PROCEEDINGS A

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Research

Article submitted to journal

Subject Areas:

xxxxx, xxxxx, xxxx

Keywords:

xxxx, xxxx, xxxx

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Rigidity for sticky disks

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We study the combinatorial and rigidity properties of disk packings with generic radii. We show that a packing of n disks in the plane with generic radii cannot have more than $2n-3$ pairs of disks in contact.

The allowed motions of a packing preserve the disjointness of the disk interiors and tangency between pairs already in contact (modeling a collection of sticky disks). We show that if a packing has generic radii, then the allowed motions are all rigid body motions if and only if the packing has exactly $2n-3$ contacts. Our approach is to study the space of packings with a fixed contact graph. The main technical step is to show that this space is a smooth manifold, which is done via a connection to the Cauchy-Alexandrov stress lemma.

Our methods also apply to jamming problems, in which contacts are allowed to break during a motion. We give a simple proof of a finite variant of a recent result of Connelly, et al. [1] on the number of contacts in a jammed packing of disks with generic radii.

1. Introduction

In this paper we study existence and rigidity properties of packings of sticky disks with fixed generic radii.

A (planar) *packing* P of $n \geq 2$ disks is a placement of the disks, with centers $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_n)$ and fixed radii $\mathbf{r} = (r_1, \ldots, r_n)$, in the Euclidean plane so that their interiors are disjoint. The *contact graph* of a packing is the graph that has one vertex for each disk and an edge between pairs of disks that are mutually tangent. Figure 1 shows an example of a packing.

Sticky disk and framework rigidity A motion of a packing, called a *flex*, is one that preserves the radii, the disjointness of the disk interiors, as well as tangency between pairs of disks with a corresponding edge in the

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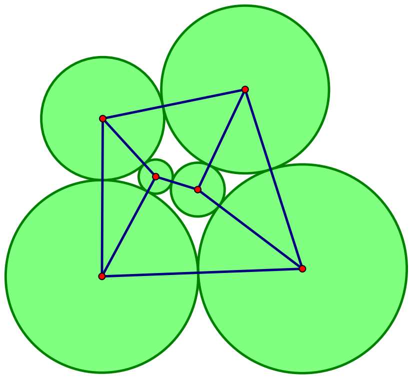


Figure 1. A rigid sticky disk packing and its underlying bar framework. Disks are the green circles with red center points / joints, the bars are blue segments.

contact graph. (The last condition makes the disks "sticky".) A packing is *rigid* when all its flexes arise from rigid body motions; otherwise, it is *flexible*.

Since any packing has a neighborhood on which the contact graph remains fixed along any flex (one must move at least some distance before a new contact can appear), the constraints on the packing are locally equivalent to preserving the pairwise distances between the circle centers. Forgetting that the disks have radii and must remain disjoint and keeping only the distance contraints between the centers, we get exactly "framework rigidity", where we have a *configuration* $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_n)$ of n points in a d-dimensional Euclidean space and a graph G; the pair $(G, \mathbf{p})$ is called a *(bar-and-joint) framework*. The allowed flexes of the points are those that preserve the distances between the pairs indexed by the edges of G. As with packings, a framework is *rigid* when all its flexes arise from rigid body motions and otherwise *flexible*. Given a packing $(\mathbf{p}, \mathbf{r})$ with contact graph G, we call the framework $(G, \mathbf{p})$ its *underlying framework*.

Motivations Rigidity of sticky disk packings, and the relationship to frameworks, has several related, but formally different motivations. The first comes from the study of colloidal matter [2], which is made of micrometer-sized particles that interact with "short range potentials" [3,4] that, in a limit, behave like sticky disks [5] (spheres in 3d).

Secondly, there is a connection to jammed packings. Here one has a "container", which can shrink uniformly and push the disks together. In this setting, we do not require any contact graph to be preserved. A fundamental problem is to understand the geometry and combinatorics of maximally dense packings (where the container can shrink no more—full definitions will be given in Section 4). In a series of papers, Connelly and co-workers [6–8] relate configurations that are locally maximally dense to the rigidity of a related tensegrity (see, e.g., [9]) over the contact graph. Notably, the recent work in [1] proves results about the number of contacts appearing in such locally maximally dense packings under appropriate genericity assumptions.

Another motivation comes from geometric constraint solving [10], where combinatorial methods are also applied to structures made of disks and spheres.

Laman's Theorem Given the relationship between disk packings and associated frameworks, it is very tempting to go further and apply the methods of *combinatorial* rigidity (see, e.g., [11]) theory to infer geometric or physical properties from the contact graph alone. Several recent works in the soft matter literature [12–14] use such an approach.

The combinatorial approach is attractive because we have a very good understanding of framework rigidity in dimension 2, provided that $\mathbf{p}$ is not very degenerate.

Definition 1.1. A vector in $\mathbb{R}^N$ is called *generic* if its coordinates are algebraically independent over $\mathbb{Q}$. A point configuration $\mathbf{p}$ of n points in $\mathbb{R}^d$ is called *generic* if the coordinates of its points (a vector in $\mathbb{R}^{dn}$) are algebraically independent over $\mathbb{Q}$. ■

Almost all configurations are generic, so generic configurations capture the general case. Moreover, all the results in the present paper remain true if we simply avoid the zero set of a specific but unspecified set of polynomials with rational coefficients (i.e., the results hold on a Zariski open subset, defined over $\mathbb{Q}$, of the appropriate configuration space).

The following combinatorial notion captures the 2-dimensional case of a counting heuristic due to Maxwell [15].

Definition 1.2. Let $G = (V, E)$ be a graph with n vertices and m edges. A graph G is *Laman-sparse* if, for every subgraph on n' vertices and m' edges, $m' \le 2n' - 3$. If, in addition, $m = 2n - 3$, G is called a *Laman graph*. ■

The following theorem, which combines results of Asimow and Roth [16] and Laman [17] characterizes rigidity and flexibility of generic frameworks in the plane.

