(V)$  in G 9: 10:  $T_1 := \emptyset$ for  $t \in \mathfrak{U}$  do 11: if  $G_1 \leq t^{-1}Vt$  then 12: $T_1 \leftarrow T_1 \cup \{t\}$ 13:compute  $\mathfrak{T}$ , a transversal of the right cosets of V in G 14:for  $k \in \{2, ..., n\}$  do 15: $T_k := \emptyset$ 16:for  $g \in \mathfrak{T}$  do 17:if  $G_k \leq g^{-1}Vg$  then 18: $T_k \leftarrow T_k \cup \{g\}$ 19:for  $t_1 \in T_1, t_2 \in T_2, ..., t_n \in T_n$  do 20: $\mathfrak{M} \leftarrow \mathfrak{M} \cup \left( \{ \{ E(R'), \{i_1\} \times t_1^{-1} V t_1 \times \{\lambda_1\}, (i_1, t_1^{-1} t_2, \lambda_2), \ldots, \right.$ 21: $(i_1, t_1^{-1}t_n, \lambda_n), (i_2, t_2^{-1}t_1, \lambda_1), \ldots, (i_n, t_n^{-1}t_1, \lambda_1)\} \Psi^{-1}$ 22: return M

from removing any of the vertices -1, -2, -3, -4, or -5 or 1, 3, 4, or 5, respectively. Thus there are 9 subsemigroups of types (R3) and (R4).

The maximal cliques in the dual of this graph corresponding to maximal subsemigroups are:

$$\{-1, -2, -3, -4, -5, 6\}, \{-1, -2, -3, -6, 4, 5\}, \{-1, -2, -3, 4, 5, 6\}, \{-2, -4, -5, -6, 2\}, \\ \{-2, -4, -5, 2, 6\}, \{-2, -6, 2, 4, 5\}, \{-2, 2, 4, 5, 6\}, \{-3, -4, -5, -6, 1, 3\}, \{-3, -4, -5, 1, 3, 6\}, \\ \{-3, -6, 1, 3, 4, 5\}, \{-3, 1, 3, 4, 5, 6\}, \{-4, -5, -6, 1, 2, 3\}, \{-4, -5, 1, 2, 3, 6\}, \{-6, 1, 2, 3, 4, 5\}.$$

The only maximal cliques in the dual of  $\Gamma(R)$  that do not correspond to maximal subsemigroups are  $\{-6, \ldots, -1\}$  and  $\{1, \ldots, 6\}$ . Hence there are 14 maximal subsemigroups of R of type (R5).

To calculate the maximal subsemigroups of R of type (R6), it is first necessary to find a normalized Rees 0-matrix semigroup that is isomorphic to R. One such normalized Rees 0-matrix semigroup is  $R' = \mathcal{M}^0[I, S_4, \Lambda; P']$ , where

$$P' = \begin{pmatrix} \text{id} & \text{id} & \text{id} & 0 & 0 & 0\\ \text{id} & 0 & (1\ 2)(3\ 4) & 0 & 0 & 0\\ 0 & \text{id} & 0 & 0 & 0\\ 0 & 0 & 0 & \text{id} & \text{id} & 0\\ 0 & 0 & 0 & \text{id} & (1\ 2\ 3\ 4) & 0\\ 0 & 0 & 0 & 0 & 0 & \text{id} \end{pmatrix}$$

and where id is the identity permutation on  $\{1, 2, 3, 4\}$ .

It is clear by inspecting  $\Gamma(R)$  that R, and hence R', have 3 connected components.

To find the maximal subsemigroups in R of type (R6), we first find the groups  $G_1, G_2, G_3$  from the definition of a normalized Rees 0-matrix semigroup. The groups  $G_1, G_2$ , and  $G_3$  are generated by the non-zero matrix entries corresponding to the relevant connected component of the graph  $\Gamma(R)$ , and so:

$$G_1 = \langle (1 \ 2)(3 \ 4) \rangle, \quad G_2 = \langle (1 \ 2 \ 3 \ 4) \rangle, \text{ and } G_3 = \mathbf{1},$$

where  $\mathbf{1}$  is the trivial subgroup of  $S_4$ .

We then determine the maximal subgroups V of G up to conjugacy, and for each V, find the maximal subsemigroups of R that arise from conjugates of V. For each such V, it suffices to find a transversal  $\mathfrak{U}$  of the right cosets of  $N_G(V)$  in G and a transversal  $\mathfrak{T}$  of the right cosets of V in G, along with the sets

$$\{g \in \mathfrak{U} : G_1 \leq g^{-1}Vg\}$$
 and  $\{g \in \mathfrak{T} : G_k \leq g^{-1}Vg\}$ , for all  $k \in \{2,3\}$ .

If, for a given choice of V, any of these sets is empty, then there are no maximal subsemigroups of R arising from conjugates of V.

Up to conjugacy, there are 3 maximal subgroups of  $S_4$ : the alternating group  $A_4$  in its natural representation; the symmetric group  $S_3$ ; and the dihedral group of order 8,  $D_4 = \langle (1 \ 2), (1 \ 3)(2 \ 4) \rangle$ . We consider each of these cases separately.

**Case 1:**  $V = A_4$ . Since  $G_2$  is not a subgroup of  $g^{-1}A_4g = A_4$  for any  $g \in G$ , there are no maximal subsemigroups of R of type (R6) arising from conjugates of  $A_4$ .

**Case 2:**  $V = S_3$ . Since  $G_1$  has no fixed points but every subgroup of every conjugate of  $S_3$  fixes at least one point, it follows that  $G_1 \not\leq g^{-1}S_3g$  for any  $g \in G$ . Hence there are no maximal subsemigroups of R arising from conjugates of  $S_3$ . **Case 3:**  $V = D_4$ . Since  $D_4$  is not a normal subgroup of  $S_4$ , it follows that  $N_{S_4}(D_4) = D_4$ . We choose the transversal:

$$\mathfrak{T} = \mathfrak{U} = \{ \mathrm{id}, (2\ 3), (2\ 4\ 3) \}.$$

All 3 conjugates of  $D_4$  contain  $G_1$ , the only conjugate of  $D_4$  that contains  $G_2$  is  $(2 \ 3)^{-1}D_4(2 \ 3)$ , and every subgroup of  $S_4$  contains  $G_3$ . Hence there are 9 maximal subsemigroups of R' (and hence R) of type (R6) that arise from conjugates of  $D_4$ .

In total there are 32 maximal subsemigroups of R.



Figure 1: The Graham-Houghton graph  $\Gamma(R)$  of R.

## 3 Arbitrary semigroups

In this section, we consider the problem of computing the maximal subsemigroups of an arbitrary finite semigroup, building on the results of Section 2.

By Lemma 1.3, if M is a maximal subsemigroup of a finite semigroup S, then  $S \setminus M \subseteq J$  for some  $\mathcal{J}$ -class J of S. By Proposition 1.4, either J is non-regular and M has the following form:

(S1)  $M = S \setminus J$  (Proposition 1.4(a)),

or J is regular and M has precisely one of the following forms:

(S2)  $M \cap J$  has non-empty intersection with every  $\mathscr{H}$ -class in J (Proposition 1.4(b));

- (S3)  $M \cap J$  is a non-empty union of both  $\mathscr{L}$  and  $\mathscr{R}$ -classes of S (Proposition 1.4(c)(i));
- (S4)  $M \cap J$  is a non-empty union of  $\mathscr{L}$ -classes of S (Proposition 1.4(c)(ii));
- (S5)  $M \cap J$  is a non-empty union of  $\mathscr{R}$ -classes of S (Proposition 1.4(c)(iii));
- (S6)  $M \cap J = \emptyset$  (Proposition 1.4(c)(iv)).

Throughout this section, S denotes an arbitrary finite semigroup with generating set X, J denotes a  $\mathscr{J}$ -class of S,  $\phi: J^* \longrightarrow R = \mathscr{M}^0[I, G, \Lambda; P]$  is an isomorphism from the principal factor  $J^*$  of J, and X' consists of those generators  $x \in X$  such that  $J_x > J$ , where  $J_x$  is the  $\mathscr{J}$ -class of x.

We will use the following straightforward lemma repeatedly in this section, for which we include a proof for completeness.

**Lemma 3.1.** Let T be a subset of S such that  $S \setminus T \subseteq J$ . Then T is a subsemigroup of S if and only if

(i)  $\langle X' \rangle \subseteq T$ ;

(ii) if  $x, y \in T \cap J$ , then  $xy \in J$  implies that  $xy \in T$ ; and

(iii) if  $x \in T \cap J$  and  $y \in \langle X' \rangle$ , then  $xy \in J$  implies that  $xy \in T$ , and  $yx \in J$  implies that  $yx \in T$ .

*Proof.* ( $\Rightarrow$ ) The second condition holds since T is a subsemigroup, and since T additionally contains X', it follows that the first and third conditions hold.

 $(\Leftarrow)$  Suppose that  $x, y \in T$  and that  $xy \in J$ . Then  $x, y \in J \cup \langle X' \rangle$ . If  $x, y \in J$  then  $xy \in T$  by (ii). If  $x, y \in \langle X' \rangle$ , then  $xy \in \langle X' \rangle \subseteq T$ . For the remaining cases,  $xy \in T$  by (iii).

## 3.1 Maximal subsemigroups arising from non-regular $\mathcal{J}$ -classes: type (S1)

In this section, we characterise those maximal subsemigroups of a finite semigroup S arising from the exclusion of a non-regular  $\mathcal{J}$ -class. Throughout this section we suppose that J is a non-regular  $\mathcal{J}$ -class of S.

**Lemma 3.2.** Either  $\langle S \setminus J \rangle = S$  or  $S \setminus J$  is a maximal subsemigroup of S.

*Proof.* Clearly,  $S \setminus J \subseteq \langle S \setminus J \rangle \subseteq S$ . If  $\langle S \setminus J \rangle \neq S$ , then  $\langle S \setminus J \rangle$  is a proper subsemigroup of S, and is therefore contained in a maximal subsemigroup of S arising from J. However, by Proposition 1.4, the only maximal subsemigroup of S that can arise from a non-regular  $\mathscr{J}$ -class is formed by removing it. Therefore  $S \setminus J$  is a maximal subsemigroup of S.  $\Box$ 

**Proposition 3.3.** Let S be a finite semigroup generated by X, let J be a non-regular  $\mathscr{J}$ -class of S, and let  $X' \subseteq X$  be the set of generators whose  $\mathscr{J}$ -classes are strictly greater than J in the  $\mathscr{J}$ -class partial order on S. Then  $S \setminus J$  is a maximal subsemigroup of S if and only if  $\langle X' \rangle \cap J = \emptyset$ .

*Proof.* ( $\Rightarrow$ ) Since  $\langle X' \rangle \leq \langle S \setminus J \rangle = S \setminus J$ , it follows that  $\langle X' \rangle \cap J = \emptyset$ .

( $\Leftarrow$ ) We prove the contrapositive. If  $S \setminus J$  is not a maximal subsemigroup, then, by Lemma 3.2,  $S \setminus J$  is a generating set for S. Hence  $\langle X' \rangle \cap J = J \neq \emptyset$ .

By Proposition 3.3, if J is a maximal non-regular  $\mathcal{J}$ -class of S, then  $S \setminus J$  is a maximal subsemigroup of S.

**Corollary 3.4.** Let J be a non-regular  $\mathscr{J}$ -class of a finite semigroup S generated by X where  $x \notin \langle X \setminus \{x\} \rangle$  for all  $x \in X$ . Then  $S \setminus J$  is not a maximal subsemigroup of S if and only if  $J \cap X = \emptyset$ .

*Proof.* ( $\Leftarrow$ ) If  $J \cap X = \emptyset$ , then  $S = \langle X \rangle \subseteq \langle S \setminus J \rangle \subseteq S$ , i.e.  $\langle S \setminus J \rangle = S$ .

 $(\Rightarrow)$  Since  $S \setminus J$  is not maximal, Lemma 3.2 implies that  $S \setminus J$  generates S. Thus  $\langle X' \rangle \cap J = J$  and so if  $x \in J \cap X$ , then  $x \in \langle X' \rangle \leq \langle X \setminus \{x\} \rangle$ , which contradicts the assumption on X. Therefore  $J \cap X = \emptyset$ .

# 3.2 Maximal subsemigroups arising from regular *J*-classes that intersect every *H*-class: type (S2)

In this section, we consider those maximal subsemigroups of a finite semigroup S that arise from the exclusion of elements in a regular  $\mathscr{J}$ -class J of S, and that intersect every  $\mathscr{H}$ -class of S non-trivially. In other words, we are considering maximal subsemigroups of type (S2). Since the principal factor  $J^*$  is isomorphic to a regular Rees 0-matrix semigroup over a group, the algorithms described in Section 2 could be used to compute the maximal subsemigroups of  $J^*$ . The purpose of this section is to characterise the maximal subsemigroups of type (S2) in terms of the maximal subsemigroups of  $J^*$ .

**Lemma 3.5.** Let S be a finite semigroup, and let T be a subset of S such that  $S \setminus T$  is contained in a regular  $\mathcal{J}$ -class J of S. Suppose that T intersects every  $\mathcal{H}$ -class of S and let E be a set consisting of one idempotent from each  $\mathcal{L}$ -class of J. Then T is a subsemigroup of S if and only if

(i)  $EX' \subseteq T$ ;

(ii) if  $x, y \in T \cap J$ , then  $xy \in J$  implies  $xy \in T$  (Lemma 3.1(ii)).

*Proof.* ( $\Rightarrow$ ) Since T is a subsemigroup, Lemma 3.1(ii) holds. Since T is finite and intersects every  $\mathscr{H}$ -class of S, it contains every idempotent of S. By assumption,  $X' \subseteq T$ , and so  $EX' \subseteq T$ .

( $\Leftarrow$ ) It suffices to show that the remaining conditions of Lemma 3.1 hold. In order to do this, we first show that  $E\langle X' \rangle \subseteq T$ . Let  $x \in E\langle X' \rangle \cap J$ . By definition, there exists an idempotent  $e_1 \in E$  and a sequence of generators  $x_1, \ldots, x_n \in X'$  such that  $x = e_1x_1 \cdots x_n$ . Since the element  $e_1$  and the product  $e_1x_1 \cdots x_n$  are both members of J, it follows that the intermediate product  $e_1x_1 \cdots x_k$  is in J for every  $k \in \{1, \ldots, n\}$ . Hence by definition of the set E, for each index k < n there exists an idempotent  $e_{k+1} \in E$  such that  $e_{k+1} \mathscr{L}e_1x_1 \cdots x_k$ , and in particular  $(e_1x_1 \cdots x_k)e_{k+1} = e_1x_1 \cdots x_k$ , since an idempotent is a right identity for its  $\mathscr{L}$ -class [24, Proposition 2.3.3]. Therefore  $x = \prod_{k=1}^n e_k x_k$ . Furthermore, for each  $k \in \{1, \ldots, n\}$  the element  $e_k x_k$  is contained in T by assumption, and is contained in J since

$$J = J_{e_k} \ge J_{e_k x_k} \ge J_x = J.$$

By repeated application of condition (ii), it follows that  $x = \prod_{k=1}^{n} e_k x_k \in T \cap J$ , and so  $E\langle X' \rangle \subseteq T$ .

Note that condition (ii) is equivalent to the statement that  $(T \cap J) \cup \{0\}$  is a subsemigroup of  $J^*$ . Since T intersects every  $\mathscr{H}$ -class of J and J is finite, it follows that T contains every idempotent of J.

To prove that condition (i) of Lemma 3.1 holds, let  $x \in \langle X' \rangle \cap J$ . Since J is a regular  $\mathscr{J}$ -class, there exists an idempotent  $f \in T \cap J$  such that fx = x. By definition of the set E, there exists an idempotent  $e \in E$  such that  $e\mathscr{L}f$ , and so x = fx = (fe)x = f(ex). We have  $ex \in T \cap J$  since  $E\langle X' \rangle \subseteq T$ , and since  $(T \cap J) \cup \{0\}$  is a subsemigroup of  $J^*$ , it follows that  $x = f(ex) \in T$ .

To prove that condition (iii) of Lemma 3.1 holds, let  $x \in T \cap J$  and  $y \in \langle X' \rangle$ . First suppose that  $xy \in J$ . By assumption, there exists an idempotent  $e \in E$  such that x = xe. Since  $xy = x(ey) \in J$  it follows that

$$J = J_e \ge J_{ey} \ge J_{x(ey)} = J_{xy} = J,$$

and so  $ey \in J$ . Furthermore,  $ey \in E\langle X' \rangle \subseteq T$ . Since  $x, ey \in T \cap J$ ,  $xy \in J$ , and  $(T \cap J) \cup \{0\}$  is a subsemigroup of  $J^*$ , it follows that  $xy = x(ey) \in T \cap J$ . Finally suppose that  $yx \in J$ . Since J is a regular  $\mathscr{J}$ -class, there exists an idempotent  $f \in T \cap J$  such that f(yx) = yx. By definition of E, there exists  $e \in E$  such that fe = f, and so yx = f(yx) = (fe)yx = f(ey)x. Note that  $E\langle X' \rangle \subseteq T$  implies that  $ey \in T$ , and  $ey \in J$  since

$$J = J_e \ge J_{ey} \ge J_{feyx} = J_{yx} = J.$$

Finally,  $f, ey, x \in T \cap J$  and  $(T \cap J) \cup \{0\}$  is a subsemigroup of  $J^*$ , and it follows that  $yx \in T$ .

Note that dual results hold if we replace E by a set consisting of one idempotent from each  $\mathscr{R}$ -class of J and we replace EX' by X'E.

Let  $\phi$  be the isomorphism from  $J^*$  to a Rees 0-matrix semigroup defined at the start of Section 3 and suppose that T is a subset of S such that  $S \setminus T \subseteq J$  and T intersects every  $\mathscr{H}$ -class of J non-trivially. Condition (ii) of Lemma 3.1 is equivalent to the statement that  $(T \cap J) \cup \{0\}$  is a subsemigroup of  $J^*$ , the principal factor of J. Since  $\phi$  is an isomorphism, this is equivalent to the statement that  $(T \cap J) \cup \{0\}$  is a subsemigroup of  $(J^*)\phi$ .

The following corollary is the main result of this section.

**Corollary 3.6.** Let S be a finite semigroup, and let T be a subset of S such that  $S \setminus T$  is contained in a regular  $\mathscr{J}$ -class J of S. Suppose that T intersects every  $\mathscr{H}$ -class of S and let E be a set consisting of one idempotent from each  $\mathscr{L}$ -class of J. Then T is a maximal subsemigroup of S if and only if  $(T \cap J)\phi \cup \{0\}$  is a maximal subsemigroup of  $(J^*)\phi$  containing  $(EX')\phi$ .