Theorem 1.3. *Let G be a graph with n vertices and $\mathbf{p}$ a generic configuration of n points in dimension 2. Then the framework $(G, \mathbf{p})$ is rigid if G contains a Laman graph as a spanning subgraph and otherwise flexible.*

Theorem 1.3 makes generic rigidity and flexibility a combinatorial property that can be analyzed very efficiently using graph-theoretic algorithms [18,19], even on large inputs. Of course, one needs to make the modeling assumption that the process generating $\mathbf{p}$ is generic. If not, then the use of combinatorial methods is not formally justified.

In fact, the genericity[1] hypothesis is essential: one may find (necessarily non-generic) $\mathbf{p}$ for which a framework $(G, \mathbf{p})$ is flexible and G is a Laman graph (see, e.g., [20]); conversely, there are non-generic frameworks that are rigid but have too few edges to be rigid with generic $\mathbf{p}$ (see, e.g., [21])[2].

Packing Laman question The starting point for this paper is the observation that configurations arising from packings with disks in contact are not generic. Thus the theory of generic bar frameworks does not apply directly to packings.

The most general situation for packings is the (extremely) "polydisperse" case in which the radii r_i are algebraically independent over $\mathbb{Q}$. Even with generic radii, the configuration of disk centers could be very degenerate relative to picking $\mathbf{p}$ freely. For an edge ij in G, the contact constraint is

$$\|\mathbf{p}_i - \mathbf{p}_j\| = r_i + r_j \tag{1.1}$$

which means that there are only n degrees of freedom (the radii) to pick the edge lengths, instead of the $2n - 3$ available to a general $(G, \mathbf{p})$ with G a Laman graph. Thus the underlying $\mathbf{p}$ of a disk packing with a Laman contact graph is certainly not generic, and so Theorem 1.3 does not apply. We need another formal justification for analyzing packings combinatorially.

Packing non-existence question Beyond rigidity, there is the general question of what graphs can appear as the contact graph of a packing with generic radii. Once we fix $\mathbf{r}$, there are $2n - 3$ non-trivial degrees of freedom in packing $\mathbf{p}$. If a graph G has n vertices, each of its m edges contributes a constraint of the type (1.1). Thus, when $m > 2n - 3$, we expect, heuristically that either no $\mathbf{p}$ exists or $\mathbf{r}$ satisfies some additional polynomial relation.

[1]Or at least restricting $\mathbf{p}$ to a Zariski open set.
[2]Many examples of both types are classically known in the engineering literature.

Figure 2. Nine disks behaving generically. There are fifteen contacts. One cannot create any more contacts. One cannot deform the configuration without breaking a contact.

(a) Main result

Our main result is a variant of Theorem 1.3 for packings that answers both the rigidity and non-existence questions under the assumption of generic radii. To state our theorem, we need one more rigidity concept, which is a linearization of rigidity.

Definition 1.4. An *infinitesimal flex* $\mathbf{p}'$ of a d-dimensional bar-and-joint framework $(G, \mathbf{p})$ is an assignment of a vector $\mathbf{p}'_i \in \mathbb{R}^d$ to each vertex of G so that for all edges ij of G,

$$(\mathbf{p}_j - \mathbf{p}_i) \cdot (\mathbf{p}'_j - \mathbf{p}'_i) = 0.$$

There is always a $\binom{d+1}{2}$-dimensional space of *trivial* infinitesimal flexes arising from Euclidean isometries. A framework is *infinitesimally rigid* if all its infinitesimal flexes are trivial. Infinitesimal rigidity implies rigidity.

An infinitesimal flex for a disk packing is simply an infinitesimal flex of its underlying bar-and-joint framework. A packing is infinitesimally rigid if its underlying bar-and-joint framework is. ■

Since rigidity does not imply infinitesimal rigidity, infinitesimal rigidity is a more stringent condition to place on a framework than rigidity. Generically, however, the two concepts coincide [16].

Here is our main result:

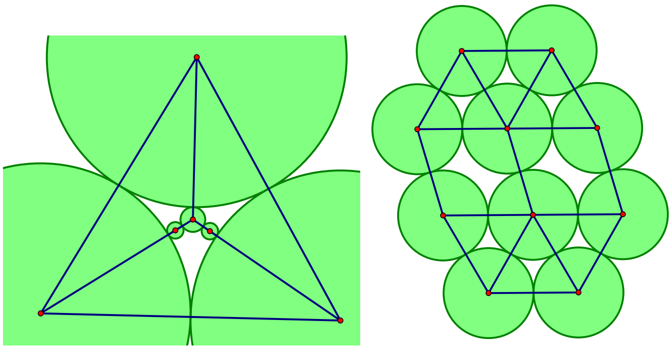
Theorem 1.5. *Let P be a packing of n disks in in the Euclidean plane $\mathbb{R}^2$ with generic radii (or even just generic radii ratios). Then the contact graph G of P has at most $2n - 3$ edges and is Laman sparse and planar. Moreover, if P has $2n - 3$ contacts, it is rigid and infinitesimally rigid. If the number of contacts is fewer, then P is flexible and infinitesimally flexible.*

Figure 2 shows a "real world" example of a packing with radii that exhibit generic behavior.

We note that the upper bound on the number of contacts does not rely on any rigidity properties of P, and indeed, a similar upper bound will appear in Section 4, even though the rigidity statement is different.

The rigidity characterization has an algorithmic consequence. Since it only requires checking the total number of contacts, generic rigidity of a graph known to be the contact graph of a disk packing with generic radii can be checked in linear time. In contrast, for general graphs, the Laman sparsity condition must be verified for all subgraphs. The best known algorithms [18,19,22] for this task have super-linear running times.

As with Theorem 1.3, genericity is essential to Theorem 1.5. If all the radii are the same, the triangular lattice packing gives a packing with more than $2n - 3$ contacts. More generally, the



(a) (b)

Figure 3. (a) This packing of sticky disks with non-generic radii has $< 2n - 3$ contacts but is rigid (See [24, Figure 8b]). (b) This packing of sticky disks with non-generic radii has $2n - 3$ contacts but is flexible.