*Proof.* ( $\Rightarrow$ ) Let U be a subset of J such that  $U \cup \{0\}$  is a maximal subsemigroup of  $(J^*)\phi$  containing  $(T \cap J)\phi \cup \{0\}$ . Then by Lemma 3.5, the set  $M = (S \setminus J) \cup U$  is a proper subsemigroup of S containing T. Since T is maximal, it follows that T = M, and  $U = T \cap J$ .

( $\Leftarrow$ ) Let M be a maximal subsemigroup of S containing T. Then  $(M \cap J)\phi \cup \{0\}$  is a proper subsemigroup of  $(J^*)\phi$  containing  $(T \cap J)\phi \cup \{0\}$ . Since the latter is a maximal subsemigroup of  $(J^*)\phi$ , it follows that  $T \cap J = M \cap J$ , and hence T = M.

To calculate the maximal subsemigroups of type (S2) arising from J, we compute the set EX', we compute an isomorphism  $\phi$  from  $J^*$  to a normalized regular Rees 0-matrix semigroup R, and then we search for the maximal subsemigroups of R containing  $(EX')\phi$  that intersect every  $\mathscr{H}$ -class of R. Computing the maximal subsemigroups of R that have non-trivial intersection with every  $\mathscr{H}$ -class was the subject of Algorithm 1. One approach would be to simply compute all of the maximal subsemigroups using Algorithm 1 and then discard those which do not contain  $(EX')\phi$ . It is possible to modify Algorithm 1 to compute directly only those maximal subsemigroups containing  $(EX')\phi$ . However, we will not do this in detail.

A generating set for any such maximal subsemigroup is given by a generating set for  $S \setminus J$ , along with the preimage under  $\phi$  of a generating set for the maximal subsemigroup of R (minus the element  $0 \in R$ ). Algorithm 1 produces generating sets for the maximal subsemigroups that it finds.

## 3.3 Maximal subsemigroups arising from regular $\mathcal{J}$ -classes that are unions of $\mathcal{H}$ -classes: types (S3)-(S6)

In this section, we consider those maximal subsemigroups of a finite semigroup S that arise from the exclusion of elements in a regular  $\mathscr{J}$ -class of S, and that are unions of  $\mathscr{H}$ -classes of S. The principal purpose of this section is to give necessary and sufficient conditions for a subset of a finite semigroup to be a maximal subsemigroup in terms of the properties of certain associated graphs. The formulation in terms of graphs makes the problem of computing maximal subsemigroups of this type more tractable. In particular, we can take advantage of several well-known algorithms from graph theory, such as those for computing strongly connected components (see [16, 33] or [32, Section 4.2]) and finding all maximal cliques [2].

A digraph is a pair (V, E) consisting of a set of vertices V and a set of edges  $E \subseteq V \times V$  such that  $(u, v) \in E$  implies  $u \neq v$ . An edge  $(u, v) \in E$  is an edge from u to v. If  $\Gamma = (V, E)$  is a digraph and W is a subset of the vertices of  $\Gamma$ , then the subdigraph induced by W is the digraph with vertices W and edges  $\{(u, v) \in E : u, v \in W\}$ . A vertex v in a digraph  $\Gamma = (V, E)$  is a source if  $(u, v) \notin E$  for all  $u \in V$ , and is a sink if  $(v, u) \notin E$  for all  $u \in V$ . If  $\Gamma = (V, E)$  is a digraph, then a path is a sequence of distinct vertices  $(v_1, \ldots, v_m), m \geq 1$ , of  $\Gamma$  such that  $(v_i, v_{i+1}) \in E$  for all  $i \in \{1, \ldots, m-1\}$ . A path  $(v_1, \ldots, v_m)$  is said to be a path from  $v_1$  to  $v_m$  in  $\Gamma$ . A vertex v of a digraph is the set consisting of all vertices u such that there is a path from u to v and a path from v to u. Note that our definition of digraphs does not allow loops. A colouring of a digraph  $\Gamma = (V, E)$  is just a function  $c : V \longrightarrow \mathbb{N}$ .

Throughout this section we suppose that J is a regular  $\mathcal{J}$ -class of the finite semigroup  $S = \langle X \rangle$ . Recall that X' consists of those generators  $x \in X$  such that  $J_x > J$ .

The following digraphs are central to the results in this section. We define  $\Gamma_{\mathscr{L}}$  to be the quotient of the digraph with vertices  $J/\mathscr{L}$  and edges

$$\{(L_a, L_b) \in J/\mathscr{L} \times J/\mathscr{L} : L_a x = L_b, \ L_a \neq L_b, \text{ for some } x \in X'\}$$

by its strongly connected components. We define a colouring c of  $\Gamma_{\mathscr{L}}$  so that any vertex V containing an  $\mathscr{L}$ -class which has non-empty intersection with  $\langle X' \rangle$  has c(V) = 1 and every other vertex U has c(U) = 0. The digraph  $\Gamma_{\mathscr{R}}$  is defined dually. Note that  $\Gamma_{\mathscr{L}}$  and  $\Gamma_{\mathscr{R}}$  are acyclic digraphs.

We require two additional graphs. The first graph,  $\Delta$  is isomorphic to a quotient of the Graham-Houghton graph of the principal factor of J. We define  $\Delta$  to have vertex set equal to the disjoint union of the vertices of  $\Gamma_{\mathscr{L}}$  and  $\Gamma_{\mathscr{R}}$ , and edges  $\{U, V\}$  if the intersection of some  $\mathscr{L}$ -class in U with an  $\mathscr{R}$ -class in V, or vice versa, is a group  $\mathscr{H}$ -class. The second graph,  $\Theta$ , has the same vertex set as  $\Delta$ , and it has an edge incident to U and V if there is an element of  $\langle X' \rangle$  in the intersection of some  $\mathscr{L}$ -class in U with an  $\mathscr{R}$ -class in V, or vice versa. Both of the graphs  $\Delta$  and  $\Theta$  are bipartite.



Figure 2: The digraphs  $\Gamma_{\mathscr{L}}$  and  $\Gamma_{\mathscr{R}}$  from Example 3.7.



Figure 3: The graph  $\Delta$  from Example 3.7.

**Example 3.7.** The *full transformation monoid*  $\mathcal{T}_n$ , for  $n \in \mathbb{N}$ , is the semigroup consisting of all mappings from  $\{1, \ldots, n\}$  to itself (called *transformations*) with the operation of composition of functions. This semigroup plays an analogous role in semigroup theory as that played by the symmetric group in group theory.

Let W be the subsemigroup of  $\mathcal{T}_7$  generated by the transformations

$x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\frac{2}{3}$	$\frac{3}{4}$	4 1	$5\\5$	$\frac{6}{5}$	$\binom{7}{5}$ ,	$x_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$\frac{2}{4}$	$\frac{3}{1}$	$\frac{4}{3}$	$\frac{5}{5}$	$\frac{6}{5}$	$\binom{7}{5}$ ,
$x_3 = \begin{pmatrix} 1\\ 3 \end{pmatrix}$	$\frac{2}{3}$	$\frac{3}{1}$	$\frac{4}{2}$	$5\\5$	$\frac{6}{5}$	$\binom{7}{5}$ ,	$x_4 = \begin{pmatrix} 1\\ 4 \end{pmatrix}$	$\frac{2}{4}$	$\frac{3}{2}$	$\frac{4}{3}$	$5\\5$	$\frac{6}{5}$	$\binom{7}{5}$ ,
$x_5 = \begin{pmatrix} 1\\ 1 \end{pmatrix}$	$\frac{2}{1}$	$\frac{3}{3}$	$\frac{4}{4}$	$5\\5$	$\frac{6}{5}$	$\begin{pmatrix} 7\\ 6 \end{pmatrix}$ ,	$x_6 = \begin{pmatrix} 1\\ 1 \end{pmatrix}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{4}{4}$	$\frac{5}{5}$	$\frac{6}{6}$	$\binom{7}{7}$ ,
$x_7 = \begin{pmatrix} 1\\ 1 \end{pmatrix}$	$\frac{2}{4}$	$\frac{3}{3}$	$\frac{4}{4}$	$\frac{5}{5}$	6 6	$\begin{pmatrix} 7\\7 \end{pmatrix}$ ,	$x_8 = \begin{pmatrix} 1\\ 1 \end{pmatrix}$	$\frac{2}{2}$	$\frac{3}{4}$	$\frac{4}{4}$	$\frac{5}{5}$	$\frac{6}{6}$	$\binom{7}{7}$ .

Let J be the  $\mathscr{J}$ -class  $J_{x_1}$ . The following calculations were performed with the GAP [17] package SEMIGROUPS [29]. The  $\mathscr{J}$ -class J is regular, and contains the generators  $x_1, x_2, x_3$ , and  $x_4$ . The remaining generators are contained in  $\mathscr{J}$ -classes that are above J in the  $\mathscr{J}$ -class partial order, and so  $X' = \{x_5, x_6, x_7, x_8\}$ . The set of  $\mathscr{L}$ -classes of J is  $J/\mathscr{L} = \{L_{x_1}, L_{x_3}, L_{x_4}, L_{x_1x_6}\}$ , and the set of  $\mathscr{R}$ -classes of J is  $J/\mathscr{R} = \{R_{x_1}, R_{x_2}, R_{x_3}, R_{x_8x_2}, R_{x_6x_2}, R_{x_7x_3}\}$ .