Köbe-Andreev-Thurston Theorem [23, Theorem 13.6.2] says that any planar graph can appear as the contact graph of some packing (which necessarily has non generic radii when $m > 2n - 3$).

One can also construct non-generic examples of packings with fewer than $2n - 3$ contacts that are rigid (see Figure 3a) and at least $2n - 3$ contacts that are flexible (see Figure 3b).

In Section 4 we also provide a bonus result characterizing the number of contacts that appear in a maximally jammed packing where the boundary is formed by three touching large exterior disks.

(b) Other related work

General questions about of whether combinatorial characterizations of rigid frameworks remain valid in the presence of special geometry have been addressed before. Notably, our "packing Laman question" is similar in flavor to the "Molecular conjecture" of Tay and Whiteley [25], which was solved by Katoh and Tanigawa [26].

2. The packing manifold

To prove our theorem, we start by defining the packing manifold of a contact graph.

Definition 2.1. Let G be a graph with n vertices and m edges. We think of a disk packing $(\mathbf{p}, \mathbf{r})$ as a point in $\mathbb{R}^{3n}$. Let $S_G \subset \mathbb{R}^{3n}$ be the set of disk packings where (only) the edges in G correspond to the circle pairs that are in contact. S_G is a semi-algebraic set defined over $\mathbb{Q}$. ■

Our goal in this section is to prove that (when not empty) S_G is a smooth manifold and of the expected dimension $3n - m$.

Let P be a point of S_G. Since exactly the pairs of disks corresponding to the edges of G are in contact, the constraints defining S_G near P are m equations of the type (1.1).

We now compute the Jacobian matrix M for these constraints at P. The matrix M is m-by-$3n$, with one row per contact edge, and three columns corresponding to each vertex i of G: two for $\mathbf{p}_i$ and one for r_i.

Differentiating (1.1) with respect to the coordinates of $\mathbf{p}_i$, $\mathbf{p}_j$, r_i and r_j in turn, we find that the row corresponding to an edge ij of G has the following pattern

$$\begin{array}{ccccccccc} \cdots & \mathbf{p}_i & \cdots & \mathbf{p}_j & \cdots & r_i & \cdots & r_j \\ \Big(\cdots 0 \cdots & 2(\mathbf{p}_i - \mathbf{p}_j) & \cdots 0 \cdots & 2(\mathbf{p}_j - \mathbf{p}_i) & \cdots 0 \cdots & -2\|\mathbf{p}_i - \mathbf{p}_j\| & \cdots 0 \cdots & -2\|\mathbf{p}_i - \mathbf{p}_j\| \Big) \end{array} \tag{2.1}$$

where the first row above labels the matrix columns. Here, we used the fact that disks i and j are in contact to make the simplification

$$-2(r_i + r_j) = -2\|\mathbf{p}_i - \mathbf{p}_j\|$$

in (2.1).

To prove that S_G is smooth, we only need to show that M has rank m at every point P. We will establish this via a connection between row dependencies in M and a specific kind of equilibrium stress in planar frameworks.

Definition 2.2. A *edge-length equilibrium stress* ω of a framework $(G, \mathbf{p})$ is a non-zero vector in $\mathbb{R}^m$ that satisfies

$$\sum_j \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j) = 0 \tag{2.2}$$

$$\sum_j \omega_{ij}\|\mathbf{p}_i - \mathbf{p}_j\| = 0 \tag{2.3}$$

for each vertex $i \in V(G)$. The sums in (2.2)–(2.3) are over neighbors j of i in G.

An edge-length equilibrium stress is *strict* if it has no zero coordinates.

■

Vectors ω satisfying only (2.2) are called *equilibrium stresses* and play a fundamental role in the theory of bar-and-joint frameworks. Equation (2.3) is the new part of the definition which is relevant to packings.

Remark 2.3. W. Lam (private communication) has shown that edge-length equilibrium stresses are equivalent to the holomorphic quadratic differentials of Köbe type from [27]. In particular, for planar frameworks without boundary (in the sense of [27]), there are none.

Interestingly, the main theorem of [27] says that a planar embedded framework has an edge-length equilibrium stress if and only if a triangulation of its dual medial graph has an equilibrium stress satisfying the formally different condition $\sum_j \omega_{ij}\|\mathbf{p}_i - \mathbf{p}_j\|^2 = 0$ at each vertex. This condition is connected to orthogonal circle patterns [27] and discrete minimal surfaces [28]. ♢

We defined edge-length equilibrium stresses because they are the co-kernel vectors of M.

Lemma 2.4. *Let G be a graph, $P \in S_G$ be a packing, and $(G, \mathbf{p})$ its underlying framework. Then ω is an edge-length equilibrium stress of $(G, \mathbf{p})$ if and only if ω is in the co-kernel of the packing constraint Jacobian M.*

Proof. Equations (2.2) and (2.3) are equivalent to $\omega^t M = 0$ written down column by column. □

Next we want to show that there can be no co-kernel vector for the underlying framework of a disk packing P. We will do this by showing a stronger statement, namely that there can be no edge-length equilibrium stress for *any* bar-and-joint framework with a planar embedding. (When $m > 2n - 3$, there will always be equilibrium stresses of any framework $(G, \mathbf{p})$. However, none of them will be an edge-length equilibrium stress, satisfying Equation (2.3) when $(G, \mathbf{p})$ has a planar embedding.)

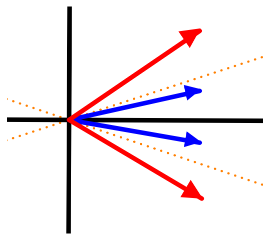


Figure 4. Illustration of the proof of Lemma 2.8. This vector configuration has only two sign changes (blue are negative signs and red are positive) and satisfies the equilibrium condition (2.2) with coefficients ± 1. This forces the sum of the lengths of the red edges to be larger than that of the blue edges, so (2.3) is violated.