There are 4 strongly connected components of  $\mathscr{L}$ -classes in J each consisting of a single  $\mathscr{L}$ -class. Hence the digraph  $\Gamma_{\mathscr{L}}$  has 4 vertices, one for each strongly connected component. There are also 4 strongly connected components of  $\mathscr{R}$ -classes: two of which have size 1, while the other two each have size 2. The digraphs  $\Gamma_{\mathscr{L}}$  and  $\Gamma_{\mathscr{R}}$  are depicted in Figure 2.

Since  $\Gamma_{\mathscr{L}}$  and  $\Gamma_{\mathscr{R}}$  each have four vertices, it follows that the bipartite graphs  $\Delta$  and  $\Theta$  each have eight vertices. These graphs are shown in Figures 3 and 4. The set of edges of  $\Delta$  was determined by computation of the idempotents in J. There are four elements in  $\langle X' \rangle \cap J$ :  $x_5^2 \in L_{x_1} \cap R_{x_3}, x_4x_1x_6 \in L_{x_1x_6} \cap R_{x_3}, x_7x_4x_1x_6 \in L_{x_1x_6} \cap R_{x_7x_3}$ , and  $x_7x_4x_1 \in L_{x_1} \cap R_{x_7x_3}$ . The  $\mathscr{L}$ - and  $\mathscr{R}$ -classes of these elements determine the edges in the graph  $\Theta$ , along with the colours of the vertices in  $\Gamma_{\mathscr{L}}$  and  $\Gamma_{\mathscr{R}}$ . The vertices of  $\Gamma_{\mathscr{L}}$  with colour 1 are  $\{L_{x_1}\}$  and  $\{L_{x_1x_6}\}$ , whilst the only vertex of  $\Gamma_{\mathscr{R}}$  with colour 1 is  $\{R_{x_3}, R_{x_7x_3}\}$ .

The following lemma is a straightforward consequence of Green's Lemma and Lemma 1.2.



Figure 4: The graph  $\Theta$  from Example 3.7.

- **Lemma 3.8.** (i) The vertex containing  $L_b \in J/\mathscr{L}$  in  $\Gamma_{\mathscr{L}}$  is reachable from the vertex containing  $L_a$  if and only if  $L_a = L_b$ , or there exists  $s \in L_a$  and  $x \in \langle S \setminus J \rangle$  such that  $sx \in L_b$ .
  - (ii) The vertex containing  $R_b \in J/\mathscr{R}$  in  $\Gamma_{\mathscr{R}}$  is reachable from the vertex containing  $R_a$  if and only if  $R_a = R_b$ , or there exists  $s \in R_a$  and  $x \in \langle S \setminus J \rangle$  such that  $xs \in R_b$ .

By Green's Lemma [Lemma 1.1] and Lemma 1.2, if T is a subsemigroup of S such that  $S \setminus T \subseteq J$ , then  $T \cap J$  contains an  $\mathscr{L}$ -class L if and only if  $T \cap J$  contains every  $\mathscr{L}$ -class in every vertex of  $\Gamma_{\mathscr{L}}$  that is reachable from the vertex of  $\Gamma_{\mathscr{L}}$ containing L. The analogous statement holds for  $\mathscr{R}$ -classes. Furthermore,  $T \cap J$  contains an  $\mathscr{H}$ -class H if and only if  $T \cap J$  contains every  $\mathscr{H}$ -class that is the intersection of an  $\mathscr{L}$ - and an  $\mathscr{R}$ -class that are contained in vertices that are reachable from the vertices containing the  $\mathscr{L}$ - and  $\mathscr{R}$ -classes containing H in  $\Gamma_{\mathscr{L}}$  and  $\Gamma_{\mathscr{R}}$ , respectively.

**Example 3.9.** Let W and J be the semigroup and the  $\mathscr{J}$ -class, respectively, from Example 3.7. Suppose that T is a subsemigroup of W such that  $W \setminus T \subseteq J$ . By analysing the digraph  $\Gamma_{\mathscr{L}}$ , depicted in Figure 2, we see that if T contains the  $\mathscr{L}$ -classes  $L_{x_4}$ , then T also contains the  $\mathscr{L}$ -classes  $L_{x_1}$  and  $L_{x_1x_6}$ , since the vertices containing these  $\mathscr{L}$ -classes are reachable in  $\Gamma_{\mathscr{L}}$  from the vertex containing  $L_{x_4}$ . Likewise, if T contains the  $\mathscr{R}$ -classes  $R_{x_8x_2}$ , then by considering the digraph  $\Gamma_{\mathscr{R}}$ , depicted in Figure 2, we see that T also contains the  $\mathscr{R}$ -classes  $R_{x_6x_2}, R_{x_3}$ , and  $R_{x_7x_3}$ . If we consider these digraphs together, then we see that T contains the  $\mathscr{R}$ -classes  $L_{x_1} \cap R_{x_3}$  if and only if T also contains the  $\mathscr{R}$ -classes  $L_{x_1} \cap R_{x_7x_3}$ ,  $L_{x_1x_6} \cap R_{x_3}$ , and  $L_{x_1x_6} \cap R_{x_7x_3}$ .

The digraphs  $\Gamma_{\mathscr{L}}$ ,  $\Gamma_{\mathscr{R}}$ , and the bipartite graphs  $\Delta$  and  $\Theta$  can be created using graph algorithms applied to the left and right Cayley graphs of the semigroup S with respect to its generating set X. The time complexity of finding  $\Gamma_{\mathscr{L}}$ ,  $\Gamma_{\mathscr{R}}$ ,  $\Delta$ , and  $\Theta$  using the left and right Cayley graphs of S is O(|S||X|). This is the same as the time complexity of determining the left and right Cayley graphs of S using, say, the Froidure-Pin Algorithm [15]. However in practice finding  $\Gamma_{\mathscr{L}}$ ,  $\Gamma_{\mathscr{R}}$ ,  $\Delta$ , and  $\Theta$  using the Cayley graphs of S will be much faster than determining the Cayley graphs themselves. For certain types of semigroups, such as semigroups represented by a generating set consisting of transformations of a finite set, the  $\mathscr{J}$ -class itself and the vertices and edges of  $\Gamma_{\mathscr{L}}$ ,  $\Gamma_{\mathscr{R}}$ , and  $\Delta$  can be determined without finding either the left or right Cayley graph of S; see [11] for further details.

In what follows, we will assume that  $\Gamma_{\mathscr{L}}, \Gamma_{\mathscr{R}}, \Delta$ , and  $\Theta$  and the colourings of  $\Gamma_{\mathscr{L}}$  and  $\Gamma_{\mathscr{R}}$  are known a priori.

Suppose that T is a subset of S such that  $S \setminus T \subseteq J$  and suppose that there exist proper subsets  $A \subsetneq J/\mathscr{L}$  and  $B \subsetneq J/\mathscr{R}$  such that  $T \cap J$  is the union of the Green's classes in A and in B. If T is a maximal subsemigroup of S, then type (S3) is when  $A \neq \emptyset$  and  $B \neq \emptyset$ ; type (S4) corresponds to  $A \neq \emptyset$  and  $B = \emptyset$ ; type (S5) is when  $A = \emptyset$  and  $B \neq \emptyset$ ; and type (S6) is when  $A = \emptyset$  and  $B = \emptyset$ .

Recall that the vertices of  $\Gamma_{\mathscr{L}}$  and  $\Gamma_{\mathscr{R}}$  are sets of  $\mathscr{L}$ - and  $\mathscr{R}$ -classes of J, respectively.

**Proposition 3.10.** Let S be a finite semigroup, and let T be a proper subset of S such that  $S \setminus T$  is contained in a regular  $\mathcal{J}$ -class J of S. Suppose that there exist proper subsets  $A \subsetneq J/\mathscr{L}$  and  $B \subsetneq J/\mathscr{R}$  such that  $T \cap J$  is the union of the Green's classes in A and in B. Then T is a subsemigroup of S if and only if the following hold:

- (i) A is a union of vertices of  $\Gamma_{\mathscr{L}}$ , and B is a union of vertices of  $\Gamma_{\mathscr{R}}$ ;
- (ii) if U and V are vertices of  $\Gamma_{\mathscr{L}}$  such that U is contained in A and there is an edge from U to V in  $\Gamma_{\mathscr{L}}$ , then V is contained in A;
- (iii) if U and V are vertices of  $\Gamma_{\mathscr{R}}$  such that U is contained in B and there is an edge from U to V in  $\Gamma_{\mathscr{R}}$ , then V is contained in B;

- (iv) if  $\{U, V\}$  is an edge in  $\Theta$ , then either U or V, or both, is contained in  $A \cup B$ ;
- (v) the vertices contained in  $A \cup B$  form an independent set of  $\Delta$ .

Proof.  $(\Rightarrow)$  As mentioned after Lemma 3.8, if  $T \cap J$  contains an  $\mathscr{L}$ -class L, then  $T \cap J$  contains every  $\mathscr{L}$ -class in every vertex of  $\Gamma_{\mathscr{L}}$  that is reachable from the vertex containing L. An analogous statement holds for  $\mathscr{R}$ -classes and  $\Gamma_{\mathscr{R}}$ . Parts (i), (ii), and (iii) follow immediately from these observations.

If  $\{U, V\}$  is an edge in  $\Theta$ , then by definition, there is an element  $x \in \langle X' \rangle$  in the intersection of some  $\mathscr{L}$ -class L in U and some  $\mathscr{R}$ -class R in V. By Lemma 3.1, since T is a subsemigroup,  $x \in T$  and so either  $L \in A$  and U is contained in A; or  $R \in B$  and V is contained in B. Therefore part (iv) holds.

If  $A = \emptyset$  or  $B = \emptyset$ , then part (v) holds immediately by the definition of  $\Delta$ , so suppose otherwise. Since T is a proper subsemigroup of S, it follows that it is contained in a maximal subsemigroup M of S. By the assumption that  $T \cap J$  is a union of non-empty sets of  $\mathscr{L}$ - and  $\mathscr{R}$ -classes, M must be of type described in Proposition 1.4(c)(i). Hence  $(M \cap J)\phi = (I \times G \times \Lambda) \setminus (I' \times G \times \Lambda')$  for some non-empty sets  $I' \subsetneq I$  and  $\Lambda' \subsetneq \Lambda$ , and  $(M \cap J)\phi \cup \{0\}$  is a maximal subsemigroup of  $(J)\phi \cup \{0\}$  of type (R5). We may assume without loss of generality that  $I = J/\mathscr{R}$  and  $\Lambda = J/\mathscr{L}$ , and that for an  $\mathscr{L}$ -class L of J and  $\mathscr{R}$ -class R of J,  $(L)\phi = I \times G \times \{L\}$  and  $(R)\phi = \{R\} \times G \times \Lambda$ .

Proposition 1.5 implies that there exist non-empty sets  $X \subsetneq I$  and  $Y \subsetneq \Lambda$  such that  $I' = I \setminus X$ ,  $\Lambda' = \Lambda \setminus Y$ , and  $X \cup Y$  is a maximal independent set in the Graham-Houghton graph of  $(J)\phi \cup \{0\}$ . Since M contains the union of the  $\mathscr{L}$ -classes in A, it follows that  $A \subseteq Y$ , and similarly,  $B \subseteq X$ .

If U and V are vertices of  $\Delta$  contained in  $A \subseteq Y$  and  $B \subseteq X$ , respectively, then, since  $\Delta$  is a quotient of the Graham-Houghton graph of  $(J)\phi \cup \{0\}$ , there is no edge in  $\Delta$  incident to U and V. Thus (v) holds.

 $(\Leftarrow)$  It suffices to show that T satisfies the conditions (i), (ii), and (iii) of Lemma 3.1.

To verify that condition (i) of Lemma 3.1 holds, let  $x \in \langle X' \rangle \cap J$ . Then there exists an edge in  $\Theta$  between the vertex U containing  $L_x$  and the vertex V containing  $R_x$  (by the definition of  $\Theta$ ). By assumption (iv) of this proposition, either  $U \subseteq A$  or  $V \subseteq B$  (or both). In the first case,  $x \in L_x \in A$  and  $A \subseteq T$  by assumption. The other case is similar.

For the second condition, suppose that  $x, y \in T \cap J$  and  $xy \in J$ . By Lemma 1.2,  $xy \in R_x \cap L_y$ . It follows by [24, Proposition 2.3.7] that the  $\mathscr{H}$ -class  $L_x \cap R_y$  contains an idempotent, and is therefore a group. By assumption (v), the vertices contained in  $A \cup B$  form an independent set of  $\Delta$ . Hence either  $L_x \notin A$  or  $R_y \notin B$ . If  $R_y \notin B$ , then, since  $T \cap J$ is a union of  $\mathscr{L}$ - and  $\mathscr{R}$ -classes of J and  $y \in T \cap J$ ,  $L_y \in A$ . By Lemma 1.2,  $L_{xy} = L_y \in A$ , and  $xy \in T$ . If  $L_x \notin A$ , then the proof is analogous.

To show that the final condition of Lemma 3.1 holds, let  $x \in T \cap J$  and  $y \in \langle X' \rangle$ . Note that since  $x \in T \cap J$ , either  $L_x \in A$  or  $R_x \in B$ . Suppose that  $xy \in J$ . If  $L_x \in A$ , then since the vertex containing  $L_{xy} = L_x y$  is reachable in  $\Gamma_{\mathscr{L}}$  from the vertex containing  $L_x$ , it follows that  $L_{xy} \in A$  and so  $xy \in T$ . Otherwise  $R_x \in B$ , and so  $R_{xy} = R_x \in B$  and  $xy \in T$ . The proof that  $yx \in J$  implies  $yx \in T$  is similar.

#### **3.3.1** Maximal subsemigroups from maximal rectangles: type (S3)

The following corollary, a slight adaptation of Proposition 3.10, gives necessary and sufficient conditions for a maximal subsemigroup of type (S3) to exist.

**Corollary 3.11.** Suppose that  $A \neq \emptyset$  and  $B \neq \emptyset$ . Then T is a maximal subsemigroup of S if and only if conditions (i)–(iv) of Proposition 3.10 hold, and the vertices contained in  $A \cup B$  form a maximal independent set of  $\Delta$ .

*Proof.* ( $\Rightarrow$ ) Since T is itself the only maximal subsemigroup of S containing T, then, as described in the proof of Proposition 3.10 (forward implication), the set  $A \cup B$  is a maximal independent set in the Graham-Houghton graph of  $(J)\phi \cup \{0\}$ . Since  $\Delta$  is a quotient of the Graham-Houghton graph, this implies the maximality of the independent set in  $\Delta$ .

( $\Leftarrow$ ) Let M be a maximal subsemigroup of S containing T. By the assumption that  $T \cap J$  is a union of non-empty sets of  $\mathscr{L}$ - and  $\mathscr{R}$ -classes, M must be of the type described in Proposition 1.4(c)(i). By Proposition 3.10, M corresponds to an independent set of  $\Delta$  that contains the vertices contained in  $A \cup B$ . Since this latter is a maximal independent set in  $\Delta$ , these sets are equal. Since the subsemigroups T and M are determined by their corresponding independent sets, it follows that T = M.

We describe, in the following steps, how to use Corollary 3.11 to compute the maximal subsemigroups of type (S3) corresponding to a given regular  $\mathcal{J}$ -class.

The first step is to determine the maximal independent sets in the bipartite graph  $\Delta$ , or equivalently to find the maximal cliques in the complement of  $\Delta$ . The Bron-Kerbosch Algorithm [2] (implemented in the GAP [17] package DIGRAPHS [1]) is a recursive algorithm for finding maximal cliques in a graph. Roughly speaking, this algorithm proceeds by attempting to extend a given clique by another vertex. By Proposition 3.10(ii) and (iii), in the search for maximal

cliques in the complement of  $\Delta$ , we are only interested in those cliques K with following property: if U is a vertex in Kand V is a vertex of  $\Delta$ , such that there is an edge in  $\Gamma_{\mathscr{R}}$  or  $\Gamma_{\mathscr{L}}$  from U to V, then V is in K also. As such, the search tree in the Bron-Kerbosch Algorithm can be pruned to exclude any branch starting at a clique containing a vertex U for which there is a vertex V in  $\Gamma_{\mathscr{R}}$ , or  $\Gamma_{\mathscr{L}}$ , which is reachable from U but does not extend the given clique, or where we have already discovered every maximal clique containing V.

We thereby produce a collection of maximal independents sets in  $\Delta$ , each of which gives rise to sets of vertices A and B that satisfy Proposition 3.10(i), (ii), and (iii). The second step is then to check which of these sets A and B satisfy part (iv) of Proposition 3.10, which is routine. Given sets A and B satisfying all the conditions in Proposition 3.10, the final step is to specify a generating set for the maximal subsemigroup T; see Proposition 3.13.

**Example 3.12.** Let W and J be the semigroup and the  $\mathcal{J}$ -class, respectively, from Example 3.7. Consider the graph  $\Delta$ , which is shown in Figure 3.

There are 7 maximal independent subsets of  $\Delta$  in total: two correspond to the vertices of  $\Gamma_{\mathscr{L}}$  and  $\Gamma_{\mathscr{R}}$ ; three further correspond to sets A and B that do not satisfy either of Proposition 3.10(ii) or (iii); the remaining two maximal independent subsets of  $\Delta$  correspond to non-empty sets A and B satisfying Proposition 3.10(i), (ii), and (iii).

The first such subset is  $\{\{L_{x_1}\}, \{L_{x_1x_6}\}, \{R_{x_1}\}\}$ , which corresponds to the sets  $A_1 = \{L_{x_1}, L_{x_1x_6}\} \subseteq J/\mathscr{L}$  and  $B_1 = \{R_{x_1}\} \subseteq J/\mathscr{R}$ ; the second such subset corresponds to the sets  $A_2 = \{L_{x_1x_6}\}$  and  $B_2 = \{R_{x_1}, R_{x_3}, R_{x_7x_3}\}$ .

There are two edges in the graph  $\Theta$ , as shown in Figure 4:  $\{\{L_{x_1}\}, \{R_{x_3}, R_{x_7x_3}\}\}$ , and  $\{\{L_{x_1x_6}\}, \{R_{x_3}, R_{x_7x_3}\}\}$ . For the first edge,  $\{L_{x_1}\} \subseteq A_1$  and  $\{R_{x_3}, R_{x_7x_3}\} \subseteq B_2$ ; for the second edge,  $\{L_{x_1x_6}\} \subseteq A_1$  and  $\{R_{x_3}, R_{x_7x_3}\} \subseteq B_2$ . In other words, the sets  $A_1$  and  $B_1$ , and the sets  $A_2$  and  $B_2$ , satisfy Proposition 3.10(iv).