Definition 2.5. We say that a framework $(G, \mathbf{p})$ is a *planar embedded framework* if all the points in the configuration $\mathbf{p}$ are distinct and correspond to the vertices of a non-crossing, straight line drawing of G in the plane. (In particular, the existence of a planar embedded framework $(G, \mathbf{p})$ implies that G is a planar graph.) ■

Definition 2.6. Given a planar embedded framework $(G, \mathbf{p})$ and a strict edge-length equilibrium stress ω, we can assign a sign in $\{+, -\}$ to each undirected edge ij using the sign of ω_{ij}. This assignment gives us a *sign vector*.

Given a sign vector on a planar embedded framework we can define the *index* I_i as the number of times the sign changes as we traverse the edges in order around vertex i. (We use the planar embedding to get the cyclic ordering of edges at each vertex.)

The index I_i is always even. ■

The following is Cauchy's index lemma, which can be proven using Euler's formula. For a proof, see e.g., [29, Lemma 5.2] or [30, Page 87].

Lemma 2.7. *Let $(G, \mathbf{p})$ be a planar embedded framework and let s be a sign vector. Then $\sum_i I_i \leq 4n - 8$. Thus there must be at least one vertex with index of either 0 or 2.*

Next, we establish the following geometric lemma.

Lemma 2.8. *Let $(G, \mathbf{p})$ be a planar embedded framework with a strict edge-length equilibrium stress vector ω. Let s be the associated sign vector and I be the associated index vector. For each vertex i, the index I_i is at least 4.*

Proof. Suppose some vertex i has fewer than 4 sign changes. If it has 0 sign changes, then it cannot satisfy Equation (2.3), as all lengths are positive. So now let's suppose that it has 2 sign changes.

With only 2 sign changes, the edges from at least one of the signs (say $-$) must be in a wedge of angle $2\theta < \pi$.

Euclidean images of $\mathbf{p}$ have the same edge-length equilibrium stresses, so we may assume that the positive part of the x-axis is the bisector of the wedge. The 2D equilibrium condition of Equation (2.2) must hold after projection along any direction, including onto the x-axis, since (2.2) is invariant under any affine transformation (see, e.g., [31]).

Let N^+ denote the neighbors of i connected by edges with positive sign and N^- the neighbors connected by negatively signed edges. Let p_i^x be the x-coordinate of the point $\mathbf{p}_i$. We then get:

$$\sum_{j \in N^+} \omega_{ij}(p_i^x - p_j^x) = \sum_{j \in N^-} -\omega_{ij}(p_i^x - p_j^x)$$

But for $j \in N^+$ (outside the wedge), we have

$$(p_i^x - p_j^x) < \cos(\theta) \, \|\mathbf{p}_i - \mathbf{p}_j\|$$

while for $j \in N^-$ (inside the wedge), we have

$$(p_i^x - p_j^x) > \cos(\theta) \, \|\mathbf{p}_i - \mathbf{p}_j\|$$

Putting these estimates together we have

$$\sum_{j \in N^+} \omega_{ij} \cos(\theta) \, \|\mathbf{p}_i - \mathbf{p}_j\| > \sum_{j \in N^+} \omega_{ij} (p_i^x - p_j^x) = \sum_{j \in N^-} -\omega_{ij} (p_i^x - p_j^x) > \sum_{j \in N^-} -\omega_{ij} \cos(\theta) \, \|\mathbf{p}_i - \mathbf{p}_j\|$$

which means that Equations (2.2) and (2.3) cannot hold simultaneously. □

See Figure 4 for an illustration of this argument. Understanding which, necessarily non-planar, frameworks can have edge-length equilibrium stresses would be interesting.

Remark 2.9. Lemma 2.8 can also be reduced to the the Cauchy-Alexandrov stress lemma (see [29, Lemma 5.2]), which says that an equilibrium stress satisfying Equation (2.2) in 3D must have at least 4 sign changes at a strictly convex vertex of a polytope.

In the reduction, we place a vertex with two sign changes at the origin and lift its neighbors from (x, y) to $(x, y, \sqrt{x^2 + y^2})$. If the 3D equilibrium equation holds (2.2) at this vertex, then both of (2.2) and (2.3) hold in the plane.

The index and stress lemmas are the two central ingredients in a proof of Cauchy's theorem on the (infinitesimal) rigidity of convex polyhedra (see [30]). ♢

Putting together the index lemma (2.7) and geometric lemma (2.8) we get:

Lemma 2.10. *Let $(G, \mathbf{p})$ be a planar embedded framework. Then it cannot have a non-zero edge-length equilibrium stress vector.*

Proof. If $(G, \mathbf{p})$ has an edge-length equilibrium stress vector, then by removing edges with 0 stress coefficients, we obtain a subframework $(G', \mathbf{p}')$ with a strict edge-length equilibrium stress vector. A strict edge-length equilibrium stress vector would form a contradiction between Lemmas 2.7 and 2.8. □

Remark 2.11. A similar statement and proof to Lemma 2.10, phrased in terms of inversive distances [32], was found independently by Bowers, Bowers and Pratt [33]. ♢

We are ready to prove the main result of this section.