Therefore there are two maximal subsemigroups of W of type (S3) arising from J: the set consisting of  $W \setminus J$  along with the union of the  $\mathscr{L}$ -classes in  $A_1$  and the union of the  $\mathscr{R}$ -classes in  $B_1$ , and the set consisting of  $W \setminus J$  along with the union of the  $\mathscr{L}$ -classes in  $A_2$  and the union of the  $\mathscr{R}$ -classes in  $B_2$ .

**Proposition 3.13.** Let T be a maximal subsemigroup of a finite semigroup S such that  $S \setminus T$  is contained in a regular  $\mathscr{J}$ -class of J of S. Suppose that there exist non-empty proper subsets  $A \subsetneq J/\mathscr{L}$  and  $B \subsetneq J/\mathscr{R}$  such that  $T \cap J$  is the union of the sets in A and in B. Then T is generated by any set consisting of:

- (i)  $X \setminus J$ ;
- (ii) a generating set for the ideal of S consisting of those  $\mathscr{J}$ -classes that lie strictly below J in the  $\mathscr{J}$ -class partial order;
- (iii) a generating set for a single group  $\mathscr{H}$ -class  $H_x$  in an  $\mathscr{L}$ -class belonging to A for some  $x \in T \cap J$ ;
- (iv) for every source vertex U in the induced subdigraph of  $\Gamma_{\mathscr{L}}$  on A, one element y such that  $y\mathscr{R}x$  and where  $L_y$  belongs to U;
- (v) for every source vertex V in the induced subdigraph of  $\Gamma_{\mathscr{R}}$  on the complement of B, one element z such that  $z\mathscr{L}x$ and where  $R_z$  belongs to V;
- (vi) a generating set for a single group  $\mathscr{H}$ -class  $H_{x'}$  in an  $\mathscr{R}$ -class belonging to B for some  $x' \in T \cap J$ ;
- (vii) for every source vertex U' in the induced subdigraph of  $\Gamma_{\mathscr{R}}$  on B, one element y' such that  $y'\mathscr{L}x'$  and where  $R_{y'}$  belongs to U';
- (viii) for every source vertex V' in the induced subdigraph of  $\Gamma_{\mathscr{L}}$  on the complement of A, one element z' such that  $z'\mathscr{R}x'$ and where  $L_{z'}$  belongs to V';
- (ix) for every source vertex U of  $\Gamma_{\mathscr{L}}$  contained in A, one element t such that  $R_t \in B$  and  $L_t \in U$ .

*Proof.* Let Y be a set of the kind described in the proposition. Clearly every element of Y is contained in T, and so  $\langle Y \rangle$  is a subsemigroup of T. Furthermore, the inclusion of the generators in (i) and (ii) implies that  $\langle Y \rangle$  contains  $S \setminus J$ . To show that  $T \leq \langle Y \rangle$ , let  $a \in T \cap J$ . By the definition of T, either  $L_a \in A$  or  $R_a \in B$ .

First suppose that  $L_a \in A$  and  $R_a \notin B$ . The vertex of  $\Gamma_{\mathscr{L}}$  containing  $L_a$  is reachable from some vertex U that is a source of the induced subdigraph of  $\Gamma_{\mathscr{L}}$  on A. Hence there exists an element  $y \in Y$  of type (iv) such that  $L_y \in U$ and  $y\mathscr{R}x$ , where x is an element in the group  $\mathscr{H}$ -class from part (iii). By Lemma 3.8, either  $L_a = L_y$ , or there exists  $r' \in S \setminus J \subseteq \langle Y \rangle$  such that  $yr' \in L_a$  and, by Lemma 1.2,  $yr' \in R_y = R_x$ . In either case, there exists an element  $r \in \langle Y \rangle$ such that  $r\mathscr{R}x$  and  $r\mathscr{L}a$ . Likewise, by using a generator of type (v), there exists an element  $s \in \langle Y \rangle$  such that  $s\mathscr{L}x$  and  $s\mathscr{R}a$ . Since  $H_x$  is a group, it follows by Green's Lemma that  $a \in H_a = sH_xr \subseteq \langle Y \rangle$ . If instead  $L_a \notin A$  and  $R_a \in B$ , then the proof that  $a \in \langle Y \rangle$  is similar. For the final case, suppose that  $L_a \in A$  and  $R_a \in B$ . The vertex of  $\Gamma_{\mathscr{L}}$  containing  $L_a$  is reachable from some vertex U that is a source of  $\Gamma_{\mathscr{L}}$ . If  $U \subseteq A$  then by (ix) there exists an element  $t \in Y$  such that  $R_t \in B$  and  $L_t \in U$ . If  $U \not\subseteq A$  then, by the previous paragraph, every element t' such that  $R_{t'} \in B$  and  $L_{t'} \notin A$  is contained in  $\langle Y \rangle$ . In either case, there exists an element  $t \in \langle Y \rangle$  such that  $R_t \in B$  and  $L_t \in U$ . Therefore, since  $L_a$  is reachable from  $L_t$  in  $\Gamma_{\mathscr{L}}$ , there exists an element  $r \in \langle Y \rangle$  such that  $r \in L_a \cap R_t$ . By the regularity of J, there exists an idempotent e such that  $e \mathscr{R}r$ , and since  $A \cup B$  corresponds to an independent set in  $\Delta$ ,  $L_e \notin A$ . By the arguments of the above paragraph, the  $\mathscr{H}$ -classes  $H_e$  and  $H_s = L_e \cap R_a$  are both contained in  $\langle Y \rangle$ . Since  $H_e$  is a group, we have  $a \in H_a = sH_e r \subseteq \langle Y \rangle$ .

Note that if there exists an idempotent  $e \in S$  such that  $L_e \notin A$  and  $R_e \notin B$  (such an idempotent exists if and only if the complement of  $A \cup B$  corresponds to a non-independent set in  $\Delta$ ), then  $a \in H_a = (L_a \cap R_e)(R_a \cap L_e)$ . Hence if such an idempotent e exists, the generators in (ix) are redundant.

### **3.3.2** Maximal subsemigroups from removing either $\mathscr{R}$ -classes or $\mathscr{L}$ -classes: types (S4) and (S5)

If the set of  $\mathscr{R}$ -classes B is empty, then the criteria for T to be a subsemigroup of S can be simplified from those in Proposition 3.10. In particular, the second part of condition (i) and condition (iii) are vacuously satisfied. Condition (v) of Proposition 3.10 is also automatically satisfied. Furthermore, the reference to the set B in condition (iv) can be removed, to obtain the following immediate corollary.

**Corollary 3.14.** Suppose that  $A \neq \emptyset$  and  $B = \emptyset$ . Then T is a (proper) subsemigroup of S if and only if conditions (i) and (ii) of Proposition 3.10 hold, and every vertex V of  $\Gamma_{\mathscr{L}}$  with c(V) = 1 is contained in A.

Thus we may give necessary and sufficient conditions for a maximal subsemigroup of type (S4) to exist.

**Corollary 3.15.** Suppose that  $A \neq \emptyset$  and  $B = \emptyset$ . Then T is a maximal subsemigroup of S if and only if the complement of A is a source of  $\Gamma_{\mathscr{L}}$  with colour 0, and there is no maximal subsemigroup of S of type (S3) whose subset of  $J/\mathscr{L}$  is equal to A.

Proof.  $(\Rightarrow)$  Since  $\Gamma_{\mathscr{L}}$  is acyclic, it follows that any induced subdigraph of  $\Gamma_{\mathscr{L}}$  is acyclic. Therefore there is a sink in the induced subdigraph of  $\Gamma_{\mathscr{L}}$  on the vertices not contained in A. Note that this vertex is not necessarily a sink in  $\Gamma_{\mathscr{L}}$  itself. Let A' be the subset of  $J/\mathscr{L}$  formed from the union of A with the set of  $\mathscr{L}$ -classes contained in this sink, and define T' to be the subset of S such that  $S \setminus T' \subseteq J$  and  $T' \cap J$  is the union of the  $\mathscr{L}$ -classes in A'. Then either  $A' = J/\mathscr{L}$ , or, by Corollary 3.14, T' is a proper subsemigroup of S that properly contains T. Since T is maximal, it follows that  $A' = J/\mathscr{L}$ , and so the complement of A forms a single vertex of  $\Gamma_{\mathscr{L}}$ . This vertex is a source of  $\Gamma_{\mathscr{L}}$  by condition (ii) of Proposition 3.10, and has colour 0 by Corollary 3.14.

Since T is a maximal subsemigroup of S, it is not contained in another maximal subsemigroup of S. Since T is not a maximal subsemigroup of type (S3), T is not contained in such a maximal subsemigroup.

 $(\Leftarrow)$  Let M be a maximal subsemigroup of S containing T. Since T contains a union of  $\mathscr{L}$ -classes of J, M must be of type (S3) or type (S4). In either case, it follows that M contains the  $\mathscr{L}$ -classes in A. However, A lacks only one vertex of  $\Gamma_{\mathscr{L}}$ , and since a maximal subsemigroup is a proper subsemigroup, it follows that M contains no additional  $\mathscr{L}$ -classes. Such a maximal subsemigroup of type (S3) does not exist by assumption, and so M is of type (S4). Therefore T = M.  $\Box$ 

Analogues of Corollaries 3.14 and 3.15 hold in the case that  $A = \emptyset$  and  $B \neq \emptyset$ , giving necessary and sufficient conditions for a maximal subsemigroup of type (S5) to exist.