Proposition 2.12. *The set S_G, if not empty, is a smooth submanifold of dimension $3n - m$.*

Proof. Let P be a point in S_G. It has the edges of G in contact and no other pairs of disks in contact in a sufficiently small neighborhood. Thus restricted to this neighborhood, S_G is exactly defined by the contact constraints of Equation 1.1. Meanwhile, the contact graph creates a planar embedded framework. From Lemmas 2.4 and 2.10 there can be no co-kernel vector, and hence the Jacobian matrix has rank m. By the implicit function theorem, S_G restricted to some neighborhood of P is a smooth manifold of dimension $3n - m$. □

Remark 2.13. The Köbe-Andreev-Thurston theorem implies that S_G is non-empty provided that G is planar. (See [23, Theorem 13.6.2].) ♢

3. Finishing the Proof

Definition 3.1. Let π be the projection $(\mathbf{p}, \mathbf{r}) \mapsto \mathbf{r}$ taking a disk packing to its vector of radii in $\mathbb{R}^n$. ■

Definition 3.2. A *π-kernel* vector of a disk packing P with contact graph G is a tangent vector to S_G of the form $(\mathbf{p}', 0)$. These vectors form the kernel of the linearization of the map π. ■

Lemma 3.3. *The π-kernel vectors of a disk packing $(\mathbf{p}, \mathbf{r})$ with contact graph G are infinitesimal flexes of the underlying bar-framework $(G, \mathbf{p})$.*

Proof. As in the proof of Proposition 2.12, we have the m-by-$3n$ Jacobian matrix M with rank m. Tangent vectors to S_G are thus (right) kernel vectors $(\mathbf{p}', \mathbf{r}')$ of M. From Equation (2.1), the tangent vectors $(\mathbf{p}', \mathbf{r}')$ to S_G are exactly the vectors satisfying

$$(\mathbf{p}_j - \mathbf{p}_i) \cdot (\mathbf{p}'_j - \mathbf{p}'_i) = (r_j + r_i)(r'_j + r'_i). \tag{3.1}$$

π-kernel vectors are defined as the tangent vectors with $\mathbf{r}' = 0$, giving us

$$(\mathbf{p}_j - \mathbf{p}_i) \cdot (\mathbf{p}'_j - \mathbf{p}'_i) = 0.$$

Thus π-kernel vectors of M are exactly the infinitesimal flexes of the underlying framework $(G, \mathbf{p})$. □

We now recall a standard definition and result from differential topology.

Definition 3.4. Let X and Y be smooth manifolds of dimension m and n, respectively. A smooth map $f : X \to Y$ is a *submersion* at a point $x \in X$ if the linearization $\mathrm{d}f_x : T_x X \to T_{f(x)} Y$ at x is surjective. A point $x \in X$ is called a *regular point* of f if f is a submersion at x; otherwise x is a *critical point* of f.

A point $y \in Y$ is called a *regular value* of f if $f^{-1}(y)$ consists of only regular points (or is empty); otherwise y is a *critical value*. ■

The following is the semi-algebraic version of Sard's theorem.

Theorem 3.5. *Let X and Y be smooth semi-algebraic manifolds of dimensions m and n defined over $\mathbb{Q}$ and $f : X \to Y$ a rational map. Then the critical values of f are a semi-algebraic subset of Y, defined over $\mathbb{Q}$, and of dimension strictly less than n.*

Proof. That the critical values are semi-algebraic and of lower dimension is [34, Theorem 9.6.2]. That the field of definition does not change follows from the fact that the critical points lie in a semi-algebraic subset defined over $\mathbb{Q}$ (by the vanishing of a determinant), and then the critical values do because quantifier elimination preserves field of definition [35, Theorem 2.62]. □

Lemma 3.6. *Let $\mathbf{r}$ be a a generic point in $\mathbb{R}^n$ and let $P := (\mathbf{p}, \mathbf{r})$ be a disk packing with m contacts and contact graph G. Then the linear space of π-kernel vectors is of dimension $2n - m$.*

Moreover, the set of packings with radii $\mathbf{r}$ and contract graph G form a smooth, semi-algebraic manifold of the same dimension $2n - m$.

Proof. Because P exists, S_G is non-empty, and so, by Proposition 2.12, is a smooth semi-algebraic manifold of dimension $3n - m$. The map $\pi : S_G \to \mathbb{R}^n$ is a polynomial map, so Theorem 3.5 applies, making all the critical values of π non-generic.

Since $\mathbf{r}$ is generic, it must be a regular value of π. Hence P is a regular point, and the linearization of π at P has rank n. Its kernel then has dimension $(3n - m) - n = 2n - m$.

Finally, the set of packings with radii $\mathbf{r}$ and contact graph G is simply $\pi^{-1}(\mathbf{r})$. The preimage theorem (see, e.g., [36, p. 21]), implies that $\pi^{-1}(\mathbf{r})$ is smooth and of the same dimension as the kernel of $\mathrm{d}\pi_P$. □

The rank of $\mathrm{d}\pi_P$ is never larger than the dimension of S_G, which is $3n - m$. If the dimension of S_G is less than n, $\mathrm{d}\pi_P$ cannot be surjective for any P in S_G, making every point in S_G a critical point. Hence, for $\mathbf{r}$ to be generic, we must have $m \leq 2n$. We will improve the preceding bound on m momentarily.

Remark 3.7. The proof of Lemma 3.6 shows that $\mathbf{r}$ does not need to be generic for the conclusion to hold. It just needs to be a regular value of π. By Theorem 3.5, the regular values of π are a Zariski open subset of $\mathbb{R}^n$, defined over $\mathbb{Q}$. ◊

We can now prove our main result.

Proof of Theorem 1.5. Since we are in dimension $d = 2$, there is a 3-dimensional space of trivial infinitesimal motions of any framework $(G, \mathbf{p})$.

The packing $P := (\mathbf{p}, \mathbf{r})$ has $\mathbf{r}$ generic by hypothesis. Lemma 3.6 then implies that the space of π-kernel vectors has dimension $2n - m$. The presence of a 3-dimensional space of trivial infimitesimal motions then implies that $2n - 3 \geq 3$, so $m \leq 2n - 3$. The same argument, applied to each subgraph of G, shows that G is Laman-sparse.

If G has $m = 2n - 3$ edges, then Lemmas 3.3 and 3.6 imply that $(G, \mathbf{p})$ has only a 3-dimensional space of infinitesimal flexes and is thus infinitesimally rigid and hence rigid. Otherwise $m < 2n - 3$, and so the $\pi^{-1}(\mathbf{r})$ has dimension at least 4 by Lemma 3.6. As the space of frameworks related to $(G, \mathbf{p})$ by rigid body motions is only 3-dimensional, it follows that $(G, \mathbf{p})$ is flexible. A flexible framework is always infinitesimally flexible.