We describe how to use Corollary 3.15 to compute the maximal subsemigroups of type (S4) corresponding to a regular  $\mathscr{J}$ -class. The first step is to compute the maximal subsemigroups of type (S3), as described above, and in doing so, to record the sets A and B that occur for each maximal subsemigroup. The second step is to search for the sources of the digraph  $\Gamma_{\mathscr{L}}$  with colour 0. For each source, we check whether its complement A occurs as the set of  $\mathscr{L}$ -classes of some maximal subsemigroup of type (S3). Given a set A satisfying the conditions in Corollary 3.15, the final step is to specify a generating set for the maximal subsemigroup T; this is described in Proposition 3.17.

**Example 3.16.** Let W and J be the semigroup and the  $\mathscr{J}$ -class, respectively, from Example 3.7. The digraph  $\Gamma_{\mathscr{L}}$ , depicted in Figure 2, contains two sources of colour 0,  $\{L_{x_3}\}$  and  $\{L_{x_4}\}$ . The complements of these vertices in  $J/\mathscr{L}$  are  $A_1 = \{L_{x_1}, L_{x_4}, L_{x_{1x_6}}\}$  and  $A_2 = \{L_{x_1}, L_{x_3}, L_{x_{1x_6}}\}$ , respectively. In Example 3.12, we found no maximal subsemigroup of W of type (S3) whose set of  $\mathscr{L}$ -classes was equal to  $A_1$  or  $A_2$ . Therefore there are two maximal subsemigroups of W of type (S4) arising from J: the set consisting of  $W \setminus J$  along with the union of the  $\mathscr{L}$ -classes in  $A_1$ , and the set consisting of  $W \setminus J$  along with the union of the  $\mathscr{L}$ -classes in  $A_2$ .

In  $\Gamma_{\mathscr{R}}$ , the sources are  $\{R_{x_1}\}$  and  $\{R_{x_2}\}$ , and they have colour 0. In Example 3.12, we found no maximal subsemigroup of W of type (S3) whose set of  $\mathscr{R}$ -classes was equal to the complement of either of these sources. Hence  $W \setminus R_{x_1}$  and  $W \setminus R_{x_2}$  are the only maximal subsemigroups of W of type (S5) arising from J. The following proposition describes generating sets for maximal subsemigroups of type (S4), the proof is similar to (but less complicated than) the proof of Proposition 3.13, and is omitted.

**Proposition 3.17.** Let T be a subsemigroup of a finite semigroup S such that  $S \setminus T$  is contained in a regular  $\mathcal{J}$ -class J of S. Suppose that there exists a non-empty proper subset  $A \subsetneq J/\mathscr{L}$  such that  $T \cap J$  is the union of the  $\mathscr{L}$ -classes in A. Then T is generated by any set consisting of:

- (i)  $X \setminus J$ ;
- (ii) a generating set for the ideal of S consisting of those  $\mathcal{J}$ -classes that lie strictly below J in the  $\mathcal{J}$ -class partial order;
- (iii) a generating set for a single group  $\mathscr{H}$ -class  $H_x$  in an  $\mathscr{L}$ -class belonging to A for some  $x \in T \cap J$ ;
- (iv) for every source vertex U in the induced subdigraph of  $\Gamma_{\mathscr{L}}$  on A, one element y such that  $y\mathscr{R}x$  and where  $L_y$  belongs to U;
- (v) for every source vertex V in  $\Gamma_{\mathscr{R}}$ , one element z such that  $z\mathscr{L}x$  and where  $R_z$  belongs to V.

Generating sets for maximal subsemigroups of type (S5) are obtained analogously.

### 3.3.3 Maximal subsemigroups from removing the $\mathcal{J}$ -class: type (S6)

If the set of  $\mathscr{L}$ -classes A and the set of  $\mathscr{R}$ -classes B are both empty, then  $T = S \setminus J$ , and the criteria for T to be a subsemigroup of S can be further simplified from those in Proposition 3.10. In particular, conditions (i), (ii), (iii), and (v) of Proposition 3.10 are vacuously satisfied, leaving only condition (iv), which permits the following corollary.

**Corollary 3.18.** The subset  $S \setminus J$  of S is a subsemigroup if and only if  $\Theta$  has no edges.

To compute the maximal subsemigroups of type (S6) corresponding to a regular  $\mathscr{J}$ -class, it is necessary to have first computed those maximal subsemigroups of types (S2), (S3), (S4), and (S5). If no such maximal subsemigroups exist, then we check the number of edges of the graph  $\Theta$ ; if  $\Theta$  has no edges, then  $S \setminus J$  is a maximal subsemigroup of S. In this case, a generating set for  $S \setminus J$  is given by the union of  $X \setminus J$  with a generating set for the ideal of S consisting of those  $\mathscr{J}$ -classes that lie strictly below J in the  $\mathscr{J}$ -class partial order.

### 3.4 The algorithm

In this section we describe the overall algorithm for computing the maximal subsemigroups of a finite semigroup S. This is achieved by putting together the procedures described in the preceding sections.

The algorithm considers, in turn, each  $\mathscr{J}$ -class that contains a generator. It is clear that it is only necessary to consider those  $\mathscr{J}$ -classes containing generators. Maximal and non-maximal  $\mathscr{J}$ -classes are then treated separately. Every maximal  $\mathscr{J}$ -class contains at least one generator. A non-regular maximal  $\mathscr{J}$ -class is necessarily trivial. For any maximal  $\mathscr{J}$ -class,  $S \setminus J$  is a subsemigroup of S. Therefore for a trivial maximal  $\mathscr{J}$ -class, the only maximal subsemigroup arising from J is  $S \setminus J$ . This leaves the non-trivial maximal  $\mathscr{J}$ -classes to consider. Such a  $\mathscr{J}$ -class is necessarily regular. Hence a maximal subsemigroup of S arising from a non-trivial maximal  $\mathscr{J}$ -class J is of the form  $(S \setminus J) \cup U$ , and occurs precisely when  $U \cup \{0\}$  is a maximal subsemigroup of  $J^*$ . Thus it suffices to compute the maximal subsemigroups of the Rees 0-matrix semigroup isomorphic to  $J^*$  of types (R3)–(R6), as described in Section 2. For the non-maximal  $\mathscr{J}$ -classes we proceed as described earlier in this section.

The algorithm is described in pseudocode in Algorithm 2. The algorithms described in this paper are fully implemented in the SEMIGROUPS package [29] for GAP [17], and the underlying algorithms for graphs and digraphs are implemented in the DIGRAPHS package [1] for GAP [17].

# 4 Performance analysis

In this section we provide some analysis of the performance of the algorithms described in this paper. These algorithms are implemented in the SEMIGROUPS package [29] for GAP [17] and the computations in this section were run on a 2.66 GHz Intel Core i7 processor with 8GB of RAM.

Given a semigroup S represented by a set of generators X, such as transformations, we compare the time taken to compute the Green's structure of S with that taken to find the maximal subsemigroups. Additionally we include the amount of time spent during the computation of the maximal subsemigroups on finding maximal cliques or maximal subgroups of group  $\mathcal{H}$ -classes, when these times are not negligible.

The semigroups considered are given below.

Algorit	<b>hm 2</b> Maximal subsemigroups of a finite semigroup	
Input:	$S = \langle X \rangle$ , a finite semigroup with generating set X.	
Output	t: the non-empty maximal subsemigroups $\mathfrak{M}$ of $S$ .	
1: M ::	$= \varnothing$	
2: <b>for</b>	$J_x \in \left\{J \in S   \mathscr{J} : J \cap X \neq \varnothing \right\}  \mathbf{do}$	
3: <b>i</b>	<b>f</b> $J_x$ is a maximal $\mathscr{J}$ -class of $S$ <b>then</b>	
4:	$\mathbf{if}  \left  J_x \right  = 1  \mathbf{then}$	$[J_x \text{ is non-regular, or is a trivial subgroup}]$
5:	$\mathfrak{M} \leftarrow \mathfrak{M} \cup \{S \setminus J_x\}$	
6:	else	$[J_x is necessarily regular]$
7:	compute $\phi$ , an isomorphism from $J_x^*$ to a normalized Re	ees 0-matrix semigroup
8:	compute $\mathfrak{J}$ , the maximal subsemigroups of $(J_x)\phi \cup \{0\}$ of	of types $(R3)$ , $(R4)$ , $(R5)$ , and $(R6)$ [Section 2]
9:	for $M \in \mathfrak{J}$ do	
10:	$\mathfrak{M} \leftarrow \mathfrak{M} \cup \left\{ (S \setminus J_x) \cup (M \setminus \{0\}) \phi^{-1} \right\}$	
11: <b>e</b>	else	$[J_x \text{ is a non-maximal } \mathcal{J}\text{ -class}]$
12:	$X' := \left\{ y \in X : J_y > J_x \right\}$	
13:	if $J_x$ is non-regular and $x \notin \langle X' \rangle$ then	
14:	$\mathfrak{M} \leftarrow \mathfrak{M} \cup \{S \setminus J_x\}$	[Type $(S1)$ , Proposition 3.3]
15:	else if $J_x$ is regular and $J_x \cap X \not\subseteq \langle X' \rangle$ then	
16:	compute the digraphs $\Gamma_{\mathscr{L}}$ and $\Gamma_{\mathscr{R}}$ and the graphs $\Delta$ and	dΘ
17:	compute $\phi$ , an isomorphism from $J_x^*$ to a normalized Re	ees 0-matrix semigroup
18:	compute the set $E$ consisting of one idempotent in each	$\mathscr{L}$ -class of $J_x$
19:	compute $\mathfrak{J}$ , the maximal subsemigroups of $(J_x)\phi \cup \{0\}$ of	of type (R6) which contain $(EX')\phi$ [Algorithm 1]
20:	for $M \in \mathfrak{J}$ do	
21:	$\mathfrak{M} \leftarrow \mathfrak{M} \cup \left\{ (S \setminus J_x) \cup (M \setminus \{0\})\phi^{-1} \right\}$	[Type (S2), Corollary 3.6]
22:	add maximal subsemigroups of type (S3) to $\mathfrak{M}$	[Section $3.3.1$ ]
23:	add maximal subsemigroups of types $(S4)$ and $(S5)$ to $\mathfrak{A}$	$\mathfrak{N}$ [Section 3.3.2]
24:	if no maximal subsemigroups have been found to arise f	from $J_x$ then
25:	$\mathfrak{M} \leftarrow \mathfrak{M} \cup \{S \setminus J_x\}$	[Type (S6), Section 3.3.3]
26: reti	ırn M	