Since rigidity properties are invariant with respect to global scaling, we only need genericity of the radii ratios. □

4. Isostatic Jamming

In this section we use the technology developed above to prove a result about "jammed packings". In light of [1], the fact that this can be proven is not surprising. Our contribution is to show how the elementary methods we used to treat sticky disks adapt easily to jamming questions.

Rigidity preliminaries for jamming Loosely speaking we consider a set of disks in the plane with a fixed set of radii. We don't allow the disks to ever overlap. The disks are not sticky. Now we suppose that there is some boundary shape surrounding these disks that is uniformly shrinking. The boundary will start pressing the disks together, and eventually the boundary can shrink no more. At this point in the process, we will say that the packing is "locally maximally dense".

An equivalent way to study this problem is to assume that the boundary shape stays fixed and that the disk radii are all scaling up uniformly, maintaining their radii ratios. We will use this interpretation.

The literature considers a number of different boundary shapes, including convex polyhedra (e.g., [7]) and flat tori (e.g., [1,12]). Here, we consider a boundary formed by three large touching exterior disks. This kind of boundary, which we will call a "tri-cusp", has the advantage that it can be modeled by the same type of constraints as those on the interior disks.

Definition 4.1. A *packing inside of a tri-cusp* is a disk packing in the plane where the first three disks are in mutual contact, and the remaining $n - 3$ "internal" disks are in the interior tri-cusp shape bounded by these first 3 disks. The packing will have some contact graph G that includes the triangle $\{1, 2, 3\}$. See Figure 5. ■

To represent the internal radii ratios we define, for $j = 4, 5, \ldots, n - 1$ the ratio $\bar{r}_j := r_j / r_n$.

Definition 4.2. We say that a disk packing in a tri-cusp is *locally maximally dense* if there is no nearby tri-cusp packing with the same $\{r_1, r_2, r_3, \bar{r}_4, \ldots, \bar{r}_{n-1}\}$ that has a higher area of coverage within the tri-cusp. The outer three disks must maintain contact. ■

Local maximal density can be studied using a notion related to infinitesimal rigidity.

Definition 4.3. Given a disk packing in a tri-cusp $(\mathbf{p}, \mathbf{r})$, with contact graph G, an *infinitesimal tensegrity flex* is a vector $\mathbf{p}'$ that satisfies

$$(\mathbf{p}_j - \mathbf{p}_i) \cdot (\mathbf{p}'_j - \mathbf{p}'_i) = 0, \tag{4.1}$$

on the three edges of the outer triangle (see Figure 5) and satisfies

$$(\mathbf{p}_j - \mathbf{p}_i) \cdot (\mathbf{p}'_j - \mathbf{p}'_i) \geq 0, \tag{4.2}$$

on the rest of the edges.

Any infinitesimal flex, in the sense of Definition 1.4, is also an infinitesimal tensegrity flex. We say that the packing is *infinitesimally collectively jammed* if the only infinitesimal tensegrity flexes $\mathbf{p}'$ are are trivial infinitesimal flexes of $(G, \mathbf{p})$. There is always a 3-dimensional space of trivial infinitesimal flexes. ■

Remark 4.4. The terminology "tensegrity flex" comes from the fact that a vector $\mathbf{p}'$ satisfying Equations (4.1)–(4.2) is an infinitesimal flex of a tensegrity structure where the edges of outer triangle are fixed-length bars, and all of the internal edges are "struts" that can can increase, but not decrease, their length during a flex. (See the notes [9] for a detailed treatment of tensegrities.) ♢

These two notions are related by the following theorem of Connelly [6].

Theorem 4.5. *If a packing in a tri-cusp is infinitesimally collectively jammed then it is locally maximally dense. If a packing in a tri-cusp is locally maximally dense, then there is a sub-packing that is infinitesimally collectively jammed.*

When the tri-cusp packing is locally maximally dense, then the maximal infinitesimally collectively jammed sub-packing forms a "spine" in which none of the disks can expand, even non-uniformly. Disks that are not part of the spine are called "rattlers" in the literature. Rattlers can expand in any desired way.

(a) Bonus result: finite isostatic theorem

The main result of this section is the following "isostatic" theorem:

Theorem 4.6. *Let $P := (\mathbf{p}, \mathbf{r})$ be a packing in a tri-cusp. Suppose that the vector $\bar{\mathbf{r}} := \{r_1, r_2, r_3, \bar{r}_4, \ldots, \bar{r}_{n-1}\}$ is generic in $\mathbb{R}^{n-1}$. Then P cannot have more than $2n-2$ contacts. If P is infinitesimally collectively jammed then it has exactly $2n-2$ contacts.*

The genericity assumption in Theorem 4.6 is only on $\bar{\mathbf{r}}$ (as opposed to $\mathbf{r}$ in Theorem 1.5), so the relative scale between the inner disks and the outer three need not be generic. The weaker genericity hypothesis will allow for the possibility of one (and only one) extra contact in the packing. The upper bound of $2n-2$ contacts does not depend on any notions of density or jamming. When there are fewer than $2n-2$ contacts, the theorem says then there must exist a a non-trivial infinitesimal tensegrity flex, thus precluding infinitesimal collective jamming.

Combining Theorems 4.6 and 4.5, we obtain:

Corollary 4.7. *Let $P := (\mathbf{p}, \mathbf{r})$ be a locally maximally dense packing in a tri-cusp with the vector $\bar{\mathbf{r}} := \{r_1, r_2, r_3, \bar{r}_4, ..., \bar{r}_{n-1}\}$ generic in $\mathbb{R}^{n-1}$. Then P has a sub-packing in a tri-cusp P' of n' disks with $2n'-2$ contacts among them that is infinitesimally collectively jammed.*

Proof of Theorem 4.6. Let us redefine S_G, in this section only, to be the set of tri-cusp packings with a fixed contact graph G. As above (Proposition 2.12), S_G (if not empty) is smooth and of dimension $3n-m$.