- The full transformation monoids  $\mathcal{T}_n$  consisting of all transformations of  $\{1, \ldots, n\}$  when  $n = 2, \ldots, 11$ . If n > 2, then  $\mathcal{T}_n$  is generated by 3 transformations, and this is the minimal number. There are  $n^n$  transformations of  $\{1, \ldots, n\}$  and the number of  $\mathcal{J}$ -classes in  $\mathcal{T}_n$  is n. See Example 3.7 for further details.
- The inverse monoids  $\mathcal{PORI}_n$  consisting of the bijective functions between subsets of  $\{1, \ldots, n\}$  which preserve or reverse the cyclic orientation of  $\{1, \ldots, n\}$  for  $n = 11, \ldots, 20$ . As an inverse monoid  $\mathcal{PORI}_n$  is minimally generated by 3 elements when n > 2. If n > 0, then

$$|\mathcal{PORI}_n| = 1 + n \binom{2n}{n} - \frac{n^2(n^2 - 2n + 3)}{2}$$

and the number of  $\mathcal{J}$ -classes in  $\mathcal{PORI}_n$  is n+1.

• The Jones monoids  $\mathcal{J}_n$ , introduced in [27], and also known as the Temperley-Lieb monoid, for n = 6, ..., 20. The definition of  $\mathcal{J}_n$  is too long to give here. A minimal generating set for  $\mathcal{J}_n$  has n-1 elements when n > 1. If n > 0, then

$$|\mathcal{J}_n| = \frac{1}{n+1} \binom{2n}{n}$$

and the number of  $\mathscr{J}$ -classes in  $\mathcal{J}_n$  is  $\left\lceil \frac{n}{2} \right\rceil$ .

- 100 subsemigroups of  $\mathcal{T}_9$  generated by 9 transformations chosen uniformly at random; see Table 4 for further information.
- 100 regular Rees 0-matrix semigroups  $R = \mathscr{M}^0[I, G, \Lambda; P]$  where  $|I| = |\Lambda| = 1, \ldots, 20, G$  was a permutation group on 10 points chosen uniformly at random from the representatives of conjugacy classes of all subgroups of the symmetric group on 10 points, and the entries of the matrix P were chosen randomly such that the Graham-Houghton graph of R had between 1 and  $|I| = |\Lambda|$  connected components. The entries of P were not chosen uniformly, nor do we claim that the resulting Rees 0-matrix semigroups represent a uniform sample of such semigroups.

We refer the interested reader to [13] and the references therein for further details about the monoids  $\mathcal{T}_n$ ,  $\mathcal{PORI}_n$ , and  $\mathcal{J}_n$  and for a characterisation of their maximal subsemigroups; see also Tables 1, 2, and 3 for the numbers of maximal subsemigroups of these monoids. The results obtained in [13] relied heavily on computational experiments performed using the algorithms described in this paper, and their implementations in GAP [17]. The monoids  $\mathcal{T}_n$ ,  $\mathcal{PORI}_n$ , and  $\mathcal{J}_n$  were chosen because they are well-studied in the literature, because they have different representations in the SEMIGROUPS package for GAP, and because they exhibit different behaviour when computing the maximal subsemigroups. In Figures 5, 6, and 7, the time taken to compute the partial order of the  $\mathscr{J}$ -classes of a given semigroup is compared to that taken to find the maximal subsemigroups. The partial order of the  $\mathscr{J}$ -classes of a semigroup was obtained using the method described in [11, Algorithm 14] and implemented in the SEMIGROUPS package [29] for GAP [17]. The range of values of n, in each case, includes the largest value where the Green's structure could be computed within the limitations of the hardware.<sup>2</sup>

We opted to include some data relating to random semigroups to highlight possible typical behaviour of the maximal subsemigroups algorithms. What constitutes a reasonable notion of a "random semigroup" is debatable, although we believe that the notions used here are somewhat meaningful. Figure 8 concerns the 100 subsemigroups of  $\mathcal{T}_9$  generated by 9 transformations chosen uniformly at random. A point on the x-axis corresponds to a single such semigroup. The points on the x-axes are sorted in increasing order according to the ratio of the time taken to compute the maximal subsemigroups and the time taken to compute the partial order of the  $\mathscr{J}$ -classes. The particular choices of 9 transformations on 9 points were made because they approach the limit of what is practical to compute. Figure 9 concerns the 2000 random Rees 0-matrix semigroups. The Green's structure of a regular Rees 0-matrix semigroup can be determined immediately from its definition, and so the time to determine this is not used for comparison in Figure 9. For each dimension considered, the mean of 100 examples is shown in Figure 9.

While there are some instances in the data presented in this section where computing the maximal subsemigroups is several orders of magnitude slower than computing the Green's structure, for the majority the times taken are roughly comparable. For  $\mathcal{T}_n$  and  $\mathcal{PORI}_n$ , the time taken to compute the maximal subsemigroups is not dominated by either the time taken to compute maximal cliques or the time to compute maximal subgroups, but rather by constructing and processing the graphs  $\Gamma_{\mathscr{L}}$ ,  $\Gamma_{\mathscr{R}}$ ,  $\Delta$ , and  $\Theta$  from Section 3. For these monoids, the graphs  $\Delta$  have 2 vertices, and as such there are no maximal cliques from which a maximal subsemigroup could arise. On the other hand, for the Jones monoids  $\mathcal{J}_n$ , the majority of the time in the computation of the maximal subsemigroups is spent finding maximal cliques.

For Rees 0-matrix semigroups, it appears from Figure 9 that the time taken to compute maximal subsemigroups approaches the time taken to compute maximal cliques as the dimension increases.

<sup>&</sup>lt;sup>2</sup>Note that  $|\mathcal{T}_{11}| = 285\ 311\ 670\ 611,\ |\mathcal{PORI}_{20}| = 2\ 756\ 930\ 503\ 801,\ \text{and}\ |\mathcal{J}_{20}| = 6\ 564\ 120\ 420.$ 

Degree $n$	2	3	4	5	6	7	8	9	10	11
Maximal subsemigroups	2	5	9	23	54	185	354	1377	3978	363905

Table 1: The number of maximal subsemigroups of the full transformation monoid  $\mathcal{T}_n$  for some small values of  $n \in \mathbb{N}$ .

Degree $n$	10	11	12	13	14	15	16	17	18	19	20
Maximal subsemigroups	9	14	7	16	11	11	5	19	7	22	9

Table 2: The number of maximal subsemigroups of  $\mathcal{PORI}_n$  for some small values of  $n \in \mathbb{N}$ .

Degree $n$	10	11	12	13	14	15	16	17	18	19	20
Maximal subsemigroups	85	129	199	311	491	781	1249	2005	3227	5203	8399

Table 3: The number of maximal subsemigroups of the Jones monoids  $\mathcal{J}_n$  for some small values of  $n \in \mathbb{N}$ .

	Min	Max	Mean	Median	Standard deviation
Size	125333	85014449	5657333	2484319	9865603
Number of maximal subsemigroups	9	640	25	13	65
Number of $\mathcal{J}$ -classes	8	89858	5336	2550	10187
Time for maximal subsemigroups (milliseconds)	8	272132	15998	5003	37944

Table 4: Information about the 100 subsemigroups of  $\mathcal{T}_9$  generated by 9 transformations chosen uniformly at random.



Figure 5: Comparison of the time taken to compute the partial order of  $\mathscr{J}$ -classes of the full transformation monoids  $\mathcal{T}_n$ ,  $n = 2, \ldots, 11$ , with the time taken to compute their maximal subsemigroups, and the time spent finding maximal subgroups of group  $\mathscr{H}$ -classes.



Figure 6: Comparison of the time taken to compute the partial order of  $\mathscr{J}$ -classes of the monoids  $\mathcal{PORI}_n$ ,  $n = 10, \ldots, 20$ , with the time taken to compute their maximal subsemigroups, and the time spent finding maximal subgroups of group  $\mathscr{H}$ -classes.



Figure 7: Comparison of the time taken to compute the partial order of  $\mathcal{J}$ -classes of the Jones monoids  $\mathcal{J}_n$ ,  $n = 10, \ldots, 20$ , with the time taken to compute their maximal subsemigroups, and the time spent finding maximal cliques.



Figure 8: Comparison of the time taken to compute the partial order of  $\mathcal{J}$ -classes of 100 random 9-generated subsemigroups of  $\mathcal{T}_9$  with the time taken to compute their maximal subsemigroups.



Figure 9: Comparison of the mean time taken to compute the maximal subsemigroups of 100 random regular Rees 0matrix semigroups of a given dimension with the mean time taken to compute the maximal cliques of the duals of their Graham-Houghton graphs.

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