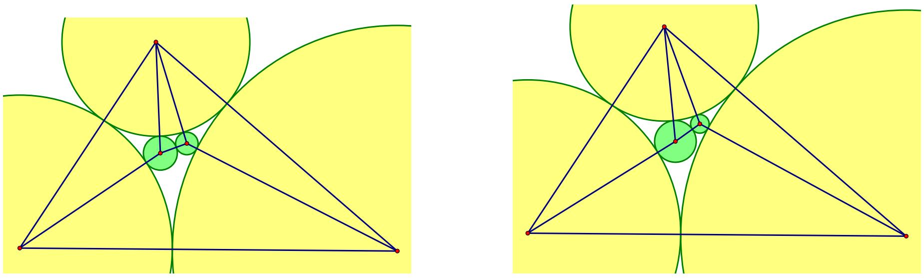


Figure 5. (a) An infinitesimally collectively jammed packing in a tri-cusp with generic $\bar{\mathbf{r}}$. The outer triangle consists of the three large yellow disks, and there are 2 green interior disks, so $n = 5$. This packing has $8 = 2n - 2$ contacts. (b) This packing in a tri-cusp also has generic $\bar{\mathbf{r}}$ and $2n - 2$ contacts but it is not infinitesimally collectively jammed or even maximally dense. The lower green disk can move slightly to the right and down, immediately breaking one contact, and then allow both green disks to expand.

Let us redefine our projection π, in this section only, to be the map from $\mathbb{R}^{3n}$ to $\mathbb{R}^{n-1}$, that maps $(\mathbf{p}, \mathbf{r})$ to $\{r_1, r_2, r_3, \bar{r}_4, \ldots, \bar{r}_{n-1}\}$. With the new definition of π, the preimage $\pi^{-1}(\bar{\mathbf{r}})$ in S_G corresponds to packings with contact graph G, where the outer three disks are fixed (up to a Euclidean isometry) and the internal radii ratios are fixed.

Next, we redefine a π-kernel vector at $(\mathbf{p}, \mathbf{r}) \in S_G$ to be a tangent vector of S_G that is in the kernel of the linearization of π. Following the proof of Lemma 3.6 above, except with an image of dimension $n-1$ instead of n, we see that the space of π-kernel vectors above a generic (and so regular) value in $\mathbb{R}^{n-1}$ will be of dimension $2n - m + 1$. Since the space of π-kernel vectors above any point contains a 3-dimensional subspace of trivial motions, we obtain $2n - m + 1 \geq 3$. Hence $m \leq 2n - 2$, giving us the first statement.

Let us now explore the implications of $m < 2n - 2$, which means that the π-kernel is of dimension at least 4. We note for later that $\partial \bar{r}_j / \partial r_j = 1/r_n$ and $\partial \bar{r}_j / \partial r_n = -r_j / r_n^2$. Also, recall, from Equation (3.1), that $(\mathbf{p}', \mathbf{r}')$ is tangent to S_G iff it satisfies

$$(\mathbf{p}_j - \mathbf{p}_i) \cdot (\mathbf{p}'_j - \mathbf{p}'_i) = (r_j + r_i)(r'_j + r'_i).$$

For a tangent vector $(\mathbf{p}', \mathbf{r}')$ to be a π-kernel vector, it must additionally satisify, for $j = 1, 2, 3$:

$$r'_j = 0$$

and, for $j = 4, \ldots, n$:

$$r'_j / r_n = r'_n (r_j / r_n^2) \tag{4.3}$$

$$r'_j r_n = r'_n r_j \tag{4.4}$$

Equations (4.3) and (4.4) imply that the r'_j are either all non-positive or all non-negative. After negating, if necessary, we conclude that, if $(\mathbf{p}', \mathbf{r}')$ is a π-kernel vector, then $\mathbf{p}'$ is an infinitesimal tensegrity flex. (Note that the converse is not true. An infinitesimal tensegrity flex vector $\mathbf{p}'$ does not necessarily have a corresponding π-kernel vector. This ties into the discussion of Section (b), below.)

Thus if the π-kernel at $(\mathbf{p}, \mathbf{r})$ is of dimension at least 4, then we will be able to find a non-trivial infinitesimal tensegrity flex. Hence, P is not infinitesimally collectively jammed. □

Remark 4.8. The lower bound aspect of Theorem 4.6, namely that infinitesimal collective jamming requires at least $2n - 2$ contacts, is already established in [7]. The results in [7] do not require genericity and generalize to higher dimensions. ◊

(b) No converse

There is a major difference between Theorems 1.5 and 4.6 in that simply having generic radii and $2n-2$ contacts does not guarantee that a packing in a tri-cusp is infinitesimally collectively jammed or even locally maximally dense. Figure 5(b) shows an example.

Such examples show an essential difference between the inequalities in the jamming setup and the equalities defining the sticky behavior of sticky disks. In fact, if the disks in Figure 5(b) are forced to be sticky (but are still allowed to grow uniformly), then the example becomes rigid and infinitesimally rigid in an appropriate sense.

Interestingly, the papers [12–14] that partially motivated our interest in these problems use counting to analyze simulated jammed packings, essentially treating them as if the disks were sticky.

(c) Relationship to the flat-torus isostatic theorem

The isostatic theorem with a flat torus as the container was proven in [1] using very general results of Guo [37] on circle patterns in piecewise-flat surfaces. Such methods could also be applied to the tri-cusp setting.

It would be interesting to know if our methods can be extended to the torus. Such an extension would require us to show that the corresponding S_G-set of n disks on a a flat torus (with flexible metric and fixed affine structure) is smooth and of the expected codimension, m. We don't know how to do that, and establishing the analogue of Lemma 2.10 for frameworks embedded in a torus might be interesting in its own right.

5. Open Problems

(a) Existence

Our paper proves certain properties about disk packings with generic radii. There certainly are classes of planar Laman graphs for which we can build packings with generic radii. For a simple example, starting with a triangle, we can sequentially add on disks on the exterior of the packing, each time using generic radii and adding two contacts. But importantly, we do not know about the existence of such packings for *all* planar Laman graphs. As in Remark 2.13, we can see that if G is any planar graph, then there must exist some disk packings with contact graph G. But this reasoning does not tell us about the genericity of the resulting radii.

Question 5.1. *Is the following claim true? Let G be planar and Laman. Then there is a packing P with contact graph G that has generic radii.*

If there is a packing with generic radii, then there will at least be an open ball of radii that can be used with G.

The generalization of Question 5.1 to ball packings in three dimensions with contact graphs that are "$3n-6$ sparse" (i.e., satisify the generalization of Maxwell's counting heuristic to dimension 3) appears to be false. The double banana graph [38, Figure 2] can appear as the packing graph of 8 balls, but it seems that such a packing will need carefully selected radii. Of course the double banana is not a generically isostatic graph. So in 3D, we can weaken the existence question and only consider contact graphs that are generically isostatic.

Here is an even stronger claim:

Question 5.2. *Is the following claim true? Let P be any disk packing where its contact graph G is planar and Laman. Then there is a nearby packing P' with generic radii and the same contact graph.*

We can give a partial answer.

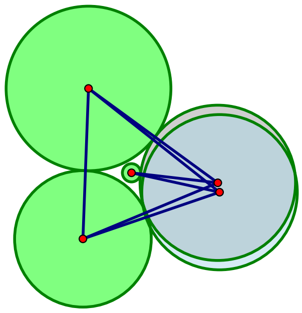


Figure 6. This disk configuration has two identical disks (separated slightly for visualization. In any nearby configuration with the same contact graph, the two gray disks must be identical. Notably, this is not an example of a packing.

Proposition 5.3. *Assuming that P is infinitesimally rigid, the answer to Question 5.2 is "yes".*

Proof sketch. If $(G, \mathbf{p})$ is an infinitesimally rigid framework of a Laman graph, then the vector $\mathbf{l}$ of m edge lengths of $(G, \mathbf{p})$ is a regular point of the map that measures edge lengths. The constant rank theorem (see, e.g., [39, Theorem 9.32]) then implies that there is a neighborhood N of $\mathbf{l}$ in $\mathbb{R}^m$ consisting of edge length measurements arising from frameworks close to $(G, \mathbf{p})$.

Next let L be the linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ defined by $L_{ij}(\mathbf{r}) := (r_i + r_j)$, where ij ranges over the edges of G. The image of L is a linear space. (Restricted to packing with contact graph G, the map L measures the edge lengths of pairs of disks in contact, but we want the more general setting.)

By assumption, $P = (\mathbf{p}, \mathbf{r})$ is infinitesimally rigid. Hence, N and $\mathbf{l}$ as in the first paragraph are defined for the underlying framework $(G, \mathbf{p})$. Since $L(\mathbf{r}) = \mathbf{l}$, the image of L intersects the interior of N. Call this intersection N'.

For a sufficiently small perturbation $\mathbf{r}'$ of $\mathbf{r}$, we have $L(\mathbf{r}')$ in N'. This means there is a $\mathbf{p}'$ close to $\mathbf{p}$ so that $(G, \mathbf{p}')$ has edge lengths $L(\mathbf{r}')$. By picking $\mathbf{r}'$ close enough to $\mathbf{r}$, $\mathbf{p}'$ can be made close enough to $\mathbf{p}$ to guarantee that $(\mathbf{p}', \mathbf{r}')$ is a packing with contact graph G (i.e., with no new contacts or disk overlaps).

Since any neighborhood of $\mathbf{r}$ contains a generic $\mathbf{r}'$, we obtain a nearby packing with generic radii. □

Remark 5.4. If P is not infinitesimally rigid, the proof of Proposition 5.3 above fails at the first step. Without infinitesimal rigidity, the neighborhood N may not exist, in which case the image of L could be tangent to the set of achievable lengths at $\mathbf{l}$. ◇

The answer to Question 5.2 is "no", if we relax the packing-overlap constraint, as shown in Figure 6, even though all the disk contacts are external.

(b) Removing Inequalities

The results of this paper rely on the fact that the underlying framework is a planar embedding, which is guaranteed by the packing inequalities.

In the *algebraic setting*, we ignore the packing inequalities and enforce only the equality constraints $\|\mathbf{p}_i - \mathbf{p}_j\|^2 = (r_i + r_j)^2$ on the edges of our graph G. The analogous object to S_G in the algebraic setting is an algebraic variety (as opposed to a semi-algebraic set).

Question 5.5. *In the algebraic setting, do the results of Theorem 1.5 still hold?*

(c) Three Dimensions

The topics of this paper can be considered in three dimensions, where disks are replaced with balls. The natural target contact number would then become $3n - 6$.

Question 5.6. *Do the results of Theorem 1.5 generalize to three dimensions?*

The authors of [10] conjecture that Question 5.6 has a positive answer.

Interestingly, it conceivable that the Maxwell counting heuristic is sufficient for generic rigidity for generic radius ball packing. Maxwell counting is not sufficient for bar frameworks, with that pesky double banana as a counter example. But it appears that the double banana cannot appear as the contact graph of a ball packing with generic radii.

Ethics. There are no relevant ethical considerations.

Data Accessibility. This article has no additional data.

Authors' Contributions. RC, SJG, and LT devised and conducted the research. RC, SJG, and LT wrote the paper.

Competing Interests. The authors have no competing interests.

Funding. RC is partially supported by NSF grant DMS-1564493. SJC is partially supported by NSF grant DMS-1564473.

Acknowledgements. We thank Meera Sitharam and Wai Yeung Lam for helpful feedback on an earlier version of this paper. The anonymous referees made a number of suggestions that improved the exposition.

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